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REFLECTION PRINCIPLE FOR FINITE-VELOCITY RANDOM MOTIONS

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Abstract

We present a reflection principle for a wide class of symmetric random motions with finite velocities. We propose a deterministic argument which is then applied to trajectories of stochastic processes. In the case of symmetric correlated random walks and the symmetric telegraph process, we provide a probabilistic result recalling the classical reflection principle for Brownian motion, but where the initial velocity has a crucial role. In the case of the telegraph process we also present some consequences which lead to further reflection-type characteristics of the motion.

Keywords: Reflection principle; telegraph process; random walks; distribution of the maximum; induction principle

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Secondary 60K99

1. Introduction

Random motions with finite velocity are a wide class of stochastic processes that preserve the natural property of moving with finite speed along the same direction. Random walks represent the discrete-time motions within this class, and they describe the position of a particle which, at each unit of time, randomly either continues at its present velocity or changes direction. On the other hand, a one-dimensional continuous-time random motion with finite speed describes the position of a particle moving on the line alternating its velocity at random times. In this paper we restrict ourselves to the case of processes having a particular 'geometric' symmetry, 'reflection-invariant' motions, in other words, where the reflection of a sample path having positive probability (density or mass) leads to another trajectory having a positive probability (density or mass). We study reflection-invariant processes behaving as follows: in the discrete case the particle performs jumps of constant size (without any loss of generality we consider unitary jumps) and in the continuous-time case the particle moves with a constant absolute speed, i.e. the possible velocities are $\pm c$, where $c > 0$.

Among the discrete processes, we focus on correlated random walks, interest in which can be traced back to [20]. The probability distribution of each step of these processes depends on the direction of the previous step. Later, the works [15] and [19] developed some generalizations, also in multidimensional spaces. Further results were presented in [4], [24], [41], and [44], where motions in presence of different types of barrier appeared. These random walks have several applications, for instance in physics [25], biology [40], for animal diffusion [42], chemistry [16], and finance [21].

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The (one-dimensional) symmetric telegraph process is the prototype of continuous-time finite-speed random motions, and was formally presented in [20]. It describes the position of a particle starting from the origin $x=0$ at time $t=0$ and moving forwards and backwards on the real line. It moves alternately with two finite constant velocities $+c$ and $-c$, where $c>0$, and the initial speed is uniformly chosen between the two possible alternatives. The changes of direction are paced by a homogeneous Poisson process $N=\{N(t)\}_{t\geq 0}$ of rate $\lambda>0$, meaning that the displacements of the particle are exponentially distributed with average length c/λ. Denoting the initial velocity of the motion by $V_0\sim \text{Unif}\{-c,+c\}$, we can define the symmetric telegraph process $\{\mathcal{T}(t)\}_{t\geq 0}$ as follows:

$$\mathcal{T}(t):=V_0\int_0^t(-1)^{N(s)}\,\mathrm{d}s=V_0\sum_{i=0}^{N(t)-1}(T_{i+1}-T_i)(-1)^i+V_0(-1)^{N(t)}(t-T_{N(t)}),\tag{1.1}$$

where T_i is the ith arrival time of the Poisson process, for $i\in\mathbb{N}$, and $T_0=0$ a.s.

Several authors have studied the symmetric telegraph process (1.1) (see e.g. [26] and [34]) as well as its generalizations. The most common one is asymmetric motion with velocities c_1 and $-c_2$, $c_1\neq c_2>0$, and two possible rates of reversal $\lambda_1\neq\lambda_2>0$, whose probability law was first obtained in [3] and has been further investigated in many papers, for instance [13], [31], and [43]. We also recall the motion with Erlang-distributed displacements (see [12]), and the motion describing a particle that uniformly chooses its velocity in the continuous set $[-c,c]$ (see [10]). Note that all these extensions belong to the class of finite-velocity random motions, but when there are more than two velocities or these are different in absolute values, we have ‘geometric’ asymmetry; the results of this paper do not apply to stochastic motions with this kind of asymmetry (in the case of a motion whose velocity takes value in $[-c,c]$ the process has some interesting symmetry properties, but our method cannot be applied to it).

Some researchers also undertook the study of telegraph-type processes in higher dimensions (see [27]). Among others, we recall [8], [36], and [37], where the authors study planar motions with orthogonal directions, [28], regarding the planar random motion with infinite possible directions, and [35], concerning random motions with finite speed in spaces of dimension $n\geq 3$.

Telegraph-type processes are suitable for describing real motions and they occur in several fields: in physics (see [33]), in finance, where they can model the stock prices or the volatility of financial markets (see [14], [29], and [38]), and in ecology, where they model the displacements of wild animals on the land (see [22]). Hence finite-speed random processes represent a realistic alternative to the widely used diffusion processes.

In this paper we present a reflection principle for finite-velocity random motions with geometric symmetry (as described above). In Section 2 we use a reflection argument to prove the bijection between the following sets of trajectories of a finite-speed random motion $X=\{X(t)\}_{t\geq 0}$ moving with velocities $\pm c$:

$$\Big\{\omega\in\Omega\colon s\mapsto X(\omega,s)\text{ s.t. }V_0(\omega)=+c,\ \max_{0\leq s\leq t}X(\omega,s)>\beta,\ X(\omega,t)=x\Big\}$$

$$\text{and}\quad\{\omega\in\Omega\colon s\mapsto X(\omega,s)\text{ s.t. }V_0(\omega)=+c,\ X(\omega,t)=x-2\beta\},\tag{1.2}$$

where $t>0$, $-ct<x<\beta$ and $(\Omega,\mathcal{F},\{\mathcal{F}_t\}_{t\geq 0},\mathbb{P})$ is a filtered probability space such that X is adapted to the filtration. By means of the above relationship concerning the sample paths, in Section 3 we present a probabilistic result for the telegraph process resembling the reflection

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principle for Brownian motion. The well-known classical reflection principle first appeared in [1] and was later presented with a more rigorous treatment, for example in [30]. Also, in the last few years some extensions of the reflection principle have been proposed, for instance regarding Lévy processes (see [2]) and hyperbolic diffusions (see [23]).

