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Mathematical models and methods for Electromagnetism on fractal geometries

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Introduction

> 𐀞𐀯𐀳𐀃𐀂𐄀 𐀕𐀪𐄀 *pa-si-te-o-i* *me-ri* VAS_{um} 1
> 𐀅𐁆𐀪𐀵𐀋𐄀 𐀡𐀴𐀚𐀈𐄀 𐀕𐀪𐄀 *da-pu$_2$-ri-to-jo* *po-ti-ni-ja* *me-ri* VAS_{um} 1
>
> To all ye Godſ: one honey amphora.
> To ye Labyrinth' Miſtreſſ: one honey amphora.
>
> Linear B tablet KN **Gg702** (Knossos)

This work summarizes the research path done by Walter Arrighetti during his three years of Doctorate of Research in Electromagnetism at Università degli Studi di Roma "La Sapienza," Rome, under the guidance of Professor Giorgio Gerosa. This work was mainly motivated by the struggle to find simpler and simpler models to introduce complex geometries (like fractal ones, for example, which are complicated but far from being 'irregular') in physical field theories like the Classical Electrodynamics, and which stand at the base of most contemporary applied research activities: from antennas (of any sizes, bandwidths and operational distances) to waveguides & resonators (for devices ranging from PC motherboards and IC boards, to high-speed fibre channel links), to magnetic resonance (NMR) devices (for both diagnostic and research purposes), all the way up to particle accelerators. All of these models need not only a solid physical base, but also a specifically crafted ensemble of mathematical methods, in order to tackle with problems which "standard-geometry" models (both in the continuum and the discrete cases) are not best-suited for.

During his previous years of study towards the Laurea degree in Electronic Engineering, the author used different approaches toward Fractal Electrodynamics, form purely-analytical, to computer-assisted numerical simulations of applied electromagnetic structures (both radiating and wave-guiding), down to algebraic-topological ones. The latter approaches, more often than not, proved to be the best way to start with, because the author found out that *self-similarity* (a property which many complicated geometries —even non-fractal ones— seem to, at least, *tend to* possess) can be easily interpreted as a *topological* symmetry, wonderfully described using "ad hoc" nontrivial algebraic languages. Whatever can be successfully described in the language of Algebra (either via numbers, symmetry groups, graphs, polynomials, etc.) is then always *simplified* (or "quotiented" — so to speak in a more strict mathematical language) and, when numerical computation takes the way towards the solution of a specific applied problem, those simplifications turn in handy to reduce the complexity of it. For example, the strict self-similarity possessed by some fractals (like those generated via an Iterated Function System — or IFS) allows to numerically store the geometrical data for a fractal object in a sequence of simpler and simpler data which are, for example, instantly recovered by a computer starting from the simplest data (like simplices, squares/cubes, circles/spheres and regular polygons/polytopes).

For the same reason, all the physical properties that depend on the geometry (or the topology — i.e. basically the number of "holes" or inner connections) of the domain can be reduced, estimated or be even completely known a priori, even before a numerical simulation is performed.

In this work, several of these methods (coming from apparently different branches of pure and applied Mathematics) are presented and finally joined with Electromagnetism equations to solve some more or less applied problems. Since many of the mathematical tools used to build the studied models and methods are advanced and generally not sufficiently known to experts in either such different fields, the first two Chapters are devoted to a brief introduction of some purely mathematical topics. In that context, the author found that the best way to accomplish this was to re-write all those different results from different branches of both pure and applied Mathematics in a formalism as more solid and unified as possible, with continuous links back and forth to different topics (and to the next more applied Chapters). That approach is seldom found in most graduate-level texts. For example, very similar mathematical objects may be even called or classified in different ways, according to the different mathematical contexts they are introduced in, which is exactly the opposite philosophy which has guided underneath in writing these first Chapters.

On the other end, simpler and more trivial mathematical definitions, formalisms or electromagnetic problems, when not elsewhere referenced to, can be found in [9], Arrighetti W., *Analisi di Strutture Elettromagnetiche Frattali*, the author's Laurea degree dissertation (currently only in Italian language).

The most original part of the work is in the last three Chapters where —always using the same "language" and helping with cross-links, as well as to the Bibliography— methods are introduced and then applied to model some electromagnetic problems (previously either unsolved — or already-known, but here solved with a different, usually simpler, or at least more elegant approach).

More specifically, the Thesis is structured like this:

- **Chapter One** deals with advanced Algebra: after a review on intermediate-level Group Theory (including an introduction to both finite, infinite and differential Galois Theory), several less-known objects are presented, such as: formal power, Laurent and Puiseux series; rooted trees and abstract Harmonic Analysis (the basic theory which Fourier and Laplace transforms —thoroughly used in this work— start from).

- **Chapter Two** introduces the reader to intermediate-level Topology, as well as to the analytical and algebraic ways to represent and study topological, smooth and analytic manifolds in one or several complex variables (Kähler and complex Riemann manifolds). This Chapter also introduces a more advanced theory of ordinary differential equations (including an introduction to Riemann-Hilbert problems), as well as many other results on and about Riemann surfaces, like the definition and properties of the (iterated) monodromy groups. Particular attention is done to wisely mix topics of either genuine 'algebraic' and 'geometrical' taste, because they are jointly to be used in the following Chapters. For example, the morphisms between Galois groups, deck transformation groups and monodromy groups is highly stressed. A few new results on the theory of Riemann surfaces are presented, where different tools coming from different mathematical branches (General Topology, Complex Analysis, Algebraic Topology, Galois Theory and Mathematical Physics) are again tied together, with their similarities and difference stressed further.

- **Chapter Three** is the first, entirely original part of the work: it introduces the methods to both synthesize and analyze special kinds of *fractal Riemann surfaces*, deeply comments their properties and employs many results from the previous Chapters in order to shed more light on those mathematical objects. In particular, the author formulates and proves the Fractal Riemann Theorem, §3.3.4, which provides a necessary and sufficient condition for a fractal Riemann surface to be "computable." At the end of the Chapter (§3.4.2), a "gallery" of fractal Riemann surfaces is shown (with obvious simplifications due to the complex-numeric nature of such objects) and commented. The first title's model is then introduced: that of **Riemann Chaos**, as the onset of deterministic chaos on a complicated-enough compact Riemann surface, such that curves traveling on it and starting arbitrarily close to each other, eventually diverge and follow paths on more and more deviating different sheets. A list of open problem regarding such fractal surfaces is also given.

- **Chapter Four**: the first paragraphs re-formulate the Electromagnetic equations (mainly Maxwell's, Helmholtz's and Poynting's), for both direct and inverse problems (Green's tensors) within the framework of differential forms; although such "exterior" formalism is well-known, it has never been extensively treated —to the author's current knowledge—in its most general form and regarding all the possible equations of Electrodynamics. Then next original part is re-formulating again the same equations in a *complexified* version (i.e. in a space-time with complex rather than real space coordinates), and then applied to solve a few well-known resonator problems. This setting for Electromagnetism was found by the author during his research on complex multi-dimensional geometries to be more natural than initially though and —although being completely equivalent to the real-variables' formulation, thus not providing any solution more than by that— allows to write most electromagnetic equations in a more compact way (basically halving the number of equations), as well as to better physically interpret the 'Duality Theorem' of Electromagnetism, and ultimately unifiying electric-derived with magnetic-derived quantities.

- **Chapter Five**: this is were all the main models can be found. The author solved, first of all, the problem of searching what kind of electromagnetic eigenfunctions are present on a self-similar geometry and how it is possible to reduce the amount of data to know about them, employing self-similarity at its best. The *diaperiodic* modes are then found out and defined; then the eigenfunctions of the (scalar) Helmholtz's equation on famous fractal domains (like the Sieprinski carpet in 2D, as well as both Sierpiński's and Menger's sponges in 3D). Afterwards the author gives a personal definition of *spectral dimension* and later proves (with the Fractal Drum Theorem, §5.2.3) that «one can hear the fractal dimension of a drum», i.e. the fractal dimension of an object can be estimated if one knows its complete (Laplace's) resonating spectrum. At last, the prefractal Riemann surfaces introduced in §3 are commented in order to be used in Electromagnetic Inverse Scattering problems: reconstructing the incident field from the scattered field by a fractal (even fractally inhomogeneous) domain.

Although a great part of the Thesis elaborates on already-known scientific topics, many new inter-connections and minor results were discovered in the course of presenting them in this unified manner. Such results are too many to be listed here, but can be found throughout the text, often including example sections too (which are marked with a leading-left grey bar).

Theorems and fundamental mathematical results are immediately titled (within a light-gray full line) with their statements sided by a leading-left grey bar; when needed, proofs immediately follow them —always included for all the major original results— and end with a '□' character on the right part of their last sentence. When an important term or scientific noun is first introduced, it is typed in either dark red, *italic* and **semi-bold** types, depending on its relevance and syntax. Italicized common words or parts of sentences are also used to put occasional emphases.

I hope that this text will be useful for both undergraduates, graduates and researchers in different fields of pure and applied sciences and allows them to either directly employ the described models, or to expand their knowledge and give, where possible, some hints to solve issues or improve knowledge by means of methods herein described. By ‘methods’ I mean, in this case, also the way which the topics are presented in, exposed in, and take inspiration and momentum from each other throughout all the text.

Walter Arrighetti

September 6,th 2007.

Protasis

Plate xxxii. Chapters 2–16.

Ȧnef	ḥrȧu-θen	[…]
Homage	to thee!	

Ari-a	χeperu	neb	er ṭaṭa	ȧ b-ȧ	e m bu neb	mer ka-a
I made	transformations	all	according to the wills	of my heart,	instead of which	my ka wanted

[…]	I-nȧ	āa	er	semeter	maāt	er	erṭāt
	Mehath come	forward	to	maketh a declaration	of right and truth,	to	placeth

ȧusu	er	āḥāu	em	χennu	ḳaȧu
the balance	upon	its supports	within	the groves of amaranth.	

Excerpt from the cxxv chapter of
The Book of the Dead
(xviii Dynasty, xvi century b.c. — Translated by W. A. Budge)

Table of Contents

1 Elements of Algebra

> Verum sine mendacio certum et uerissimum // Quod est inferius est sicut quod est superius, et quod est superius est sicut quod est inferius, ad penetranda miracula rei Unius.
>
> Thruth, without falsehood, is certain and authentic. // What is below equals what is above, and what is above equals what is below; this allows penetration into the wonders of Unity.
>
> Hermes Trismegistus, *Tabula Smaragdina*, incipit.

```
[~]# init 3
```

gnu/Linux sh console terminal.

1.1 Intermediate Group Theory

Most of the Algebra-related definitions and facts described in this Chapter are extensions of those already exposed in the Appendix A of the author's Laurea degree thesis [9], *Analisi di Strutture Elettromagnetiche Frattali* (in Italian language), whereas functional-analytic considerations (especially those in §4.2) were introduced in the Appendix B of the same text.

1.1.1. Basic group theory

Recall that two subgroups $H,K \leq G$ of a group $G(\circ)$ (hereinafter considered to have a neutral element e — unless otherwise stated) are said to be **permuting** with each other if and only if their mutual cosets[1] commute, i.e. $H\circ K=K\circ H$ (which is equivalent to state $H\circ K\leq G$). A non-trivial group $N<G$ [$N\leq G$] which permutes with any other subgroup of G is said to be **permutable** (or *quasi*normal). N is a ***normal*** subgroup of G if, $\forall g\in G,\ g^{-1}\circ N\circ g=N \Leftrightarrow g\circ N=N\circ g$. In this case N-in-G's inclusion is written $N\lhd G$ [$N\unlhd G$]. If G admits **no** normal [and *abelian*] subgroups (except for the trivial one and itself) it is said to be a ***simple*** [***semi**simple*] group. Trivially, every subgroup of an abelian group is normal.

The ***commutator*** and **anti**-commutator of any two elements $g,h\in G$ are the elements

$$\begin{cases}[g|h]\overset{\text{def}}{=}g^{-1}\circ h^{-1}\circ g\circ h\\ \{g|h\}\overset{\text{def}}{=}g\circ h^2\circ g\end{cases}\tag{1.1}$$

The commutator *group*[2] of two subgroups $H,K\leq G$ is $[H|K]:=\{[h|k]\ |\ h\in H,\ k\in K\}$, i.e. the subgroup of all its commutators. The anti-commutator *set* is defined accordingly as $\{H|K\}:=\{\{h|k\}\ |\ h\in H,\ k\in K\}$.

G is said to be a **solvable** (or soluble) group if there exists a finite inclusive sequence of normal subgroups $\{G_n\}_{0\leq n\leq S}$ such that

$$\{e\}=G_0\lhd G_1\lhd\ldots\lhd G_{S-1}\lhd G_S=G,$$

and whose factor groups G_{n+1}/G_n are all abelian, $0\leq n<S$. Every subgroup of a solvable group is solvable; furthermore, if $N\lhd G$ then G is solvable if and only if both N and G/N are.

---

[1] A left [right] ***coset*** of a subgroup $H\leq G$ and $g\in G\backslash H$ is the subgroup $g\circ H$ [$H\circ g$]. All the cosets of H have the same cardinality and there are as many left cosets as the right ones: this number is indicated as $[G:H]:=\operatorname{ord}G/\operatorname{ord}H$ and called the *index* of H *in* G.

[2] G is abelian if and only if $[G|G]$ is trivial. G is called ***perfect*** if $[G|G]=G$ instead.

> The p^{th} ***homology group*** of a path-connected and locally path-connected topological space X is the quotient group of its p-dimensional homotopy group[3] by its own commutator subgroup, i.e.:
>
> $$\mathrm{H}_p(X) \stackrel{\text{def}}{=} \frac{\pi_p(X)}{\left[\pi_p(X)\middle|\pi_p(X)\right]} \tag{1.2}$$
>
> (where (1.2) is well-posed, since these quotient groups do not depend on base points). If X is either a simplicial complex or a chain group ([9]–§2, [81]) then $\mathrm{H}_p(X)=\mathrm{Z}_p(X)/\mathrm{B}_p(X)$, i.e. its elements are (equivalence classes of) p-cycles which are not p-borders of X and are invariant by "integration" over homologue paths, i.e.
>
> $$\langle c|\phi\rangle=\langle c'|\phi\rangle\in\mathbb{Z}\,, \quad \forall c,c'\in[\![c]\!]\in\mathrm{H}_p(X),\ \forall\phi\in\mathrm{Z}^p(X)\ \ (\forall[\![\phi]\!]\in\mathrm{H}^p(X)).$$

The centre of any element $g\in G$ [of a subgroup of its $H\leq G$] is[4] the normal subgroup

$$\begin{gathered}\zeta_G g \stackrel{\text{def}}{=} \left\{h\in G \;\middle|\; [h|g]=e\right\} \trianglelefteq G \\ \left[\quad \zeta_G H \stackrel{\text{def}}{=} \bigcup_{h\in H\setminus\{e\}} \zeta_G h \trianglelefteq G \quad\right];\end{gathered} \tag{1.3}$$

trivially $\zeta_G G=G$ if and only if G is abelian whereas, by definition, $\zeta_G e\equiv G$. The centre of a non-abelian group $G(\circ)$ is used to build its associated ***projective** group* $G/\zeta G$.

---

[3] Recall the n-dimensional **homotopy** group $\pi_n(X,x)$ to be the (equivalence class of) homotopies $\mathbb{S}^n\to X$ whose preimage of $g\in X$ is a point of $\mathbb{S}^n$. For more Homotopy Theory, cfr. [37] and §2.3.1.

[4] Centre operator may be written 'ζ' whenever the group which it is computed on is unambiguous.

> The matrix projective linear [special] groups over a field K, defined by observing that $\zeta\mathrm{GL}_n(K)=\{\alpha I_n\in\mathrm{GL}_n(K) \mid \forall\alpha\in K^*\}\cong K$ (*scalar* matrices' subfield), are:
>
> $$\mathrm{PGL}_n(K)\stackrel{\text{def}}{=}\mathrm{GL}_n(K)/\zeta\mathrm{GL}_n(K)$$
> $$\left[\ \mathrm{PSL}_n(K)\stackrel{\text{def}}{=}\mathrm{SL}_n(K)/\zeta\mathrm{SL}_n(K)\ \right];$$
>
> these fields can be accordingly defined also for linear [and bounded] operators on [topological] vector spaces $\mathbf{V}_K$: $\mathrm{PGL}_K(\mathbf{V})$ and $\mathrm{PSL}_K(\mathbf{V})$ [$\mathfrak{PGL}_K(\mathbf{V})$ and $\mathfrak{PSL}_K(\mathbf{V})$].
>
> They also provide example of normal subgroups, in fact $\mathrm{SL}_n(K)\lhd\mathrm{GL}_n(K)$, $\mathfrak{SL}_K(\mathbf{V})\lhd\mathfrak{GL}_K(\mathbf{V})$ and $\mathfrak{SLC}_K(\mathbf{V})\lhd\mathfrak{GLC}_K(\mathbf{V})$.

The group G is said to be ***nilpotent*** if there exists a finite inclusive sequence of subgroups $\{G_n\}_{0\leq n\leq N}$ such that

$$\{e\}=G_0<G_1<\ldots<G_{N-1}<G_N=G,$$

and whose factor groups G_{n+1}/G_n are contained in the centres of G/G_n, i.e. $G_{n+1}/G_n\leq\zeta(G/G_n)$, respectively for $0\leq n<N$; the smallest $N\in\mathbb{N}$ such that this finite sequence exists is called the **nilpotency index** of G, which is equivalent to state that $\forall g,h\in G,\ g\neq h,\ \exists N\in\mathbb{N}$ such that $[g|h]^N=e$.

Also recall a similar definition for moduli and vector spaces. To every R-modulo $\mathbf{M}_R$, the associated **projective R-modulo** $\mathbb{P}_R\mathbf{M}$ (or simply $\mathbb{P}\mathbf{M}$, when $R(+,\cdot)$ is unambiguous) is the R-modulo $(\mathbf{M}(+)\setminus\{\mathbf{0}\})/R(\cdot)$, with $\dim_R\mathbb{P}\mathbf{M}=\dim_R\mathbf{M}-1$ and whose equivalence classes are: $\mathbb{P}\mathbf{M}\ni[\![\mathbf{v}]\!]=\{\lambda\mathbf{v} \mid \mathbf{v}\in\mathbf{M},\ \forall\lambda\in R^*\}$.

> The **real** *projective n-plane* $\mathbb{R}^n\mathbb{P}$ [the complex one $\mathbb{C}^n\mathbb{P}$] is $\mathbb{P}\mathbb{R}^{n+1}$ [$\mathbb{P}\mathbb{C}^{n+1}$ respectively] and representatives of their equivalence classes indicated as the $(n+1)$-tuplets $(z_0{:}z_1{:}z_2{:}\ldots{:}z_n)\equiv(\frac{z_1}{z_0},\frac{z_2}{z_0},\ldots,\frac{z_n}{z_0})$. More on that will be given in §2.2.4 and (1.193).
>
> From the topological point of view the connected sum of $\kappa\in\mathbb{N}$ real [complex] projective planes ($n=1$) is called a *κ-fold* real [complex] projective plane, written as
>
> $$\mathbb{R}\mathbb{P}_\kappa \stackrel{\text{def}}{=} \overset{\kappa}{\#}\,\mathbb{R}^1\mathbb{P} \qquad \left[\ \mathbb{C}\mathbb{P}_\kappa \stackrel{\text{def}}{=} \overset{\kappa}{\#}\,\mathbb{C}^1\mathbb{P}\ \right].$$

For every group $G(\circ)$ let $G^*(\circ)$ denote the ***conjugate*** *group* of G defined as $G^* \stackrel{\text{def}}{=} \mathrm{Hom}(G, \mathrm{Aut}\, G)$, where the group operation '$\circ$' is inherited from the one in G and from the composition of functions '$\circ$' in $\mathrm{Aut}\, G$: $\forall g\in G\ \ \exists\cdot^g\in G^*$:

$$G\ni x \ \mapsto\ x^g \stackrel{\text{def}}{=} g^{-1}\circ x\circ g, \tag{1.4}$$

which is called an **inner automorphism** induced by g. The subgroup of inner automorphisms of G, indicated as $\mathrm{Inn}\, G$, satisfies the following:

$$G/\zeta G\cong \mathrm{Inn}\, G \trianglelefteq \mathrm{Aut}\, G; \tag{1.5}$$

where $\mathrm{Inn}\, G=\mathrm{Aut}\, G \Leftrightarrow G\cong G^*$.[5]

As a last notion, let $G(\circ)$ be a finite cyclic group[6] of generator $b\in G$, i.e. $\mathrm{Card}\, G=n$ and $G=\langle b\rangle(\circ)\cong\mathbb{Z}_n(+)$, then the base-$b$ ***logarithm*** in G is the function

$$\log_b\in\mathrm{Hom}(G,\mathbb{Z}_n):$$

$$g=b^k \ \mapsto\ \log_b g \stackrel{\text{def}}{=} [\![k]\!]_n\in\mathbb{Z}_n\,. \tag{1.6}$$

---

[5] As for topological linear spaces [68], attention can be focused on **continuous** homomorphisms between topological groups; in this case the *conjugate group* $G^\dagger := \mathrm{CHom}(G, \mathrm{CAut}\, G)$ is defined.

[6] G is isomorphic to either $\mathbb{Z}_n(+)$ or $\mathbb{Z}(+)$ [or $\mathbb{T}(\cdot)$] according whether it is finite or countable [uncountable], respectively. For simplicity's sake, let $\mathrm{Card}\, G=n$, so $G\cong\mathbb{Z}_n$ in this case.

The group-theoretic logarithm satisfies all the arithmetic logarithm's basic properties, like $\log_b(g\circ h)=\log_b g+\log_b h$ and $\log_c g=(\log_b c)(\log_b g)$, $\forall g,h\in G$, $\langle c\rangle=\langle b\rangle$. If $G(\circ)=\mathbb{Z}_p^*(\cdot)$ with $p\in\mathbb{P}$ prime,[7] then $\log :\equiv \log_{[\![1]\!]_p}$ is called discrete p-logarithm.

1.1.2. Permutation groups

Let $A\neq\varnothing$ be any non-empty set; the subset $\mathrm{Sym}A\subset A^A$ of all bijective maps $A\leftrightarrow A$ is a group with respect to functions' composition '$\circ$' and inversion '$^{-1}$' and is called the symmetric group of A. If A is at most countable (i.e. $\mathrm{Card}A\leq\aleph_0$) its elements are also called permutations of A and, if $\mathrm{Card}A=n\in\mathbb{N}$, then $\mathrm{Sym}A$ is trivially isomorphic to $\mathbb{S}_n:=\mathrm{Sym}\{1,2,\ldots,n\}$, called *the* n^{th} symmetric group, which is abelian for $n\leq 2$ and $\mathrm{ord}\,\mathbb{S}_n=n!$. Any permutation $\pi_{\mathbf{i}}\in\mathbb{S}_n$ (and, by isomorphism, any permutation of a finite set) is represented by either a $2\times n$ matrix

$$\begin{pmatrix}1 & 2 & \cdots & n\\ i_1 & i_2 & \cdots & i_n\end{pmatrix},$$

or simply by a multi-index $\mathbf{i}=(i_1,i_2,\ldots,i_n)$ (cfr. §1.3), where $\{i_1,i_2,\ldots,i_n\}\equiv\{1,2,\ldots,n\}$, meaning $k\mapsto\pi_{\mathbf{i}}(k)=i_k$, $1\leq k\leq n$. Permutations can be inverted, combined or iterated:

$$\pi_{\mathbf{i}}^{-1}=\begin{pmatrix}i_1 & i_2 & \cdots & i_n\\ 1 & 2 & \cdots & n\end{pmatrix}\qquad\Rightarrow\qquad\pi_{\mathbf{i}}^{-1}(i_k):=k\,;\tag{1.7}$$

$$\pi_{\mathbf{i}}\left((k_l)_{l\in\mathbb{N}}\right):=(\pi_{\mathbf{i}}(k_l))_{l\in\mathbb{N}}=(k_{i_l})_{l\in\mathbb{N}}\,,\qquad\forall(k_l)_{\mathbb{N}}\in\mathbb{N}^{\mathbb{N}};\tag{1.8}$$

$$k\mapsto\pi_{\mathbf{i}}(k)=i_k\mapsto\pi_{\mathbf{i}}^2(k)=\pi_{\mathbf{i}}(i_k)=i_{i_k}\mapsto\pi_{\mathbf{i}}^3(k)=i_{i_{i_k}}\mapsto\cdots.\tag{1.9}$$

The sign of a permutation $\pi_{\mathbf{i}}\in\mathbb{S}_n$ is defined as

[7] The set of ***prime* numbers** is indicated with $\mathbb{P}\subset\mathbb{N}$, whereas $\mathbb{P}_1:=\mathbb{P}\cup\{1\}$ and $\mathbb{P}^*:=\mathbb{P}\setminus\{2\}$.

$$\operatorname{sign}\pi_{\mathbf{i}} \overset{\text{def}}{=} \prod_{1\leq h<k\leq n} \frac{\pi_{\mathbf{i}}(h)-\pi_{\mathbf{i}}(k)}{h-k} \in \{-1,1\}, \tag{1.10}$$

with $\operatorname{sign}(\pi_{\mathbf{i}}\circ\pi_{\mathbf{j}})=\operatorname{sign}\pi_{\mathbf{i}}\operatorname{sign}\pi_{\mathbf{j}}$ and $\operatorname{sign}\pi_{(1,2,\dots,n)}=1$, with $\pi_{(1,2,\dots,n)}\equiv\mathbb{1}$ being the identity element of $\mathbb{S}_n(\circ)$. A permutation is thus either termed *even* $(+)$ or *odd* $(-)$ according to its sign, respectively. Under this property the Levi-Civita symbol may also be used:

$$\begin{cases} \varepsilon^{\mathbf{i}}_{\mathbf{j}} :\equiv \varepsilon^{i_1,i_2,\dots,i_n}_{j_1,j_2,\dots,j_n} \overset{\text{def}}{=} \operatorname{sign}(\pi_{\mathbf{i}}\circ\pi_{\mathbf{j}}) \\ \varepsilon_{\mathbf{j}} :\equiv \varepsilon^{1,2,\dots,n}_{j_1,j_2,\dots,j_n} \equiv \operatorname{sign}\pi_{\mathbf{i}} \end{cases} \tag{1.11}$$

which is also an antisymmetric (n,n)-tensor of $\operatorname{skw}\mathbb{Z}^{(n,n)}$ cfr. [9]–§A.4. Permutations on (a subset of) integer numbers are just indicated as sequences of permuted numbers between symbols '⌈' and '⌋', for example:

$$\lceil i_1,i_2,\dots,i_n \rfloor \overset{\text{def}}{=} \begin{pmatrix} 1 & 2 & \cdots & n \\ i_1 & i_2 & \cdots & i_n \end{pmatrix}.$$

The even permutations of $\mathbb{S}_n$ for $n>2$ form the n^{th} ***alternating*** group $\mathbb{A}_n$, of order $n!/2$, which is simple $\forall n\in\mathbb{N}\backslash\{1,2,4\}$ and is the only normal subgroup of the symmetric group for $n\neq 4$, i.e.:

$$\exists! \mathbb{A}_n \lhd \mathbb{S}_n, \qquad \forall n\in\mathbb{N}\backslash\{1,4\}.$$

The following Theorem needs advanced Galois' Theory in order to be proven (some beginning insights of which are discussed in §1.1.10) and is the main Corollary of it.

Cayley's Theorem on permutation groups

For any non-trivial group $G(\circ)$ there exists a set $A\neq\varnothing$ such that:

$$G \lessapprox \operatorname{Sym} A\,.$$

The n^{th} ***dihedral*** group, $\forall n\in\mathbb{N}$ (and identity element e), is defined with presentation

$$\begin{aligned}\mathbb{D}_{2n} &\stackrel{\text{def}}{=} \left\langle \rho,\phi \,\middle|\, \rho^n = \phi^2 = e,\ \phi\rho\phi = \rho^{-1} \right\rangle \\ &\equiv \left\langle x,y \,\middle|\, x^2 = y^2 = (xy)^n = e \right\rangle.\end{aligned} \tag{1.12}$$

The n^{th} dihedral group is the symmetries' group of a regular n-gon, where element ρ 'is' a $(2\pi/n)$-radiants rotation around its centre and ϕ 'is' the reflection around one of its axes. $\operatorname{ord}\mathbb{D}_{2n} = 2n$ and for all[8] $n > 2$ it is non-abelian; furthermore $\mathbb{D}_6 \cong \mathbb{S}_3$ $(n=3)$ and $\mathbb{D}_{2n} < \mathbb{S}_n$ for $n > 3$.

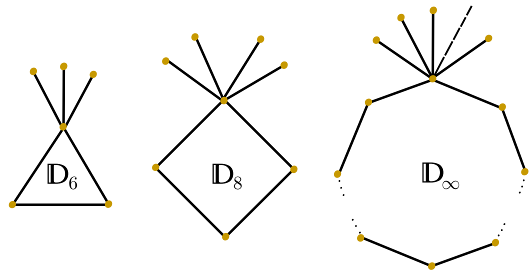

Figure 1 Graphs representing the topology of $\mathbb{D}_6$, $\mathbb{D}_8$ and $\mathbb{D}_\infty$.

The ***infinite dihedral* group** is defined accordingly as

$$\begin{aligned}\mathbb{D}_{\infty} &\stackrel{\text{def}}{=} \left\langle \rho,\phi \,\middle|\, \phi^2 = e,\ \phi\rho\phi = \rho^{-1} \right\rangle \\ &\equiv \left\langle x,y \,\middle|\, x^2 = y^2 = e \right\rangle,\end{aligned} \tag{1.13}$$

i.e. the symmetries' group of a countable cycle[9] with $\mathbb{D}_\infty \cong \operatorname{Sym}\mathbb{Z}$. See Figure 1 for an inductive visualization of finite and infinite dihedral groups.

The ***Braid* group** of order $n \in \mathbb{N}$ is the fundamental group of $\mathbb{S}_n$ with respect to its trivial element and the discrete topology and is indicated as $\mathbb{B}_n \stackrel{\text{def}}{=} \pi_1(\mathbb{S}_n, \mathbb{1})$; its elements are called n-**braids** and they are 'paths' where each permutation is changed into another a finite, ordered number of times. Braids made by one simple permutation between the i^{th} and the $(i+1)^{\text{th}}$ element only are indicated as

---

[8] In fact $\mathbb{D}_2 \cong \mathbb{Z}_2$ and the Klein group $\mathbb{D}_4 \cong \mathbb{Z}_2 \oplus \mathbb{Z}_2$, the one 4^{th}-order group other than $\mathbb{Z}_4$.

[9] Cycle graphs in Graph Theory are indicated by corresponding dihedral groups' symbols.

$\lceil \ldots, i-1, i+1, i, i+2, \ldots \rfloor = \sigma_i \in \mathbb{S}_n$, for $1 \leq i < n$. Due to the continuity in the definition of path in $\mathbb{S}_n$ they are such that any braids containing either $\sigma_i\sigma_j$ (for $|j-i|>1$) or $\sigma_i\sigma_{i+1}\sigma_i$ as consecutive permutations (i.e. a path 'crossing by' those points) are homotopically equivalent to a path containing $\sigma_j\sigma_i$ or $\sigma_{i+1}\sigma_i\sigma_{i+1}$, respectively (these are also called *Artin* relations). Such homotopy classes are the same braid or —as it is frequently said in Knot Theory— they represent *isotopic* braids. All in all, the braid groups $\mathbb{B}_n$, for $n>2$, have the following presentations:

$$\mathbb{B}_n = \Big\langle \sigma_i \,\Big|\, |i-j|>1 \Rightarrow \sigma_i\sigma_j = \sigma_j\sigma_i;\ \sigma_i\sigma_{i+1}\sigma_i = \sigma_{i+1}\sigma_i\sigma_{i+1} \Big\rangle_{1\leq i<n}. \quad (1.14)$$

$\mathbb{B}_n$ is torsion-free (i.e. $\mathrm{Tor}\mathbb{B}_n = \{e\}$) and, for $n>1$, is also infinite, i.e. all its nontrivial elements have infinite order; moreover $\mathbb{B}_0 \cong \mathbb{B}_1 \cong \{e\}$, whereas $\mathbb{B}_2 \cong \mathbb{Z}$ and, for $n>2$, $\zeta\mathbb{B}_n = \langle (\sigma_1\sigma_{n-1})^n \rangle$. Furthermore, $\mathbb{B}_n \lhd \mathbb{B}_{n+1}\ \forall n \in \mathbb{N}$.

1.1.3. Basic ring and field theory

The ***characteristic*** of an arithmetic ring $R(+,\cdot)$ is the nilpotency index of 1 as element of the group $R(+)$, i.e the least integer $\mathrm{char}R \in \mathbb{N}_0$ such that $(\mathrm{char}R)1 = 0$ (where $(\mathrm{char}R)1 \equiv 1+1+\ldots+1$, $\mathrm{char}R$ times); if 1 is not nilpotent, then $\mathrm{char}R := 0$ and $\mathrm{char}R = 0 \Rightarrow \mathrm{Card}R \geq \aleph_0$. The only rings with characteristic 1 are the trivial ones; those with no divisors of 0 always have a prime characteristic, i.e. $\mathrm{char}R \in \mathbb{P}$. Every sub-ring shares the same characteristic of its super-ring, whereas $\mathrm{char}\mathbb{Z}_n = n$.

Taylor's Theorem — polynomials

For every $P \in K[x]$ in a zero-characteristic field K, and $\forall x,y \in K$:

$$P(x+y) = P(x) + \sum_{k=0}^{\deg P} \frac{\mathrm{D}^k P(x)}{k!} y^k. \quad (1.15)$$

The above result can be easily proven using the map $P(x)\mapsto P(x+y)$ in $\mathrm{Hom}(K^n[x],K^n[x,y])$ and using the properties of polynomials' formal derivatives.[10]

The characteristic of fields becomes an important index to estimate some of their algebraic properties, [71]. In particular, prime-characteristic rings are finite fields with order a power of that prime, i.e. $\mathrm{char}R=p\in\mathbb{P} \;\Rightarrow\; \mathrm{ord}R=p^n$; furthermore $(x+y)^p\equiv x^p+y^p$ in the polynomial field $R^p[x,y]$. A most remarkable result in this case is that $\mathbb{Z}_p\lessapprox R$ and R is a n-dimensional $\mathbb{Z}_p$-vector space, i.e. $\dim_{\mathbb{Z}_p}R=n$.

Every ordered field has zero characteristic (which is a sufficient condition, in fact $\mathrm{char}\,\mathbb{C}=0$, which is a non-ordered field); every finite field's order is a power of a prime number and all the equipotent finite fields are isomorphic with each other. The prototype field of order p^n $(p\in\mathbb{P},\ n\in\mathbb{N})$ is universally addressed to as $\mathbb{F}_{p^n}(+,\cdot)$. In particular, $\mathrm{char}\,\mathbb{F}_{p^n}=p$, $\mathbb{F}_p\cong\mathbb{Z}_p$ and[11] $\mathbb{F}_{p^m}\leq\mathbb{F}_{p^n} \;\Leftrightarrow\; m|n$.

The size of any *finite*-dimensional vector space $\mathbf{V}_{\mathbb{F}_{p^n}}$ over the finite field $\mathbb{F}_{p^n}$ is itself a prime power: $\dim_{\mathbb{F}_{p^n}}\mathbf{V}=N \;\Rightarrow\; \mathrm{Card}\mathbf{V}=p^{nN}$. The principal matrix groups (linear, special, orthogonal, etc.) on finite fields are usually indicated with alternate notations; for example for the special linear group, $\mathrm{SL}(m,p^n)\overset{\mathrm{def}}{=}\mathrm{SL}_m(\mathbb{F}_{p^n})$.

Finite fields are usually represented and computed on as either polynomials with coefficients in their characteristic ring, or digit representation, via the isomorphism $\mathbb{F}_{p^n}\cong\mathbb{Z}_p^n[x]\cong\mathbb{Z}_p^{n+1}$; operations on finite fields are equivalent to the corresponding addition and multiplication over $\mathbb{Z}_p^n[x]$.

The finite fields of characteristic 2, $\mathbb{F}_{2^n}$, are called Galois fields, where addition '$+$' is replaced by the logical ***XOR*** '$\uplus$' and multiplication '$\cdot$' by the logical ***AND*** '$\Cap$'. Operations on Galois fields are then performed using the Boolean algebra $\mathbb{Z}_2[x]$ of binary numbers — consult [9]–§A.2 for more information on Boole's Algebra.

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[10] For properties of *derivation* endomorphisms, cfr. [9]–§A.2. The ***formal*** *derivative* $\mathrm{D}:R^n[x]\to R^{n-1}[x]$ is defined as: $\mathrm{D}(a_0+a_1x+a_2x^2+\ldots+a_nx^n):=a_1+2a_2x+3a_3x^2+\ldots+na_nx^{n-1}$.

[11] Note that $\mathbb{F}_{p^n}\ncong\mathbb{Z}_{p^n}$ for $n>1$ (neither $\mathbb{Z}_{p^n}$ are fields); for example, $\mathbb{F}_4\cong\mathbb{D}_4$.

> Polynomials $x^7+x^6+x^3+x$ and x^6+x^4+x+1 in $\mathbb{Z}_2[x]$ could be represented —as common in Computer Science— as either binary numbers 11001010_2 and 1010011_2, or hexadecimal numbers CA_{16} and 53_{16}, respectively. Their polynomial sum $(x^7+x^4+x^3+1)\in\mathbb{Z}_2[x]$ is computed, according to these equivalences, as:
>
> $$\begin{array}{c|ccccc} \mathbb{Z}_2^7[x] & x^7+x^6+x^3+x & + & x^6+x^4+x+1 & = & x^7+x^4+x^3+1 \\ \text{bin.} & 11001010_2 & \uplus & 01010011_2 & = & 10011001_2 \\ \text{hex.} & \text{CA}_{16} & \uplus & 53_{16} & = & 99_{16} \end{array}$$

Let $N(+)$ be an abelian monoid (with neutral element *0*) and $R(+,\cdot)$ be finite field such that, as additive group $R(+)$, is expressed as a direct sum (cfr. §1.1.4) of a sequence of N-indexed subgroups $(R_n)_{n\in N}$ such that:

$$R = \bigoplus_{n\in N} R_n; \qquad R_m\cdot R_n\subset R_{m+n}, \quad \forall m,n\in N.$$

If those conditions are met R is said to be a N-*graded* ring (with R_0 being a subring itself and the elements of subgroups R_n, $\forall n\in N$, called homogeneous elements of *degree* n).

> Polynomial (of any degree) rings, like $R[x]$, are among the simplest examples of graded rings, with their homogeneous elements being the additive subgroups of monomials of any degree (with the null-polynomial as common neutral elements).

Let $\mathbf{A}_K(+,\cdot)$ be a K-algebra with an inclusive sequence of vector subspaces of its $(\mathbf{A}_n)_{n\in\mathbb{N}_0}$ such that:

$$\mathbf{A}_0\subset\mathbf{A}_1\subset\mathbf{A}_2\subset\ldots, \qquad \mathbf{A} = \bigcup_{n\in\mathbb{N}_0} \mathbf{A}_n; \qquad \mathbf{A}_m\cdot\mathbf{A}_n\subset\mathbf{A}_{m+n}, \ \forall m,n\in\mathbb{N}_0.$$

$\mathbf{A}$ is then said to be a **filtered** algebra (with the sequence $(\mathbf{A}_n)_{n\in\mathbb{N}_0}$ being a filtration of it). If the second identity above holds as orthogonal decomposition, i.e.

$\mathbf{A}=\bigoplus_{n\in\mathbb{N}_0}\mathbf{A}_n$, then $\mathbf{A}$ is said to be a K-***graded*** algebra (other than being a K-graded ring) with $\mathbf{A}$ happening to be a $\mathbf{A}_0$-algebra.

> The exterior algebra $\Lambda(\mathbf{V})(+,\wedge)$ of any vector space $\mathbf{V}$ (cfr. [9]–§A.5 and §2.1.1) —as well as its tensor and symmetric algebras (both with respect to $\otimes$)— are trivial examples of graded ones. In the exterior algebra case, multilinear p-forms are the homogeneous p^{th}-degree elements.

1.1.4. Direct and semi-direct products

Let $H,K<G$ be two distinct subgroups of $G(\circ)$ such that $\forall g\in G\ \exists!h_g\in H,\ \exists!k_g\in K$: such that $g=h_g\circ k_g$; this unique decomposition is indicated as $G=H\circledcirc K$, with H and K said to be *direct factors* of G, [9]–§A.3. This definition can be applied by induction up to any cardinality, such that $\bigodot_{i\in I}G_i$, where $G_i<G\ \forall i\in I\neq\varnothing$, and

$$\operatorname{ord}\bigodot_{i\in I}G_i=\prod_{i\in I}\operatorname{ord}G_i. \tag{1.16}$$

If $N\lhd G$ and $G=N\circledcirc H$ [$G=H\circledcirc N$], then G is said to be a **internal** *left* [right] *semi-direct product* of N and H and this may be written:

$$G=N\rtimes H \qquad \left[\ G=H\ltimes N\ \right]. \tag{1.17}$$

Let $(I;\preccurlyeq)$ be an ordered set and $\{G_i\}_{i\in I}$ be a non-empty family of groups $G_i(\circ_i)$, whose neutral elements are $e_i\ \forall i\in I$, in general all different with each other (i.e. not subgroups of a unique group). The ***direct product*** of such groups (respect to induced ordering '$\preccurlyeq$') is the group indicated with symbol

$$\bigodot_{i\in I}G_i \tag{1.18}$$

(or as $G_1\circledcirc G_2\circledcirc G_3\circledcirc\ldots$ if I is at most countable, e.g. $I\subseteq\mathbb{N}$ respect to natural well-ordering '$\leq$') whose base set is the cartesian product of the family

$$\bigtimes_{i\in I} G_i$$

and whose group operation $\circ{:}(\circledcirc_{i\in I}G_i)\times(\circledcirc_{i\in I}G_i)\to\circledcirc_{i\in I}G_i$ is defined as

$$\begin{cases}(g_i)_{i\in I}\circ(h_i)_{i\in I}\stackrel{\text{def}}{=}(g_i\circ_i h_i)_{i\in I}\\(g_i)^{-1}_{i\in I}\stackrel{\text{def}}{=}(g_i^{-1})_{i\in I}\end{cases}\tag{1.19}$$

and whose neutral element is $(e_i)_{i\in I}$. Clearly $\circledcirc_{i\in I}G_i$ contains $\mathrm{Card}\,I$ normal subgroups, each isomorphic to the every group of the family.

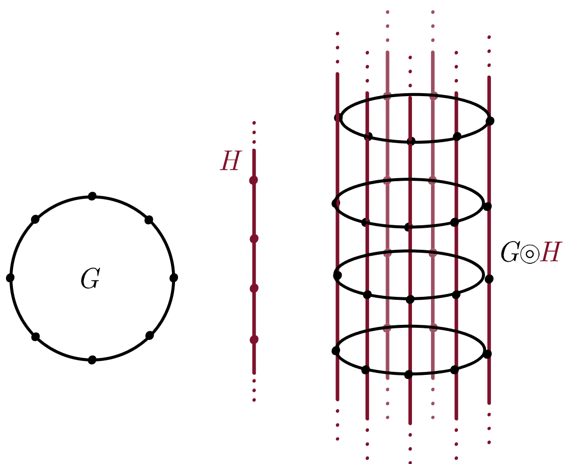

Figure 2 Schematics of two groups' direct product (e.g. direct sum of cyclic groups $G=\mathbb{Z}_8$ and $H=\mathbb{Z}$ gives rise to abelian free group $\mathbb{Z}_8\oplus\mathbb{Z}$).

Let $G=G_1\circledcirc G_2\circledcirc\ldots\circledcirc G_n$, and $H_i<G_i$ for $1\le i\le n$; then the subgroups' composition is automatically (a) direct (product), i.e. $H_1\circ H_2\circ\ldots\circ H_n=H_1\circledcirc H_2\circledcirc\ldots\circledcirc H_n$ and:

$$\frac{G}{\displaystyle\mathop{\circledcirc}_{i=1}^{n}H_i}\cong\mathop{\circledcirc}_{i=1}^{n}\frac{G_i}{H_i};\qquad G/G_j\cong\mathop{\circledcirc}_{\substack{i=1,\\ i\neq j}}^{n}G_i.\tag{1.20}$$

An endomorhism $f \in \mathrm{End}\, G$ is said to be ***Jordan*** if a f-invariant[12] decomposition $G = N \circledcirc D$ exists such that $f|_N$ is nilpotent, $f|_D$ is f-invariant and $\left[f|_N \middle| f|_D \right] = \emptyset$.

If all the groups are additive [9]–§A.2, $G_i(+)$, their direct product is also called a direct **sum**, indicated with the alternative symbols (used also for direct factors of the same additive group), see Figure 2 :

$$\bigoplus_{i \in I} G_i \qquad \text{or} \qquad G_1 \oplus G_2 \oplus G_3 \oplus \ldots.$$

If $I = \mathbb{N}$ the elements of $\bigoplus_{i \in I} G_i$ can also be written as $e_1 \oplus e_2 \oplus e_3 \oplus \ldots$.

> Direct sums of finite cyclic groups satisfy the "LCM relation," i.e.:
>
> $$\mathbb{Z}_m \oplus \mathbb{Z}_n \cong \mathbb{Z}_{\mathrm{LCM}(m,n)}, \qquad \forall m,n \in \mathbb{N}_0;$$
>
> a "dual" isomorphism holds for tensor products of cyclic groups, i.e.:
>
> $$\mathbb{Z}_m \otimes \mathbb{Z}_n \cong \mathbb{Z}_{\mathrm{GCD}(m,n)}, \qquad \forall m,n \in \mathbb{N}_0.$$

Now let $G(\circ)$ and $H(\square)$ be any two groups and $\alpha \in \mathrm{Hom}(G, \mathrm{Inn}\, H)$; the ***semi*-direct** product of G and H respect to α, *as a set*, is either one of the cartesian products,

$$\begin{cases} G \ltimes_\alpha H = G \times H \\ H \rtimes_\alpha G = H \times G \end{cases}$$

equipped with the following group operation[13] '$_\alpha$' and inversion '$^{-\alpha}$' (here defined for $G \ltimes_\alpha H$ only: you just have to invert couples $(h,g) \in H \rtimes_\alpha G$ otherwise):

$$(g,h)_\alpha(g',h') := \left(g \circ g',\, h^{\alpha(g')} \square h' \right) \equiv \left(g \circ g',\, \alpha(g')^{-1} \square h \square \alpha(g') \square h' \right),$$

$$(g,h)^{-\alpha} := \left(g^{-1},\, (h^{-1})^{\alpha(g)^{-1}} \right) \equiv \left(g^{-1},\, \alpha(g) \square h^{-1} \square \alpha(g)^{-1} \right). \qquad (1.21)$$

---

[12] Let $f \in A^A$; f-invariant decompositions are partitions of A whose subsets E are invariant under f.
[13] This operation is due to the fact that in any group $G(\cdot)$, $x \cdot y \cdot z \cdot w = x \cdot z \cdot y^z \cdot w$, $\forall x,y,z,w \in G$.

Both $G,H \lesssim G \ltimes_\alpha H$ and H is isomorphic to a normal subgroup of $G \ltimes_\alpha H$ (which is one internal semi-direct factor of); for this reason one often refers to external semi-direct products just using symbols '$\ltimes$' and '$\rtimes$'.

> Examples of groups generated by semi-direct products include:
>
> - finite dihedral groups admit a semi-direct decomposition via cyclic groups, in fact $\mathbb{D}_{2n} \cong \mathbb{Z}_n \rtimes \mathbb{Z}_2 \cong \mathrm{Aut}(\mathbb{Z}_n \oplus \mathbb{Z}_2)$, $\forall n \in \mathbb{N}$;
> - for the infinite dihedral group: $\mathbb{D}_\infty \cong \mathbb{Z} \rtimes \mathbb{Z}_2 \cong \mathrm{Aut}\mathbb{Z}$;
> - every alternating group is a semi-direct factor of its symmetric group by the group of simple permutations, i.e.:
>
> $$\mathbb{S}_n = \mathbb{A}_n \rtimes \mathbb{Z}_2, \qquad \forall n \in \mathbb{N} \setminus \{1,4\}.$$
>
> - for special linear groups (cfr. examples in §1.2.2): $\mathrm{GL}_n(K) = \mathrm{SL}_n(K) \rtimes K^*$.

From the considerations in §1.1.3, it is now easy to build graded rings starting from an abelian monoid $N(+)$ and any sequence of N-indexed additive groups $(G_n)_{n \in N}$: first, let $G_0(+,\cdot)$ be a commutative ring and maps $G_m \times G_n \to G_{m+n}$ be defined, $\forall m,n \in N$, such that compositions under them are associative and G_0-linear. Then the direct sum $G := \bigoplus_{n \in N} G_n$ is a ring by component-wise inheritance of the multiplication as:

$$\left(\sum_{m \in N} x_m \right) \cdot \left(\sum_{n \in N} y_n \right) \stackrel{\text{def}}{=} \sum_{l \in N} \sum_{\substack{m,n \in N, \\ m+n=l}} x_m \cdot y_n .$$

In the case where N is either $\mathbb{Z}$ or $\mathbb{Z}_2$, another kind of multiplication can be defined, called *super product* $\odot : G \otimes G \to G$ (both associative and bilinear), which is $x \odot y := (-1)^{mn} xy$, $\forall x \in G_m$, $\forall y \in G_n$ ($\forall m,n$).

> Let $\left(\mathrm{H}^p(X;R)\right)_{p\in\mathbb{N}_0}$ be the sequence of R-cohomology groups of a topological space X with respect to an abelian ring $R(+,\cdot)$.The *cup* product $\cup\colon \mathrm{H}^p(X;R)\times\mathrm{H}^q(X;R)\to\mathrm{H}^{p+q}(X;R)$ allows to introduce the R-**cohomology** ring of X as the graded algebra built on the above sequence via the aforementioned method — cfr. [9]–§2.3 and [81].

Let $\mathbf{A}$ be a graded algebra over either $\mathbb{Z}$ and $\mathbb{Z}_2$ (which means that $\mathbf{A}$ is also a graded ring, cfr. §1.1.3); with respect to the super-product defined above, $\mathbf{A}$ takes the name of a ***super* algebra**, its homogeneous elements being either even (0^{th}-degree) *or* odd (1^{st}-degree) — and is sometimes denoted as $\mathbf{A}_{\mathrm{su}}:\equiv\mathbf{A}(+,\odot)$, to better differentiate it from $\mathbf{A}(+,\cdot)$.

1.1.5. Group actions

Let $X\neq\varnothing$ be a set, then a left [right] group ***action*** of G on X is a binary operation $\bullet\colon G\times X\to X$ [$\bullet\colon X\times G\to X$] such that $g\bullet(h\bullet x)=(g\circ h)\bullet x$ and $e\bullet x=x$ [$(x\bullet g)\bullet h=x\bullet(g\circ h)$ and $x\bullet e=x$], $\forall g,h\in G$, $\forall x\in X$. Group actions (e.g. left ones) can be classified as being

- Transitive: $\forall x,y\in X\ \ \exists g\in G\colon\quad g\bullet x=y$;
- Faithful: $\forall g\in G\setminus\{e\}\ \ \exists x\in X\colon\quad g\bullet x\neq x$;
- Free: $\exists x\in X\colon\ g\bullet x=x\quad\Rightarrow\quad g=e$;
- N-transitive ($N\in\mathbb{N}\setminus\{1\}$): $\forall x_1,x_2,\ldots,x_N,y_1,y_2,\ldots,y_N\in X\colon\ x_i\neq x_j$, $y_i\neq y_j$, $1\leq i<j\leq N\quad\Rightarrow\quad\exists g\in G\colon\ g\bullet x_k=y_k$, $1\leq k\leq N$;
- Regular: transitive *and* free, i.e. $\forall x,y\in X\ \ \exists! g\in G\colon\quad g\bullet x=y$.

Every free action is faithful. Every left or right group action is clearly an element of $\mathrm{Hom}(G,\mathrm{Sym}X)$ or $\mathrm{Hom}(\mathrm{Sym}X,G)$, respectively, thus the concepts of semi-direct products can be applied to group actions rather than groups' couples: whenever there is neither confusion on the homomorphisms nor on the left [right] group action (and X is not a group by itself), $G\ltimes X$ $[X\rtimes G]$ will denote $G\ltimes \mathrm{Sym}X$ $[\mathrm{Sym}X\rtimes G]$.

Let $\bullet\colon G\times X\to X$ be a left group action, the ***orbit*** of any $x\in X$ [the orbiset of $Y\subseteq X$] is

$$\begin{gathered} G{\bullet}x \overset{\text{def}}{=} \left\{ g{\bullet}x\in X \;\middle|\; \forall g\in G \right\} \\ \left[\quad G{\bullet}Y \overset{\text{def}}{=} \bigcup_{y\in Y} G{\bullet}y \quad \right] \end{gathered} \tag{1.22}$$

(similar definition holds for the orbits $x{\bullet}G$ [orbisets $Y{\bullet}G$] of a right action). Any subset $Y\subseteq X$ is invariant under G if $G{\bullet}Y=Y$ and fixed if $g{\bullet}y=y, \forall y\in Y, \forall g\in G$. Every orbit is an invariant subset on which G acts transitively, i.e. $G{\bullet}(G{\bullet}x)\equiv G{\bullet}x$. '$\bullet$' is transitive if and only if there is only one orbit (i.e. $G{\bullet}x=X, \forall x\in X$).

Let $\mathcal{E}_\bullet\subseteq X^2$ be the equivalence relation [9]–§A.1 such that $x\mathcal{E}_\bullet y \Leftrightarrow \exists g\in G\colon g{\bullet}x=y$; the quotient set $X/\mathcal{E}_\bullet$, written X/G if '$\bullet$' is unambiguous, is called the *orbispace* of X respect to G and its equivalence classes are the orbits of its representatives, i.e. $[\![x]\!]_{\mathcal{E}_\bullet} :\equiv [\![x]\!]_G \equiv G{\bullet}x, \forall x\in X$.

The *stabilizer* (or **isotropy** group) of $x\in X$ under G is $\mathrm{stab}_G x := \{g\in G \mid g{\bullet}x=x\}\leq G$. '$\bullet$' is free if and only if all its stabilizers are trivial subgroups of G.

Orbit Stabilizer Theorem

For any (left) group action $\bullet\colon G\times X\to X$ and $\forall x\in X$:

$$\begin{aligned} G{\bullet}x &\cong G/\mathrm{stab}_G x\,; \\ \mathrm{ord}\, G{\bullet}x &= \frac{\mathrm{ord}\, G}{\mathrm{ord}\,\mathrm{stab}_G x}; \\ \mathrm{Card}\, X/G &= \frac{1}{\mathrm{ord}\, G}\sum_{g\in G} \mathrm{Card}\left\{ x\in X \;\middle|\; g{\bullet}x=x \right\}. \end{aligned} \tag{1.23}$$

A *one-parameter* group (or subgroup) of $G(\circ)$ is any $\mathbb{R}(+)$-homomorphism $\mathbb{R}\to G$; their (left) actions on sets are also flows, which are usually indicated by explicitly showing the real parameter of the corresponding subgroup. That is, let an action be $\bullet: G\times X\to X$ and a one-parameter subgroup be $\mathfrak{g}\in\mathrm{Hom}(\mathbb{R},G)$, the associated flow is the "lifted action" $\mathfrak{g}_\bullet:\mathbb{R}\times X\to X$ such that:

$$\begin{cases}\mathfrak{g}_{t+s}x = \mathfrak{g}_t\mathfrak{g}_s x \\ \mathfrak{g}_{-t}x = \mathfrak{g}_{-t}^{-1}x, \\ \mathfrak{g}_0 x = x\end{cases} \qquad \forall t,s\in\mathbb{R},\quad \forall x\in X.$$

Orbits and orbisets of flows are indicated accordingly as $\mathfrak{g}_\bullet x$ and $\mathfrak{g}_\bullet Y$ respectively ($\forall Y\subseteq X$).

1.1.6. Wreath products

Let $G(\circ)$ and $H(\square)$ be groups, let X and Y two non-empty sets which either groups respectively act on (e.g. the left: $\bullet: G\times X\to X$ and $\blacksquare: H\times Y\to Y$). The direct product $G\circledcirc H$ trivially acts on the cartesian product $X\times Y$ as

$$\bullet\wr\blacksquare: (G\circledcirc H)\times(X\times Y)\longrightarrow X\times Y,$$

$$((g,h),(x,y)) \mapsto (g,h)\bullet\wr\blacksquare(x,y) \overset{\text{def}}{=} (g\bullet x, h\blacksquare y). \tag{1.24}$$

Consider $\forall y\in Y$ the left action $\bullet_y:(G\times H)\times(X\times Y)\to X\times Y$ defined as $(g,h)\bullet_y(x,y'):=\begin{cases}(g\bullet x,y'), & \text{if } y=y' \\ (x,y'), & \text{if } y\neq y'\end{cases}$; consider maps $y\mapsto\bullet_y$ and $(x,y)\mapsto \mathbb{1}\wr\blacksquare$, then

$$G\wr H \overset{\text{def}}{=} \left\langle \bullet_y,\ \mathbb{1}\wr\blacksquare \;\middle|\; \forall y\in Y\right\rangle \tag{1.25}$$

is called the ***wreath product*** of G and H (or, better, of actions '$\bullet$' and '$\blacksquare$'), indicated as $G\wr H$, and specifically acts on $X\times Y$ (i.e. it is a permutation group of it).

Wreath products are associative, i.e. $(G\wr H)\wr K \cong G\wr(H\wr K) :\cong G\wr H\wr K$, for any other group $K(\diamond)$ acting on another set Z (e.g. via $\blacklozenge: K\times Z\to Z$).

For any two groups $G(\circ)$, $H(\square)$ and a left [right] action $\blacksquare: H\times Y\to Y$ [$\blacksquare: Y\times H\to Y$],

$$H\wr G \cong H\ltimes \bigodot_Y G \qquad \left[\ H\wr G \cong \left(\bigodot_Y G\right)\rtimes H\ \right], \tag{1.26}$$

where $\circledcirc_Y G$ is the direct product of G with itself directed by all elements of Y. In the left action case then, $h{\blacksquare}(g_{y_1}\circledcirc g_{y_2}\circledcirc g_{y_3}\circledcirc\ldots):=g_{h\blacksquare y_1}\circledcirc g_{h\blacksquare y_2}\circledcirc g_{h\blacksquare y_3}\circledcirc\ldots\in\circledcirc^{\mathrm{Card}\,Y}G$ (with abuse of notation),[14] $\forall h\in H$, $Y=\{y_1,y_2,y_3,\ldots\}$ and $\forall g_{y_1},g_{y_2},g_{y_3},\ldots\in G$. The group operation '$\cdot$' on $H\wr G$ is then, $\forall f,g\in\circledcirc_Y G$ and $\forall h,k\in H$:

$$\left(h{\blacksquare}\circledcirc_{y\in Y} g_y\right)\cdot\left(k{\blacksquare}\circledcirc_{y\in Y} f_y\right)\stackrel{\text{def}}{=}(h\square k){\blacksquare}\circledcirc_{y\in Y}(g_{k\blacksquare y}\circ f_y)\,. \tag{1.27}$$

An example of group acting as a wreath product is $\mathbb{D}_4\cong\mathbb{Z}_2\wr\mathbb{Z}_2$.

More generally, since every group acts on itself in three ways — $h\mapsto g\circ h$ (left, regular), $h\mapsto h\circ g$ (right) and[15] $h\mapsto h^g=g^{-1}\circ h\circ g$ (the action of G^* on G) — the wreath and the semi-direct product of any group with itself are isomorhphic, i.e.:

$$G\wr G\cong G\ltimes G=G\rtimes G=G\circledcirc G.$$

For any $m,n\in\mathbb{N}$, the wreath product $\mathbb{Z}_m\wr\mathbb{S}_n$ is called *generalized (m,n)-symmetric group* and has the following properties:

- it is trivial for either $m=1$ and $n=1$, i.e. $\mathbb{Z}_1\wr\mathbb{S}_n\cong\mathbb{S}_n$, $\mathbb{Z}_m\wr\mathbb{S}_1\cong\mathbb{Z}_m$;
- the 1st homology group is $\mathrm{H}_1\left(\mathbb{Z}_m\wr\mathbb{S}_n\right)\cong\mathbb{Z}_m\oplus\mathbb{Z}_2\,$;
- the 2nd homology group depends on n and the parity of m only, i.e.:

$$\mathrm{H}_2\left(\mathbb{Z}_{2m+1}\wr\mathbb{S}_n\right)\cong\mathrm{H}^2(\mathbb{S}_n)\cong\begin{cases}\{e\}, & n<4\\ \mathbb{Z}_2, & n\geq 4\end{cases}$$

$$\mathrm{H}_2\left(\mathbb{Z}_{2m+1}\wr\mathbb{S}_n\right)\cong\mathrm{H}^2\left(\mathbb{Z}_2\wr\mathbb{S}_n\right)\cong\begin{cases}\{e\}, & n=0,1\\ \mathbb{Z}_2, & n=2\\ \mathbb{Z}_2\oplus\mathbb{Z}_2, & n=3\\ \mathbb{Z}_2\oplus\mathbb{Z}_2\oplus\mathbb{Z}_2, & n\geq 4\end{cases}$$

---

[14] The abuse stands as the writing is not correct if Y is uncountable, despite (1.26) always holds.

[15] This action is useful to slim out some lengthy group arithmetics; also $[h|\,g]=h^{-1}\circ h^g$ holds.

1.1.7. Direct and inverse limits

Let $(I;\preccurlyeq)$ be a directed set (i.e. $\mathrm{Card}\,I\geq\aleph_0$ and $\nexists\max_{\preccurlyeq}I$), $\{G_i\}_{i\in I}$ a family of groups and $\{\phi_i^j\}_{i,j\in I,\, i\preccurlyeq j}$ a semigroup with $\phi_i^j\in\mathrm{Hom}(G_i,G_j)$ [or $\phi_i^j\in\mathrm{Hom}(G_j,G_i)$] with respect to composition such that, for $i\preccurlyeq j\preccurlyeq k$:

$$\phi_j^k\circ\phi_i^j=\phi_i^k;\qquad \phi_{i,i}\equiv\mathbb{1}. \tag{1.28}$$

The structure $(G_i,\phi_i^j)_{i,j\in I,i\preccurlyeq j}$ is called a direct [inverse] system of groups.

Now consider the disjoint union $\bigsqcup_{i\in I}G_i$ and quotient it respect to '$\sim$', the following equivalence relation: $(i,g)\sim(j,h)\ \Leftrightarrow\ \exists k\in I$: $\phi_i^k(g)=\phi_j^k(h)$ $(g\in G_i,\ h\in G_j)$; let $[\![g_i]\!]_\sim$ denote the equivalence classes of $\bigsqcup_{i\in I}G_i\big/\sim$, represented by (i,g_i), $g\in G_i$. For any $[\![g_i\sim g_{i'}]\!]_\sim$ and $[\![g_j\sim g_{j'}]\!]_\sim$, $\exists k\succcurlyeq i,i',j,j'$ such that

$$\phi_i^k(g_i)\circ_k\phi_j^k(g_j)=\phi_{i'}^k(g_{i'})\circ_k\phi_{j'}^k(g_{j'}); \tag{1.29}$$

a pairing can be defined as $[\![g_i]\!]_\sim[\![g_j]\!]_\sim:=\left[\!\!\left[\phi_i^k(g_i)\circ_k\phi_j^k(g_j)\right]\!\!\right]_\sim$ and $[\![g_i]\!]_\sim^{-1}:=[\![g_i^{-1}]\!]_\sim$. This pairing is a group operation in $\bigsqcup_{i\in I}G_i\big/\sim$ (whose neutral element is $[\![e_i]\!]_\sim$), which is called ***direct limit*** of $\{G_i\}_{i\in I}$ and is indicated with the equivalent symbols:[16]

$$\varinjlim_{i\in I}G_i\qquad\text{or}\qquad \lim_{\preccurlyeq}\{G_i\}_{i\in I}.$$

It can be proven that the direct limit $\varinjlim_{i\in I}G_i$ is unique, in the sense that it does not depend on the choice of the semigroup $\{\phi_i^j\}_{i,j\in I,\ i\preccurlyeq j}$ (which always exist since the directedness of I).

The first example of direct limits is that of a sequence of groups $\{G_n(\circ_n),\sigma_n\}_{n\in\mathbb{N}}$, where $\sigma_n\in\mathrm{Hom}(G_n,G_{n+1})$, and the homomorphisms' semigroup $\{\phi_m^n\}_{m,n\in\mathbb{N},m\leq n}$ is

$$\phi_m^n:=\sigma_n\circ\sigma_{n-1}\circ\ldots\circ\sigma_{m+1}\circ\sigma_m \tag{1.30}$$

[16] The direct limit algebraic operator '$\varinjlim$' is not to be confused with topological limit one '$\lim$', although their resulting sets can be isomorphic for topological groups' sequences, cfr. §1.1.7.

and there is a proper inclusion $G_n \lnapprox G_{n+1}$, $\forall n\in\mathbb{N}$. The direct limit of sequence $(G_n)_{n\in\mathbb{N}}$ is then $\varinjlim_n G_n$.

The second example of direct limits is related to homological algebra [81], [9]–§2.5: let $C=\{C_p,\partial_p\}_{p\in\mathbb{Z}}$ be an additive chain group and $C^*=\{C^*_p,\delta_p\}_{p\in\mathbb{Z}}$ be its conjugate one; customarily, let $\partial_p\partial_{p+1}\equiv 0_{p-1}$ and $\delta_p\delta_{p-1}\equiv 0_{p+1}$ (where $0_p\in C_p$ is the netural element), $\forall p\in\mathbb{Z}$. Two homomorphisms' semigroups can be defined, $\{\phi^q_p\}_{p,q\in\mathbb{Z},p\geq q}$ and $\{\psi^q_p\}_{p,q\in\mathbb{Z},p\leq q}$, with:

$$\phi^q_p = \begin{cases} \partial_p, & p=q-1 \\ 0, & p\neq q-1 \end{cases} \qquad \psi^q_p = \begin{cases} \delta_p, & p=q+1 \\ 0, & p\neq q+1 \end{cases} \tag{1.31}$$

Then both a *direct* and an ***inverse* limit** group exist (unique up to isomorphism),

$$\varinjlim C_p := \lim_{\leq}\{C^*_p\}_{p\in\mathbb{Z}}, \qquad \varprojlim C_p := \lim_{\geq}\{C_p\}_{p\in\mathbb{Z}}, \tag{1.32}$$

where both limits are trivial in the case of finite-rank groups $C_p(+)$.

- $\overleftarrow{\mathbb{Z}}_p \overset{\text{def}}{=} \varprojlim_n \mathbb{Z}_{p^n}$, which is the set of ***p*-adic integers**, $\forall p\in\mathbb{P}$.
- $\varprojlim_n \mathbb{Z}_n = \bigoplus_{p\in\mathbf{P}} \mathbb{Z}_p$.

1.1.8. Free products and free groups

The *free product* of the groups' family $\{G_i\}_{i\in I}$ is the quotient subset indicated as

$$\mathop{\ast}_{i\in I} G_i \overset{\text{def}}{=} \bigcup_{n\in\mathbb{N}}\left(\bigsqcup_{i\in I} G_i\right)^n \Big/ \sim \tag{1.33}$$

(or as $G_1 * G_2 * G_3 * \ldots$ if I is at most countable, e.g. $I\subseteq\mathbb{N}$). $\ast_{i\in I}G_i$ contains any *finite-length* **words** of elements of groups $\{G_i\}_{i\in I}$ (an r-tuple of any finite length r) simply indicated via the *juxtaposition*

$$g_1 g_2 g_3 \cdots g_r ::\equiv ((i_1, g_1), (i_2, g_2), \ldots, (i_r, g_r)) \in \left(\bigsqcup_{i \in I} G_i \right)^r, \qquad \forall r \in \mathbb{N},$$

where $i_1, i_2, \ldots, i_r \in I$ such that

$$(i_j, g_j) \in \bigsqcup_{i \in I} G_i, \qquad 1 \le j \le r.$$

So (1.33) is the set of all finite-length words composed of characters of the alphabet $\bigsqcup_{i\in I} G_i$; for any word represented by $g_1 g_2 \cdots g_r$, each g_k $(1 \le k \le r)$ is called a syllable, which is a diphthong if $g_k \notin \bigcup_{i\in I} G_i$ (i.e. it is not a letter of the alphabet).

The zero-length word '1' is unique[17] whereas $*_{i\in I} G_i$ becomes a group respect to juxtaposition of sequences and the equivalence relation '$\sim$', both acting as:

- juxtaposition of words $(g_1 g_2 \cdots g_r,\ h_1 h_2 \cdots h_s) \mapsto g_1 g_2 \cdots g_r h_1 h_2 \cdots h_s$, such that, for any juxtapositions $(g,h) \mapsto gh$ for which $\exists i \in I$: $g,h \in G_i$, then $gh \sim g \circ_i h$;
- inversion of tuplets $(g_1 g_2 \cdots g_r)^{-1} \sim g_r^{-1} g_{r-1}^{-1} \cdots g_2^{-1} g_1^{-1}$ (inversion being carried out by induction down to single group elements which are inverted respect to their own group inversions);
- $1g \sim g1 \sim g$, implying both $1^{-1} \sim 1$ and $1e_i \sim e_i 1 \sim e_j \sim 1$, $\forall i,j \in I$ (e_i being the neutral element of G_i);

$\forall g_1, g_2, \ldots, g_r, h_1, h_2, \ldots, h_s \in *_{i\in I} G_i$. This means that, $g_1 g_2 \cdots g_r \sim h_1 h_2 \cdots h_s$ if and only if $g_1 g_2 \cdots g_r$ becomes *equal* to $h_1 h_2 \cdots h_s$ after a finite number (in general non-unique) of the above equivalences, i.e. performing any finite number of the operations below:

- inserting or deleting any number of 1's in any positions;

[17] Also indicated with either a pre-defined symbol like '0', '0' or '1', '1', depending on whether the groups are additive or multiplicative, respectively.

- replacing two or more consecutive elements beloging to the *same* group G_i of the family with their combined element respect to '$\circ_i$' (contraction) or vice versa (expansion).

A word $g_1g_2\cdots g_r \in \ast_{i\in I}G_i$ is said to be reduced if both $g_h \notin \{1, e_i\}_{i\in I}$ and for $|h-k|=1 \Rightarrow g_h, g_k \notin G_h$ hold, $1\leq h\leq k\leq r$, i.e. if the word $g_1g_2\cdots g_r$ neither contains neutral elements, nor consecutive elements belonging to the same group. Every element of $\ast_{i\in I}G_i$ admits (i.e. is equivalent to) a reduced form and this does not depend on the cardinality of I (which is a powerful achievement since many problems usually arise when dealing with the Algebra of uncountable structures). For any words, their reduced equivalences, despite not being unique, have all the same length.

Elementary properties of free-product groups include the fact that free product is both an associative and commutative functor, i.e. $(G\ast H)\ast K = G\ast(H\ast K) = G\ast H\ast K$ and $G\ast H = H\ast G$.