We are able to suitably adapt our reflection result to discrete processes as well. We point out that the interesting work [21] recently presented another reflection principle for symmetric correlated random walks.

As shown in (1.2), the initial velocity V_0 represents one of the main differences between the classical reflection and the one presented here. In fact, unlike diffusion processes, the velocity of the first displacement has a crucial role in finite-speed random motions. By simply observing definition (1.1) of the telegraph process, it is clear that the initial velocity is important in the probabilistic analysis of the motion. Remember that the transition density of the telegraph process is a solution to the telegraph equation

$$\frac{\partial^2 u}{\partial t^2} + 2\lambda \frac{\partial u}{\partial t} = c^2 \frac{\partial^2 u}{\partial x^2},$$

which is a hyperbolic differential equation and requires two initial conditions to be solved: the initial position and the initial velocity of the particle (whereas for a standard parabolic diffusion equation we only need the starting position).

We conclude Section 3 by showing some consequences of the reflection principle for the telegraph process. In particular, we focus on some interesting relationships arising for the conditional distributions of the telegraph process at time $t > 0$ and its maximum up to t. In the case of a negative initial velocity, we show another reflection property for the motion.

1.1. Intuition for the reflection principle

The intuition for the reflection principle arises from the results stated in Theorems 3.1 and 3.2 of [6] concerning the conditional distribution of the maximum of the symmetric telegraph process, $\mathcal{T} = \{\mathcal{T}(t)\}_{t\geq 0}$: for $n \in \mathbb{N},\ \beta \in (0, ct)$,

$$\mathbb{P}\Big\{\max_{0\leq s\leq t} \mathcal{T}(s) \in \mathrm{d}\beta \mid V_0 = c,\ N(t) = n\Big\} = 2\,\mathbb{P}\{\mathcal{T}(t) \in \mathrm{d}\beta \mid N(t) = n\}.$$

Now let us put $M(t) := \max_{0\leq s\leq t} \mathcal{T}(s),\ t > 0$, and consider the following notation for the conditional probability measure:

$$\mathbb{P}_n^{\pm}\{\,\cdot\,\} := \mathbb{P}\{\cdot \mid V_0 = \pm c,\ N(t) = n\} \tag{1.3}$$

for integer $n \geq 0$ and real $t > 0$. We have

$$\begin{aligned}
\mathbb{P}_n^{+}\{\mathcal{T}(t) > \beta\} + \mathbb{P}_n^{-}\{\mathcal{T}(t) > \beta\} &= 2\,\mathbb{P}\{\mathcal{T}(t) > \beta \mid N(t) = n\} \\
&= \mathbb{P}_n^{+}\{M(t) > \beta\} \\
&= \mathbb{P}_n^{+}\{M(t) > \beta,\ \mathcal{T}(t) > \beta\} + \mathbb{P}_n^{+}\{M(t) > \beta, \mathcal{T}(t) \leq \beta\}
\end{aligned}$$

for $n \in \mathbb{N},\ \beta \in [0, ct)$, and thus

$$\mathbb{P}_n^{+}\{M(t) > \beta,\ \mathcal{T}(t) \leq \beta\} = \mathbb{P}_n^{-}\{\mathcal{T}(t) > \beta\} = \mathbb{P}_n^{+}\{\mathcal{T}(t) < -\beta\}, \tag{1.4}$$

where the last equality follows from the property of the symmetric telegraph process $\mathcal{T}$ (see definition (1.1)).

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Note that if $V_0 = c$, by conditioning on $N(t) = n$, each path of the telegraph process is uniquely determined by the Poisson times $T_1, \ldots, T_n$. Their joint random variable is uniformly distributed on the simplex since N is a homogeneous Poisson process. This means that each trajectory of the process has the same probability (density). Thus relation (1.4) suggests that for each trajectory of the set on the right-hand side there exists a trajectory in the set on the left-hand side. With this at hand, it is reasonable to investigate whether there exists a one-to-one correspondence between the following sets of trajectories of the telegraph process: for $n \in \mathbb{N}$ and $\beta \in [0, ct)$,

$$\Big\{\omega \in \Omega\colon V_0(\omega) = +c,\ N(\omega, t) = n,\ \max_{0\le s\le t} \mathcal{T}(\omega, s) > \beta,\ \mathcal{T}(\omega, t) \le \beta\Big\}$$

$$\text{and} \quad \{\omega \in \Omega\colon V_0(\omega) = +c,\ N(\omega, t) = n,\ \mathcal{T}(\omega, t) < -\beta\},$$

where $(\Omega, \mathcal{F}, \{\mathcal{F}_t\}_{t\ge 0}, \mathbb{P})$ is a filtered probability space and N, $\mathcal{T}$ are adapted to the filtration.

We point out that the first paper dealing with the maximum of the telegraph process was [34], and then [17] and [18] presented explicit results for the first passage time. Recent papers concerning these issues, also in asymmetric cases, include [7], [9], [31], [32], and [39].

2. Reflection principle for finite-velocity random motions

The above intuition follows from a probabilistic argument. Here we present a more general deterministic result about reflection that we later apply to sets of trajectories of random motions with finite velocity.

Theorem 2.1. (Reflection principle.) *Let $c, t > 0$, $\beta \in (0, ct)$ and $-ct < x < \beta$. There exists a one-to-one correspondence between*

$$\begin{aligned}\mathcal{F}_{t,x,\beta} = \bigcup_{n\in\mathbb{N}} \mathcal{F}_{n,t,x,\beta} = \bigcup_{n\in\mathbb{N}} \Big\{ & f\colon [0, t] \longrightarrow (-ct, ct)\colon f(0) = 0, f(t) = x,\\ & \exists\, 0 = t_0 < t_1 < \cdots < t_{n+1} = t \text{ s.t. } f(s) = f(t_i) + (-c)^i (s - t_i),\\ & s \in [t_i, t_{i+1}],\ i = 0, \ldots, n,\ \text{and } \exists\, h \in \{1, \ldots, n\} \text{ s.t. } f(t_h) > \beta \Big\}\end{aligned} \tag{2.1}$$

and

$$\begin{aligned}\mathcal{F}_{t,x-2\beta} = \bigcup_{n\in\mathbb{N}} \mathcal{F}_{n,t,x-2\beta} = \bigcup_{n\in\mathbb{N}} \Big\{ & f\colon [0, t] \longrightarrow (-ct, ct)\colon f(0) = 0, f(t) = x - 2\beta,\\ & \exists\, 0 = t_0 < t_1 < \cdots < t_{n+1} = t \text{ s.t. } f(s) = f(t_i) + (-c)^i (s - t_i),\\ & s \in [t_i, t_{i+1}],\ i = 0, \ldots, n \Big\}.\end{aligned} \tag{2.2}$$

In words, the set $\mathcal{F}_{n,t,x,\beta}$ in (2.1) contains all the functions obtained by concatenating $n + 1$ segments which alternate the sign of the slope c, starting with a positive orientation, and such that $f(0) = 0, f(t) = x < \beta$ and $f(s) > \beta$ for some $s \in (0, t)$. The set $\mathcal{F}_{n,t,x-2\beta}$ in (2.2) contains all the functions obtained by concatenating $n + 1$ segments which alternate the sign of the slope c, starting with a positive orientation, and such that $f(0) = 0, f(t) = x - 2\beta$.

Proof. We prove that for $n \in \mathbb{N}$ there exists a bijection ($\longleftrightarrow$) between $\mathcal{F}_{n,t,x,\beta}$ and $\mathcal{F}_{n,t,x-2\beta}$. Obviously, for $n \in \mathbb{N}$, the set $\mathcal{F}_{n,t,x-2\beta}$ in (2.2) is bijective to the set

$$\mathcal{F}^-_{n,t,2\beta-x} = \{f^-\colon [0, t] \longrightarrow (-ct, ct)\colon f^- = -f, f \in \mathcal{F}_{n,t,x-2\beta}\},$$

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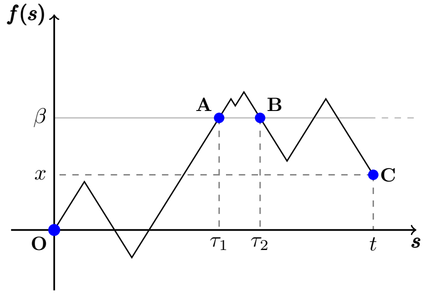

FIGURE 1. Graph of $f \in \mathcal{F}$, with $n = 7$.

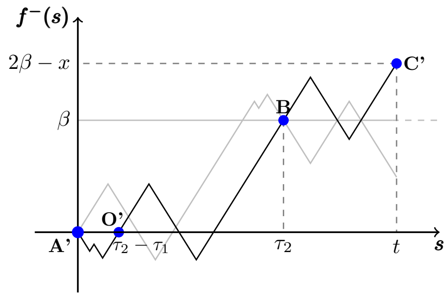

FIGURE 2. Graph of $f^- \in \mathcal{F}^-$, with $n = 7$, the negatively reflected function of f.

meaning that $f^-(t_1) = -ct_1$ and $f^-(t) = 2\beta - x$ if $f^- \in \mathcal{F}^-_{n,t,2\beta-x}$. We now prove that $\mathcal{F}_{n,t,x,\beta} \longleftrightarrow \mathcal{F}^-_{n,t,2\beta-x}$, and we call this relationship the *negative reflection principle* because of the inversion of the slope (from c to $-c$) of the first segment of the function. For the sake of simplicity, put $\mathcal{F} = \mathcal{F}_{n,t,x,\beta}$ and $\mathcal{F}^- = \mathcal{F}^-_{n,t,2\beta-x}$.

Let $f \in \mathcal{F}$; then there exists $0 < \tau_1 = \inf\{s \geq 0 : f(s) = \beta\} < \tau_2 = \inf\{s > \tau_1 : f(s) = \beta\} < t$, respectively the first and the second passage abscissa where f crosses level β. The *negatively reflected* function of f is graphically obtained as follows (see Figures 1 and 2).

(1) Reflect the graph from time τ_1 to time τ_2 across level β (see the polygonal curve from A to B in Figure 1).

(2) Shift the reflected poly-line at point 1 to the origin of the axes, so it now starts at (0,0) and ends at the point $(\tau_2 - \tau_1, 0)$ (see the sample from A′ to O′ in Figure 2).

(3) Consider the original graph in the time interval $[0, \tau_1]$ (poly-line from O to A in Figure 1), and shift it horizontally by the vector $(\tau_2 - \tau_1, 0)$, so it starts at the point $(\tau_2 - \tau_1, 0)$, i.e. where the poly-line in steps 1–2 ends, and ends at (τ_2, β) (see the poly-lines from O′ to B in Figure 2).

(4) Reflect the (original) graph in the time interval $[\tau_2, t]$ across level β (see the broken line from B to C in Figure 1 and its counterpart from B to C′ in Figure 2).

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This procedure produces one and only one negatively reflected function of $f \in \mathcal{F}$. By applying the method in reverse, we can build one and only one function in $\mathcal{F}$ starting from an element $f^- \in \mathcal{F}^-$. Notice that for f^-, $\tau_2 - \tau_1$ represents the first crossing abscissa through level 0 and τ_2 represents the first crossing abscissa through β. Figures 1 and 2, respectively, illustrate a function $f \in \mathcal{F}$ and its negatively reflected counterpart $f^- \in \mathcal{F}^-$, in the case $n = 7$.

Note that $\mathcal{F}$ and $\mathcal{F}^-$ are sets of broken lines, so their functions can be characterized by the values $f(t_i)$, $i = 1, \ldots, n+1$, where we use the notation in sets (2.1) and (2.2), meaning that t_i, $i = 1, \ldots, n+1$, are the abscissas of junction, i.e. the points where the slope changes. Then we have $\mathcal{F} \longleftrightarrow \mathcal{V}$ and $\mathcal{F}^- \longleftrightarrow \mathcal{V}^-$, where

$$\mathcal{V} = \{v \in \mathbb{R}^{n+1} : \exists f \in \mathcal{F} \text{ s.t. } v_i = f(t_i),\, i = 1, \ldots, n+1\}$$

and

$$\mathcal{V}^- = \{v \in \mathbb{R}^{n+1} : \exists f \in \mathcal{F}^- \text{ s.t. } v_i = f(t_i),\, i = 1, \ldots, n+1\},$$

with $0 = t_0 < t_1 < \cdots < t_n < t_{n+1} = t$ still being the abscissas of the junctions of the $n+1$ broken lines describing the graph of the functions (recall that n is fixed). Hence we only need to prove that $\mathcal{V} \longleftrightarrow \mathcal{V}^-$.