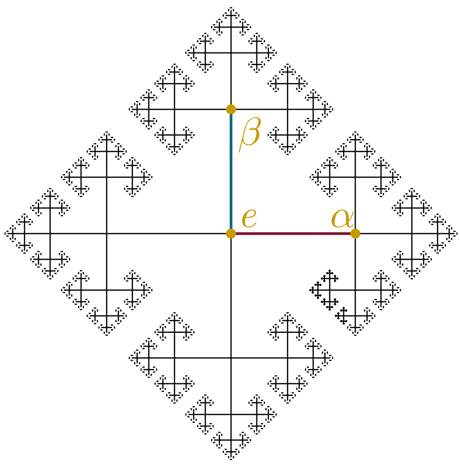

Figure 3 Cayley graph of the free group $\mathbb{F}_2$ generated by a two-elements alphabet $\{\alpha, \beta\}$ (and neutral element the empty word e). Source: [104].

In particular, the inverse limit of a directed groups' family $\{G_i\}_{i\in I}$ (with respect to a directed set $(I;\preccurlyeq)$, cfr. §1.1.7) is always isomorphic to a subgroup of its free product:

$$\varprojlim_{i\in I} G_i \lessapprox \mathop{\ast}_{i\in I} G_i .$$

Free groups can be generated starting from any set $S\neq\varnothing$ (or even from other groups) via group presentations [9]–§A.3, which are nothing but a way to represent group elements as words whose alphabet is given by elements of S together with relations among them. In particular, free groups generated by a finite set of N elements are unique (up to isomorphism) and indicated as[18] $\mathbb{F}_N(\circ)$, $\forall N\in\mathbb{N}$. That topic goes beyond the scope of this work, but it is worth noting mentioning some examples below — and also see [71].

Furthermore:

$$\operatorname{rk}\mathop{\ast}_{i\in I} F_i = \sum_{i\in I}\operatorname{rk}F_i, \tag{1.34}$$

for any family of *free* groups $\{F_i\}_{i\in I}$.

A presented group $G=\langle S|\mathcal{R}_i\rangle_{i\in I}$ is *free* if and only if $I=\varnothing$, i.e. there are no relations defining the group (respect to the generating set S); otherwise G is said to be either finitely- or infinitely-generated [-presented] according to $\operatorname{Card}S$ [$\operatorname{Card}I$]. The generating set S is "symmetric" if $s^{-1}\in S< G\,\forall s\in S$.

Also recall that, if $H=\langle T|\mathcal{Q}_j\rangle_{j\in J}$ is another group, then

$$G\circledcirc H=\langle S\cup T\,|\,\mathcal{R}_i,\,\mathcal{Q}_j,\,[S|T]\rangle_{i\in I,j\in J};$$

$$G\ast H=\langle S\cup T\,|\,\mathcal{R}_i,\mathcal{Q}_j\rangle_{i\in I,j\in J}. \tag{1.35}$$

To any group $G=\langle S|\mathcal{R}_i\rangle_{i\in I}$ the following graphs[19] are defined:

---

[18] Free groups $\mathbb{F}_N$ must not be confused with fields of order a prime power $\mathbb{F}_{p^n}$, defined in §1.1.3.

[19] Recall that for a **graph** $(N,S,@)$ the node-set is N, the side-set is S and $@\subseteq S\times N^2$ is the ***incidence*** relation. If the graph has no cycles, it is a **tree**; for more definitions on graphs, cfr. [9]–§A.1, [16].

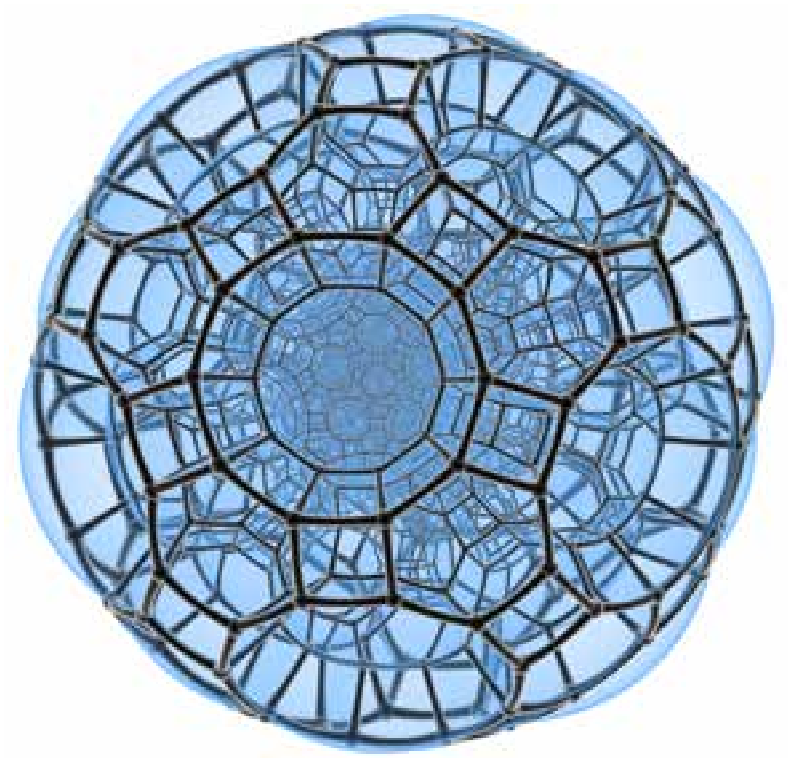

Figure 4 Cayley graph of $\mathbb{Z}_{10}*\mathbb{Z}_6*\mathbb{Z}_4$, projected sterepographically onto a sphere.

- The *Cayley* graph of G is the oriented, connected graph $(G,S,@)$, where $g@h \Leftrightarrow \exists s\in S$: $h=g\circ s$, $\forall g,h\in G$. It has $\mathrm{Card}\,I$ cycles[20] (it has as many cycles as the relations defining G) and every node has incidence index $2\mathrm{Card}S$ (it has as many leading in *and* leading out sides as the elements of S). If S is symmetric the graph is non-oriented. See Figure 3 , Figure 4 , Figure 5 .

- If S is symmetric and G acts on X (on the left) the *Schreier* graph of G is the oriented graph $(X,S,@)$, where $x@y \Leftrightarrow \exists s\in S$: $y=s\bullet x$, see Figure 9 .

[20] Inspecting a Cayley graph tells how many relations $\mathcal{R}_i$ are needed for the presentation of G.

Examples of free-product groups include[21] $\mathbb{D}_\infty \cong \mathbb{Z} \rtimes \mathbb{Z}_2 \cong \mathbb{Z}_2 * \mathbb{Z}_2$ and $\mathrm{PSL}_2(\mathbb{Z}) \cong \mathbb{Z}_2 * \mathbb{Z}_3$. Figure 4 and Figure 5 show the Cayley graphs of $\mathbb{Z}_{10} * \mathbb{Z}_6 * \mathbb{Z}_4$ and of $\mathrm{SL}(2,5)$, respectively.

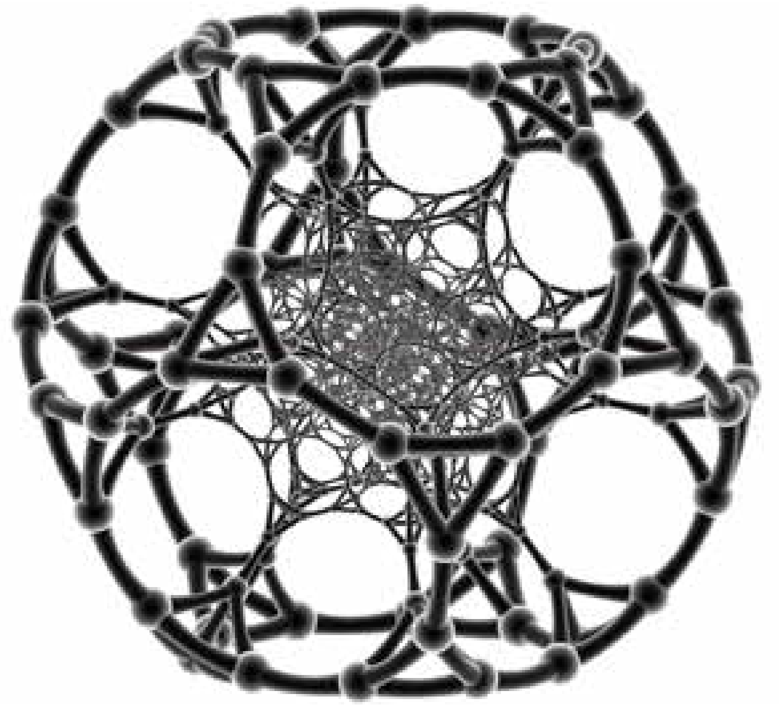

Figure 5 Cayley graph of $\mathrm{SL}(2,5)$, sterepographically projected onto a sphere.

It can be proven that the isotopy conditions let the braid groups (cfr. §1.1.2) be, simply, the free groups of presentations, $\forall n \in \mathbb{N}$:

$$\mathbb{B}_n := \left\langle \sigma_i \in \mathbb{S}_{n-1} \;\middle|\; \sigma_i \sigma_{i+1} = \sigma_{i+1} \sigma_i,\ \sigma_{i+1} \sigma_i \sigma_{i+1} = \sigma_i \sigma_{i+1} \sigma_i \right\rangle_{1 \le i < n}. \qquad (1.36)$$

In particular, $\mathbb{B}_1 = \{\mathbb{1}\}$, $\mathbb{B}_2 \cong \mathbb{Z}$, whereas $\mathbb{B}_n$ is both infinite and non-abelian for $n \geq 2$.

[21] In this case let ρ be the "adding machine" $\rho(n) = n+1$, and $\phi(n) = -n$, $\forall n \in \mathbb{Z}$; cfr. §1.1.1.

1.1.9. Growth rates

Let S be a fixed symmetric set of generators of a free group G, then every element of $g\in G$ is a word written as a juxtaposition of a finite number of generators; g always admits a reduced word representation, i.e. is generated by a minimal number of generators (the length of the reduced word, $\mathrm{l}_S(g)$, which depends on the generating set S). Every free group is thus a metric space with respect to the S-***word metric*** $\mathrm{w}_S\colon G\times G\to \mathbb{N}_0$:

$$\mathrm{w}_S(g,h):=\mathrm{l}_S(g^{-1}h), \qquad \forall g,h\in G. \tag{1.37}$$

> By using reduced-word representation for braid groups defined in (1.36) and using their elementary braids $\sigma_1,\sigma_2,\sigma_3,\ldots$ as generators 'weighting' $+1$ and their inverse as 'weighting' -1, the braids' length becomes a homomorphism $\mathrm{l}_{\mathbb{B}}\in\mathrm{Hom}(\mathbb{B}_n,\mathbb{Z})$, where for example letting $\beta\in\mathbb{B}_n$ such that $\beta=\sigma_3\sigma_1\sigma_2\sigma_1^{-1}\sigma_4^{-1}\sigma_3^2\sigma_2^{-1}$, leads to:
>
> $$\mathrm{l}_{\mathbb{B}}(\beta):=1+1+1-1-1+2-1=2.$$

The open balls $\mathbf{B}_N(g)\subseteq G$, $\forall g\in G$, $\forall N\in\mathbb{N}_0$ —always depending on the generating set S— are the subsets of words whose "S-word distance" from g is lower than N. It can be proven that any two word metrics on G are equivalent (i.e. for any couples of symmeyric generating sets); the S-**growth** function $\mathrm{Card}\,\overline{\mathbf{B}_N(e)}$ is defined as depending on variable $N\in\mathbb{N}_0$. It is strictly monotone-increasing; $\mathrm{Card}\,\overline{\mathbf{B}_N(e)}\leq(2\mathrm{Card}\,S+1)^N$ and its asymptotic behaviour as $N\to\infty$ partitions every finitely-generated group in three classes:

Generalized *Gromov*'s Theorem

The ***growth rate*** of any (infinite) finitely-generated group G does not depend on the choice of its generating set and is of three types:

- polynomial (P), if $\mathrm{Card}\,\overline{\mathbb{B}_N(e)} \sim N^d$ for some $d \in \mathbb{R}_+$;
- exponential (E), if $\mathrm{Card}\,\overline{\mathbb{B}_N(e)} \sim e^N$;
- intermediate (I), if it is neither polynomial nor exponential (and in this case $N^d \lesssim \mathrm{Card}\,\overline{\mathbb{B}_N(e)} \lesssim e^N,\ \forall d \in \mathbb{R}_+$).

Furthermore G has polynomial growth if and only if it has a nilpotent subgroup of finite index (see notes [1–2]).

The Theorem proves that "superpolynomial" and "subexponential" growth groups —like $\mathrm{O}(N^N)$, $\mathrm{O}(N^{\log\log N})$ or $\mathrm{O}(e^{N/\log N})$, $\mathrm{O}(e^{\sqrt{N}})$, respectively— fall in the same category.

The infinite *Grigorčuk* group $\mathbb{G}(\cdot)$ has endomorphic presentation

$$\mathbb{G} \stackrel{\text{def}}{=} \left\langle a,b,c,d \,\middle|\, a^2, b^2, c^2, d^2, bcd, \tau^n(ad)^4, \tau^n(adacac)^4; \forall n \in \mathbb{N} \right\rangle,$$

$$\tau \in \mathrm{End}\big(\mathrm{free}(a,b,c,d)\big):\qquad \begin{cases} a \mapsto aca \\ b \mapsto d \\ c \mapsto b \\ d \mapsto c \end{cases} \tag{1.38}$$

(where $\mathrm{free}(a,b,c,d) \equiv a*b*c*d \cong \mathbb{F}_4$, the free group generated by 4 elements, cfr. §1.1.8). It is a self-similar infinite 2-group[22] (of neutral element e), has intermediate growth and every element $g \in \mathbb{G}$ is a word composed by $m \in \mathbb{N}$ juxtaposed diphthongs as_i of the form

$$s_{\mathrm{b}}as_1as_2a\cdots s_{m-1}as_mas_{\mathrm{f}},$$

[22] A *p*-group $G(\cdot)$ has all the elements nilpotent of index a power of p, i.e. $\forall g \in G\ \exists n \in \mathbb{N}:\ g^{p^n} = e$.

with $s_i \in \{b,c,d\}$, $1 \leq i \leq m$ and $s_b, s_f \in \{e,b,c,d\}$.

The Grigorčuk group was discovered in 1984 in the course of proving the existence of intermediate-growth groups, [57]; it also provides a self-similar action on rooted trees (cfr. §1.4.2) and was the first group to be studied with the new methods of self-similar groups [82], [83].

Groups may also have "superexponential" growth rates such as $\mathrm{O}(\boxed{n})$. The symbol $\boxed{n}$ is the ***expofactorial*** of $n \in \mathbb{N}_0$, recursively defined $\forall n \in \mathbb{N}$ as:

$$\begin{cases} \boxed{0} := 0 \\ \boxed{n} \overset{\text{def}}{=} n^{\boxed{n-1}} \end{cases} \tag{1.39}$$

The expofactorial sequence $(\boxed{n})_{n\in\mathbb{N}_0} = (0, 1, 2, 9, 262\,144, \ldots)$ is the "factorial analogue" to a power tower (thus it is also called a **factorial power tower**) and is present in [95]–A049384. The fifth expofactorial $\boxed{5} = 5^{4^{3^2}} \equiv 5^{262\,144} \simeq 6.20 \cdot 10^{183230}$ is already "quite a large" number with $183\,231$ decimal digits.

1.1.10. Basic Galois Theory

First of all recall that, let F and K be two fields such that $K < F$ (or $K \lesssim F$), then F is said to be an **extension field** of K, and the quotient field F/K is called a **field extension** of K. If $E \subset F \backslash K$ then the smallest sub-field[23] $K(E) \leq F$ such that $K, E \subset K(E)$ is proven to be the following extension field:

$$K(E) = \bigcup_{n=1}^{\infty} \Big\{ f(e_1, e_2, \ldots, e_n) \;\Big|\; \forall f \in K\{x_1, x_2, \ldots, x_n\},\ \ \forall e_1, e_2, \ldots, e_n \in E \Big\},$$

with $K(\{e_1, e_2, e_3, \ldots\}) \equiv: K(e_1, e_2, e_3, \ldots)$. A *radical* extension field $K(e_1, e_2, e_3, \ldots)$ is such that $\exists n_1, n_2, n_3, \ldots \in \mathbb{N} \backslash \{1\}$, with $e_1^{n_1} \in K$ and $e_k^{n_k} \in K(e_1, e_2, \ldots, e_{k-1})$, $\forall k \in \mathbb{N} \backslash \{1\}$ —cfr. §1.5.3 for more information— and if $n_1 = n_2 = n_3 = \ldots = n$ the extension is

[23] $K(E)$ [resp. $K[E]$] is the intersection of all the sub-fields [-rings] of F containing both K and E.

called n-radical. If $r\in F\backslash K$, then $K(r)$ is called a simple extension field [or a cyclotomic extension field if r is a n^{th} root of unity, i.e. $\exists n\in\mathbb{N}\backslash\{1\}$ such that $r^n=1$ (and $r^{n-1}\neq 1$)]. Simple and radical field extensions are related to rational functions, as well as with formal power and Laurent series, more extensively treated in §1.3: in fact, for any field extension $K(\mathbf{e})$ (with $\mathbf{e}\equiv(e_1,e_2,\dots,e_n)\in F^n$)

$$K(\mathbf{e}) = \left\{\frac{N(\mathbf{e})}{D(\mathbf{e})}\ \middle|\ \forall\frac{N(\mathbf{x})}{D(\mathbf{x})}\in K\{\mathbf{x}\},\quad D(\mathbf{e})\neq 0\right\}, \tag{1.40}$$

i.e. it is the set (field) of rational functions $K\{\mathbf{x}\}$ *evaluated* in the n-tuple $\mathbf{e}\in F^n$. Since field extensions (and extension fields) are vector spaces over K, their dimensions are called the degrees of field extensions: $[F:K]\stackrel{\text{def}}{=}\dim_K F/K$.

- The fields of p-adic numbers $\mathbb{Q}_p:=\operatorname{Frac}\varprojlim_n\mathbb{Z}_{p^n}$ for any $p\in\mathbb{P}$ are non-ordered extension fields of $\mathbb{Q}$.
- The field of rational functions $K\{x\}:=\operatorname{Frac}K[x]$, §1.3.1, is an extension field of its base field K, with $[K\{x\}:K]=\aleph_0$.
- $\mathbb{Q}(\iota)=\mathbb{Q}+\iota\mathbb{Q}<\mathbb{R}$ is an extension field of $\mathbb{Q}$ for any *irrational* number $\iota\in\mathbb{R}\backslash\mathbb{Q}$, with $[\mathbb{Q}(\iota):\mathbb{Q}]=2$ and generators $\{1,\iota\}$.
- $\mathbb{C}$ is a extension field of both $\mathbb{R}$ and the Gaussian numbers'[24] field $\mathbb{Q}(\imath)$: both are intermediate fields (but via independent paths) respect to $\mathbb{Q}$. In particular, $\mathbb{C}\equiv\mathbb{R}(\imath)\ \Rightarrow\ [\mathbb{C}:\mathbb{R}]=[\mathbb{Q}(\imath):\mathbb{Q}]=2$ (because they have two generators, $\{1,\imath\}$), whereas $[\mathbb{C}:\mathbb{Q}]=[\mathbb{R}:\mathbb{Q}]=\mathfrak{c}$.
- Let $f=N/D\in K\{x\}\backslash K$, with $N,D\in K[x]$ and $\operatorname{GCD}(N,D)=1$. Then $K(f)\cong K\{f(x)\}$ is a finite [simple] extension field of $K\{x\}$ [of K], and:

$$\Big[K\{f\{x\}\}:K\{x\}\Big] = \max\Big\{\deg N,\deg D\Big\}. \tag{1.41}$$

[24] Not to be confounded with its subring of Gaussian integers, $\mathbb{Z}(\imath)=\mathbb{Z}+\imath\mathbb{Z}$.

- If $H \leq K \leq F$ then $[F{:}H] = [F{:}K][K{:}H]$. As a Corollary, $[F{:}H] \in \mathbb{P}$ implies that there are *no* intermediate extension fields $H < K < F$.

Any element of F which is [not] a root of polynomials in $K[x]$ is said to be an *algebraic* [*transcendent*] element over K. If $e_1, e_2, \ldots, e_n \in F \setminus K$ are algebraic over K, then $K[e_1, e_2, \ldots, e_n]$ is a finite-dimensional field extension of K and:

$$\Big[K[e_1, e_2, \ldots, e_n] : K \Big] \leq \prod_{j=1}^{n} \Big[K(e_j) : K \Big]. \tag{1.42}$$

Every field K admits an ***algebraic** closure* as the smallest extension field $K^{\natural} \gtrapprox K$ that is algebraic over K; moreover K is said to be **algebraically** closed if $K = K^{\natural}$. Every irreducible polynomial $P \in K[x]$ has *exactly* $\deg P$ roots (counted with their multiplicities) in $K^{\natural}$; this result is known as the Fundamental Theorem of Algebra, although its original statement (and proof) just related to a specific field:

Fundamental Theorem of Algebra historical statement

The field of complex numbers $\mathbb{C}$ is algebraically closed.

The algebraic closure of any finite field is equal to the union of all the finite fields with the same characteristic, cfr. §1.1.3, i.e.:

$$\mathbb{F}^{\natural}_{p^N} \cong \bigoplus_{n=0}^{\infty} \mathbb{F}_{p^n}, \qquad \forall p \in \mathbb{P},\ \forall N \in \mathbb{N}.$$

- If $H < K < F$ and both F/K and K/H are algebraic[25] field extensions, then F/H is algebraic too.
- $\mathbb{C}$ is the algebraic closure of both $\mathbb{R}$, $\mathbb{Q}(\imath)$ (as an intermediate fields, but via independent paths) and $\mathbb{Q}$:

$$\mathbb{C} = \mathbb{R}^\natural \equiv \mathbb{Q}(\imath)^\natural \equiv \mathbb{Q}^\natural \qquad \Leftrightarrow \qquad \begin{tikzcd} & \mathbb{Q}(\imath) \arrow[dr, "\natural"] & \\ \mathbb{Q} \arrow[rr, "\natural"] & & \mathbb{C} \\ & \mathbb{R} \arrow[ur, "\natural"'] & \end{tikzcd} .$$

A field extension F/K is also said to be:

- **Normal**: if every irreducible polynomial in $K[x]$ which has a root in F can be factored into a simple polynomial of $F[x]$. Every field admits a normal closure $K^\mathfrak{n}$, i.e. a minimal extension field respect to which $K^\mathfrak{n}/K$ is a normal field extension.
- **Separable**: if the minimal polynomial[26] $\mathrm{Min}_y^K(x) \in K[x]$ of any $y \in F \setminus K$ is separable[27] in $F[x]$. Every finite separable field extension is simple.
- **Galois**: if it is both normal and separable.

If F/K is a Galois extension, then all the automorphisms of F which leave K invariant, also *fix* K (cfr. §1.1.5) and such group $\mathrm{Aut}(F/K)$ is said to be the ***Galois group*** of field extension F/K, usually indicated as either $\mathrm{Gal}(F/K)$ or $\mathrm{Gal}_K F$, cfr. [79]. When F/K is either infinite and *not* Galois, then one considers the **Galois** closure of F, i.e. the smallest extension field $F^\mathfrak{G} \gtrapprox F$ such that $F^\mathfrak{G}/K$ is a Galois field extension of K — whenever such a field exists. In this case $\mathrm{Gal}(F/K)$ and $\mathrm{Gal}_K F$

[25] If all the elements of F are algebraic over K, the former is an **algebraic** extension over the latter.

[26] The ***minimal*** *polynomial* in $R[x]$ of any $y \in K$ (where R and K are homomorphic algebraic structures) is the monic, minimum-degree polynomial $\mathrm{Min}_y^R \in R[x]$ such that $\mathrm{Min}_y^R(y) = 0$.

[27] A polynomial $P \in K[x]$ is called **separable** if it is simple (i.e. it has simple roots only) in $K^\natural[x]$. P is separable if and only if it is co-prime with its formal derivative (cfr. §1.1.3), i.e. iff $\mathrm{GCD}(P, P') = 1$.

automatically refer to this Galois-closure extension, which is by definition a *compact* group, $\mathrm{Gal}(F/K) :\equiv \mathrm{Aut}(F^{\mathfrak{G}}/K)$ which is a different group from $\mathrm{Aut}(F/K)$ — but that is *not* always common practice and sometimes $\mathrm{Gal}(F/K) = \mathrm{Aut}(F/K)$ is simply used in Literature. In general $\mathrm{ord}\,\mathrm{Aut}(F/K) \leq [F{:}K]$, with the equality holding if and only if F/K is Galois, i.e.:

$$\mathrm{ord}\,\mathrm{Gal}(F/K) = [F{:}K]. \tag{1.43}$$

Examples of Galois groups include:

- $\mathrm{Gal}(\mathbb{Q}(\iota)/\mathbb{R}) = \{\mathbb{1}, \iota\mathbb{1}\},\ \forall \iota \in \mathbb{R} \backslash \mathbb{Q}$;
- $\mathrm{Gal}(\mathbb{R}/\mathbb{Q}) = \{\mathbb{1}\}$;
- $\mathrm{Gal}(\mathbb{C}/\mathbb{R}) = \{\mathbb{1},\ \boldsymbol{i} \equiv \imath\mathbb{1}\}$ (cfr. §2.1.1 for automorphisms like '$\boldsymbol{i}$');
- $\mathrm{Gal}(\mathbb{C}/\mathbb{Q}) \cong \mathbb{R}(+)$, a typical example of *transcendental* extensions.
- $\mathrm{Gal}(\mathbb{F}_{p^n} / \mathbb{F}_p) \cong \mathbb{Z}_n$. In particular every extension of $\mathbb{F}_{p^n}$ is separable and $\mathbb{F}_{p^n}$ is the unuique n^{th}-degree extension of $\mathbb{F}_p$, $\forall n \in \mathbb{N}$.

Fundamental Theorem of (finite) Galois Theory

For any finite field extension F of K there is a 1-to-1 "inclusion-reversing" correspondence between intermediate fields extensions $K < H < F$ and the subgroups $G_H \leq \mathrm{Gal}(F/K)$. If $H \cong G_H$ (as groups):

$$[F{:}H] = \mathrm{ord}\, G_H,$$

$$[H{:}K] = \mathrm{ord}\,\frac{\mathrm{Gal}(F/K)}{G_H}, \tag{1.44}$$

and $G_H \lhd \mathrm{Gal}(F/K)$ if and only if H is a Galois extension over K, in which case also

$$\mathrm{Gal}(H/K) \cong G/G_H,$$

The Fundamental Theorem of finite Galois Theory states that every intermediate field of a finite Galois field extension F/K is isomorphic (as a group) to a subgroup of its Galois group, and that this correspondence between subgroups of the Galois group and the associated field sub-extensions is "inclusion-reversing," that is:

$$K < H' < H'' < F \qquad \Rightarrow \qquad \mathrm{Gal}(F/H'') \leq \mathrm{Gal}(F/H') \leq \mathrm{Gal}(F/K). \quad (1.45)$$

By this property, the process of constructing larger and larger extensions fields of a given field K (up to the largest needed, F) corresponds to finding smaller and smaller subgroups of the Galois group:

> *Abel-Ruffini*'s Theorem
>
> The symmetric group $\mathbb{S}_n$ with $n \geq 5$ is *not* solvable; [86], [79].

Abel's Theorem was born (and still finds its major applications) in the Galois' Theory of polynomials: in this context it states that there exist polynomials of degree greater than 4 which are not solvable 'by radicals,' i.e. by expressing all of their roots as closed-form radical functions of their coefficients — cfr. [79] and §1.5.4 for further considerations on those topics.

Let $P \in K[x]$ be an n^{th}-degree irreducible polynomial; if $\exists \alpha \in K$ and $x_1, x_2, \ldots, x_n \in F$ such that $P(x) = \alpha \prod_{1 \leq j \leq n} (x - x_j) \in F[x]$, then it is said that P splits over F. Similarly, the extension field $K(\ker P) \equiv K(x_1, x_2, \ldots, x_n)$ (i.e. the extension made with the roots of a split polynomial P) is called the ***splitting* field** of P in F; its extension field and the associated Galois group are usually indicated as:

$$K[x]/P :\equiv \frac{K[x]}{P(x)} \stackrel{\mathrm{def}}{=} \frac{K[x]}{K(\ker P)}; \quad (1.46)$$

$$\mathrm{Gal}_K P \stackrel{\mathrm{def}}{=} \mathrm{Gal}\, \frac{K(\ker P)}{K}. \quad (1.47)$$

An equivalent condition for a field extension F/K to be Galois is that F has to be the splitting field of a separable monic polynomial $P\in K[x]$, that P is reducible in F and its roots $\ker P$ generate F over K, i.e. by (1.47): $F=K(\ker P)$ or, more compactly:

$$F/K \text{ Galois} \quad \Rightarrow \quad \exists P\in K[x]\backslash K: \quad F=K(\ker P). \tag{1.48}$$

Galois' Theorem

Let $P\in K[x]$ be a polynomial in a zero-characteristic field K; P is solvable 'by radicals' (i.e. all its roots can be computed in closed form as depending on P's coefficients) if and only if the Galois group associated to its splitting extension field, $\mathrm{Gal}(K[x]/P)$, is solvable.

The application of Galois Theory to a polynomial $P\in K[x]$ reduces to computing whether the roots of P all lie in a radical extension field of K or not. If such a field exists, it is exactly the splitting field of P, which it can be completely factored in.

Corollary

A polynomial $P\in K[x]$ is (completely) 'solvable by radicals' over an extension field $F\geq K$ if and only if all the Galois' subgroups of their intermediate extensions are cyclic.

Since $\mathrm{Gal}(K[x]/P) \cong \mathbb{S}_{\deg P}$ (cfr. §1.5.4) and, by Abel's Theorem formulated in §1.1.10, $\mathbb{S}_n$ is not solvable for $n\geq 5$, then it follows that the generic n^{th}-degree polynomial is *not* solvable by radicals if $\deg P>4$.

Consider the polynomial $(x^4-5x^2+6)=(x^2-2)(x^2-3)\in\mathbb{Q}^4[x]$, whose four roots (all simple and in $\mathbb{R}$) are $\mp\sqrt{2}$ and $\mp\sqrt{3}$. $\mathbb{Q}(\sqrt{2},\sqrt{3})$ is its splitting and Galois field over $\mathbb{Q}$ and is 4^{th} degree,

$$\left[\mathbb{Q}(\sqrt{2},\sqrt{3}):\mathbb{Q}\right]=\dim_{\mathbb{Q}}\frac{\mathbb{Q}[x]}{x^4-5x^2+6}=4,$$

with the four Galois automorphisms —just described as their action on the roots, represented as permutations of $\mathbb{S}_4$— being:

$$(\sqrt{2},\sqrt{3})\xrightarrow{\mathbb{1}}(\sqrt{2},\sqrt{3}),\qquad(\sqrt{2},\sqrt{3})\xrightarrow{\sigma}(-\sqrt{2},\sqrt{3}),$$
$$(\sqrt{2},\sqrt{3})\xrightarrow{\tau}(\sqrt{2},-\sqrt{3}),\qquad(\sqrt{2},\sqrt{3})\xrightarrow{\sigma\tau}(-\sqrt{2},-\sqrt{3});$$

so $\mathrm{Gal}\left(\mathbb{Q}(\sqrt{2},\sqrt{3})/\mathbb{Q}\right)\equiv\{\mathbb{1},\sigma,\tau,\sigma\tau\}\cong\mathbb{D}_4$. The intermediate fields are $\mathbb{Q}(\sqrt{2})$, $\mathbb{Q}(\sqrt{3})$ and $\mathbb{Q}(\sqrt{6})$, exactly as there are 3 "intermediate" subgroups of its Galois group: $\langle\sigma\rangle$, $\langle\tau\rangle$ and $\langle\sigma\tau\rangle$, all isomorphic to $\mathbb{Z}_2$.

Suppose K is a field with a derivation[28] $\mathrm{D}\in\mathrm{D}[K]<\mathrm{End}K$, cfr. [9]–§A.1; the subfield $\mathrm{con}_{\mathrm{D}}K:=\mathrm{Ker}\,\mathrm{D}$ is the subfield of constants[29] of D in K. A simple, transcendental extension field $K(\alpha)$ is said to be a *exponential* [*logarithmic*] extension of K if $\exists\beta\in K$ such that $\mathrm{D}\alpha=\alpha\mathrm{D}\beta$ [respectively $\mathrm{D}\alpha=\beta^{-1}\mathrm{D}\beta$], i.e. $\alpha\notin K$ is a formal analogue to a exponential [logarithm] of the element β. An *Liouvillian* differential extension of K is a finite chain of field extensions of its such that each one is either a simple algebraic, logarithmic or exponential extension of the previous one (which is a "differential" analogue to radical field extensions introduced above).

Fundamental Theorem of Differential Galois Theory

Let F be a finite simple differential field extension of K with the same constants, i.e. $\mathrm{con}_{\mathrm{D}}F=\mathrm{con}_{\mathrm{D}}K$ and $[F{:}K]=n$; let $\alpha\in F$ and $\beta\in K$ such that $\mathrm{D}\alpha=\beta$. Then $\exists c_1,c_2,\ldots,c_n\in\mathrm{con}_{\mathrm{D}}K$ and $\exists\varphi_1,\varphi_2,\ldots,\varphi_n,\psi\in K$ and:

$$\int^{\mathrm{D}}\beta:=\alpha\qquad\beta=\sum_{k=1}^{n}c_k\frac{\mathrm{D}\varphi_k}{\varphi_k}+\mathrm{D}\psi\,.\tag{1.49}$$

Theorem above states that the only elements of K with an "elementary" *anti*derivation α (i.e. formal analogue to Calculus' anti-derivatives) are of the form

[28] The **derivations** of a ring $R(+,\cdot)$ is $\mathrm{D}[R]:=\{f\in\mathrm{End}R\mid f(x\cdot y)=f(x)\cdot y+x\cdot f(y),\ \forall x,y\in R\}$, i.e. those endomorphisms whose properties are formally equivalent to those of Calculus' derivatives.
[29] E.g. $\mathrm{con}\mathbb{C}\{z\}=\mathbb{C}$, i.e. complex numbers are constants for the holomorphic derivative $\mathrm{D}=\mathrm{d}/\mathrm{d}z$.

above, i.e. linear combinations of "simple" elements (i.e. elements with antiderivation in K itself) and logarithms of them. The antiderivation associated to $\mathrm{D}\in\mathrm{D}[K]$ is thus also indicated as ${}^{\mathrm{D}}\!\!\int$ (analogue to the Calculus' indefinite integral).

1.2 Topological groups

Topological algebraic structures (topological groups and linear spaces in particular) are the best sets where notions from Algebra and Topology get merged together, so powerful analytical tools can be developed (almost as much as happens on the numeric spaces of Calculus, which also fall into these categories). This paragraph is just a summary of the tools used throughout this work and is not intended to be even slightly exhaustive at all. Linear vector spaces are introduced in [9]–§B.1; more on Topology can be found in [59] and [80], whereas Lie Theory is developed in [89].

1.2.1. Topological algebraic structures

Let $(A;\circ_1,\circ_2,\circ_3,\ldots;\bullet_1,\bullet_2,\bullet_3,\ldots)$ be any algebraic structure and (A,T) a topological structure (thus A is also a topological space whose chosen topology is T); if A is Hausdorff and all the algebraic operations —both binary and unary— are *continuous* with respect to topology T, e.g.[30] $\circ_1,\circ_2,\circ_3,\ldots\in\mathrm{C}^0(A\times A;A)$ and $\bullet_1,\bullet_2,\bullet_3,\ldots\in\mathrm{C}^0(A;A)$, then A is said to be a topological algebraic structure and $(A;\circ_1,\circ_2,\circ_3,\ldots;\mathsf{T};\bullet_1,\bullet_2,\bullet_3,\ldots)$ is the corresponging algebro-topological structure. For example one can talk about «paracompact, semisimple (topological) groups», «dense, separate, abelian (topological) rings», «complete, topologically- but *not* algebraically-closed fields», «connected and locally path-connected, commutative, graded algebras», etc.

Continuous homomorphisms of algebraic structures (as well as epi-/mono-/iso-/endo-/automorphisms) are elements of both C^0- and Hom-type categories: their

[30] The definition is to be completed requesting external operations $\bullet_1,\bullet_2,\bullet_3,\ldots$ to be continuous with respect to their external algebraic *and* topolgical structures too.

classification goes far beyond the scopes of this work. Let suffice to say that a map $f\colon A\to B$ between two topological algebraic structures which is both a *homo*- and a *homeo*morphism, is simply called a *morhpism* of $A\asymp B$ and, since A (being Hausdorff) is a topological manifold too, local charts can be operated on via algebraic operations of the structure, the same way as happens to neighbourhoods (see below the topological groups' case). If A and B are also a r-differentiable manifolds ($r\in\mathbb{N}\cup\{\infty,\omega\}$ — thus smooth or analytical manifolds too [36], [88]) their morphisms get enriched with differentiable algebraic properties. In particular, a *smooth* algebraic structure (i.e. which is compatible with the C^∞ structure) which is *not* an algebra is called a Lie algebraic structure. Topological algebraic structures which are both Hausdorff, totally disconnected and compact are termed *profinite*.

A topological group is any group G whose group operation '$\circ$' and inversion '$^{-1}$' are functions of $C^0(G\times G;G)$ and $C^0(G;G)$ respect to G's pre-defined topology T. It follows that any system of neighbourhoods $\boldsymbol{\mathcal{U}}(e)$ of the neutral element $e\in G$ can be used to generate neighbourhoods' systems for any other group element, i.e.[31] $\forall g\in G$: $g\circ\mathcal{N}(e),\mathcal{N}(e)\circ g\in\boldsymbol{\mathcal{U}}(g)\quad\forall\mathcal{N}(e)\in\boldsymbol{\mathcal{U}}(e)$ and $\mathcal{N}(g)^{-1}\in\boldsymbol{\mathcal{U}}(g^{-1})\ \forall\mathcal{N}(g)\in\boldsymbol{\mathcal{U}}(g)$. Let $H\leq G$:

- if H is open, it is also closed;
- if H is closed and of finite index, it is also open (cfr. note [1]);
- if G is compact, H is open if and only if it is closed of finite index.

The identity component of G is the connected component $\mathrm{id}\,G$ containing the neutral element.

Profinite groups, in particular, are equivalent to compact, Hausdorff and totally disconnected groups, and the following is fundamental result hold:

[31] In this case one can speak about **left** $g\circ\mathcal{N}(e)$ and **right** neighbourhoods $\mathcal{N}(e)\circ g$, respectively.

Theorem

The inverse limit of any finite groups' sequence (cfr. §1.1.7) is always, at least, a profinite group, [71].

A ***Lie*** *group* $\mathcal{G}(\circ)$ (of neutral element e) is a topological group whose smooth structure is compatible with the algebraic one, i.e. composition and inversion of elements of $\mathcal{G}$ are C^{∞}-functions. A Lie group whose structure is complex and analytic (i.e. of class $\mathcal{O}$ — cfr. §2.1.3) is called a *complex* Lie group — its elements' inversion and composition being [bi-]holomorphic functions.

Theorem

All the compact, connected, complex Lie groups are always abelian.

***Cartan*'s Theorem**

Any topologically-closed subgroup of a Lie group is a Lie group itself.

A ***Lie*** *algebra* is an algebra $\mathfrak{A}_K$ (over a field K) such that its internal product '$\cdot$' satifises the Jacobi identity, cfr. [89]:

$$(\mathbf{u}\cdot\mathbf{v})\cdot\mathbf{w} + (\mathbf{v}\cdot\mathbf{w})\cdot\mathbf{u} + (\mathbf{w}\cdot\mathbf{u})\cdot\mathbf{v} = \mathbf{0}, \qquad \forall \mathbf{u},\mathbf{v},\mathbf{w}\in\mathfrak{A}. \tag{1.50}$$

The commutator '$[\cdot|\cdot]$' in a Lie algebra (with respect to its addition '$+$') is called *Lie* bracket; Lie-algebra morphisms are algebra homomorphisms which are commutator-invariant too, i.e. $f([\mathbf{u}|\mathbf{v}]) = [f(\mathbf{u})|f(\mathbf{v})]$, $\forall \mathbf{u},\mathbf{v}\in\mathfrak{A}$.

Any sub-space closed with respect to the Lie bracket is called a *sub-Lie* algebra; a *ideal* of a Lie algebra is any sub-Lie algebra of its $\mathfrak{I}\leq\mathfrak{A}$ such that $[\mathfrak{A}|\mathfrak{I}]\leq\mathfrak{A}$. Lie algebras *with* non-trivial commutator and *without* any non-trivial ideals are called *simple* Lie algebras.

Examples of Lie algebras include:

- Every associative algebra $\mathbf{A}(\cdot)$ with respect to its commutator '$[\cdot|\cdot]$', $[\mathbf{A}]:=\mathbf{A}([\cdot|\cdot])$, is called a *commutator* algebra; in this case the Jacobi identity reads, $\forall\mathbf{u},\mathbf{v},\mathbf{w}\in\mathbf{A}$:

$$\left[\mathbf{u}\middle|[\mathbf{v}|\mathbf{w}]\right]+\left[\mathbf{v}\middle|[\mathbf{w}|\mathbf{u}]\right]+\left[\mathbf{w}\middle|[\mathbf{u}|\mathbf{v}]\right]=\mathbf{0}.$$

- Every commutator algebra of derivations $[\mathrm{D}(\mathbf{A})]$; in this case the Lie bracket is a derivation itself and is called a *Poisson* bracket (in a sense which is not going to be further treated here, cfr. [37], [89]).
- The Euclidean space $\mathbb{R}^3$ endowed with ordinary **cross** product '$\times$'.
- The tangent bundle of a Lie group $\mathcal{G}$'s neutral element $\mathsf{T}_e\mathcal{G}$, with respect to its vector fields' commutator (a Poisson bracket itself).

In particular, the Lie algebra associated to a [complex] Lie group $\mathcal{G}$, which is unique and over $\mathbb{R}$ [over $\mathbb{C}$], is indicated as either $\mathrm{Lie}_{\mathbb{R}}\mathcal{G}$ [$\mathrm{Lie}_{\mathbb{C}}\mathcal{G}$] or with a 'gothic', lower-case lettering analogue to the Lie group's name — in this case $\mathfrak{g}$. Note that a Lie algebra may be associated, up to isomorphisms, to *more* than one Lie group. In particular, abelian connected Lie groups derive Lie algebras with *trivial* Lie brackets (and thus non-simple). For every finite-dimensional Lie algebra there is always at least one Lie group which derives it (that is not always true for infinite-dimensional ones).

1.2.2. Continuous group actions

Let $G(\circ)$ be a topological group, X be a topological space and $\bullet\in\mathrm{C}^0(G\times X;X)$ a (left)[32] *continuous* group action of G on X — cfr. §1.1.5. Of course the associated orbispace X/G inherits the quotient topology from X and, if X is simply connected too, then:

[32] Only left group actions will be considered here, extensions to right ones being straightforward.

$$\pi_1(X/G) \cong G. \tag{1.51}$$

Continuous group actions can be further classified as follows:

- Properly Discontinuous: $\forall x{\in}X\ \exists\mathcal{N}(x){\subset}X\colon\quad \big(g{\bullet}\mathcal{N}(x)\big){\cap}\mathcal{N}(x){\neq}\varnothing \Rightarrow\ g{=}e$;
- Co-compact: X is locally compact and $C{\subset}X$ is a compact subset such that $G{\bullet}C{=}X$. If '$\bullet$' is also properly discontinuous, then X/G is compact.
- Strongly Continuous: Letting ${\bullet}{\in}\mathrm{C}^0(\mathbf{G}{\times}\mathbf{V};\mathbf{V})$ be a group action of a topological linear space $\mathbf{G}$ on a topological linear space $\mathbf{V}$, if maps $\mathbf{g}\mapsto\mathbf{g}{\bullet}\mathbf{v}$, are continuous with respect to both spaces' topologies, i.e. are in $\mathrm{C}^0(\mathbf{G};\mathbf{V})$, $\forall\mathbf{v}{\in}\mathbf{V}$.

The *exponential* (map) associated with a Lie group $\mathcal{G}(\circ)$ with neutral element e is

$$\exp\colon \operatorname{Lie}\mathcal{G} \longrightarrow \operatorname{id}\mathcal{G}$$

such that, $e^{t\mathbf{v}} :\equiv \exp(t{\bullet}\mathbf{v}) \overset{\text{def}}{=} \mathfrak{g}_t\mathbf{v}$ and $e^{0\mathbf{v}} \equiv e^{t\mathbf{0}} :\equiv e^0 = e$, $\forall\mathbf{v}{\in}\mathsf{T}_e\mathcal{G}$, where $\mathfrak{g}_{\bullet}$ is the unique flow of $\operatorname{Lie}\mathcal{G}$ ($\mathfrak{g}$ being the corresponding one-parameter group of $\mathcal{G}$, cfr. §1.1.5) whose tangent vector in e is $\mathbf{v}$. The exponential thus maps neighbourhoods of $\mathbf{0}{\in}\mathsf{T}_e\mathcal{G}$ into neighbourhoods of $e{\in}\mathcal{G}$ (i.e. almost-neutral to almost-neutral elements).

Trivial properties of the exponential map include $e^{(t+s)\mathbf{v}} \equiv e^{t\mathbf{v}}e^{s\mathbf{v}}$, $e^{-t\mathbf{v}} \equiv \big(e^{t\mathbf{v}}\big)^{-1}$, $\exp{\in}\mathrm{C}^\infty(\operatorname{Lie}_{\mathbb{R}}\mathcal{G}, \operatorname{id}\mathcal{G})$ or equivalently $\exp{\in}\mathcal{O}(\operatorname{Lie}_{\mathbb{C}}\mathcal{G}, \operatorname{id}\mathcal{G})$ for complex Lie groups — stating that the exponential is a smooth [biholomorphic] diffeomorphism (cfr. §2.1.4) between a corresponding Lie algebra and the Lie group— and the analogicity property $\mathrm{D}e \equiv \mathbb{1}$ (because the Frechét derivative is an operator in $\mathrm{C}^\infty(\operatorname{Lie}\mathcal{G}, \operatorname{Lie}\mathcal{G})$).

For every matrix Lie group, the exponential map is the common matrix exponential.[33] For Lie subgroups of matrix algebras like $\mathbb{M}_n(\mathbb{C})$ the exponential is a matrix in the corresponding Lie algebra:

- For $\mathrm{GL}_n(\mathbb{C})$ as a Lie group with respect to the matrix commutator, $\mathfrak{gl}_n(\mathbb{C}) \equiv \mathrm{Lie}\,\mathrm{GL}_n(\mathbb{C}) = \mathbb{M}_n(\mathbb{C})$ and, since algebraic properties of matrix functions, the same holds on $\mathbb{M}_n(\mathbb{C})/\mathbb{C}$ because $e^{C^{-1}AC} = C^{-1}e^{A}C,\quad \forall C\in\mathrm{GL}_n(\mathbb{C})$ (and the same as well for real matrices). All in all, that means every invertible matrix is the exponential of some (in general, singular) matrix.

- The n^{th} *special* linear group in a field K is the closed, connected, subspace $\mathrm{SL}_n(K)\lhd\mathrm{GL}_n(K)$ of matrices with unitary determinant; The corresponding Lie algebra is the subspace of *traceless* matrices, i.e. $\mathfrak{sl}_n(K):=\mathrm{Lie}\,\mathrm{SL}_n(K)\equiv\{A\in\mathbb{M}_n(K)\ \mid\ \mathrm{Tr}A=0\}$, thus every special-linear matrix is the exponential of a traceless matrix.

- The n^{th} [*special*] unitary group in a (hyper-)complex body $B\cong\mathbb{C}^d$ is the connected Lie group $\mathrm{U}_n(B)$ [$\mathrm{SU}_n(B)$] whose unitary matrices Y [satisfying $\det Y=1$].[34] Its Lie algebra $\mathfrak{u}_n:=\mathrm{Lie}\,\mathrm{U}(n)$ is the algebra of *anti-hermitian* matrices [and $\mathfrak{su}_n:=\mathrm{Lie}\,\mathrm{SU}(n)$ of the *traceless* anti-hermitian ones], with respect to matrix commutator.

- The n^{th} [*special*] orthogonal group is the closed subgroup $\mathrm{O}_n(K)\leq\mathrm{GL}_n(K)$ [$\mathrm{SO}_n(K)$] of matrices A such that $AA^{\mathsf{T}}=I_n$ [and $\det A=1$].[35] The associated Lie algebra [for both] is $\mathfrak{o}_n(K):\equiv\mathfrak{so}_n(K)$, the algebra of *skew-symmetric* matrices, with respect to the matrix commutator.[36]

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[33] The tame real [complex] numbers' exponential is a morphism $\mathbb{R}(+)\to\mathbb{R}^*(\cdot)$ $[\mathbb{C}(+)\to\mathbb{C}^*(\cdot)]$.

[34] Since every [special] unitary group is isomorphic to one in $\mathbb{C}$, then their prototypes are the n^2-dimensional space $\mathrm{U}(n):=\mathrm{U}_n(\mathbb{C})$ and the (n^2-1)-dimensional space $\mathrm{SU}(n):=\mathrm{SU}_n(\mathbb{C})$.

[35] In particolar, $\mathrm{SL}_n(\mathbb{C})$, like $\mathrm{SU}(n)$, is simply-connected (whereas $\mathrm{SL}_n(\mathbb{R})$, like $\mathrm{SL}_n(\mathbb{R})$, is not) and $\dim_{\mathbb{C}}\mathrm{SL}_n(\mathbb{C})=n^2-1$, whereas $\dim\mathrm{SL}_n(\mathbb{R})=(n+1)(n-1)/2$. Furthermore, $\zeta(\mathrm{SU}(n))=\mathbb{Z}_n$.

[36] The hyperspherical symmetry group $\mathrm{O}_n(\mathbb{R})\equiv\mathrm{U}_n(\mathbb{R})$ has in particular two connected components, whose locus is $\det A=\pm1$, the positive one being orientation-preserving group $\mathrm{SO}_n(\mathbb{R})$, with $\pi_1(\mathrm{SL}_n(\mathbb{R}))=\pi_1(\mathrm{SO}_n(\mathbb{R}))\cong\mathbb{Z}_2$ for $n\geq2$ (otherwise $\mathbb{Z}$ for $n=2$).

- The n^{th} unitary group is the Lie algebra associated to the group of imaginary matrices, $\imath\mathbb{M}_n(\mathbb{R})$; in fact $e^{\imath A}\in \mathrm{U}_n(\mathbb{C})$, $\forall A\in\mathbb{M}_n(\mathbb{R})$. In a formula, $\mathrm{U}_n(\mathbb{C})=\mathrm{Lie}_{\mathbb{C}}(\imath\mathbb{M}_n(\mathbb{R}))$.

The examples above prove that matrix exponentials map matrices with certain algebraic and topological properties into matrices with usually different ones, and vice versa: invertible matrices have traceless exponentials, Hermitian matrices have Hermitian exponentials, anti-Hermitian [skew-symmetric] matrices have unitary [orthogonal] exponentials.

Theorem

Every one-parameter subgroup of $\mathrm{GL}_n(\mathbb{C})$ (as well as of $\mathrm{GL}_n(\mathbb{R})$) is an exponential map, i.e. $\exists A\in\mathbb{M}_n(\mathbb{C})$ for it such that $t\mapsto e^{tA}$, $\forall t\in\mathbb{R}$.

Compact Lie groups have non-injective exponential maps, whereas non-connected or non-simply-connected Lie groups have non-surjective exponential maps.

1.2.3. Profinite fields and Infinite Galois Theory

This paragraph briefly introduces some concepts regarding *infinite* Galois extensions (whose meaning and finite-degree analogues are discussed in §1.1.10) which —as often happens when dealing with infinite-dimensional Algebra— need a proper topology to be successfully handled.

Consider a (possibly infinite) field extension F/K along with its intermediate fields in F. Let the Krull topology K of its Galois group $\mathrm{Gal}(F/K)$ be the left cosets' collection

$$\mathsf{K}\stackrel{\text{def}}{=}\left\{\bigcup_{\substack{K\le H\le F:\\ [H:K]<\aleph_0}}\sigma\bullet\mathrm{Gal}(F/H)\;\middle|\;\forall\sigma\in\mathrm{Gal}(F/K)\right\}. \tag{1.52}$$

It is proven that the Galois group $\mathrm{Gal}(F/K)$ —which is profinite with respect to K, §1.2.1— is the inverse limit (as defined in §1.1.7) of its intermediate subgroups:

$$\mathrm{Gal}(F/K) = \varprojlim_{[H:K]<\aleph_0} \mathrm{Gal}(H/K)\,; \tag{1.53}$$

- if $[F{:}K]<\aleph_0$, then K is just the discrete topology of $\mathrm{Gal}(F/K)$ and[37] the whole theory exposed here just adheres to Finite Galois Theory;
- open subgroups of $\mathrm{Gal}(F/K)$ are precisely those with a finite number of left cosets;
- closed subgroups of $\mathrm{Gal}(F/K)$ are precisely of type $\mathrm{Gal}(F/H)$, for extension fields $K\leq H\leq F$;
- if $G\leq\mathrm{Gal}(F/K)$ and $H_G\leq F$ is the intermediate field which is fixed by G, then (with respect to K):

$$\overline{G} = \mathrm{Gal}(F/H_G). \tag{1.54}$$

Fundamental Theorem of Infinite Galois Theory

For any field extension F/K there is a 1-to-1 "inclusion-reversing" correspondence between intermediate extension fields $K<H<F$ and the *closed* subgroups $G_H\leq\mathrm{Gal}(F/K)$. If $H\cong G_H$ (as groups) then

$$\mathrm{ord}\,\frac{\mathrm{Gal}(F/K)}{G_H}<\aleph_0 \quad\Leftrightarrow\quad [F{:}H]<\aleph_0 \quad\Leftrightarrow\quad H \text{ is open, and:}$$

$$[H{:}K] = \mathrm{ord}\,\frac{\mathrm{Gal}(F/K)}{G_H} = [G:G_H]. \tag{1.55}$$

Moreover, if G_H is closed, $G_H\triangleleft\mathrm{Gal}(F/K)$ if and only if H is Galois over K, then the structures below are *both* isomorphic and homeomorphic:

$$\mathrm{Gal}(H/K) \cong \frac{\mathrm{Gal}(F/K)}{G_H}:$$

[37] For finite Galois extensions in fact, $\{e\}$ is open and all the subgroups of $\mathrm{Gal}(F/K)$ are closed.

In other words F/K is Galois if and only if the following, equivalent conditions hold:

- F/K is a separable, normal and algebraic field extension;
- F is the union of all the finite Galois extension fields of K;
- $K = F$,$^{\mathrm{Gal}(F/K)}$ and $\mathrm{Gal}(F/K)$ is a compact subgroup of $\mathrm{Aut}\,F$.
- F is the splitting field of monic separable polynomials in K;

As far as intermediate field extensions H/K are concerned, the following propositions are equivalent — thus also satisfying (1.55):

- H/K is a finite extension;
- G_H is open;
- $[G{:}G_H] < \aleph_0$, that is G_H has finite index in G.

> **Theorem**
>
> Every profinite group is the Galois group of some field extension.

Infinite and Differential Galois Theory are useful disciplines when studying integral and/or differential equations, where their solutions are expressible in closed-form (i.e. the equations are said to be integrable) when they are written as either linear combinations, products (and quotients) and (anti-)derivatives of “elementary” functions, i.e. functions within the function space of the equation’s coefficients.

1.2.4. Harmonic Analysis on groups

Abstract Harmonic Analysis studies the properties of measures (and thus integration) on groups —locally compact topological groups in particular— merging both Functional Analysis and Group Theory in one discipline.

A measure μ on a group $G(\circ)$ is said to be [right-] left-invariant if $\mu(g\circ E)=\mu(E)$ [$\mu(E\circ g)=\mu(E)$] for every μ-measurable $E\subset G$ and $\forall g\in G$; a simultaneously left- and right-invariant measure is said to be an *invariant measure* on G. Due to the fact that a "reversed measure" $\mu^{\Leftarrow}$ exists for every μ such that $\mu^{\Leftarrow}(E):=\mu(E^{-1})$, a [left-] right-invariant measure also exists for every [right-] left-invariant one.

It can be proven that, let G be a locally compact topological group, all the countably-additive, invariant measures on $\mathfrak{B}(G)$ (which exist)[38] are equivalent up to multiplication by a positive constant[39] and such [left-/right-] invariant measure is called [left/right] *Haar measure* on G.

> Examples of Haar measures on groups, defined on according to their Lebesgue measures, include: $x^{-1}\mu_{\mathrm{L}}(x)$ in $\mathbb{R}_{+}(\cdot)$; $|\det A|^{-n}\mu_{\mathrm{L}}(A)$ in $\mathrm{GL}_n(\mathbb{C})$; $|\star\varpi|\mu_{\mathrm{L}}$ in any n-dimensional Lie group $\mathcal{G}$ (of identity e, cfr §1.2.1), $\forall\varpi\in\Lambda^n(\mathcal{G})\setminus\{e\}$.

The Lebesgue spaces on G respect to its Haar measure are simply indicated as $\mathrm{L}^p(G)$, $\forall p\in]0,+\infty[$; respect to the left [right] Haar measure they are indicated as $\mathrm{L}^p_{\blacktriangleleft}(G)$ $\left[\ \mathrm{L}^p_{\blacktriangleright}(G)\ \right]$. In particular, $\mathrm{L}^1(G)$ is a commutative Banach algebra respect to the *convolution* operator $\circledast:\mathrm{L}^1(G)\times\mathrm{L}^1(G)\to\mathrm{L}^1(G)$:

$$f,g\in\mathrm{L}^1(G) \quad\mapsto\quad (f\circledast g)(x)\overset{\text{def}}{=}\int_G f(x\circ y^{-1})g(y)\mathrm{d}y \tag{1.56}$$

(where $\mathrm{d}y$ formally indicates integration of variable y with respect to G's Haar measure). Usually $\mathrm{L}^1(G)$ is without a unity[40] but either its conjugate $\mathrm{L}^1(G)^\dagger$ has one, or there exists a sequence $(u_n)_{n\in\mathbb{N}}\in\mathrm{L}^1(G)^{\mathbb{N}}$ that weakly converges to a formal unity, $u_n\cdots\!\to\delta_G\notin\mathrm{L}^1(G)$ (which, in the distributions' sense, means that $\lim_n f\circledast u_n=f$; cfr.

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[38] Recall that if G is **discrete** (i.e. $\mathrm{Card}\,G\leq\aleph_0$) then, by the discrete topology, $\mathfrak{B}(G)\equiv\wp(G)$.

[39] For this reason, once this constant is fixed (i.e. one measure is defined), there is usually no need to continually specify the measure μ any more for that given group.

[40] $\delta_G\in\mathrm{L}^1(G)$ such that $f\circledast\delta_G\equiv f$, $\forall f\in\mathrm{L}^1(G)$, that is a *non*-distributional analog to **Dirac's delta**.

notes [40] and [45]). Another interesting feature of convolution regards the integration with respect to the Haar measures, since the integral of a convolution is the product of the integrated functions, i.e.:

$$\int_G (f \circledast g)(x)\mathrm{d}x = \left(\int_G f(x)\mathrm{d}x\right)\left(\int_G g(x)\mathrm{d}x\right), \qquad \forall f,g \in \mathrm{L}^1(G). \tag{1.57}$$

The convolution of a locally-integrable function and a compactly-supported one is well-defined and continuous, i.e. $\circledast\colon \mathrm{L}^1_0(G) \times \mathrm{L}^1_{\mathrm{loc}}(G) \to \mathrm{C}^0(G)$.

The convolution operator can be continued to be an exterior product between specific couples of Lebesgue spaces: let $p,q \in [1,+\infty[$, let $r = (p^{-1}+q^{-1}-1)^{-1}$ and $f \in \mathrm{L}^p(G)$, $g \in \mathrm{L}^q(G)$; then the convolution is also integrable, $f \circledast g \in \mathrm{L}^r(G)$, and

$$\left\| f \circledast g \right\|_{\mathrm{L}^r(G)} \leq \| f \|_{\mathrm{L}^p(G)} \| g \|_{\mathrm{L}^q(G)}, \qquad \frac{1}{r} + 1 = \frac{1}{p} + \frac{1}{q}. \tag{1.58}$$

$\mathrm{L}^2(G)$ is also a pre-Hilbert space with respect to the inner product:

$$f,g \in \mathrm{L}^2(G) \quad \mapsto \quad \langle f \,|\, g \rangle_{\mathrm{L}^2(G)} \stackrel{\mathrm{def}}{=} \int_G f(x)\overline{g(x)}\mathrm{d}x\,. \tag{1.59}$$

If either G is compact or has finite measure then $\mathrm{L}^q(G) \leq \mathrm{L}^p(G)$ for any $1 \leq p < q$ (the equality trivially holding *if* G is finite and *only if* G is discrete).

Pontryagin's Theorem

The subset $\hat{G} \leq \mathrm{Hom}(G,\mathbb{T})$ of all the continuous homomorphisms χ (called *characters* of G) between a locally compact abelian group $G(\circ)$ and[41] $\mathbb{T}(\cdot)$ is called the *Pontryagin's dual* group of G and is a locally compact abelian group itself respect to uniform convergence on G and point-wise multiplication on $\mathbb{T}$ (complex conjugation as inverse).

The *Fourier transform* is the linear operator $\mathcal{F} \in \mathfrak{L}\left(\mathrm{L}^1(G), \mathrm{C}_0^0(\hat{G})\right)$, i.e. the continuous algebra homomorphism $\mathcal{F}: \mathrm{L}^1(G) \to \mathrm{C}_0^0(\hat{G})$

$$\mathcal{F}[f](\chi) :\equiv \hat{f}(\chi) \stackrel{\text{def}}{=} \int_G f(x)\overline{\chi(x)}\mathrm{d}x\,. \tag{1.60}$$

Furthermore Pontryagin's dual groups have the following algebro-topological properties as depending on the ones of G:

- Pontryagin's duality is distributive respect to direct product of groups, i.e. $\widehat{G_1 \circledcirc G_2} = \hat{G}_1 \circledcirc \hat{G}_2$ (with respect to complex-numbers multiplication in $\mathbb{T}(\cdot)$ for the characters);
- $\hat{G}$ is a discrete group if and only if G is compact;
- if G is compact, then $\hat{G}$ is torsion-free (i.e. $\mathrm{Tor}\,\hat{G}$ is trivial)[42] if and only if G is also connected;
- if G is separable then $\hat{G}$ is metrizable;
- if H is a closed subgroup of G, i.e. $H < G$ and $\overline{H} = H$, then $\exists o_H \in \hat{G}$ such that $o_H^{-1}(1) = H$ (*Hahn-Banach*'s Theorem on groups);

[41] In this contest the diffeomorphic version of $\mathbb{T}$ used is the multiplicative one, i.e. the complex circumference (of radius 1 and centre 0) $\mathbb{T}(\cdot) \asymp \partial\mathrm{B}_1(0) \subset \mathbb{C}$.

[42] The **torsion group** $\mathrm{Tor}\,G$ is the subgroup of the cyclic groups (nilpotent elements) of G, cfr. [93].

- if G is compact then $\hat{G}$ is a complete orthonormal basis of $\mathrm{L}^2(\hat{G})$, i.e. $\langle\chi|\gamma\rangle_{\mathrm{L}^2(\hat{G})}=0$ (and trivially $\|\chi\|_{\mathrm{L}^2(\hat{G})}=1$) $\forall\chi,\gamma\in\hat{G},\ \chi\neq\gamma$;
- *meta*duality: $\forall A\in\mathrm{End}\,G\quad\exists!\,A\hat{}\in\mathrm{End}\,\hat{G}$ such that $A\hat{}(\chi):=\chi\circ A$.

The Fourier transform has several properties, among which the most relevant one is that it preserves the Banach-algebra structure between domain and codomain, since convolution on $\mathrm{L}^1(G)$ is lifted to point-wise product on $\mathrm{C}^0_0(\hat{G})$ via $\mathcal{F}$ and vice versa:

$$\mathcal{F}[f\circledast g]=\mathcal{F}[f]\mathcal{F}[g],\qquad\forall f,g\in\mathrm{L}^1(G).\tag{1.61}$$

Corollary

Pontryagin's duality is always involutory, i.e. $\hat{\hat{G}}\cong G$ using complex conjugated characters. If G is finite then $G\cong\hat{G}$ but this condition, though sufficient, is *not* necessary (see examples).

The Fourier transform $\mathrm{L}^1(\hat{G})\to\mathrm{C}^0_0(\hat{\hat{G}})$ is called inverse Fourier transform when it acts on G via the Corollary isomorphism, i.e. $\mathcal{F}^{-1}\colon\mathrm{L}^1(\hat{G})\to\mathrm{C}^0_0(G)$, although this $\mathcal{F}^{-1}$ is *not* properly the inverse operator of $\mathcal{F}$:

$$\mathcal{F}^{-1}[f](x):\equiv\breve{f}(x)\overset{\mathrm{def}}{=}\int_{\hat{G}}f(\chi(x))\,\chi(x)\mathrm{d}x,\tag{1.62}$$

with respect to the Haar measure of $\hat{G}$.

Examples of Potryagin's dualities include:

- $\widehat{\mathbb{Z}_p}(+)=\mathbb{Z}_p(\cdot)$ and vice versa, with characters $[\![k]\!]_p\mapsto e^{\imath k\frac{m}{p}}$ and $[\![m]\!]_p\mapsto e^{\imath k\frac{m}{p}}$, $\forall[\![k]\!]_p,[\![m]\!]_p\in\mathbb{Z}_p$;
- $\widehat{\mathbb{Z}^n}(+)=\mathbb{T}^n(\cdot)$ and vice versa, with characters $\mathbf{m}\mapsto\boldsymbol{\theta}^{\mathbf{m}}$ and $\boldsymbol{\theta}\mapsto\boldsymbol{\theta}^{\mathbf{m}}$ respectively, $\forall\mathbf{m}\in\mathbb{Z}^n$, $\forall\boldsymbol{\theta}\in\mathbb{T}^n$ (where $\boldsymbol{\theta}^{\mathbf{m}}=\theta_1^{m_1}\theta_2^{m_2}\cdot\ldots\cdot\theta_n^{m_n}$).
- $\widehat{\mathbb{R}^n}(+)=\mathbb{R}^n(+)$, with characters $\mathbf{x}\mapsto e^{\imath\mathbf{x}\cdot\boldsymbol{\xi}}$ and $\boldsymbol{\xi}\mapsto e^{\imath\mathbf{x}\cdot\boldsymbol{\xi}}$, $\forall\mathbf{x},\boldsymbol{\xi}\in\mathbb{R}^n$;
- $\widehat{\mathrm{U}(n)}(\cdot)=\mathrm{U}(n)(\cdot)$ with characters $U\mapsto\det UY$ and $Y\mapsto\det UY$, $\forall U,Y\in\mathrm{U}(n)$ (cfr. examples in §1.2.2); the subgroup $\mathrm{O}(n)$ is dual to subgroup $\mathbb{Z}_2\lesssim\mathbb{T}$ (both $\mathrm{O}(n)$ and $\mathrm{SO}(n)$ are closed and $o_{\mathrm{SO}(n)}=\det\in\widehat{\mathrm{U}(n)}$).

Since the Haar measures are equivalent up to a multiplicative constant there always exists a *normalized* Haar measure such that $\mathcal{F}$ is an isometry;[43] respect to the starting measure μ in $\mathbb{C}$ this constant equals[44] $\mu(\mathbb{T})^{-\dim G}$.

Plancherel's Theorem

If G is compact there is an isomorphism[45] $\mathrm{L}^2(\hat{G})\cap\mathrm{C}_0^0(\hat{G})\cong\mathrm{L}^2(G)$ via the normalized Fourier transform restricted to $\mathrm{L}^2(G)$, which becomes a unitary operator $\mathcal{F}\in\mathfrak{U}\left(\mathrm{L}^2(G)\right)$.

In this case the inverse Fourier transform maps L^2-functions into L^2-functions and (as unitary operator) also preserves the inner products, being defined as the adjoint operator $\mathcal{F}^{-1}\overset{\text{def}}{=}\mathcal{F}|_{\mathrm{L}^2(G)}^{\dagger}$,

[43] This means that $\|\mathcal{F}\|=1$ in $\mathfrak{BC}\left(\mathrm{L}^1(G),\mathrm{C}_0^0(\hat{G})\right)$, the bounded linear operators' Banach space.

[44] For example, it is equal to $(2\pi)^{-n}$ for the ordinary Fourier transform on $\mathbb{R}^n$ since, as for note [41], $\mathbb{T}(+)=\mathbb{R}/2\pi\mathbb{Z}$. It is p^{-1} for the Fourier transform on $\mathbb{Z}_p$ where $\mathbb{T}=\mathbb{R}/\mathbb{Z}$ is a common choice.

[45] This is done proving that the separable space of compact-support, countinuous, square-integrable functions on the Pontryagin's dual group is canonically isomorphic to $\mathrm{L}^2(G)^{\dagger}$.

$$\langle f \,|\, g \rangle_{\mathrm{L}^2(G)} = \left\langle \hat{f} \,\middle|\, \hat{g} \right\rangle_{\mathrm{L}^2(\hat{G})}, \qquad \forall f, g \in \mathrm{L}^2(G), \tag{1.63}$$

which the well-known *Plancherel*'s equalities follow from:

$$\|\mathcal{F}\| = 1 \qquad \Leftrightarrow \qquad \begin{cases} \|f\|_{\mathrm{L}^1(G)} = \|\mathcal{F}[f]\|_{\mathrm{C}^0(\hat{G})} \\ \|f\|_{\mathrm{L}^2(G)} = \|\mathcal{F}[f]\|_{\mathrm{L}^2(\hat{G})} \end{cases} \tag{1.64}$$

The Fourier transform can also be extended to spaces $\mathrm{L}^p(G)$ for $p \in]1,2[$; that is because of the *Orlicz*'s Theorem [31] and the fact that, for every μ-measurable set E, $\mathrm{L}^p(E,\mu) = \mathrm{L}^1(E,\mu) \oplus \mathrm{L}^2(E,\mu)$.

Any $\mathbb{R}$-linear functional $m \in \mathrm{L}^\infty_{\mathbb{R}}(G)^\dagger$ that maps both $\mathbb{1} \mapsto 1$ and non-negative functions to $\mathbb{R}_+$ is called a mean on G. If G admits a left- or right-invariant mean, then it is said to be an *amenable* group. Examples of amenable groups include:

- finite groups (their mean being a probability measure);
- abelian groups;
- subgroups and direct products of amenable groups, cfr. §1.1.9;
- groups whose finitely-generated subgroups are all amenable (i.e. locally amenable groups are amenable);
- finitely-generated groups of intermediate growth, cfr. §1.1.9;
- solvable groups, cfr. §1.1.1;
- compact groups (whose normalized Haar measure is such a mean).