We now show analytically that $\mathcal{V} \longleftrightarrow \mathcal{V}^-$ and that the graphical procedure above is bijective. Let $f \in \mathcal{F}$ and let $0 < \tau_1 < \tau_2 < t$, defined as above. Then there exist integer numbers $1 \le h < l \le n$ such that $t_{h-1} < \tau_1 < t_h$ and $t_{l-1} < \tau_2 < t_l$, meaning that f crosses β for the first time along the hth segment and for the second time along the lth segment. Let $v \in \mathcal{V}$ be the vector counterpart of f. By suitably applying the constructive (graphical) method above, we obtain the negatively reflected vector $v^- \in \mathcal{V}^-$, which uniquely describes the negatively reflected function $f^- \in \mathcal{F}^-$. In detail, the vectors v, v^- are given by

$$v = \begin{pmatrix} \left.\begin{matrix} 0 < x_1 \\ \cdot \\ x_{h-1} \end{matrix}\right\} < \beta \\ \left.\begin{matrix} x_h \\ \cdot \\ x_{l-1} \end{matrix}\right\} > \beta \\ x_l < \beta \\ \cdot \\ \cdot \\ x_{n+1} = x < \beta \end{pmatrix}, \quad v^- = \begin{pmatrix} \left.\begin{matrix} y_1 = \beta - x_h \\ \cdot \\ y_i = \beta - x_{h-1+i} \\ \cdot \\ y_{l-h} = \beta - x_{l-1} \end{matrix}\right\} < 0 \\ \left.\begin{matrix} 0 < y_{l-h+1} = x_1 \\ \cdot \\ y_{l-h+j} = x_j \\ \cdot \\ y_{l-1} = x_{h-1} \end{matrix}\right\} < \beta \\ y_l = 2\beta - x_l > \beta \\ \cdot \\ \cdot \\ y_{n+1} = 2\beta - x > \beta \end{pmatrix},$$

where the elements $y_1, \ldots, y_{l-h}$ are consequences of points 1–2 in the building procedure (which work on $x_h, \ldots, x_{l-1}$), elements $y_{l-h+1}, \ldots, y_{l-1}$ are consequences of point 3 (which works on $x_1, \ldots, x_{h-1}$), and $y_l, \ldots, y_{n+1}$ are consequences of point 4 (which works on

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$x_l, \dots, x_{n+1})$. Hence we have the following bijective affine relationship $R\colon \mathbb{R}^{n+1} \longrightarrow \mathbb{R}^{n+1}$ between v and v^-:

$$v^- = R(v) = \begin{pmatrix} 0 & -I_{l-h} & 0 \\ I_{h-1} & 0 & 0 \\ 0 & 0 & -I_{n-l+2} \end{pmatrix} v + \begin{pmatrix} \beta_{l-h} \\ 0_{h-1} \\ 2\beta_{n-l+2} \end{pmatrix},$$

where I_k is the $k \times k$ identity matrix, and β_k and 0_k are k-dimensional vectors of all β and 0 respectively.

Note that f and v are characterized by the pair (h, l), which identifies the segments of the graph where the function crosses level β for the first and second time, respectively. On the other hand, the negatively reflected function of f, f^-, and its vector counterpart v^- cross level 0 for the first time along the $(l-h+1)$th segment and cross level β for the first time along the lth segment. Thus we can characterize them with the pair $(l-h+1, l)$. Now, each function $f^- \in \mathcal{F}^-$ with pair $(l-h+1, l)$, $1 \le h < l \le n$, has a unique vector form $v^- \in \mathcal{V}^-$ which has a unique negatively reflected vector $v = R^{-1}(v^-) \in \mathcal{V}$, with pair (h, l). Finally, since $g(h, l) = (l - h+1, l)$ is an automorphism, the negatively reflected vector of $v \in \mathcal{V}$, with pair (h,l), is different from the negatively reflected vector $v' \in \mathcal{V}$ with pair $(h', l') \neq (h, l)$. Therefore $\mathcal{V} \longleftrightarrow \mathcal{V}^-$, and this concludes the proof of the theorem. □

Note that the request of a constant coefficient $c > 0$ ensures a smooth junction of the segments after the reflections.

Remark 2.1. (*Reflection principle: discrete case.*) Theorem 2.1 describes a reflection involving continuous functions. However, it can be easily adapted to the case of sequences moving with unit step, i.e. $\{s_n\}_{n\in\mathbb{N}}$ such that $|s_n - s_{n+1}| = 1$ for all n. In particular, with $N \in \mathbb{N}$ and integer numbers $0 < \beta < N$, $-N < x \le \beta$, there exists a one-to-one correspondence between

$$\begin{aligned} \mathcal{S}_{N,x,\beta} = \bigcup_{n<N} \mathcal{S}_{n,N,x,\beta} = \bigcup_{n<N} \Big\{ & s\colon \{0, \dots, N\} \longrightarrow \{-N, \dots, N\}\colon s_0 = 0,\ s_N = x, \\ & \exists \text{ integers } 0 = t_0 < t_1 < \cdots < t_{n+1} = N \text{ s.t. } s_m = s_{t_i} + (-1)^i (m - t_i), \\ & t_i \le m \le t_{i+1},\ i = 0, \dots, n, \text{ and } \exists\, h \in \{1, \dots, n\} \text{ s.t. } s_{t_h} > \beta \Big\} \end{aligned} \tag{2.3}$$

and

$$\begin{aligned} \mathcal{S}_{N,x-2\beta} = \bigcup_{n<N} \mathcal{S}_{n,N,x-2\beta} = \bigcup_{n<N} \Big\{ & s\colon \{0, \dots, N\} \longrightarrow \{-N, \dots, N\}\colon s_0 = 0,\ s_N = x - 2\beta, \\ & \exists \text{ integers } 0 = t_0 < t_1 < \cdots < t_{n+1} = N \text{ s.t. } s_m = s_{t_i} + (-1)^i (m - t_i), \\ & t_i \le m \le t_{i+1},\ i = 0, \dots, n \Big\}. \end{aligned} \tag{2.4}$$

The set (2.3) includes the sequences (with unit steps) starting with a positive step ($s_1 = 1$), crossing level β before N and such that $s_N = x < \beta$. On the other hand, the set (2.4) contains the sequences (with unit steps) starting with a positive step and such that $s_N = x - 2\beta$.