1.2.5. Appendix: Dirichlet series and the Laplace transform

Let $\mathbf{B}_{\mathbb{C}}$ be a Banach space, $(\mathbf{v}_n)_{n\in\mathbb{N}} \in \mathbf{B}^{\mathbb{N}}$ and $(\lambda_n)_{n\in\mathbb{N}} \in \mathbb{R}_+^{\mathbb{N}}$ an increasing, diverging sequence of non-negative real numbers; the associated *Dirichlet series* is then

$$\mathcal{D}[(\mathbf{v}_n)_{\mathbb{N}};(\lambda_n)_{\mathbb{N}}](z) \stackrel{\text{def}}{=} \sum_{n=1}^{\infty} e^{-\lambda_n z}\mathbf{v}_n\,; \tag{1.65}$$

in the case where $(\lambda_n)_{n\in\mathbb{N}}=(\log n)_{n\in\mathbb{N}}$, this is also called an *ordinary* Dirichlet series,

$$\mathcal{D}[(\mathbf{v}_n)_{\mathbb{N}}](z) := \sum_{n=1}^{\infty}\frac{\mathbf{v}_n}{n^z}.$$

A Dirichlet series converging at $z_0\in\mathbb{C}$ it also converges to a holomorphic function in $\{z\in\mathbb{C}\mid \operatorname{Re}z>\operatorname{Re}z_0\}$, uniformly in the closed angle $|\operatorname{Arg}(z-z_0)|\leq\frac{\pi}{2}$. The [absolute] convergence abscissa of a Dirichlet series is the *least* abscissa $c,a\in\mathbb{R}$ such that the series [absolutely] converges for $\operatorname{Re}z>c$ $[\operatorname{Re}z>a]$. According to the behaviour of the Fourier series $\sum_n\mathbf{v}_n$ such abscissæ are computable as:

- if $\sum_n\mathbf{v}_n$ is convergent, $c=\overline{\lim_n}\,\lambda_n^{-1}\log\left\|\sum_{k=n+1}^{\infty}\mathbf{v}_k\right\|$;
- if $\sum_n\mathbf{v}_n$ is absolutely convergent, $a=\overline{\lim_n}\,\lambda_n^{-1}\log\sum_{k=n+1}^{\infty}\|\mathbf{v}_k\|$;
- if $\sum_n\mathbf{v}_n$ is divergent, $c=\overline{\lim_n}\,\lambda_n^{-1}\log\left\|\sum_{k=1}^{n}\mathbf{v}_k\right\|$;
- if $\sum_n\|\mathbf{v}_n\|$ is divergent, $a=\overline{\lim_n}\,\lambda_n^{-1}\log\sum_{k=1}^{n}\|\mathbf{v}_k\|$.

More generally, $0\leq a-c\leq\overline{\lim_n}\dfrac{\log n}{\lambda_n}$ and, if the latter is 0, then $c=\overline{\lim_n}\dfrac{\log\|\mathbf{v}_n\|}{\lambda_n}$.

Furthermore, Dirichlet series always diverge for $\operatorname{Re}z<c$.

Recall that the (unilateral) *Laplace transform* of any function $f\in\mathrm{L}^1_{\mathrm{loc}}(\mathbb{R}_+)$ (with respect to the Lebesgue measure on $\mathbb{R}_+$ for simplicity's sake, cfr. [9]–§B.3) is

$$\mathcal{L}[f](s)\stackrel{\text{def}}{=}\int_0^{\infty}e^{-st}f(t)\mathrm{d}t\,, \tag{1.66}$$

which is holomorphic in $\mathbb{C}\backslash U$, where $U\subseteq\mathbb{C}^-$ and $\mathbb{C}^-:=\{s\in\mathbb{C}\mid\operatorname{Re}s\leq 0\}$. The Laplace abscissa of f is the *least* abscissa $l_f\leq 0$ in $\mathbb{C}$ such that $\mathcal{L}[f]$ is holomorphic at least inside the hyperbolic plane $\{s\in\mathbb{C}\mid\operatorname{Re}s>l_f\}$, i.e.:

$$l_f \stackrel{\text{def}}{=} \inf\left\{ l\in\mathbb{R}_- \;\middle|\; f\in\mathcal{O}\Big(\mathbb{C}\backslash(l+\mathbb{C}^-)\Big)\right\},$$

The Laplace transform is a linear continuous operator and has many properties similar or identical to the Fourier transform; in particular $\forall f,g\in\mathrm{L}^1_{\mathrm{loc}}(\mathbb{R}_+)$, $\forall\alpha,\beta\in\mathbb{C}$, $\forall\tau\in\mathbb{R}$, $\forall n\in\mathbb{N}_0$, and δ_{-1} being the Heaviside *step* function $\delta_{-1}(t) \stackrel{\text{def}}{=} \begin{cases}1, & t\geq 0\\ 0, & t<0\end{cases}$:

- $\mathcal{L}[f(t-\tau)\delta_{-1}(t-\tau)](s) = e^{-\tau s}\mathcal{L}[f](s)$;
- $\mathcal{L}\big[t^n e^{\alpha t} f(\beta t)\big](s) = \dfrac{(-1)^n}{|\beta|}\dfrac{\mathrm{d}^n}{\mathrm{d}s^n}\mathcal{L}[f]\left(\dfrac{s-\alpha}{\beta}\right)$;
- $\mathcal{L}[f'](s) = s\mathcal{L}[f](s) - f(0^+)$;
- $\mathcal{L}[f \circledast g] = \mathcal{L}[f]\mathcal{L}[g]$;
- $f(0^+) = \lim\limits_{s\to\infty} s\mathcal{L}[f](s)$ and $f(+\infty) = \lim\limits_{s\to 0} s\mathcal{L}[f](s)$;
- if $\exists p\in[1,2]$: $f\in\mathrm{L}^p(\mathbb{R}_+)\cap\mathrm{L}^1_{\mathrm{loc}}(\mathbb{R}_+) \Rightarrow \mathcal{L}[f](\imath\omega) \underset{\mathrm{L}^2}{\equiv} \hat{f}(\omega)$ a.e. for $\omega\in\mathbb{R}$.

1.3 Algebra of Power Series

In order to proceed with the analysis of fractal Riemann surfaces, one has to properly characterize the algebraic and topological aspects of power series first, in order to stress the geometric correspondance between every branched surface and the singular and/or multi-valued functions "living" on them.

Power series are to $\mathbb{N}_0$ like Laurent series are to $\mathbb{Z}$, like transfinite and Poiseux series are to $\mathbb{Q}$: the more the index-set of such formal series enriches algebraically (but staying well-ordered), the more complicated becomes the equivalence of such series, which is what is needed to use the for describing "entangled" geometrical objects like compact Riemann surfaces (cfr. §2.4). Basically, as one goes deeper into the language of formal series, more operations become invertible to them: formal terms of Laurent series can be reciprocated (i.e. exchanging symbols "x" with "$1/x$"); formal terms of Puiseux series can be rooted (i.e. exchanging symbols "x^n" with "$\sqrt[n]{x}$") —

everywhere keeping the algebraics well-posed and analogue to common arithmetical operations.

First of all let us recall a bit of multi-index formalism applied to rational expressions.

$\mathbb{Z}^n$ is a complete $\mathbb{Z}$-modulo respect to scalar product induced by $\mathbb{R}^n$, preferably equipped with Hölder 1-norm [9]–§B.2. For an integer n-tuplet $\mathbf{m}\equiv(m_1,m_2,\dots,m_n)\in\mathbb{Z}^n$, generally referred to as a multi-index, the semi-norm[46] $|\mathbf{m}|:=m_1+m_2+\dots+m_n\in\mathbb{Z}$ is of practical use: recall the factorial and *multifactorial* of $\mathbf{m}\in\mathbb{N}_0^n$:

$$\mathbf{m}! \stackrel{\text{def}}{=} \prod_{k=1}^{n} m_k\,! \,; \qquad \mathbf{m}¡ \stackrel{\text{def}}{=} \frac{|\mathbf{m}|!}{\mathbf{m}!} \equiv \frac{(m_1+m_2+\ldots+m_n)!}{m_1!\,m_2!\cdot\ldots\cdot m_n!}\,; \tag{1.67}$$

Every (monic) polynomial expression $P\in R[\mathbf{x}]$ in n variables $\mathbf{x}\equiv(x_1,x_2,\dots,x_n)$ can be written in a formal, "vector-like" way depending on the vector $\mathbf{y}\equiv(y_1,y_2,\dots,y_n)\in R^n$ of the n elements which are the P's roots in the case[47] $\mathbf{x}=x\mathbf{1}_n$, as well as their multiplicities' multi-index $\mathbf{m}=(m_1,m_2,\dots,m_l)\in\mathbb{N}^n$, thus

$$P(\mathbf{x}) = \prod_{k=1}^{n}(x_k-y_k)^{m_k} \ ::= (\mathbf{x}-\mathbf{y})^{\mathbf{m}}\,. \tag{1.68}$$

Also, every rational function $f\in R\{x\}$ is subject to the same formalism, where $\mathbf{y}$ is the vector of its n special points (zeroes and poles) and $\mathbf{m}\in\mathbb{Z}^{*n}$ is the vector of their orders (taken conventionally positive for zeroes and negative for poles), thus:

$$f(x) = (x\mathbf{1}_n-\mathbf{y})^{\mathbf{m}}\,. \tag{1.69}$$

1.3.1. Fractions

Let $R(\circ,\bullet)$ be a ring (of zero 0 and unity 1), let $a,b,m\in R$ and let us say that

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[46] The Hölder 1-norm $\|\mathbf{m}\|_1=|m_1|+|m_2|+\ldots+|m_n|$ trivially coincides with semi-norm $|\mathbf{m}|$ on $\mathbb{N}_0^n$.
[47] Also remember the following integer l-tuplets of $\mathbb{Z}^l$: $\mathbf{0}_l:=(0,0,\dots,0)$ and $\mathbf{1}_l:=(1,1,\dots,1)$.

$$a \equiv b \bmod m$$

if $\exists k \in R$ such that $a = (k \bullet b) \circ m$. This is the definition of *Euclidean division* with **remainder** (m) for arbitrary rings and it can be easily proven that it has all the property of arithmetic congruence relations. In particular, $\forall p \in R$, let quotient rings

$$R/pR = \left\{ a \in R \,\middle|\, a \equiv p \bmod 0 \right\}$$

be defined. Prototypes of these quotients for finite rings are in fact $\mathbb{Z}_p := \mathbb{Z}/p\mathbb{Z}$, which are fields with respect to addition and multiplication modulo p if and only if $p \in \mathbb{P}^*$, cfr. §1.1.3.

Chinese reminders' Theorem

Let $\mathbf{m} \equiv (m_1, m_2, \ldots, m_n) \in \mathbb{N}^n$, $\mathbf{q} \equiv (q_1, q_2, \ldots, q_n) \in \mathbb{N}_0^n$ such that $\text{GCD}(m_i, m_j) = 1$, $1 \le i < j \le n$, then $\exists! x \in \mathbb{N}_0$ such that:

$$x\mathbf{1}_n \equiv \mathbf{q} \bmod \mathbf{m} \quad \Leftrightarrow \quad \begin{cases} x \equiv q_1 \bmod m_1 \\ x \equiv q_2 \bmod m_2 \\ \quad \vdots \\ x \equiv q_n \bmod m_n \end{cases}$$

$$x \equiv 0 \bmod \mathbf{m}^{\mathbf{1}_n} \quad \Leftrightarrow \quad m_1 m_2 \cdot \ldots \cdot m_n \mid x.$$

Let the following equivalence relation on $R^2 \times R^2$ hold: $(m,n),(p,q) \in R^2$ are equivalent **fractions** iff $\exists k \in R$: $(m,n) = (k \bullet p, k \bullet q)$ or $(k \bullet m, k \bullet n) = (p,q)$. An equivalence class of fractions will thus be indicated in the following, equivalent ways, according to one of its representatives, e.g. (m,n):

$$\frac{m}{n}, \qquad m/n, \qquad {}^{m}\!/_{n}. \tag{1.70}$$

Let the quotient ring of fractions of R indicated as $\mathrm{Frac}R$: it becomes a body (a field if R is abelian) let the following operations, inherited from R itself, be defined:

$$\circ : \mathrm{Frac}R \times \mathrm{Frac}R \to \mathrm{Frac}R \tag{1.71}$$

$$\frac{m}{n}\circ\frac{p}{q}\overset{\text{def}}{=}\frac{(m\bullet q)\circ(n\bullet p)}{n\bullet q};$$

$$\bullet\colon \mathrm{Frac}R\times\mathrm{Frac}R\to\mathrm{Frac}R \tag{1.72}$$

$$\frac{m}{n}\bullet\frac{p}{q}\overset{\text{def}}{=}\frac{m\bullet p}{n\bullet q}.$$

Together with the trivial exterior multiplication $R\times\mathrm{Frac}R\to\mathrm{Frac}R$, $(k,m/n)\mapsto k(m/n):=(k\bullet m)/n$, $\mathrm{Frac}R$ also becomes a R-modulo.

> The most important examples of fraction fields are the field of **rational** numbers $\mathbb{Q}:=\mathrm{Frac}\mathbb{Z}$ and the body of rational functions with coefficients in an ring R, defined as the R-modulo of fractions of its polynomial ring: $R\{x\}\overset{\text{def}}{=}\mathrm{Frac}R[x]$.
>
> Another example, introduced in §2.1.3, is that of meromorphic functions $\mathcal{K}(\Omega)$ on a open set $\Omega\subseteq\mathbb{C}$ as the fractions' ring of homlomorphic functions, i.e. $\mathcal{K}(\Omega):=\mathrm{Frac}\,\mathcal{O}(\Omega)$.

If K is any field, then $K\{x\}$ is a countable extension field of K itself (and, naturally, $K\leq K[x]\leq K\{x\}$), cfr. §1.1.10. Let $f\in K\{x\}$ be a rational function over field K and let $y=f(x)$; then $K\{y\}$ (more simply indicated as $K\{f(x)\}$) is naturally an extension field of $K\{x\}$, i.e.

$$K\{x\}\leq K\{f(x)\},\qquad \forall f\in K\{x\}. \tag{1.73}$$

A Galois closure of $K\{y\}$ always exists, and the Galois group $\mathrm{Gal}_{K\{x\}}K\{y\}$, called the ***monodromy group*** of $K\{f(x)\}$ (or monodromy group of $K\{x\}$ *in* f) is indicated as:

$$\mathrm{MG}_f K\{x\}\overset{\text{def}}{=}\mathrm{Gal}\frac{K\{f(x)\}}{K\{x\}}\equiv\mathrm{Aut}\frac{K\{y\}^{\mathfrak{G}}}{K\{x\}}. \tag{1.74}$$

As far as Galois Theory is concerned, field extensions of the form $\mathbb{C}\{z\}[\sqrt[n]{z}]$, $\forall n\in\mathbb{N}\setminus\{1\}$, are equivalent to the splitting field of polynomial $w^n - z$, i.e.:

$$\mathbb{C}\{z\}[\sqrt[n]{z}] \cong \frac{\mathbb{C}\{z\}[w]}{w^n - z} \cong \mathbb{C}\{w\}. \tag{1.75}$$

The monodromy group (1.74) of the map $z \mapsto \sqrt[n]{z}$ is then, by definition, the Galois group associated to the (multi-valued) arithmetic n^{th} root function $w = \sqrt[n]{z}$:

$$\mathrm{MG}_{\sqrt[n]{z}}\mathbb{C}\{z\} = \mathrm{Gal}\,\frac{\mathbb{C}\{\sqrt[n]{z}\}}{\mathbb{C}\{z\}} \equiv \mathrm{Gal}_{\mathbb{C}\{z\}}\mathbb{C}\{w\}\,. \tag{1.76}$$

Finally, as a particular of what exposed on Galois groups in §1.1.10, the following holds by:

Corollary

Every finite group G admits a finite Galois extension field F of $\mathbb{C}\{z\}$ whose Galois group is exactly G, that is $G\cong\mathrm{Gal}(F/\mathbb{C}\{z\})$.

Which means that every finite group can be represented as the symmetries' group of *either* some subfield *or* some finitely-extended field of complex rational functions.

1.3.2. Formal power series

Let $R(+,\cdot)$ be an ideal abelian ring as defined in §1.1.3; the completition of $R[x]$ respect to its ideal topology[48] (or the standard one if R is a topological ring, cfr. §1.1.3) is the topological abelian R-modulo of *formal power series* such that $R[\![x]\!] \overset{\text{def}}{=} \overline{R[x]} \cong R^{\mathbb{N}_0}$. It is a metric space whose metric, inherited from $R^{\mathbb{N}}$, is $\mathrm{d}\big((a_n)_{\mathbb{N}}, (b_n)_{\mathbb{N}}\big) := 2^{-\inf\{k\in\mathbb{N}_0 \;|\; a_{k+1}\neq b_{k+1}\}}$.

[48] Let I be the ideal of $R[x]$ generated by x; its ***ideal*** (or I-adic) *topology* is such that any subset $A\subseteq R$ is defined to be *open* if and only if $\forall x\in A\ \exists n\in\mathbb{N}_0$ such that $x+I^n\in A$.

Two sequences have distance either 0, $k\in\mathbb{N}$ or ∞ if and only if they are identical, have the same elements up to the k^{th} one, or begin with different ones (i.e. $a_0\neq b_0$), respectively. Formal power series are thus "near" if their low-order terms agree (up to the first different one): this regards just equality of terms and not their "arithmetic" value IN SE; practically, nothing more what happens when evaluating the asymptotic behaviour of Taylor series of, say, $\mathbb{R}[\![x]\!]$ or $\mathbb{C}[\![z]\!]$ via Landau symbols 'O' and 'o'. The elements of $R[\![x]\!]$ are written as [79]

$$R^{\mathbb{N}_0}\ni(a_n)_{n\in\mathbb{N}_0}\quad\mapsto\quad\sum_{n\in\mathbb{N}_0}a_nx^n\in R[\![x]\!],\tag{1.77}$$

whose sum and product $\cdot\colon R\times R[\![x]\!]\to R[\![x]\!]$ inherited from $R[x]$ are component-wise, whereas the *Cauchy product* (discrete **convolution**) $\circledast\colon R[\![x]\!]\times R[\![x]\!]\to R[\![x]\!]$, inherited from $R^{\mathbb{N}_0}$ is such that, $\forall A(x)=\sum_n a_nx^n$, $\forall B(x)=\sum_n b_nx^n$,

$$(A\circledast B)(x)\colon\equiv A(x)B(x)\overset{\text{def}}{=}\sum_{n\in\mathbb{N}_0}\left(\sum_{k=0}^{n}a_kb_{n-k}\right)x^n\tag{1.78}$$

and turns $R[\![x]\!]$ into an R-algebra. $R^{\mathbb{N}_0}$ and $R[\![x]\!]$ are thus isomorphic: their zero and unity being sequences $(0)_{\mathbb{N}_0}=(0,0,0,\ldots)$ and $(\delta_{1,n})_{\mathbb{N}_0}=(1,0,0,\ldots)$, respectively.

All the algebraic properties of R (locality, Nœther, domain-integrality, etc.) are reflected on $R[\![x]\!]$. As an additive group $R[\![x]\!]$ is infinitely-generated, whereas $R[x]$ is finitely-generated (since every polynomial is the sum of a finite number of monomials). Topologically, $R[\![x]\!]$ is compact if and only if R is finite. If K is a field then $K[\![x]\!]$ is a discrete-valuation complete local ring, as well as an infinite-dimensional topological K-algebra, Banach if K has any valuation.

To any $A(x)=\sum_n a_nx^n$ such that a_0 has "reciprocal" element $a_0^{-1}\in R$, such that $a_0a_0^{-1}=1$ (this trivially includes the case where R is a body or a field) a reciprocal power series $\exists A^{-1}\in R[\![x]\!]$ is associated, such that $A\cdot A^{-1}=A^{-1}\cdot A\equiv 1$. For example,[49] since $1\cdot 1=1$:

$$(1-A)^{-1}=\sum_{n\in\mathbb{N}_0} A^n.$$

$\forall A,B\in R[\![x]\!]$ with B homogenous (i.e. $b_0=0$) the composition operator '$\circ$' is well defined in $R[\![x]\!]$ and the composed power series $A\circ B=A\big(B(x)\big)$ has its n^{th} coefficient being computable as a *finite* sum of A's and B's coefficients, called *Faà di Bruno*'s Formula:

$$\frac{1}{n!}\sum_{\substack{\{1,2,\dots,n\}=\bigcup_{j=1}^{k}B_j:\\ B_i\cap B_j=\varnothing,\ i\neq j}} b_k\bullet a_{\prod_{j=1}^{k}\operatorname{Card}B_j}. \tag{1.79}$$

The formal derivative operator $\mathrm{D}\in\mathrm{D}\big[R[\![x]\!]\big]$ is the derivation of the component-wise formal derivatives of monomials $a_n x^n$, $\forall n\in\mathbb{N}_0$, cfr. note [10]:

$$\mathrm{D}A(x):\equiv A'(x)\overset{\text{def}}{=}\sum_{n\in\mathbb{N}} n a_n x^{n-1}. \tag{1.80}$$

Power series of power series are equivalent to power series in several variables *just algebraically*, not topologically. By induction on $d\in\mathbb{N}$, in fact, $R[\![x_1,x_2,\dots,x_{d-1}]\!][\![x_d]\!]\cong R[\![x_1,x_2,\dots,x_d]\!]$ holds since convergence in $R[\![x_1,x_2,\dots,x_{d-1}]\!]$, inherited from ideal topology of $\Lambda^{d-1}\big(R^{\mathbb{N}_0}\big)$, is different from the one in $\Lambda^{d}\big(R^{\mathbb{N}_0}\big)$.

Let $\mathbf{x}=(x_1,x_2,\dots,x_d)$ and $\mathbf{k}=(k_1,k_2,\dots,k_d)\in\mathbb{N}_0^d$ be a multi-index; let $A,B\in R[\![\mathbf{x}]\!]$ of coefficients $a_{\mathbf{k}}$ and $b_{\mathbf{k}}$ respectively and A written as:[50]

$$A(\mathbf{x})=\sum_{\mathbf{k}\in\mathbb{N}_0^d} a_{\mathbf{k}}\mathbf{x}^{\mathbf{k}}\equiv\sum_{n\in\mathbb{N}_0}\sum_{\substack{\mathbf{k}\in\mathbb{N}_0^d:\\ |\mathbf{k}|=n}} a_{\mathbf{k}}\mathbf{x}^{\mathbf{k}}.$$

---

[49] A generalization of the *cyclotomic* decomposition: $1-x^n=(1-x)(1+x+x^2+\dots+x^{n-1})$.

[50] With formal expressions for monomials $\mathbf{x}^{\mathbf{k}}::=x_1^{k_1}x_2^{k_2}\cdot\ldots\cdot x_d^{k_d}$ derived from (1.68).

They are "near" in the metric inherited from $R^{\mathbb{N}_0}$ if their low-order coefficients agree, i.e. $a_{\mathbf{k}}=b_{\mathbf{k}}$ for sufficiently low $|\mathbf{k}|$'s.

1.3.3. Formal Laurent Series

The R-modulo of ***formal Laurent series*** over the ring $R(+,\cdot)$ is the following topological completition (in the sense of §1.4.1):

$$R[\![x]\!] \overset{\text{def}}{=} \overline{\mathrm{Frac}R[x]}\,. \tag{1.81}$$

If K is a field, it can be shown that $K[\![x]\!]\cong\overline{K\{x\}}\cong\mathrm{Frac}K[\![x]\!]$, it is both the field of fractions of formal power series than the topological closure of the rational functions's field. Formal Laurent series are isomorphic to a submodulo of $R^{\mathbb{Z}}$ and equivalent to formal power series with a finite (or zero) number of negative terms:

$$f(x)=\sum_{n=-N}^{\infty} a_n x^n, \qquad a_{-N}\neq 0. \tag{1.82}$$

The equivalence is with the submodulo $\left\{(a_n)_{n\in\mathbb{Z}}\in R^{\mathbb{Z}} \;\middle|\; \exists N\in\mathbb{N}_0\colon\ n>N \Rightarrow\ a_{-n}=0\right\}$; under this condition, f can be also, more simply, written as $f(x)=\sum_{n\in\mathbb{Z}}a_n x^n$. Product, quotient and composition are well-posed since the formal Laurent series only carry a finite number of negative powers and that the quotient formula for derivatives adds to Leibnitz's and chain rules for the formal derivative operator (cfr. note [10]), i.e. $\mathrm{D}(f/g)=(f'g-fg')/g^2,\ \forall f,g\in R[\![x]\!]$.

The **residue** of the formal Laurent series $f\in R[\![x]\!]$ is the ring element $\mathrm{Res} f \overset{\text{def}}{=} a_{-1}\in R$. The "manageability" (aka, actual computability) of formal Laurant series is assured by the well-orderedness of its index set.

From the topological point of view $R[\![x]\!]$ is neither compact nor complete for 0-characteristic rings R. In this case the topological closure of $R[\![x]\!]$ is the algebra of formal bilateral power series written as

$$f(x) = \sum_{n\in\mathbb{Z}} a_n x^n :\equiv \sum_{n=-\infty}^{\infty} a_n x^n . \tag{1.83}$$

This structure is algebraically "poorer" than $R[\![x]\!]$, corresponding in Complex Analysis to the fact that the topological closure of $\mathbb{C}[\![z]\!]$ contains essential singularities too (cfr. §2.4); for an algebraically "richer" closure of formal Laurent series, cfr. next paragraph.

1.3.4. Transfinite and formal Puiseux Series

Further generalization of formal series leads to the **transfinite series** over a field K:

$$K^{\mathbb{Q}}[x] \overset{\text{def}}{=} \left\{ f(x) = \sum_{\alpha\in Q} a_\alpha x^\alpha \;\middle|\; \forall Q \subset \mathbb{Q}\colon \exists \min_{\mathbb{Q}} Q \right\}. \tag{1.84}$$

$K^{\mathbb{Q}}[x] \cong \overline{K^{\mathbb{Q}}}$, where $K^{\mathbb{Q}}$ is the vector space of $\mathbb{Q}$-indexed sequences. Furthermore $(K^\natural)^{\mathbb{Q}}[x] \cong K^{\mathbb{Q}}[x]^\natural$; so if K is algebraically closed, so is $K^{\mathbb{Q}}[x]$, cfr. §1.1.3. The directed sets Q of (1.84) are the support of a transfinite series and are defined as $\operatorname{supp} f := \{\alpha\in\mathbb{Q} \mid a_\alpha \neq 0\}$.

The elements of $K[\![x]\!]^\natural$, the algebraic closure[51] of $\operatorname{Frac} K[\![x]\!]$, are called ***formal Puiseux series***. If K has characteristic 0 then, formally,

$$K[\![x]\!]^\natural \cong \bigoplus_{n\in\mathbb{N}} K[\![\sqrt[n]{x}\,]\!] \cong K[\![x]\!]\left(\left\{\sqrt[n]{x} \;\middle|\; n\in\mathbb{N}\right\}\right), \tag{1.85}$$

and the formal Puiseux series get a best-known representation as "formal fractional-power series." The righ-hand side of (1.85) expresses this series as a radical, infinite field extension, like those introduced in §1.1.10; this fact is further stated in (1.89) in the case of complex Puiseux series.

Let $f\in K[\![x]\!]^\natural$; then $\exists p\in\mathbb{N}\setminus\{1\}$, $\exists (a_n)_{n\in\mathbb{Z}}\in K^{\mathbb{Z}}$ and $\exists N\in\mathbb{N}_0$ such that:

---

[51] It is important to stress that even if K is algebraically close, $K[\![x]\!]$ might not.

$$f(x) = \sum_{n=-N}^{\infty} a_n \sqrt[p]{x^n}, \qquad a_{-N} \neq 0. \tag{1.86}$$

If K is algebraically closed, then $K[\![x]\!]^{\natural} \lessapprox K^{\mathbb{Q}}[x]$, thus formal Puiseux series are a specific type of transfinite series —a particular case for $K=\mathbb{C}$ given by (1.89) below.

Of particular importance are the (complex) Puiseux series, $\mathbb{C}[\![z]\!]^{\natural}$ where summation and roots have precise meanings in $\mathbb{C}$ which, together with its topology and the Riemannian metric inherited from the branches of multi-valued algebraic functions (cfr. §1.1.3), allow to define convergence radii $\rho_c(f)$, $\forall f\in\mathbb{C}[\![z]\!]^{\natural}$ as is done for complex Laurent and power series. In particular, the role of Puiseux series is given by the following

Puiseux's Theorem

Let $P\in\mathbb{C}[\![z]\!]^n[w]$ be a n^{th}-degree irreducible polynomial over $\mathbb{C}[\![z]\!]$:

$$P(w,z) = \sum_{k=0}^{n} w^{n-k} a_k(z) = \\ w^n + a_1(z)w^{n-1} + a_2(z)w^{n-2} + \ldots + a_{n-2}(z)w^2 + a_{n-1}(z)w + a_n(z).$$

There exists a Laurent series $f(z)=\sum_{k=-N}^{\infty} c_k z^k \in\mathbb{C}[\![z]\!]$ such that

$$P(f(z), z^n) \equiv 0; \tag{1.87}$$

i.e. the solution of polynomial equation $P(w,z)=0$ is a Puiseux series

$$f(\sqrt[n]{z}) = \sum_{k=-N}^{\infty} c_k z^{\frac{k}{n}} \in \mathbb{C}[\![z]\!]^{\natural}. \tag{1.88}$$

When all the coefficients of P are holomorphic, i.e. $a_k\in\mathbb{C}[\![z]\!] \cong \mathcal{O}(\mathbb{C})$ for $0\leq k<n$, then also $f(z)$ is an entire function, i.e. a MacLaurin series $f\in\mathbb{C}[\![z]\!]$. In such a case the polynomial $P\in\mathbb{C}[\![z]\!]^n[w]$ used in the above Theorem is clearly a 2-variables' Weierstrass polynomial (cfr. §2.1.3) of trascendental variable z, since $\mathbb{C}[\![z]\!][w] \cong \mathcal{W}_1(\mathbb{C}, w)$.

Corollary

The complex map $z \mapsto z^n$ ($\forall n \in \mathbb{N} \setminus \{1\}$) is a homomorphism such that $\mathbb{C}[\![z^n]\!]$ is a n^{th}-degree extension field of $\mathbb{C}[\![z]\!]$ whose basis is, §1.1.3:

$$\mathbb{C}[\![z^n]\!] / \mathbb{C}[\![z]\!] \;=\; \operatorname{span}_{\mathbb{C}}\{z^k\}_{0 \le k < n}.$$

The Puiseux series $f(\sqrt[n]{z})$, considered as being evaluated on all the n branches of the arithmetic root, is (or, better, *are*) themselves all the distinct roots of P, i.e. $P\left(f(e^{\frac{2\pi ik}{n}} \sqrt[n]{|z|}), z^n\right) = 0$, for $0 \le k < n$. So $\mathbb{C}[\![z^n]\!]$ is the splitting field of P in $\mathbb{C}[\![z]\!]][w]$, that is $\mathbb{C}[\![z^n]\!] = \mathbb{C}[\![z]\!](\ker P)$, cfr. §1.1.10.

From (1.75)–(1.76) and it follows that, $\forall n \in \mathbb{N} \setminus \{1\}$, it follows that the minimal polynomial of $\sqrt[n]{z} \in \mathbb{C}[\![z]\!][\sqrt[n]{z}]$ in $\mathbb{C}[\![z]\!]$ is simply $\operatorname{Min}_{\sqrt[n]{z}}^{\mathbb{C}\{z\}}(w) = w^n - z \in \mathbb{C}[w, z]$, whose roots are $\left\{e^{\frac{2\pi ik}{n}} \sqrt[n]{|z|}\right\}_{0 \le k < n}$ and

$$\operatorname{Gal}(\mathbb{C}[\![z]\!](\sqrt[n]{z}) \,/\, \mathbb{C}[\![z]\!]) \cong \mathbb{Z}_n.$$

First of all $\mathbb{C}[\![z]\!]^{\natural}$ is, by (1.85), the union of all the finite field extensions $\mathbb{C}[\![z]\!][\sqrt[n]{z}]$, $\forall n \in \mathbb{N} \setminus \{1\}$. Then $(\mathbb{Z}_n, \phi_m^n)_{m,n \in \mathbb{N},\, m \le n}$ is an inverse system of groups, with $\phi_m^n([\![a]\!]_n) = [\![a]\!]_m$, $\forall a \in \mathbb{Z}$ and $m \le n$.

Theorem

The field of complex Puiseux series $\mathbb{C}[\![z]\!]^{\natural}$ is an infinite Galois extension of complex Laurent series $\mathbb{C}[\![z]\!]$; its corresponding Galois group is the following inverse limit, cfr. §1.1.7:

$$\operatorname{Gal}(\mathbb{C}[\![z]\!]^{\natural} / \mathbb{C}[\![z]\!]) \cong \varprojlim_n \mathbb{Z}_n = \bigoplus_{p \in \mathbf{P}} \mathbb{Z}_p. \tag{1.89}$$

Dealing with (positive-characteristic) algebraically closed fields instead, formal transfinite series provide a better ground than formal Puiseux series when extending the field of their formal Laurent series.

1.4 Fractal Group Theory

1.4.1. Algebra and Topology of rooted trees

A tree $A^{\intercal}$ associated to an **alphabet** $A=\{a_1,a_2,\ldots,a_L\}$ is the rooted, L-ary tree whose node-set is $\bigsqcup_{n\in\mathbb{N}_0} A^n$ (with n^{th}-level node-set being A^n, $\forall n\in\mathbb{N}_0$) and whose adjacency relation '@' is [9]–§A.1:

$$u\,@\,v \quad\Leftrightarrow\quad \exists n\in\mathbb{N}_0,\, \exists a\in A\text{: } u\in A^n,\ v\in A^{n+1},\, (1,a)\sqcup(n,u)=(n+1,v) \tag{1.90}$$

For simplicity's sake the node-set's elements will be written by juxtaposition of symbols one after another (just like was done in §1.1.8) without using either disjoint union symbol '⊔' and formalism; e.g. let $v::=au$ in (1.90). The root (0^{th}-level node) of $A^{\intercal}$ is the empty word $\varnothing::=(0,\varnothing)$, whereas the trees with node-sets

$$\bigsqcup_{n=0}^{N} A^n\,,\qquad \bigsqcup_{n=0}^{\infty} A^n\,,$$

representing words with at most N "characters" and words of *any* finite lengths, respectively, are the prototypes of all regular, either finite or infinite, rooted trees. In the former case, using $A^{\intercal}$'s discrete topology, $\partial A^{\intercal}=A^N$ (i.e. all the N-characters words). In the latter case it can be proven that $\overline{A^{\intercal}}=A^{\intercal}\cup A^{\mathbb{N}}$, where $A^{\mathbb{N}}$ is the complete Cantor set (cfr. Figure 15 and example of §2.3.2) of all the "infinite-length words." Figure 6 represents the infinite binary tree $\{0,1\}^{\intercal}$. **Bi-infinite** words are also introduced as $A^{\mathbb{Z}}$.

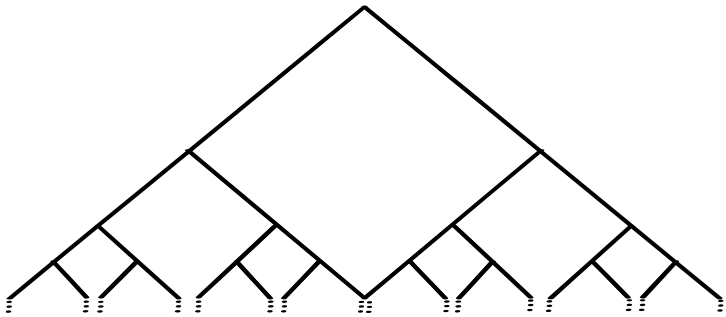

Figure 6 Example of an infinite binary rooted tree (*not* compactified, as the boundarying Cantor set is missing).

A function $f{:}A^{\top}\to B^{\top}$ between rooted trees is a tree morphism if and only if $f(\varnothing)=\varnothing$ and $\forall a\in A\ \exists b\in B$: $f(au)=bf(u)\ \forall u\in A^{\top}$. As usually, if f is bijective it is said to be a tree isomorphism and the two trees are isomorphic with each other, $A^{\top}\cong B^{\top}$; if $\mathrm{Card}A=\mathrm{Card}B$ (not necessarily $A=B$, since in that case alphabets can be just replaced with each other without changing their trees' algebra and topology) then f is a tree endomorphism [automorphism, if f is bijective — see Figure 7]; $\mathrm{Hom}(A^{\top},B^{\top})$, $\mathrm{End}A^{\top}$ and $\mathrm{Aut}A^{\top}$ denote the sets of all tree homo-/endo-/automorphisms between rooted trees, respectively. Also, $\mathrm{Aut}A^{\top}<\mathrm{Sym}A^{\top}$, i.e. the symmetric group of any free permutation of the tree's nodes (even the non-level-preserving ones).

$\forall v\in A^{\top}$ the mapping $A^{\top}\ni w\mapsto vw$ trivially belongs to $\mathrm{End}A^{\top}$, is onto and maps $A^{\top}$ to $vA^{\top}\cong A^{\top}$, which is the infinite rooted tree of root v and $vA^{\top}$ is the subtree of nodes "beginning" with the word v; the inverse endomorphism $vA^{\top}\to A^{\top}$ is $vw\mapsto w$. So the natural topology of compact set $\overline{A^{\top}}$ is $\mathsf{T}=\{(wA^{\top})\cup(wA^{\mathbb{N}})\mid\forall w\in A^{\top}\}$.

To any $f\in\mathrm{End}A^{\top}$ an endomorphism $f|_v\in\mathrm{End}A^{\top}$ is defined such that $f(vw):=f(v)f|_v(w)$, and it is called the **restriction of f in v**.

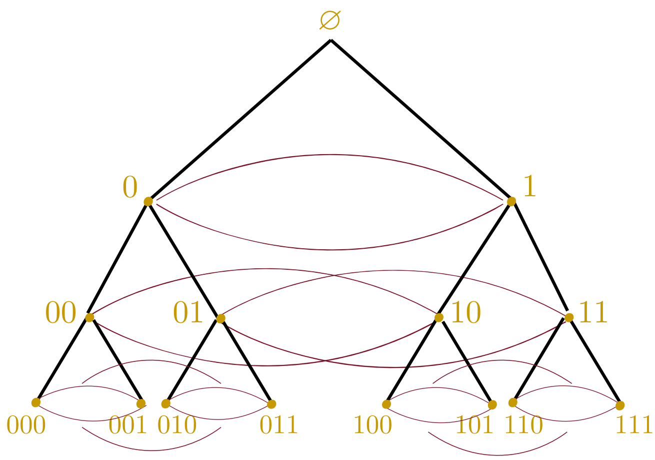

Figure 7 Example of a tree automorphism (in blue) on the 3^{rd}-order binary rooted tree $\mathbb{Z}_2 \sqcup \mathbb{Z}_2^2 \sqcup \mathbb{Z}_2^3$ of Figure 6 (in black, with its words on the alphabet $\{0,1\}$ shown).

Tree automorphisms, in particular, have a simple interpretation: $\mathrm{Aut}\,A^{\top}$ is in fact (respect on functions' composition '$\circ$') the permutation group acting on $A^{\top}$ and preserving words' length (i.e. whose invariant sets are A^n, $\forall n \in \mathbb{N}_0$), cfr. § o and Figure 7 . For more information on groups acting on rooted trees, cfr. [83] and [27].

A (sub-)group of tree automorphisms $G \leq \mathrm{Aut}\,A^{\top}$ is said to be level-**transitive** if it is transitive on every level A^n, $\forall n \in \mathbb{N}_0$; in analogy with group-theoretic stabilizers defined in § o , the vertex-stabilizer of $v \in A^{\top}$ is $\mathrm{stab}_G v := \{ g \in G \mid g(v) = v \} \leq G$, whereas the *$n^{th}$-level* vertex-stabilizer is $\mathrm{stab}_n G := \bigcap_{v \in A^n} \mathrm{stab}_G v$, $\forall n \in \mathbb{N}_0$. The **rigid** [$n^{\mathrm{th}}$-level] vertex-stabilizer is $\mathrm{STAB}_G v := \{ g \in G \mid g(u) = u, \quad \forall u \in A^{\top}, \quad u \notin vA^{\top} \}$ $\left[\ \mathrm{STAB}_n G := \bigcap_{v \in A^n} \mathrm{STAB}_G v \ \right]$.

Proposition

Either all the rigid and rigid level stabilizers are infinite, or all but a finite number of rigid level stabilizers are trivial; furthermore:

$$\mathrm{STAB}_n \mathrm{Aut}\,A^{\top} \equiv \mathrm{stab}_n \mathrm{Aut}\,A^{\top}, \quad \forall n \in \mathbb{N}_0. \tag{1.91}$$

> Subgroups $\{\mathrm{stab}_n \mathrm{Aut} A^{\intercal}\}_{n\in\mathbb{N}}$ form a fundamental system of neighbourhoods of $\mathbb{1}\in\mathrm{Aut}A^{\intercal}$ which, thanks to the pointwise convergence on $A^{\intercal}$, makes $\mathrm{Aut}A^{\intercal}$ a homeomorphisms' group.

This Proposition proves that the trees and their automorphisms can be given proper topologies such that operations between more general topological spaces (like orbifolds, cover spaces, smooth manifolds) can be represented by tree symmetries via the Topology language too.

1.4.2. Self-similar groups

The (left) *faithful* action of a group $G\leq\mathrm{Aut}A^{\intercal}$ on $A^{\intercal}$ is said to be self-similar (and G act self-similarly on tree $A^{\intercal}$ — see Figure 8) if $\forall g\in G$ and $\forall a\in A$, $\exists h_g\in G$ and $\exists b_{g,a}\in A$:

$$g\bullet(aw)=b_{g,a}(h_g\bullet w)\,,\qquad \forall w\in A^{\intercal}. \tag{1.92}$$

Then it is clear that, if '$\bullet$' is such a self-similar action, then $\forall v\in A^{\intercal}$:

$$g\bullet(vw)=(g\bullet v)\left(g|_v\bullet w\right).$$

From Cayley's Theorem (§1.1.1) we know that every group is a permutation group, but it could not be —in general— a group of tree automorphisms: it can be shown that any faithful action $\bullet\colon G\times A\to A$ on the alphabet A can be turned into a transitive action $\bullet\colon G\times A^{\intercal}\to A^{\intercal}$ via juxtaposed restrictions on any level of the tree, i.e. $g\bullet(ab\cdots z):=(g\bullet a)(g\bullet b)\cdots(g\bullet z)\in A^{\intercal}$ $\forall a,b,\dots,z\in A$. From considerations in §1.4.1 and (1.26), it follows that $\mathrm{Aut}A^{\intercal}\cong\mathrm{Sym}A\wr\mathrm{Aut}A^{\intercal}$, where an element of $\mathbb{S}_L\wr\mathrm{Aut}A^{\intercal}$ is the following permutation:

$$\pi_{\mathbf{a}}(f|_a f|_b f|_c\cdots)=f\Big|_{\pi_{\mathbf{a}}(a)} f\Big|_{\pi_{\mathbf{a}}(b)} f\Big|_{\pi_{\mathbf{a}}(c)}\cdots\in\overset{L}{\circledcirc}\mathrm{Aut}A^{\intercal}\,, \tag{1.93}$$

$\forall f\in\mathrm{Aut}A^{\intercal}$ and $\forall\pi_{\mathbf{a}}\in\mathrm{Sym}A\cong\mathbb{S}_L$ with $\mathbf{a}\in A^L$. This roughly means that permuting the symbols of the alphabet with each other, $A^{\intercal}$ is equivalent to applying a tree automorphism, i.e. all the tree automorphism are described via permutations of the alphabet's symbols. This procedure is called **wreath recursion** on $\mathrm{Sym}\,A \wr \mathrm{Aut}\,A^{\mathsf{T}}$.

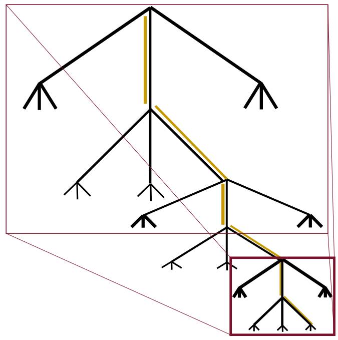

Figure 8 Example of self-similar action on a ternary tree: h_g and $b_{g,a}$ are represented according to (1.92).

To be more precise, let $G(\circ)=\langle g_i\rangle_{1\leq i\leq r}$ be a finitely generated automorphisms' subgroup ($\mathrm{rk}\,G=r$); any wreath recursion on $\mathrm{Sym}\,A \wr G$ is represented by a matrix $H\equiv(h_{i,j})_{i,j}\in\mathbb{M}_{r,L}(A)$ where $h_{i,j}:=g_i|_{a_j}$. If $\pi_1,\pi_2,\ldots,\pi_r\in\mathrm{Sym}\,A$ are fixed permutations of the alphabet then, from (1.93), the generic $g\in G\leq\mathrm{Aut}\,A^{\mathsf{T}}$ is generated for a choice of coefficients $c_1,c_2,\ldots,c_r\in\mathbb{N}_0$ such that:

$$g=\bigcirc_{i=1}^{r} g_i^{c_i} \qquad \text{and} \qquad \begin{cases} g_1=\pi_1(h_{1,1}h_{1,2}\cdots h_{1,L}) \\ g_2=\pi_2(h_{2,1}h_{2,2}\cdots h_{2,L}) \\ \quad\vdots \\ g_r=\pi_r(h_{r,1}h_{r,2}\cdots h_{r,L}) \end{cases}. \tag{1.94}$$

The matrix H contains all the information on the words' permutations in the alphabet generating the automorphism. So, as long as G is finitely generated and acts on a finite set (alphabet), the definition of self-similar group is well-posed and is a pure feature of the group, not of the associated rooted-tree action.

Definition

$G(\circ)$ is said to be a ***self-similar** group* if and only if $\exists n \in \mathbb{N} \setminus \{1\}$:

$$G \cong \mathbb{S}_n \wr G. \tag{1.95}$$

that is G acts on a non-trivial finite set A and $G \cong \mathrm{Sym}\, A \wr G$, i.e. $g|_w \in G,\ \ \forall g \in G, \forall w \in A^\intercal$ (after isomorphism $G \lessapprox \mathrm{Aut}\, A^\intercal$ is given).

The "self-similarity" property in a such a group is thus exactly (1.95), which means

$$G \cong \mathbb{S}_n \wr (\mathbb{S}_n \wr G) \cong \mathbb{S}_n \wr (\mathbb{S}_n \wr (\mathbb{S}_n \wr G)) \cong \ldots \cong \mathbb{S}_n \wr \Big(\mathbb{S}_n \wr (\mathbb{S}_n \wr (\mathbb{S}_n \wr \cdots)) \Big) \stackrel{\text{def}}{=} \wr^{\infty} \mathbb{S}_n .$$

A more complete statement can be formulated whenever $\mathrm{rk}\, G = \mathrm{Card}\, A$:

Wreath recursion Theorem

Let $A = \{a_j\}_{1 \leq j \leq N}$ be an alphabet and $G(\circ) = \langle g_i \rangle_{1 \leq i \leq N}$ be a finitely generated group acting faithfully on $A^\intercal$, i.e. $G \lessapprox \mathbb{S}_N \wr G$.

If G is self-similar the matrix $H \in \mathbb{M}_N(A)$ where $h_{i,j} = g_i|_{a_j}$, $1 \leq i, j \leq N$ uniquely determines the wreath recursion (1.94) and whence the action of G on $\mathrm{Aut}\, A^\intercal$ or —vice versa if (1.94) is given but G is unknown— a self-similar G can be built by computing its generators.

As a Corollary, the union of all the finitely generated self-similar groups acting on $A^\intercal$ is a countable group itself (respect to group actions' composition).

Although this discipline is relatively new, earlier applications of self-similar actions on rooted trees regarded Number Theory and Fractal Geometry (Complex Dynamics [86] and IFS Theory [27] in particular, where both disciplines are strictly related), especially when the alphabet is used as a base for a numbering system (for example $A = \{0, 1, \ldots, b-1\}$ for base-b integers, $\forall b \in \mathbb{N} \setminus \{1\}$).

The group of affine transformations of an n-dimensional R-modulo (for any arithmetic ring $R(+,\cdot)$), e.g. its prototype R^n, is the additive group $\mathrm{GL}_n(R)\ltimes R^n$. Affine transformations are the group actions

$$R^n \ni \mathbf{x} \mapsto A\mathbf{x}+\mathbf{b}, \tag{1.96}$$

where $A\in\mathrm{GL}_n(R)$, $\mathbf{b}\in R^n$ and whose inverse acts as $\mathbf{x} \mapsto A^{-1}(\mathbf{x}-\mathbf{b})$.

The group $\mathrm{GL}_n(\mathbb{Z})\ltimes\mathbb{Z}^n$, in particular, is self-similar and acts continuously on $\overline{\mathbb{Z}_2^{\mathbb{I}}}$. For the binary (i.e. base-2) natural numbers' semi-group, $\mathbb{N}_0 \cong \mathbb{Z}_2^{\mathbb{N}} \cong \mathbb{Z}_2[\![2]\!]$ holds; in fact let the following binary multi-index decomposition $\forall\mathbf{m}\in\mathbb{N}_0^n$:

$$\mathbf{m}=\left(\sum_{k=0}^{\infty} m_{h,k}2^k\right)_{1\leq h\leq n}, \qquad m_{h,k}\in\{0,1\}\asymp\mathbb{Z}_2,\ 1\leq h\leq n, \forall k\in\mathbb{N}_0.$$

Then consider juxtaposition $\mathbb{N}_0^n \leftrightarrow \mathbb{Z}_2^{\mathbb{I}}$ of multi-index coordinates

$$\mathbf{m} \mapsto m_{0,1}m_{0,2}\cdots m_{0,n}m_{1,1}m_{1,2}\cdots m_{1,n}m_{2,1}m_{2,2}\cdots m_{2,n}\cdots\cdots;$$

composing this with (1.96), generates the action of $\mathrm{GL}_n(\mathbb{Z})\ltimes\mathbb{Z}^n$ on $\mathbb{Z}_2^{\mathsf{T}}$, whereas topologically closing the tree into $\overline{\mathbb{Z}_2^{\mathsf{T}}}$ corresponds to passing from $\mathbb{Z}_2^{\mathbb{N}}$ to formal binary power series[52] $\mathbb{Z}_2[\![2]\!]$, cfr. §1.4.1.

The action of $\mathrm{GL}_n(\mathbb{Z})\ltimes\mathbb{Z}^n$ on $\mathbb{Z}_2^{\mathsf{T}}$ is self-similar since any infinite word can always be trivially "factored" into an infinite number of syllables of length n (i.e. a sequence in $(\mathbb{Z}_2^n)^{\mathbb{N}}$, by inverse juxtaposition) to which each automorphism (1.96) acts.

Since $\mathrm{Sym}\mathbb{Z}_2\equiv\mathrm{S}_2\cong\mathbb{Z}_2$, then $\mathrm{GL}_n(\mathbb{Z})\ltimes\mathbb{Z}^n\cong\mathbb{Z}_2\wr(\mathrm{GL}_n(\mathbb{Z})\ltimes\mathbb{Z}^n)$.

As far as the growth rate of a self-similar groups is concerned, one still has to resort on their actions on rooted trees: if G is a finitely-generated group (of generating set S) acting on A^{T} its contraction coefficient and growth degree are the limits:

$$\rho(G,A)\overset{\text{def}}{=}\overline{\lim_n}\sqrt[n]{\sup_{g\in G}\max_{w\in A^{\mathsf{T}}}\frac{\mathrm{l}_S(g|_w)}{\mathrm{l}_S(g)}}; \tag{1.97}$$

$$\gamma(G,A)\overset{\text{def}}{=}\sup_{w\in\overline{A^{\mathsf{T}}}}\overline{\lim_{\delta\to\infty}}\frac{\mathrm{Card}\left\{g(w)\;\middle|\;\mathrm{l}_S(g)\leq\delta\right\}}{\log\delta}. \tag{1.98}$$

If $\rho(G,A)<1$ the action is said to be **contracting**. If G acts self-similarly, (1.97) and (1.98) are proven to exist, finite and do not depend on the generating set S.

The first self-similar group example is the Grigorčuk group acting on the infinite binary tree, introduced in §1.4.1, [57] and [83].

The Grigorčuk group $\mathbb{G}$ is amenable and acts self-similarly on $\overline{\mathbb{Z}_2^{\mathsf{T}}}$; its generators are $a,b,c,d\in\mathrm{Aut}\mathbb{Z}_2^{\mathsf{T}}$ acting according to the following table:

$$\begin{array}{c|cccc} & a & b & c & d \\ \hline 0w & 1w & 0a(w) & 0a(w) & 0w \\ 1w & 0w & 1c(w) & 1d(w) & 1b(w) \end{array} \qquad \forall w\in\mathbb{Z}_2^{\mathsf{T}}. \tag{1.99}$$

[52] The fact that R^{T}'s tree-topology equals ideal topology on R is far more general than this $\mathbb{Z}_2$ case.

In the end then, $\mathbb{G} \cong \mathbb{Z}_2 \wr \mathbb{G}$, Q.E.D.

For any contracting action the following subset $\mathcal{N}_G \subseteq G$, called the ***nucleus*** of the action, is non-empty and finite:

$$\mathcal{N}_G \overset{\text{def}}{=} \bigcup_{g\in G} \bigcap_{n\in\mathbb{N}_0} \left\{ g\big|_v \in G \;\middle|\; v\in A^{\mathsf{T}},\ |v|\geq n \right\}, \tag{1.100}$$

i.e. $\mathcal{N}_G$ is the minimal set of group elements which *definitely* fix the words under G's action. So a self-similar action is contracting if it definitely acts in a finite number of ways on long enough words. Consider the similarity between the Krull topology (1.52) introduced in §1.2.3, and the definition of nucleus of a contracting action (1.100): in fact, (Infinite) Galois Theory could be formulated using the language of (Galois) group actions on the open and closed sets of the maximal extension field; this alternative approach lends itself to beautiful geometrical speculations, like those in §3.3.4 for a certain class of Riemann surfaces.

The growth degree of a contracting action is upper-bounded:

$$\gamma(G,A) \leq -\frac{\log \operatorname{Card} A}{\log \rho(G,A)}. \tag{1.101}$$

The 3rd-order ***Sierpiński* group** $\breve{\mathbb{S}}_3 := \left\langle \blacktriangle_n \,\middle|\, \blacktriangle_m(n) = m + l\delta_{m,n} \right\rangle_{l,m,n\in\mathbb{Z}_3}$ is defined by its action on $\mathbb{Z}_3^{\mathsf{T}}$ (where $\mathbb{Z}_3 \asymp \{0,1,2\}$) by following table:

$$\begin{array}{c|ccc} & \blacktriangle_0 & \blacktriangle_1 & \blacktriangle_2 \\ \hline 0w & 0w & 2w & 1w \\ 1w & 2w & 1w & 0w \\ 2w & 1w & 0w & 2w \end{array} \qquad \forall w\in\mathbb{Z}_3^{\mathsf{T}}. \tag{1.102}$$

It can be proven that $\breve{\mathbb{S}}_3 \cong \mathbb{Z}_3 \wr \breve{\mathbb{S}}_3$, whose Schreier graph is depicted in Figure 9 whereas combinatorial computation leads to $\rho(\breve{\mathbb{S}}_3, \mathbb{Z}_3) = \frac{1}{2}$ and, finally, to

$$\gamma(\breve{\mathbb{S}}_3, \mathbb{Z}_3) = \frac{\log 3}{\log 2} = \dim_{\mathrm{H}} \check{S}_{\triangle 2}. \tag{1.103}$$

> In this case the 3rd-order Sierpiński group's growth rate equals the Hausdorff dimension of the well-known **Sierpiński gasket** $\check{S}_{\triangle 2}\subset\mathbb{R}^2$, depicted in Figure 10 (also cfr. [9]–§1.8 and [26]), which it is the symmetry group of. Further analysis of $\breve{\mathbb{S}}_3$ is carried out in the example of §1.4.3.

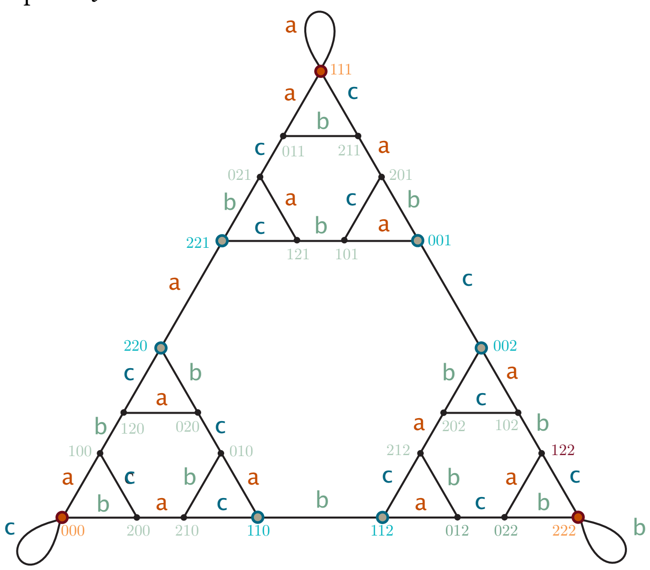

Figure 9 Schreier graph of the 3rd-order Sierpiński group $\breve{\mathbb{S}}_3$ acting on $\mathbb{Z}_3^3$ with $\blacktriangle_0\equiv$ a, $\blacktriangle_1\equiv$ b and $\blacktriangle_2\equiv$ c for simplicity's sake.

The growth degree has the "flavour" of a fractal dimension [9]–§1.5 or a Lyapunov exponent [6], [7], [51], [92], in its definition and meaning; it is in fact something similar, let the group action be properly described within the dynamical systems' setting, cfr. §1.4.3 and [82].

1.4.3. Limit spaces and self-similar graphs

We want to define useful 'limit' operators on the closed tree $\overline{A^\intercal}$ and extend the properties explained in the previous paragraphs to it. Let us consider now the orbispace $A^\intercal/G$ and the induced transitive action $\bullet : G\times A^\intercal\to A^\intercal$; the following limits are equal $\forall\xi=x_1x_2x_3\cdots\in\partial A^\intercal\equiv A^{\mathbb{N}}$, with respect to both the topology of $\overline{A^\intercal}$ defined in §1.4.1 on the left and the word metric for $\mathbb{1}\in G\lessapprox\mathrm{Aut}A^\intercal$ defined in §1.1.9 on the right:

$$\lim_{x\to\xi}x=\lim_n x_1x_2\cdots x_n. \tag{1.104}$$

This extends in fact the concept of limit word, under the action of G. Two infinite words $x_1x_2x_3\cdots,y_1y_2y_3\cdots\in\partial A^\intercal$ are **asymptotically** equivalent if $\exists(g_n)_{\mathbb{N}}\in\ell^\infty(G)$ bounded sequence, i.e. with $\mathrm{Card}\{g_n\}_{\mathbb{N}}<\aleph_0$, such that $g_n(x_1x_2\cdots x_n)=y_1y_2\cdots y_n$, $\forall n\in\mathbb{N}$ and are written as $x_1x_2x_3\cdots\sim y_1y_2y_3\cdots$. In this sense, they asymptotically belong to the same orbit, i.e. $\lim_n G(x_1x_2\cdots x_n)=\lim_n y_1y_2\cdots y_n:=G(x_1x_2x_3\cdots)$ and, from (1.23), the same holds for asymptotic stabilizers' definition:

$$\mathrm{stab}_G(x_1x_2x_3\cdots)\stackrel{\text{def}}{=}\lim_n\mathrm{stab}_G(x_1x_2\cdots x_n):=\bigcap_{n=1}^{\infty}\mathrm{stab}_G(x_1x_2\cdots x_n) \tag{1.105}$$

$$\Downarrow$$

$$\exists g\in\mathrm{stab}_G(x_1x_2x_3\cdots)\leq G:\qquad \left.\begin{array}{r}\lim_n g(x_1x_2\cdots x_n)\\ \lim_{\xi\to x_1x_2x_3\cdots} g(\xi)\end{array}\right\}=y_1y_2y_3\cdots. \tag{1.106}$$

The orbispace $A^\intercal/G$ admits a compactification, indicated as $\overline{A^\intercal}/G$, which is the topological quotient of $\overline{A^\intercal}/\sim$ under the asymptotical equivalence relation '$\sim$'. Its invariant set $(\partial A^\intercal)/\sim$, called the ***limit space*** $\mathcal{J}_G$ for the action of G, is often a "fractal" set, being a portrait for the self-similarity of G.

Theorem

The limit space $\mathcal{J}_G$ is a compact, metrizable invariant[53] orbiset of G; it does not depend on the choice of the contracting action and $\dim\mathcal{J}_G < \mathrm{Card}\mathcal{N}_G$. It is [locally] connected if G is finitely-generated and acts [level-]transitively; it is both path- and locally path-connected if G acts recurrently upon it.

Further topological properties of limit spaces $\mathcal{J}_G$, together with the definition of *orbifold*, will be investigated in §2.3 in order to introduce the concept of iterated monodromy groups for either self-similar groups and Riemann surfaces.

Due to the invariance property of $\mathcal{J}_G$, a topological dynamical system can be introduced as $(\mathcal{J}_G;\mathsf{s})$, where s is the *shift* map $x_1x_2x_3\cdots \mapsto x_2x_3x_4\cdots$, with $\mathrm{Card}\,\mathsf{s}^{-1}(\mathcal{J}_G) \le L$ (i.e. pre-images of s have at most $\mathrm{Card}A$ points). The shift map $\mathsf{s}:\mathring{\mathcal{J}}_G \to \mathcal{J}_G$ is a partial self-covering of the limit set and if G is finitely generated, $\mathcal{J}_G$ is its own universal covering (by previous Theorem), [82].

[53] The limit set is a G-invariant set, i.e. $G\mathcal{J}_G = G$ (that is why it is also a self-covering, cfr. §2.3.2).

The “farthest” nodes of the infinite ternary (base-3) tree $\overline{\mathbb{Z}_3^{\top}}$ induced by the action of $\breve{\mathbb{S}}_3$ —the 3rd-order Sierpiński group defined as (1.102)— are $000\cdots_3$, $111\cdots_3$ and $222\cdots_3$ (cfr. Figure 9), whereas the pending asymptotic equivalences are: $1000\cdots_3 \sim 2000\cdots_3$, $0111\cdots_3 \sim 2111\cdots_3$ and $0222\cdots_3 \sim 1222\cdots_3$.

The connections with the Sierpiński gasket $\check{S}_{\triangle 2}$ are higher than (1.103) in fact the orbispace $\mathcal{J}_{\breve{\mathbb{S}}_3}$ is homeomorphic to the Sierpiński gasket itself, i.e. $\mathcal{J}_{\breve{\mathbb{S}}_3} \asymp \check{S}_{\triangle 2}$, see Figure 9 and Figure 10 .

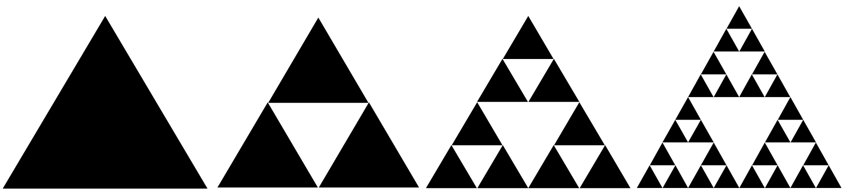

Figure 10 The Sierpiński gasket's prefractal $\check{S}_{\triangle 2,N}$ of triangular initiator, for $0 \leq N \leq 3$. See Figure 9 for topological comparison with 3rd-order Sierpiński group $\breve{\mathbb{S}}_3$.

As far as *self-similar Lie* groups are concerned, let $\bullet \in \mathrm{C}^0(\mathcal{G} \times X; X)$ be a continuous, self-similar action of a Lie group $\mathcal{G}(\circ)$ (of neutral element e) on a topological space $X \neq \varnothing$; $\mathcal{G}$ is self-similar from §1.4.2 if $\mathcal{G} \cong X \wr \mathcal{G}$.

Let $\mathfrak{A}_{\mathbb{C}}(\cdot)$ be a Lie algebra and $\mathcal{G}(\circ) = \mathbb{C}^*(\cdot)$ the multiplicative complex (Lie) group acting on $\mathfrak{A}$ as the exponential maps

$$\mathfrak{A} \times \mathbb{C}^* \ni (\mathbf{a}, w) \quad \mapsto \quad w^{\mathbf{a}} \stackrel{\text{def}}{=} \sum_{k=0}^{\infty} \frac{(\mathbf{a} \log w)^k}{k!};$$

it is clear that $\mathbb{C}^*$ acts self-similarly on any formal power series $\sum_{n\in\mathbb{N}} \mathbf{a}_n z^n \in \mathfrak{A}[\![z]\!]$ of elements of $\mathfrak{A}$, in fact

$$w^{\sum\limits_{n\in\mathbb{N}} \mathbf{a}_n z^n} = \prod_{n\in\mathbb{N}} e^{\mathbf{a}_n z^n}.$$

Since (at least from $\mathfrak{A}$'s discrete topology point of view) $\mathfrak{A}^{\mathsf{T}} \cong \mathfrak{A}[\![z]\!]$, $\mathbb{C}^*$ is a (quite trivial) self-similar Lie group because $\mathbb{C}^* \cong \mathfrak{A}[\![z]\!] \wr \mathbb{C}^*$.

1.5 Appendix – Alternative number systems

This Appendix is related to some less-known numbering representation schemes, particularly useful for advanced numbers' investigation: continued fraction and nested radicals in particular. In the last part of the Appendix further considerations on Galois' Theory (introduced in §1.1.10) will be given.

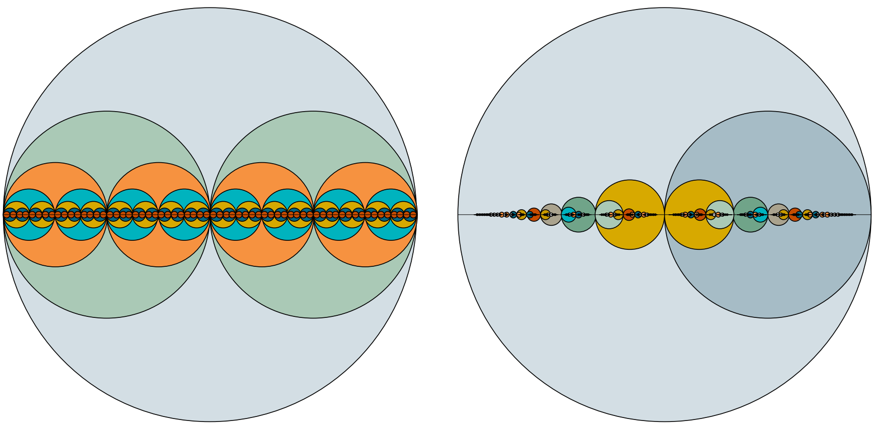

Figure 11 Approximating sets on the unit interval $[0,1]$ for a binary digit expansion (left) and a continued fraction expansion (right).

1.5.1. Continued fractions

Let $R(+,\cdot)$ be an arithmetic abelian ring and $N\in\mathbb{N}$, let $\mathbf{a}=(a_1,a_2,\ldots,a_N)$ and $\mathbf{b}=(b_1,b_2,\ldots,b_N)$ two vectors of R^N such that $a_N+b_N\neq 0$. The following element of the fractions' ring $\operatorname{Frac}R$,

$$\bigcirc\!\!\!\!\!\mathrm{C}\,\frac{\mathbf{b}}{\mathbf{a}}:\equiv \underset{n=1}{\overset{N}{\bigcirc\!\!\!\!\!\mathrm{C}}}\,\frac{b_n}{a_n}:\equiv \bigcirc\!\!\!\!\!\mathrm{C}\,\frac{b_1,b_2,\ldots,b_N}{a_1,a_2,\ldots,a_N}\overset{\text{def}}{=}b_1+\cfrac{a_1}{b_2+\cfrac{a_2}{\ddots+\cfrac{a_{n-1}}{b_n+a_n}}}, \tag{1.107}$$

is defined, with $\mathbf{a}$ [$\mathbf{b}$] being the vector of its partial numerators [denominators]. Extending the continued fraction definition (1.107) AD LIBITUM (i.e. for infinite sequences of partial numerators and denominators) leads to a new ordered set, larger than $\operatorname{Frac}R$, whose every subset admits a supremum 'sup' and an infimum 'inf' element; such set is the *Dedekind completition* of R, which is not discussed further: only the Dedekind competition of $\mathbb{Q}$ as being $\mathbb{R}$ will be described here as an example.

Let $(a_n)_{n\in\mathbb{N}},(b_n)_{n\in\mathbb{N}}\in\mathbb{C}^{\mathbb{N}}$ with $a_n\neq 0\ \ \forall N\in\mathbb{N}$, then the *generalized* ***continued fraction*** associated to them,

$$\bigcirc\!\!\!\!\!\mathrm{C}\,\frac{(b_n)_{\mathbb{N}}}{(a_n)_{\mathbb{N}}}:\equiv \underset{n=1}{\overset{\infty}{\bigcirc\!\!\!\!\!\mathrm{C}}}\,\frac{b_n}{a_n}\overset{\text{def}}{=}b_1+\cfrac{a_1}{b_2+\cfrac{a_2}{b_3+\cfrac{a_3}{\ddots}}}, \tag{1.108}$$

either converges to a complex number $\zeta\in\mathbb{C}$ or absolutely diverges. Since

$$\underset{n=1}{\overset{\infty}{\bigcirc\!\!\!\!\!\mathrm{C}}}\,\frac{b_n}{a_n}=\lim_n \bigcirc\!\!\!\!\!\mathrm{C}\,\frac{b_1,b_2,\ldots,b_n}{a_1,a_2,\ldots,a_n}, \tag{1.109}$$

the n^{th} partial continued fraction in the right-hand side of (1.109) is called the *n^{th} convergent* ζ_n of both (1.108) and ζ, $\forall n\in\mathbb{N}_0$. Trivially, $\zeta=\lim_n\zeta_n$.

If $\mathbf{a},\mathbf{b}\in\mathbb{Z}^N$ [$(a_n)_{\mathbb{N}},(b_n)_{\mathbb{N}}\in\mathbb{Z}^{\mathbb{N}}$] the associated continued fraction exists, finite, and its convergents' sequence $(\zeta_n)_{\mathbb{N}}$ is composed by rational numbers $\zeta_n=A_n/B_n\in\mathbb{Q}$ (with $A_n,B_n\in\mathbb{Z}$, $B_n\neq 0$, $\forall n\in\mathbb{N}_0$), computed with the recursive formula, $\forall n\in\mathbb{N}\setminus\{1\}$:

$$\begin{cases} A_{n+1}=a_{n+1}A_{n-1}+b_{n+1}A_n\,, \\ B_{n+1}=a_{n+1}B_{n-1}+b_{n+1}B_n\,, \\ A_1=b_1\,, \qquad B_1=1; \end{cases}$$

$$\Downarrow$$

$$\begin{pmatrix} A_{n+1} \\ B_{n+1} \end{pmatrix} = \begin{pmatrix} A_{n-1} & A_n \\ B_{n-1} & B_n \end{pmatrix}\begin{pmatrix} a_{n+1} \\ b_{n+1} \end{pmatrix} \tag{1.110}$$

and $A_{n-1}B_n-A_nB_{n-1}=(-1)^n$. In the case of a finite generalized continued fraction, the N^{th} convergent ζ_N trivially coincides with ζ itself.