Note that the condition $f(0) = 0$ in sets (2.1) and (2.2) or $s_0 = 0$ in sets (2.3) and (2.4) can be replaced by assuming a general starting level $x_0 \in \mathbb{R}$; the reflection principle holds with trivial changes in both discrete and continuous cases.

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Theorem 2.1 permits us to determine a one-to-one correspondence between the sets of sample paths of a wide class of one-dimensional stochastic motions with finite velocity. The following corollary states this result in the case of continuous-time stochastic motions.

Corollary 2.1. *Let* $(\Omega, \mathcal{F}, \{\mathcal{F}_t\}_{t\geq 0}, \mathbb{P})$ *be a filtered probability space. Let* V_0 *be a random variable taking values in* $\{\pm c\}$, $c>0$. *Let* $N=\{N(t)\}_{t\geq 0}$ *be a non-decreasing process counting isolated events (meaning that it increases by one unit at a time) and let* $X=\{X(t)\}_{t\geq 0}$ *such that* $X(t)=V_0\int_0^t(-1)^{N(s)}\,\mathrm{d}s$, $t\geq 0$, *with N and X adapted to the given filtration. There exists a one-to-one correspondence between*

$$\begin{aligned}&\mathcal{W}_{t,x,\beta}\\ &=\bigcup_{n\in\mathbb{N}}\mathcal{W}_{n,t,x,\beta}\\ &=\bigcup_{n\in\mathbb{N}}\Big\{\omega\in\Omega\colon s\mapsto X(\omega,s)\ s.t.\ V_0(\omega)=+c,\ N(\omega,t)=n,\ \max_{0\leq s\leq t}X(\omega,s)>\beta,\ X(\omega,t)=x\Big\}\end{aligned} \tag{2.5}$$

and

$$\begin{aligned}&\mathcal{W}_{t,x-2\beta}\\ &=\bigcup_{n\in\mathbb{N}}\mathcal{W}_{n,t,x-2\beta}\\ &=\bigcup_{n\in\mathbb{N}}\{\omega\in\Omega\colon s\mapsto X(\omega,s)\ s.t.\ V_0(\omega)=+c,\ N(\omega,t)=n,\ X(\omega,t)=x-2\beta\}.\end{aligned} \tag{2.6}$$

It is worth emphasizing that Corollary 2.1 concerns a bijection between sets of trajectories and says nothing about the probabilistic structure, that is, there is not really a role for the probability measure in the statement.

Note that if N is a non-exploding process, i.e. $N(t)<\infty$ a.s. for $t>0$ (meaning that the motion can perform only a finite number of switches in a finite time interval), then sets (2.5) and (2.6) coincide with the sets in (1.2).

Proof. The corollary immediately follows by observing that the sample paths of X are broken lines and the sets of trajectories $\mathcal{W}_{n,t,x,\beta}$ and $\mathcal{W}_{n,t,x-2\beta}$ respectively coincide with the sets $\mathcal{F}_{n,t,x,\beta}$ and $\mathcal{F}_{n,t,x-2\beta}$, in (2.1) and (2.2), for all $n\in\mathbb{N}$. □

Remark 2.2. Recalling Remark 2.1, it is obvious that Corollary 2.1 can be easily extended to random walks $S=\{S_n\}_{n\in\mathbb{N}_0}$ such that $S_n=S_0+\sum_{i=1}^n X_i$, where $X_i\in\{\pm 1\}$ a.s. and $S_0=0$ a.s.

As explained above, the requirement that the random processes start at level 0 at time $t=0$ can be easily generalized in both continuous and discrete cases.

Also note that Theorem 2.1 follows by performing spatial reflections only (meaning that only the space position is involved in the transformation; graphically this means that the

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reflection is applied with respect to lines parallel to the space axis), so there are no 'time reversals' in the manipulation of the sample paths.

3. Applications of the reflection principle

It is important to underline that the reflection principle presented in Corollary 2.1 and Remark 2.2 concerns trajectories only. However, in some cases we obtain equality between the probabilities of the sets (2.5) and (2.6), in particular when all the sample paths of the process have the same probability. Here we present two examples.

Example 3.1. (*Symmetric correlated random walk.*) Let $S = \{S_n\}_{n\in\mathbb{N}_0}$ be a symmetric correlated random walk starting from level 0, i.e. $S_n = S_0 + \sum_{i=1}^{n} X_i$, where $S_0 = 0$ a.s., $X_i \in \{\pm 1\}$ a.s., $i \in \mathbb{N}$, X_1 is a uniformly distributed random variable, and $\mathbb{P}\{X_{i+1} = X_i \mid X_0, \ldots, X_i\} = p \in (0, 1)$ for all i. It is well known that each trajectory of S up to a fixed time $n \in \mathbb{N}$ depends on the number of turns that occur (each one occurring with probability $1 - p$ independently from the previous movements). The inversions are given by $\sum_{i=1}^{n-1} \mathbb{1}(X_{i+1} \neq X_i)$. The probability mass of each sample path with $k < n$ switches is equal to $(1 - p)^k p^{n-k-1}/2$. Thus, with $n \geq 2$, from Remark 2.2 and Corollary 2.1 we obtain equality between the probability of the events $\{\max_{0\leq k\leq n} S_k > \beta, S_n = x\}$ and $\{S_n = x - 2\beta\}$ given that $\{S_1 = 1\}$, with $\beta \in \{0, 1, \ldots, n - 2\}$ and $x \in \{2(\beta + 1) - n, \ldots, \beta\}$.