Partial derivatives of a finite generalized continued fraction $\copyright\frac{\mathbf{a}}{\mathbf{b}} = b_1+\cfrac{a_1}{b_2+\frac{a_2}{\ddots}}$ are:

$$\begin{aligned} \frac{\partial}{\partial a_m}\copyright\frac{\mathbf{a}}{\mathbf{b}} &= \frac{(-1)^{m-1}}{a_m}\left(\mathop{\copyright}_{k=m+1}^{N}\frac{a_k}{b_k}\right)^{-1}\prod_{h=1}^{m}a_h\left(\mathop{\copyright}_{k=h}^{N}\frac{a_k}{b_k}\right)^{-2} \\ &= \frac{(-1)^{m-1}a_1a_2\cdot\ldots\cdot a_{m-1}}{\left(b_2+\cfrac{a_2}{b_3+\cfrac{a_3}{\cdots+\cfrac{a_{N-1}}{b_N+a_N}}}\right)^2\left(b_3+\cfrac{a_3}{b_4+\cfrac{a_4}{\cdots+\cfrac{a_{N-1}}{b_N+a_N}}}\right)^2\cdot\ldots\cdot\left(b_{m+1}+\cfrac{a_{m+1}}{b_{m+2}+\cfrac{a_{m+2}}{\cdots+\cfrac{a_{N-1}}{b_N+a_N}}}\right)}; \end{aligned}$$

$$\begin{aligned} \frac{\partial}{\partial b_m}\copyright\frac{\mathbf{a}}{\mathbf{b}} &= \frac{(-1)^{m-1}}{a_m}\prod_{h=1}^{m}a_h\left(\mathop{\copyright}_{k=h}^{N}\frac{a_k}{b_k}\right)^{-2} \\ &= \frac{(-1)^{m-1}a_1a_2\cdot\ldots\cdot a_{m-1}}{\left(b_2+\cfrac{a_2}{b_3+\cfrac{a_3}{\cdots+\cfrac{a_{N-1}}{b_N+a_N}}}\right)^2\left(b_3+\cfrac{a_3}{b_4+\cfrac{a_4}{\cdots+\cfrac{a_{N-1}}{b_N+a_N}}}\right)^2\cdot\ldots\cdot\left(b_m+\cfrac{a_{m+1}}{b_{m+1}+\cfrac{a_{m+1}}{\cdots+\cfrac{a_{N-1}}{b_N+a_N}}}\right)^2}. \end{aligned}$$

From the above formulæ the differential analogues to (1.110) follow, for $1\leq m\leq N$:

$$\frac{\partial}{\partial a_m}\copyright\frac{\mathbf{a}}{\mathbf{b}} = \frac{\partial}{\partial b_m}\copyright\frac{\mathbf{a}}{\mathbf{b}}\left(\mathop{\copyright}_{k=m+1}^{N}\frac{a_k}{b_k}\right)^{-1};$$

$$\begin{cases} \dfrac{\partial}{\partial a_{m+1}} \copyright\dfrac{\mathbf{a}}{\mathbf{b}} = -a_m \left(\underset{k=m}{\overset{N}{\text{\large Ⓒ}}} \dfrac{a_k}{b_k} \right)^{-1} \left(\underset{k=m+1}{\overset{N}{\text{\large Ⓒ}}} \dfrac{a_k}{b_k} \right)^{-1} \dfrac{\partial}{\partial a_m} \copyright\dfrac{\mathbf{a}}{\mathbf{b}} \\ \dfrac{\partial}{\partial b_{m+1}} \copyright\dfrac{\mathbf{a}}{\mathbf{b}} = -a_m \left(\underset{k=m+1}{\overset{N}{\text{\large Ⓒ}}} \dfrac{a_k}{b_k} \right)^{-2} \dfrac{\partial}{\partial b_m} \copyright\dfrac{\mathbf{a}}{\mathbf{b}} \end{cases} \tag{1.111}$$

$$\Downarrow$$

$$\begin{cases} \dfrac{\partial}{\partial a_{m+1}} \copyright\dfrac{\mathbf{a}}{\mathbf{b}} = -a_m \left(\underset{k=m}{\overset{N}{\text{\large Ⓒ}}} \dfrac{a_k}{b_k} \right)^{-1} \left(\underset{k=m+1}{\overset{N}{\text{\large Ⓒ}}} \dfrac{a_k}{b_k} \right)^{-2} \dfrac{\partial}{\partial b_m} \copyright\dfrac{\mathbf{a}}{\mathbf{b}} \\ \dfrac{\partial}{\partial b_{m+1}} \copyright\dfrac{\mathbf{a}}{\mathbf{b}} = -a_m \left(\underset{k=m+1}{\overset{N}{\text{\large Ⓒ}}} \dfrac{a_k}{b_k} \right)^{-1} \dfrac{\partial}{\partial a_m} \copyright\dfrac{\mathbf{a}}{\mathbf{b}} \end{cases}$$

When $\mathbf{a}=\mathbf{1}_N$ [$(a_n)_{\mathbb{N}}=\mathbf{1}_{\mathbb{N}}$] and $\mathbf{b}\in\mathbb{Z}^N$ [$(b_n)_{\mathbb{N}}\in\mathbb{Z}^{\mathbb{N}}$] then (1.108) is called a (*simple*) **continued fraction**.

$$\copyright\mathbf{b} \coloneqq \underset{n=0}{\overset{N}{\text{\large Ⓒ}}} b_n \coloneqq [b_0;b_1;\cdots;b_N] \overset{\text{def}}{=} b_0 + \cfrac{1}{b_1 + \cdots \cfrac{1}{b_{N-1} + \cfrac{1}{b_N}}} \tag{1.112}$$

$$\left[\underset{n=0}{\overset{\infty}{\text{\large Ⓒ}}} b_n \coloneqq [b_0;b_1;b_2;\cdots] \overset{\text{def}}{=} b_0 + \cfrac{1}{b_1 + \cfrac{1}{b_2 + \cfrac{1}{b_3 + \ddots}}} \right]$$

and its convergents' sequence $(\zeta_n)_{\mathbb{N}}$ is computed via the II-order difference equation

$$\zeta_n \equiv \frac{A_n}{B_n} = \frac{a_n A_{n-1} + A_{n-2}}{a_n B_{n-1} + B_{n-2}}; \qquad \begin{cases} A_{-1}=1, & B_{-1}=0 \\ A_0=a_0, & B_0=1 \end{cases}. \tag{1.113}$$

Simple *complex* continued fractions are defined with $(b_n)_{\mathbb{N}}$ sequence being Gaussian integers, i.e. $b_n\in\mathbb{Z}(\imath)$, $\forall n\in\mathbb{N}_0$ — cfr. note.[24] Simple [complex] continued fractions are proven to always converge to a unique real [complex] number $\xi\in\mathbb{R}$ [$\zeta\in\mathbb{C}$], but in general the same number has *multiple* continued fraction expansions.

- The following two finite continued fractions of lengths $N+1$ and $N+2$ are *equivalent*, i.e. lead to the same element of R: $[b_0;b_1;\cdots;b_{N-1};b_N+1]=[b_0;b_1;\cdots;b_{N-1};b_N;1]$.
- The convergent $\xi\in\mathbb{R}$ is rational if and only if its simple continued fractions are finite, i.e. $\xi\in\mathbb{Q} \Leftrightarrow \exists N\in\mathbb{N}$: $b_n=0\ \forall n>N$.
- Simple continued fraction expansions of rational numbers are unique modulo the "trailing '1'" equivalence above.
- Simple continued fraction expansions of irrational numbers are *unique*, i.e. they provide a bijection $\mathbb{Z}^{\mathbb{N}} \leftrightarrow \mathbb{R}$ which is different from the best-known "digit" expansions of real numbers.
- A *infinite* simple continued fraction is said to be periodic (of period $N\in\mathbb{N}$ —*trivially* if $N=1$— and tail of length $L\in\mathbb{N}_0$) if $b_{L+N+n}=b_{L+n}$, $\forall n\in\mathbb{N}_0$.
- Periodic simple continued fractions are bijections to the quadratic irrationals, i.e. to irrational roots of quadratic polynomials in $\mathbb{Z}^2[x]$.
- The n^{th} convergent $\xi_n=A_n/B_n\in\mathbb{Q}$ of a simple continued fraction is the *best* rational approximation of the real number $\xi=\lim_n\xi_n$, $\forall n\in\mathbb{N}_0$; this means that ξ_n is the best-approximating rational whose denominator is less than B_n.

Some $\mathbb{N}\to\mathbb{R}$ functions (i.e. some real numbers' sequences) have extremely regular continued fractions (as functions of $n\in\mathbb{N}$); a few, [104], are shown below.

$$\frac{1}{\sqrt{e}-1}=\underset{n=0}{\overset{\infty}{\text{C}}}\frac{2n}{2n+1}\equiv 1+\cfrac{2}{3+\cfrac{4}{5+\cfrac{6}{7+\ddots}}}.$$

$$
\begin{aligned}
e \stackrel{\text{def}}{=} \lim_n \left(1+\frac{1}{n}\right)^n &= 2+1\Big/ \underset{k=0}{\overset{\infty}{\copyright}} \left[\left(\tfrac{2}{3}k-1\right)\delta_{k\equiv 0 \bmod 3}+1\right] \\
&\equiv [2;1;2;1;1;4;1;1;6;1;1;8;1;1;10;1;1;12;\cdots]; \\
e^{\frac{1}{n}} &= 1+\underset{k=0}{\overset{\infty}{\copyright}} \left[\left(\tfrac{2}{3}kn+n-2\right)\delta_{k\equiv 0 \bmod 3}+1\right] \\
&\equiv \left[1;n-1;1;1;3n-1;1;1;5n-1;1;1;7n-1;1;1;\cdots\right].
\end{aligned}
$$

$$
\begin{aligned}
\tan 1 &= 1+\underset{k=0}{\overset{\infty}{\copyright}} (2k\delta_{k\equiv 0 \bmod 2}+1) \equiv [1;1;1;3;1;5;1;7;1;9;1;11;1;\cdots]; \\
\tan\frac{1}{n} &= \underset{k=1}{\overset{\infty}{\copyright}} \left[(kn-3)\delta_{k\equiv 1 \bmod 2}+\delta_{k,1}+1\right] \\
&\equiv \left[0;n-1;1;3n-2;1;5n-2;1;7n-2;1;\cdots\right]; \\
\tanh\frac{1}{n} &= \underset{k=1}{\overset{\infty}{\copyright}} (2k-1)n \equiv [0;n;3n;5n;7n;\cdots].
\end{aligned}
$$

Another famous continued fraction expansion is that of the *golden ratio*, i.e. the irrational number $g := (1+\sqrt{5})\big/2$, cfr. §1.5.3 and (1.118). Finally, the simple continued fraction expansions of irrational square roots (which are all quadratic irrationals — some of shown in the examples below) are periodic with a length-1 tail and a period (of least length 2) composed of a symmetric integer-numbers' string followed by the integer number which is twice the leading coefficient of the expansion. Conventionally, the tail and period of the expansions in the examples below are emphasized with a convention on digits' typing — much like it is done for periodic digit expansions: $1/3 = 0.\overline{3}_{10} \stackrel{\text{def}}{=} 0.333\ldots$.

- $\sqrt{7}=[2;\overline{1;1;1;4}]\overset{\text{def}}{=}[2;1;1;1;4;1;1;1;4;1;1;1;4;\cdots]$; $\qquad\sqrt{2}=[1;\overline{2}]$;
- $\sqrt{14}=[3;\overline{1;2;1;6}]$; $\qquad\sqrt{34}=[5;\overline{1;4;1;10}]$;
- $\sqrt{178}=\left[13;\overline{2;1;12;1;2;26}\right]$;
- $\sqrt{179}=\left[13;\overline{2;1;1;1;3;5;13;5;3;1;1;1;2;26}\right]$.

Derivating a finite simple continued fraction is a simpler task derived from (1.111),

$$\begin{aligned}\frac{\partial ©\mathbf{b}}{\partial b_m} &= (-1)^m\prod_{h=1}^{m}\left(\underset{k=h}{\overset{N}{©}} b_k\right)^{-2} = -\left(\underset{k=m}{\overset{N}{©}} b_k\right)^{-2}\frac{\partial ©\mathbf{b}}{\partial b_{m-1}}\\ &= \frac{(-1)^m}{\left(b_1+\cfrac{1}{b_2+\cfrac{1}{\cdots+\cfrac{1}{b_N}}}\right)^2\left(b_2+\cfrac{1}{b_3+\cfrac{1}{\cdots+\cfrac{1}{b_N}}}\right)^2\cdot\ldots\cdot\left(b_m+\cfrac{1}{b_{m+1}+\cfrac{1}{\cdots+\cfrac{1}{b_N}}}\right)^2},\end{aligned}$$

whereas for the infinite continued fraction case

$$\begin{aligned}\frac{\partial ©(b_n)_{n\in\mathbb{N}_0}}{\partial b_m} &= (-1)^m\prod_{h=1}^{m}\left(\underset{k=h}{\overset{\infty}{©}} b_k\right)^{-2} = -\left(\underset{k=m}{\overset{N}{©}} b_k\right)^{-2}\frac{\partial ©(b_n)_{n\in\mathbb{N}_0}}{\partial b_{m-1}}\\ &= \frac{(-1)^m}{\left(\cfrac{1}{b_1+\cfrac{1}{b_2+\cfrac{1}{b_3+\ddots}}}\right)^2\left(\cfrac{1}{b_2+\cfrac{1}{b_3+\cfrac{1}{b_4+\ddots}}}\right)^2\cdot\ldots\cdot\left(\cfrac{1}{b_m+\cfrac{1}{b_{m+1}+\cfrac{1}{b_{m+2}+\ddots}}}\right)^2}.\end{aligned}$$

1.5.2. Fractal fractions

Let $R(+,\cdot)$ be an arithmetic abelian ring, let $N\in\mathbb{N}$ and either $\mathbf{a}=(a_1,a_2,\ldots,a_N)\in R^N$ or $(a_n)_{n\in\mathrm{N}}\in R^{\mathrm{N}}$; the asso.ciated *fractal fraction* is defined as:

$$\left.\begin{array}{c}\circledR\mathbf{a}\\ \underset{n\in\mathbb{N}}{\circledR}\overline{a_n}\end{array}\right\}\overset{\text{def}}{=}-a_1+\cfrac{-a_2+\cfrac{-a_3+\cfrac{-a_4+\frac{\iddots}{a_1}}{-a_1+\frac{\iddots}{a_1}}}{-a_1+\cfrac{-a_2+\frac{\iddots}{a_1}}{-a_1+\frac{\iddots}{a_1}}}}{-a_1+\cfrac{-a_2+\cfrac{-a_3+\frac{\ddots}{a_1}}{-a_1+\frac{\ddots}{a_1}}}{-a_1+\cfrac{-a_2+\frac{\ddots}{a_1}}{-a_1+\frac{\ddots}{a_1}}}}, \tag{1.114}$$

where, in the first case $\circledR\mathbf{a}$, one conventionally sets $a_n = 0$ for every $n > N$. If the fractal fraction is stopped at the N^{th} nested fraction (both at the numerators and denominators), it is called a *pre*-fractal fraction, indicated as:

$$\underset{n=1}{\overset{N}{\circledR}}\overline{a_n}\overset{\text{def}}{=}-a_1+\cfrac{-a_2+\cfrac{-a_3+\cdots\cfrac{\iddots+\frac{a_N}{a_1}}{\ddots+\frac{a_{N-1}}{a_1}}}{-a_1+\cdots\cfrac{\iddots+\frac{a_{N-1}}{a_1}}{\ddots+\frac{a_{N-2}}{a_1}}}}{-a_1+\cfrac{-a_3+\cdots\cfrac{\iddots+\frac{a_{N-1}}{a_1}}{\ddots+\frac{a_{N-2}}{a_1}}}{-a_1+\cdots\cfrac{\iddots+\frac{a_{N-3}}{a_1}}{\ddots+\frac{a_{N-4}}{a_1}}}};$$

the n^{th} prefractal fraction of a linear array of symbols $\mathbf{a}$ can be computed in *MATHEMATICA®* via custom function `FractalFraction`, defined by recursion as:

```
FractalFraction[a_List,n_Integer]:= FractalFraction[a,a,n];
FractalFraction[a_List,b_List,n_Integer]:= -First[a]+
    If[(Length[a] ≤ 1)||(n ≤ 1),0, FractalFraction[Drop[a,1],b,n-1] / FractalFraction[b,b,n-1]];
```
(1.115)

Let $P(x) = x^N + a_{N-1}x^{N-1} + a_{N-2}x^{N-2} + \ldots + a_1 x + a_0$ be a monic heterogeneous ($a_0 \neq 0$) polynomial and $Q(x) = b_N x^N + b_{N-1}x^{N-1} + \ldots + b_2 x^2 + b_1 x + 1$ another polynomial, both N^{th}-degree, i.e. $P, Q \in R^N[x] \setminus R^{N-1}[x]$ and, in general, with non-zero coefficients of any degree, let $\mathbf{a} = (a_{N-1}, a_{N-2}, \ldots, a_0)$, $\mathbf{b} = (b_1, b_2, \ldots, b_N) \in R^N$. If $R = \mathbb{R}$, then the following fractal fractions (or those with opposite-signed vectors $-\mathbf{a}$ and $-\mathbf{b}$) are *related* to either a maximum-modulus root of P and a minimum-modulus root of Q, respectively:

$$x_{\max}=\circledR\overline{\mathbf{a}}\,, \qquad\qquad x_{\min}=(\circledR\overline{\mathbf{b}}\,)^{-1}\,. \tag{1.116}$$

Let $P(x)=x^2-2x-1$, whose roots —all simple and real— are $1\mp\sqrt{2}$. Its maximum root happens to be exactly $x_{\max}=\circledR\overline{(-2,-1)}$ by (1.116):

$$x_{\max} = 2+\cfrac{1+\cfrac{0+\frac{0+\frac{0\iddots}{-2\ddots}}{2+\frac{-1\iddots}{-2\ddots}}}{2+\frac{1+\frac{0\iddots}{-2\ddots}}{2+\frac{-1\iddots}{-2\ddots}}}}{2+\cfrac{1+\frac{0+\frac{0\iddots}{-2\ddots}}{2+\frac{-1\iddots}{-2\ddots}}}{2+\frac{1+\frac{0\iddots}{-2\ddots}}{2+\frac{-1\iddots}{-2\ddots}}}} = 2+\cfrac{1}{2+\cfrac{1}{2+\cfrac{1}{2+\ddots}}} \equiv \overset{\infty}{\copyright}2 = 1+\sqrt{2}\,;$$

it is easily seen that $x_{\max}$ is the trivially-periodic simple continuous fraction expansion of a quadratic irrational — cfr. §1.5.1.

Now let $Q(x)=(x+1)^3-2\equiv x^3+3x^2+3x-1$, whose (simple) roots are $\left\{\sqrt[3]{2}-1,-1-2^{-\frac{2}{3}}\left(1\mp\imath\sqrt{3}\right)\right\}$. Its *minimum* root —which is also real— happens to be exactly $x_{\min}=\sqrt[3]{2}-1$, computable by (1.116) as:

$$x_{\min}=\frac{1}{\circledR\overline{(-3,-3,-1)}}\equiv\cfrac{1}{3+\cfrac{3+\frac{1+\frac{0+\frac{0\iddots}{-3\ddots}}{3+\frac{-3\iddots}{-3\ddots}}}{3+\frac{3+\frac{-1\iddots}{-3\ddots}}{3+\frac{-3\iddots}{-3\ddots}}}}{3+\cfrac{3+\frac{1+\frac{0\iddots}{-3\ddots}}{3+\frac{-3\iddots}{-3\ddots}}}{3+\frac{3+\frac{-1\iddots}{-3\ddots}}{3+\frac{-3\iddots}{-3\ddots}}}}}=\cfrac{1}{3+\cfrac{3+\frac{1}{3+\frac{3+\iddots}{3+\ddots}}}{3+\frac{3+\frac{1}{3+\ddots}}{3+\frac{3+\iddots}{3+\ddots}}}}=\sqrt[3]{2}-1\,.$$

It is important to note, however, that the above statements are *neither* necessary *nor* sufficient to find the maximum- and minimum-modulus roots of a given polynomial in R, especially if R is not algebraically closed and either one is *not* in R. For example $\text{Ⓡ}\overline{-\mathbf{a}} \neq \text{Ⓡ}\mathbf{a}$, whereas the sign of $\mathbf{a}$ must be an invariant for the roots of P.

1.5.3. Nested radicals

The n-fold [*infinite*] ***nested*** p^{th} ***radical*** ($p\in\mathbb{N}\backslash\{1\}$) associated to vectors $\mathbf{a},\mathbf{b}\in\mathbb{R}^n$, $\mathbf{p}\in\mathbb{N}^n$ [to sequences $(a_n)_{\mathbb{N}},(b_n)_{\mathbb{N}}\in\mathbb{R}^{\mathbb{N}}$, $(p_n)_{\mathbb{N}}\in\mathbb{N}^{\mathbb{N}}$], [30] is the complex number [the limit — whenever it exists]:

$$\sqrt[\mathbf{p}]{\mathbf{a}}^{\mathbf{b}} \stackrel{\text{def}}{=} \overbrace{\sqrt[p_1]{a_1+b_1\sqrt[p_2]{a_2+b_2\sqrt[p_3]{\cdots+b_{n-1}\sqrt[p_n]{a_n+b_n}}}}}^{n\text{ times}}\;; \tag{1.117}$$

$$\left[\quad \sqrt[(p_n)_{\mathbb{N}}]{(a_n)_{\mathbb{N}}}^{(b_n)_{\mathbb{N}}} \stackrel{\text{def}}{=} \sqrt[p_1]{a_1+b_1\sqrt[p_2]{a_2+b_2\sqrt[p_3]{a_3+b_3\sqrt[p_4]{\ddots}}}} \quad\right].$$

If either $\mathbf{a}=\mathbf{1}_n$ [$\mathbf{a}=\mathbf{1}_{\mathbb{N}}$], and $\mathbf{p}=p\cdot\mathbf{1}_n$ [$\mathbf{p}=p\mathbf{1}_{\mathbb{N}}$], the corresponding vectors [sequences] are just not indicated and one talks about *telescopic* n-fold [infinite] nested p^{th} radicals (with the sole «radical» indication rather than «p^{th} radical» whenever $p=2$).

Here is some example ($g\in\mathbb{R}\backslash\mathbb{Q}$ is the golden ratio — or *AUREA PROPORTIO*, or **golden mean**):

$$\begin{aligned} g \stackrel{\text{def}}{=} \frac{1+\sqrt{5}}{2} &= \sqrt{\mathbf{1}_{\aleph_0}} \equiv \sqrt{1+\sqrt{1+\sqrt{1+\sqrt{\ddots}}}} \simeq 1.6180339 \\ &= \underset{n\in\mathbb{N}}{\text{Ⓒ}}\, 1 \equiv [1;1;\cdots;1] \equiv 1+\cfrac{1}{1+\cfrac{1}{1+\cfrac{1}{1+\ddots}}}\;; \end{aligned} \tag{1.118}$$

$$\sqrt{\pm a\cdot\mathbf{1}_{\mathbb{N}}}^{\,b\cdot\mathbf{1}_{\mathbb{N}}} \equiv \sqrt{a\pm b\sqrt{a\pm b\sqrt{a\pm b\sqrt{\ddots}}}} = \frac{\pm b+\sqrt{b^2-4a}}{2}.$$

When nested p^{th} radicals are generalized to complex numbers, then they are always "multi-valued" and an n-fold one (like that in the example of §3) takes p^n different values. Nested radicals are important to both Number Theory and Algebra, since they used to write down the solutions to polynomial equations (cfr. §1.1.3 and §1.5.4); they are also related to continuous fraction expansions, like the AUREA PROPORTIO (1.118) is.

From the algebraic point of view, every finite nested radical maps to a radical field extension of a given numeric field, cfr. §1.1.3. For example, starting from $\mathbb{Q}$, $\mathbf{a}\equiv(a_1,a_2,\ldots,a_r)\in\mathbb{Q}^r$ and $\mathbf{p}\equiv(p_1,p_2,\ldots,p_r)\in\mathbb{N}^r$ the radical extension field

$$\mathbb{Q}(\sqrt[\mathbf{p}]{\mathbf{a}}) \cong \mathbb{Q}(\sqrt[p_r]{a_r}, \sqrt[p_{r-1}]{a_{r-1}}, \ldots, \sqrt[p_1]{a_1}),$$

§1.1.3, is generated.

1.5.4. Galois Theory from the radicals' point of view

The Galois' Theory, [79], [71], [93], was born from and is anyway directly applied to the problem of finding the roots of a polynomial (with coefficients in an arbitrary field) via radical expressions, i.e. as combinations of elements of the field by using the "five arithmetic operations" (the fifth being the extraction of arithmetic roots). But more than that, since Abel-Ruffini's Theorem (cfr. §1.1.10, previously-cited references, as well as [86] and [69]), it is involved in finding what polynomials allow closed-form computations of their roots and what not: the very essence of CALCULUS! This is essentially done via an elegant, symmetry-oriented, Group Theory.

The action of a Galois group $\mathrm{Gal}(F/K)$ over its polynomial ring $K[x]$ is to permute its elements' roots in F. Let $\mathbf{x}\equiv(x_1,x_2,\ldots,x_d)\in F^d$, $f\in K\{\mathbf{x}\}$ and $\pi_{\mathbf{i}}\in\mathbb{S}_d$ and $\mathbf{i}\equiv(i_1,i_2,\ldots,i_d)\in\{1,2,\ldots,d\}^d$ then

$$\pi_{\mathbf{i}} f := f(x_{i_1}, x_{i_2}, \ldots, x_{i_d}); \tag{1.119}$$

under this definition it is easy to prove that $\mathbb{S}_d \lessapprox \mathrm{Aut}\,K\{\mathbf{x}\}$ and, in particular $\mathbb{S}_d$ is the Galois group fixing the *symmetric* rational functions with coefficients in K — the invariant subfield of $K\{\mathbf{x}\}$ under $\mathbb{S}_d$, i.e. $\{f\in K\{\mathbf{x}\} \mid \mathbb{S}_d f = f\}$:

$$\pi_{\mathbf{i}} f \equiv f(x_{i_1}, x_{i_2}, \ldots, x_{i_d}) = f(x_1, x_2, \ldots, x_d), \qquad \forall \mathbf{x}\in K^d.$$

The sub-field defined above is easily proven to be the extension field generated by the n^{th}-degree **symmetric** polynomials of d variables, for every $1\leq n\leq d$, where

$$K^n[\mathbf{x}] \ni \mathrm{sy}_n(\mathbf{x}) \overset{\text{def}}{=} \sum_{\substack{i_1, i_2, \ldots, i_n = 1,\\ i_1\neq i_2\neq \ldots \neq i_n}}^{d} \prod_{k=1}^{n} x_{i_k}. \tag{1.120}$$

$\mathrm{sy}_n \in \mathrm{sym}(K^d)^{\otimes n}$ is a symmetric n-tensor of both K^d and F^d, whereas a basis of $\mathrm{sym}(K^d)^{\otimes n}$ is given by each term of (1.120) and $\mathrm{span}\{\mathrm{sy}_n\}_{0\leq n\leq d} = \mathrm{sym}\bigotimes^{\infty} K^d$ with

$$\begin{cases} \mathrm{sy}_0(x_1, x_2, \ldots, x_d) = 1 \\ \mathrm{sy}_1(x_1, x_2, \ldots, x_d) = x_1 + x_2 + \ldots + x_d \\ \mathrm{sy}_2(x_1, x_2, \ldots, x_d) = x_1x_2 + x_1x_3 + \ldots + x_1x_d + x_2x_3 + x_2x_4 + \ldots + x_{d-1}x_d \\ \quad\vdots \\ \mathrm{sy}_{d-1}(x_1, x_2, \ldots, x_d) = x_2x_3x_4\cdot\ldots\cdot x_d + x_1x_3x_4\cdot\ldots\cdot x_d + \ldots + x_1x_2x_3\cdot\ldots\cdot x_{d-1} \\ \mathrm{sy}_d(x_1, x_2, \ldots, x_d) = x_1x_2\cdot\ldots\cdot x_d \end{cases}$$

Thus, for the above Galois group:

$$\mathbb{S}_d \cong \mathrm{Gal}\frac{K\{\mathbf{x}\}}{K\{\mathrm{sy}_1(\mathbf{x}), \mathrm{sy}_2(\mathbf{x}), \ldots, \mathrm{sy}_d(\mathbf{x})\}}. \tag{1.121}$$

This means that every function which is fully symmetric in the d-tuplet $\mathbf{x}$ is a rational function whose coefficients are symmetric polynomials.

The factorized and the coefficient-based expression of any monic, n^{th}-degree polynomial $P\in K^n[x]$ whose n roots (counted with multiplicities) are $x_1, x_2, \ldots, x_n \in K^\natural$:

$$P(x) = \prod_{j=1}^{n}(x-x_j) = \sum_{k=0}^{n} a_k x^k \qquad (a_n{=}1),$$

are inter-related as:

$$a_{n-k} = (-1)^k \,\mathrm{sy}_k(x_1,x_2,\ldots,x_n), \qquad 1\leq k\leq n, \tag{1.122}$$

whereas the converse formula is not generally available in closed form for $n\geq 5$ (by Abel-Ruffini's Theorem, §1.1.10).

> Roots of the polynomial $(z^p-z-\alpha)\in\mathbb{C}^p[z]$ (for $p\in\mathbb{N}\setminus\{1\}$ and $\alpha\in\mathbb{C}^*$) can be found by solving the transcendental (fixed-point) equation
>
> $$z = \sqrt[p]{z+\alpha}\,,$$
>
> which trivially admits a closed-form and multi-valued expansion as the infinite nested p^{th} radical (cfr. §1.5.3)
>
> $$\sqrt[p]{\alpha\mathbf{1}_{\mathbb{N}}} \equiv \sqrt[p]{\alpha+\sqrt[p]{\alpha+\sqrt[p]{\alpha+\ddots}}}.$$

In particular, if $K<H<F$ —which means that $\mathrm{Gal}(F/K) \gtrsim \mathrm{Aut}(H/K) \gtrsim \{\mathbb{1}\}$ by (1.45)— and $\mathrm{Aut}(H/K)=\{h^n\}_{0\leq n<N}$ (i.e. the subgroup corresponding to the intermediate field extension H/K is cyclic of order N) then H is a *radical* extension field and its elements are expressed as combinations involving the N^{th} arithmetic root of some element(s) in K. If all such groups are cyclic, then P can be solved by radicals in F. That is the statement of the Fundamental Theorem of (finite) Galois Theory, (1.44).

When all such subgroups are cyclic, by the previous Theorem, the 'maximal' splitting field F is exactly associated to the polynomial's Galois' group $\mathrm{Gal}_K P$.

> Consider the polynomial $P(z)=(z^3+z-1)\in\mathbb{Q}^3[z]$, whose roots are:
>
> $$z_1 = \frac{\sqrt[3]{2(9+\sqrt{93})}-2\sqrt[3]{\frac{3}{9+\sqrt{93}}}}{6^{\frac{2}{3}}},$$

$$z_{2/3}=\frac{1\mp\imath\sqrt{3}}{2^{\frac{2}{3}}\sqrt[3]{3(9+\sqrt{93})}}-(1\pm\imath\sqrt{3})\frac{\sqrt[3]{\frac{1}{2}(9+\sqrt{93})}}{2\cdot 3^{\frac{2}{3}}}.$$

Complex-conjugated roots z_2 and z_3 can be exchanged with each other, so permutation $\lceil 1,3,2\rfloor$ is allowed for P, which corresponds to simple 2-radical extension field $\mathbb{Q}(\sqrt{-3})$. The symmetry for P's 0^{th}-degree coefficient is $\mathrm{sy}_3(z_1,z_2,z_3)\equiv -z_1z_2z_3=-1$. Furthermore, $\mathrm{sy}_1(z_1,z_2,z_3)\equiv z_1+z_2+z_3=0$ and $\mathrm{sy}_2(z_1,z_2,z_3)\equiv z_2z_3+z_1z_3+z_1z_2=1$, corresponding to the simple, 3-radical extension fields $\mathbb{Q}(\sqrt[3]{2})$ and $\mathbb{Q}(\sqrt[3]{3})$. These three field extensions all have degree 2.

From these considerations it is clear that any algebraic equation involving P's roots is invariant by freely permuting z_1, z_2 and z_3 with each other, which corresponds to $\{\mathbb{1},\lceil 1,3,2\rfloor\}\oplus\{\mathbb{1},\lceil 2,1,3\rfloor,\lceil 3,2,1\rfloor\}$, i.e. to the symmetric group $\mathbb{S}_3\cong\mathbb{Z}_6\cong\mathbb{Z}_3\rtimes\mathbb{Z}_2\cong\mathbb{D}_3$ — cfr. §1.1.4. The other symmetry equations satisfied by the roots of P are:

$$\begin{cases} z_1^{2n-1}+(z_2+z_3)^{2n-1}=0 \\ z_1^{2n}-(z_2+z_3)^{2n}=0 \end{cases}\qquad \forall n\in\mathbb{N},$$

which all depend on the ones for $n=1$, that is $z_1+z_2+z_3= =z_1^2-(z_2+z_3)^2=0$; the former being a combination of the ones for 1^{st}- and 2^{nd}-degree coefficients, the latter corresponding to the 3^{rd}-degree, simple, 3-radical extension field $\mathbb{Q}(\sqrt[3]{9+\sqrt{93}})$.

The associated Galois' group is then isomorphic to $\mathrm{Gal}(\mathbb{Z}_3\oplus\mathbb{Z}_2)=\mathbb{S}_3$.

2 Complex Geometry and Topology

> When I was a child, I was seized by the idea that everyone has an invisible tail at his back. This tail [...] trails behind as we walk, from birth to death. The tail extends far away, beyond our world, and originates in the Land of the Dead. When we walk, the tail is reeled out from its source. When we have pulled out the entire tail, we die.
>
> The strategy for a long life, therefore, is to try to avoid pulling out your tail. In order to do this, walk null-homotopic paths as much as possible. When you return, follow the same route. Never go through a tunnel. (My ideal world was a universal covering space of the real world (if there is such a thing).)
>
> Michio Kuga (1928–1990),
> *Galois' Dream*, 12th week, [69].

2.1 Complexified vector spaces

Some real vector spaces —even- and (countably) infinite-dimensional ones in particular— can be re-formulated using the field of complex numbers' as base field, cfr. [9]–§A.2. Just as the complex numbers are somewhat "universal" due to algebraic closure (cfr. §1.1.3), a complexified vector space benefits from more structures than its real counterpart, both as Linear Algebra (including the metric theory of inner-space products), Topology and Geometry are concerned.

From the applications' point of view, complexified spaces play a fundamental role in many classical, relativistic and above all quantum field theories [36]–[37] (as well as [33]–[34] concerning pure Quantum Physics).

2.1.1. Algebraic theory

Recall that any real vector space $\mathbf{V}_{\mathbb{R}}$ admitting an ***imaginary*** automorphism $\boldsymbol{\imath}\in\mathrm{Aut}_{\mathbb{R}}\mathbf{V}$ (or $\boldsymbol{\imath}\in\mathfrak{GL}_{\mathbb{R}}(\mathbf{V})$ if $\mathbf{V}$ is a topological linear space) such that $\boldsymbol{\imath}^2=-\mathbb{1}$ is called an ***almost-complex*** space; it is always orientable and can be complexified as $\mathbf{V}^{\mathbb{C}}\overset{\mathrm{def}}{=}\mathbf{V}\otimes_{\mathbb{R}}\mathbb{C}$. Its $\mathbb{C}$-modulo structure is $(x+\imath y)\mathbf{v}:=x\mathbf{v}+y\boldsymbol{\imath}\mathbf{v},\ \forall x,y\in\mathbb{R}, \forall\mathbf{v}\in\mathbf{V}$. On a general $\mathbf{V}^{\mathbb{C}}$, $\mathrm{spec}\,\boldsymbol{\imath}=\{\imath,-\imath\}$, whereas its eigenspaces are $\mathbf{V}^{1,0}:=\mathrm{Ker}(\boldsymbol{\imath}-\imath\mathbb{1})$ and $\mathbf{V}^{0,1}:=\mathrm{Ker}(\boldsymbol{\imath}+\imath\mathbb{1})$ respectively, with[54] $\mathbf{V}^{\mathbb{C}}=\mathbf{V}^{1,0}\oplus\mathbf{V}^{0,1}$. Besides, $\boldsymbol{\imath}$ is trivially a $(1,1)$-tensor of $\mathbf{V}$, i.e. $\boldsymbol{\imath}\in\mathbf{V}^{(1,1)}\equiv\mathbf{V}^*\otimes\mathbf{V}$, cfr. [9]–§A.4.

As far as dual spaces[55] are concerned, $\mathbf{V}^{\mathbb{C}*}\cong\mathbf{V}^{*\mathbb{C}}$ with $\langle\boldsymbol{\imath}\mathbf{v}|\upsilon\rangle=\langle\mathbf{v}|\boldsymbol{\imath}^*(\upsilon)\rangle, \forall\upsilon\in\mathbf{V}^*, \forall\mathbf{v}\in\mathbf{V}$ and $\boldsymbol{\imath}^*\in\mathrm{Aut}_{\mathbb{R}}\mathbf{V}^*$ such that $\boldsymbol{\imath}^*\boldsymbol{\imath}^*\equiv-1$; this can be naturally extended to $\mathbf{V}^{\mathbb{C}}$'s exterior algebra $\Lambda(\mathbf{V}^{\mathbb{C}})=\bigoplus_{p\in\mathbb{N}_0}\Lambda^p(\mathbf{V}^{\mathbb{C}})\cong\Lambda(\mathbf{V})^{\mathbb{C}}$. Complexified spaces anyway, benefit from another type of Exterior Algebra [9]–§A.5:

Proposition

Let $\mathbf{V}^{\mathbb{C}}$ be a complexified vector space and $p,q\in\mathbb{N}_0$; then the multi-vectors of biorder (p,q), also called *(p,q)-vectors*, are the elements of vector space

$$\begin{cases}\Lambda_{p,q}(\mathbf{V})\overset{\mathrm{def}}{=}\Lambda^p(\mathbf{V}^{1,0})\otimes\Lambda^q(\mathbf{V}^{0,1})\\ \Lambda_{\mathbb{C}}(\mathbf{V})\overset{\mathrm{def}}{=}\Lambda(\mathbf{V}^{\mathbb{C}})\end{cases}\tag{1.123}$$

---

[54] **Warning:** On $\mathbf{V}^{\mathbb{C}}$ both $\boldsymbol{\imath}$ and $\imath\mathbb{1}$ are different imaginary automorphisms, with symmetries $\boldsymbol{\imath}(\mathbf{V}^{1,0})=\imath\mathbf{V}^{1,0}$ and $\boldsymbol{\imath}(\mathbf{V}^{0,1})=-\imath\mathbf{V}^{0,1}$. Also, $\mathbf{V}$, $\mathbf{V}^{1,0}$ and $\mathbf{V}^{0,1}$ are mutually isomorphic with each oher via $\mathbf{v}\mapsto\mathbf{v}\otimes_{\mathbb{R}}1$ for $\mathbf{V}\cong\mathbf{V}^{1,0}$, via $\mathbf{v}\mapsto\mathbf{v}\otimes_{\mathbb{R}}\imath$ for $\mathbf{V}\cong\mathbf{V}^{0,1}$ and via $\mathbf{v}\mapsto\boldsymbol{\imath}(\mathbf{v})\equiv\imath\mathbf{v}$ for $\mathbf{V}^{1,0}\cong\mathbf{V}^{0,1}$.

[55] Similarly, $(\mathbf{V}^{1,0})^*=(\mathbf{V}^*)^{1,0}\cong\mathrm{Hom}_{\mathbb{C}}(\mathbf{V}_{\mathbb{R}},\mathbb{C})$ and $(\mathbf{V}^{0,1})^*=(\mathbf{V}^*)^{0,1}$.

The elements of the dual exterior algebra $\Lambda_{p,q}(\mathbf{V}^*)$ are called complex multlinear forms of biorder (p,q), also called *(p,q)-forms* and:

$$\begin{cases} \Lambda^{p,q}(\mathbf{V}) \stackrel{\text{def}}{=} \Lambda_{p,q}(\mathbf{V}^*) \cong \Lambda_{p,q}(\mathbf{V})^* \\ \Lambda^*_{\mathbb{C}}(\mathbf{V}) \stackrel{\text{def}}{=} \Lambda(\mathbf{V}^{\mathbb{C}*}) \cong \Lambda(\mathbf{V}^{\mathbb{C}})^* \end{cases} \tag{1.124}$$

Both exterior algebras have the following properties:

- Complex conjugation swaps the orders of multilinear forms;[56] $\overline{\Lambda^{p,q}(\mathbf{V})} = \Lambda^{q,p}(\mathbf{V})$, thus $\Lambda^{p,q}(\mathbf{V}) \cong \Lambda^{q,p}(\mathbf{V})$;
- $\dim_{\mathbb{C}} \Lambda^k(\mathbf{V}^{\mathbb{C}}) = \binom{2n}{k}$ and $\dim_{\mathbb{C}} \Lambda^{p,q}(\mathbf{V}^{\mathbb{C}}) = \binom{n}{p}\binom{n}{q}$;
- $\Lambda^0(\mathbf{V}^{\mathbb{C}}) \equiv \Lambda_{0,0}(\mathbf{V}) \cong \mathbf{V}^{\mathbb{C}}$ and $\Lambda^0(\mathbf{V}^{\mathbb{C}*}) \equiv \mathbf{V}^{\mathbb{C}*} \cong \Lambda^{0,0}(\mathbf{V})$;
- $\Lambda^k(\mathbf{V}^{\mathbb{C}}) = \bigoplus_{\substack{p,q\in\mathbb{N}_0,\\ p+q=k}} \Lambda^{p,q}(\mathbf{V})$ and $\Lambda^*_{\mathbb{C}}(\mathbf{V}) = \bigoplus_{p,q\in\mathbb{N}_0} \Lambda^{p,q}(\mathbf{V})$;
- The wedge product is biadditive respect to the multivectors' biorders, i.e. $\underline{\mathbf{u}}\wedge\underline{\mathbf{v}}\in\Lambda_{p+r,q+s}(\mathbf{V})$, $\forall\underline{\mathbf{u}}\in\Lambda_{p,q}(\mathbf{V})$, $\forall\underline{\mathbf{v}}\in\Lambda_{r,s}(\mathbf{V})$.

The value of any (p,q)-form $\phi\in\Lambda^{p,q}(\mathbf{V})$ computed on $p+q$ vectors $\mathbf{u}_1,\mathbf{u}_2,\dots,\mathbf{u}_p,\mathbf{v}_1,\mathbf{v}_2,\dots,\mathbf{v}_q\in\mathbf{V}^{\mathbb{C}}$ is indicated as $\langle\mathbf{u}_1,\mathbf{u}_2,\dots,\mathbf{u}_p;\mathbf{v}_1,\mathbf{v}_2,\dots,\mathbf{v}_q|\,\phi\rangle$.

The natural projector $\mathbb{I}\in\mathrm{Aut}_{\mathbb{C}}\Lambda^*(\mathbf{V})$, which is *not* an imaginary automorphism, is

$$\mathbb{I} \stackrel{\text{def}}{=} \sum_{p,q\in\mathbb{N}_0} \imath^{p-q}\, \mathrm{proj}_{\Lambda^{p,q}(\mathbf{V})} \,. \tag{1.125}$$

The complexified exterior algebra $\Lambda_{\mathbb{C}}(\mathbf{V})$ [the real exterior algebra $\Lambda(\mathbf{V})$] and its dual, are transcendental extension fields of $\mathbf{V}^{\mathbb{C}}$ [of $\mathbf{V}$] and its dual, cfr. §1.1.3.

The metric possessed by $\mathbb{C}$ is consistent with the complex Exterior Algebra over any almost-complex pre-Hilbert space $(\mathbf{H}_{\mathbb{R}},\langle\cdot|\cdot\rangle)$; $\mathbf{H}^{\mathbb{C}}$ admits a *real* $(1,1)$-form

[56] In this case *only* the overbar ¯ stands for complex conjugation of sets, ***not*** for topological closure.

$\varpi\in\Lambda^{1,1}(\mathbf{H})\cap\Lambda^2(\mathbf{H}^*)$ such that $\langle\mathbf{u};\mathbf{v}|\varpi\rangle=-\mathbf{u}\cdot\imath\mathbf{v}=(\imath\mathbf{u})\cdot\mathbf{v}$, $\forall\mathbf{u},\mathbf{v}\in\mathbf{H}$ and is called a ***fundamental form*** over $\mathbf{H}$ (or $\mathbf{H}^{\mathbb{C}}$). Any almost-complex pre-Hilbert space can be complexified to a complex Euclidean space with *orthogonal* decomposition respect to inner '$\langle\cdot,\cdot\rangle$' and hermitian products[57] '$\langle\cdot|\cdot\rangle_{\mathbb{C}}$', $\forall\mathbf{u},\mathbf{v}\in\mathbf{H}$ (and $\forall\lambda,\mu\in\mathbb{C}$):

$$\mathbf{H}^{\mathbb{C}}=\mathbf{H}^{0,1}\oplus\mathbf{H}^{1,0} \tag{1.126}$$

$$\langle\mathbf{u},\mathbf{v}\rangle\overset{\text{def}}{=}\langle\mathbf{u}|\mathbf{v}\rangle-\imath\langle\mathbf{u};\mathbf{v}|\varpi\rangle;\qquad\langle\mathbf{u}\otimes_{\mathbb{R}}\lambda\mid\mathbf{v}\otimes_{\mathbb{R}}\mu\rangle_{\mathbb{C}}\overset{\text{def}}{=}\lambda\overline{\mu}\langle\mathbf{u}|\mathbf{v}\rangle. \tag{1.127}$$

Scalar, inner and hermitian products are equivalent metrics on $\mathbf{H}^{1,0}$ (cfr. note [54]) and mutually define the fundamental form (and vice versa), in fact, $\forall\mathbf{u},\mathbf{v}\in\mathbf{H}\cong\mathbf{H}^{1,0}$:

$$\langle\mathbf{u}|\mathbf{v}\rangle_{\mathbb{C}}=\frac{1}{2}\langle\mathbf{u},\mathbf{v}\rangle;\qquad\begin{cases}\langle\mathbf{u};\mathbf{v}|\varpi\rangle=-\operatorname{Im}\langle\mathbf{u},\mathbf{v}\rangle\\\langle\mathbf{u}|\mathbf{v}\rangle=\operatorname{Re}\langle\mathbf{u},\mathbf{v}\rangle\end{cases} \tag{1.128}$$

If $\mathbf{V}$ is a topological **separable** linear space (for example $\dim_{\mathbb{R}}\mathbf{V}=\aleph_0=\dim_{\mathbb{C}}\mathbf{V}^{\mathbb{C}}$) and $\mathcal{B}\{\mathbf{x}_1,\mathbf{y}_1,\mathbf{x}_2,\mathbf{y}_2,\mathbf{x}_3,\mathbf{y}_3,\dots\}$ is a **complete** basis for $\mathbf{V}_{\mathbb{R}}$, then a imaginary automorphism $\imath$ can always be defined such that $\imath\mathbf{x}_n=\mathbf{y}_n$, $\imath\mathbf{y}_n=-\mathbf{x}_n$ and the *natural*, complete, complexified basis[58] to $\mathbf{V}^{\mathbb{C}}$ is $\mathcal{B}^{\mathbb{C}}\{\mathbf{z}_1,\overline{\mathbf{z}}_1,\mathbf{z}_2,\overline{\mathbf{z}}_2,\mathbf{z}_3,\overline{\mathbf{z}}_3,\dots\}$ with:

$$\mathbf{z}_n:=\frac{\mathbf{x}_n-\imath\mathbf{y}_n}{2},\qquad\overline{\mathbf{z}}_n:=\frac{\mathbf{x}_n+\imath\mathbf{y}_n}{2},\qquad\forall n\in\mathbb{N}. \tag{1.129}$$

The following transformation rule holds for $\mathbf{V}^{\mathbb{C}}$ and $\mathbf{V}^{\mathbb{C}}$'s $(p,0)$-forms, $p\le\dim_{\mathbb{R}}\mathbf{V}$:

$$\begin{cases}(-2\imath)^p\displaystyle\bigwedge_{j=1}^{p}\mathbf{z}_j\wedge\overline{\mathbf{z}}_j=\bigwedge_{j=1}^{p}\mathbf{x}_j\wedge\mathbf{y}_j\\\left(\dfrac{\imath}{2}\right)^p\displaystyle\bigwedge_{j=1}^{p}\mathbf{z}^j\wedge\overline{\mathbf{z}}^j=\bigwedge_{j=1}^{p}\mathbf{x}^j\wedge\mathbf{y}^j\end{cases} \tag{1.130}$$

Recall that, in the separable case, $\mathbf{V}\cong\mathbb{R}^{\mathbb{N}}$, then $\mathbf{V}^{\mathbb{C}}\cong\mathbb{C}^{\mathbb{N}}\cong\mathbb{R}^{\mathbb{N}}\otimes_{\mathbb{R}}\mathbb{C}$.

---

[57] This means that both $\langle\mathbf{u}|\mathbf{v}\rangle=\overline{\langle\mathbf{v}|\mathbf{u}\rangle}$, $\mathbf{v}\cdot\mathbf{v}\ge0$ and $\langle\mathbf{v}|\mathbf{v}\rangle_{\mathbb{C}}\ge0$, with equalities holding iff $\mathbf{v}=\mathbf{0}$.

[58] That is $\mathcal{B}^{1,0}\{\mathbf{z}_n\}_{n\in\mathbb{N}_0}$ is the canonical basis for $\mathbf{V}^{0,1}$ and $\mathcal{B}^{0,1}\{\overline{\mathbf{z}}_n\}_{n\in\mathbb{N}_0}$ for $\mathbf{V}^{1,0}$.

Finally, a finite-dimensional space $\mathbf{V}$ is almost-complex if and only if $\dim_{\mathbb{R}}\mathbf{V}=2n$, so for its complexification $\dim_{\mathbb{C}}\mathbf{V}^{\mathbb{C}}=2n$ still holds. Complexification is *different* from just using $\mathbb{C}$ as base field for any real, even-dimensional $\mathbf{V}$, for which $\mathbf{V}_{\mathbb{C}}$ has halved dimension, $\dim_{\mathbb{C}}\mathbf{V}=n$ and its canonical basis is build starting from $\mathbb{R}^{2n}\cong\mathbb{C}^n$ as:

$$\left.\begin{aligned}(\mathbf{x},\mathbf{y})\in\mathbb{R}^{2n}\\ \mathbf{z}\in\mathbb{C}^n\end{aligned}\right\}\quad\mapsto\quad\begin{cases}\mathbf{x}+\imath\mathbf{y}\in\mathbb{C}^n\\ (\operatorname{Re}\mathbf{z},\operatorname{Im}\mathbf{z})\in\mathbb{R}^{2n}\end{cases}\tag{1.131}$$

The prototype for such spaces is still $\mathbb{R}^{2n}$, with $(\mathbb{R}^{2n})^{\mathbb{C}}=\mathbb{R}^{2n}\otimes_{\mathbb{R}}\mathbb{C}\cong\mathbb{C}^{2n}$, whose natural orientation is given by the canonical basis $\mathcal{B}^{\mathbb{C}}\{(\delta_{h,k-n}+\imath\delta_{h-n,k})_{1\le k\le 2n}\}_{1\le h\le 2n}$ respect to which $\{\mathbf{v},\imath\mathbf{v}\}$ is a left-handed couple (since the orientability).

In this case the associated natural basis for $\mathbb{C}^{2n}$, written coordinate-wise for $(\mathbf{x},\mathbf{y})\equiv(x_1,x_2,\ldots,x_n,y_1,y_2,\ldots,y_n)\in\mathbb{R}^{2n}$, is given by the complex $2n$-tuple $(\mathbf{z},\overline{\mathbf{z}})\equiv(z_1,z_2,\ldots,z_n,\overline{z}_1,\overline{z}_2,\ldots,\overline{z}_n)\in\mathbb{C}^{2n}$, with $z_j=\frac{1}{2}(x_j-\imath y_j)$ and $\overline{z}_j=\frac{1}{2}(x_j+\imath y_j)$, $1\le j\le n$, as in (1.131).

2.1.2. Metric theory

Let $\mathbb{E}_{\mathbb{R}}$ be an even-dimensional Euclidean space, with $\dim_{\mathbb{C}}\mathbb{E}^{1,0}=n$ and $\mathcal{B}$ be one of its orthonormal bases respect to which its scalar product '$\cdot$' is represented by the metric (positive-definite hermitian matrix) $G=\frac{1}{2}(g_{i,j})_{1\le i\le j\le n}$, coordinating vectors $\mathbf{u}(\boldsymbol{\xi})$ and $\mathbf{v}(\boldsymbol{\eta})$, with $\boldsymbol{\xi}=(\xi_1,\xi_2,\ldots,\xi_n)$, $\boldsymbol{\eta}=(\eta_1,\eta_2,\ldots,\eta_n)\in\mathbb{C}^n$, then:

$$\mathbf{u}\cdot\mathbf{v}\;=\;\overline{\boldsymbol{\eta}}^{\mathsf{T}}G\,\boldsymbol{\xi}\;\equiv\;\frac{1}{2}\sum_{i,j=1}^{n}\xi_i g_{i,j}\overline{\eta}_j\;\equiv\;\sum_{1\le i\le j\le n}g_{i,j}\xi_i\overline{\eta}_j;\tag{1.132}$$

using $\mathbb{C}^{2n}$ as a prototype, the fundamental form is decomposed as:

$$\begin{aligned}\varpi&=\frac{\imath}{2}\sum_{i,j=1}^{n}g_{i,j}\mathbf{z}^i\wedge\overline{\mathbf{z}}^j\;\equiv\;\imath\sum_{1\le i\le j\le n}g_{i,j}\mathbf{z}^i\wedge\overline{\mathbf{z}}^j\\&=\frac{\imath}{2}\sum_{j=1}^{n}\mathbf{z}^j\wedge\overline{\mathbf{z}}^j\;=\;\sum_{j=1}^{n}\mathbf{x}^j\wedge\mathbf{y}^j.\end{aligned}\tag{1.133}$$

Let μ be a measure on $\mathbb{E}$, then $\exists!\mathrm{vol}_\mu\in\Lambda^n(\mathbb{E})$ is the μ-volume element such that:

$$\int_E \mathrm{vol}_\mu = \mu(E), \tag{1.134}$$

for every μ-measurable subset $E\subseteq\mathbb{E}$. There also exists a unique n-form $\mathrm{vol}\in\Lambda^n(\mathbb{E})$ such that (1.134) holds respect to the Peano-Jordan measure and, by extension, to the Lebesgue measure on $\mathbb{E}$, cfr. [68], [9]–§1, [29]. If $\dim_{\mathbb{R}}\mathbb{E}=n$ and $\mathcal{B}_\perp\{\mathbf{e}_1,\mathbf{e}_2,\ldots,\mathbf{e}_n\}$ is a left-handed orthonormal basis, then:

$$\mathrm{vol} = \mathbf{e}_1\wedge\mathbf{e}_2\wedge\ldots\wedge\mathbf{e}_n \tag{1.135}$$

and, for every countably-additive measure μ on $\mathbb{E}$ there exists $\exists!c_\mu\in\mathbb{R}_+^*$ such that $\mathrm{vol}_\mu=c_\mu\mathrm{vol}$. If the measure μ_G is inherited by a different metric G (1.132), then $c_{\mu_G}=\sqrt{|\det G|}$.

> Every Lebesgue-measurable set $E\subseteq\mathbb{R}^n$ (with n-dimensional Lebesgue measure $0<\mu_{\mathrm{L}}^n(E)<+\infty$) is also Hausdorff-measurable with respect to the n-dimensional Hausdorff measure 'μ_{H}^n' and $c_{\mu_{\mathrm{H}}^n}=\mu_{\mathrm{L}}^n\left(\mathbb{B}_1^n(\mathbf{0})\right)$:
>
> $$\mathrm{vol}_{\mathrm{H},n} := \mathrm{vol}_{\mu_{\mathrm{H}}^n} = \left\lfloor\frac{n}{2}\right\rfloor!\left(\frac{2}{\sqrt{\pi}}\right)^n \mathrm{vol}\,. \tag{1.136}$$
>
> For definition and properties of *fractional* Hausdorff measures and dimensions 'μ_{H}^d' $(\forall d\in\mathbb{R}_+)$, cfr. [9]–§1.4, [39]; for applications to fractals, cfr. also [40].

Under these assumptions, the *Hodge "star"* **operator** $\star:\Lambda(\mathbb{E})\to\Lambda(\mathbb{E})$ is known to be defined coordinate-wise on every subspace $\Lambda^p(\mathbb{E})\to\Lambda^{n-p}(\mathbb{E})$ as $(1\leq j_1<j_2<\ldots<$ $<j_p\leq n)$:

$$\star\bigwedge_{h=1}^{p}\mathbf{e}_{j_h} \overset{\mathrm{def}}{=} \frac{1}{p!}\sum_{1\leq j_{p+1}<j_{p+2}<\ldots<j_n\leq n}\varepsilon_{j_1,j_2,\ldots,j_n}\bigwedge_{k=p+1}^{n}\mathbf{e}_{j_k}, \tag{1.137}$$

with $\star 1 = \mathbf{e}_1 \wedge \mathbf{e}_2 \wedge \ldots \wedge \mathbf{e}_n \equiv \mathrm{vol}$ and $\star\mathrm{vol} \equiv \star(\mathbf{e}_1 \wedge \mathbf{e}_2 \wedge \ldots \wedge \mathbf{e}_n) = 1$. The μ-Hodge-$\star$ operator $\star_\mu := c_\mu \star$ and the inner product $\langle\cdot|\cdot\rangle_\mu \in \Lambda^p(\mathbb{E})^{\otimes 2}$ are thus defined, $0 \le p \le n$ as satisfying

$$\underline{\mathbf{u}} \wedge \star_\mu \underline{\mathbf{v}} = \langle \underline{\mathbf{u}} \,|\, \underline{\mathbf{v}} \rangle_\mu \, \mathrm{vol}_\mu, \qquad \forall \underline{\mathbf{u}}, \underline{\mathbf{v}} \in \Lambda(\mathbb{E}); \tag{1.138}$$

under this definition $\|\mathrm{vol}_\mu\| = c_\mu$, $\|\mathrm{vol}\| = 1$ and $\langle \underline{\mathbf{u}} \,|\, \underline{\mathbf{v}} \rangle = c_\mu^{-1} \star \big(\underline{\mathbf{u}} \wedge \star_\mu \underline{\mathbf{v}} \big)$. The Hodge-$\star$ also has the following properties on $\mathbb{E}_{\mathbb{R}}$:

- $\langle \underline{\mathbf{u}} \,|\, \star \underline{\mathbf{v}} \rangle = (-1)^{\mathrm{ord}\,\underline{\mathbf{u}}\,\mathrm{ord}\,\underline{\mathbf{v}}} \langle \underline{\mathbf{v}} \,|\, \star \underline{\mathbf{u}} \rangle, \quad \forall \underline{\mathbf{u}} \in \Lambda^p(\mathbb{E}), \forall \underline{\mathbf{v}} \in \Lambda^{n-p}(\mathbb{E});$
- Considering $\star \in \mathbf{U}_{\mathbb{R}}(\Lambda(\mathbb{E}))$, it is a anti-involutory[59] isometry, i.e.: $\|\star\| = 1$ and $\star\star\big|_{\Lambda^p(\mathbb{E})} = (-1)^{p(n-p)} \mathbb{1}$.

For the μ-Hodge-$\star$ operator $\|\star_\mu\| = c_\mu$ holds. The *Lefshetz* and *μ-dual* Lefshetz operators $L, \varLambda_\mu \colon \Lambda^*(\mathbb{E}) \to \Lambda^*(\mathbb{E})$ are defined to be conjugate one another, as

$$\alpha \mapsto L\alpha \overset{\text{def}}{=} \varpi \wedge \alpha; \qquad \varLambda_\mu \overset{\text{def}}{=} \star_\mu^{-1} L \star_\mu; \tag{1.139}$$

the dual Lefshetz operator, $\varLambda$, is the one with respect to the Peano-Jordan (and to the Lebesgue) measure on $\mathbb{E}$.

It follows immediately that $\langle \varLambda_\mu \alpha \,|\, \beta \rangle_\mu = \langle \alpha \,|\, L\beta \rangle_\mu$, $\forall \alpha, \beta \in \Lambda^*(\mathbb{E})$ and that:

$$\begin{aligned} L\big|_{\Lambda^p(\mathbb{E})} &: \Lambda^p(\mathbb{E}) \to \Lambda^{p+2}(\mathbb{E}) \\ \varLambda\big|_{\Lambda^p(\mathbb{E})} &: \Lambda^p(\mathbb{E}) \to \Lambda^{p-2}(\mathbb{E}) \\ [L \mid \varLambda] &= \sum_{k=0}^{2n} (k-n)\, \mathrm{proj}_{\Lambda^k(\mathbb{E})} \end{aligned} \tag{1.140}$$

After this short review of (real) algebraic Hodge theory, let us introduce its complex analogue: the Lefshetz theory.[60] Thanks to (1.126) and since complex multilinear

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[59] Since $p(n-p) \equiv 1 \bmod 2 \;\Leftrightarrow\; (n,p)^\intercal \equiv (0,1)^\intercal \bmod 2$, the Hodge-$\star$ is involutory for all $\{n,p\}$ combinations **but** n even and p odd. In particular, $\star \in \mathrm{Aut}_{\mathbb{R}} \Lambda^p(\mathbb{E}) \;\Leftrightarrow\; \binom{n}{p} = p \;\Leftrightarrow\; n = 3$.

[60] For the rest of the chapter only the Lebesgue measure will be considered, since extensions to arbitrary measures μ can be extended easily as it was done with Hodge-$\star$ operators.

forms [multi-vectors] have two degrees, operators' couples acting on either/both of them are needed. Focusing the theory to pre-Hilbert rather than Euclidean spaces $\mathbf{H}$ implicitly means considering conjugate spaces $\mathbf{H}^\dagger$ rather than dual ones $\mathbf{H}^*$, i.e. imposing [multi-]linear forms to be also continuous respect to the topology. That is specifically important when extensions to infinite-dimensional (pre-)Hilbert spaces are needed where the Hodge-$\star$ operator is defined as $\star\in\mathfrak{U}_{\mathbb{R}}(\Lambda(\mathbf{H}))$ — also cfr. §2.4.7.

Let $(\mathbf{H}_{\mathbb{R}},\langle\cdot|\cdot\rangle)$ be an almost-complex pre-Hilbert space (for which both a fundamental form $\varpi\in\Lambda^{1,1}(\mathbf{H})\cap\Lambda^2(\mathbf{H}^\dagger)$ and an imaginary automorphism $\imath\in\mathfrak{GL}_{\mathbb{R}}(\mathbf{H})$ exist); the Lefschetz operators (1.140) are well-defined in $\Lambda^\dagger_{\mathbb{C}}(\mathbf{H})$ and inherit similar properties:

$$L\big|_{\Lambda^{p,q}(\mathbf{H})}:\Lambda^{p,q}(\mathbf{H})\to\Lambda^{p+1,q+1}(\mathbf{H})$$
$$\varLambda\big|_{\Lambda^{p,q}(\mathbf{H})}:\Lambda^{p,q}(\mathbf{H})\to\Lambda^{p-1,q-1}(\mathbf{H})$$

The hermitian product '$\langle\cdot|\cdot\rangle_{\mathbb{C}}$' in $\mathbf{H}^{\mathbb{C}}$ defined in (1.127) and the Hodge-$\star$ operator can also be extended to complex multilinear forms, and follow similar properties, e.g.:

$$\begin{cases}\phi\wedge\star\overline{\vartheta}=\langle\phi|\vartheta\rangle_{\mathbb{C}}\ \mathrm{vol}\\ \langle\varLambda\varphi|\vartheta\rangle_{\mathbb{C}}=\langle\varphi|L\vartheta\rangle_{\mathbb{C}}\end{cases}\qquad\forall\phi,\vartheta\in\Lambda^\dagger_{\mathbb{C}}(\mathbf{H}).\tag{1.141}$$

When $\Lambda(\mathbf{H})$ contains real-valued differential forms and its complexification $\Lambda(\mathbf{H})^{\mathbb{C}}$, containing complex-valued forms, is considered, the Lefschetz operators interchange the real and the imaginary parts of such forms with each other, in a way which will be clarified in §2.2.3, and in §2.4.7 as an example of infinite-dimensional Lefshetz theory.

2.1.3. Analytic theory

This paragraph somewhat deviates from the statements in the rest of the Chapter but introduces basic concepts of Complex Analysis (i.e. the study of complex-valued functions of one or —as specifically in this case— several complex variables). For a base reading on Complex Analysis in one variable, cfr. [72], [77].

First of all recall that, let $U\subseteq\mathbb{C}^n$ be an open set, a function $f\colon U\to\mathbb{C}$ is said to be **holomorphic** if it is holomorphic with respect to each variable of $\mathbf{z}\equiv(z_1,z_2,\dots,z_n)\in U$, separately. This is equivalent for f to satisfy either the ***Cauchy-Riemann* equations**

$$\begin{cases} \dfrac{\partial \operatorname{Re} f}{\partial \operatorname{Re} z_j} = \dfrac{\partial \operatorname{Im} f}{\partial \operatorname{Im} z_j} \\[2ex] \dfrac{\partial \operatorname{Re} f}{\partial \operatorname{Im} z_j} = -\dfrac{\partial \operatorname{Im} f}{\partial \operatorname{Re} z_j} \end{cases} \qquad 1\le j\le n, \tag{1.142}$$

or the *Cauchy*'s integral formula

$$\oint_{+\gamma} f(\mathbf{z})\mathrm{d}\boldsymbol{\gamma} \stackrel{\text{def}}{=} \int_a^b f\big(\boldsymbol{\gamma}(t)\big)\dot{\boldsymbol{\gamma}}(t)\mathrm{d}t = 0, \tag{1.143}$$

for *every* closed regular curve $\gamma\in\mathrm{Z}_1(U)$ whose parametrization is given by $\boldsymbol{\gamma}\in\mathrm{C}^\infty([a,b])^n$ — and the superscript dot '$\cdot$' denotes ordinary differentiation with respect to $t\in[a,b)$.

Let the real [anti-]*holomorphic* **derivative** operator for every variable[61] $z\in\mathbb{C}$ be

$$\begin{aligned} \frac{\partial}{\partial z} &\stackrel{\text{def}}{=} \frac{1}{2}\left(\frac{\partial}{\partial \operatorname{Re} z} - \imath\frac{\partial}{\partial \operatorname{Im} z}\right) \equiv \frac{\mathrm{d}}{\mathrm{d}z} \\ \left[\quad \frac{\partial}{\partial \bar{z}} \right. &\stackrel{\text{def}}{=} \left. \frac{1}{2}\left(\frac{\partial}{\partial \operatorname{Re} z} + \imath\frac{\partial}{\partial \operatorname{Im} z}\right) \quad\right] \end{aligned} \tag{1.144}$$

and the [anti-]*holomorphic* **gradient** be

$$\begin{aligned} \frac{\partial f}{\partial \mathbf{z}} &\stackrel{\text{def}}{=} \left(\frac{\partial f}{\partial z_1},\frac{\partial f}{\partial z_2},\dots,\frac{\partial f}{\partial z_n}\right) =: \operatorname{grad}_{\mathbb{C}} f \\ \left[\quad \frac{\partial f}{\partial \bar{\mathbf{z}}} \right. &\stackrel{\text{def}}{=} \left. \left(\frac{\partial f}{\partial \bar{z}_1},\frac{\partial f}{\partial \bar{z}_2},\dots,\frac{\partial f}{\partial \bar{z}_n}\right) \quad\right]. \end{aligned} \tag{1.145}$$

---

[61] $\partial/\partial z$ coincides with the standard derivative $\mathrm{D}f(z) :\equiv f'(z) \stackrel{\text{def}}{=} \lim_{z\to z_0}\big(f(z)-f(z_0)\big)/(z-z_0)$.

Cauchy-Riemann equations (1.142) can be compactly re-written as: $\dfrac{\partial f}{\partial \overline{\mathbf{z}}} \equiv \mathbf{0} \in \mathbb{C}^n$.

The algebra of holomorphic functions on an open set $U \subseteq \mathbb{C}^n$ is indicated as $\mathcal{O}(U)$. The value of holomorphic functions is often computed via integral representations:

Cauchy's Integral Theorem

Let $f \in \mathcal{O}(U)$, then $\forall \mathbf{w} \in U$ and $\forall \boldsymbol{\varepsilon} \in \mathbb{R}^n_+$ such that[62] $\overline{\mathbb{B}_{\boldsymbol{\varepsilon}}(\mathbf{w})} \subset U$:

$$f(\mathbf{z}) = \frac{1}{(2\pi\imath)^n} \oint\limits_{+\partial \mathbb{B}_{\boldsymbol{\varepsilon}}(\mathbf{w})} \frac{f(\boldsymbol{\zeta})}{(\boldsymbol{\zeta} - \mathbf{z})^{\mathbf{1}_n}} \mathrm{d}\boldsymbol{\zeta}, \qquad \forall \mathbf{z} \in \mathbb{B}_{\boldsymbol{\varepsilon}}(\mathbf{w}). \quad (1.146)$$

The same equation holds on if $f \in \mathcal{O}(\mathbb{B}_{\boldsymbol{\varepsilon}}(\mathbf{w})) \cap \mathrm{C}^0\left(\overline{\mathbb{B}_{\boldsymbol{\varepsilon}}(\mathbf{w})}\right)$ and can be generalized further as follows: (1.146) still holds if the $(n-1)$-dimensional circulation is done on the boundary of any Lebesgue-measurable set V such that $\overline{\mathbb{B}_{\boldsymbol{\varepsilon}}(\mathbf{w})} \subseteq V \subset U$; furthermore, $\forall \mathbf{m} \in \mathbb{N}_0^n$:

$$\frac{\partial^{|\mathbf{m}|} f}{\partial \mathbf{z}^{\mathbf{m}}} \equiv f^{(\mathbf{m})}(\mathbf{z}) = \frac{\mathbf{m}!}{(2\pi\imath)^n} \oint\limits_{+\partial V} \frac{f(\boldsymbol{\zeta}) \mathrm{d}\boldsymbol{\zeta}}{(\boldsymbol{\zeta} - \mathbf{z})^{\mathbf{m}+\mathbf{1}_n}}. \qquad (1.147)$$

It is straightforward to prove that in such a $\mathbb{B}_{\boldsymbol{\varepsilon}}(\mathbf{w})$, as a generalization of $n=1$ case, f coincides with its **Taylor series** in $\mathbf{w}$, i.e. with the following power series of $\mathbb{C}[\![\mathbf{z}-\mathbf{w}]\!]$:

$$f\Big|_{\mathbb{B}_{\boldsymbol{\varepsilon}}(\mathbf{w})}(\mathbf{z}) = \sum_{\mathbf{m} \in \mathbb{N}_0^n} a_{\mathbf{m}} (\mathbf{z} - \mathbf{w})^{\mathbf{m}}, \qquad (1.148)$$

where the series is absolutely converging inside the polydisc and the polynomials in (1.146)–(1.148) were —and will be hereinafter— formally written as $(\mathbf{z} - \mathbf{w})^{\mathbf{m}} ::= \prod_{k=1}^{n} (z_k - w_k)^{m_k}$ and

$$a_{\mathbf{m}} = \frac{1}{\mathbf{m}!} \frac{\partial^{|\mathbf{m}|} f}{\partial \mathbf{z}^{\mathbf{m}}}\Bigg|_{\mathbf{w}}, \qquad \forall \mathbf{m} \equiv (m_1, m_2, \dots, m_n) \in \mathbb{N}_0^n; \qquad (1.149)$$

cfr. §1.3 for formal power series' formalism and properties.

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[62] The ***polydisc*** of centre $\mathbf{w} \equiv (w_1, w_2, \dots, w_n)$ and **axes** $\boldsymbol{\varepsilon} \equiv (\varepsilon_1, \varepsilon_2, \dots, \varepsilon_n)$ is $\mathbb{B}_{\boldsymbol{\varepsilon}}(\mathbf{w}) := \times_{j=1}^{n} \mathbb{B}_{\varepsilon_j}(w_j)$.

If $f(\mathbf{z})$ is *not* holomorphic, but yet "well-behaved" in $U\subseteq\mathbb{C}^n$ (e.g. almost everywhere differentiable with respect to $\overline{\mathbf{z}}$, or integrable with respect to some measure in $\mathbb{C}^n$), then the ***Generalized*** Cauchy integral formula still holds, $\forall\mathbf{z}\in V$:

$$f(\mathbf{z}) = \frac{1}{(2\pi\imath)^n}\left(\oint\limits_{+\partial V}\frac{f(\boldsymbol{\zeta})}{(\boldsymbol{\zeta}-\mathbf{z})^{\mathbf{1}_n}}\mathrm{d}\boldsymbol{\zeta} + \int\limits_{V}\frac{\partial f}{\partial\overline{\boldsymbol{\zeta}}}\frac{\mathrm{d}\boldsymbol{\zeta}\wedge\mathrm{d}\overline{\boldsymbol{\zeta}}}{(\boldsymbol{\zeta}-\mathbf{z})^{\mathbf{1}_n}}\right). \tag{1.150}$$

Consider the complex **Dirac's delta** "function" $\delta\in\mathrm{C}_0^\infty(\mathbb{C}^n)^*$ such that, $\forall f\in\mathrm{C}^\infty(\mathbb{C}^n)$, $\langle f|\delta\rangle = -(2\imath)^n f(\mathbf{0})$, that is, $\forall\mathbf{z}_0\in\mathbb{C}^n$:

$$\int\limits_V f(\mathbf{z})\delta(\mathbf{z}-\mathbf{z}_0)\mathrm{d}\mathbf{z}\wedge\mathrm{d}\overline{\mathbf{z}} = (2\imath)^n f(\mathbf{z}_0). \tag{1.151}$$

Combining the above Cauchy formulæ with (1.151) one has, for $f\in\mathcal{O}(U)$:

$$(2\pi\imath)^n f(\mathbf{z}) = \oint\limits_{+\partial V}\frac{f(\boldsymbol{\zeta})}{(\boldsymbol{\zeta}-\mathbf{z})^{\mathbf{1}_n}}\mathrm{d}\boldsymbol{\zeta} = -\int\limits_V f(\mathbf{z})\frac{\partial}{\partial\overline{\boldsymbol{\zeta}}}\left[\frac{1}{(\boldsymbol{\zeta}-\mathbf{z})^{\mathbf{1}_n}}\right]\mathrm{d}\boldsymbol{\zeta}\wedge\mathrm{d}\overline{\boldsymbol{\zeta}}, \tag{1.152}$$

which is equivalent to state that the rational function

$$(\pi^n\mathbf{z}^{\mathbf{1}_n})^{-1} \equiv \frac{1}{\pi^n z_1 z_2\cdot\ldots\cdot z_n} \equiv \frac{1}{\pi^n\mathrm{sy}_n(\mathbf{z})}\in\mathbb{C}\{\mathbf{z}\} \tag{1.153}$$

(which is the reciprocal of π^n times the highest-degree symmetric polynomial of the n-tuple $\mathbf{z}$ — cfr. §1.5.4) is the Green's function of the antiholomorhic gradient[63] $\overline{\partial}$ (1.145) and (1.171), with vanishing boundary condition for $\|\mathbf{z}\|\to\infty$. In fact, $\forall\mathbf{z}_0\in\mathbb{C}^n$, the following identity holds in a "weak" sense, cfr. [47], [68] and [9]–§B.1:

$$\frac{\partial}{\partial\overline{\mathbf{z}}}\frac{1}{\pi^n(\mathbf{z}-\mathbf{z}_0)^{\mathbf{1}_n}} = \delta(\mathbf{z}-\mathbf{z}_0). \tag{1.154}$$

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[63] For $n=1$ this reads $1/\pi z$ as the Green's function of the antiholomorphic derivative $\partial/\partial\overline{z}$, with vanishing boundary condition for $|z|\to\infty$, a fundamental fact for non-holomorphic functions.

The Green's functions will be re-introduced in §4.2 for applications in Electrodynamics, whereas their complex counterpart (especially in the one-variable case $n=1$), will be used in §5.3.1 to solve the (scalar) Riemann-Hilbert problem.

Hartogs' Theorem

Let $n>1$, $\mathbf{z}_0\in\mathbb{C}^n$ and $\boldsymbol{\varepsilon},\boldsymbol{\delta}\in\mathbb{R}^n_+$ such that[64] $\boldsymbol{\varepsilon}<\boldsymbol{\delta}$. Then any function $f\in\mathcal{O}\left(\mathbb{B}_{\varepsilon}(\mathbf{z}_0)\setminus\overline{\mathbb{B}_{\delta}(\mathbf{z}_0)}\right)$ has one analytic continuation[65] in $\mathcal{O}\left(\mathbb{B}_{\varepsilon}(\mathbf{z}_0)\right)$.

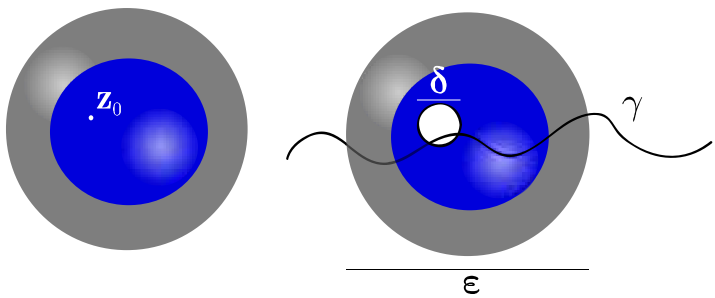

Figure 12 Analytic continuation of a several-variables' holomorphic function around any isolated singularity $\mathbf{z}_0$ (on the left), whereas no continuation is possible —for example— along a singular curve γ (on the right).

The Hartogs' Theorem basically tells that holomorphic functions of several variables neither posses isolated singularities nor, by inversion of complex variable $z\mapsto 1/z$, isolated zeroes — see **Figure 12** and the Monodromy Theorem of §2.4.1 — that is the main difference with holomorphic functions of one variable.

The following constitutes an example of a famous family of multiply-periodic (and almost-periodic) several-variables functions, which will be used again in §2.4.4.

The Siegel's g-plane is the g^2-dimensional hyperbolic half-plane

$$\mathbb{H}_g:=\{B\in\mathrm{sym}_g(\mathbb{C})\mid \mathrm{Im}B>0\},$$

[64] In this case $\varepsilon<\boldsymbol{\delta}$ means that, for $\varepsilon\equiv(\varepsilon_1,\varepsilon_2,\ldots,\varepsilon_n)$ and $\boldsymbol{\delta}\equiv(\delta_1,\delta_2,\ldots,\delta_n)$, then $\varepsilon_j<\delta_j$ for $1\leq j\leq n$.

[65] $F\in\mathcal{O}(\Omega)$ is said to be an ***analytic** continuation* of $f\in\mathcal{O}(U)$, with $U\subset\Omega$, if $F|_U\equiv f$.

of symmetric matrices with positive-definite imaginary parts. The (0^{th}-order) *Riemann **theta** function* $\theta{:}\mathbb{C}^g\times\mathbb{H}_g\to\mathbb{C}$,

$$\theta(\mathbf{z}\,|\,\varPi) \stackrel{\text{def}}{=} \sum_{\mathbf{n}\in\mathbb{Z}^g} \exp\left[2\pi\imath\left(\mathbf{n}\cdot\mathbf{z}+\frac{1}{2}\mathbf{n}^{\intercal}\varPi\mathbf{n}\right)\right], \tag{1.155}$$

is well-defined to be an absolutely convergent Fourier series, uniformly on every compact subset of $\mathbb{C}^g\times\mathbb{H}_g$, possessing the following periodicity condition, $\forall\mathbf{l},\mathbf{m}\in\mathbb{Z}^g$:

$$\theta(\mathbf{z}+\varPi\mathbf{m}+\mathbf{l}\,|\,\varPi) = e^{-\imath\pi(\mathbf{m}^{\intercal}\varPi\mathbf{m}+2\mathbf{m}\cdot\mathbf{z})}\theta(\mathbf{z}\,|\,\varPi). \tag{1.156}$$

From (1.156) it follows that θ is a even and $\mathbf{1}_g$-periodic function with respect to $\mathbf{z}\in\mathbb{C}^g$, in fact $\theta(\mathbf{l}\pm\mathbf{z}\,|\,\varPi)=\theta(\mathbf{z}\,|\,\varPi)$, $\forall\mathbf{l}\in\mathbb{Z}^g$. Furthermore $\theta\in\mathcal{O}(\mathbb{C}^g\times\mathbb{H}_g)$, with its derivatives uniformly satisfying the tensor equation

$$\nabla_{\mathbf{z}}^2\theta = 2\pi(\mathbb{1}^{\otimes 2}+\mathbf{1}_g\otimes\mathbf{1}_g){:}\nabla_{\varPi}\theta, \tag{1.157}$$

which means that, letting $\varPi = (\pi_{i,j})_{1\le i\le j\le g}$, (1.157) is a relation between θ's *Hessian tensor* $\nabla_{\mathbf{z}}^2\theta$ over $\mathbb{C}^g$ (left-hand side) and its gradient over the Siegel g-plane, with $[\nabla_{\varPi}\theta]\equiv\frac{\partial\theta}{\partial\varPi}$, i.e. in vector form:

$$\frac{\partial^2\theta}{\partial\mathbf{z}^2} = 2\pi(I_g+[\mathbf{1}_g\otimes\mathbf{1}_g])\frac{\partial\theta}{\partial\varPi};$$

whereas coordinate-wise, (1.157) becomes:

$$\frac{\partial\theta}{\partial\pi_{i,j}} = \frac{1}{2\pi(\delta_{i,j}+1)}\frac{\partial^2\theta}{\partial z_i\partial z_j}, \qquad 1\le i\le j\le g.$$

Riemann theta functions are indeed Pontryagin's characters of $\mathbb{C}^g\times\mathbb{H}_g$, that is $\theta\in\widehat{\mathbb{C}^g\oplus\mathbb{H}_g}$ — cfr. §1.2.4.

The ring of *Weierstrass polynomials* of degree at most $d\in\mathbb{N}_0$ is $\mathcal{O}(\mathbb{C}^{d-1})^d[w]$. Its d^{th}-*degree* elements are the following functions of $(w,\mathbf{z})\equiv(w,z_1,z_2,\dots,z_{d-1})\in\mathbb{C}^d$ which can be decomposed as d^{th}-degree polynomials in the undeterminate $w\in\mathbb{C}$ and whose coefficients are holomorphic functions $a_0,a_1,\dots,a_d\in\mathcal{O}(\mathbb{C}^{d-1})$ of $d-1$ variables, i.e.:

$$P(w,\mathbf{z})\equiv P_{\mathbf{z}}(w)=\sum_{k=0}^{d} w^{d-k}a_k(\mathbf{z})\,;\qquad \deg P_{\mathbf{z}}:=d.$$

The graded ring of Weierstrass polynomials (cfr. §1.1.3) is trivially defined as:

$$\mathcal{O}(\mathbb{C})[w]\stackrel{\text{def}}{=}\sum_{d=0}^{\infty}\mathcal{O}(\mathbb{C}^{d-1})^d[w]\equiv\bigcup_{d=0}^{\infty}\mathcal{O}(\mathbb{C}^{d-1})^d[w]\,. \tag{1.158}$$

Weierstrass' Division Theorem

Let $f\in\mathcal{O}(\mathbb{C}^{n+1})$ and $\mathbf{z}\in\mathbb{C}^n$, then for every $Q_{\mathbf{z}}\in\mathcal{O}(\mathbb{C}^n)[w]$ there exist a unique holomorphic function $\exists!\,g\in\mathcal{O}(\mathbb{C}^{n+1})$ and a unique Weierstrass polynomial $\exists!\,R_{\mathbf{z}}\in\mathcal{O}(\mathbb{C}^n)[w]$ with $\deg R_{\mathbf{z}}<\deg Q_{\mathbf{z}}$ such that $f(w,\mathbf{z})=g(w,\mathbf{z})Q_{\mathbf{z}}(w)+R_{\mathbf{z}}(w)$, $\forall(w,\mathbf{z})\in\mathbb{C}^{n+1}$ and this is written:

$$f\ \equiv\ R_{\mathbf{z}}\bmod Q_{\mathbf{z}}\,. \tag{1.159}$$

A quotient space of entire functions by Weierstrass polynomials is then naturally defined[66] as $\mathcal{W}_0(\mathbb{C}^{n+1},w):=\mathcal{O}(\mathbb{C}^{n+1})/\mathcal{O}(\mathbb{C})[w]$.

This Theorem proves first that, locally, every holomorphic function of several complex variables has a 'polynomial' behaviour with respect to each single variable (whereas it may behave 'transcendentally' with respect to the other ones). Second, as it was done for rational functions in §1.1.3, it is now possible to build the ring of fractions of holomorphic functions to define the algebra of *meromorphic* functions on any open connected subset $U\subseteq\mathbb{C}^{n+1}$. In fact, (1.159) implies the existence of an equivalence class of fractions (i.e. Weierstrass polynomials in one variable) such that:

[66] Subscript '0' denotes the 0^{th} coordinate of the $(n+1)$-tuple $(w,\mathbf{z})$ to be the 'polynomial' one. Isomorphic spaces $\mathcal{W}_j(\mathbb{C}^n,w)$ are defined, $1\leq j\leq n$, where the polynomial coordinate is the j^{th} one: i.e. $(z_1,z_2,\dots,z_{j-1},w,z_j,\dots,z_n)\in\mathbb{C}^n$.

$$f(w,\mathbf{z}) = \frac{g(w,\mathbf{z})}{Q_{\mathbf{z}}(w)} + R_{\mathbf{z}}(w), \qquad \forall (w,\mathbf{z}) \in U; \tag{1.160}$$

from this notion of Euclidean division with remainder between integer functions in several complex variables —cfr. §1.3.1— the following one follows, for every $0 \leq j \leq r$:

$$\mathcal{K}(U) \stackrel{\text{def}}{=} \operatorname{Frac} \mathcal{O}(U) \cong \operatorname{Frac} \mathcal{W}_j(U,w). \tag{1.161}$$

A set $\{f_i\}_{i\in I}$ of holomorphic [meromorphic] functions in an open set $U \subseteq \mathbb{C}^n$ is said to be a *normal* family if it is *relatively* **sequentially** compact with respect to the usual metric in $\mathcal{O}(U)$ [$\mathcal{K}(U)$], [68], [40], i.e. if every sequence therein admits a subsequence uniformly converging (in $\tilde{\mathbb{C}}$) on every compact subset of U. That also means, by General Topology (*Montel*'s Lemma), that if $\{f_i\}_{i\in I}$ is closed, then it is also compact in $\mathcal{O}(U)$ [$\mathcal{K}(U)$].

The meromorphic functions of $\mathcal{K}(U)$ are equivalence classes of fractions of two holomorphic functions and, among their representatives, a GCD (greatest common divisor) and a LCM (less common multiple) functions can be defined, both being Weierstrass polynomials. Zeroes and poles of a $f \in \mathcal{K}(U)$ such that $f \equiv \frac{g}{h}$, $h \not\equiv 0$ and $\operatorname{GCD}(g,h) = 1$ on every connected component of U, are the zeroes of g and h, respectively (since $\operatorname{GCD}(g,h) = 1 \Rightarrow \ker g \cap \ker h = \varnothing$). Furthermore, by the Hartog's Theorem, zeroes and poles of meromorphic function are *no*where-*dense* subsets of U, which means that they are both, if $n > 1$, at-most-countable sets of curves which *can* accumulate in single points; if $n = 1$ instead, they are at most a countable number of isolated points accumulating in one point of $\tilde{\mathbb{C}}$.

From the previous Theorem, also, it is clear that meromorphic functions are locally equal to rational functions of n variables, so $\mathcal{O}(\mathbb{C}^n)$ locally behaves like $\mathbb{C}[\mathbf{z}]$ as well as $\mathcal{K}(\mathbb{C}^n)$ locally behaves like $\mathbb{C}\{\mathbf{z}\}$ (whereas —more globally— they behave as $\mathbb{C}[\![\mathbf{z}]\!]$ and $\mathbb{C}[\![\mathbf{z}]\!]$, respectively): that is why the global theory of functions of several complex variables is deeply rooted into Algebraic Geometry rather then Complex Analysis (cfr. the GAGA Theorem, [78]).

Meromorphic functions of several variables are subject to Laurent series expansion, which leads to results formally equivalent to those obtained in §1.3.3. For more properties of meromorphic functions (in one variable), such as the Residue Theorem, cfr. §2.4.1, (1.222), or [42].

2.1.4. Holomorphic Functional Calculus

The concept of holomorphy can be extended to arbitrary objects within the Functional Analysis framework; in this context let $(\mathbf{B}_{\mathbb{C}}, \|\cdot\|_{\mathbf{B}})$ be a complex Banach space (even a complexified one, like those in §2.1.1), and $\Omega \subseteq \mathbb{C}$ an open subset of the complex plane. The space $\mathcal{O}(\Omega;\mathbf{B}) \leq \mathrm{C}^0(\Omega;\mathbf{B})$ contains Banach-space-valued functions f of one complex variable (and that could be easily extended to several variables, like in §2.1.3) being **holomorphic**, in the sense that, $\forall z \in \Omega$, the following limit exists, which equals the existence of the Fréchet derivative of f in Ω (i.e. the limit below exists with finite $\mathbf{B}$'s norm):

$$f'(z) \stackrel{\text{def}}{=} \lim_{\zeta \to z} \frac{f(\zeta) - f(z)}{\zeta - z}. \tag{1.162}$$

On such functions the (Lebesgue) circulation along any cycle of $\gamma \in \mathrm{Z}^1(\mathbf{B})$ (rectifiable closed curve) is zero, which generalizes Cauchy's Integral Theorem (1.143):

$$\oint_{+\gamma} f(z)\mathrm{d}\boldsymbol{\gamma} \equiv \int_a^b f(\boldsymbol{\gamma}(t))\,\dot{\boldsymbol{\gamma}}(t)\mathrm{d}t = 0, \qquad \forall \boldsymbol{\gamma} \in \mathrm{C}^0([a,b];\mathbf{B}); \tag{1.163}$$

the above Lebesgue integral's infimum being computed with respect to the weak operator topology[67] on $\mathbf{B}$.

Instead f is said to be ***weakly*** holomorphic if it is pulled back to a (classic) holomorphic function by any continuous linear functionals, i.e. $f \in o(\Omega;\mathbf{B}) \Leftrightarrow$

[67] A sequence $(T_n)_{\mathbb{N}} \in \mathfrak{L}\mathfrak{C}(\mathbf{B})^{\mathbb{N}}$ converges to $T \in \mathfrak{L}\mathfrak{C}(\mathbf{B})$ with respect to the ***uniform*** [*weak*] *operator* topology if $\lim_n \|T_n - T\| = 0$ [if $(T_n\mathbf{v})_{\mathbb{N}}$ is a weakly convergent sequence $\forall \mathbf{v} \in \mathbf{B}$, i.e. $\lim_n \langle T_n\mathbf{v} | \phi \rangle = \langle T\mathbf{v} | \phi \rangle, \ \forall \phi \in \mathbf{B}^\dagger$]. Vector-wise convergence leads to *strong* operator topology.

$\langle f|\phi\rangle\in\mathcal{O}(\Omega),\ \forall\phi\in\mathbf{B}^{\dagger}$. As usual, it can be proven that every holomorphic function is also weakly holomorphic (but not the reverse).

Seemingly, any bounded linear operator $T\in\mathfrak{L}_{\mathbb{C}}(\mathbf{B})\otimes\mathbb{C}^{\Omega}$ is said to be **holomorphic** (with respect to the latter variable $z\in\Omega$) if

$$\oint_{+\gamma} T(\mathbf{v},z)\mathrm{d}z \stackrel{\text{def}}{=} \int_a^b T\left(\mathbf{v},\gamma(t)\right)\dot{\gamma}(t)\mathrm{d}t = 0, \tag{1.164}$$

for any $\gamma\in\mathrm{Z}_1(\Omega)$ whose parametrization is $\gamma\in\mathrm{C}^{\infty}([a,b])$, much like is derived from (1.143) for the complex variable(s) case.

Residuals' Theorem — infinite-dimensional case

Let $f\in\mathcal{O}(\Omega)$ be a holomorphic function and $T\in\mathfrak{L}_{\mathbb{C}}(\mathbf{B})$ a bounded linear operator whose spectrum lies inside Ω's holomorphy open, i.e. $\operatorname{Spec}T\subset\Omega\subseteq\mathbb{C}$; then the resolvent of T is holomorphic itself (with respect to the spectral parameter λ) and bounded, outside the spectrum of T, i.e. $(\lambda\mathbb{1}-T)^{-1}\in\mathfrak{L}(\mathbf{B})\otimes\mathcal{O}(\Omega\backslash\operatorname{Spec}T)$.

Under the previous assumptions the operator $f(T)\in\mathfrak{L}_{\mathbb{C}}(\mathbf{B})$ is defined to be:

$$f(T) \stackrel{\text{def}}{=} \frac{1}{2\pi\imath}\oint_{+\partial D} f(\zeta)(\zeta\mathbb{1}-T)^{-1}\mathrm{d}\zeta, \tag{1.165}$$

Spectral Mapping Theorem

Under the above assumptions (and since $\operatorname{Spec}T$ is compact if T is bounded) $\operatorname{Spec}f(T)=f(\operatorname{Spec}T)$.

with the domain $\operatorname{Spec}T\subset D\subset\Omega$ such that $\partial D\cap\operatorname{Spec}T=\varnothing$, whch means ∂D fully encircles and does not intersect the operator's spectrum; this way (1.165) is proven not to depend on the integration path.

The exponential of such a bounded linaer operator, $\exp T$, can be easily proven to be computable as

$$\exp T \equiv e^{T} = \sum_{n=0}^{\infty} \frac{T^n}{n!} = \mathbb{1} + T + \tfrac{1}{2}T^2 + \tfrac{1}{6}T^3 + \ldots.$$

Moreover the exponential-like operator $e^{\imath T} \in \mathfrak{U}_{\mathbb{C}}(\mathbf{B})$ for operators on real Banach spaces $T \in \mathfrak{L}_{\mathbb{R}}(\mathbf{B})$ is *unitary*, due to elementary Lie algebra facts, as sketched in §1.2.2, i.e. $\mathfrak{U}_{\mathbb{C}}(\mathbf{B}) = \mathrm{Lie}_{\mathbb{C}} \mathfrak{L}_{\mathbb{R}}(\mathbf{B})$.

To complement definitions given in §1.1.4, a linear operator $T \in \mathfrak{L}(\mathbf{H})$ on a pre-Hilbert space $\mathbf{H}_K$ is Jordan if a orthogonal decomposition $\mathbf{H} = \mathbf{N} \oplus \mathbf{D}$ of T-invariant subspaces (cfr. note [12]) exists such that, $T|_{\mathbf{N}}$ is quasi-nilpotent,[68] $T|_{\mathbf{D}} \in \mathfrak{NC}(\mathbf{H})$ (i.e. it is a compact, normal operator), and $\left[T|_{\mathbf{N}} \,\middle|\, T|_{\mathbf{D}} \right] = \mathbb{O}$.

Dunford Decomposition Theorem

Every linear operator $T \in \mathfrak{L}(\mathbf{H})$ on a separable pre-Hilbert space $\mathbf{H}_K$ on any algebraically closed field K is Jordan.

As regards the infinite-dimensional analogue to the Jordan decomposition, here is a Conjecture formulated in 1999 by the author of this work himself.

Arrighetti's Conjecture

Let $T \in \mathfrak{LC}(\mathbf{H})$ be a compact operator on a separable pre-Hilbert space $\mathbf{H}_K$ on any algebraically closed field K complete. There exists a complete orthogonal basis for $\mathbf{H}$, $\mathcal{B}\{\mathbf{u}_n\}_{n \in \mathbb{N}}$, as long as a binary sequence $(\upsilon_n)_{\mathbb{N}} \in \{0,1\}^{\mathbb{N}}$ such that, letting $\{\lambda_n\}_{\mathbb{N}} = \mathrm{spec}\, T$, the following *infinite* Jordan decomposition holds for T, with respect to the basis:

$$T\mathbf{u}_n = \lambda_n \mathbf{u}_n + \upsilon_{n+1} \mathbf{u}_{n+1}, \quad \forall n \in \mathbb{N}. \tag{1.166}$$

[68] A endomorphism $F \in \mathrm{End}_K \mathbf{V}$ [or $F \in \mathfrak{L}_K(\mathbf{V})$] is *quasi*-nilpotent if [$\mathrm{Spec} F \equiv$] $\mathrm{spec} F = \{0\}$.

This is stating that, upon the above conditions (and upon finding the appropriate representation), a compact operator's action may be interpreted much like what happens for Jordan canonical decomosition in finite-dimensional vector spaces. On the right-hand side of (1.166) the sum is an orthogonal decomposition of the operator: $T|_{\mathbf{D}}$ acting like a diagonal operator ($\lambda_n \mathbf{u}_n$) and $T|_{\mathbf{N}}$ acting as a nilpotent operator ($\upsilon_{n+1}\mathbf{u}_{n+1}$); this means that (1.166), represented as a infinite matrix, is:

$$[T] = \begin{pmatrix} \lambda_1 & \upsilon_1 & 0 & 0 & \cdots \\ 0 & \lambda_2 & \upsilon_2 & 0 & \cdots \\ 0 & 0 & \lambda_3 & \upsilon_3 & \vdots \\ \vdots & & 0 & \ddots & \ddots \\ \vdots & & & & \ddots \end{pmatrix}. \tag{1.167}$$

2.2 Complex and Riemann manifolds

Complex Geometry is the study of complexified topological and, above all, smooth manifolds, along with their complex or quasi-complex structures (which can be of Riemann or of Kähler type). This subject is vast and yet definitive results still lack in more subfields than real manifolds' theory. In this paragraph basic definitions and properties of complex Riemann and Kähler manifolds will be introduced, in order to be applied, at a higher (and often less rigid) appoach, in the next Chapters. More complete references on the subject can be found in [62], then also in [32] and [78].

2.2.1. Complexified bundles

Every (real) smooth $2n$-manifold $\mathcal{M}$ is almost-complex in the sense that its tangent and cotangent bundles get complexified as vector spaces, thus inheriting all the properties carefully defined in §2.1.1 to be as general as possible. For every point $p \in \mathcal{M}$ the tangent fibre $\mathsf{T}_p\mathcal{M}$ with local tangent basis

$\mathcal{B}\left\{\frac{\partial}{\partial x_1},\frac{\partial}{\partial x_2},\ldots,\frac{\partial}{\partial x_n},\frac{\partial}{\partial y_1},\frac{\partial}{\partial y_2},\ldots,\frac{\partial}{\partial y_n}\right\}$ is an almost-complex n-dimensional space via the imaginary automorphism $\imath\in\mathfrak{GL}_{\mathbb{R}}(\mathsf{T}_p\mathcal{M})$ acting as $\frac{\partial}{\partial x_i}\mapsto\frac{\partial}{\partial y_i}$ and $\frac{\partial}{\partial y_i}\mapsto-\frac{\partial}{\partial x_i}$, $1\leq i\leq n$. So from §2.1.1, the complexified fibre $\mathsf{T}_p^{\mathbb{C}}\mathcal{M}\overset{\text{def}}{=}\mathsf{T}_p\mathcal{M}\otimes_{\mathbb{R}}\mathbb{C}=\mathsf{T}_p^{1,0}\mathcal{M}\oplus\mathsf{T}_p^{0,1}\mathcal{M}$ has local tangent basis $\mathcal{B}_{\mathbb{C}}\left\{\frac{\partial}{\partial z_1},\frac{\partial}{\partial z_2},\ldots,\frac{\partial}{\partial z_n},\frac{\partial}{\partial \overline{z}_1},\frac{\partial}{\partial \overline{z}_2},\ldots,\frac{\partial}{\partial \overline{z}_n}\right\}$ defined as in (1.129) and (1.144) as:

$$\frac{\partial}{\partial z_i}:=\frac{1}{2}\left(\frac{\partial}{\partial x_i}-\imath\frac{\partial}{\partial y_i}\right),\qquad\frac{\partial}{\partial \overline{z}_i}:=\frac{1}{2}\left(\frac{\partial}{\partial x_i}+\imath\frac{\partial}{\partial y_i}\right).\tag{1.168}$$

Similarly, the cotangent fibre $\mathsf{T}_p^*\mathcal{M}$ with local cotangent basis $\mathcal{B}^\dagger\{\mathrm{d}x_1,\mathrm{d}x_2,\ldots,\mathrm{d}x_n,\mathrm{d}y_1,\mathrm{d}y_2,\ldots,\mathrm{d}y_n\}$ admits the imaginary automorphism $\imath^*(\mathrm{d}x_j)=\mathrm{d}y_j$ and $\imath^*(\mathrm{d}y_j)=-\mathrm{d}x_j$ so, letting $\mathrm{d}z_j:=\mathrm{d}x_j+\imath\mathrm{d}y_j$ and $\mathrm{d}\overline{z}_j:=\mathrm{d}x_j-\imath\mathrm{d}y_j$, $1\leq j\leq n$, $\mathsf{T}_p^{*\mathbb{C}}\mathcal{M}$ has local cotangent basis $\mathcal{B}_{\mathbb{C}}^\dagger\{\mathrm{d}z_1,\mathrm{d}z_2,\ldots,\mathrm{d}z_n,\mathrm{d}\overline{z}_1,\mathrm{d}\overline{z}_2,\ldots,\mathrm{d}\overline{z}_n\}$. Any tangent fibre on a $p_0\in\mathcal{M}$ is isomorphic to some subspace of power series in the $\dim_{\mathbb{C}}\mathcal{M}$ local variables, the Taylor series converging in p_0, i.e. $\mathsf{T}_{p_0}\mathcal{M}\lessapprox\mathbb{C}[\![p-p_0]\!]$.

The *complex* [**co-**]**tangent bundle** $\mathsf{T}_{\mathbb{C}}\mathcal{M}$ $[\mathsf{T}_{\mathbb{C}}^*\mathcal{M}]$ is defined accordingly as

$$\begin{gathered}\mathsf{T}_{\mathbb{C}}\mathcal{M}\overset{\text{def}}{=}\bigsqcup_{p\in\mathcal{M}}\mathsf{T}_p^{\mathbb{C}}\mathcal{M}\cong(\mathsf{T}\mathcal{M})^{\mathbb{C}}\\ \left[\;\mathsf{T}_{\mathbb{C}}^*\mathcal{M}\overset{\text{def}}{=}\bigsqcup_{p\in\mathcal{M}}\mathsf{T}_p^{*\mathbb{C}}\mathcal{M}\cong(\mathsf{T}^*\mathcal{M})^{\mathbb{C}}\;\right]\end{gathered}\tag{1.169}$$

The *differential (p,q)-forms* are the tangent bundle's multilinear forms $\forall p,q\in\mathbb{N}_0$, i.e.,

$$\Lambda^{p,q}(\mathcal{M})\overset{\text{def}}{=}\Lambda^{p,q}(\mathsf{T}_{\mathbb{C}}\mathcal{M})\equiv\Lambda^p(\mathsf{T}^{1,0}\mathcal{M})\otimes\Lambda^q(\mathsf{T}^{0,1}\mathcal{M})\,;\tag{1.170}$$

and of course, $\Lambda_{\mathbb{C}}(\mathcal{M})\overset{\text{def}}{=}\Lambda_{\mathbb{C}}^*(\mathsf{T}_{\mathbb{C}}\mathcal{M})\cong\Lambda_{\mathbb{C}}(\mathsf{T}_{\mathbb{C}}^*\mathcal{M})$. Into them the holomorphic and anti-holomorphic differential derivations[69] $\partial,\overline{\partial}\in\mathrm{D}[\Lambda_{\mathbb{C}}(\mathcal{M})]$ are defined:

$$\partial\overset{\text{def}}{=}\sum_{p,q\in\mathbb{N}_0}\mathrm{proj}_{\Lambda^{p+1,q}(\mathcal{M})}\,\mathrm{d}\,,\qquad\overline{\partial}\overset{\text{def}}{=}\sum_{p,q\in\mathbb{N}_0}\mathrm{proj}_{\Lambda^{p,q+1}(\mathcal{M})}\,\mathrm{d}\,,\tag{1.171}$$

---

[69] $\partial(\phi\wedge\vartheta)=\partial\phi\wedge\vartheta+(-1)^{p+q}\phi\wedge\partial\vartheta$ and $\overline{\partial}(\phi\wedge\vartheta)=\overline{\partial}\phi\wedge\vartheta+(-1)^{p+q}\phi\wedge\overline{\partial}\vartheta$ $\forall\phi\in\Lambda^{p,q}(\mathcal{M})$ $\forall\vartheta\in\Lambda_{\mathbb{C}}(\mathcal{M})$; they are the manifold analogue to [anti-]holomorphic derivatives in §2.1.3.

with $\mathrm{d} = \partial + \overline{\partial}$, $\left[\partial \,|\, \overline{\partial}\right] = \partial^2 = \overline{\partial}^2 \equiv 0$, $\overline{\partial\phi} \equiv \overline{\partial}\,\overline{\phi}$ $\forall\phi\in\Lambda(\mathcal{M})$, and the order-incrementing domain/codomain conditions $\partial|_{\Lambda^{p,q}(\mathcal{M})}\colon \Lambda^{p,q}(\mathcal{M}) \to \Lambda^{p+1,q}(\mathcal{M})$ and $\overline{\partial}\big|_{\Lambda^{p,q}(\mathcal{M})}\colon \Lambda^{p,q}(\mathcal{M}) \to \Lambda^{p,q+1}(\mathcal{M})$.

Their actions on the local coodinates $(\mathbf{z}, \overline{\mathbf{z}}) \equiv (z_1, z_2, \ldots, z_n, \overline{z}_1, \overline{z}_2, \ldots, \overline{z}_n) \in \mathbb{C}^{2n}$ of any cotangent fibre are, respectively:

$$\partial = \sum_{h=1}^{n} \frac{\partial}{\partial z_h} \mathrm{d}z_h \wedge \mathbb{1}; \qquad \overline{\partial} = \sum_{k=1}^{n} \frac{\partial}{\partial \overline{z}_k} \mathrm{d}\overline{z}_k \wedge \mathbb{1}. \tag{1.172}$$

Recall that $\mathcal{M}$ is a [pseudo-]Riemann $2n$-manifold if $\exists \underline{g} \in \mathsf{T}\mathcal{M}^{\otimes 2}$ which is a symmetric, positive [non-negative] definite 2-tensor called [pseudo-]*Riemann metric*. $\underline{g}$ is said to be compatible with the almost-complex structure of $\mathcal{M}$ if on local-chart point $p\in U\subseteq\mathcal{M}$ the vector-field scalar product induced by $\underline{g}$ satisfies:

$$\langle X \,|\, Y \rangle_{\underline{g}(p)} = \langle \boldsymbol{i} X \,|\, \boldsymbol{i} Y \rangle_{\underline{g}(p)}, \qquad \forall X, Y \in \mathsf{T}_p\mathcal{M}. \tag{1.173}$$

Let μ be a measure on the $\mathcal{M}$, with $\dim_{\mathbb{C}}\mathcal{M} = n$; then for the n-volume element $\mathrm{vol}_\mu \in \Lambda^n(\mathcal{M})$, as per (1.134) and (1.135), $\exists M \in C^\infty(\mathcal{M}; \mathbb{R}_+)$ such that:

$$\mathrm{d}\mu = \imath^n M \left(\bigwedge_{h=1}^{p} \mathrm{d}z_h \right) \wedge \left(\bigwedge_{k=1}^{q} \mathrm{d}\overline{z}_k \right); \tag{1.174}$$

the associated fundamental (1,1)-form $\varpi \in \Lambda^{1,1}(\mathcal{M}) \cap \Lambda^2(\mathcal{M})$ to such a measure is defined as:

$$\varpi = \imath \partial \overline{\partial} \log M. \tag{1.175}$$

Whatever is its origin (either "algebraic" or "metric," as in §2.1.2), from any real fundamental form $\varpi \in \Lambda^{1,1}(\mathcal{M}) \cap \Lambda^2(\mathcal{M})$ defines a ***Hermitian* metric** $\underline{g}_{\mathbb{C}} \in \mathsf{T}_{\mathbb{C}}\mathcal{M}^{\otimes 2}$ as:

$$\langle X; Y \,|\, \varpi \rangle := \langle \boldsymbol{i} X \,|\, Y \rangle_{\underline{g}}, \quad \forall X, Y \in \mathsf{T}\mathcal{M}$$

$$\Downarrow \tag{1.176}$$

$$\langle Z \,|\, W \rangle_{\underline{h}} \stackrel{\text{def}}{=} \left\langle \mathrm{proj}_{\mathsf{T}^{0,1}\mathcal{M}} Z \,\middle|\, \mathrm{proj}_{\mathsf{T}^{0,1}\mathcal{M}} W \right\rangle_{\underline{g}} - \imath \langle Z; W \,|\, \varpi \rangle, \quad \forall Z, W \in \mathsf{T}_{\mathbb{C}}\mathcal{M};$$

cfr. (1.127), (1.128). Then on every local chart:

$$\varpi(\mathbf{z}) = \frac{\imath}{2}\sum_{j=1}^{n} \mathrm{d}z_j \wedge \mathrm{d}\overline{z}_j = \sum_{1\le i<j\le n} \left\langle i\frac{\partial}{\partial x_i}{}_* \middle| \frac{\partial}{\partial x_j}{}_* \right\rangle_G \mathrm{d}z_i \wedge \mathrm{d}\overline{z}_j, \qquad (1.177)$$

where

$$G(\mathbf{z}) = \frac{1}{2}\big(g_{i,j}(\mathbf{z})\big)_{1\le i\le j\le n} = \left(\left\langle \frac{\partial}{\partial x_i} \middle| \frac{\partial}{\partial x_j} \right\rangle_{\underline{g}}\right)_{1\le i\le j\le n} \qquad (1.178)$$

and

$$H(\mathbf{z}) = \frac{1}{2}\big(h_{i,j}(\mathbf{z})\big)_{1\le i\le j\le n} = G(\mathbf{z}) - \imath\left(\left\langle \frac{\partial}{\partial x_i}; \frac{\partial}{\partial x_j} \middle| \varpi \right\rangle\right)_{1\le i\le j\le n}. \qquad (1.179)$$

are the local representative matrices of $\underline{g}$ and its Hermitian extension $\underline{g}_{\mathbb{C}}$, respectively.

Any almost-complex n-manifold $\mathcal{M}^{\mathbb{C}}$ (i.e. any real, smooth $2n$-manifold) whose change-of-coordinates functions are holomorphic (which is proven to be independent from the atlases' choice) is called a **complex** n-manifold. Any group $G(\circ)$ acting freely and properly on a complex manifold $\mathcal{M}$ is such that $\mathcal{M}/G$ is also naturally a complex manifold, whereas the canoncial projection $\mathcal{M}\rightarrow\mathcal{M}/G$, [9]–§A.1 is a holomorphic function.

Complex manifolds and almost-complex Riemann manifolds are, in general, different: the equivalent notion to that of Riemann manifold for the complexified case is given by Kähler manifolds, cfr. §2.2.3.

2.2.2. Functional Analysis on complex manifolds

Let us recall that for any Riemann n-manifold $\mathcal{M}$ and measure μ over it, for every $q\in\mathbb{N}_0$ and $p\in]0,+\infty[$, $\mathrm{L}^q_p(\mathcal{M},\mu)$ is the vector space of q-forms whose proper coordinates are functions[70] of $\mathrm{L}^p(\mathcal{M},\mu)$. Thus if $\phi\in\Lambda^q(\mathcal{M})$ such that, locally,

$$\phi = \sum_{1\leq j_1<j_2<\ldots<j_q\leq n} \varphi_{j_1,j_2,\ldots,j_q} \bigwedge_{k=1}^{q} \mathrm{d}x_k,$$

$\phi\in\mathrm{L}^q_p(\mathcal{M},\mu)$ if and only if $\varphi_{j_1,j_2,\ldots,j_q}\in\mathrm{L}^p(\mathcal{M},\mu)$ for every $1\leq j_1<j_2<\ldots<j_q\leq n$; in this case the following semi-norm is defined:

$$|\phi|_{\mathrm{L}^p(\mathcal{M},\mu)} \overset{\text{def}}{=} \sqrt[p]{\sum_{1\leq j_1<j_2<\ldots<j_q\leq n} \|\varphi_{j_1,j_2,\ldots,j_q}\|^p_{\mathrm{L}^p(\mathcal{M},\mu)}}. \tag{1.180}$$

These spaces are Banach [31] for $p\in[1,+\infty[$; their semi-norms coincide with the L^p-norms $\|\phi\|_{\mathrm{L}^p(\mathcal{M},\mu)}$ equivalent to that of a $\binom{n}{p}$-dimensional vector (1.180).

Hereinafter the indication of the generic measure μ will be suppressed as long as there can be not any confusion; by default the locally lifting on $\mathcal{M}$ by the Lebesgue measure on $\mathbb{R}^2\cong\mathbb{C}$ will be used.

If $\mathcal{M}$ is compact, $\mathrm{L}^q_p(\mathcal{M})\cong\ell^{1-\frac{1}{p}}\left(\Lambda^{\binom{n}{q}}(\mathbb{C})\right)$ are separable spaces and every p-integrable differential q-form on $\mathcal{M}$ is represented by a $(1-p^{-1})^{-1}$-summable sequence of linear q-forms of complex numbers.

***Riesz-Hodge*'s Theorem** real manifolds

Let $\mathrm{L}^q_2(\mathcal{M},\mu)$ be a Hilbert space ($q\in\mathbb{N}_0$) whose inner product is

$$\langle\phi|\vartheta\rangle_{\mathrm{L}^2(\mathcal{M},\mu)} \overset{\text{def}}{=} \int_{\mathcal{M}} \phi\wedge\star_\mu\overline{\vartheta}. \tag{1.181}$$

[70] Of course, trivially, $\mathrm{L}^p_0(\mathcal{M},\mu)\equiv\mathrm{L}^p(\mathcal{M},\mu)$.

The same definition holds for $\mathrm{L}_2^q(\mathcal{M},\underline{g})$, where $\mathcal{M}$ is a (pseudo-) Riemann manifold of metric $\underline{g}$. If $\mathcal{M}$ is locally compact, then a duality by inner product exists for every $\forall\vartheta\in\mathrm{L}_2^q(\mathcal{M},\mu)$, i.e.:

$$|\vartheta\rangle_\mu\in\mathrm{L}_2^p(\mathcal{M},\mu)^\dagger\cong\mathrm{L}_2^{1-\frac{1}{p}}(\mathcal{M},\mu),$$
$$\mathrm{L}_2^p(\mathcal{M},\mu)\ni\phi\mapsto\langle\phi|\vartheta\rangle_{\mathrm{L}^2(\mathcal{M},\mu)}. \tag{1.182}$$

It is also worth to note that $\langle\phi|\vartheta\rangle_{\mathrm{L}^2(\mathcal{M},\mu)}=c_\mu\langle\phi|\vartheta\rangle_{\mathrm{L}^2(\mathcal{M})}$ and $|\vartheta\rangle_\mu\equiv c_\mu|\vartheta\rangle$, $\forall\phi,\vartheta\in\mathrm{L}_2^p(\mathcal{M},\mu)\cap\mathrm{L}_2^p(\mathcal{M})$, where $c_\mu\in\mathbb{R}_+$ was defined in §2.1.2, (1.136).

If $\mathcal{M}$ is an almost-complex n-manifold, then $\forall p,q\in\mathbb{N}_0$ and $\forall s\in]0,+\infty[$, and for every measure μ on it, the same definitions apply for the spaces of s-integrable (p,q)-forms, $\mathrm{L}_s^{p,q}(\mathcal{M},\mu)$ and $\mathrm{L}_s^{p,q}(\mathcal{M},g)$, the only "addition," by (1.128) and (1.176), being:

Riesz-Hodge's Theorem — Riemann manifolds

If $\mathcal{M}$ is a (complex) Riemann manifold $\mathrm{L}_2^{p,q}(\mathcal{M},\underline{g}_\mathbb{C})$ is a Hilbert space whose Hermitian product is:

$$\langle\phi|\vartheta\rangle_{\mathrm{L}^2(\mathcal{M},\mu)}\overset{\text{def}}{=}\int_{\mathcal{M}}\langle\phi|\overline{\vartheta}\rangle_\mathbb{C}\star 1. \tag{1.183}$$

In a similar fashion to Lebesgue spaces (1.180), also Sobolev and, in particular, Hilbert-Sobolev spaces $H_k^p(\mathcal{M},\mu)$ and $H_k^{p,q}(\mathcal{M},\mu)$ can be defined $\forall p,q,k\in\mathbb{N}_0$ —cfr. [74], [9]–§B.3 and [92] for basic definitions and properties of these spaces— as:

$$\begin{aligned}\|\phi\|_{H^k(\mathcal{M},\mu)}&\overset{\text{def}}{=}\sqrt{\sum_{1\le j_1<j_2<\ldots<j_p\le n}\|\varphi_{j_1,j_2,\ldots,j_p}\|^2_{H^k(\mathcal{M},\mu)}}\\&=\sqrt{\sum_{1\le j_1<j_2<\ldots<j_p\le n}\ \sum_{\substack{\mathbf{j}\in\mathbb{N}_0^n:\\|\mathbf{j}|\le k}}\left\|\partial^{\mathbf{j}}\varphi_{j_1,j_2,\ldots,j_p}\right\|^2_{\mathrm{L}^2(\mathcal{M},\mu)}}.\end{aligned} \tag{1.184}$$

Hilbert-Sobolev spaces are useful because, when the complex manifold is a complex Lie group $\mathcal{G}(\circ)$ (or a quotient manifold $\mathcal{M}/G$ as defined in §2.2.1), the topology can be recovered from its Pontryagin's dual, §1.2.4, e.g. the norms are computed —by Plancherel's Theorem of §1.2.4 with respect to the manifold's Haar measure— using their Fourier transforms (possibly extending them to the *fractionary*-order spaces $H^p_s(\mathcal{G})$, $\forall s\in\mathbb{R}_+$):

$$\|\phi\|_{H^s(\mathcal{G})} = \sqrt{\sum_{1\leq j_1<j_2<\ldots<j_p\leq n}\ \sum_{\substack{\mathbf{k}\in\mathbb{N}_0^n:\\ |\mathbf{k}|\leq\lceil s\rceil}} \left(1+|\mathbf{k}|^2\right)^{2s} \|\hat{\varphi}_{j_1,j_2,\ldots,j_p}\|^2_{\mathrm{L}^2(\hat{\mathcal{G}})}}. \quad (1.185)$$

2.2.3. Dolbeault cohomology

Of particular interest is the "duality" between the subspace of closed differential forms in relation to the space of cycles on a given complex manifold:

Theorem

The following bilinear mapping $\langle\cdot|\cdot\rangle\colon \mathrm{Z}_p(\mathcal{M})\times\mathrm{Z}^{p,0}(\mathcal{M})\to\mathbb{C}$

$$\langle c|\phi\rangle \stackrel{\text{def}}{=} \oint_{+c}\phi \quad (1.186)$$

is a symmetric tensor of $\mathrm{Z}_p(\mathcal{M})\otimes\mathrm{Z}^{p,0}(\mathcal{M})$ and induces a pairing $\langle\cdot|\cdot\rangle\colon \mathrm{H}_p(\mathcal{M})\times\mathrm{H}^{p,0}(\mathcal{M})\to\mathbb{C}$.

Kernels and images of the holomorphic and anti-holomorphic differential (1.171) are defined, $\mathrm{Z}^{p,q}(\mathcal{M}) \stackrel{\text{def}}{=} \mathrm{Ker}\bar{\partial}\big|_{\Lambda^{p,q}(\mathcal{M})}$ and $\mathrm{B}^{p,q}(\mathcal{M}) \stackrel{\text{def}}{=} \mathrm{Im}\bar{\partial}\big|_{\Lambda^{p,q-1}(\mathcal{M})}$ respectively.

The (p,q)-forms in $\mathrm{Z}^{p,q}(\mathcal{M})$ [$\mathrm{B}^{p,q}(\mathcal{M})$] are called (p,q)-**closed** [-**exact**] forms.[71] The $(p,q)^{\text{th}}$ ***Dolbeault cohomology space*** is then the quotient linear topological space

$$\mathrm{H}^{p,q}(\mathcal{M}) \stackrel{\text{def}}{=} \frac{\mathrm{Z}^{p,q}(\mathcal{M})}{\mathrm{B}^{p,q}(\mathcal{M})}. \quad (1.187)$$

[71] Another terminology is p-closed, q-**co**closed [p-exact, q-**co**exact] forms (cfr. §2.4.2).

Let $p,q\in\mathbb{N}_0$ and $k=p+q$, let $s\in[1,+\infty[$ and $\mathrm{B}^{p,q}_s(\mathcal{M},\mu)\overset{\text{def}}{=}\overline{\mathrm{L}^k_q(\mathcal{M},\mu)^\circ\cap\mathrm{B}^{p,q}(\mathcal{M})}$, i.e. $\mathrm{B}^{p,q}_s(\mathcal{M},\mu)$ is the L^k-closure of the Banach subspace of s-integrable, compact-supported exact (p,q)-forms with respect to measure μ;[72] similar definition holds for $\mathrm{Z}^{p,q}_s(\mathcal{M},\mu)$. The chain groups $\left\{\mathrm{L}^{p,q}_s(\mathcal{M},\mu),\overline{\partial}_{p,q}\right\}_{p,q\in\mathbb{Z}}$ constitute the so-called ***s*-integrable** Dolbeault cohomology (with respect to measure μ) on the manifold $\mathcal{M}$; for the algebraic interpretation of such chain groups, cfr. §1.1.7 and [9]–§2.5.

Since $\mathcal{M}$ is analytic, if μ is a "regular enough" measure, then (1.187) is isomorphic to $\mathrm{Z}^{p,q}_s(\mathcal{M},\mu)/\mathrm{B}^{p,q}_s(\mathcal{M},\mu)$ by the Riesz-Hodge's Theorem (1.183): the Dolbeault cohomology is purely topological, although it can be computed using differential forms with different integrability properties. Turning back to the real cohomology case, this fact is similar to half-integer-order Hilbert-Sobolev spaces $H^p_{1/2}(\partial\mathcal{M},\mu)$ of orientable Riemann manifolds (which are also called "trace spaces" since they are the function spaces where boundary conditions of weak differential systems in $\mathcal{M}$ live, [9]–§B.3, [74]) being isomorphic to 1st DeRham's cohomology spaces.

In analogy to the real case, the adjoint operators to [holomorphic and anti-holomorphic] differentials are defined,

$$\delta\overset{\text{def}}{=}(-1)^{n(k+1)+1}\star\mathrm{d}\star\colon\ \Lambda^k(\mathcal{M})\to\Lambda^{k-1}(\mathcal{M})$$
$$\left[\begin{array}{l}\partial^*\overset{\text{def}}{=}-\star\partial\star\colon\ \Lambda^{p,q}(\mathcal{M})\to\Lambda^{p-1,q}(\mathcal{M})\\ \overline{\partial}^*\overset{\text{def}}{=}-\star\overline{\partial}\star\colon\ \Lambda^{p,q}(\mathcal{M})\to\Lambda^{p,q-1}(\mathcal{M})\end{array}\right]\tag{1.188}$$

(with $\partial^\dagger$ being the L^2-extension of ∂^*, and similar argument for $\overline{\partial}^\dagger$). The associated *Laplace* operators $\Lambda^k(\mathcal{M})\to\Lambda^k(\mathcal{M})$ [$\Lambda^{p,q}(\mathcal{M})\to\Lambda^{p,q}(\mathcal{M})$]:

$$\Delta:\equiv\Delta_{\mathrm{d}}\overset{\text{def}}{=}\delta\mathrm{d}+\mathrm{d}\delta$$
$$\left[\begin{cases}\Delta_\partial\overset{\text{def}}{=}\partial^\dagger\partial+\partial\partial^\dagger\\ \Delta_{\overline{\partial}}\overset{\text{def}}{=}\overline{\partial}^\dagger\overline{\partial}+\overline{\partial}\overline{\partial}^\dagger\end{cases}\right]\tag{1.189}$$

[72] These subspaces should not be confused with **Besov spaces** $B^{p,q}_r(E,\mu)$ having similar symbols, but representing spaces of p-integrable, q-times weakly differentiable spline functions, [9]–§B.3.

these operators decompose and commute according to the following formulæ:

$$\delta = \partial^\dagger + \overline{\partial}^\dagger, \qquad \left[\partial^\dagger \middle| \overline{\partial}^\dagger\right] = \partial^{\dagger 2} = \overline{\partial}^{\dagger 2} = 0; \tag{1.190}$$

furthermore Δ commutes with every other operator: $\star, \partial, \partial^\dagger, \overline{\partial}, \overline{\partial}^\dagger, L$ and Λ.

2.2.4. Kähler manifolds

A complex Riemann n-manifold $\mathcal{M}$ is said to be a ***Kähler manifold*** if its fundamental form (1.176) —called for this reason a **Kähler form** on $\mathcal{M}$— is closed, i.e. $\varpi \in \mathrm{Z}^{1,1}(\mathcal{M})$, and can be locally written as

$$\varpi = \frac{\imath}{2} \sum_{i,j=1}^{n} h_{i,j} \mathrm{d}z_i \wedge \mathrm{d}\overline{z}_j, \tag{1.191}$$

and $H = (h_{i,j})_{1 \le i \le j \le r} \in \mathrm{sym}_n^+ \mathcal{O}(\varphi(U))$ is the hermitian, positive-definite matrix representing the associated metric $\underline{h} \in \mathsf{T}_{\mathbb{C}}\mathcal{M}^{\otimes 2}$ on any local chart $U \subseteq \mathcal{M}$.

On a Kähler manifold the differential operators defined in §2.2.3 satisfy (1.189)–(1.190) and the following *symmety* properties, inherited from the vector spaces' case (cfr. §2.1.1):

$$\begin{gathered} [\partial | L] = [\overline{\partial} | L] = \left[\partial^\dagger \middle| \Lambda\right] = \left[\overline{\partial}^\dagger \middle| \Lambda\right] = 0; \\ \Delta \equiv 2\Delta_\partial \equiv 2\Delta_{\overline{\partial}}. \end{gathered} \tag{1.192}$$

> The complex projective n-space $\mathbb{C}^n\mathbb{P}$ introduced in §1.1.1 is a typical example of Kähler manifold whose canonical atlas has n charts
>
> $$U_i = \left\{\mathbf{z} \equiv (z_0, z_1, \ldots, z_n) \in \mathbb{C}^{n+1} \middle| \; z_i \neq 0\right\}, \qquad 1 \le i \le n,$$

maps $\varphi_i : U_i \to \mathbb{C}^n$ with $(z_0, z_1, \dots, z_n) \mapsto \mathbf{w} = \left(\frac{z_0}{z_i}, \dots, \frac{z_{i-1}}{z_i}, \frac{z_{i+1}}{z_i}, \dots, \frac{z_n}{z_i}\right)$, and which the **Fubini-Study** metric is well-known for. Its Kähler form, on the i^{th} chart of coordinates $\mathbf{w} \equiv (w_1, w_2, \dots, w_n)$, is locally written as:

$$\begin{aligned}\varpi_{\mathrm{FS}} &= \frac{\imath}{2\pi}\partial\bar{\partial}\log\left(\sum_{j=0}^{n}\left|\frac{z_j}{z_i}\right|^2\right) = \frac{\imath}{2\pi}\partial\bar{\partial}\log\sum_{j=1}^{n}|w_j|^2 \\ &= \frac{\imath}{2\pi}\sum_{j=1}^{n}\frac{w_i\bar{w}_j - (1+\|\mathbf{w}\|^2)\delta_{i,j}}{(1+\|\mathbf{w}\|^2)^2}\mathrm{d}w_i\wedge\mathrm{d}\bar{w}_j,\end{aligned} \tag{1.193}$$

which can easily be proven since $\partial\bar{\partial}\log\|\mathbf{z}\| = 0$ and

$$\begin{aligned}\int_{\mathbb{C}^n\mathbf{P}}\varpi_{\mathrm{FS}} &= \frac{\imath}{2\pi}\int_{\mathbb{C}}\frac{\mathrm{d}w\wedge\mathrm{d}\bar{w}}{(1+|w|^2)^2} = \\ &= \frac{1}{\pi}\iint_{\mathbb{R}^2}\frac{\mathrm{d}x\wedge\mathrm{d}y}{\left(1+\|(x,y)\|^2\right)^2} = 2\int_0^\infty\frac{\rho\,\mathrm{d}\rho}{(1+\rho^2)^2} = 1.\end{aligned}$$

On a Riemann manifold $\mathcal{M}$ with hermitian metric $\underline{g}_{\mathbb{C}}$, the operators $\partial^\dagger, \bar{\partial}^\dagger$ are self-commuting[73] with $\partial, \bar{\partial}$ with respect to the $\mathrm{L}_2^{p,q}(\mathcal{M}, \underline{h})$ topology (1.183) —so are d and δ with each other— and

$$\begin{cases}\mathrm{H}_{\partial}^{p,q}(\mathcal{M}) \stackrel{\text{def}}{=} \mathrm{Ker}\Delta_\partial = \mathrm{Ker}\bar{\partial}\cap\mathrm{Ker}\bar{\partial}^\dagger \\ \mathrm{H}_{\bar{\partial}}^{p,q}(\mathcal{M}) \stackrel{\text{def}}{=} \mathrm{Ker}\Delta_{\bar{\partial}} = \mathrm{Ker}\partial\cap\mathrm{Ker}\partial^\dagger\end{cases} \tag{1.194}$$

The elements of $\mathrm{H}_{\partial}^{p,q}(\mathcal{M})$ and $\mathrm{H}_{\bar{\partial}}^{p,q}(\mathcal{M})$ are called **harmonic** and ***co*harmonic** forms, respectively;[74] by (1.198), no matter whether they are (p,q)-forms or k-forms with $k=p+q$; their spectrum obeys the following — either with respect to the Hermitian metric $\underline{g}_{\mathbb{C}}$ or a measure μ on $\mathcal{M}$, cfr. (1.183), $\forall\alpha\in\mathrm{H}_{\partial}^{p,q}(\mathcal{M})$ $\left[\ \forall\alpha\in\mathrm{H}_{\bar{\partial}}^{p,q}(\mathcal{M})\ \right]$:

---

[73] Two *self*-commuting operators (i.e. commuting, self-adjoint) have the same spectrum and are mutually diagonalizable by the same complete orthogonal eigenvectors' basis [9]–§3.7,§B.4, [52].

[74] Another terminology, [62], is ∂-harmonic and $\bar{\partial}$-harmonic forms, respectively.

$$\begin{cases} \langle \Delta_{\partial}\alpha \,|\, \alpha\rangle_{\mathbb{C}} = \|\partial\alpha\|_{\mathbb{C}}^2 + \|\partial^*\alpha\|_{\mathbb{C}}^2 \\ \langle \Delta_{\overline{\partial}}\alpha \,|\, \alpha\rangle_{\mathbb{C}} = \|\overline{\partial}\alpha\|_{\mathbb{C}}^2 + \|\overline{\partial}^*\alpha\|_{\mathbb{C}}^2 \end{cases};$$

$$\begin{cases} \langle \Delta_{\partial}\alpha \,|\, \alpha\rangle_{\mathrm{L}^2(\mathcal{M})} = \|\partial\alpha\|_{\mathrm{L}^2(\mathcal{M})}^2 + \|\partial^\dagger\alpha\|_{\mathrm{L}^2(\mathcal{M})}^2 \\ \langle \Delta_{\overline{\partial}}\alpha \,|\, \alpha\rangle_{\mathrm{L}^2(\mathcal{M})} = \|\overline{\partial}\alpha\|_{\mathrm{L}^2(\mathcal{M})}^2 + \|\overline{\partial}^\dagger\alpha\|_{\mathrm{L}^2(\mathcal{M})}^2 \end{cases}. \tag{1.195}$$

Hodge Decomposition Theorem — complex manifolds

For every *compact* Riemann manifold $\mathcal{M}$ the following two orthogonal decompositions hold $\forall p,q\in\mathbb{N}_0$:

$$\begin{aligned} \Lambda^{p,q}(\mathcal{M}) &= \partial\Lambda^{p-1,q}(\mathcal{M}) \oplus \partial^\dagger\Lambda^{p+1,q}(\mathcal{M}) \oplus \mathrm{H}^{p,q}_{\partial}(\mathcal{M}) \\ &= \overline{\partial}\Lambda^{p,q-1}(\mathcal{M}) \oplus \overline{\partial}^\dagger\Lambda^{p,q+1}(\mathcal{M}) \oplus \mathrm{H}^{p,q}_{\overline{\partial}}(\mathcal{M}) \end{aligned} \tag{1.196}$$

where $\mathrm{H}^{p,q}_{\partial}(\mathcal{M})$ and $\mathrm{H}^{p,q}_{\overline{\partial}}(\mathcal{M})$ are finite-dimensional. If $\mathcal{M}$ is Kähler then $\mathrm{H}^{p,q}_{\partial}(\mathcal{M}) = \mathrm{H}^{p,q}_{\overline{\partial}}(\mathcal{M}) \cong \mathrm{H}^{p,q}(\mathcal{M})$, the Dolbeault cohomologies.

The Hodge Decomposition Theorem for *real* manifolds can be found in [37], [88] (as well as in [9]–§§2.5, 3.6, 3.7 using simplicial cohomology on $\mathbb{Z}$ instead of DeRham's on $\mathbb{R}$); both theorems are the continuum (field) extension of the Kronecker's Basis Theorem, [9]–§A.3, on the spaces of real differential forms on a compact Riemann manifold.

On a compact Kähler manifold $\mathcal{M}$ instead, the integer numbers

$$\mathrm{h}_{p,q}(\mathcal{M}) \stackrel{\mathrm{def}}{=} \dim_{\mathbb{C}} \mathrm{H}^{p,q}(\mathcal{M}) \tag{1.197}$$

are called **Hodge** numbers of $\mathcal{M}$, visualized as the $2\binom{n}{p}$-components *Hodge diamond*.

$$
\begin{array}{ccccccccc}
 & & & & \mathrm{h}_{0,0} & & & & \\
 & & & \mathrm{h}_{1,0} & & \mathrm{h}_{0,1} & & & \\
 & & \mathrm{h}_{2,0} & & \mathrm{h}_{1,1} & & \mathrm{h}_{0,2} & & \\
 & \iddots & & & \vdots & & & \ddots & \\
\mathrm{h}_{n,0} & & & & \boxed{\text{Serre}} & & & & \mathrm{h}_{0,n} \\
 & \ddots & & & \vdots & & & \iddots & \\
 & & \mathrm{h}_{n,n-2} & & \mathrm{h}_{n-1,n-1} & & \mathrm{h}_{n-2,n} & & \\
 & & & \mathrm{h}_{n,n-1} & & \mathrm{h}_{n-1,n} & & & \\
 & & & & \mathrm{h}_{n,n} & & & &
\end{array}
\quad \updownarrow \text{Hodge}
$$

$$\xleftrightarrow[\text{conjugation}]{}$$

∂∂̄-Lemma

Every closed (d-closed) form $\alpha\in\mathrm{Z}_{\mathbb{C}}^{p+q}(\mathcal{M})$ on a compact Kähler manifold satisfies the following equivalent conditions:

- α is d-exact: $\exists\beta\in\Lambda_{\mathbb{C}}^{p+q-1}(\mathcal{M}) : \quad \alpha=\mathrm{d}\beta\,;$
- α is ∂-exact: $\exists\beta\in\Lambda^{p-1,q}(\mathcal{M}) : \quad \alpha=\partial\beta\,;$
- α is $\bar\partial$-exact: $\exists\beta\in\Lambda^{p,q-1}(\mathcal{M}) : \quad \alpha=\bar\partial\beta\,;$
- α is $\partial\bar\partial$-exact: $\exists\beta\in\Lambda^{p-1,q-1}(\mathcal{M}) : \quad \alpha=\partial\bar\partial\beta\,.$

As a Corollary to the $\partial\bar\partial$-Lemma it follows that (always on a compact Kähler manifold) the Dolbeault cohomology spaces provide perfect orthogonal decomposition for the complexified DeRham's cohomology spaces; the analogue of the *Poincaré*'s duality ([9]–§3.5, [37], [88]) to the Kähler case is the so-called ***Serre* duality** [62]:

$$\mathrm{H}_{\mathbb{C}}^{k}(\mathcal{M}) \overset{\text{def}}{=} \mathrm{H}^{k}(\mathcal{M})\otimes_{\mathbb{R}}\mathbb{C} = \bigoplus_{\substack{p,q\in\mathbb{N}_0:\\ p+q=k}} \mathrm{H}^{p,q}(\mathcal{M})\,; \tag{1.198}$$

$$\overline{\mathrm{H}^{p,q}(\mathcal{M})} = \mathrm{H}^{p,q}(\mathcal{M}) \cong \mathrm{H}^{n-p,n-q}(\mathcal{M})^{\dagger}. \tag{1.199}$$

The geometry and, of course, the topology of a Kähler manifold do not depend specifically on the Kähler form itself $\varpi \in \mathrm{Z}^{1,1}(\mathcal{M}) \cap \mathrm{Z}^2(\mathcal{M})$ but, rather on its **Kähler class** (which is a cohomology equivalence): $[\![\varpi]\!] \in \mathrm{H}^{1,1}(\mathcal{M}) \cap \mathrm{H}^2(\mathcal{M})$.

From all these considerations it follows that Kähler n-manifolds are always *triangulable*, i.e. homeomorphic to a (real) $2n$-dimensional simplicial complex, cfr. [81] and [9]–§2.1.

Hodge-Riemann bilinear relation

Let $\mathcal{M}$ be a Kähler n-manifold and $[\![\varpi]\!] \in \mathrm{H}^{1,1}(\mathcal{M}) \cap \mathrm{H}^2(\mathcal{M})$ its Kähler class; then $\forall \omega \in [\![\varpi]\!]$ and $\forall [\![\phi]\!] \in \mathrm{H}^{p,q}(\mathcal{M})$ $(0 \leq p, q \leq n)$:

$$(-1)^{\frac{(p+q)(p+q-1)}{2}} \imath^{p-q} \int_{\mathcal{M}} \phi \wedge \overline{\phi} \wedge \omega^{n-(p+q)} > 0\,. \tag{1.200}$$

Hard *Lefshetz* Theorem

Lefshetz operators restricted to a Kähler n-manifold's (p,q)-forms

$$L\colon \mathrm{H}^{p,q}(\mathcal{M}) \to \mathrm{H}^{p+1,q+1}(\mathcal{M}),$$
$$\Lambda\colon \mathrm{H}^{p,q}(\mathcal{M}) \to \mathrm{H}^{p-1,q-1}(\mathcal{M}),$$

only depend on the Kähler class $[\![\varpi]\!]$ and decompose the (real) k^{th} DeRham's cohomology space ($\forall k \in \mathbb{N}_0$ and recalling $\dim_{\mathbb{R}} \mathcal{M} = 2n$) as:

$$\begin{aligned} L^{n-k}\Big|_{\mathrm{H}^k(\mathcal{M})} &\colon \mathrm{H}^k(\mathcal{M}) \to \mathrm{H}^{n-2k}(\mathcal{M}) \ ; \\ \mathrm{Ker}\, L\Big|_{\mathrm{H}^k(\mathcal{M})} &= \bigoplus_{h=0}^{\infty} L^h \mathrm{H}^{k-2h}(\mathcal{M}) \ . \end{aligned} \tag{1.201}$$

2.3 Topology, Monodromy, Fractals

This paragraph deviates from Complex Geometry to mainly introduce wider-validity concepts in General Topology [80] such as covering and self-covering spaces, and some basic Homotopy Theory [38]; then it proceeds with analyzing the topology of limit spaces generated by self-similar groups (that were introduced and analyzed from the algebraic point of view in §1.4) [82]. Finally, the concept of "monodromy" and the definition of the Iterated Monodromy Group is given: the latter tools will be treated in the Riemann surfaces' context in §2.3.3 and then applied for the analysis of fractal Riemann surfaces in §3.

2.3.1. Covering spaces

A ***covering*** (map) of a topological space X is a continuous, surjective function $c\in\mathrm{C}^0(C;X)$ where C is a Hausdorff space (called a covering **space** of X) such that $\forall x\in X$ an (open) neighbourhood $\mathcal{N}(x)\subseteq X$ and a non-empty $I\neq\varnothing$ exist such that:

$$c^{-1}\left(\mathcal{N}(x)\right)=\bigcup_{i\in I}U_i,\qquad U_i\cap U_j=\varnothing,\quad \forall i,j\in I,\ i\neq j;\tag{1.202}$$

that is the $\mathcal{N}(x)$'s preimage under c is the union of mutually disjoint subsets of C whose images (always under c) coincide with $\mathcal{N}(x)$ itself. Sets $\{U_i\}_{i\in I}$ are called ***sheets*** (lying) over x and their union $\bigcup_{i\in I}U_i$ a ***fibre*** (lying) over x. Coverings have the following properties:

- $\forall y\in C\ \exists\mathcal{N}(y)\subset C$: $c|_{\mathcal{N}(y)}$ is a homeomorphism, i.e. $c(\mathcal{N}(y))\asymp X$.
- Each fibre of c is a discrete set of sheets and its cardinality $(\mathrm{Card}\,I)$ is invariant on every connected component of X: if X is connected, then c is said to be a $(\mathrm{Card}\,I)$-cover of X.
- If $C\subseteq\mathring{X}$ (or C is embedded into $\mathring{X}$) then c is said to be a partial ***self-covering*** of X;

- Every path $\gamma \in \mathrm{C}^0([0,1];X)$ admits a unique path $\gamma \uparrow c$ on C, called its **lift** on C, such that $c \circ (\gamma \uparrow c) = \gamma$ and $\gamma \uparrow c$ is said to be "lying over" γ.

If C is simply connected (i.e. every path in C is null-homotopic) c is called a ***universal* cover** of X (and C a universal covering space of X); only path-connected *and* locally path-connected spaces admit universal covering spaces, which are all homeomorphic with each other. That is, if two universal covers $c \in \mathrm{C}^0(C;X)$ and $c' \in \mathrm{C}^0(C';X)$ exists, then $C \asymp C'$.

Let $c \in \mathrm{C}^0(C;X)$ be the cover map and $G(\circ)$ a discrete group freely acting on X; this means that every fibre corresponds to an orbit of the group, i.e. $G \circ x = c^{-1}(x)\ \forall x \in X$ and that $G \circ x$ does not contain accumulation points. In this case, by the Orbit Stabilizer Theorem (1.23), X/G is a new space and the just-defined map $X \to X/G$ is an isomorphic cover to c, which is said to be **regular** (as an action of G — cfr. §1.1.5).