In the case of a simple symmetric random walk, meaning when $p = 1/2$, an application of the classical reflection principle is sufficient for a direct proof of this probabilistic result. If $n \geq 2$ and x, β as above, we have

$$\begin{aligned}\mathbb{P}\Big\{\max_{0\leq k\leq n} S_k > \beta, S_n = x \mid S_1 = 1\Big\} &= \mathbb{P}\Big\{\max_{0\leq k\leq n-1} S_k > \beta - 1, S_{n-1} = x - 1\Big\} \\ &= \mathbb{P}\{S_{n-1} = 2\beta - x + 1\} = \mathbb{P}\{S_n = 2\beta - x \mid S_1 = -1\} \\ &= \mathbb{P}\{S_n = x - 2\beta \mid S_1 = 1\}.\end{aligned}$$

The interested reader can refer to the paper [21] for another reflection principle for symmetric correlated random walks.

3.1. Symmetric telegraph process

Let $\mathcal{T} = \{\mathcal{T}(t)\}_{t\geq 0}$ be a symmetric telegraph process. Let $t > 0$. From definition (1.1) we observe that by conditioning on $N(t) = n \in \mathbb{N}$ and the initial velocity V_0, each path of the motion in $[0,t]$ is uniquely determined by the Poisson times $T_1, \ldots, T_n$. It is well known that these random variables are uniformly distributed on the simplex since N is a homogeneous Poisson process and therefore each trajectory of the telegraph process has the same probability (density). Now, Corollary 2.1 is sufficient to claim that, for suitable x, β, the events $\{V_0 = c, N(t) = n, M(t) > \beta, \mathcal{T}(t) = x\}$ and $\{V_0 = c, N(t) = n, \mathcal{T}(t) = x - 2\beta\}$ have equal probability (remember notation (1.3)). The next theorem states this result rigorously and we also provide a direct derivation.

Note that in the case of a particle moving velocities $\pm c$ but with different rates of reversals, each trajectory of the process has a different probability (density) since N is not a homogeneous Poisson process. This means that Corollary 2.1 still holds, but we cannot establish a relationship concerning probabilities.

Theorem 3.1. (Reflection principle for symmetric telegraph process.) *Let* $\{\mathcal{T}(t)\}_{t\geq 0}$ *be a symmetric telegraph process. Let* $n\in\mathbb{N}$ *and* $x\in(-ct, ct)$*. Then*

$$\mathbb{P}_n^+\{M(t)\leq\beta,\ \mathcal{T}(t)\in \mathrm{d}x\}$$
$$=\begin{cases}0 & \textit{if } \beta<\max\{0,x\},\\ \mathbb{P}_n^+\{\mathcal{T}(t)\in\mathrm{d}x\}-\mathbb{P}_n^+\{\mathcal{T}(t)\in\mathrm{d}x-2\beta\} & \textit{if } \max\{0,x\}\leq\beta<(ct+x)/2,\\ \mathbb{P}_n^+\{\mathcal{T}(t)\in\mathrm{d}x\} & \textit{if } \beta\geq(ct+x)/2.\end{cases} \tag{3.1}$$

Before proving the theorem it is necessary to recall the conditional distributions of the position of the telegraph particle at time $t>0$. Let $x\in(-ct, ct)$ and $v=\pm c$. Then

$$\mathbb{P}\{\mathcal{T}(t)\in\mathrm{d}x \mid V_0=v,\ N(t)=2k\}=\frac{(2k)!}{k!(k-1)!}\frac{(c^2t^2-x^2)^{k-1}(ct+\mathrm{sign}(v)x)}{(2ct)^{2k}}\,\mathrm{d}x \tag{3.2}$$

for $k\in\mathbb{N}$, and

$$\begin{aligned}\mathbb{P}\{\mathcal{T}(t)\in\mathrm{d}x \mid V_0=v,\ N(t)=2k+1\}&=\mathbb{P}\{\mathcal{T}(t)\in\mathrm{d}x \mid N(t)=2k+1\}\\ &=\mathbb{P}\{\mathcal{T}(t)\in\mathrm{d}x \mid N(t)=2k+2\}\\ &=\frac{(2k+1)!}{k!^2}\frac{(c^2t^2-x^2)^k}{(2ct)^{2k+1}}\,\mathrm{d}x\end{aligned} \tag{3.3}$$

for $k\in\mathbb{N}_0$; see [6] and [11] for the proof of (3.2) and (3.3).

Proof. When $N(t)=1$, distribution (3.1) reduces to

$$\mathbb{P}_1^+\{M(t)\leq\beta,\ \mathcal{T}(t)\in\mathrm{d}x\}=\begin{cases}0 & \text{if } \beta<(ct+x)/2,\\ \mathbb{P}_1^+\{\mathcal{T}(t)\in\mathrm{d}x\}=\dfrac{\mathrm{d}x}{2ct} & \text{if } \beta\geq(ct+x)/2,\end{cases} \tag{3.4}$$

since the random variable $\mathcal{T}(t)$ is uniformly distributed in $(-ct, ct)$ if one Poisson event occurs in the time interval $[0,t]$; see (3.3) with $k=0$. To prove (3.4) we observe that the process reaches level β at time β/c, then changes direction and keeps moving with speed $-c$ until time t, where it will be located in $\beta-c(t-\beta/c)=2\beta-ct=x$. Thus

$$\mathbb{P}_1^+\{M(t)=(ct+\mathcal{T}(t))/2,\ \mathcal{T}(t)\in\mathrm{d}x\}=\mathbb{P}_1^+\{\mathcal{T}(t)\in\mathrm{d}x\}.$$

The first and third cases of (3.1) are trivial for $n\geq 2$. We focus on the second case, which can be written as follows: for $\beta\in[0, ct)$ and $x\in(2\beta-ct, \beta]$,

$$\mathbb{P}_n^+\{M(t)\leq\beta,\ \mathcal{T}(t)\in\mathrm{d}x\}=\mathbb{P}_n^+\{\mathcal{T}(t)\in\mathrm{d}x\}-\mathbb{P}_n^-\{\mathcal{T}(t)\in 2\beta-\mathrm{d}x\}. \tag{3.5}$$

For $n=2$, letting T_j denote the jth Poisson arrival time, we consider that

$$\mathbb{P}_2^+\{M(t)\leq\beta,\ \mathcal{T}(t)\in\mathrm{d}x\}=\int_0^{\beta/c}\mathbb{P}\left\{T_1\in\mathrm{d}t_1,\ T_2\in\frac{ct-\mathrm{d}x}{2c}+T_1 \mid N(t)=2\right\}=\frac{\beta}{c^2t^2}\,\mathrm{d}x,$$

which coincides with result (3.5) when $n=2$.