If $c \in \mathrm{C}^0(C;X)$ and $c' \in \mathrm{C}^0(C';X)$ are two covers of X, any map $f: C' \to C$ such that either $c = f \circ c'$ or $c' = f \circ c$ is said to be **fibre-preserving**, i.e. every fibre lying over $x \in X$ is mapped into another fibre (in the other covering space) lying over x. Fibre-preserving homeomorphisms between covering spaces of a given topological space X are called ***deck*** (or covering) *transformations*, and form a topological group under mapping composition, which is indicated as $\mathrm{Deck}\,X$. Furthermore, let $c \in \mathrm{C}^0(C;X)$ be such a cover, then [45]

$$\mathrm{Deck}(C \xrightarrow{c} X) \stackrel{\text{def}}{=} \left\{ f \in \mathrm{Sym}\,C \;\middle|\; f \text{ is fibre-preserving} \right\}; \tag{1.203}$$

which can also be written as $\mathrm{Deck}(C,X)$ and is a subgroup of $\mathrm{Deck}\,X$. The cover c is said to be *Galois* cover if a deck transformation $f \in \mathrm{Deck}(C;X)$ exists such that

$$y_1, y_2 \in C,\ \ c(y_1) = c(y_2) \qquad \Rightarrow \qquad f(y_1) = y_2. \tag{1.204}$$

The deck transformation group basically contains permutations of fibres and, if $f\neq\mathbb{1}$ and X is path-connected, f has *no* fixed points and every couple of points lying over $x\in X$ belong to different fibres of x, so $\mathrm{Deck}(C\xrightarrow{c}X)$ acts *freely* (and properly) *discontinuously* on them, cfr. §§1.1.5, 1.2.2. If is c a universal covering, then $\mathrm{Deck}(C\xrightarrow{c}X)\cong\pi_1(X)$ and $X\asymp C/\mathrm{Deck}(C,X)$.

The Homotopy theory, which particularly investigates the topological spaces' properties respect to continuous homeomorphisms' semigroups and cover maps, is treated in more detail in [38], [7] and [6]. Since this argument, as well as that of monodromy groups, is really vast if treated from the more general viewpoint of Topology, the reader is encouraged to be acquainted with its language in suggested, as well as in other sources: just the application to Riemann surfaces will be investigated here.

If $\mathcal{X}$ and $\mathcal{C}$ are two C^1-differentiable n-manifolds, cfr. [9]–§3.1, and $c\in\mathrm{C}^1(\mathcal{C};\mathcal{X})$ is [*almost*] *regular* covering (i.e. its Jacobian $\det\partial c$ is [almost-]nowhere zero), then $\mathrm{Card}\,I<\aleph_0$ (although finite covers do exist for topological spaces without any differentiable structure). $B=\ker(\det\partial c)$, the zero-level set of c's Jacobian, is called *ramification locus*, and its points are the **ramification** points of $\mathcal{X}$. The map $\mathrm{b}_c:\mathcal{X}\to\mathbb{N}$,

$$\boldsymbol{p}\;\mapsto\;\mathrm{b}_c(\boldsymbol{p})\stackrel{\mathrm{def}}{=}\mathrm{Card}\,c^{-1}(\boldsymbol{p})\,,$$

is called the ramification order (of point $\boldsymbol{p}\in\mathcal{X}$).

If $\dim B<\dim\mathcal{X}-1$ —that is $c(B)$ does not split $\mathcal{X}$— then $c|_{\mathcal{C}\setminus c^{-1}(c(B))}:\mathcal{C}\setminus c^{-1}\big(c(B)\big)\to\mathcal{X}\setminus c(B)$ is called a ***ramified* cover** —or a ramified $(\mathrm{Card}\,I)$-cover— of $\mathcal{X}$, whereas $\mathcal{C}\setminus c^{-1}\big(c(B)\big)$ is proven to be connected.

Riemann-Hurwitz Formula

Every ramified N-cover $c\in\mathrm{C}^1(\mathcal{R};\mathcal{S})$ between smooth surfaces interrelates their Euler-Poincaré characteristics according to the following:[75]

$$\chi(\mathcal{R}) = N\chi(\mathcal{S}) + \sum_{p\in\mathcal{S}} (\mathrm{b}_c(p) - 1). \tag{1.205}$$

The Riemann-Hurwitz formula can be proven just using topological considerations on the ramification points' count and is useful to compute the topological invariants of surfaces ramifying one over another (such as Riemann surfaces, where (1.205) will be reformultated in §3.1.1 and then applied throughout §3.2). Other proofs of (1.205) can be performed by using triangulations of compact surfaces and using the Euler-Poincaré constant's definition as an alternate-signs sum of Betti numbers, [9]–§2.1 — for surfaces: $\chi(\mathcal{S})=\beta_0(\mathcal{S})-\beta_1(\mathcal{S})+\beta_2(\mathcal{S})=2-\beta_1(\mathcal{S})\equiv 2-\dim\mathrm{H}_1(\mathcal{S})$ (since for any surface $\beta_0(\mathcal{S})=1$ whereas $\beta_2(\mathcal{S})$ is either 1 or 0, depending whether $\mathcal{S}$ is compact or not, respectively).

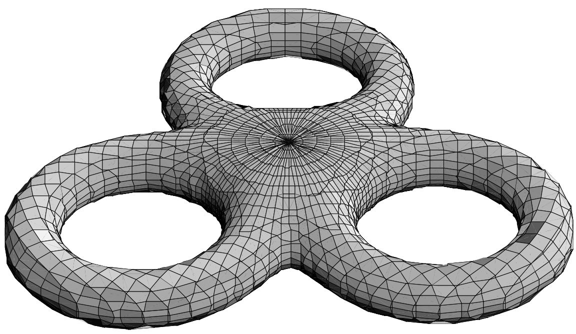

Figure 13 Compact, orientable surface of genus $\kappa=6$ (i.e. $g=3$), which is a 3-handled sphere $\mathbb{T}_3$.

A classification of compact smooth surfaces is long-time available: every compact, non-orientable surface is homeomorphic to a κ-fold real projective plane $\mathbb{RP}_\kappa$ (cfr.

[75] Note that the sum over every points $p\in\mathcal{S}$ is well-posed since for smooth surfaces, $\mathrm{Card}\,B<\aleph_0$.

§1.1.1), whereas every orientable one is homeomorphic to a g-fold torus[76] $\mathbb{T}_g$. In either cases κ and g are called the surface's non-orientable and orientable *genus*, respectively, which are fundamental topological invariants. Compact orientable surfaces are, in particular, homeomorphic to a g-handled sphere, i.e. a sphere with g handles "attached" (i.e. by connected sum surgery[77] '$\#$', [59]) and g represents the number of 'holes' in the surface, cfr. Figure 13 Since the connected sum of two real projective planes is homeomorphic to a torus,

$$\mathbb{RP}_{2g} \asymp \mathbb{T}_g, \qquad \forall g \in \mathbb{N}, \tag{1.206}$$

then every compact, non-orientable κ-genus surface is homeomorphic to a $\lfloor \frac{\kappa}{2} \rfloor$-fold torus with one real projective plane attached, i.e. $\mathbb{RP}_{2g+1} \asymp \mathbb{T}_g \# \mathbb{RP}$. This also means that $\mathbb{RP}_\kappa$ is orientable if and only if the non-orientable genus κ is even.

Homotopy and homology groups for such fundamental topological spaces are listed here for completeness' sake; a more detailed list (with proofs and variations) can be found in any complete General Topology book, such as [59], [80] and [65].

surface	π_1	H^1
$\mathbb{T}^n$	$\mathbb{Z}^n$	$\Lambda(\mathbb{Z}^n)$
$\mathbb{T}_g$	$\overset{g}{\ast}\mathbb{Z}$	$\mathbb{Z}^{2g}$
$\mathbb{RP}_\kappa$	$\mathbb{Z}_2$	$\mathbb{Z}$

As far as the Riemann-Hurwitz Formula (1.205) is concerned, recall also that $\chi(\mathbb{RP}_\kappa) = 2 - \kappa$ and $\chi(\mathbb{T}_g) = 2 - 2g$.

---

[76] The g-fold n-**torus** is the connected sum of g n-dimensional tori: $\mathbb{T}_g^n \overset{\text{def}}{=} \bigvee^g \mathbb{T}^n$.

[77] The connected sum of topological spaces X and Y satisfies $\chi(X \# Y) = \chi(X) + \chi(Y) - 2$, whereas $\chi(X) = 0 \Leftrightarrow X \asymp \mathbb{S}^2$: the sphere is neutral element to topological spaces' grupoid by connected sums.

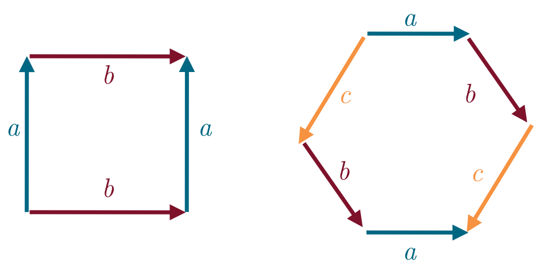

Figure 14 Fundamental square and hexagon of a 1-fold and a 3-fold tori, $\mathbb{T}_1$ and $\mathbb{T}_3$, respectively.

Any orientable 2κ-genus [non-orientable $(2\kappa+1)$-genus], compact surface is topologically equivalent to a (regular) even-sided **fundamental polygon** with as many sides as four times its genus, with any of the 2κ couples of consecutive sides mutually identified 2-by-2 with the other 2κ (same orientations) and the spare couple of sides —in the non-orientable case— identified with itself (opposite orientation), as shown in Figure 14 . For further details cfr. [37] and [42].

2.3.2. Orbifolds and self-coverings

A manifold is a topological space locally behaving like a Euclidean space, whereas an orbifold locally behaves like the orbispace of a topological group, e.g. a quotient space like $\mathbb{R}^n/\Gamma$, cfr. §1.1.5.

An n-dimensional topological **orbifold** (or n-orbifold) $\mathcal{O}$ is given by a topological manifold[78] $\mathcal{O}$ with an additional quotient structure: respect to an atlas of its $\{U_i,\varphi_i\}_{i\in I}$, for every neighbourhood $\mathcal{N}(p)\subseteq U_i$ of any point $p\in U_i$ both a finitely-generated group $\Gamma_p \lessapprox \mathrm{Aut}\mathbb{R}^n \cong \mathrm{GL}_n(\mathbb{R})$ acts —i.e. $\Gamma_p(\circ)$ acts continuously and effectively on $\varphi_i(\mathcal{N}(p))\subseteq\mathbb{R}^n$— and a homeomorphism θ_p:$\varphi_i(\mathcal{N}(p))/\Gamma_p\to\varphi_i(\mathcal{N}(p))$ must exist such that, letting π_p:$\varphi_i(\mathcal{N}(p))\to\varphi_i(\mathcal{N}(p))/\Gamma_p$ be the canonical projector:

$$\varphi_i\Big|_{\mathcal{N}(p)} = \theta_p \circ \pi_p \,. \tag{1.207}$$

[78] The orbifold $\mathcal{O}$, being a manifold, is thus a paracompact Hausdorff topological space.

Any couple of points on $\mathcal{O}$ (as a topological space) $p\in U_i$ and $q\in U_j$ are equivalent if and only if $\varphi_i(p)=\varphi_j(q)$, which means that a homeomorphism $\psi_{i,j}:\mathcal{N}(p)\leftrightarrow\mathcal{N}(q)$ exists such that:

$$\varphi_i\Big|_{\mathcal{N}(p)}=\varphi_j\Big|_{\mathcal{N}(q)}\circ\psi_{i,j}.$$

A differentialble, smooth or analytic orbifold is a manifold carrying a C^r structure ($r\in\mathbb{N}\cup\{\infty,\omega\}$) compatible with the finitely-generated groups. Of course a manifold is always an orbifold, though of trivial nature — whenever Γ_p are trivial groups $\forall p\in\mathcal{O}$. The most tractable orbifold structures are those whose Γ_p are isomorphic to either each other or a finite small groups' set, for [almost] every $p\in\mathcal{O}$.

As far as orbifolds' coverings are concerned, of particular importance are their *partial* self-coverings $c\in\mathrm{C}^0(\mathcal{O}_1;\mathcal{O})$: in this case a sub-orbifolds' sequence $(\mathcal{O}_n)_{n\in\mathbb{N}}\in\wp(\mathcal{O})^{\mathbb{N}}$ can be constructed by induction such that $\mathring{\mathcal{O}}\supseteq\mathring{\mathcal{O}}_n\supseteq\mathring{\mathcal{O}}_{n+1}$ $\forall n\in\mathbb{N}$ and the partial self-coverings $c^n\in\mathrm{C}^0(\mathcal{O}_n;\mathcal{O})$ where, with a notation's abuse, c^n is properly the composition of $c\big|_{\mathcal{O}_n}$ with itself n times, that is

$$c^n ::= c\big|_{\mathcal{O}_n}\circ c\big|_{\mathcal{O}_{n-1}}\circ\cdots\circ c\big|_{\mathcal{O}_1}:\mathcal{O}_n\longrightarrow\mathcal{O}\,. \tag{1.208}$$

Such a sequence $(c^n)_{\mathbb{N}}$ is then called a (partial) self-covering of $\mathcal{O}$.

> Let $\mathfrak{C}\subset[0,1]$ be the middle-3rd ***Cantor set***, [9]–§1.8, [14], see **Figure 15** ;
> it has several equivalent definitions:
>
> - The subset of $[0,1]$ whose real numbers have *ternary* representation (i.e. in base[79] 3) lacking digit 1_3, i.e. $\mathfrak{C}\cong\partial\{0_3,2_3\}^{\intercal}$; cfr. §1.4.1:
>
> $$x\in\mathfrak{C}\quad\Leftrightarrow\quad x=\sum_{k=1}^{\infty}3^{-k}\tau_k\equiv(0.\tau_1\tau_2\tau_3\cdots)_3\,,\quad\tau_k\in\{0,2\}\ \forall k\in\mathbb{N}.$$

[79] $\mathbb{R}$ is a specific limit space of the alphabet $\mathbb{Z}_b$, $b\in\mathbb{N}\backslash\{1\}$, i.e. is represented by *bi*-infinite words $(\cdots\beta_1\beta_0\beta_{-1}\cdots)_b:\equiv(\beta_n)_{n\in\mathbb{Z}}\in\{0,1,\ldots,b-1\}^{\mathbb{Z}}$, [83]. Any number $x\in\mathbb{R}$ **in base** b is the bi-infinite word $(\cdots\beta_1\beta_0\beta_{-1}\cdots)_b$ such that $x=\sum_{\mathbb{Z}}\beta_k b^k$ and $\beta_k\in\mathbb{Z}_b$, $\forall k\in\mathbb{Z}$.

- The attractor (i.e. the limit set) of the IFS $\left\{[0,1];\frac{x}{3},\frac{x+2}{3}\right\}$.

$\mathcal{C}$ is a perfect, uncountable, compact, complete, totally-disconnected, isolated, nowhere-dense metric space[80] of zero Lebesgue measure. $\mathbb{Z}_2$ acts on $[0,1]$ by swapping every digit 1_3 with 1_2 in the next place of its binary representation, and has $\mathcal{C}$ as invariant orbispace, cfr. §1.1.5:

$$\mathbb{Z}_2[0,1]=\mathcal{C}.$$

It is easy to prove that this gives $\mathcal{C}$ an orbifold structure; consider the surjective map $\mathsf{c}\in C^0(\mathcal{C};[0,1])$ defined as

$$\mathsf{c}\left(\sum_{k=1}^{\infty}\frac{\tau_k}{3^k}\right)\overset{\text{def}}{=}\sum_{k=1}^{\infty}\frac{\tau_k}{2^{k+1}},$$

e.g. $\mathsf{c}(0.20200222\cdots_3)=0.010100111\cdots_2$. It is the limit $\lim_n\mathsf{c}_n(x)$ of partial self-coverings and is, itself, a self-covering of $[0,1]$ representing the shift map for the dynamical system $([0,1],\mathsf{c})$ — cfr. §1.4.3.

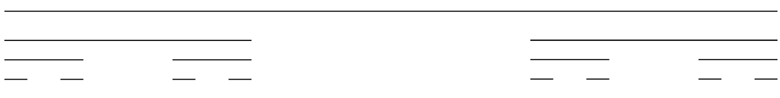

Figure 15 Initiator and first four prefractals of the middle-3rd Cantor set $\mathcal{C}$.

A path on a orbifold $\mathcal{O}$ is a finite set of couples $\{(\gamma_j,g_j)\}_{0\leq j<N}$, where $\gamma_j\in C^0([t_j,t_{j+1}];\mathcal{O})$ are paths on $\mathcal{O}$ and $g_j\in\Gamma_{\gamma_j(t_j)}$ such that $g_j\bullet\gamma_j(t_{j+1})=\gamma_{j+1}(t_{j+1})$, $0\leq j<N$; furthermore $\{[t_j,t_{j+1}]\}_{0\leq j<N}$ is a partition of the unit interval such that $[0,1]=\bigcup_j[t_j,t_{j+1}]$ with $0=t_0<t_1<\ldots<t_N=1$. Such paths are represented by 'mixed words' (of paths and group elements) such as either

$$g_1\gamma_1g_2\gamma_2\cdots g_N\gamma_N \quad\text{or}\quad \gamma_Ng_N\gamma_{N-1}g_N\cdots\ \gamma_1g_1 \tag{1.209}$$

[80] Every non-empty, perfect, totally disconnected, compact metric space is homeomorphic to $\mathcal{C}$ and called a *Cantor-like set* (that is why $\mathcal{C}$ is **the** middle-3rd *Cantor set*). Metric spaces whose Hausdorff dimension, [39], [9]–§1.4, is less than 1 are called Cantor dusts due to similar properties.

(depending whether groups $\{\Gamma_{\gamma_j(t_j)}\}_{0\leq j<N}$ act on either the left or the right).

> The 2-torus $\mathbb{T}^2$ is homeomorphic to the additive quotient group $(\mathbb{R}^2/\mathbb{Z}^2)(+)$, where a couple $(x,y)\in\mathbb{R}^2$ corresponds to the couple $(\mathrm{frac}\,x, \mathrm{frac}\,y) \equiv (x \bmod 1, y \bmod 1)\in[0,1[^2$ via the trivial equivalence relation/projection. A path on the orbifold $\mathbb{R}^2/\mathbb{Z}^2$ is, for example, a finite set of curves in $[0,1[^2$ such that their intersections with the boundarying square $\partial[0,1[^2$ are 'glued together' (i.e. it represents a continuous curve in $\mathbb{T}^2$).
>
> $\mathbb{R}^2/\mathbb{Z}^2$, as such an orbifold, is the phase space of the famous *Arnold*'s cat dynamical system, [7], [8].

Two paths in $\mathcal{O}$ are equivalent if one can pass from a representation to another by either taking homeotopic curves to $\{\gamma_j\}_{0\leq j<M}$ such that each ending and starting points of them coincides and adding a new element (γ'_j,g'_j) "in-between" (i.e. where $t'_j\in]t_j,t_{j+1}[$). This way the fundamental groups $\pi_1(\mathcal{O},x)$ of points $x\in\mathcal{O}$ are defined accordingly and a proper topology/homotopy theory can be formulated on orbifolds.

2.3.3. Monodromy groups

Let $c\in\mathrm{C}^0(C;X)$ be a cover of a connected and path-connected topological space X. In general, for very loop $\gamma\in\mathrm{C}^0([0,1];X)$, i.e. every continuous, closed curve in X:

$$(\gamma\uparrow c)(1) \neq (\gamma\uparrow c)(0), \tag{1.210}$$

i.e. starting from any point $x_0=\gamma(0)\in X$, the loop's lift $\gamma\uparrow c$ on the covering space is not closed anymore, but still $(\gamma\uparrow c)(1)\in c^{-1}(\gamma_0)$, that is the endpoint belongs to the same fibre. The lifted path $\gamma\uparrow c\in\mathrm{C}^0([0,1];C)$ induces a permutation of sheets which is proven to be just depending on the homotopy class $[\![\gamma]\!]\in\pi_1(X,x_0)$: it is a right action $F_{x_0}\in\mathrm{Hom}\big(\mathrm{Sym}\,c^{-1}(x_0),\pi_1(X,x_0)\big)$ on the sheets ramifying from x_0, called monodromy of γ (or of $[\![\gamma]\!]$ in x_0). The images of $\pi_1(X,x)$ under F_x,

$$\mathrm{MG}_x X \stackrel{\text{def}}{=} F_x(\pi_1(X,x)) \leq \operatorname{Sym} c^{-1}(x), \qquad \forall x \in X, \tag{1.211}$$

are the ***monodromy groups*** of X, whose neutral element is the '**null-homotopic** monodromy' action '$\circlearrowleft$'. If c is a p-cover, for some $p \in \mathbb{N}$, then $\mathrm{MG}_x X \lessapprox \mathbb{S}_p$, i.e. it is some permutations' group of p objects.

Lemma

For every universal cover $c \in \mathrm{C}^0(C;X)$ defined via a discrete group $G(\circ)$'s action, since C is simply connected (thus $\pi_1(C)$ is trivial),

$$\pi_1(X) \cong G.$$

It is worth to note that two group actions on the fibres of a topological space have been introduced: the deck transformation group (§2.3.1) acting on the left, and the monodromy grup acting on the right, i.e. for any base point $\forall [\![\gamma]\!] \in \pi_1(X,x_0)$:

$$f(F[\![\gamma]\!]) = f(F)[\![\gamma]\!], \qquad \forall f \in \operatorname{Deck}(C,X),\ \forall F \in \mathrm{MG}_x X.$$

It can be proven, in fact, that the deck transformation and the monodromy group of a path-connected and locally path-connected topological space are isomorphic — and both isomorphic to the fundamental group if the topological space is simply connected, cfr. the above Lemma and §2.3.1.

Monodromy groups re-appear, with a slight different flavour, in the theory of linear differential equations on Riemann surfaces, whose suggested references are [1] and [69].

Monodromy Theorem

If $c\in\mathrm{C}^0(C;X)$ is a regular cover with respect to $G(\circ)$, then the monodromy groups $\mathrm{Sym}\pi_1(X,x)$ are isomorphic to fibres' symmetric groups $\mathrm{Sym}\,c^{-1}(x)$ and[81]

$$\mathrm{MG}_x X\cong\frac{\pi_1(X,c(y))}{c_*\pi_1(C,y)},\qquad\forall y\in C.\tag{1.212}$$

(1.212) is easily proven since the map $c_*:\pi_1(C,y)\to\pi_1(X,c(y))$ is injective $\forall y\in C$. Its "denominator" group is the kernel of the monodromy action $F_x\in\mathrm{MG}_xX$, which will be indicated as $\mathrm{Ker}F_x$.

If $\mathcal{R}$ is a compact, connected g-genus surface and admits an homotopy basis $\{[\![\omega_i]\!]_w\}_{1\leq i\leq g}$ on $w\in\mathcal{R}$, the permutations $\{F_w([\![\omega_i]\!])\}_{1\leq i\leq g}$ are called fundamental monodromies of $\mathcal{R}$ in w. If $\mathcal{R}$ has p sheets, the monodromy action is often considered to be just the group homomorphism $F_w:\pi_1(\mathcal{R},w)\to\mathbb{S}_p$. More on the monodromy groups of Riemann surfaces, with examples, will be discussed in §2.4.5.

Ramified coverings and monodromy groups are used to compute the topology of certain spaces (like Riemann surfaces and, more generally, complex manifolds); below are some classic examples:

---

[81] Let X and Y be two topological spaces, $x\in X$ and $f\in\mathrm{C}^0(X;Y)$; then a continuous map $f_*\in\mathrm{Hom}(\pi_1(X,x),\pi_1(Y,f(x)))$ is defined accordingly to be $f_*([\![\gamma]\!]_x\circ[\![\gamma']\!]_x)=[\![\gamma\uparrow f]\!]_{f(x)}\circ[\![\gamma'\uparrow f]\!]_{f(x)}$.

- $\pi_1(\mathbb{S}^1)\cong\mathbb{Z}$ and the map $\mathbb{R}\to\mathbb{S}^1$ defined, when $\mathbb{S}^1\asymp\partial\mathbb{B}_1(0)\subset\mathbb{C}$, as $x\mapsto e^{ix}$ is a cover of $\mathbb{S}^1$ under the action of the discrete group $\mathbb{Z}(+)$.
- $\pi_1(\mathbb{RP}_n)\cong\mathbb{Z}_2$ for $n\in\mathbb{N}\backslash\{1\}$, and $\mathbb{Z}_2$ acts as $[\![\mathbf{x}]\!]\mapsto[\![-\mathbf{x}]\!]$ in $\mathbb{S}^n\subset\mathbb{R}^{n+1}$ (since $\mathbb{S}^n$ is simply connected for $n>1$);
- $\pi_1(\mathbb{T}^n)\cong\widehat{\mathbb{T}^n}\cong\mathbb{Z}^n$, cfr. example of §1.2.4;
- more generally $\pi_1(\mathrm{SO}_n(\mathbb{R}))\cong\mathbb{Z}_2$ and since $\mathrm{U}(n)\asymp\mathbb{S}^1\times\mathrm{SU}(n)$, then $\pi_1(\mathrm{U}(n))=\pi_1(\mathbb{S}^1)+\pi_1(\mathrm{SU}(n))\cong\mathbb{Z}$ (since $\pi_3(\mathrm{SU}(n))\cong\mathbb{Z}$ is the only non-trivial homotopy group among special unitary groups for $n>1$, [37] and §1.1.1)

2.3.4. The Iterated Monodromy Group

From considerations in the previous paragraphs the monodromy groups of an orbifold $\mathcal{O}$ —whose quotient group will be considered to be one, namely $\Gamma(\circ)$— can be defined. When such an orbifold admits a self-covering $(c^n)_{\mathbb{N}}$ though, the monodromy action shows some degree of (purely topological) self-similarity; for this reason, instead of computing several monodromy groups for every (equivalence class of) points in the orbifold, one group can be defined to incorporate both algebraic and topological properties of the orbifold structure within itself, the ***iterated monodromy group*** (abbreviated in IMG) [82], [83]. Let $(\mathcal{O}_n)_{\mathbb{N}}$ be the inclusive sequence of self-covering orbifolds, and $\mathrm{Ker}F_x^n\leq\pi_1(\mathcal{O}_n,x)$ be the monodromy actions' kernels on the n^{th} partial self-covering $c^n\colon\mathcal{O}_n\to\mathcal{O}$ (in $x\in\mathcal{O}_n$ and $\forall n\in\mathbb{N}$), then the IMG of $\mathcal{O}$ is the quotient group:

$$\mathrm{IMG}\,\mathcal{O}\stackrel{\text{def}}{=}\frac{\pi_1(\mathcal{O})}{\bigcap\limits_{n\in\mathbb{N}}\mathrm{Ker}\,F^n}\cong\varprojlim_n\frac{\pi_1(\mathcal{O})}{\mathrm{Ker}\,F^n}\tag{1.213}$$

The isomorphism (1.213) defining IMG $\mathcal{O}$'s quotients as inverse limits is proven via some technicalities using the inverse limit's definition given in §1.1.7 with respect to an equivalence relation on $\mathcal{O}$'s paths (as they were defined in §2.3.2). The topological closure of the iterated monodromy group —which is in fact proven to be a Hausdorff, compact, totally disconnected topological IMG $\mathcal{O}$-dense subgroup (cfr. §1.4.1)— is called the ***profinite*** IMG, is indicated as $\overline{\mathrm{IMG}}\mathcal{O} \stackrel{\mathrm{def}}{=} \pi_1(\mathcal{O}) \Big/ \bigcap_{n\in\mathbb{N}} \mathrm{Ker}\, F^n$ and IMG $\mathcal{O}$ is, by definition, dense in it.

Of course, if $\mathcal{O}$ is simply connected too, then $\pi_1(\mathcal{O}/\mathrm{IMG}\,\mathcal{O}) \cong \mathrm{IMG}\,\mathcal{O}$ by (1.51), i.e. every iterated monodromy actions is in a "continuously 1-to-1" correspondence with the homotopy classes of $\mathcal{O}$ which preserve the monodromy at each level (and are by this reason called "iteratively **isomonodromic**").

Trivially, IMGs of topologically conjugate self-coverings (and mutually homeomorphic orbifolds) are isomorphic with each other, for this reason they are often referred to as depending from the starting partial self-covering $c{:}\mathcal{O}_1 \to \mathcal{O}$, i.e.:

$$\begin{cases} \mathrm{IMG}\, c :\equiv \mathrm{IMG}\, \mathcal{O} \\ \overline{\mathrm{IMG}}\, c :\equiv \overline{\mathrm{IMG}}\, \mathcal{O} \end{cases}$$

A [d-fold] covering of a path-connected and locally path-connected topological manifold $\mathcal{M}$ by a path-connected open submanifold $\mathcal{M}_1 \subseteq \mathcal{M}$ is simply a covering of its, $c \in \mathrm{C}^0(\mathcal{M}_1;\mathcal{M})$, such that, $\forall x \in \mathcal{M}$, $c^{-1}(\mathcal{N}(x))$ is a disjoint union of [exactly $d \in \mathbb{N}$] open subsets of $\mathcal{M}_1$ which are homeomorphically mapped *onto* $\mathcal{N}(x)$ by c. By this property the covering can be iterated arbitrarily $n \in \mathbb{N}$ times as $c^n{:}\mathcal{M}_n \to \mathcal{M}$ [such that c^n is a d^n-fold partial self-covering], and the IMG $\mathcal{M} \equiv$ IMG c be considered.

A [d-regular] rooted tree $\mathrm{T}_c(x)$ can be constructed out of every point $x \in \mathcal{M}$, its root being x and its n^{th} level being $c^{-n}(x)$, $\forall n \in \mathbb{N}_0$:

$$\mathrm{T}_c(x) = \bigsqcup_{n=0}^{\infty} c^{-n}(x) \tag{1.214}$$

(this way every node $y \in c^{-(n-1)}(x)$ is adjacent to the nodes in $c^{-n}(x)$). Every monodromy action F_o^n associated to loops $[\![\gamma]\!] \in \pi_1(\mathcal{M},x)$ starting and ending in

$x \in \mathcal{M}$ pulls back to tree automorphisms of $\mathrm{T}_c(x)$ once the γ's, lifted under c, $^{-n}$ become permutations of $\mathrm{T}_c(x)$'s n^{th}-level; for this reason the n^{th} monodromy actions all belong to $\mathrm{Aut} \bigsqcup_{k=0}^{n} c^{-k}(x)$. More generally, [83], if $\gamma(1) \in c^{-n}(x)$ then both $c(\gamma(1)) \in c^{-(n-1)}(x)$ and $c(\gamma(1)) \in c^{-(n-1)}(x)$, so the action of $\mathrm{IMG}\mathcal{M}$ on $\mathrm{T}_c(x)$ is faithful. See Figure 16 for a graphical representation of such regular rooted trees.

> Let $f(z) = z^2 \in \mathbb{C}[z]$, which is a 2-fold self-covering of $\mathbb{C}^*$, and $\gamma = \partial \mathbb{B}_1(0) \in \mathrm{Z}_1(\mathbb{C}^*)$ the loop $[0,1) \to \mathbb{C}^*$ be defined as $t \mapsto e^{2\pi \imath t}$. The n^{th}-level of f's binary rooted tree is $f^{-n}(1) = \ker\left(z^{2^n} - 1\right)$ — the set of all the $(2^n)^{\text{th}}$ roots of unity, $\forall n \in \mathbb{N}_0$. The monodromy action for f^{-n} is thus a cyclic permutation of such roots,
>
> $$F_1^n\left(e^{\frac{2\pi \imath m}{2^n}}\right) = e^{\frac{2\pi \imath (m+1)}{2^n}}, \qquad 0 \le m < 2^n,$$
>
> whereas $\mathrm{MG}\, f^{-n}(\mathbb{C}^*) \cong \mathbb{Z}_{2^n}$, the well-known Galois field, cfr. §1.1.3. It is clear that the action of $\pi_1(\mathbb{C}^*)$ on $\mathrm{T}_f(1)$ is faithful and
>
> $$\mathrm{IMG}\, \mathbb{C}^* = \frac{\pi_1(\mathbb{C}^*)}{\bigcap^{\infty} \mathbb{1}} \cong \mathbb{Z}. \tag{1.215}$$

By the self-covering property of c, all the iterated lifts lying over γ, pulled back on $\mathrm{T}_c(x)$, satisfy the identity $\gamma \uparrow c^n = c(\gamma) \uparrow c^{n-1}$.

The monodromy actions computed on whole sub-trees —like for example the pre-images of F_γ^n — are precisely the iterated monodromy actions of c.

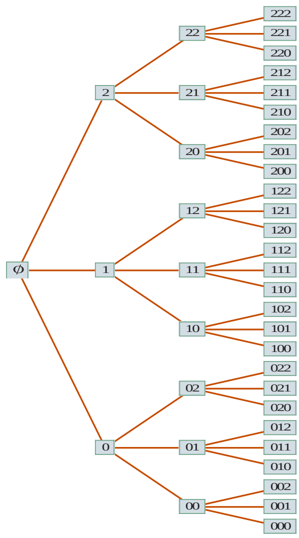

Figure 16 Example of a 3rd-order ternary rooted tree $\mathbb{Z}_3 \sqcup \mathbb{Z}_3^2 \sqcup \mathbb{Z}_3^3$ with its words represented on the alphabet $\{0,1,2\}$.

> The limit space $\mathcal{J}_G$ is a compact, metrizable invariant[82] orbiset of G; it does not depend on the choice of the contracting action and $\dim \mathcal{J}_G < \mathrm{Card}\,\mathcal{N}_G$. It is [locally] connected if G is finitely-generated and acts [level-]transitively; it is both path- and locally path-connected if G acts recurrently.

From the previous example it is clear that the IMG contains, naturally, all the monodromy actions of any finite-level rooted tree, thus it is quite a “universal” algebraic tool from this point of view. The fundamental relation between IMGs and their associated limit sets is given by the following Theorem, [82].

[82] The limit set is a G-invariant set, i.e. $G\mathcal{J}_G = G$ (that is why it is also a self-covering, cfr. §2.3.2).

IMG-Theorem

The limit space $\mathcal{J}_{\mathrm{IMG}c}$ induced by the iterated monodromy action of an orbifolds' partial self-covering $c{:}\mathcal{O}_1 \to \mathcal{O}$ is homeomorphic to the orbifold $\mathcal{O}/\mathrm{IMG}\,\mathcal{O}$ — the quotient of the orbifold itself by its IMG.

In the case of an infinite p-rooted tree, then $\mathcal{J}_{\mathrm{IMG}\mathcal{A}_\infty} \asymp \overline{\mathbb{Z}_p^{\mathsf{T}}} \big/ \mathrm{IMG}\mathcal{A}_\infty$.

2.4 Riemann Surfaces

A Riemann surface is a complex one-dimensional Riemann manifold, i.e. it is a smooth complex *curve*; all the results exposed in §2.1.2 are thus valid for Riemann surfaces as the $n=1$ case, which is the simplest one since we're only dealing with complex 0-, 1- and 2-forms. As most advanced results regading higher-dimensional manifolds are either too complicated to be shown here or lacking, the ones to be used in this thesis will be directly shown for the Riemann surfaces only.

The reason why Riemann surfaces are often considered a topic apart in Mathematics is that they can be approached more easily from several disciplines than their higher-dimensional analogues, which less, little or nothing is known of yet. They are considered complexified surfaces in Differential Geometry; algebraic curves in Algebraic Geometry; sheaves in Differential Topology; one-dimensional ramified coverings in Topology; multi-valued functions in Calculus. This also creates some confusion of "Plato-Aritsotelic taste" as regards their treatment in different text books: in Topology texts a multi-valued function locally associated to a Riemann surface is either its germs' Lie algebra or its sheaf over one of its points; in Algebraic Geometry texts it is the locus of zeroes of a two-variables irreducible polynomial, i.e. the subset of points $(w,z)\in\mathbb{C}^2$ such that $P(w,z)=0$, for $P\in\mathbb{C}[w,z]$; and so on.

Where no ambiguity is present, points of a Riemann surface will be indicated as either points of an abstract manifold (like *p*), couples of complex numbers (like

(w,z), if $\mathcal{R}\subseteq\mathbb{C}^2$ such that $\mathcal{R}=\ker P$, cfr. above) or complex numbers themselves (like z, independent variable of explicit multivalud equation $w=f(z)$).

2.4.1. Local theory: Calculus and Complex Dynamics

Just to make the point on the differential calculus on a Riemann surface $\mathcal{R}$ of complexified local coordinates $(z,\overline{z})$ —which is always Kähler— the differential operators act this way on 0-, 1- and 2-forms f, $\phi=\varphi\mathrm{d}z+\psi\mathrm{d}\overline{z}$ and $\varpi=\omega\mathrm{d}z\wedge\mathrm{d}\overline{z}$ respectively, all of class C^2:

$$\begin{cases}\partial = \dfrac{\partial}{\partial z}\mathrm{d}z\\ \overline{\partial} = \dfrac{\partial}{\partial \overline{z}}\mathrm{d}\overline{z}\end{cases} \quad\Rightarrow\quad \begin{cases}\mathrm{d}f \equiv (\partial+\overline{\partial})f = \dfrac{\partial f}{\partial z}\mathrm{d}z + \dfrac{\partial f}{\partial \overline{z}}\mathrm{d}\overline{z}\\ \mathrm{d}\phi = \left(\dfrac{\partial\psi}{\partial z} - \dfrac{\partial\varphi}{\partial \overline{z}}\right)\mathrm{d}z\wedge\mathrm{d}\overline{z}\end{cases}$$

$$\begin{cases}\star f = f\,\mathrm{d}z\wedge\mathrm{d}\overline{z}\\ \star\phi = -\imath\psi\mathrm{d}z + \imath\varphi\mathrm{d}\overline{z}\\ \star\varpi = \omega\end{cases}$$

$$\Delta f\equiv\Delta_{\partial}f\equiv\Delta_{\overline{\partial}}f\equiv\delta f\equiv\begin{cases}-2\imath\overline{\partial}\partial f = -2\imath\frac{\partial^2 f}{\partial z\partial\overline{z}}\mathrm{d}z\wedge\mathrm{d}\overline{z}\\ \mathrm{d}\star\mathrm{d}f = \imath\left(\frac{\partial^2 f}{\partial z^2}+\frac{\partial^2 f}{\partial \overline{z}^2}\right)\mathrm{d}z\wedge\mathrm{d}\overline{z}\end{cases}$$

A q-form is harmonic if and only if it is closed and coclosed simltaneously, i.e. $\mathrm{H}^q(\mathcal{R})=\mathrm{Z}^{q,0}(\mathcal{R})\cap\mathrm{Z}^{0,q}(\mathcal{R})$; it is holomorphic only if it is of kind $\phi=\alpha+\imath\star\alpha$, $\alpha\in\mathrm{H}^q(\mathcal{R})$ or, equivalently, $\star\phi=-\imath\phi$, $\phi\in\mathrm{Z}^{q,0}(\mathcal{R})$. A 1-form ϕ is harmonic [holomorphic] if and only if, locally, $\phi=\mathrm{d}f$ with $f\in\mathrm{C}^2(\mathbb{C})$ being a harmonic [holomorphic] function.

Let on a local chart $U\subseteq\mathcal{R}$ be $\phi(\boldsymbol{p})=\varphi(z,\overline{z})\mathrm{d}z+\psi(z,\overline{z})\mathrm{d}\overline{z}$ and let $\gamma\in\mathrm{C}^0([a,b],\mathcal{R})$ be a path completely contained within the chart ($\gamma\subset U$); the integral of ϕ along γ is:

$$\langle\gamma|\phi\rangle = \int_{+\gamma}\phi \equiv \int_a^b\left(\varphi(\gamma(t),\overline{\gamma(t)})\dot{\gamma}(t)+\psi(\gamma(t),\overline{\gamma(t)})\overline{\dot{\gamma}(t)}\right)\mathrm{d}t\,.$$

Meromorphic differential forms are the most important tools to develop a local theory of Riemann surfaces: as happens for functions, only meromorphic 0- and 1-forms matter so, since local theorems for meromorphic functions (0-forms) were introduced in §2.1.3, **Abelian differentials** (i.e. meromorphic linear forms) will be briefly reviewed here.

Theorem

The Abelian differentials of a Riemann surface $\mathcal{R}$ are the elements of $\mathrm{Frac}\Lambda^{1,0}(\mathcal{R})$. On any local chart (U,φ) of $\mathcal{R}$ of local coordinate $z\in\varphi(U)$ every $\omega\in\mathrm{Frac}\Lambda^{1,0}(\mathcal{R})$ is the following 1-form of $\Lambda^1(\mathcal{K}(\varphi(U)))$, i.e. locally:

$$\omega\big|_U \equiv \omega(p(z)) = f(z)\mathrm{d}z, \tag{1.216}$$

where $f\in\mathcal{K}(\varphi(U))$ is a meromorphic function of $\varphi(U)\subseteq\mathbb{C}$.

From all these results it is clear that $(1,0)$-forms on Riemann surfaces are locally represented by complex 1-forms of kind $\omega(p)=f(z)\mathrm{d}z$, where f is either a continuous, holomorphic or meromorphic function. In the latter case, the *residue* of an Abelian differential at point $p_0\in\mathcal{R}$ which is coordinated by $z_0\in\mathbb{C}$ on the local chart is defined, along with its order, as[83]

$$\mathrm{Res}_\omega p_0 \stackrel{\text{def}}{=} \frac{\langle\gamma|\omega\rangle}{2\pi\imath} \equiv \frac{1}{2\pi\imath}\oint_{+\gamma}\omega = \mathrm{Res}_f z_0, \tag{1.217}$$

$$\mathrm{ord}_\omega p_0 \stackrel{\text{def}}{=} \left\{\begin{matrix} \mathrm{Res}_{\frac{\mathrm{d}\omega}{\omega}} p_0 \\ \Updownarrow \\ \mathrm{Res}_{\mathrm{D}\log f} z_0 \end{matrix}\right\} = \mathrm{ord}_f z_0,$$

83 The poles of the ***logarithmic*** *derivative* $\mathrm{D}\log f(z) \equiv f'(z)/f(z)$ are simple, coincide with zeroes and poles of $f(z)$ and their residues are $\mathrm{Res}_{\mathrm{D}\log f} z = \pm\mathrm{ord}_f z$, respectively, $\forall z\in\mathbb{C}$.

for every loop (1-cycle) $\gamma \in Z^1(\mathcal{R})$ winding once around p_0 and no other singularities of ω. The ***winding number*** of ω along the 1-cycle γ is instead the sum of all the poles' and branch points' orders enclosed by γ:

$$\begin{aligned}\operatorname{wind}_\omega \gamma &\overset{\text{def}}{=} \frac{\langle \gamma \,|\, \arg \omega \rangle}{2\pi} \equiv \frac{1}{2\pi} \oint_{+\gamma} \arg \omega = \frac{1}{2\pi} \oint_{+\gamma} \arg f(z) \mathrm{d}z \\ &\equiv \frac{\langle \gamma \,|\, \frac{\mathrm{d}\omega}{\omega} \rangle}{2\pi\imath} \equiv \frac{1}{2\pi\imath} \oint_{+\gamma} \frac{\mathrm{d}\omega}{\omega} = \frac{1}{2\pi\imath} \oint_{+\gamma} \mathrm{D} \log f(z) \mathrm{d}z.\end{aligned} \tag{1.218}$$

Theorem — Argument's Principle

Let $\mathcal{R}$ be a non-compact Riemann surface and $\omega \in \mathrm{Frac}\Lambda^{1,0}(\mathcal{R})$ a meromorphic form without zeroes and poles in $\partial\mathcal{R}$, then the winding number of ω along $\partial\mathcal{R}$ is exactly the difference between its zeroes and poles in $\mathcal{R}$ (counted with their respective multiplicities), i.e.:

$$\operatorname{wind}_\omega \partial\mathcal{R} = \sum_{p \in \mathcal{R}} (\operatorname{mul}_\omega p - \operatorname{ord}_\omega p)\,. \tag{1.219}$$

Lemma

If $f \in C^\omega(\mathcal{R};\mathcal{S})$ is an analytic map between Riemann surfaces, then $\forall \omega \in \mathrm{Frac}\Lambda^{1,0}(\mathcal{S})$ and $\forall p \in \mathcal{R}$.

$$\operatorname{ord}_{f_*\omega} p := (1 + \operatorname{ord}_\omega f(p)) \operatorname{mul}_f p - 1\,. \tag{1.220}$$

Equation (1.217) is proven to show that, if z is a local coordinate such that $z(p_0) = z_0$, then $\operatorname{ord}_\omega p_0 = \operatorname{ord}_f z_0$, f has the same type of singularity of ω in p_0 and its residue coincides with that of meromorphic function f in z_0; thus a Residue Theorem (1.222) cfr. [9]–§3.3, can be easily formulated for Riemann surfaces too, with consequences for their Algebraic Topology (see Corollary below).

Figure 17 shows the simplest, non-trivial example of Riemann surface: that of the square-root function $z \mapsto \sqrt{z}$, which is homeomorphic to the simple torus $\mathbb{T}_1$.

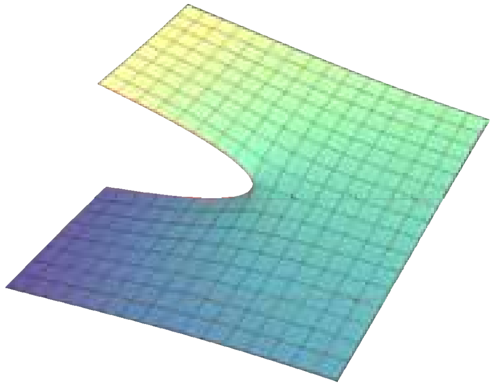

Figure 17 Plot of $\operatorname{Im}\sqrt{z}$ to show one of two sheets of the square-root function. The canonical branch cut $›0,\infty‹ = \mathbb{R}_{-}$ is represented by the red curve.

Lemma

For every finite number of distinct points of a Riemann surface $p_1, p_2, \ldots, p_N \in \mathcal{R}$ there exists an Abelian differential $\omega \in \operatorname{Frac}\Lambda^{1,0}(\mathcal{R})$ which is holomorphic in $\mathcal{R} \setminus \{p_j\}_{1 \le j \le N}$, has simple poles in $\{p_j\}_{1 \le j \le N}$ only, i.e. $\operatorname{ord}_\omega p_j = -1$, for $1 \le j \le N$, and:

$$\sum_{j=1}^{N} \operatorname{Res}_\omega p_j = 0\,. \tag{1.221}$$

Residue Theorem

Every Abelian differential $\omega \in \operatorname{Frac}\Lambda^{1,0}(\mathcal{R})$ satisfies

$$\sum_{p \in \mathcal{R}} \operatorname{Res}_\omega p = 0 \qquad \text{and} \qquad \sum_{p \in \mathcal{R}} \operatorname{ord}_\omega p = 0\,. \tag{1.222}$$

Corollary

Every Riemann surface is triangulable.

Sketch of proof: Let $f \in \mathcal{K}(\mathcal{R})$ a non-constant meromorphic function (which exists by Residue Theorem) and $B \subset \mathcal{R}$ its ramifications points' set; let a simplicial complex Σ

triangulating $\tilde{\mathbb{C}}$ be built, such that $B = \mathrm{smp}_0\Sigma$ (i.e. its vertices are the ramification points of f) and $f\big|_{f^{-1}(\sigma)^\circ}$ is injective $\forall\sigma\in\mathrm{smp}_1\Sigma$. Then $f_\blacktriangle^{-1}(\Sigma)$ is a triangulation of $\mathcal{R}$. See [9]–§2.1 and [81] for terminology and properties of simplicial complexes. □

Abelian differentials thus admit Laurent series' representations, which are usual complex Laurent series (with all of their properties, [42], [72]) once coordinate systems are fixed: in particular the Laurent series of $\alpha\in\mathrm{Frac}\Lambda^{1,0}(\mathcal{R})$ around $p_0\in\mathcal{R}$ is a formal Laurent series of $\Lambda^{1,0}(\mathcal{R})[\![p_0]\!]$ (whose formal residue defined in §1.3.3 coincides with $\mathrm{Res}_\omega p_0$). Such series has no negative-degree terms if p_0 is either a removable singularity or a regular point.

With respect to the Zariski topology, [71], the sets $\ker P$ for any irreducible $P\in\mathbb{C}[w,z]$, provide closed-set topologies for the Riemann surfaces defined by implicit equations $P(w,z)=0$. Locally, for every $w\in\tilde{\mathbb{C}}$ but a finite set, cfr. [72], these equations can be explicited as $w=f(z)$, with $f\in\mathcal{O}(\tilde{\mathbb{C}})$ and $f(\tilde{\mathbb{C}})$ representing one sheet. In this context the couple of complex coordinates (w,z) is equivalent to local-coordinates $(z,\overline{z})$.

Let $B=\left\{(w,z)\in\mathbb{C}^2 \mid \nexists P^{-1}\right\}$ and for every different $(w,z),(w,z')\in\mathbb{C}^2$ $[(w,z),(w',z)\in\mathbb{C}^2]$, w $[z]$ is a ramification point, with z and z' $[w$ and $w']$ its corresponding ***branch** points*. The multiplicity of a branch point z_0 [the branching number of ramification point w_0], is the number of ramification [branch] points corresponding to it. For example in the case of explicit equation $w=f(z)$, let $\mathrm{b}_f(z_0):=\mathrm{Card}\, f^{-1}(z_0)$. If $\mathrm{b}_f(z_0)>1$ is either finite or infinite z_0 is called an **algebraic** or a **logarithmic** branch point, respectively, the latter of which is sampled in Figure 18

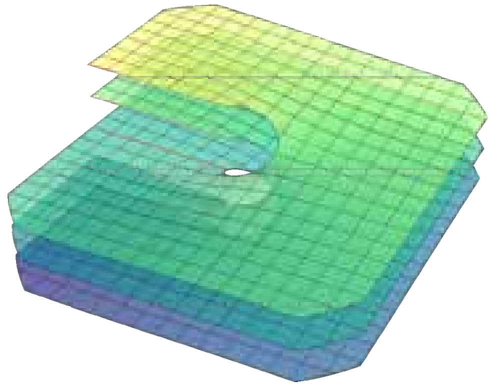

Figure 18 Plot of $\operatorname{Im}\log z \equiv \arg\log z$ to show three of the logarithm's infinite sheets. The canonical branch cut $]0,\infty[=\mathbb{R}_-$ is represented by the red curve (the logarithmic branch point $z=0$ was punctured to avoid showing its singular behaviour).

> Consider polynomial $w^7 - z(z-1)^2 \in \mathbb{C}[w,z]$ and the Riemann surface defined by implicit equation $w^7 - z(z-1)^2 = 0$. It is clear that, locally, it can be explicited to a single-valued equation $w = f(z) = \sqrt[7]{z(z-1)^2}$ and the surface has three branch points: 0, 1 and ∞, all *algebraic* of multiplicity 7, i.e. $\mathrm{b}_f(0) = \mathrm{b}_f(1) = \mathrm{b}_f(\infty) = 7$. The ramification points are instead 0 and ∞, whose branching numbers are $\mathrm{b}_{f^{-1}}(0) = \mathrm{b}_{f^{-1}}(\infty) = 3$ (since 1 is a double root).

Let $Q_1, Q_2, \ldots, Q_n \in \mathbb{C}^n[z]$ mutually co-prime polynomials, i.e. with $\mathrm{GCD}_{1\leq j\leq n}(Q_j) = 1$; then $P(w,z) = \sum_k w^k Q_k(z) \in \mathbb{C}^n[w,z]$ is an **irreducible** Weierstrass polynomial (cfr. §2.1.3) and consider the n-sheeted Riemann surface $\mathcal{R} \subset \mathbb{C}^2$ defined as $P(w,z) = w^n + w^{n-1}Q_1(z) + w^{n-2}Q_2(z) + \ldots + w^2 Q_{n-2}(z) + wQ_{n-1}(z) + Q_n(z) = 0$.

The 2-variables polynomial $P(w,z)$ induces an n-cover of $\tilde{\mathbb{C}}$ made up by local inverse functions of P with respect to either w or z; let such local functions be $w = f(z)$ (being the explicit equations *locally* defining $\mathcal{R}$).

If its ramification locus $B\subset\mathcal{R}$ is removed, the restricted covering map $\mathcal{R}\backslash f^{-1}(B)\to\tilde{\mathbb{C}}\backslash f(B)$ is indeed a ramified cover of $\tilde{\mathbb{C}}$. B is the locus of points satisfying the following system of algebraic equations [37]:

$$\begin{cases} P(w,z)=0 \\ \dfrac{\partial P}{\partial w}=0 \end{cases} \tag{1.223}$$

Furthermore, such a Riemann surface does admit neither essential singularities nor logarithmic branch points, cfr. [72].

> If $\deg Q_k\leq k$, $1\leq k\leq n$, then (1.223) is a non-degenerate system of equations and $\mathcal{R}$ has exactly $n(n-1)$ ramification points and a ramified n-cover of $\tilde{\mathbb{C}}$ has been constructed.
>
> If $P(w,z)=w^n-Q(z)$ the branch points of $\mathcal{R}$ are the roots of Q (and possibly ∞, cfr. §3.1.1), whereas the ramification points are their preimages with respect to w: Q induces a ramified n-cover of $\tilde{\mathbb{C}}$.

Polynomials are very important holomorphic functions on Riemann surfaces, not only for their algebraic properties (like those discussed in §1), but also thanks to the

> **_Runge_ Approximation Theorem**
>
> Let $\mathcal{R}$ be a (non-compact) Riemann surface and $\Omega\subset\mathcal{R}$ an open subset such that Ω^c has no compact connected components. Then for every $f\in\mathcal{O}(\Omega)$ there exist a sequence $(f_n)_{n\in\mathbb{N}}\in\mathcal{O}(\mathcal{R})^{\mathbb{N}}$ such that f can be *uniformly* approximated by them (for example polynomials — thus by Taylor series), on every compact subset of Ω, which reads:
>
> $$f_n \underset{\Omega}{\rightrightarrows} f\,.$$

This Theorem is important for non-compact Riemann surfaces only (because it is trivial for compact ones), and even more since it can be proven that even non-compact Riemann surfaces have a countable topology, i.e. they are paracompact (Hausdorff, like any other topological manifold). On non-compact Riemann surfaces, in particular, the following holds

Theorem

Let $\mathcal{R}$ be a *non*-compact Riemann surface and $(z_n)_{\mathrm{N}}\in\mathcal{R}^{\mathrm{N}}$ accumulating [nowhere] on $\mathcal{R}$); $\forall(w_n)_{\mathrm{N}}\in\mathcal{R}^{\mathrm{N}}$ $\exists f\in\mathcal{O}(\mathcal{R})$ such that $f(z_n)=w_n$, $\forall n\in\mathbb{N}$.

The above Theorem can also be extended to sequences $(z_n)_{\mathrm{N}}\in\ell^1(\mathcal{R})$; in this case the limit point is a logarithmic branch point, as defined above and shown in **Figure 18** From the above considerations, the Monodromy Theorem (1.212) takes a nice "analytic" form for holomorphic functions of one variable (which is complementary to the Hartog's Theorem for the several variables' case, cfr. §2.1.3 and note [65]).

Monodromy Theorem — Analytic Continuation

Let $f\in\mathcal{O}(U)$ be a holomorphic function with $U\subset\Omega\subseteq\mathbb{C}$ both connected and Ω simply connected — i.e. $\mathrm{H}_0(U)\cong\mathrm{H}_0(\Omega)\cong\mathrm{H}_1(\Omega)=$ $=\{\circlearrowleft\}$. If f can be analytically continued along two different paths $\gamma,\gamma'\in\mathrm{C}^0([0,1];\Omega)$ to a point in Ω (i.e. $\gamma(0)=\gamma'(0)\in U$ and $\gamma(1)=$ $=\gamma'(1)\in\Omega\backslash U$) such that the region enclosed by γ and γ' contains, at most, *isolated* singularities, that analytic continuation is unique.

Isolated singularities (either removable, polar or essential, as further reviewed in §2.4.2) are "isolated" from the topological point of view. Non-isolated singularities $S\subseteq\tilde{\mathbb{C}}$ are instead points with a non-empty limit-point set, $\mathcal{D}S\neq\varnothing$ (where $\overline{S}=S\cup\mathcal{D}S$). Some authors consider **branch cut**s as non-isolated singularities for multi-valued functions; they are rather "arbitrary" for Riemann surfaces indeed (and not necessarily invariant by deck transformations). Non-isolated singularities for functions of one complex variable are usually of two kinds:

- ***Cluster points***— accumulation points of isolated singularities, such that *no* Laurent series exist in any of their neighbourhoods, [72].

> A typical case is when $(z_n)_{n\in\mathbb{N}}\in\ell^1(\tilde{\mathbb{C}})$ is a converging sequence such that $z_n\to z_\infty\in\tilde{\mathbb{C}}$ and $f(z)$ is synthesized to have polar singularities in $(z_n)_{\mathbb{N}}$; then z_∞ is a cluster point for f. For example consider the function $\tan(z^{-1})$, which is meromorphic with simple poles in $z_n=(\frac{\pi}{2}+n\pi)^{-1}, \forall n\in\mathbb{N}_0$; since $z_n\to 0$ every punctured disc in 0 contains an infinite number of singularities, so no Laurent series expansion is available for $\tan\frac{1}{z}$ around 0.

- ***Natural boundary***— any curve which functions can not be analytically continued around (or outside them if they are closed curves in $\tilde{\mathbb{C}}$);

More on natural boundaries for complex functions will be discussed in §5.3.1.

> Function $f\in\mathcal{O}(\mathbb{B}_1(0))$ shown in Figure 19 is defined via a Maclaurin series,[84] converging for $|z|<1$, and has $\partial\mathbb{B}_1(0)$ as natural boundary.
>
> $$f(z)=\sum_{n=0}^{\infty}z^{2^n}\in\mathbb{C}[\![z]\!].$$

[84] f is *not* analytic in $\partial\mathbb{B}_1(0)$ because $f(z^2)=f(z)-z$, hence it is singular on every $(2^n)^{\text{th}}$ roots of unity, $f^{-1}(\infty)=\bigcup_{n\in\mathbb{N}_0}\ker(z^{2^n}-1)$, $\forall n\in\mathbb{N}_0$, which are dense in $\partial\mathbb{B}_1(0)$ — also cfr. (1.215).

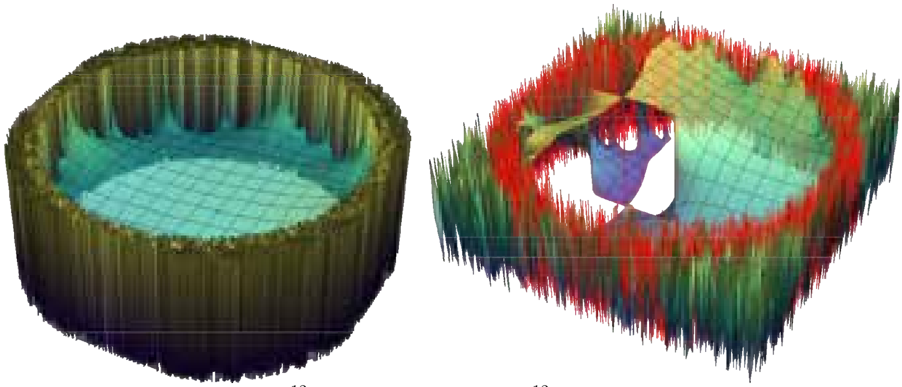

Figure 19 $\mathrm{Re}\sum_{n=0}^{12} z^{2^n}$ (left) and $\mathrm{Arg}\sum_{n=0}^{12} z^{2^n}$ (right) to show an approximation of the natural boundary $\partial\mathbb{B}_1(0)$ (red curve). The function increases approaching the 2^n roots of unity as faster as n is large, for $1\leq n\leq 12$ — cfr. note[84]

When complex functions are used to define self-coverings of a Riemann surface (like $\tilde{\mathbb{C}}$ or $\mathbb{C}$ or their punctured versions), i.e. the iterations $f^n(z)$ of some $f\in\mathcal{K}(U)$ for $U\subseteq\tilde{\mathbb{C}}$ are considered, several sets are defined in Complex Dynamics:

- the ***Fatou* set** of f as the set of points where the sequence $(f^n(z))_{n\in\mathbb{N}_0}$ admits a convergent sub-sequence (i.e. where the "tame" behaviour of $f^n(z)$ occurs when $n\to\infty$),

$$\mathrm{F}(f) \stackrel{\text{def}}{=} \left\{ z\in U \;\middle|\; \exists (n_k)_{k\in\mathbb{N}}\in\mathbb{N}^{\mathbb{N}}:\ \exists \lim_k f^{n_k}(z) \right\};$$

- the ***Julia* set** $\mathrm{J}(f):=\mathrm{F}(f)^c$, which is its complement, and where all the "wild" behaviour of $f^n(z)$ occurs when $n\to\infty$;

- the ***filled* Julia set** $\mathrm{FJ}(f)$, which is in general a subset of the Fatou set, where $(f^n(z))_{\mathbb{N}_0}$ remains bounded, that is in ℓ^∞:

$$\mathrm{FJ}(f) \stackrel{\text{def}}{=} \left\{ z\in U \;\middle|\; \lim_n |f^n(z)|<\infty \right\};$$

- the **escaping set**, $\mathrm{FJ}(f)^c$, where $(f^n(z))_{\mathbb{N}_0}$ actually diverges.

In any case when $f \in \mathcal{K}(\mathbb{C})$, both the Julia and the Fatou sets are *completely* invariant under f, i.e. $f(\mathrm{J}(f)) = f^{-1}(\mathrm{J}(f)) = \mathrm{J}(f)$ and $f(\mathrm{F}(f)) = f^{-1}(\mathrm{F}(f)) = \mathrm{F}(f)$. The Julia set is also perfect.[85] If $f \in (\mathcal{O}(\mathbb{C}) \setminus \mathbb{C}[z]) \cup \mathcal{K}(\mathbb{C})$ —i.e. f is *either* an entire transcendent *or* a meromorphic function— the Julia set is also the boundary of the escaping set and includes some repelling points, i.e. $\mathrm{J}(f) = \partial(\mathrm{FJ}(f)^{c})$ and $\mathrm{J}(f) \cap (\mathrm{FJ}(f)^{c}) \neq \varnothing$. Furthermore, the Julia set can be equivalently defined as:

- the smallest, closed set containing at least three points which is completely invariant under f;
- the closure of any repelling periodic points;
- for entire functions, $f \in \mathcal{O}(\mathbb{C})$, it is is the boundary of the escaping set, i.e. $\mathrm{J}(f) = \partial(\mathrm{FJ}(f)^{c})$;
- for polynomials, $f \in \mathbb{C}[z]$, it is also the boundary of the filled Julia set, i.e. $\mathrm{J}(f) = \partial \mathrm{FJ}(f)^{c}$.

A point $z_0 \in \tilde{\mathbb{C}}$ is said to be N-*periodic* under f ($N \in \mathbb{N}$) if $f^{N}(z_0) = z_0$ and $f^{N-1}(z_0) \neq z_0$ (trivially it is a fixed point if $N=1$); similartly, it is N-*preperiodic* if $f^{-N}(z_0) = z_0$. In a fashion much like was done for group actions[86] in §1.1.5, the forward [backward] **orbit** of z_0 under f is the (ordered) set $\mathrm{W}_f^{\pm}(z_0) := \{ f^{\pm n}(z_0) \}_{n \in \mathbb{N}_0}$ (which is finite if z_0 is [pre-]periodic).

The ***Lyapunov exponent*** of f with starting point $z_0 \in \tilde{\mathbb{C}}$ is the following number

$$\lambda_f(z_0) \overset{\text{def}}{=} \lim_{n} \frac{1}{n} \log |\mathrm{D} f^{n}(z_0)|, \tag{1.224}$$

which measures the rate of divergence for the orbits of starting points arbitrarily close to z_0. It is a powerful tool used in the dynamical systems' theory to measure

[85] In Topology a set $X \neq \varnothing$ is **perfect** if it is closed without isolated points, i.e. $\overline{X} = X$ and $\mathcal{I}X = \varnothing$.

[86] In fact in some cases (e.g. when f is rational), the action of f on $\mathbb{C}$ is "isomorphic" to a **group** action, so that algebraic language can be applied to prove interesting symmetry results.

local stability, and is the main indicator of *chaos*, §3.4, [84], [105], classically intended as "sensitive dependence on initial conditions." When $\lambda_f(z_0) > 1$ orbits starting close to each other tend to exponentially diverge, whereas if $\lambda_f(z_0) < 1$ they are bound to converge (the latter case called "Lyapunov stability"). The limiting case $\lambda_f(z_0) = 1$ is quite difficult to be generally characterized; as a mean behaviour the orbits associated to a unitary Lyapunov exponent remain close to each other, or at least periodically do that. When computed as the supremum on all the possible starting points $z_0 \in \tilde{\mathbb{C}}$, it is called the *topological* Lyapunov exponent of the map f.

> The topological Lyapunov exponent of $P \in \mathbb{C}^d[z]/\mathbb{C}$ with $\deg P = d$, *if* the corresponding Julia set $\mathrm{J}(P)$ is connected, is bounded as
>
> $$\sup_{z \in \tilde{\mathbb{C}}} \lambda_P(z) \leq \sup_{z \in \mathrm{J}(P)} \left| P'(z) \right| \leq 2 \log d \,.$$

When either f and f^{-1} are multi-valued (as in the cases throughout §3), they are better topologically handled as branched and ramified covers. In this case their orbits are defined as rooted trees as was done in §§1.4.1, 2.3.4, i.e. by means of disjoint unions (1.214), i.e.:

$$\mathrm{W}_f^{\pm}(z_0) \stackrel{\text{def}}{=} \mathrm{T}_{f^{\mp 1}}(z_0) = \bigsqcup_{n=0}^{\infty} f^{\pm n}(z_0) \,. \tag{1.225}$$

Forward and backward orbits are invariant in the sets defined above — e.g. orbits of points in the Fatou [[filled] Julia] set, remain inside the Fatou [[filled] Julia] set. In particular, forward orbits of "nearby" points in the Fatou set tend to stay "near" to each other; those of nearby points in the escaping set all asymptotically converge to ∞ by definition. Forward orbits of "nearby" points in the Julia set, instead, feature a *chaotic* behaviour. Orbits of nearby points in the Julia set *either* evolve along different finite periods (arbitrarily close periodic points always having different periods), *or* densely fill the Julia set itself in different ways, as is stated by the Theorem below.

Theorem

The Julia set of $f\in\mathcal{K}(\mathbb{C})$ is the closure of any of its forward orbits $\mathrm{J}(f)=\overline{\mathrm{W}_f^+(z)}$ of any of its *non*-periodic points $z\in\mathrm{J}(f)$.

Non-periodic points in the Julia set are thus also called ***quasi**-periodic* points; that is the name given to orbits of generic n-dimensional dynamical systems (both continuous and discrete) which densely fill a region of dimension $n-2$, cfr. [105] and [84].

When $f\in\mathcal{K}(\mathbb{C})\setminus\mathbb{C}\{z\}$ then it has infinitely many periodic points z_0, of any period $\forall N\in\mathbb{N}\setminus\{1\}$, although such N may not be *minimal* (i.e. there might also be $f^{N-1}(z_0)=z_0$), whereas it could have *no* fixed points. This is consistent with the fact that the transcendental meromorphic functions (on the Riemann plane) are a countably infinite extension field of rational ones, i.e. $[\mathcal{K}(\mathbb{C}):\mathbb{C}\{z\}]=\aleph_0$. Since $[\mathbb{C}\{z\}:\mathbb{C}]=\aleph_0$, then $[\mathcal{K}(\mathbb{C}):\mathbb{C}]=\mathfrak{c}$ — cfr. §§1.1.3, 1.1.10.

Local theory: Algebra

From the Lemma (1.221) of §2.4.1 it follows that meromorphic functions on a compact Riemann surface $\mathcal{R}$ are locally represented by Laurent series, i.e. $\mathrm{Frac}\Lambda^{1,0}(\mathcal{R})$ is locally isomorphic (as a field) to $\mathbb{C}[\![z]\!][w]$. The topology of $\mathcal{R}$ deeply influences the "size" of $\mathcal{K}(\mathcal{R})$ since there always exists a maximum number of linearly independent meromorphic functions over different compact Riemann surfaces. For example on the Riemann sphere $\mathcal{R}=\tilde{\mathbb{C}}$ the only meromorphic functions with a fnite number of poles are rational functions (cfr. the GAGA Theorem, [78], [45]); that is the key point allowing, for example, to re-state meromorphic functions on $\mathcal{R}$ as holomorphic maps $\mathcal{R}\leftrightarrow\tilde{\mathbb{C}}$, i.e. local (non-constant) holomorphic functions defined over the sheets of $\mathcal{R}$ and with values in $\mathbb{C}\cup\{\infty\}$ (with pre-images of ∞ corresponding to the polar singularities of the meromorphic function).

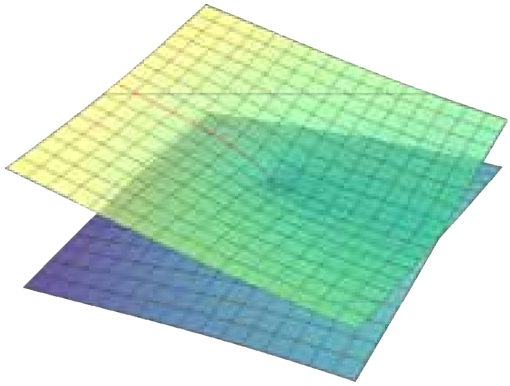

Figure 20 Plot of $\mathrm{Im}\sqrt{z}$ to show the gluing of the square-root function's *two* sheets, this time along the branch cut $]0,\infty[=\mathbb{R}_+$ (***not*** the red curve).

If $p_0\in\mathcal{R}$ is a regular point or a polar singularity locally coordinated by $z_0\in\mathbb{C}$ and $\omega\in\mathrm{Frac}\Lambda^{1,0}(\mathcal{R})$ is such a meromorphic 1-form, then inside an annulus-neighbourhood of p_0 there exists a Laurent series $f\in\mathbb{C}[\![z-z_0]\!]$ (whose outer and inner convergence radii are equal to the radii of such annulus) such that:

$$\omega(p) = f(z)\mathrm{d}z = \sum_{n=-N}^{\infty} a_n(z-z_0)^n\mathrm{d}z \qquad (a_{-N}\in\mathbb{C}^*) \tag{1.226}$$

(or $f(z)=\sum_{n=N}^{\infty} a_n z^{-n}$ if $z_0=\infty$ and $f\in\mathbb{C}[\![z^{-1}]\!]$); then also, $\omega\in\mathcal{K}(\mathcal{R})$.

In this case the residual in z_0 is proven to be simply the leading-negative coefficient of the Laurent series, i.e. $\mathrm{ord}_f z_0 = N$ and:

$$\mathrm{Res}_f z_0 = \frac{1}{(N-1)!}\lim_{z\to z_0}(z-z_0)^N\frac{\mathrm{d}^{N-1}f}{\mathrm{d}z^{N-1}} = a_{-1}. \tag{1.227}$$

If p_0 is (also) a branch point of order $\nu-1$ then the same isomorphism holds, but this time Puiseux series come into play: $f\in\mathbb{C}[\![z-z_0]\!]^\natural$ and

$$\omega(p) = f(z)\mathrm{d}z = \sum_{n=-N}^{\infty} a_n(z-z_0)^{\frac{n}{\nu}}\mathrm{d}z \tag{1.228}$$

(or $f(z)=\sum_{n=N}^{\infty} a_n z^{-\frac{n}{\nu}}$ if $z_0=\infty$ and $f\in\mathbb{C}[\![z^{-1}]\!]^\natural$).

Figure 20 shows the square-root function's two sheets glued together; its single branch is also depicted in Figure 17

Of course local representations (1.226)–(1.228) are equivalent once a different coordinate system is chosen, for example by simply letting $\zeta=(z-z_0)^{\nu}_{,}$ which also turns multi-valued representation (1.228) in the single-valued one (1.226). This choice is always possible in a small-enough neighbourhood of z_0, [42], [45].

Let $z\mapsto f(z)$ a complex (analytical) multi-valued function and $z_b\in\widetilde{\mathbb{C}}$ a branch point of its. Regardless of its branching number, it can be considered to act, algebraically, as an automorphy $A_{z_b}\in\operatorname{Aut}\mathbb{C}$ between each and every of its single-valued branches. That is expressed with the symbolism

$$f(z \overset{m}{\curvearrowleft} z_b) = A^m_{z_b} f(z), \tag{1.229}$$

where symbols '$\curvearrowleft$' or '$\curvearrowright$' are used to indicate whether the automorphy acts whenever the path winds either counter-clockwise or clockwise around z_b and the optional parameter $m\in\mathbb{N}_0$ accounts for the number of times such a winding occurs (not indicated if $m=1$). When such automorphy purely behaves as either a multiplication or an addition, then it is usually called **automorphy factor** or addend, respectively.

- The multi-valued function $z \mapsto \sqrt[p]{z}$, whose algebraic branch point (of order $p-1$) is 0, has automorphy factor $\sqrt[p]{\curvearrowleft 0} = e^{\frac{2\pi\imath}{p}}\cdot$.
- A multi-valued function $z\mapsto f(z)$ whose automorphy **factor** for a point $\zeta\in\mathbb{C}$ is equal to $\alpha\in\mathbb{C}\setminus\{0,1\}$ (let $g(z)$ be an *un*branched function in ζ) is factored as: $f(z) = (z-\zeta)^{\frac{\log\alpha}{2\pi\imath}} g(z)$.

- The logarithmic branch point of the complex logarithm $z \mapsto \log z \equiv \equiv \operatorname{Log} z + 2\pi\imath \arg z \equiv \log|z| + 2\pi\imath(\operatorname{Arg} z + k)$, $\forall k \in \mathbb{Z}$, has automorphy addend $\log(\curvearrowleft 0) = 2\pi\imath +$.

- A multi-valued function $z \mapsto f(z)$ whose automorphy **addend** for a point $\zeta \in \mathbb{C}$ is equal to $\alpha \in \mathbb{C}^*$ (let $g(z)$ be *un*branched as before) is factored as:

$$f(z) = \frac{\log \alpha}{2\pi\imath} \log(z - \zeta) + g(z).$$

In elementary functions, algebraic [logarithmic] branch points usually provide automorphy factors [addends], but exceptions are easily found (cfr. [72]).

It is also important to stress that branch points for specific values of the dependent variable z do not necessarily exist on every sheet of the Riemann surface: they can actually be branch points (or other types of singularities) on some sheets, as well as not (or different-order branch-points) on some other, as pointed in the example below. In this case a proper **sheet transition** has to be associated *with* every **branch point winding**, which also depends on the Riemann sheet the winding starts on. With this phenomenon in mind, one actually uses a relation symbol like '$h \rightsquigarrow k$' for a winding like $z \curvearrowleft z_0$, meaning that a counterclockwise winding around the (branch) point z_0 on the h^{th} sheet leads to a travel towards the k^{th} sheet of the Riemann surface (once that a customary ordering of all its sheets was carried over).

Consider the 3-sheeted Riemann surface defined by implicit equation $w^3-3w-z+1=0$. It has three simple branch points in $z=-1$, $z=3$ and $z=\infty$, but not all of them are branch points on every sheet: for example, the branching locus is $\{-1,\infty\}$ on the first sheet, $\{-1,3\}$ on the second and $\{\infty,3\}$ on the third sheet. Following the previously introduced formalism, all the possible independent travels on the Riemann surface are:

$$\begin{matrix} z & \curvearrowleft & -1 \\ 1 & \rightsquigarrow & 2 \end{matrix}, \quad \begin{matrix} z & \curvearrowleft & 3 \\ 2 & \rightsquigarrow & 3 \end{matrix}, \quad \begin{matrix} z & \curvearrowleft & \infty \\ 1 & \rightsquigarrow & 3 \end{matrix}.$$

So it is globally possible to go from any sheet to every other, but only through winding around appropriate branch points (or, which is equivalent, by crossing appropriate branch cuts): -1 joins I and II sheets, 3 joins II and III sheets, whereas ∞ joins III and I sheets; equivalently, I sheet has branching locus $\{-1,\infty\}$, II sheet has branching locus $\{-1,3\}$, and III has branching locus $\{3,\infty\}$. More on that in §2.4.5.

As an example of this fact, let $\mathbf{F}_{\mathbb{C}}$ be an algebra of functions, in one variable $z\in\mathbb{C}$ for simplicity's sake (for example $\mathbf{F}=\mathbb{C}^d[z]$, $\mathbf{F}=\mathbb{C}[\![z]\!]$ or $\mathbf{F}=\mathcal{K}(\mathbb{C})$). To any matrix function $A\in\mathbb{M}_n(\mathbf{F})$ the associated characteristic polynomial $\det(wI_n-A(z))\in\mathbf{F}^n[w]$ —as a function of w and z— defines a Riemann surface $\mathcal{S}_A$, called ***spectral* curve** (or spectral 1-manifold) of A, which is the geometric locus of its solutions, i.e. is defined by the explicit equation

$$\det(wI_n-A(z))=0. \tag{1.230}$$

If the elements of $\mathbf{F}$ are holomorphic or meromorphic functions, then the spectral curve is a compact, n-sheeted Riemann surface whose branch/ramification points correspond to couples $(w,z)\in\mathcal{S}_A\subset\mathbb{C}^2$ such that (1.230) has multiple roots. The splitting field (1.46) of A's characteristic polynomial, $\mathbf{F}(\ker(\det(wI_n-A)))$, is an extension field of $\mathbf{F}$ —whose further extension field would be $\mathcal{K}(\mathcal{S}_A)$— and the solvability of the corresponding Galois group $\mathrm{Gal}_{\mathbb{C}}\det(wI_n - A)$, (1.47), corresponds to the possibility to write A's functional eigenvalues by radicals, cfr. §1.1.10. Such explicit single-valued equations locally defining the spectral curve are the *local* closed-form expressions of each single eigenvalue of $A(z)$ (as a function of either z or w). If for example there exists a chart $U \subseteq \mathbb{C}$ such that $A(z) \sim \mathrm{diag}(\lambda_1(z), \lambda_2(z), \dots, \lambda_n(z))\ \forall z \in U$, then the extension field is

$$\mathbf{F}(\lambda_1, \lambda_2, \dots, \lambda_n),$$

and each sheet of the spectral curve is defined by explicit equation $w = \lambda_j(z)$, $1 \le j \le n$. A similar argument holds whenever $A(z)$ has a local fixed-blocks Jordan form, [66], and versal deformations ([7], [23]) have to be accounted when transitions from a block structure to another happen (corresponding to transitions from a sheet to another of the spectral curve). For example, when $A(z)$ is the dynamical matrix of a system of differential equations, cfr. §2.4.6, such Jordan-block transitions, interpreted as the dynamical influence of a parameter w, represent the so-called phase transitions (e.g. the "period-doubling" found in the standard map, [23], [49], [51], [66] [84], [92]) since the dynamical, and possibly topological, behaviour of the solution drastically changes — which may also be referred to as a catastrophe, [5].

From the above considerations, one has not to resort to complcated function algebras for $\mathbf{F}$ (unless essential singularities are to be included, cfr. §2.4.2), since $\mathbb{C}\{z\}[w]$ is enough to represent compact Riemann surfaces. As an example, let $A \in \mathbb{M}_n(\mathbb{C}[z]) \cong \mathbb{M}_n(\mathbb{C})[z]$ be a polynomial matrix function, of degree d, then:

$$A(z) = \sum_{k=0}^{d} A_k z^k, \qquad \det(wI_n - A(z)) = \sum_{k=0}^{n} a_k(z) w^k,$$

with $\det(wI_n - A) \in \mathbb{C}\{z\}[w]$. What is more important about the rational coefficients $a_1, a_2, \dots, a_n \in \mathbb{C}\{z\}$ of the characteristic polynomial is that they are invariant in

$\mathbb{C}\{z\}/\mathbb{C}$ under similarity transformations (since the functional spectrum of $A(z)$ would not change), thus the definition of $\mathcal{S}_A$ via the sole $a_1, a_2, \ldots, a_n$ is well-posed.

Lemma

The projective linear group $\mathrm{PGL}_n(\mathbb{C})$, §1.1.1, acts freely on $\mathbb{M}_n(\mathbb{C}[z])$ as $[\![C]\!] \mapsto C^{-1}A(z)\,C,\ \forall [\![C]\!] \in \mathrm{PGL}_n(\mathbb{C})$.

Among all the deformations such that $A(z)$ is **isospectral**, i.e. $\frac{\partial \det(wI_n - A(z))}{\partial z} \equiv 0$, a special type exists:

Lax Pair Theorem

Let $A, B \in \mathbb{M}_n(\mathbb{C}[z]) \cong \mathbb{M}_n(\mathbb{C})[z]$ such that

$$\frac{\mathrm{d}A}{\mathrm{d}z} = [A|B], \tag{1.231}$$

then $A(z)$ is isospectral, and the ordered couple of matrix polynomials (A,B) is said to be a ***Lax pair***.

The Lax Pair Theorem still holds for a broader class of function matrices, and in infinite dimensions too, when A and B are (nonlinear) evolution operators in a suitable Banach algebra — then the derivative in (1.231) is usually exchanged with a total derivative with respect to real time t for the evolution operators. A *generalized* Lax pair is, simplty, a couple of elements in a function algebra $\mathbf{a}, \mathbf{b} \in \mathbf{A}_K$ such that (1.231) holds with respect to their commutator algebra, i.e. $\exists d \in \mathrm{D}[\mathbf{A}]$ that $d\mathbf{a} = [\mathbf{a}|\mathbf{b}]$, cfr. [9]–§A.3.

Consider the (scalar) nonlinear Schrödinger equation $(\alpha \in \mathbb{R}_+)$:

$$\begin{cases} \dfrac{\partial^2 u}{\partial x^2} + \imath \dfrac{\partial u}{\partial t} + \alpha u(x,t)|u(x,t)|^2 = 0 \\ u(x,0) = f(x) \end{cases} \qquad f \in \mathrm{C}_0^\infty(\mathbb{R}).$$

By setting $\kappa^2 = 1 - 2\alpha^{-1}$ and defining two nonlinear operators

$$\begin{cases} L = \imath(I_2 + \kappa J_2)\dfrac{\partial}{\partial x} + \begin{pmatrix} 0 & \overline{u} \\ u & 0 \end{pmatrix} \\ M = \imath\kappa\dfrac{\partial^2}{\partial x^2} + \begin{pmatrix} -\frac{\imath|u|^2}{1+\kappa} & \overline{\frac{\partial u}{\partial x}} \\ -\frac{\partial u}{\partial x} & \frac{\imath|u|^2}{1-\kappa} \end{pmatrix} \end{cases}$$

all the solutions to the above PDE are equivalent to those such that L and M are a Lax pair, i.e. $\frac{\partial L}{\partial t} + [L|M] = 0$, cfr. (1.231)

When local representations (1.226)–(1.228) are explicited with respect to complex coordinates (w,z) it can be proven that —since the algebraic definition of meromorphic functions (1.216), thanks to (1.241) and the Riemann-Roch Theorem (1.242)— a non-compact Riemann surface can be represented (along with its singularities) as a polynomial with meromorphic coefficients in the algebra $\mathcal{K}(\mathbb{C})[w]$. Any *compact* Riemann surface is instead represented by polyomials with rational coefficients in the algebra $\mathbb{C}\{z\}[w]$, respectively. The elements of the former two polynomial algebras are always, clearly, Weierstrass polynomials in $\mathcal{W}_0(\mathbb{C}^2,w)$, cfr. §2.1.3, which the Weierstrass Division Theorem (1.159) is applied to in order to quotient different Riemann surfaces described by kernels of $\mathcal{W}_0(\mathbb{C}^2,w)$ (i.e. roots of Weierstrass' polynomials).

More generally refer to the Puiseux's Theorem (1.87), §1.3.4, that states, among other functional-oriented facts, the correspondence between explicit and implicit definitions for Riemann surfaces. Of crucial importance from the algebraic viewpoint are the following results, [45]:

Lemma

Let $\mathcal{R}$ be a Riemann surface, $P(w,z)\in\mathcal{K}(\mathcal{R})[z]/\mathbb{C}$ be a monic, n^{th}-degree polynomial. There exists a Riemann surface $\mathcal{S}$, a branched holomorphic n-cover $\pi\in C^0(\mathcal{S},\mathcal{R})$ and a meromorphic function $f\in\mathcal{K}(\mathcal{S})$ such that:

$$(\pi^* P)(f,z) = P\big(f(z),\pi^{-1}(z)\big) \equiv 0. \tag{1.232}$$

The identification of the algebraic function $(\pi^* P)(f,z)$ (as a function of z), as well as that of $\mathcal{S}$, is unique, in the sense that if $\mathcal{T}$ and $g\in\mathcal{K}(\mathcal{T})$ have the same properties of $\mathcal{S}$ and f repsectively, then $\exists!\,\tau\in C^0(\mathcal{T},\mathcal{S})$ fibre-preserving such that $\tau^* f = g$ and $P(\tau^* f, \tau^*\pi^*) = P\big(f(\tau^{-1}(z)), \tau^{-1}(\pi^{-1}(z))\big) \equiv 0$.

Theorem

Let $\mathcal{R}$ be a Riemann surface, $P\in\mathcal{K}(\mathcal{R})[z]/\mathbb{C}$ be a monic, n^{th}-degree polynomial with meromorphic coefficients $c_0, c_1, \ldots, c_n\in\mathcal{K}(\mathcal{R})$ and $\mathcal{S}$ the n-sheeted Riemann surface defined by the previous Lemma as:

$$P(w,z) = \sum_{k=0}^{n} c_k(z) w^k = 0. \tag{1.233}$$

Then $\mathcal{K}(\mathcal{S}) \cong \mathcal{K}(\mathcal{R})[z]/P$ is an n^{th}-degree extension field of $\mathcal{K}(\mathcal{R})$, thanks to the monomorphism $\pi^*\in\mathrm{Hom}(\mathcal{K}(\mathcal{R}),\mathcal{K}(\mathcal{S}))$ induced by the branched holomorphic n-cover $\pi\in C^0(\mathcal{S},\mathcal{R})$ in (1.232). The dual map $f\mapsto \tau^* f$ is a group action of $\mathrm{Deck}(\mathcal{S},\mathcal{R})$ on $\mathcal{K}(\mathcal{S})$ which *fixes* $\mathcal{K}(\mathcal{R})$, such that the following isomorphism holds:

$$\mathrm{Deck}(\mathcal{S},\mathcal{R}) \cong \mathrm{Aut}\,\frac{\mathcal{K}(\mathcal{S})}{\mathcal{K}(\mathcal{R})}. \tag{1.234}$$

> Finally, a cover π is Galois if and only if $\mathcal{K}(\mathcal{S})/\mathcal{K}(\mathcal{R})$ is a Galois field extension.

This Theorem is an extension of the Weierstrass Division Theorem (1.159) in the one-dimensional case $n=1$, and $\mathcal{W}_0(\mathbb{C}^2,w)$ defined in (1.161) is an isomorphic vector space to $\mathcal{K}(\mathcal{R})[z]$, whereas the field extension $\mathcal{K}(\mathcal{R})[z]/P$ contains (the equivalence classes of) those elements fixing $\sigma_* f$, cfr. §2.1.3.

2.4.2. Global theory: Cohomology

For every 1-cycle $c \in \mathrm{Z}_1(\mathcal{R})$ of a Riemann surface (i.e. any closed regular curve on it) and any δ-body[87] around it ($\forall \delta > 0$, which is of course split in two semi-annuli by c) a compact-supported, real-valued closed 1-form $\eta_c = \mathrm{d} e_c \in \mathrm{Z}^{1,0}(\mathcal{R})$ exists such that it is zero on c and outside its δ-body;[88] furthermore

$$\langle c | \alpha \rangle = \langle \alpha | \star \eta_c \rangle_{\mathrm{L}^2(\mathcal{R})}, \quad \forall \alpha \in \mathrm{Z}_2^{1,0}(\mathcal{R}). \tag{1.235}$$

> **Lemma**
>
> A 1-form $\alpha \in \mathrm{L}_2^1(\mathcal{R})$ is exact [coexact] if it inner-multiplies to zero with every compact-supported, coclosed [closed] 1-forms, i.e.:
>
> $$\left.\begin{matrix} \alpha \in \mathrm{B}_2^{1,0}(\mathcal{R}) \\ \alpha \in \mathrm{B}_2^{0,1}(\mathcal{R}) \end{matrix}\right\} \quad \Leftrightarrow \quad \langle \alpha | \beta \rangle_{\mathrm{L}^2(\mathcal{R})} = 0 \quad \forall \beta \in \left\{\begin{matrix} \mathrm{Z}_2^{0,1}(\mathcal{R}) \\ \mathrm{Z}_2^{1,0}(\mathcal{R}) \end{matrix}\right. .$$

> ***Helmholtz*'s Decomposition Theorem**
>
> If $\mathcal{R}$ is a compact Riemann surface, then:
>
> $$\mathrm{L}_2^1(\mathcal{R}) = \mathrm{B}_2^{1,0}(\mathcal{R}) \oplus \mathrm{B}_2^{0,1}(\mathcal{R}) \oplus \mathrm{H}^1(\mathcal{R}). \tag{1.236}$$

---

[87] A **δ-body** of a metric space set $A \subset S$ is the closed set $\mathrm{body}_\delta A := \{x \in S \mid \mathrm{d}(x,A) \leq \delta\}$ [9]–§1.6.

[88] Thus e_c is non-constant on either one of two semi-annuli and takes two values on c's "sides".

The Helmholtz's Decomposition Theorem for compact Riemann surfaces states that every 1-form can be uniquely decomposed as a sum of an exact, a coexact and an harmonic 1-form; also note that, for non-compact surfaces, it still holds, although it is not a (unique) orthogonal decomposition any longer (i.e. you have to replace vector-space direct sums '$\oplus$' with vector-space sums '$+$', cfr. §1.1.4, §2.2.4 and [9]–§§3.7, 4.5).

If $\mathcal{R}$ is a compact surface of genus $g\in\mathbb{N}_0$, then $\mathrm{H}_1(\mathcal{R})\cong\mathbb{Z}^{2g}$ and $\pi_1(\mathcal{R})\cong\mathbb{Z}^g$. It is well known [42] that, let $\mathrm{H}_1(\mathcal{R})=\langle[\![a_k]\!],[\![b_k]\!]\rangle_{1\leq k\leq g}$, a common choice for the homology basis is letting 1-cycles $a_k,b_k\in\mathrm{Z}_1(\mathcal{R})$ commute this way:

$$a_i\circ a_j=b_i\circ b_j=0,\qquad a_i\circ b_j=\delta_{i,j},\qquad 1\leq i,j\leq g. \tag{1.237}$$

Such 1-cycles a_k and b_k are exactly the sides of the fundamental polygon introduced in §2.3.1 for any compact (in this case orientable) Riemann surface, Figure 14 . To each couple of 1-cycles (a_k,b_k) the meromorphic (1,0)-forms $\omega_k\in\Lambda^{1,0}(\mathcal{A}_1)$ exist and their local expression is

$$\omega_k(z)=\frac{z^{k-1}\mathrm{d}z}{w}\equiv\frac{z^{k-1}}{f(z)}\mathrm{d}z\,,\qquad 1\leq k\leq g. \tag{1.238}$$

The cohomology space is $\mathrm{H}^1(\mathcal{R};\mathbb{R})=\langle\mathrm{Re}\,\omega_k,\ \mathrm{Im}\,\omega_k\rangle_{1\leq k\leq g}$. The *period* of any 1-form $\omega\in\Lambda^1(\mathcal{R})$ over a cycle $c\in\mathrm{Z}_1(\mathcal{R})$, [36], is the value of the counter-clockwise circulation of ω around c, i.e.:

$$\langle c|\omega\rangle=\oint_{+c}\omega\,.$$

It can be proven that the period of any fundamental form (1.238) over the fundamental cycles (1.238) is always a *Gaussian integer* in $\mathbb{Z}(\imath)$, whereas:

$$\oint_{\bigcirc_{j=1}^{g}\left(a_jb_ja_j^{-1}b_j^{-1}\right)}\phi\wedge\vartheta=\sum_{j=1}^{g}\left([\![\phi]\!]_{a_j}[\![\vartheta]\!]_{b_j}-[\![\vartheta]\!]_{a_j}[\![\phi]\!]_{b_j}\right),\ \forall\phi,\vartheta\in\mathrm{Z}^1(\mathcal{A}_1). \tag{1.239}$$

The following is an equivalent formulation of the Riemann-Hurwitz formula (1.205) for compact Riemann surfaces defined, for example, via explicit multi-valued equation(s), which is shown here in order to better clarify the importance of certain holomorphic tools in the topology of Riemann surfaces.

Riemann-Hurwitz Formula Riemann surfaces

Let $f \in C^{\omega}(\mathcal{R};\mathcal{S})$ be a holomorphic map between two compact Riemann surfaces and let $\omega \in \Lambda^{1,0}(\mathcal{R})$ and $\varpi = f_* \omega \in \Lambda^{1,0}(\mathcal{S})$, then $\forall z \in \mathcal{R}$

$$\mathrm{ord}_{\omega} z = (\mathrm{b}_f(z)+1)\,\mathrm{ord}_{\varpi} f(z) + \mathrm{b}_f(z)\,,$$

$$\chi(\mathcal{R}) = \sum_{z \in \mathcal{R}} \mathrm{ord}_{\omega} z$$

$$\Downarrow$$

$$\begin{aligned} \chi(\mathcal{S})\deg f &= \sum_{z \in \mathcal{R}} (\mathrm{b}_f(z)+1)\,\mathrm{ord}_{\varpi} f(z) \\ &= \deg f \sum_{w \in \mathcal{S}} \mathrm{ord}_{\varpi} w \end{aligned} \quad . \tag{1.240}$$

It is important to note that f is in general a ramified $(\deg f)$-cover whereas ϖ lies over ω on $\mathcal{R}$ (i.e. $\omega = \varpi \uparrow f$) such that, if $w \in \mathcal{S}$ is a local coordinate with $w \to 0$ for $z \to z_0$ converging to a ramification point z_0 (so that $0 \in \mathcal{S}$ is a branch point), then asymptotically $\lim_{w \to 0} w \sim \mathrm{o}\left((z-z_0)^{\mathrm{b}_f(z_0)+1}\right)$.

If $f \in C^{\omega}(\mathcal{R};\mathcal{S})$ is ***biholomorphic*** (i.e. f is an analytic homeomorphism with f^{-1} holomorphic), it is also called an isomorphism of Riemann surfaces; the isomorpshisms of a Riemann surface $\mathcal{R}$ to itself are called automorphisms of $\mathcal{R}$ and their space indicated as $\mathrm{Aut}\mathcal{R} :\equiv C^{\omega}(\mathcal{R};\mathcal{R})$ which, if $\mathcal{R}$ is compact of genus $g \geq 2$, is proven to be a finite group (with respect to composition) with:

$$\mathrm{ord}\,\mathrm{Aut}\mathcal{R} \leq 84(g-1). \tag{1.241}$$

Biasing the attention from holomorphic to meromorphic functions and forms, the basis decomposition for the Abelian differentials on a (compact) Riemann surface is one of the most basic “global” problems.

Nœther’s “Gap” Theorem

If $\mathcal{R}$ is compact and orientable of genus $g \in \mathbb{N}_0$, then there exist g integer numbers $1 = n_1 < n_2 < \ldots < n_g \leq 2g$ such that *no* meromorphic functions with one pole of order n_k exist on $\mathcal{R}$, for every $1 \leq k \leq g$, i.e. $\nexists f \in \mathcal{O}(\mathcal{R} \backslash \{z_0\})$ such that $\mathrm{ord}_f z_0 = n_k$.

Riemann-Roch Theorem — Abelian differentials

Let $\mathcal{R}$ be a compact, g-genus Riemann surface and let $z_1, z_2, \ldots, z_r \in \mathcal{R}$ be distinct points of its, then the maximum number of linearly independent meromorphic 1-forms on $\mathcal{R}$ is:

$$\dim_{\mathbb{C}} \mathrm{H}^1\Big(\mathcal{R} \backslash \{z_j\}_{1 \leq j \leq r}\Big) = 2g + r - 1 \qquad (1.242)$$

This is not the general formulation of the Riemann-Roch Theorem (which can be found in [62]), but rather one of its corollaries regarding the maximum number of meromorphic differentials (1-forms); this choice was made in order to avoid the introduction of the extremely useful concept of Riemann surface’s *divisors*, which are not specifically required in this work.

As a last topic, if a Riemann surface has at least one logarithmic branch point z_o (cfr. §2.4.1), then it is *not* globally represented by any *algebraic* implicit or explicit equation (that is why the finite-multiplicity branch points are called algebraic) and *no* meromorphic functions/forms may exist on any neighbourhoods z_o, which is shown by the analytic theory to be often also an ***essential** singularity* of the surface.

In the presence of an essential singularity local representations (1.226)–(1.228) may still hold, but $N = \infty$ — i.e. respectively (let $\nu = 1$ when z_o is *not* a branch point too):

$$f(z) = \sum_{n=-\infty}^{+\infty} a_n (z - z_\text{o})^{\frac{n}{\nu}} \qquad \left(\text{or } f(z) = \sum_{n\in\mathbb{Z}} a_n z^{\frac{n}{\nu}} \text{ if } z_\text{o}=\infty \right), \quad (1.243)$$

and the residual of f in z_o is still indirectly defined to be the leading-negative coefficient of the formal power series, i.e. $\mathrm{Res}_f z_\text{o} := a_{-1}$.

Logarithmic branch points which behave as singularities must be in fact essential singularities: any Laurent series expansion of $\mathbb{C}[\![z-z_\text{o}]\!]$ fails since infinite negative terms are present (cfr. §1.3.3 and [72]): the presence of infinite ramified points corresponds to the fact that, locally, the ramified cover is not defined in the branch point. The generic behaviour of holomorphic functions "near" essential singularities is governed by the Great Picard's Theorem:

Little *Picard*'s Theorem

The image of every non-constant entire function $f\in\mathcal{O}(\mathbb{C})$ is equal to the whole complex plane except one point, i.e.: $\exists! z_f\in\mathbb{C}$: $f(\mathbb{C})=\mathbb{C}\setminus\{z_f\}$.

Great *Picard*'s Theorem

If $z_\text{o}\in\Omega\subseteq\mathbb{C}$ and $f\in\mathcal{O}(\Omega\setminus\{z_\text{o}\})$ is a holomorphic function with an essential singularity in z_o, then in every neighbourhood $\mathcal{N}(z_\text{o})\subseteq\Omega$, f assumes *all* the complex values with, at most, a single exception, *infinitely often*, i.e.: $\exists! z_f\in\mathbb{C}$: $\mathbb{C}\setminus\{z_f\} \subseteq f(\mathcal{N}(z_\text{o})\setminus\{z_\text{o}\}) \subseteq \mathbb{C}$.

Essential singularities and logarithmic branch points occur as either isolated singularities, like $z=0$ for both $\exp(z^{-1})$ and $Y_n(z)$ (cfr. §2.4.6), or as accumulation points for a sequence of polar singularities and/or branch points, like $z=0$ in $\tan\log z$, cfr. [1], [72]. It is noteworthy that, despite analytic functions have no Puiseux series' expansion near their logarithmic branch points, it is still possible to expand them as a series of either logarithmic powers (cfr. [72]) or logarithmic iterations like:

$$f(z) = \sum_{n=0}^{\infty} a_n \left(\frac{\log(z-z_0)-\alpha}{\log(z-z_0)+\alpha} \right)^n, \qquad \alpha \in \mathbb{C}_+, \tag{1.244}$$

or

$$f(z) = \sum_{n=0}^{\infty} a_n \log^n(z-z_0). \tag{1.245}$$

In both cases $f \in \mathbb{C}^{\mathbb{Q}}[z]$ are still transfinite series, cfr. §1.3.4 and in the latter $\log^n z$ is the n-fold iteration of the logarithm function — $\log(\log(\cdots \log z))$ (n times). Infinite-sheeted Riemann surfaces have often infinite genus too, although extreme care has to be posed to the local structure of the surface (i.e. the existence of holomorphic differentials and the density of its branch points) so that "branching-free" open subsets do exist. In general, simpler expressions than (1.244) or (1.245) exist, like in the following examples, directly inspired from [72].

2.4.3. Hardy spaces

For completition's sake before treating a more analytical aspect of Riemann surfaces is to "turn back" to holomorphic functions on $\mathbb{C}$ and define there the complex analogues to the L^p spaces of Functional Analysis, [9]–§B.3, and will be used again in §2.4.7. Let $p \in \mathbb{R}_+$ and $\Omega \subset \mathbb{C}$ be a open domain of genus 0. The *Hardy p-space* of Ω with respect to a measure μ on $\mathbb{C}$, is

$$\mathcal{H}^p(\Omega,\mu) \stackrel{\text{def}}{=} \left\{ f \in \mathcal{O}(\Omega) \;\middle|\; \|f\|_{\mathcal{H}^p(\Omega,\mu)} := \overline{\lim_{t\to 1^-}} \sqrt[p]{\frac{1}{2\pi} \oint_{+\mathfrak{h}_\gamma(t)} |f(z)|^p \,\mathrm{d}\mu(z)} < \infty \right\}. \tag{1.246}$$

where $\mathfrak{h}_\gamma : [0,1] \to \mathrm{B}_1(\overline{\Omega})$ is any homotopy from a boundary loop $\gamma \subset \Omega$ to its boundary, i.e. $\mathfrak{h}_\gamma(0) = \gamma$ and $\mathfrak{h}_\gamma(1) = \partial\Omega$, $\forall \gamma \in \mathrm{B}_1(\Omega)$. With respect to the Lebesgue measure on $\mathbb{C}$, they are simply indicated as $\mathcal{H}^p(\Omega)$.

Hardy spaces are semi-normed, and Banach if $p \geq 1$ (of norm $\|f\|_{\mathcal{H}^p} :\equiv |f|_{\mathcal{H}^p}$); they are also separable if Ω is limited and, if $0 < p < q < \infty$, then $\mathcal{H}^q(\Omega,\mu) < \mathcal{H}^p(\Omega,\mu)$.

As a simpler case, also let $\rho \in]0,+\infty[$ and $z_0 \in \mathbb{C}$, then Hardy p-space of the disc $\mathbb{B}_\rho(z_0)$ is indicated[89] as $\mathcal{H}^p_\rho(z_0) :\equiv \mathcal{H}^p\big(\mathbb{B}_\rho(z_0)\big)$ and its semi-norm computed as:

$$|f|_{\mathcal{H}^p_\rho} = \sup_{r \in]0,\rho[} \sqrt[p]{\frac{1}{2\pi} \oint\limits_{+\partial\mathbb{B}_r(z_0)} |f(z)|^p \mathrm{d}z} = \sup_{r \in]0,\rho[} \sqrt[p]{\frac{1}{2\pi} \int_0^{2\pi} \left| f(re^{\imath\theta} - z_0) \right|^p \mathrm{d}\theta} \,.$$

$\mathcal{H}^2(\Omega,\mu)$ are Hilbert spaces with the inner product

$$\langle f \,|\, g \rangle_{\mathcal{H}^2(\Omega,\mu)} := \overline{\lim_{t \to 1^-}} \frac{1}{2\pi} \oint\limits_{+\mathfrak{h}_\gamma(t)} f(z)\overline{g(z)} \mathrm{d}\mu(z) \tag{1.247}$$

and the following estimate holds in $\mathcal{H}^2_\rho(z_0)$:

$$|f(\zeta - z_0)| \leq \frac{\|f\|_{\mathcal{H}^2}}{\sqrt{1 - |\zeta|^2}}, \qquad \forall \zeta \in \mathbb{B}_\rho(0). \tag{1.248}$$

From (1.248) it follows that the $\mathcal{H}^2$-norm induces the uniform convergence on its functions, i.e. if $(f_n)_{n\in\mathbb{N}} \in \mathcal{H}^2(\Omega)^\mathbb{N}$ is a Cauchy sequence, then $f_n \underset{\Omega}{\rightrightarrows} f \in \mathcal{H}^2(\Omega)$.

By choosing different measures, isolated singularities in $\mathcal{D}\Omega \setminus \Omega$ can be included in Hardy-space functions; if the spherical measure

$$\mathrm{d}\mu_\mathbb{S}(z) \overset{\text{def}}{=} \frac{4\mathrm{d}z}{\left(1 - |z|^2\right)^2}$$

induced by spherical Poincaré metric

$$\underline{g}_\mathbb{S}(z,\overline{z}) :\equiv \underline{g}_\mathbb{S}(z) \frac{\partial}{\partial z} \otimes \frac{\partial}{\partial \overline{z}} \overset{\text{def}}{=} \frac{4}{(1 - z\overline{z})^2} \frac{\partial}{\partial z} \otimes \frac{\partial}{\partial \overline{z}} \tag{1.249}$$

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[89] $\mathcal{H}^p_\rho(z_0)$ is the Hardy p-space of centre z_0 and radius ρ. When either $\rho = 1$ and $z_0 = 0$ they are omitted in both the space symbol and the (semi-)norm, i.e. $\mathcal{H}^p$ and $|f|_{\mathcal{H}^p}$, respectively.

is used (cfr. [36], [9]–§3.3, and §2.1.2), such singularities are exactly poles, and meromorphic functions are included in Hardy spaces, i.e.: $\exists f\in\mathcal{K}(\Omega)$, $\exists p\in]0,+\infty[$: $f\in\mathcal{H}^p(\Omega,\mu_{\$})$.

2.4.4. Theta functions and period matrices

In analogy to the Riemann theta functions defined as an example to §2.1.3, The 1st-order Riemann theta function is the extension $\theta:\mathbb{M}_{2,g}(\mathbb{C})\times\mathbb{C}^g\times\mathbb{H}_g \to \mathbb{C}$, with analogue properties to 0th-order theta functions (1.155), is defined as [41]

$$\theta_{(\varepsilon,\eta)}(\mathbf{z}\,|\,\Pi) \overset{\text{def}}{=} \sum_{\mathbf{n}\in\mathbb{Z}^g} \exp\left\{\pi\imath\left[2\left(\mathbf{n}+\frac{\varepsilon}{2}\right)\cdot\left(\mathbf{z}+\frac{\eta}{2}\right)+\left(\mathbf{n}+\frac{\varepsilon}{2}\right)^{\mathsf{T}}\Pi\left(\mathbf{n}+\frac{\varepsilon}{2}\right)\right]\right\},$$

which depends on the two-rows matrix $\begin{pmatrix}\varepsilon\\ \eta\end{pmatrix}$, whereas $\varepsilon,\eta\in\mathbb{C}^g$ are the **characteristics** of the theta function. Trivially, $\theta_{O_{2,g}}\equiv\theta$ (*zero* characteristic) and, from (1.156):

$$\theta_{(\varepsilon,\eta)}(\mathbf{z}\,|\,\Pi) = e^{-\pi\imath\varepsilon^{\mathsf{T}}\left(\frac{1}{4}\Pi\varepsilon+\frac{1}{2}\eta+\mathbf{z}\right)}\theta\left(\mathbf{z}+\frac{\Pi\varepsilon}{2}+\frac{\eta}{2}\,\middle|\,\Pi\right). \tag{1.250}$$

From the Riemann surfaces' point of view, the most useful theta functions are 1st-order with either *integer* ($\varepsilon,\eta\in\mathbb{Z}^g$), *half*-integer ($\varepsilon,\eta\in\frac{1}{2}\mathbb{Z}^g$) or *normal* characteristics ($\varepsilon,\eta\in[0,1[^g$), [42]. For integer characteristics, in particular, $\forall\varepsilon,\eta,\mathbf{l},\mathbf{m}\in\mathbb{Z}^g$:

$$\theta_{(\varepsilon,\eta)}(\mathbf{z}+\Pi\mathbf{m}+\mathbf{l}\,|\,\Pi) = e^{-\pi\imath[\mathbf{m}^{\mathsf{T}}\Pi\mathbf{m}-\varepsilon\cdot\mathbf{l}+2\mathbf{m}\cdot(\mathbf{z}+\eta)]}\theta_{(\varepsilon,\eta)}(\mathbf{z}\,|\,\Pi); \tag{1.251}$$

$$\theta_{\begin{pmatrix}\varepsilon+2\varepsilon'\\ \eta+2\eta'\end{pmatrix}}(\mathbf{z}\,|\,\Pi) = (-1)^{\varepsilon'\cdot\eta'}\,\theta_{(\varepsilon,\eta)}(\mathbf{z}\,|\,\Pi),\qquad \forall\varepsilon',\eta'\in\mathbb{Z}^g. \tag{1.252}$$

1st-order, integer-characteristics theta functions are either even or odd with respect to $\mathbb{C}^g$ since $\theta_{(\varepsilon,\eta)}(-\mathbf{z}\,|\,\Pi) = (-1)^{\varepsilon\cdot\eta}\,\theta_{(\varepsilon,\eta)}(\mathbf{z}\,|\,\Pi)$. The *parity* of a theta function with integer characteristics is simply that of $\varepsilon\cdot\eta\in\mathbb{Z}$.

Theorem

There are exactly 2^{2g} integer-characteristics 1st-order theta functions on $\mathbb{C}^g\times\mathbb{H}_g$: $2^{g-1}(2^g+1)$ of them are even, the remaining $2^{g-1}(2^g-1)$ are odd.

To any g-genus compact Riemann surface whose homology and cohomology bases have been defined according to (1.237) and (1.238), the 1st and 2nd **period matrices** $A,B\in\mathbb{M}_g(\mathbb{Z}(\imath))$, whose elements are Gaussian integers by (1.239), are defined as:

$$A=(\langle a_j|\,\omega_i\rangle)_{1\leq i,j\leq g}\,,\qquad B=(\langle b_j|\,\omega_i\rangle)_{1\leq i,j\leq g}. \tag{1.253}$$

In particular $A\in\mathrm{GL}_g(\mathbb{Z}(\imath))$, whereas $B\in\mathbb{H}_g$.

Now let the following lattice $\Gamma(+):=\langle\mathbf{g}_k\rangle_{1\leq k\leq 2g}<\mathbb{C}^g$ be considered such that

$$\mathbf{g}_k=\begin{cases}\mathbf{e}_k, & 1\leq k\leq g\\ B\mathbf{e}_k, & g<k\leq 2g\end{cases};\qquad \mathbf{e}_k=(\delta_{j,k})_{1\leq j\leq g}$$

and $\mathcal{B}\{\mathbf{e}_1,\mathbf{e}_2,\dots,\mathbf{e}_g\}$ is the canonical basis of $\mathbb{C}^g$. Any quotient group like $\mathcal{R}/\Gamma_{\mathcal{R}}\asymp\mathbb{T}_g$ is called *Jacobi variety* (or Jacobi g-torus) associated to $\mathcal{R}$ and it is both a commutative Lie group and a complex orbifold — cfr. §1.2.1 and §2.3.2 respectively.

It can be proven that there is isomorphism between Jacoby varieties defined via such a column-block matrix

$$\begin{pmatrix}\mathbf{g}_1 & \mathbf{g}_2 & \cdots & \mathbf{g}_{2g}\end{pmatrix}=\begin{pmatrix}I_g & \Pi\end{pmatrix}\in\mathbb{M}_{g,2g}(\mathbb{C}),$$

and the $g\times 2g$ matrix $\begin{pmatrix}A & B\end{pmatrix}\sim\begin{pmatrix}I_g & \Pi\end{pmatrix}$ composed by the above period matrices. In particular when the first $g\times g$ column block is the 2nd period one B, such rectangular matrix is said to be in *canonical form*, whereas the first block is the identity I_g (and the other block is then in $\mathbb{H}_g$) it is in *Riemann form*. Whatever is the form which the matrix $\begin{pmatrix} A & B \end{pmatrix} \sim \begin{pmatrix} I_g & \Pi \end{pmatrix}$ is in, such change of basis is achieved via a integer symplectic matrix[90] $C \in \mathrm{Sp}_{2g}(\mathbb{Z})$, i.e.:

$$\begin{pmatrix} A \\ B \end{pmatrix} = C \begin{pmatrix} I_g \\ \Pi \end{pmatrix},$$

from which follows that, if $\begin{pmatrix} I_g & \Pi' \end{pmatrix} \sim \begin{pmatrix} I_g & \Pi \end{pmatrix}$ then $\exists \left(\begin{smallmatrix} A & B \\ C & D \end{smallmatrix}\right) \in \mathrm{Sp}_{2g}(\mathbb{Z})$:

$$\Pi' = (C + D\Pi)(A + B\Pi)^{-1}, \tag{1.254}$$

which is the matrix-analogue of a Möbius (conformal) map. Furthermore, whatever form the period matrices (A, B) are, they always satisfy the Riemann bilinear relations

$$\begin{cases} [A^\intercal B \,|\, B^\intercal A] = O_g \\ \imath \left[B^\intercal \overline{A} \,\middle|\, A^\intercal \overline{B} \right] \in \mathrm{H}_g^+(\mathbb{C}) \end{cases} \tag{1.255}$$

(i.e. the matrix $\imath(B^\intercal \overline{A} - A^\intercal \overline{B})$ is Hermitian and positive-definite). The 2^{2g} integer-characteristics 1st-order theta functions of the previous Theorem correspond 1-to-1 to the 2^{nd}-order points of the Jacobi g-torus $\mathbb{C}^g / \langle I_g, \Pi \rangle$, or $\mathbb{C}^g / \langle A, B \rangle$ for any period matrices' non-Riemann couples — say A and B.

The Abel maps $A_k : \mathcal{R} \to \mathbb{C}$:

$$z \mapsto A_k(z) \overset{\text{def}}{=} \int_{z_0}^{z} \frac{\zeta^k \mathrm{d}\zeta}{\sqrt{P(\zeta)}} \equiv \int_{z_0}^{z} \frac{\zeta^k \mathrm{d}\zeta}{\sqrt{\prod_{j=1}^{d} (\zeta - z_j)}}, \tag{1.256}$$

$0 \le k < g$, where P is any simple d^{th}-degree polynomial and $g = \left[\frac{d}{2}\right] - 1$ are defined, whereas the Abel map $z \mapsto \mathbf{A}(z) := (A_0(z), A_1(z), \ldots, A_{g-1}(z)) \in \mathbb{C}^g / \Gamma_{\mathcal{R}}$ is an automorphism of the Jacobi variety itself, i.e. $\mathbf{A} \in \mathrm{Aut}(\mathcal{R}/\Gamma_{\mathcal{R}})$. The inversion of Abel maps is best known as Jacobi inversion problem [42], its integrands being called

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[90] The ***symplectic*** $2n$***-group*** in a ring R is $\mathrm{Sp}_{2n}(R) := \{ A \in \mathbb{M}_{2n}(R) \mid A J_{2n}^\intercal A = I_{2n} \}$, [9]–§c.1.

hyperelliptic functions after him. For an accurate analysis of Riemann surfaces defined by implicit denominator equations of (1.256)'s integrand see §3, where such functions —even in the case of non-simple polynomials P— are treated.

Riemann theta functions and Jacobi varieties are two important tools for the analysis of compact Riemann surfaces. In particular, a 2-sheeted Riemann surface is **hyperelliptic** [45], and the following holds:

Theorem

Let $\mathcal{R}\subseteq\mathbb{C}^2$ be the g-genus hyperelliptic Riemann surface defined by implicit equation $w^2-P(z)=0$, with $P\in\mathbb{C}[z]$ can always be chosen as simple with $2g+1\leq\deg P\leq 2g+2$.

Let B be the 2nd period matrix of $\mathcal{R}$ (for any fixed homology basis) and let $\mathbf{A}:\mathcal{R}\to\mathcal{R}/\Gamma_{\mathcal{R}}$ and $\theta(\mathbf{z}|B)$ be (vector) Abel map and the Riemann theta function associated to $\mathcal{R}$, respectively; then the following alternative exists for the function $\theta\circ\mathbf{A}:\mathcal{R}\to\mathcal{R}$:

- either $\theta(\mathbf{A}(\mathbf{z})|B)\equiv 0$ on $\mathcal{R}$, or
- $\theta(\mathbf{A}(\mathbf{z})|B)$ has exactly g distinct zeroes $w_1,w_2,\dots,w_g\in\sqrt{P}(\mathcal{R})$.

Generally, the composed "Abel-theta map" $\theta\circ\mathbf{A}$ is a multi-valued function in $\mathcal{R}$ (especially when it is defined via explicit equation), because the integrals defininf the proper Abel maps depend on the integration path. Every time a branch point is winded around, the integer characteristic of the theta functions changes, thus adding a multiplicative factor as given by formulæ (1.251)–(1.252). As will be clear in the next paragraph, such characters (in the Prontryagin's duality sense of §1.2.4) often provide an (almost) closed-form structure for the monodromy action on $\mathcal{R}$.

Riemann theta functions are usually employed to express the equations defining spectral curves introduced in §0 when a certain expression for the Lax pair (1.231) defining their isospectrality is used.

2.4.5. The Monodromy group of a Riemann surface

Let $\mathcal{R}$ be a compact Riemann surface of genus $g\in\mathbb{N}_0$ defined, *globally*, by an explicit equation $w=f(z)$, where $f\in\mathcal{K}(\mathbb{C})$ is unformally a meromorphic, *multi-valued* function with p branches and r distinct branch points $B=\{z_1,z_2,\ldots,z_r\}\subset\tilde{\mathbb{C}}$. Then $f:\mathcal{R}\to\tilde{\mathbb{C}}$ is a p-cover of $\mathbb{C}$ (whereas $f|_{\mathcal{R}\backslash B}:\mathcal{R}\backslash B\to\tilde{\mathbb{C}}$ is a *ramified* p-cover, cfr. §2.3.1), with its branches corresponding to the sheets of $\mathcal{R}$. The arbitrary choice of branch cuts for the function f is the "analytic" analogue to a choice of fibres for the covering map f. From §2.3.1 it is known that compact, g-genus Riemann surfaces are quotient groups of kind $\mathbb{C}/G$, i.e. Jacobi varieties (cfr. §2.4.4). More generally, and also in the case of non-compact Riemann surfaces, the following holds:

Uniformization (*Riemann* Mapping) Theorem

Connected Riemann surfaces $\mathcal{R}$ are **conformal** (i.e. holomorphically homeomorphic) to simpler ones of three kinds, and are classified in different ways according to such topological equivalence, as:

- Elliptic— the complex projective plane $\mathbb{C}^1\mathbb{P}$, i.e. the Riemann sphere $\tilde{\mathbb{C}}$,
- Parabolic— the complex plane (or $\mathbb{C}/\mathbb{Z}$, or $\mathbb{C}/\mathbb{Z}^2$),
- Hyperbolic— the unit disc $\mathbb{B}_1(0)\subset\mathbb{C}$, or a quotient group $\mathbb{B}_1(0)/G$, where $G(\circ)$ is a discrete group acting freely on $\mathbb{B}_1(0)$, which is anyway the universal covering of $\mathcal{R}$.

The monodromy action can also be interpreted as a function

$$\mathrm{MG}f\;:\;\tilde{\mathbb{C}}\backslash B\to\mathbb{S}_p,$$

$$z \mapsto F_z \in \mathrm{Hom}\big(\mathbb{S}_p, \pi_1(\mathcal{R}, z)\big). \tag{1.257}$$

If the monodromy action F_z is regular $\forall z \in \tilde{\mathbb{C}} \setminus \{z_j\}_{1\leq j\leq r}$ (i.e. $\mathrm{MG}f$ is injective with respect to the surface point $z\in\mathcal{R}$) then $\mathrm{MG}_z\mathcal{R} \cong \mathbb{S}_p$, i.e. the monodromy groups are all *maximal* (and isomorphic with each other); but since $\mathrm{MG}_z f$ grow large as p increases (recall $\mathrm{ord}\,\mathbb{S}_p = p!$, cfr. §1.1.2), the monodromy groups are not always as easy to compute as for their action. A sufficient —but not necessary— condition for the monodromy groups to be solvable is

Monodromy Theorem — Riemann surfaces

Every Riemann surface defined globally by a (multi-valued) explicit equation admitting almost everywhere a Puiseux series' expansion in $\mathbb{C}[\![z]\!]^\natural$ (cfr. §1.3.4) has solvable and isomorphic monodromy groups.

This Theorem [37] roughly states that if $\mathcal{R}$'s definition is, globally, of type $w = f(z)$, where f is composed of rational functions and arithmetic roots like '$\sqrt[p]{\ }$' (even for different $p\in\mathbb{N}$) only, then its monodromy groups are solvable and do not depend on the base point z. Every path on its covering space lifts to one whose ending point is computable closed-form (as a permutation of complex-points 'fixing' a branch for all the visited sheets).

The "topological" monodromy group defined in §2.3.3 is connected to the "algebraic" one defined in §1.3.1 for such Riemann surfaces. Since $\mathbb{C}\{f(z)\}$ is always a Galois extension field of $\mathbb{C}\{z\}$, $\forall f \in \mathbb{C}\{z\}$ (thanks to $\mathbb{C}$ being algebraically closed — cfr. §1.1.3), for every Riemann surface which is *locally*, explicitly defined by a multi-valued rational function like $w = f(z)$,

$$\mathrm{MG}f \cong \mathrm{MG}_f\mathbb{C}\{z\}. \tag{1.258}$$

This fact actually holds globally on every Riemann surface: in fact, from the final Theorem in §0 applied to a branched holomorphic *self*-covering of $\mathcal{R}$ given by a meromorphic function $f \in \mathcal{K}(\mathcal{R})$ —which is always a (branched) Galois cover— the isomorphism (1.234) proves the diagram below.

$$\begin{array}{ccc} \mathrm{Gal}\,\dfrac{\mathcal{K}(f(\mathcal{R}))}{\mathcal{K}(\mathcal{R})} & \equiv & \mathrm{MG}_f\mathcal{K}(\mathcal{R}) \\ \|\wr & & \wr\| \\ \mathrm{Deck}(f(\mathcal{R}),\mathcal{R}) & \cong & \mathrm{MG}f \end{array} \tag{1.259}$$

Furthermore, by the Fundamental Theorem of Galois Theory, cfr. §1.1.10, if f is a holomorphic n-cover, then also $[\mathcal{K}(f(\mathcal{R})):\mathcal{K}(\mathcal{R})] = \mathrm{ord}\,\mathrm{MG}f = n$.

The right part of (1.259) is the global equivalent of (1.258), whereas its upper part is the algebraic definition (1.74) of the monodromy group of field $\mathcal{K}(\mathcal{R})$, as given in §1.3.1. The lower part is also a rough proof of the isomorphism between monodromy and deck transformation groups sketched in §2.3.3, for the Riemann surfaces' case. As a Corollary, the Puiseux's Theorem of §1.3.4 immediately descends, as $\pi^{-1}(z)$ in (1.232) is in fact a multi-valued function which, locally developed into a Puiseux series, implies (1.87).

From the viewpoint of §0, the nontriviality of the a Riemann surface's monodromy group depends, locally, on the fact that different branch points are present on a subset of sheets only, rather than on all; this actually prohibits some 'direct' travels among sheets, constraining them to the case where the path winds around specific branch points (or crosses specific branch cuts), also depending on which sheet the path actually starts from.

> The 3-sheeted Riemann surface treated in the example of §0 and defined by the implicit equation $w^3 - 3w - z + 1 = 0$ has a trivial monodromy group $\mathbb{D}_6 \cong \mathbb{S}_3$, despite not featuring the same branch points on every sheet.

The following is a 'structure Theorem' for the monodromy groups of compact Riemann surfaces.

Theorem

Let $c \in \mathrm{C}^1(\mathcal{C},\mathcal{R})$ be the universal cover of a compact Riemann surface and $\omega \in \mathrm{Z}^{1,0}(\mathcal{R})$ (such that $\mathrm{d}\omega = 0$ and $\langle \gamma | \omega \rangle \in \mathbb{Z}$ on every 1-cycle $\gamma \in \mathrm{Z}_1(\mathcal{R})$ of an homology basis, cfr. §2.4.2). Then $c_*\omega := \omega \circ c$ is exact, i.e. $c_*\omega \in \mathrm{B}^{1,0}(\mathcal{C})$ and locally $c_*\omega = \mathrm{d}f$, where $f \in \mathcal{O}(\mathcal{C})$ is a holomorphic function with

$$f(z) = \int_{z_0}^{z} c_*\omega \tag{1.260}$$

(i.e. the integral does ***not*** depend on the path joining z_0 to z on $\mathcal{C}$).

Among all the loops $c \in \mathrm{Z}_1(\mathcal{R})$ their homology class $[\![c]\!] \in \mathrm{H}_1(\mathcal{R})$ such that $\langle c | \omega \rangle = 0$ contains in fact $[\pi_1(\mathcal{R}) | \pi_1(\mathcal{R})]$ by (1.2), and the minimal subgroup satisfying this condition is exactly the group $c_*\pi_1(\mathcal{C}, z_0)$.

The monodromy groups are exactly $\mathrm{MG}_z\mathcal{R} = \pi_1(\mathcal{R},z)/c_*\pi_1(\mathcal{C})$, by (1.212). Furthermore, if $\mathrm{Tor}\mathrm{H}_1(\mathcal{R}) = \{e\}$ (i.e. $\mathcal{R}$ has torsion-free homology), then $\mathrm{MG}\mathcal{R}$ is abelian free, thus the Kronecker's Basis Theorem applies to it for decomposition purposes [9]–§A.3:

$$\mathrm{MG}\mathcal{R} \cong \mathbb{Z}^{\beta(\mathrm{MG}\mathcal{R})}. \tag{1.261}$$

By (1.260) it follows that the monodromy group of a compact Riemann surface is computable as quotient group (1.212) once that the minimal subgroup $c_*\pi_1(\mathcal{C}, z_0) \le \pi_1(\mathcal{R}, z)$ for its universal covering has been found, i.e. the function $f \in \mathcal{O}(\mathcal{C})$ (which is represented by a multi-valued function of $\mathbb{C}$) satisfying (1.260) has been built.

Consider the n-dimensional torus $\mathbb{T}^n$ (compact and connected) whose unique local chart is given by multi-angle $\boldsymbol{\phi}\equiv(\phi_1,\phi_2,\ldots,\phi_n)\in[0,2\pi[^n$ (with respect to Lie-group isomorphism $\mathbb{T}^n\leftrightarrow\mathbb{R}^n/2\pi\mathbb{Z}^n$). Let $\mathbf{I}\equiv(I_1,I_2,\ldots,I_n)\in\mathbb{R}^n$ and consider the (real) 1-form $\mathbf{I}\mathrm{d}\boldsymbol{\phi}\in\mathrm{Z}^1(\mathbb{T}^n)$, [8],

$$\mathbf{I}\mathrm{d}\boldsymbol{\phi}\equiv\sum_{k=1}^{n}I_k\mathrm{d}\phi_k.$$

The monodromy group $\mathrm{MG}\mathbb{T}^n$ is abelian free and

$$\mathrm{rk}\,\mathrm{MG}\mathbb{T}^n=\dim_{\mathbb{Q}}\mathrm{span}\,\mathbf{I}. \tag{1.262}$$

2.4.6. Linear differential equations on a Riemann surface

By now all the tools for linear ordinary differential equations to be treated on Riemann surfaces have been set.

Theorem

Let $\mathcal{R}$ be a Riemann surface and $c\in\mathrm{C}^0(\mathcal{C};\mathcal{R})$ its universal covering; let[91] $A\in\mathbb{M}_n(\mathcal{O}(\mathcal{R}))$, $z_0\in\mathcal{C}$ and $\mathbf{w}_0\in\mathbb{C}^n$. There exists a unique holomorphic vector field $\mathbf{w}\in\mathcal{O}(\mathcal{C})^n$ such that, [45]:

$$\begin{cases}\mathrm{d}\mathbf{w}=(c^*A)\mathbf{w}\\ \mathbf{w}(z_0)=\mathbf{w}_0\end{cases} \tag{1.263}$$

If $A\,\mathrm{d}z\in\mathbb{M}_n(\Lambda^{1,0}(\mathcal{R}))$ and $\mathcal{R}$ is simply connected then $\mathcal{C}\equiv\mathcal{R}$ and (1.263) can be rewritten as:

$$\begin{cases}\mathrm{d}\mathbf{w}(z)=A(z)\mathbf{w}(z)\mathrm{d}z\\ \mathbf{w}(z_0)=\mathbf{w}_0\end{cases}\qquad\Rightarrow\qquad\mathbf{w}'(z)=A(z)\mathbf{w}(z).$$

[91] The functional matrix $A(z)$ is also called the ***dynamical matrix*** of the ODE above.

As in the Theory of linear ODE on Euclidean spaces [7], the solutions to equations of type (1.263) on $\mathcal{C}$ form a n-dimensional vector space. If n linearly independent n-tuplets $\mathbf{w}_{0,1},\mathbf{w}_{0,2},\dots,\mathbf{w}_{0,n}\in\mathbb{C}^n$ are chosen (i.e. $\dim_{\mathbb{C}}\mathrm{span}(\mathbf{w}_{0,j})_{1\le j\le n}=n$), then a solutions' basis for such initial conditions is given by $\mathbf{w}_1,\mathbf{w}_2,\dots,\mathbf{w}_n\in\mathcal{O}(\mathcal{C})^n_,$ with $\mathbf{w}_j(z_0)=\mathbf{w}_{0,j}$, $1\le j\le n$. The fundamental system of solutions is the invertible column-block matrix $\Phi=\begin{pmatrix}\mathbf{w}_1 & \mathbf{w}_2 & \cdots & \mathbf{w}_n\end{pmatrix}\in\mathrm{GL}_n(\mathcal{O}(\mathcal{C}))$ which then satisfies:

$$\begin{cases}\mathrm{d}\Phi = (c^*A)\Phi\\ \Phi(z_0) = (\mathbf{w}_{0,j})^{\mathsf{T}}_{1\le j\le n}\end{cases}$$

For any covering transformation $\sigma\in\mathrm{Deck}(\mathcal{C}\xrightarrow{c}\mathcal{R})\cong\pi_1(\mathcal{R})$ the matrix $\sigma^*\Phi=\Phi\circ\sigma$ is still a fundamental system, together with its change-of-basis matrix $M_\sigma\in\mathrm{GL}_n(\mathbb{C})$, called a ***monodromy matrix*** of Φ. Monodromy matrices of a linear ODE are the natural tools to express the automorphy factors of its multi-valued solution, as they were introduced in (1.229), §§0–2.4.2. Since $M_{\sigma\tau}=M_\sigma M_\tau$, $\forall\sigma,\tau\in\mathrm{Deck}(\mathcal{C},\mathcal{R})$, the map $\sigma\mapsto M_\sigma$ is a homomorphism $\pi_1(\mathcal{R})\to\mathrm{GL}_n(\mathbb{C})$, it depends on the homotopy classes in $\mathcal{R}$, so it can also be indicated as $M_{[\![\gamma]\!]}\in\mathrm{GL}_n(\mathbb{C})$, for $[\![\gamma]\!]\in\pi_1(\mathcal{R})$:

$$M_\sigma\equiv M_{[\![\gamma]\!]}\ \text{ for }\ \sigma\leftrightarrow[\![\gamma]\!]:\qquad \sigma^*\Phi=\Phi M_{[\![\gamma]\!]}. \tag{1.264}$$

From this follows that the matrix function $(\mathrm{d}\Phi)\Phi^{-1}\in\mathbb{M}_n(\Lambda^{1,0}(\mathcal{C}))$, that is $(\mathrm{D}\Phi)\Phi^{-1}\in\mathbb{M}_n(\mathcal{O}(\mathcal{C}))$, is invariant by covering transformations, so $\exists A\in\mathbb{M}_n(\mathcal{O}(\mathcal{R}))$ such that $c^*A\equiv(\mathrm{D}\Phi)\Phi^{-1}$ and Φ is also a fundamental system of solutions of the ODE $\mathrm{d}\mathbf{w}=A\mathbf{w}$ on the non-simply connected Riemann surface $\mathcal{R}$.

In this linear case, this also means that the dynamical matrix is the logarithmic derivative of the solution (as well as $(\mathrm{d}\Phi)\Phi^{-1}$ is its logarithmic differential), i.e.:

$$A(c(z)) = \mathrm{D}\log\Phi(z) \equiv \Phi'(z)\Phi(z)^{-1}_{,} \tag{1.265}$$

which proves that, for a linear (system of) ODE(s), zeroes and poles (as well as the branch points) of fundamental solutions are exactly the simple poles of the dynamical matrix — which constitute the differential operator's *fixed* singularities.

Consider the following ODE on the punctured disc $\mathbb{B}_\rho(0)\setminus\{0\}$:

$$\mathbf{w}'(z) = \frac{A}{z}\mathbf{w}(z), \qquad A\in\mathbb{M}_n(\mathbb{C}),$$

whose dynamical matrix is $z^{-1}A$. The fundamental system of solutions is given by[92] $\Phi(z)=e^{A\log z}$ on the universal covering of the punctured disc, whose deck transformation group is isomorphic to $\pi_1(\mathbb{B}_\rho(0))\setminus\{0\})\cong\mathbb{Z}(+)$, since $\sigma^n\log z=\log z+2\pi\imath n$, $\forall n\in\mathbb{Z}$. Then the (unique) associated monodromy matrix is $M=e^{2\pi\imath A}\in\mathrm{GL}_n(\mathbb{C})$, which is trivially invariant on any logarithmic branch.

Matrix $M=e^{2\pi\imath A}$ of previous example is such that, each time a solution $\mathbf{w}(z)$ winds $k\in\mathbb{Z}$ times[93] around the simple pole 0, it is transformed into $M\mathbf{w}(z)$. Let $[\![\gamma]\!]\in\pi_1(\mathbb{B}_\rho(0)\setminus\{0\})$ be such a (homotopy class of) curve(s), parametrized with $t\in[0,1]$, then:

$$\mathbf{w}(z+\gamma(1)) = e^{2\pi\imath A}\mathbf{w}(z+\gamma(0)), \qquad \forall z\in\mathbb{B}_\rho(0)\setminus\{0\}. \tag{1.266}$$

This is easy to understand since the single-valued map $\log^{-1}:\mathbb{C}\to\mathbb{B}_\rho(0)\setminus\{0\}$ is the universal cover of the punctured disc and, in fact, the fundamental system of solutions $\Phi(z)$ in $\mathbb{C}$ is invariant by the monodromy matrix, in fact:

$$\left.\begin{aligned}\Phi(z)M^k &= e^{A\log z}e^{2\pi\imath kA}\\ \sigma^k * \Phi(z) &= e^{A\sigma^k\log z}\end{aligned}\right\} = e^{A(\log z+2\pi\imath k)} \equiv \Phi(z).$$

Also note that, in this simple case, $\mathrm{MG}(\mathbb{B}_\rho(0))\setminus\{0\})=\pi_1(\mathbb{B}_\rho(0))\setminus\{0\})\cong\mathbb{Z}(\cdot)$.

It is a well-known Calculus technique to reduce any [system of] linear, n^{th}-order ODE(s) into a larger, linear, 1^{st}-order system by a linear change of variables, so the

[92] For any $A\in\mathbb{M}_n(\mathcal{O}(\mathcal{R}))$ recall that $\det e^A=e^{\mathrm{Tr}A}$ and, if $[A\,|\,A']\equiv O_n$, then $\mathrm{D}(e^A)=e^AA'\equiv A'e^A$.

[93] Conventionally it is k times clockwise if $k\geq 0$, or $|k|$ times counter-clockwise if $k<0$, cfr. §3.2.1.

result of the Theorem above still holds for any [system of] higher-order linear ODE(s) on Riemann surfaces.

> Consider the ODE on the *punctured* disc $\mathbb{B}_\rho(0)\backslash\{0\}$, for $a,b\in\mathcal{O}(\mathbb{B}_\rho(0))$:
>
> $$z^2w''(z)+za(z)w'(z)+b(z)w(z)=0\,,$$
>
> which has two linearly independent solutions, $w_1(z)$ and $w_2(z)$. By the change of variables $\mathbf{w}=(w,zw')$, the dynamical matrix is
>
> $$A(z)=\frac{\begin{pmatrix}0 & 1\\ -b(z) & 1-a(z)\end{pmatrix}}{z}\,,$$
>
> whereas the fundamental system is $\Phi(z)=z^n e^{B\log z}\Psi(z)$, with $n\in\mathbb{Z}$,
>
> $$\Phi(z)=\begin{pmatrix}w_1(z) & w_2(z)\\ zw_1'(z) & zw_2'(z)\end{pmatrix},$$
>
> $\Psi\in\mathrm{GL}_2(\mathcal{O}(\mathbb{B}_\rho(0)))$, $B\in\mathbb{M}_2(\mathbb{C})$. Since B can have two different Jordan normal forms —either $B\sim\mathrm{diag}(\lambda_1,\lambda_2)$ or $B\sim J_{\lambda,2}$, cfr. [9]–§C.2— then respectively $e^{B\log z}\sim\mathrm{diag}(z^{\lambda_1},z^{\lambda_2})$ or $e^{B\log z}\sim z^\lambda J_{\log z,2}$, so for any two independent solutions $\psi_1,\psi_2\in\mathcal{O}(\mathbb{B}_\rho(0))$ either
>
> $$\mathbf{w}(z)=z^n\begin{pmatrix}z^{\lambda_1}\psi_1(z)\\ z^{\lambda_2}\psi_2(z)\end{pmatrix},\qquad\text{or}\qquad \mathbf{w}(z)=z^{n+\lambda}\begin{pmatrix}\psi_1(z)\\ \psi_1(z)\log z+\psi_2(z)\end{pmatrix}.$$

An application to the previous example is reported in §2.4.8, which is an overwiew of the Bessel-type equations.

The ODE theory on Riemann surfaces (as well as the PDE theory on Kähler manifolds, cfr. §2.2.4) becomes even more interesting and less tied to the Euclidean theory when several polar singularities of the dynamical matrix $A(z)$ are present; only a brief review of the problem will be treated, which will be then applied throughout §3.

Let us now consider a slightly more general case, where the dynamic matrix has l distinct simple poles $z_1,z_2,\ldots,z_l\in\mathbb{C}$, i.e. on the Riemann surface $\mathcal{R}=\mathbb{C}\backslash\{z_1,z_2,\ldots,z_l\}$:

$$A(z) = \sum_{j=1}^{l} \frac{A_j}{z - z_j}, \tag{1.267}$$

where[94] $A_1,A_2,\ldots,A_l\in\mathbb{M}_n(\mathbb{C})$. This is not the most general form, due to the singular points being simple polar and the fact that they are common to all of $A(z)$'s elements. It can be proven that there exist exactly l monodromy matrices $M_1,M_2,\ldots,M_l\in\mathrm{GL}_n(\mathbb{C})$, such that if $\sigma,\tau\in\mathrm{C}^0([0,1];\mathcal{R})$ are two equivalent paths winding $k_j\in\mathbb{Z}$ times around the j^{th} pole z_j, $1\leq j\leq l$, and supposing the winding order follows the natural ordering (first k_1 times around z_1, then k_2 times around z_2, … and finally k_l times around z_l), then

$$M_{[\![\sigma]\!]}=M_{[\![\tau]\!]}=M_l^{k_l}\,M_{l-1}^{k_{l-1}}\cdot\ldots\cdot M_2^{k_2}\,M_1^{k_1}\,. \tag{1.268}$$

Furthermore, let $\gamma\in\mathrm{Z}_1(\mathcal{R})$ be a loop winding once around z_j, then $M_{[\![\gamma]\!]}=M_j$ and, as a generalization of (1.266),

$$\mathbf{w}(z+\gamma(1)) = M_j\mathbf{w}(z+\gamma(0)), \qquad \forall z\in\mathcal{R}. \tag{1.269}$$

If the composition of the l paths is null-homotopic (and they are all based on the same regular point, otherwise non-intersecting), then $M_lM_{l-1}\cdot\ldots\cdot M_2M_1\sim I_n$.

Monodromy Theorem — Linear ODEs

The group generated by the monodromy matrices of the ODE system:

$$\mathbf{w}'(z) = \sum_{j=1}^{l} \frac{A_j}{z - z_j}\mathbf{w}(z), \qquad z\in\mathcal{R}, \tag{1.270}$$

[94] Trivially, the residue matrices are $\mathrm{Res}_A z_k = A_k$, $1\leq k\leq l$.

(modulo similarities) is isomorphic to the monodromy group (1.257) of the underlying Riemann suface $\mathcal{R}$, in the following sense:

$$\begin{aligned} \langle M_j \rangle_{1\leq j\leq l}(\cdot) &\cong \pi_1(\mathcal{R}); \\ \langle [\![M_j]\!] \rangle_{1\leq j\leq l} &\cong \mathrm{MG}\mathcal{R}. \end{aligned} \tag{1.271}$$

It can be proven that all the singular points of the solution $\mathbf{w}(z)$ of (1.270) are **fixed** (i.e. they do not depend on the integration constant $\mathbf{w}(z_0)=\mathbf{w}_0$) and correspond to the singularities of the dynamical matrix — in this case the distinct points $z_1,z_2,\ldots,z_l\in\mathbb{C}$, [1].

From the Galois Theory's point of view, cfr. §1.1.10, the monodromy group of $\mathcal{R}$, as being generated by the (equivalence classes of) monodromy matrices of the ODE system, is equivalent to the Galois group of a differential field extension of the functions' field which the dynamical matrix belongs to. Roughly, by the Fundamental Theorem of Differential Galois Theory, let $A\in\mathbb{M}_n(\mathbf{K}_\mathcal{R})$, where $\mathbf{K}_\mathcal{R}$ is a functions' field on $\mathcal{R}$; let $\mathbf{w}\in\mathbf{F}^n_\mathcal{R}$, where $\mathbf{F}_\mathcal{R}$ is a elementary differential extension field of $\mathbf{K}_\mathcal{R}$, and

$$\mathrm{MG}\mathcal{R} \cong \mathrm{Gal}\,\frac{\mathbf{F}_\mathcal{R}}{\mathbf{K}_\mathcal{R}}, \tag{1.272}$$

whose elements are the cover transformations which fix the functions of $\mathbf{K}_\mathcal{R}$ used in $A(z)$, i.e. the integration constants of the differential operator (its constants as a derivation of $\mathbf{K}_\mathcal{R}$), which lie in $\mathrm{con}_\mathrm{d}\mathbf{K}_\mathcal{R}=\mathrm{con}_\mathrm{d}\mathbf{F}_\mathcal{R}$. In the above case (1.272), $\mathbf{w}(z)$ is the anti-derivative computed using the logarithmic extensions for $\log(z-z_j)$ and the matrices A_j as constants, i.e. $A_j\in\mathrm{con}_{\mathrm{d}/\mathrm{d}z}\mathbb{M}_n(\mathcal{O}(\mathcal{R}))$, all for $1\leq j\leq l$.

Every change in the position of the singularities which is invariant under $\mathrm{MG}\mathcal{R}$ —i.e. that leaves the monodromy matrices $M_1,M_2,\ldots,M_l$ unchanged in $\mathrm{GL}_n(\mathbb{C})/\sim$ by the Monodromy Theorem (1.271)— is called a ***isomonodromic* transformation**.