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We now prove (3.5) for all natural $n > 2$ by means of an induction argument. The case above, $n = 2$, represents the induction base for n even. Let natural $n > 3$; at time $T_2 = t_2 < t$ we have two possible scenarios (keep in mind that $\mathcal{T}(t_2) = 2ct_1 - ct_2$):

Case 1. The motion has enough time to cross level β and reach x at time t, so

$$c(t - t_2) \geq (\beta - 2ct_1 + ct_2) + (\beta - x) \quad \text{if and only if} \quad t_2 \leq (ct - 2\beta + x)/(2c) + t_1.$$

Case 2. The motion does not have enough time to cross level β but it has time to reach x at time t, so

$$t_2 > (ct - 2\beta + x)/(2c) + t_1 \quad \text{and} \quad c(t - t_2) \geq |x - 2ct_1 + ct_2|.$$

Thus we can write the following recurrence relationship: for natural $n \geq 3$,

$$\begin{aligned}
&\mathbb{P}_n^+\{M(t) \leq \beta, \mathcal{T}(t) \in \mathrm{d}x\} \\
&= \int_0^{\beta/c} \int_{t_1}^{\frac{ct-2\beta+x}{2c}+t_1} \mathbb{P}_{n-2}^+\{M(t-t_2) \leq \beta - 2ct_1 + ct_2, \mathcal{T}(t-t_2) \in \mathrm{d}x - 2ct_1 + ct_2\} \\
&\quad \times \mathbb{P}\{T_1 \in \mathrm{d}t_1, T_2 \in \mathrm{d}t_2 \mid N(t) = n\} \\
&\quad + \int_0^{\beta/c} \int_{\frac{ct-2\beta+x}{2c}+t_1}^{\frac{ct-x}{2c}+t_1} \mathbb{P}_{n-2}^+\{\mathcal{T}(t-t_2) \in \mathrm{d}x - 2ct_1 + ct_2\} \mathbb{P}\{T_1 \in \mathrm{d}t_1,\ T_2 \in \mathrm{d}t_2 \mid N(t) = n\}.
\end{aligned} \tag{3.6}$$

Let $k \in \mathbb{N}$. We assume that distribution (3.5) holds for $n = 2k$ (induction hypothesis). Hence, for $n = 2k + 2$, formula (3.6) reads

$$\begin{aligned}
&\mathbb{P}_{2k+2}^+\{M(t) \leq \beta, \mathcal{T}(t) \in \mathrm{d}x\} \\
&= \int_0^{\beta/c} \int_{t_1}^{\frac{ct-x}{2c}+t_1} \mathbb{P}_{2k}^+\{\mathcal{T}(t-t_2) \in \mathrm{d}x - 2ct_1 + ct_2\} \mathbb{P}\{T_1 \in \mathrm{d}t_1,\ T_2 \in \mathrm{d}t_2 \mid N(t) = 2k+2\} \\
&\quad - \int_0^{\beta/c} \int_{t_1}^{\frac{ct-2\beta+x}{2c}+t_1} \mathbb{P}_{2k}^-\{\mathcal{T}(t-t_2) \in 2\beta - \mathrm{d}x - 2ct_1 + ct_2\} \\
&\quad \times \mathbb{P}\{T_1 \in \mathrm{d}t_1,\ T_2 \in \mathrm{d}t_2 \mid N(t) = 2k+2\} \\
&= \mathrm{d}x \int_0^{\beta/c} \mathrm{d}t_1 \int_{t_1}^{\frac{ct-x}{2c}+t_1} \frac{(ct - x + 2ct_1 - 2ct_2)^{k-1}(ct + x - 2ct_1)^k}{[2c(t-t_2)]^{2k}} \frac{(2k+2)!}{k!(k-1)!} \frac{(t-t_2)^{2k}}{t^{2k+2}} \,\mathrm{d}t_2 \\
&\quad - \mathrm{d}x \frac{(2k+2)!}{k!(k-1)!} \int_0^{\beta/c} \mathrm{d}t_1 \int_{t_1}^{\frac{ct-2\beta+x}{2c}+t_1} \frac{(ct - 2\beta + x + 2ct_1 - 2ct_2)^k (ct + 2\beta - x - 2ct_1)^{k-1}}{(2c)^{2k} t^{2k+2}} \,\mathrm{d}t_2 \\
&= \frac{(2k+2)!}{(k+1)!\,k!} \frac{\mathrm{d}x}{(2ct)^{2k+2}} \Big[\big(c^2t^2 - x^2\big)^k (ct + x) - \big[c^2t^2 - (2\beta - x)^2\big]^k [ct - (2\beta - x)] \Big] \\
&= \mathbb{P}_{2k+2}^+\{\mathcal{T}(t) \in \mathrm{d}x\} - \mathbb{P}_{2k+2}^-\{\mathcal{T}(t) \in 2\beta - \mathrm{d}x\},
\end{aligned}$$

which concludes the proof of the theorem for n even.

In the case of n odd the proof works in the same way and therefore it is omitted. We note only that in formula (3.6) for $n = 3$, by considering (3.4), the first term is equal to 0 and it is trivial to see that (3.5) holds when $n = 3$. This is the induction base in the case of n odd. □

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Theorem 3.1 permits us to obtain the joint distributions of the telegraph process at time $t > 0$ and its maximum up to t under all the possible conditions of both the initial speed and the number of switches in $[0, t]$. Below we report some of the main consequences of the reflection principle for the symmetric telegraph process.

The joint density immediately follows by using the fact that, for natural $n \geq 2$, $\beta \in (0, ct)$ and $x \in (2\beta - ct, \beta)$, we have

$$\mathbb{P}_n^+\{M(t) \in \mathrm{d}\beta,\ \mathcal{T}(t) \in \mathrm{d}x\}/\mathrm{d}\beta = \frac{\partial}{\partial \beta}\mathbb{P}_n^+\{M(t) \leq \beta, \mathcal{T}(t) \in \mathrm{d}x\} = -\frac{\partial}{\partial \beta}\mathbb{P}_n^+\{\mathcal{T}(t) \in \mathrm{d}x - 2\beta\}.$$

Note that the case of $n = 1$ was derived at the beginning of the proof of Theorem 3.1.

It is interesting to observe that if the number of changes of direction is even, the last displacement is positively oriented and it may happen that the maximum coincides with the ending position. In this case, for $k \in \mathbb{N}$ and $\beta \in (0, ct)$, we have

$$\begin{aligned}\mathbb{P}_{2k}^+\{M(t) = \mathcal{T}(t) \in \mathrm{d}\beta\} &= \mathbb{P}_{2k}^+\{\mathcal{T}(t) \in \mathrm{d}\beta\} - \mathbb{P}_{2k}^+\{M(t) > \beta, \mathcal{T}(t) \in \mathrm{d}\beta\} \\ &= \mathbb{P}_{2k}^+\{\mathcal{T}(t) \in \mathrm{d}\beta\} - \mathbb{P}_{2k}^+\{\mathcal{T}(t) \in -\mathrm{d}\beta\}, \end{aligned} \tag{3.7}$$

where in the last equality we applied Theorem 3.1.

Let F_β be the first passage time of $\mathcal{T}$ across level β. Then some algebra shows the following relationship (see [5] for further details):

$$\mathbb{P}_{2k}^+\{M(t) = \mathcal{T}(t) \in \mathrm{d}\beta\} = \frac{\beta}{ct}\mathbb{P}_{2k}^+\{M(t) \in \mathrm{d}\beta\} = \frac{\mathrm{d}\beta}{c\,\mathrm{d}t}\mathbb{P}_{2k}^+\{F_\beta \in \mathrm{d}t\}.$$

Recalling that if the particle starts with positive velocity, then the first passage time through level $\beta > 0$ can occur at time t only if the particle is moving with positive velocity, i.e. only if $N(t)$ is even. The interested reader can find the explicit form of the conditional distributions of F_β in [18].

Theorem 3.1 also leads to interesting relationships between the distributions involving the maximum conditionally on a different starting speed, $V_0 = c$ or $V_0 = -c$ (maintaining $\beta > 0$). Recall that if the particle starts moving with a negative velocity, it may spend all the time interval $[0, t]$ on the negative semiaxis, i.e. $M(t) = 0$ (see [6] for a compete analysis). Now, by using the explicit form of (3.7), one can prove that

$$\mathbb{P}_{2k}^+\{M(t) = \mathcal{T}(t)\} = \int_0^{ct} \mathbb{P}_{2k}^+\{M(t) = \mathcal{T}(t) \in \mathrm{d}\beta\} = \binom{2k}{k}\frac{1}{2^{2k}} = \mathbb{P}_{2k}^-\{M(t) = 0\}, \tag{3.8}$$

where the last equality derives from known results on the maximum of the telegraph process; see [6]. Equation (3.8) shows that the probability of an initially positively oriented particle reaching its maximum position at the end of the time interval $[0, t]$ is equal to the probability that a particle starting to move with negative velocity never crosses level 0 before t (and it is independent of t).

By assuming the occurrence of an even number of Poisson events up to time t, a further study of the conditional joint distribution of $\mathcal{T}(t)$ and $M(t)$ shows an even stronger connection between the probability laws of a particle starting with positive and negative velocity.

Corollary 3.1. *Let $\{\mathcal{T}(t)\}_{t\geq 0}$ be a symmetric telegraph process. Let $k \in \mathbb{N}$. For $x \in (-ct, 0]$ we have*

$$\mathbb{P}_{2k}^-\{M(t) = 0,\ \mathcal{T}(t) \in \mathrm{d}x\} = \mathbb{P}_{2k}^+\{M(t) = \mathcal{T}(t) \in -\mathrm{d}x\},$$

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and for $\beta \in (0, ct),\ x \in (2\beta - ct, \beta)$ *we have*

$$\mathbb{P}^-_{2k}\{M(t) \in \mathrm{d}\beta,\ \mathcal{T}(t) \in \mathrm{d}x\} = \mathbb{P}^+_{2k}\{M(t) \in \mathrm{d}\beta,\ \mathcal{T}(t) \in \mathrm{d}x\}.$$

The last consequence of the reflection principle for the telegraph process we present concerns a reflection property holding in the case of an initial negative speed.

Corollary 3.2. *Let* $\{\mathcal{T}(t)\}_{t\geq 0}$ *be a symmetric telegraph process. Let* $k \in \mathbb{N}$. *For* $x \in (-ct, ct)$ *and* $\beta \geq 0$ *we have*

$$\mathbb{P}^-_{2k}\{M(t) \leq \beta,\ \mathcal{T}(t) \in \mathrm{d}x\} = \begin{cases} \mathbb{P}^-_{2k}\{\mathcal{T}(t) \in \mathrm{d}x\} - \mathbb{P}^-_{2k}\{\mathcal{T}(t) \in 2\beta - \mathrm{d}x\} & \text{if } \max\{0, x\} \leq \beta < (ct + x)/2, \\ \mathbb{P}^-_{2k}\{\mathcal{T}(t) \in \mathrm{d}x\} & \text{if } \beta \geq (ct + x)/2. \end{cases} \tag{3.9}$$

Note that probability (3.9), except for the dependence on the initial velocity, has the same form as the reflection principle for Brownian motion. We emphasize that Theorem 3.1 of [6] provided our first intuition for result (3.9).

For a complete review of the consequences of the reflection principle for symmetric telegraph processes we refer to [5], where the derivation of the above corollaries also appears.

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