Schlesinger's Theorem

The isomonodromy condition for (1.270) is::

$$\begin{cases} \dfrac{\partial A_i}{\partial z_j} = \dfrac{[A_i \,|\, A_j]}{z_i - z_j}, & 1 \leq i < j \leq l, \\ \displaystyle\sum_{j=1}^{l} \dfrac{\partial A_j}{\partial z_j} = O_n. \end{cases} \tag{1.273}$$

In the above case the isomonodromy condition is expressed as isospectrality of the dynamical matrix, as a consequence of the Lax Pair Theorem (1.231).

Even more interesting (and often just partially solved) results turn up if one extends the theory to *nonlinear* differential equations. Being such topic highly advanced and extending beyond the scope of this work (despite being strictly related with problems that will be opened in the next Chapters), only references will be given thus far, such for example [1], [7] and [61].

2.4.7. Hilbert transform and Riemann-Hilbert problems

Let $\gamma \subset \tilde{\mathbb{C}}$ be a closed, regular, simple curve on the Riemann sphere — dividing it into hemispheres, D_+ and D_-, such that $\gamma = \partial D_\pm$, $D_+ \cap D_- = \varnothing$ and $\tilde{\mathbb{C}} = D_+ \cup D_- \cup \gamma$. A function $f: \tilde{\mathbb{C}} \setminus \gamma \to \tilde{\mathbb{C}}$ satisfying a certain property $\mathcal{P}$ independently on both D_+ and D_- is said to γ-*sectionally* satisfy $\mathcal{P}$ if f has limits in every point of γ defined when $z \to \gamma$ along paths entirely contained within either D_+ or D_-. Continuing f along γ in these two (usually different) ways the functions $f_\pm : \overline{D_\pm} \setminus \{\infty\} \to \tilde{\mathbb{C}}$ are the function defined on $\overline{D_\pm}$ and satisfy $\mathcal{P}$ on the relating hemisphere only (and possibly on its boundary γ from the "same-signed" side). Conventionally, D_+ is the left-handed hemisphere with respect to γ (unless f is not defined in ∞ and $\infty \notin \gamma$, in which case let $D_- \ni \infty$).

For example if $f \in \mathcal{O}(\tilde{\mathbb{C}} \setminus \gamma)$ or $f \in C^0(\overline{D_+}) \cup C^0(\overline{D_-})$ it is said to be γ-sectionally holomorphic or γ-sectionally continuous, respectively; in particular f retains the holomorphicity [continuity] condition approaching towards γ from either D_+ *or* D_-, although the limit behaviour may be different. These definitions can be extended (but are significant to certain properties $\mathcal{P}$ only) to open curves too; in such a case the hemispheres fail to be disconnected with each other, but different *limit* values $f_\pm(\tau)$ may exist $\forall \tau \in \mathring{\gamma}$, when $z \to \tau$ approaching from either sides of the curve. In fact, the functions need not to be defined at the end-points of γ any more, with singularities or branch points appearing in them (that is why $\mathring{\gamma}$ is used in this case).

The Hilbert Transform — real functions

The **Hilbert transform** is the imaginary unitary operator $\mathcal{H} \in \mathfrak{U}(\mathrm{L}^p(\mathbb{R}))$, $p \in [1, +\infty[$ (which is also bounded for $p > 1$), defined by means of a Cauchy principal value integral '$⨍$' (also written as a principal-value convolution):

$$\mathcal{H}[f](x) \stackrel{\text{def}}{=} \frac{1}{\pi} ⨍_{-\infty}^{+\infty} \frac{f(\xi)}{x - \xi} \mathrm{d}\xi = \frac{1}{\pi x} \circledast f(x) \,. \tag{1.274}$$

If $p \in [1,2]$ then the relation between the Fourier and the Hilbert transforms of f is that, using the former transform's convolution rule, as well as (1.274) in the convolution's sense,

$$\mathcal{F}[\mathcal{H}[f]](\xi) = -\imath \hat{f}(\xi) \operatorname{sign} \xi, \tag{1.275}$$

i.e. the positive [negative] complex spectral components of $\mathcal{F}[\mathcal{H}[f]]$ are equal to those of $\hat{f}$ rotated by $\pi/2$ radiants [counter-]clockwise, cfr. §1.2.4.

Since $\mathcal{H}^2 = -\mathbb{1}$, it follows that $\mathcal{H}$ is an imaginary automorphism in $\mathrm{L}^p(\mathbb{R})$ like those defined in §2.1.1, that is $\mathcal{H}[\mathcal{H}[f]] = -f$ in $\mathrm{L}^p(\mathbb{R})$: complex-valued p-integrable functions are complexifications of real-valued ones, whose imaginary parts are Hilbert transform of their real parts, which means $\mathrm{L}^p(\mathbb{R}) = \mathrm{L}^p_{\mathbb{R}}(\mathbb{R})^{\mathbb{C}}$. Formally this fact —that, if applied to the Fourier transforms' real and imaginary part, is called

Kramers-Krönig relation, [63]— is a particular case of the **Plemelj Formula** below. $\mathcal{H}$ also maps even [odd] functions to odd [even] ones. For real-valued functions $f\in\mathrm{L}^p_{\mathbb{R}}(\mathbb{R})$ in particular, the complexified function $f(x)\pm\imath\mathcal{H}[f](x)$ can be analytically continued to the upper [lower] half-plane $\mathbb{C}_\pm$ into a $\mathbb{R}$-sectionally holomorphic function $f_\pm\in\mathcal{H}^p(\mathbb{C}_\pm)$ decaying at infinity (off the real axis $\mathbb{R}$, which is its natural boundary), i.e. $\lim_{z\to\infty} f_\pm(z)=0$, cfr. §2.4.1.

Furthermore $\forall f,g\in\mathrm{L}^p(\mathbb{R})$, $\mathcal{H}[g\mathcal{H}[f]+f\mathcal{H}[g]]=\mathcal{H}[f]\mathcal{H}[g]-fg$ and, if $\exists\alpha\in]0,1]$ with $f\in\mathrm{C}^{1,\alpha}(\mathbb{R})\cup\mathrm{L}^p(\mathbb{R})$, then $\mathcal{H}[f](x)=\pi^{-1}(f'(x)\circledast\log|x|)=\pi^{-1}\mathrm{D}\left[f(x)\circledast\log|x|\right]$.

Lastly, the Hilbert transform commutes with any (weak) differential operator, i.e.:

$$\mathcal{H}\left[\mathbb{D}^h\right]=\mathbb{D}^h\mathcal{H},\qquad \forall f\in\mathrm{W}^{k,p}(\mathbb{R}),\quad h\leq k. \tag{1.276}$$

Consider the real ODE

$$\dot{u}-u\mathcal{H}[u]=0\,,$$

which models the vorticity of inviscid, incombressible fluid flows. Let $w(t):=u(t)+\imath\mathcal{H}[u](t)$, the above ODE is equivalent to the complex-valued one $\dot{w}-\frac{1}{2}w^2=0$. Since $\mathcal{H}[w^n]=-\imath w^n$ ($\forall n\in\mathbb{N}$) then, more generally, $\mathcal{H}[e^w]=-\imath e^w$, as well as

$$\mathcal{H}\left[\sum_{n\in\mathbb{N}_0}a_n w^n\right]=-\imath\sum_{n\in\mathbb{N}_0}a_n w^n\quad\Leftrightarrow\quad\mathcal{H}\left[f(w)\right]=-\imath f(w),$$

for any uniformly converging power series in $\mathbb{C}[\![z]\!]$ (i.e. any $f\in\mathcal{O}(\mathbb{C})$). Many reducible models for nonlinear integro-differential equations are then derived using this Hilbert-transform method, such as:

$$\left.\begin{aligned}\dot{u}&=u^3-2u\mathcal{H}[u]^2\\ \dot{u}&=e^{\mathcal{H}[u]}\sin u\\ \frac{\partial u}{\partial t}&=\frac{\partial^2 u}{\partial x^2}+2\frac{\partial}{\partial x}u\mathcal{H}[u]\\ \frac{\partial u}{\partial t}&=\frac{\partial^2\mathcal{H}[u]}{\partial x^2}+2\frac{\partial}{\partial x}u\mathcal{H}[u]\end{aligned}\right\}\xleftrightarrow{\ \mathcal{H}\ }\left\{\begin{aligned}\dot{w}&=w^3\\ \dot{w}&=\imath e^{-\imath w}\\ \frac{\partial w}{\partial t}&=\frac{\partial^2 w}{\partial x^2}-\imath\frac{\partial^2}{\partial x^2}w^2\\ \frac{\partial w}{\partial t}&=-\imath\left(\frac{\partial^2 w}{\partial x^2}+2w\frac{\partial w}{\partial x}\right)\end{aligned}\right.$$

The Hilbert transform is also defined for functions of (one) complex variable(s), cfr. §2.4.1: in this case its definition depends on the simple, closed path which principal-value integration is computed on. In particular, analytical continuation results exposed above can be extended to the case of real-valued functions of a complex variable.

The Hilbert Transform complex functions

Let $\gamma\subset\mathrm{Z}_1(\tilde{\mathbb{C}})$ be a closed, regular, simple curve, $f\in\mathcal{K}(D_\pm)\cup\mathrm{C}^{0,\alpha}(\gamma)$ be a meromorphic function defined in either one of the two hemispheres D_+ and D_- which γ divides the Riemann sphere into, which is also α-Hölder on γ itself, for some $\alpha\in]0,1]$. The Hilbert transform of f with respect to γ is the function defined, on either D_+ and D_- respectively, as the Cauchy principal value circulation

$$\mathcal{H}[f](z)\overset{\text{def}}{=}\frac{1}{2\pi\imath}\oint\limits_{+\gamma}\frac{f(\zeta)}{\zeta-z}\,\mathrm{d}\zeta\,,\qquad\forall z\in\overline{D_\pm}.\tag{1.277}$$

The Hilbert transform of a function $f\in\mathrm{L}^2_{\mathbb{R}}(\partial D_+)$ defined as (1.277) is a square-integrable function in the closed hemisphere, i.e. $\mathcal{H}[f]\in\mathrm{L}^2(\overline{D_+})$, and the two functions combine to a Hardy function on the curve ∂D_+, i.e. $f+\imath\mathcal{H}[f]\in\mathcal{H}^2(\gamma)$, cfr. §2.4.7. If $f\in\mathcal{O}(\Omega)$ such that $D_+\subset\Omega$ is connected and simply connected, (1.277) allows to rewrite the generalized Cauchy integral formula (1.150) for $n=1$ as:

$$f(z)=\frac{1}{2\pi\imath}\left(\oint\limits_{+\gamma}\frac{f(\zeta)}{\zeta-z}\,\mathrm{d}\zeta+\int\limits_{D_+}\frac{f(\zeta)}{\zeta-z}\,\mathrm{d}\zeta\wedge\mathrm{d}\overline{\zeta}\right),\qquad\forall z\in D_+.$$

In other words the Hilbert transform, acting as an imaginary automorphism for $\mathcal{O}(D_+)$, provides a **continuous** orthogonal decomposition of the complexified functional space; such representation is given by a linear functional like (1.424), expressed as $(2\pi\imath)^{-1}$ times the integral of the $(1,1)$-form of complex-conjugated coordinates $\mathrm{d}z\wedge\mathrm{d}\overline{z}\in\mathrm{B}^{1,1}(\mathbb{C})$, provides an ***un*countably** analogue to (1.129), and is for this reason called completeness relation (more on that in §4.2.1):

$$f(z)-\imath\mathcal{H}[f](z)=\frac{1}{2\pi\imath}\int\limits_{D_+}\frac{f(\zeta)}{\zeta-z}\mathrm{d}\zeta\wedge\mathrm{d}\overline{\zeta},\qquad\forall z\in D_+.\tag{1.278}$$

As an example, let $\gamma=\partial\mathbb{B}_1(0)$ and let the Hilbert transform be considered on the unit circumference (as a periodic, complex-valued function of a real variable):

$$\mathcal{H}_{\partial\mathbb{B}_1(0)}[f](e^{\imath\theta})\overset{\text{def}}{=}\frac{1}{2\pi}\int\limits_{-\pi}^{\pi}f\left(e^{\imath(\theta-\vartheta)}\right)\mathrm{Im}\frac{1+e^{\imath\vartheta}}{1-e^{\imath\vartheta}}\mathrm{d}\vartheta\ .$$

Plemelj Theorem

Let $\gamma\subset\tilde{\mathbb{C}}$ a regular simple curve (either closed or open — in which latter case next results are not valid in its endpoints) and $\phi\in\mathrm{C}^{0,\alpha}(\gamma)$ be a α-Hölder function for some $\alpha\in]0,1]$, cfr. [9]–§B.3, and let

$$f(z)=\mathcal{H}[\phi](z)\equiv\frac{1}{2\pi\imath}\int\limits_{+\gamma}\frac{\phi(\zeta)}{z-\zeta}\mathrm{d}\zeta\ ,\qquad\forall z\in\tilde{\mathbb{C}}\backslash\gamma\ .\tag{1.279}$$

Then f is γ-sectionally holomorphic in $\tilde{\mathbb{C}}\backslash\gamma$ along with its limit functions $f_\pm\in\mathcal{O}(D_\pm)$ such that, on the curve itself:

$$f_\pm(\tau)=f(\tau)\pm\frac{1}{2}\phi(\tau),\qquad\forall\tau\in\mathring{\gamma}.$$

The behaviour of f at infinity (provided $\infty\notin\gamma$) is a linear decay as

$$f(z)=\mathrm{O}(z^{-1})\sim-\frac{1}{2\pi\imath z}\oint\limits_{+\gamma}\phi(\zeta)\mathrm{d}\zeta.$$

An immediate consequence of the above is that any Hölder function on such a curve γ can always be written as the difference of two γ-sectionally holomorphic functions,

$$f_+(\tau)-f_-(\tau)=\phi(\tau),\qquad\forall\tau\in\mathring{\gamma}.\tag{1.280}$$

Let $\gamma\subset Z_1(\tilde{\mathbb{C}})$ be a closed, regular, simple curve such that $0\in D_+$ and $G\equiv(g_{i,j})_{1\leq i,j\leq n}\in\mathbb{M}_n(\mathcal{K}(\mathbb{C}\backslash\gamma)\cup C^{0,\alpha}(\gamma))$ be a meromorphic matrix function in either of the two emispheres $D_\pm$, which is both α-Hölder and non-zero in γ for some $\alpha\in]0,1]$. The homogeneous vector *Riemann-Hilbert* problem (or *RH* problem) is the computation of a γ-sectionally holomorphic vector function $\phi\in\mathcal{O}(\mathbb{C}\backslash\gamma)^n$ —with at most a polar singularity at ∞— such that according to previously-introduced notation:[95]

$$\phi_+(\tau)=G(\tau)\phi_-(\tau),\qquad\forall\tau\in\gamma.\tag{1.281}$$

Let $\kappa=\mathrm{wind}_{\det G}\gamma\in\mathbb{Z}$ be the winding number (1.218) of $\det G(z)$, as a 0-form, around γ, i.e. the maximal winding number of (1.281)'s solutions: $\kappa=\max\mathrm{wind}_\phi\gamma$.

In general, vector and matrix RH problems cannot be solved in closed form except for a few cases which require iterating solutions of certain Fredholm integral equations, [1]; when $n=1$ instead, a solution of the *scalar* RH problem $\phi_+(\tau)=g(\tau)\phi_-(\tau)$ is $\phi(z)=\xi(z)P(z)$, where $P\in\mathbb{C}[z]$ is an arbitrary polynomial, $\kappa\equiv\mathrm{wind}_g\gamma=\mathrm{wind}_\phi\gamma$, and $\xi\in\mathcal{O}(\mathbb{C}\backslash\gamma)$ is called the **RH** problem's **fundamental** solution, cfr. §2.4.6, given by:[96]

$$\xi_\pm(z):=z^{-\kappa\delta_{\pm,-}}\exp\left[\frac{1}{2\pi\imath}\oint_{+\gamma}\frac{\log(\zeta^{-\kappa}g(\zeta))}{\zeta-z}\mathrm{d}\zeta\right],\qquad\forall z\in\overline{D_\pm}.\tag{1.282}$$

If $\kappa>0$ there exist exactly κ linearly independent solutions vanishing at infinity, and $\deg P=\kappa-1$; in other words, $\phi(z)\in\xi(z)(\mathbb{C}^{\kappa-1}[z])$. Otherwise, if $\kappa\leq0$, one solution with a $(|\kappa|+1)^{\text{th}}$-order pole at ∞ exists.

---

[95] Matrix RH problems are formulated with respect to a matrix solution $\Xi\in\mathbb{M}_n(\mathcal{O}(\mathbb{C}\backslash\gamma))$, such that $\Xi_+(\tau)\Xi_-^{-1}(\tau)=G(\tau)$, $\forall\tau\in\mathring{\gamma}$.

[96] The Kronecker delta $\delta_{\pm,-}$ in (1.282) is 0 for ξ_+ and 1 for ξ_-; on *opposite* hemispheres, $\xi_-(z)$ has the same definition as $\xi_+(z)$ times $z^{-\kappa}$ but $\xi_\pm(z)\neq z^{\pm\kappa}\xi_\mp(z)$ because they have different domains. Monomial $z^{-\kappa}$ is introduced in order for the function $z^{-\kappa}g(z)$ to have zero winding number, so that its logarithm is Hölder again, the Plemelj formula is applicable, and $\xi_-(z)\sim O(z^{-\kappa})$ for $z\to\infty$.

Riemann-Hilbert problems can be generalized to either inhomogeneous cases, to non-simply connected domains (i.e. non-simple γ) *and* open curves γ (equivalent to considering $0\in G(\gamma')$ on a closed curve γ' containing γ, such that $\operatorname{supp} G|_{\gamma'}=\gamma$). Only the inhomogeneous scalar RH problem is treated here, for $h\in\mathcal{K}(\mathbb{C}\backslash\gamma)\cup C^{0,\alpha}(\gamma)$:

$$\phi_+(\tau)=g(\tau)\phi_-(\tau)+h(\tau),\qquad \forall\tau\in\gamma; \tag{1.283}$$

it still has a closed-form general solution depending on the associated homogeneous problem (1.282) which —on the hemisphere D_+— is given by

$$\phi(z) = \xi(z)\left(P(z) + \psi(z)\right),\qquad \forall z\in D_+, \tag{1.284}$$

with $P\in\mathbb{C}[z]$, $\kappa=\operatorname{wind}_\phi\gamma$ and

$$\psi(z) := \frac{1}{2\pi\imath}\oint_{+\gamma}\frac{h(\zeta)\mathrm{d}\zeta}{\xi_+(\zeta)(\zeta-z)},\qquad \forall z\in D_+.$$

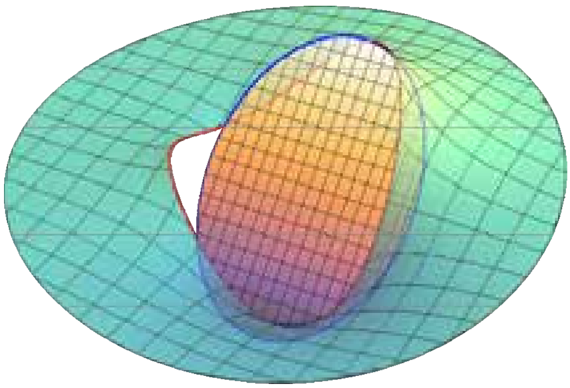

Figure 21 Plot of $\operatorname{Re}\phi_+(\tau)=\tau$ (blue) and $\operatorname{Re}\phi_-(\tau)=$ $=-\tau^{-2}(\tau+1)$ (red) for $\tau\in\partial\mathbb{B}_{1.2}(0)$. Hot- and cold-tinted surfaces around are $\operatorname{Re}\phi_+(z)$ and $\operatorname{Re}\phi_-(z)$, respectively, from the inhomogeneous scalar RH problem reported in the example below.

Furthermore if $\kappa\leq 0$ then simply $P\equiv 0$ and the unique solution is $\phi\equiv\xi\psi$; but if $\kappa<0$, then (1.283) is solvable if and only if

$$\oint_{+\gamma} \frac{z^n h(z)}{\xi_+(z)} \mathrm{d}z = 0\,, \qquad \text{for } 0{\le}n{<}{-\kappa}.$$

Let $\{0,\pm1\}\subset D_+$ and consider the inhomogeneous scalar RH problem whose solution is depicted in Figure 21 for $D_+{=}\mathbb{B}_{1.2}(0)$:

$$\phi_+(\tau) = \frac{1}{\tau-1}\left(\frac{\tau\phi_-(\tau)}{\tau+1}+\frac{1}{\tau}\right)+\tau, \qquad \forall\tau{\in}\partial D_+,$$

with $g(z){=}z/(z^2{-}1)$ (then $\kappa{=}{-}1$), $h(z){=}(z^3{-}z^2{+}1)/(z^2{-}z)$ and $\xi_+/\xi_-{\equiv}g{\in}\mathbb{C}\{z\}$ — whose the solution(s) vanish at ∞ in D_- because $g{\in}\mathcal{O}(D_-)$. The fundamental solution $\xi(z)$ of the associated *homogeneous* RH problem $\phi_+(\tau)=\frac{\tau}{\tau^2-1}\phi_-(\tau)$ is then $\xi_+(z){=}1$ and $\xi_-(z){=}(z^2{-}1)/z$.

The inhomogeneous problem has unique solution $\phi(z){=}\xi(z)\,\psi(z)$, with $\oint_{+\gamma}\left(z{+}\frac{1}{z(z-1)}\right)\mathrm{d}z{=}0$, $\psi_+(z){=}z$ and $\psi_-(z){=}{-}[z(z{-}1)]^{-1}$ so:

$$\phi_+(z) = z, \qquad \phi_-(z) = -\frac{z+1}{z^2}.$$

2.4.8. Appendix: Bessel and Airy functions and equations

This section shows the solution of a special ODE which often models wave equations of Mathematical Physics in polar/cylindrical coordinate systems, cfr. §0 and [63]. But first the special functions which constitute a solution to it need to be introduced.

Recall that J_ν and Y_ν are the ***Bessel functions*** of first and second kind, of order $\nu{\in}\mathbb{C}$, having several equivalent *multi-valued* definitions, [2], such as:[97]

[97] The definition for Y_ν is to be considered with the limit $\lim\limits_{\nu\to n}$ when ν approaches $n{\in}\mathbb{Z}$.

$$J_\nu(z) \stackrel{\text{def}}{=} \left(\frac{z}{2}\right)^\nu \sum_{k=0}^{\infty} \frac{(-1)^k}{k!\,\Gamma(\nu+k+1)} \left(\frac{z}{2}\right)^{2k};$$
$$Y_\nu(z) \stackrel{\text{def}}{=} \frac{J_\nu(z)\cos(\pi\nu) - J_{-\nu}(z)}{\sin(\pi\nu)}. \tag{1.285}$$

An equivalent integral representation for 1st-kind Bessel functions J_ν, $\forall\nu\in\mathbb{C}$, is:

$$J_\nu(z) = \frac{1}{\sqrt{\pi}\,\Gamma(\nu+\frac{1}{2})}\left(\frac{z}{2}\right)^\nu \int_0^\pi \cos(z\cos\theta)(\sin\theta)^{2\nu}\,\mathrm{d}\theta$$

and $J_\nu(-z) = (-z)^\nu z^{-\nu} J_\nu(z)$,[98] $\forall(z,\nu)\in\mathbb{C}^2$; for integer orders $n\in\mathbb{Z}$, the above become:

$$J_n(z) = \frac{1}{\pi}\int_0^\pi \cos(z\sin\theta - n\theta)\mathrm{d}\theta = \frac{1}{\imath^n\pi}\int_0^\pi e^{\imath z\cos\theta}\cos(n\theta)\mathrm{d}\theta\,;$$
$$Y_n(z) = \frac{1}{\pi}\int_0^\pi \sin(z\sin\theta - n\theta)\mathrm{d}\theta\,.$$

Both kinds of (single-valued branches of) Bessel functions are holomorphic in $(\mathbb{C}^*)^2$ and their behaviour with respect to argument and/or order conjugation is:

$$J_\nu(z) = \overline{J_{\overline{\nu}}(\overline{z})}, \qquad Y_\nu(z) = \overline{Y_{\overline{\nu}}(\overline{z})}.$$

With respect to variables $z\in\mathbb{C}^*$ and $\nu\in\mathbb{C}$ separately, they are *both* holomorphic with **essential** singularities in $z=\infty$ and $\nu=\infty$ — **entire** with respect to ν. They also feature a ***branching*** *essential singularity* in $z=\infty$ and another branch point in $z=0$, which is a branch **zero** for $J_\nu(z)$ ($\forall\nu\in\mathbb{C}\backslash\mathbb{Z}$) and a branch **singularity** for $Y_\nu(z)$ ($\forall\nu\in\mathbb{C}$), cfr. §2.4.2, and is:

- **logarithmic** branch point for $\nu\in\mathbb{C}\backslash\mathbb{Q}$,
- **algebraic** branch point of order $q-1$ for $\nu=p/q\in\mathbb{Q}$ and $q>1$ (i.e. $\mathrm{b}_{J_{\frac{p}{q}}}(0) = \mathrm{b}_{Y_{\frac{p}{q}}}(0) = q$),

---

[98] For integer orders $n\in\mathbb{Z}$ the last one becomes $J_n(-z) = (-1)^n J_n(z)$ and $Y_n(-z) = (-1)^n Y_n(z)$.

- unbranched point for $\nu = n \in \mathbb{Z}$, being thus
  - a *removable* singularity for $J_n(z)$, where it is set $J_n(0) := 0$;
  - an *essential* singularity for $Y_n(z)$.

The Bessel functions' automorphies[99] for $z \curvearrowleft 0$ (cfr. §0) using the standard branch cut $\rangle 0,\infty\langle = \mathbb{R}_-$ are different for those of 1st and 2nd kind $\forall \nu \in \mathbb{C}$, i.e., cfr. [2] and §3.1.1 for formalism (for $m \in \mathbb{N}_0$):

- $J_\nu(z \overset{m}{\curvearrowleft} 0) = e^{2m\pi\imath\nu} J_\nu(z)$;
- $Y_\nu(z \overset{m}{\curvearrowleft} 0) = e^{-2m\pi\imath\nu} Y_\nu(z) - 2\imath \sin(2m\pi\nu) \cot(\pi\nu) J_\nu(z)$;
- $Y_\nu(z \curvearrowleft 0) = e^{-2\pi\imath\nu} Y_\nu(z) + 4\imath \cos(\pi\nu)^2 J_\nu(z)$;

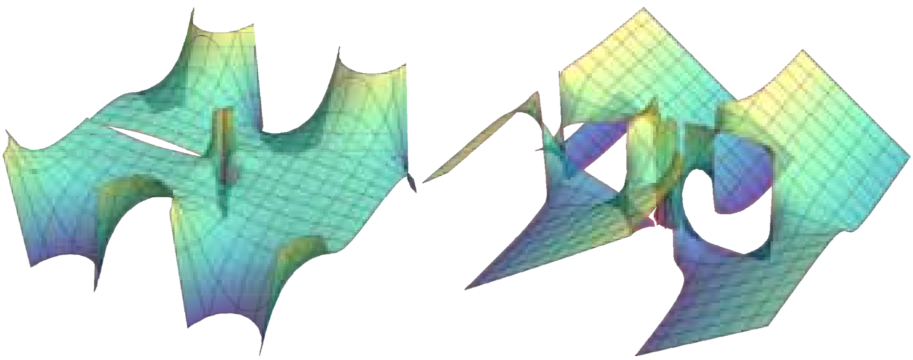

Figure 22 Plot of $\operatorname{Im} Y_3(z)$ (left) and $\operatorname{Arg} Y_3(z)$ (right) for $z \in (1+\imath)[-4,4]$. The branch cut is represented by the red curve, starting from singularity at $z=0$.

The analytic continuations above have particular equivalent formulæ when ν assumes specific values; function J_n is, in fact, single-valued, whereas:

- $Y_n(z \curvearrowright 0) = Y_n(z) - 4\imath J_n(z), \quad \forall n \in \mathbb{Z}$;
- $Y_\nu(z \curvearrowright 0) = e^{-2\pi\imath\nu} \cot(\pi\nu) J_\nu(z) - e^{2\pi\imath\nu} \csc(\pi\nu) J_{-\nu}(z), \quad \forall \nu \in \mathbb{C} \setminus \mathbb{Z}$.

[99] Thus J_ν has an automorphy **factor** of $e^{2\pi\imath\nu}\cdot$, whereas Y_ν has a linear automorphy $(e^{-2\pi\imath\nu}\cdot)+$ $+2\imath\cot(\pi\nu)J_\nu(z)\cdot\sin(2\pi\nu)$, where the sine function has increased angle at each successive winding.

Figure 22 shows an example branching of a single sheet of $Y_3(z)$, when explored *clockwise* in a neighbourhood of $z=0$. Overall, integer-order Bessel functions of 1st kind $J_n\in\mathcal{O}(\mathbb{C})$ are **entire**, $\forall n\in\mathbb{Z}$, whose partition function (1.491) is

$$\exp\frac{z(w-1)}{2w}=\sum_{n=-\infty}^{+\infty} w^n J_n(z)\in\mathcal{O}(\mathbb{C})\{w\}$$

which, for $w=e^{\imath\theta}\ \forall\theta\in 2\pi\mathbb{T}$, becomes $e^{\imath z\sin\theta}=\sum_{n\in\mathbb{Z}} e^{\imath n\theta}J_n(z)$.

When $\nu\in\mathbb{R}$ then J_ν, J_ν', Y_ν and Y_ν' all have a finite number of complex-conjugated zeroes, as well as an infinite number of real zeroes (which are all simple with possibly the exception of 0); the real positive zeroes of 1st- and 2nd-kind functions are mutually interlaced with each other; furthermore, the distribution of the complex zeroes of Y_ν and Y_ν' periodically depends on $\operatorname{frac}\nu$. If $\nu\in[-1,+\infty[$, the zeroes of J_ν are all real; otherwise, if $\nu\in]-\infty,-1[\setminus\mathbb{Z}$ then J_ν has twice $\lfloor-\nu\rfloor$ complex-conjugated zeroes, $\lfloor-\nu\rfloor\bmod 2$ of which purely imaginary. When $\nu\in\mathbb{R}_+$ also J_ν' has real zeroes only.

In such a case ($\nu\in\mathbb{R}_+$), the 1st-kind Bessel function and its derivative have the following infinite product expansions, letting $j_{\nu,s},j_{\nu,s}'\in\mathbb{R}_+^*$ be their sth positive zeroes:

$$J_\nu(z)=\frac{\left(\frac{1}{2}z\right)^\nu}{\Gamma(\nu+1)}\prod_{s=1}^{\infty}\left(1-\frac{z^2}{j_{\nu,s}^2}\right);\qquad J_\nu'(z)=\frac{\left(\frac{1}{2}z\right)^{\nu-1}}{2\Gamma(\nu)}\prod_{s=1}^{\infty}\left(1-\frac{z^2}{j_{\nu,s}'^2}\right).$$

Furthermore, some additional differential and approximation formulæ follow:

$$|J_\nu(z)|\le\begin{cases}\dfrac{e^{|\operatorname{Im} z|}}{\Gamma(\nu+1)}\left|\dfrac{z}{2}\right|^\nu, & \nu\ge-\frac{1}{2}\\ 1, & \nu\ge 0\\ \dfrac{1}{\sqrt{2}}, & \nu\ge 1\end{cases}$$

$$\frac{\partial^m J_\nu}{\partial\nu^m}=\sum_{k=0}^{\infty}\frac{(-1)^k}{k!}\left(\frac{z}{2}\right)^{2k}\frac{\partial^m}{\partial\nu^m}\frac{(z/2)^\nu}{\Gamma(\nu+k+1)};$$
$$\frac{\partial Y_\nu}{\partial\nu}=\cot(\pi\nu)\left(\frac{\partial J_\nu}{\partial\nu}-\pi Y_\nu(z)\right).$$

Products of different-order, 1st-kind Bessel functions are computed, $\forall\mu,\nu\in\mathbb{C}$, as:

$$J_\mu(z)J_\nu(z) = \left(\frac{z}{2}\right)^{\mu+\nu}\sum_{k=0}^{\infty}\frac{(-1)^k\,\Gamma(\nu+\mu+2k+1)}{k\,!\,\Gamma(\nu+k+1)\Gamma(\mu+k+1)\Gamma(\nu+\mu+k+1)}\left(\frac{z}{2}\right)^{2k}.$$

A continued fraction expansion for ratio of 1st-kind Bessel functions (cfr. §1.5.1) is:

$$\frac{J_\nu(z)}{J_{\nu-1}(z)} = \mathop{©}_{k=1}^{\infty}\frac{2(\nu+k-1)}{z} = \cfrac{1}{\frac{2\nu}{z}+\cfrac{1}{\frac{2(\nu+1)}{z}+\cfrac{1}{\frac{2(\nu+2)}{z}+\ddots}}}.$$

The Bessel functions of ***third*** **kind**, also called ***Hankel functions***, of order $\nu\in\mathbb{C}$, are following linear combinations of first- and second-kind ones, $H_\nu^{(1)}$ and $H_\nu^{(2)}$ respectively (called **Hankel** functions of first and second **kind** respectively):

$$\begin{aligned} H_\nu^{(1)}(z) &\overset{\text{def}}{=} J_\nu(z) + \imath Y_\nu(z), \\ H_\nu^{(2)}(z) &\overset{\text{def}}{=} J_\nu(z) - \imath Y_\nu(z), \end{aligned}$$

or[100] $H_\nu^{(1/2)}(z) = J_\nu(z)\pm\imath Y_\nu(z) = \pm\imath\csc(\pi\nu)\big(e^{\mp\imath\pi\nu}J_\nu(z)-J_{-\nu}(z)\big)$, where:

$$H_\nu^{(1/2)}(z) = \overline{H_{\overline{\nu}}^{(2/1)}(\overline{z})} = e^{\mp\pi\imath\nu}H_{-\nu}^{(1/2)}(z).$$

As far as analytic continuation is concerned, the automorphy factors for $z\curvearrowleft 0$, using the standard branch cut $\rangle 0,\infty\langle = \mathbb{R}_-$, are:

$$H_\nu^{(1/2)}(z\curvearrowright 0) = -e^{\mp 2\pi\imath\nu}H_{-\nu}^{(2/1)}(z). \tag{1.286}$$

$$\begin{aligned} H_\nu^{(1)}(z\curvearrowleft 0) &= -e^{-\pi\imath\nu}H_\nu^{(2)}(z) = -H_{-\nu}^{(2)}(z) \\ H_\nu^{(2)}(z\curvearrowright 0) &= -e^{\pi\imath\nu}H_\nu^{(1)}(z) = -H_{-\nu}^{(1)}(z) \end{aligned}$$

$$H_\nu^{(1/2)}(z\overset{m}{\curvearrowright}0) = \big(e^{2m\pi\imath\nu}\pm 2\sin(m\pi\nu)\cot(\pi\nu)\big)J_\nu(z)\pm\imath e^{-2m\pi\imath\nu}Y_\nu(z);$$

100 The second definition (that with the cosecant and 1st-kind Bessel function) holds $\forall\nu\in\mathbb{C}\setminus\mathbb{Z}$ only.

$$H_\nu^{(1/2)}(z \overset{m}{\curvearrowleft} 0) = \mp \frac{\sin[(m \mp 1)\pi\nu] H_\nu^{(1/2)}(z) + e^{\mp\pi\imath\nu}\sin(m\pi\nu) H_\nu^{(2/1)}(z)}{\sin(\pi\nu)}.$$

In particular, since on every double winding around 0, $H_\nu^{(1/2)}(z \overset{2}{\curvearrowleft} 0) = -e^{\pm 2\pi\imath\nu} H_{-\nu}^{(2/1)}(z \curvearrowleft 0) = e^{\pm 4\pi\imath\nu} H_\nu^{(1/2)}(z)$, it follows that a Riemann surface can be defined having an even-order branch point around 0. More exactly, it is a *double* branch point for both integer and complex orders, i.e. for $\forall \nu \in (\mathbb{C} \backslash \mathbb{Q}) \cup \mathbb{Z} = \mathbb{C} \backslash (\mathbb{Q} \backslash \mathbb{Z})$; it is a branch point of order $2(q-1)$ for non-integer, rational orders, i.e. $\forall \nu = p/q \in \mathbb{Q} \backslash \mathbb{Z}$).

Let $C_\nu(z)$, $\forall (z,\nu) \in \mathbb{C}^2$ be any linear combination of 1st-, 2nd- and 3rd-kind, νth-order Bessel functions; that is subject to the following identities (along with its holomorphic derivatives, $\forall m \in \mathbb{N}_0$), cfr. [2]:

$$\begin{aligned}
C_\nu(z) &= \frac{z}{2\nu}\left(C_{\nu-1}(z) + C_{\nu+1}(z)\right) \\
&= \frac{2(\nu \pm 1)}{z}\left(C_{\nu\pm1}(z) - C_{\nu\pm2}(z)\right); \\
C'_\nu(z) &= \frac{1}{2}\left(C_{\nu-1}(z) - C_{\nu+1}(z)\right) \\
&= \pm C_{\nu\mp1}(z) \mp \frac{\nu}{z} C_\nu(z); \\
C_\nu^{(m)}(z) &= \frac{1}{2^m}\sum_{k=0}^{m}(-1)^k \binom{m}{k} C_{\nu-m+2k}(z); \\
\left(\frac{1}{z}\frac{\mathrm{d}}{\mathrm{d}z}\right)^m z^{\pm\nu} C_\nu(z) &= (-1)^{\frac{m}{2}\mp\frac{m}{2}} z^{\pm\nu-m} C_{\nu\mp m}(z).
\end{aligned}$$

Furthermore, other general identities follow:

$$C_\nu(\lambda z) = \lambda^{\pm\nu}\sum_{k=0}^{\infty}\left(\mp\frac{z}{2}\right)^k \frac{(\lambda^2-1)^k}{k!} C_{\nu\pm k}(z)$$

(wich holds $\forall (z,\nu,\lambda) \in \mathbb{C}^3$ if $C_\nu \equiv J_\nu$, otherwise it holds only for $|\lambda^2 - 1| < 1$);

$$C_\nu(z \pm w) = \sum_{k\in\mathbb{Z}} C_{\nu\mp k}(z) J_k(w)$$

(wich holds $\forall (z,w) \in \mathbb{C}^2$ if $\nu = n \in \mathbb{Z}$ and $C_n \equiv J_n$, otherwise it holds only for $|w| < |z|$);

$$C_\nu\left(\sqrt{z^2+w^2-2zw\cos\varphi}\right) = \left(\frac{z-we^{\imath\varphi}}{z-we^{-\imath\varphi}}\right)^{\frac{\nu}{2}}\sum_{k\in\mathbb{Z}} C_{\nu+k}(z)J_k(w)e^{\imath k\varphi} \qquad \text{(Graf's)}$$

(with holds $\forall(z,w)\in\mathbb{C}^2$ only if $|we^{\pm\imath\varphi}|<|z|$). Linear combinations of 1st- and 2nd-kind, integer-order Bessel functions, $\forall n\in\mathbb{Z}$, also satisfy $C_{-n}(z)=(-1)^n C_n(z)$. For $\nu\in\mathbb{R}_+$ positive (real) zeroes of $C_\nu(z)$ and $C_{\nu+1}(z)$ are mutually interlaced with each other.

> The Bessel ODE on the punctured plane $\mathbb{C}^*$, $\forall\nu\in\mathbb{C}$,
>
> $$z^2w''(z)+zw'(z)+(z^2-\nu^2)w(z)=0, \qquad (1.287)$$
>
> has two linearly independent solutions, $H_\nu^{(1)}$ and $H_\nu^{(2)}$. If $\nu\in\mathbb{C}\backslash\mathbb{Z}$ then another set of linearly independent solutions is given by $J_\nu(z)$ and $J_{-\nu}(z)$; if $\nu=n\in\mathbb{Z}$, then they are $J_n(z)$ and $Y_n(z)$, because $J_{-n}(z)=(-1)^nJ_n(z)$ simply holds. The associated Wronskians are:
>
> $$\begin{vmatrix} H_\nu^{(1)}(z) & H_\nu^{(2)}(z) \\ H_\nu'^{(1)}(z) & H_\nu'^{(2)}(z)\end{vmatrix} = -\frac{4\imath}{\pi z}; \quad \begin{vmatrix} J_\nu(z) & Y_\nu(z) \\ J_\nu'(z) & Y_\nu'(z)\end{vmatrix} = \frac{2}{\pi z}; \quad \begin{vmatrix} J_{-\nu}(z) & J_\nu(z) \\ J_{-\nu}'(z) & J_\nu'(z)\end{vmatrix} = \frac{2\sin(\pi\nu)}{\pi z};$$
>
> such determinants are then proportional to the Green's function of the antiholomorphic derivative operator $\mathrm{d}/\mathrm{d}\bar{z}$, as in (1.154). When the Bessel equation is defined on the whole complex plane $\mathbb{C}$, then it can be shown that it has only *one* linearly independent solution $J_\nu(z)$ (or $H_\nu^{(1/2)}(z)$), $\forall\nu\in\mathbb{C}$.

The ***Airy*** «***A***» and «***B***» *functions* are *entire*, single-valued functions most-easily defined as particular cases of $(1/3)$rd-order Bessel functions (of either 1st and 2nd kind, or of 3rd kind, respectively):

$$
\begin{aligned}
\operatorname{Ai} z &\stackrel{\text{def}}{=} \frac{\sqrt{-z}}{3}\left(J_{\frac{1}{3}}\left(-\tfrac{2}{3}z^{\frac{3}{2}}\right) + J_{-\frac{1}{3}}\left(-\tfrac{2}{3}z^{\frac{3}{2}}\right)\right) \\
&= \frac{1}{2}\sqrt{\frac{-z}{3}}\left(e^{\frac{\pi\imath}{6}}H^{(1)}_{\frac{1}{3}}\left(-\tfrac{2}{3}z^{\frac{3}{2}}\right) + e^{-\frac{\pi\imath}{6}}H^{(2)}_{\frac{1}{3}}\left(-\tfrac{2}{3}z^{\frac{3}{2}}\right)\right); \\
\operatorname{Bi} z &\stackrel{\text{def}}{=} \frac{\sqrt{-z}}{3}\left(J_{\frac{1}{3}}\left(-\tfrac{2}{3}z^{\frac{3}{2}}\right) - J_{-\frac{1}{3}}\left(-\tfrac{2}{3}z^{\frac{3}{2}}\right)\right) \\
&= \frac{\imath}{2}\sqrt{\frac{-z}{3}}\left(e^{\frac{\pi\imath}{6}}H^{(1)}_{\frac{1}{3}}\left(-\tfrac{2}{3}z^{\frac{3}{2}}\right) - e^{-\frac{\pi\imath}{6}}H^{(2)}_{\frac{1}{3}}\left(-\tfrac{2}{3}z^{\frac{3}{2}}\right)\right);
\end{aligned}
$$

the Airy «A» function can also be defined as the circulation

$$
\operatorname{Ai} z = \frac{1}{2\pi\imath}\oint_{+\gamma}\exp\left(\frac{\zeta^3}{3} - z\zeta\right)\mathrm{d}\zeta,
$$

where $\gamma\in\mathrm{Z}_1(\tilde{\mathbb{C}})$ is any closed path passing by ∞ approaching it from along the line of argument $-\pi/3$ towards the line of argument $\pi/3$.

The previous formulæ can be of course inverted to obtain:

$$
\begin{aligned}
J_{\pm\frac{1}{3}}\left(\tfrac{2}{3}z^{\frac{3}{2}}\right) &= \frac{1}{2}\sqrt{\frac{3}{z}}\left(\sqrt{3}\operatorname{Ai}(-z) \mp \operatorname{Bi}(-z)\right); \\
H^{(\ddot{1}/2)}_{\dot{\pm}\frac{1}{3}}\left(\tfrac{2}{3}z^{\frac{3}{2}}\right) &= e^{\ddot{\mp}\dot{\pm}\frac{\imath\pi}{5}}\sqrt{\frac{3}{z}}\left(\operatorname{Ai}(-z) \ddot{\mp}\, \imath\operatorname{Bi}(-z)\right);
\end{aligned}
$$

> The Airy ODE on the complex plane $\mathbb{C}$,
>
> $$ w''(z) - zw(z) = 0\,, \tag{1.288} $$
>
> has two linearly independent solutions, $\operatorname{Ai} z$ and either $\operatorname{Bi} z$ or $\operatorname{Ai}(e^{\pm 2\pi\imath/3}z)$; their associated Wronskians are
>
> $$ \begin{vmatrix} \operatorname{Ai} z & \operatorname{Bi} z \\ \operatorname{Ai}' z & \operatorname{Bi}' z \end{vmatrix} = \frac{1}{\pi}; \qquad \begin{vmatrix} \operatorname{Ai} z & \operatorname{Ai}(e^{\pm\frac{2\pi\imath}{3}}z) \\ \operatorname{Ai}' z & \operatorname{Ai}'(e^{\pm\frac{2\pi\imath}{3}}z) \end{vmatrix} = \frac{e^{\mp\frac{\pi\imath}{6}}}{2\pi}. $$

3 Riemann Chaos

The universe (which others call 'the Library') is composed of an indefinite and perhaps infinite number of hexagonal galleries, with vast air shafts between […]. From any of the hexagons one can see, interminably, the upper and lower floors. […] 20 shelves, 5 long shelves per side, cover all the sides except two; […] One of the free sides leads to a narrow hallway which opens onto another gallery, […] In the hallway there is a mirror […] Men usually infer from this mirror that the Library is not infinite […] Let it suffice now for me to repeat the classic dictum: *The Library is a sphere whose exact centre is any one of its hexagons and whose circumference is inaccessible*.

There are 5 shelves for each of the hexagon's walls; each shelf contains 35 books of uniform format; each book is of 410 pages; each page, of 40 lines; each line, of some 80 letters […] First: "The Library exists *ab æterno*." […] Second: "*The orthographical symbols are 25 in number*." This finding made it possible, 300 years ago, to formulate a General Theory of the Library and solve satisfactorily the problem which no conjecture had deciphered: the formless and chaotic nature of almost all the books. One […] was made up of the letters MCV, perversely repeated from the first line to the last. Another […] is a mere labyrinth of letters, but the next-to-last page says «*Oh time thy pyramids*».

[…] *In the vast Library there are no two identical books* […], its shelves register all the possible combinations of the 20-odd orthographical symbols: […] the archangels' autobiographies, the faithful catalogues of the Library, thousands and thousands of false catalogues, the demonstration of the fallacy of those catalogues, the demonstration of the fallacy of the true catalogue […], the commentary on that gospel, the commentary on the commentary on that gospel, the true story of your death, the translation of every book in all languages, the interpolations of every book in all books. […] the possibility of a man's finding […] can be computed as zero.

To speak is to fall into tautology. […] (An n number of possible languages use the same vocabulary; in some of them, the symbol *library* allows the correct definition […] You who read me, are You sure of understanding my language? […] Those who imagine it to be without limit forget that the possible number of books does have such a limit. I venture to suggest this solution to the ancient problem: *The Library is unlimited and cyclical*. […] My solitude is gladdened by this elegant hope[4]. […]

[4] Letizia Álvarez de Toledo has observed that […] a single volume would be sufficient […] containing an infinite number of infinitely thin leaves (in the early XVII century, Cavalieri said that all solid bodies are the superimposition of an infinite number of planes). […] each apparent page would unfold into other analogous ones; the inconceivable middle page would have no reverse.

J. L. Borges (1899–1986)
The Library of Babel (1941; CIV, 21)

3.1 Initiator Riemann surfaces: polynomial case

Fractals in complex geometry arose since the beginning of XX century, thanks to the pioneering works of Gaston M. Julia (1893–1978) and Pierre J.L. Fatou (1878–1929), more or less the forefathers of Complex Dynamics. The Riemann surfaces studied throughout the whole Chapter are used to generate complex fractal and prefractal manifolds, whose topology and monodromy is studied with mixed algebraic and analytic tools to derive quite a general mathematical model of transition toward chaos which will be called Riemann Chaos.

3.1.1. Algebraic Topology

Let $P\in\left(\mathbb{C}^d[z]\backslash\mathbb{C}^{d-1}[z]\right)/\mathbb{C}$ for some $d\in\mathbb{N}\backslash\{1\}$ (i.e. P be a complex-coefficients, monic, d^{th}-degree polynomial) with $1\leq r\leq d$ different roots $z_1,z_2,\ldots,z_r\in\mathbb{C}$ such that $\mathrm{mul}_P z_j=m_j$ (i.e. z_j has multiplicity m_j) $1\leq j\leq r$, and $m_1+m_2+\ldots+m_r=d$. That is:

$$P(z)=\prod_{j=1}^{r}(z-z_j)^{m_j}=(z\mathbf{1}_r-\mathbf{z})^{\mathbf{m}}, \tag{1.289}$$

where right-side of (1.289) is the usual formal way to define a polynomial as depending on its roots' and multiplicities' vectors as in §1.3, (1.69), with $\mathbf{z}=(z_1,z_2,\ldots,z_r)\in\mathbb{C}^r$.

Now, fixed $p\in\mathbb{N}\backslash\{1\}$, let $\boldsymbol{\mu}=(\mu_1,\mu_2,\ldots,\mu_r)$ and $\boldsymbol{\pi}=(\pi_1,\pi_2,\ldots,\pi_r)$ in $\mathbb{N}^r$ such that:

$$\begin{aligned} m_j &= \mu_j\,\mathrm{GCD}(m_j,p) \\ p &= \pi_j\,\mathrm{GCD}(m_j,p) \end{aligned} \qquad 1\leq j\leq r \tag{1.290}$$

and $\mu_\infty,\pi_\infty\in\mathbb{N}$ such that $d=\mu_\infty\,\mathrm{GCD}(d,p)$ and $p=\pi_\infty\,\mathrm{GCD}(d,p)$. Finally consider the complex mapping

$$z\mapsto w=f(z)=\sqrt[p]{P(z)}, \tag{1.291}$$

where multi-valued function $f(z)$, can be written as:

$$f(z) = \sqrt[p]{\prod_{j=1}^{r}(z-z_j)^{m_j}} \equiv \prod_{j=1}^{r}(z-z_j)^{\frac{\mu_j}{\pi_j}} = (z\mathbf{1}_r - \mathbf{z})^{\frac{\boldsymbol{\mu}}{\boldsymbol{\pi}}}, \qquad (1.292)$$

with the additional formal definition: $\frac{\boldsymbol{\mu}}{\boldsymbol{\pi}} ::= \left(\frac{\mu_1}{\pi_1}, \frac{\mu_2}{\pi_2}, \ldots, \frac{\mu_r}{\pi_r}\right) \equiv p^{-1}\mathbf{m} \in \mathbb{Q}^r$, which $\left|\frac{\boldsymbol{\mu}}{\boldsymbol{\pi}}\right| = \frac{\mu_\infty}{\pi_\infty}$ and $\pi_\infty = \boldsymbol{\pi}^{\mathbf{1}_r} / \mathrm{LCM}\,\boldsymbol{\pi}$ are easily derived from. Before getting involved into the study of iterations of complex functions, let us first accomplish both a topological and geometric analysis on the Riemann surface $\mathcal{A}_1$ defined by implicit equation

$$w^p - P(z) = 0, \qquad (1.293)$$

where $w^p - P(z) \in \mathbb{C}^{\max(p,d)}[w,z]$ is a two-variables Weierstrass polynomial, cfr. §2.1.3. Equation (1.293) can also be written as $w = f(z)$, i.e. it can be explicated with respect to w, but this just happens locally for $z \in \mathbb{C}$, since the arithmetic p^{th} root function is multi-valued (it is exactly 1-to-p). The k^{th} branch of the arithmetic root $z \mapsto \sqrt[p]{z}$, $0 \le k < p$, is defined as $e^{2\pi\imath\frac{k}{p}}\sqrt[p]{|z|}$, where we conventionally draw the corresponding branch cut $]0,\infty[$ to be the negative real half-line $\mathbb{R}_-$ (this will come in handy later, cfr. §3.4.2).

Equation (1.293), though, may have in general less than p local explicit solutions of kind $w = f(z)$. It is well known, in fact, that multi-valued function f (the universal cover of $\mathcal{A}_0$) has $\mathrm{LCM}\,\boldsymbol{\pi} \equiv \mathrm{LCM}(\pi_1, \pi_2, \ldots, \pi_r)$ branches [72].

Lemma

The compact Riemann surface $\mathcal{A}_1$ defined by (1.293) has exactly $\mathrm{LCM}\,\boldsymbol{\pi}$ sheets, whereas its branch points, the same on *every* sheets, are among the roots of P and $\infty \in \tilde{\mathbb{C}}$: z_j is a branch point of order $\pi_j - 1$ ($1 \le j \le r$) and ∞ is a branch point of order $\pi_\infty - 1$, with[101]

$$|\mathbf{m}| + \mu_\infty \equiv 0 \bmod \pi_\infty. \qquad (1.294)$$

[101] This means that $[\![|\mathbf{m}|]\!] = [\![-\mu_\infty]\!] \in \mathbb{Z}_{\mathrm{LCM}\,\boldsymbol{\pi}}$.

Surface $\mathcal{A}_1$ has genus

$$\kappa_1 = |\boldsymbol{\pi}| + \pi_\infty - 2\,\mathrm{LCM}\,\boldsymbol{\pi} - r + 1, \tag{1.295}$$

i.e. it is homeomorphic to $\mathbb{RP}_{\kappa_1}$, the connected sum of κ_1 real projective planes (cfr. classification of compact Riemann surfaces, §2.4.2).

Let $\mathrm{LCM}\boldsymbol{\pi} = p$. If $\pi_\infty = p$ then $\kappa_1 = (p-1)(r-1)$; whereas if $\pi_\infty = 1$ (∞ *not* being a branch point), it becomes $\kappa_n = (p-1)(r-2)$.

It has to be noted that $\mathrm{LCM}\boldsymbol{\pi} < p$ if and only if $m_1, m_2, \ldots, m_r \mid p$, i.e. the same surface is described by (1.293) replacing p with $p/\mathrm{LCM}(\mathbf{m}, p)$. In every other case $\mathrm{LCM}\boldsymbol{\pi} = p$. Only the case $\mathrm{LCM}\boldsymbol{\pi} = p$ will be treated here for simplicity's sake, otherwise p can be replaced with $\mathrm{LCM}\boldsymbol{\pi}$ almost everywhere throughout the rest of this Chapter. On the contrary the case $\mathrm{LCM}\boldsymbol{\pi} < p$ is interesting when studying multi-valued functions like

$$z \mapsto f(z) = \prod_{j=1}^{r} \sqrt[\pi_j]{(z - z_j)^{\mu_j}} ::= \sqrt[\boldsymbol{\pi}]{(z\mathbf{1}_r - \mathbf{z})^{\boldsymbol{\mu}}}.$$

If $\mathcal{A}_1$ is orientable κ_1 is even then, letting $g_1 = \kappa_1/2$, $\mathcal{A}_1 \asymp \mathbb{T}_{g_1}$, i.e. $\mathcal{A}_1$ is homeomorphic to the g_1-handled torus. In this case one resorts for free:

$$\begin{cases} \pi_1(\mathcal{A}_1) \cong \overset{g}{\ast}\mathbb{Z} \\ \mathrm{H}^1(\mathcal{A}_1) \cong \mathbb{Z}^{2g_1} \end{cases} \tag{1.296}$$

> Let $f(z)=\sqrt[3]{(z-1)(z^2+1)}$, where $p=r=3$, $\mathbf{z}=(1,-\imath,\imath)$ and $\mathbf{m}=\mathbf{1}_3$, then $\pi_\infty=1$, $\boldsymbol{\mu}=\mathbf{1}_3$ and $\boldsymbol{\pi}=3\cdot\mathbf{1}_3$. $\mathcal{A}_0$ is thus a simple torus $\mathbb{T}_1$, since $\kappa_1=2g_1=2$ by (1.295). See Figure 23 and Figure 51 for a local, one-sheeted representation of $f(z)$ showing its branching structure.

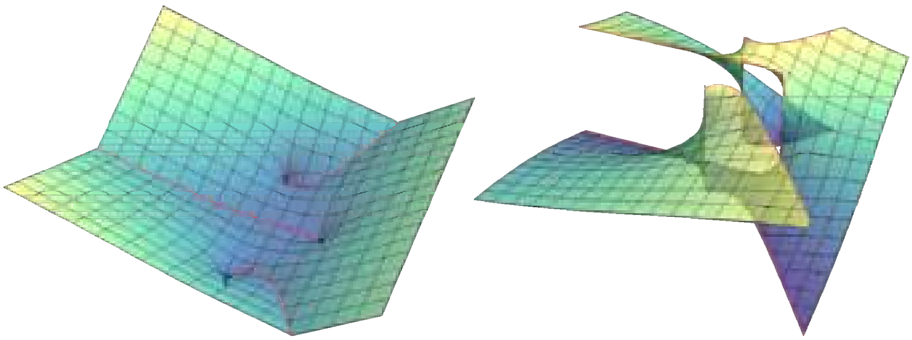

Figure 23 Real (left) and imaginary parts (right) of one single-valued branch of $\sqrt[3]{(z-1)(z^2+1)}$, with its three sheets glued along the three branch cuts, shown as red curves.

In order to use the language of coverings (cfr. §2.3.1) instead of that of ramified covers, let the starting Riemann surface be defined as the Riemann sphere punctured on the branch points, i.e. $\mathcal{A}_0{:=}\tilde{\mathbb{C}}\setminus\{z_1,z_2,\ldots,z_r,\infty \mid \pi_1,\pi_2,\ldots,\pi_r,\pi_\infty>1\}$: it is a compact surface of genus $r-\delta_{0,\pi_\infty}-\sum_{j=1}^{r}\delta_{0,\pi_j}+1\leq r+1$ and (1.291) will be, from time to time, be considered as either a universal LCM$\boldsymbol{\pi}$-cover $f^{-1}{:}\mathcal{A}_1\to\mathcal{A}_0$ or a ramified LCM$\boldsymbol{\pi}$-cover $f^{-1}{:}\mathcal{A}_1\to\tilde{\mathbb{C}}$. The inverse function $f{:}\mathcal{A}_0\to\mathcal{A}_1$, is instead what is called a *branched* LCM$\boldsymbol{\pi}$-cover (in analogy with the distinction between ramification and branch points, §2.4). Equation (1.295) is proven, for example, applying the Riemann-Hurwitz Theorem to the ramified cover $f^{-1}{:}\mathcal{A}_1\to\tilde{\mathbb{C}}$.

One can easily be convinced of that considering a closed-curve parametrization $\gamma\in C^0([0,1];\mathcal{A}_0)$, i.e. such that $\gamma(0)=\gamma(1)$ and it does not intersect any of the branch points on the z-plane: its lift $\gamma\!\uparrow\! f$ on the w-plane (which is indeed $\mathcal{A}_1$), also indicated as $f(\gamma)$ with notation's abuse, changes continuously branch (sheet) as soon as γ winds around every and each branch points.

From the topological point of view, since the same branch points are present on every sheet, the same ramification points $f(\tilde{\mathbb{C}}\setminus\mathcal{A}_0)$ are removed from every fibre of $\mathcal{A}_1$; for this reason, $\pi_1(\mathcal{A}_1;f(z))\equiv\pi_1(\mathcal{A}_1)\ \ \forall z\in\mathcal{A}_0$ and the same holds for the monodromy groups, i.e.:

$$\mathrm{MG}_z f = \mathrm{MG}_{z'} f := \mathrm{MG} f\,, \qquad \forall z,z'\in\mathcal{A}_0. \tag{1.297}$$

To any loop $\gamma\in\mathrm{Z}_1(\mathcal{A}_0)$ can be associated a winding number $\mathrm{wind}_z\gamma\in\mathbb{Z}\ \ \forall z\in\tilde{\mathbb{C}}$, which is the number of times γ "winds" counter-clockwise around z (positive if the total winding is counter-clockwise, negative if it is clockwise). This definition coincides with (1.218) when $\omega\in\mathrm{Frac}\Lambda^{1,0}(\mathcal{A}_0)$ has, at most, a singularity (or a branch point) in $z=0$ and is in fact a direct way to compute it without explicitly knowing the analytic expression of curve γ. Winding is relevant, to the Riemann surfaces' purposes, just as long as it happens around branch points, since this means that the lift $\gamma\!\uparrow\! f$ likely underwent some sheet transition.

One has to stress that, once a choice of branch cuts is done, the winding number's definition extends to generic (open) paths $\gamma\in\mathrm{C}^0([0,1],\mathcal{A}_0)$ too, not only loops: every time γ crosses a branch cut ›z',z''‹ (which happens in particular whenever it winds completely around either branch points z' *or* z'') its winding numbers around z' and z'' are summed to ±1 (either $+1$ or -1 depending on the relative orientation of the path respect to the branch point).

Let us introduce a compact formalism for travels on a Riemann surface, as it was briefly sketched in (1.229): any loop $c\in\mathrm{Z}_1(\mathcal{R})$ [any non-closed path $c\in\mathrm{C}^0([0,1],\mathcal{R})$ — once a branch cuts' choice is fixed] which winds around z_1 first, then around z_2, then around z_3, …and so on up to winding around[102] z_N [and having started from the

[102] Note that points z_1, z_2, z_3,…, z_N need not be distinct (each '$\curvearrowright$'/'$\curvearrowleft$' standing for just *one* winding).

base-point z_0] will be indicated by a "chronological" juxtaposition of the branch points with either a symbol '$\curvearrowright$' or '$\curvearrowleft$' before it (depending whether the winding is respectively clockwise or counter-clockwise); in this case, supposing clockwise contours for z_1 and z_N, and counter-clockwise for z_2 and z_3:

$$[\, z_0 \,] \curvearrowright z_1 \curvearrowleft z_2 \curvearrowleft z_3 \ldots \curvearrowright z_N$$

$$\text{(multiple windings abbreviated to } \overset{m}{\curvearrowright} z_1 := \overbrace{\curvearrowright z_1 \curvearrowright z_1 \curvearrowright \ldots \curvearrowright z_1}^{m \text{ times}} \text{).}$$

It is clear by trivial simplification rules, that these juxtapositions are analogue to those introduced for the free products of groups, cfr. §1.1.8.

As far as sheet transitions are concerned (for a given branch cuts' choice), c starting on sheet k_1, then travelling along sheet k_2, … and so on up to 'landing' on sheet k_M (for any $k_1,k_2,\ldots,k_M \in \mathbb{N}$) is indicated as: $k_1 \rightsquigarrow k_2 \rightsquigarrow \ldots \rightsquigarrow k_M$.

On a general Riemann surface any branch point may exist on *some* sheets but not on others: this fact, having deep algebraic consequences discussed in §3.2, means that the winding numbers of γ around branch points lose topologic meaning, since a sheet transition eventually happens to $\gamma \uparrow f$ only as long as it winds around ramification points (i.e. lying on the sheets containing them), as γ winds around specific branch points.

But the sheet transition of any curve lift $\gamma \uparrow f$ on $\mathcal{A}_1$ does not depend which sheets the curve actually starts on since —again— every branch point is present on every sheet of $\mathcal{A}_1$ (and with the same branching numbers). The set of winding numbers associated to any loop γ around $\mathcal{A}_0$'s branch points (corresponding 1-to-LCM$\boldsymbol{\pi}$ to $\mathcal{A}_1$'s ramification points) is enough to represent the homotopy of any curve in $\mathcal{A}_0$.

Let a coding for the sheets of $\mathcal{A}_1$ be defined first: it is clear that every branch of f maps to a multi-index $\mathbf{k} \equiv (k_1,k_2,\ldots,k_r) \in \bigtimes_{j=1}^{r} \mathbb{Z}_{\pi_j}$ such that, respect to the first branch of each of the r arithmetic p^{th} roots:

$$f(z) = e^{2\pi\imath\mathbf{k}\cdot\frac{\boldsymbol{\mu}}{\boldsymbol{\pi}}}\prod_{j=1}^{r}\left|z - z_j\right|^{\frac{\mu_j}{\pi_j}} ::= e^{2\pi\imath\mathbf{k}\cdot\frac{\boldsymbol{\mu}}{\boldsymbol{\pi}}}\left|z\mathbf{1}_r - \mathbf{z}\right|^{\frac{\boldsymbol{\mu}}{\boldsymbol{\pi}}}. \qquad (1.298)$$

To every γ whose winding vector respect to finite branch points[103] $\mathbf{z}$ an integer r-tuple $\mathbf{k} = \operatorname{wind}\gamma \stackrel{\text{def}}{=} (\operatorname{wind}_{z_j}\gamma)_{1\leq j\leq r} \in \mathbb{Z}^r$; it is evident from (1.298) that as γ winds $k_j \in \mathbb{Z}$ times around the j^{th} branch point $z_j \in \mathbb{C}$, then $\gamma\uparrow f$ "jumps" up (or down, depending on $\operatorname{sign}k_j$) by $k_j m_j \bmod p$ sheets; in particular, if γ winds any multiple of π_j times around z_j (it does not matter whether it is done consecutively around the same branch point or not), $\gamma\uparrow f$ lands on a sheet which is homotopically invariant (i.e. lands back to the same sheet). For this reason the set of sheets can be ordered in a non-trivial way (unless $\mathcal{A}_1$ is hyperelliptic).

> Let $f(z) = \sqrt[5]{z(z^6-1)}$, where $p=5$, $r=7$, $\mathbf{m}=\mathbf{1}_7$; then $\pi_\infty=6$, $\boldsymbol{\mu}=\mathbf{1}_7$ and $\boldsymbol{\pi}=6\cdot\mathbf{1}_7$, with the branch points being 0 and the 6^{th} roots of unity (all simple) due to one factor being the 6^{th}-degree cyclotomic polynomial. $\mathcal{A}_0$ is thus a sphere with 12 handles $\mathbb{T}_{12}$, since $\kappa_1 = 2g_1 = 24$ by (1.295);. See Figure 24 for a local, one-sheeted representation of $f(z)$ showing its branching structure.

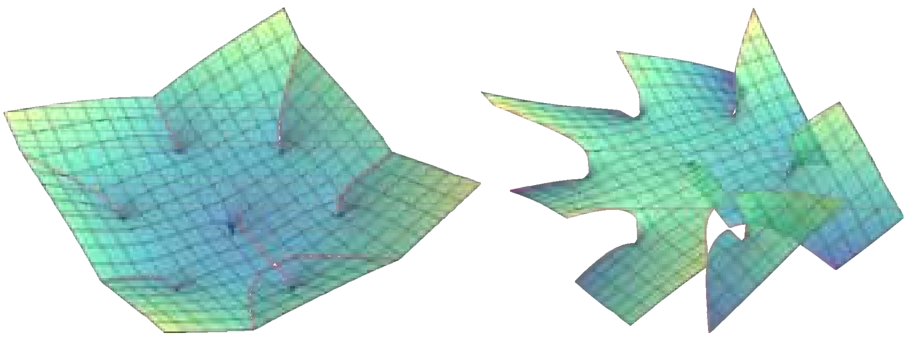

Figure 24 Real (left) and imaginary parts (right) of $f(z)$ in the above example. The five sheets come together along 'cyclotomic branch cuts' shown as red curves.

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[103] The winding number around ∞ is computed from the others as: $\operatorname{wind}_\infty\gamma = \sum_{j=1}^{r}\operatorname{wind}_{z_j}\gamma \equiv |\mathbf{k}|$.

In particular, any γ winding $k_\infty \in \mathbb{Z}$ times around branch point ∞ is equivalent to a loop winding $-k_\infty$ times around *each and every* finite branch point, whereas $\gamma{\uparrow}f$ "jumps" up by $k_\infty d \bmod p$ sheets, or down by $k_\infty|\mathbf{m}| \bmod p$ sheets: this is where (1.294) comes from.

First of all let the following groups be defined (as per §1.1.4):

$$\mathbb{Z}_{\boldsymbol{\pi}} \stackrel{\text{def}}{=} \bigtimes_{j=1}^{r} \mathbb{Z}_{\pi_j}; \qquad \mathbb{Z}_{\boldsymbol{\pi}}^{\oplus} \stackrel{\text{def}}{=} \bigoplus_{j=1}^{r} \mathbb{Z}_{\pi_j} \cong \mathbb{Z}_{\mathrm{LCM}\boldsymbol{\pi}}; \tag{1.299}$$

$$\tfrac{\boldsymbol{\mu}}{\boldsymbol{\pi}}\cdot\mathbb{Z}^r \stackrel{\text{def}}{=} \sum_{j=1}^{r} \frac{\mu_j}{\pi_j}\mathbb{Z} = \left\{ \mathbf{k}\cdot\tfrac{\boldsymbol{\mu}}{\boldsymbol{\pi}} \in \mathbb{Q} \;\middle|\; \forall \mathbf{k} \in \mathbb{Z}^r \right\} < \mathbb{Q}; \tag{1.300}$$

$$\tfrac{\boldsymbol{\mu}}{\boldsymbol{\pi}}\cdot\mathbb{Z}_{\boldsymbol{\pi}} \stackrel{\text{def}}{=} \sum_{j=1}^{r} \frac{\mu_j}{\pi_j}\mathbb{Z}_{\pi_j} = \left\{ \mathbf{k}\cdot\tfrac{\boldsymbol{\mu}}{\boldsymbol{\pi}} \in \mathbb{Q} \;\middle|\; \forall [\![\mathbf{k}]\!] \in \mathbb{Z}_{\boldsymbol{\pi}} \right\} < \tfrac{\boldsymbol{\mu}}{\boldsymbol{\pi}}\cdot\mathbb{Z}^r. \tag{1.301}$$

Of course (1.300) and (1.301) are both additive subgroups of $\mathbb{Q}(+)$; in particular $\operatorname{ord}\mathbb{Z}_{\boldsymbol{\pi}}^{\oplus} = \operatorname{ord}\left(\tfrac{\boldsymbol{\mu}}{\boldsymbol{\pi}}\cdot\mathbb{Z}_{\boldsymbol{\pi}}\right) = \mathrm{LCM}\boldsymbol{\pi}$, whereas (1.300) is infinite and is a $\mathbb{Z}$-modulo of $\mathbb{Q}$. Since $\kappa_1 = |\boldsymbol{\pi}| + \pi_\infty - 2\,\mathrm{LCM}\boldsymbol{\pi} - r + 1 > 0$, it can be proven that $\operatorname{ord}\boldsymbol{\mu}\cdot\mathbb{Z}_{\boldsymbol{\pi}} = \kappa_1$.

As every loop $\gamma \in \mathrm{C}^0([0,1];\mathcal{A}_0)$ can be mapped to a winding vector $\mathbf{k} \in \mathbb{Z}^r$, so its homotopy class $[\![\gamma{\uparrow}f]\!] \in \pi_1(\mathcal{A}_1)$ is mapped to a modular vector $[\![\mathbf{k}]\!] \in \mathbb{Z}_{\boldsymbol{\pi}}$. Putting these considerations in mathematical language, the former mapping 'wind' acting as

$$\mathrm{Z}_1(\mathcal{A}_0) \xleftrightarrow{\text{wind}} \mathbb{Z}^r, \tag{1.302}$$

is a isomorphism, with topological summation over loops (1-cycles) mapped to lattice sum in $\mathbb{Z}^r$. A quotient over the respective spaces induces the isomorphisms

$$\begin{aligned} \mathrm{H}_1(\mathcal{A}_0) &\xleftrightarrow{\text{wind}} \tfrac{\boldsymbol{\mu}}{\boldsymbol{\pi}}\cdot\mathbb{Z}_{\boldsymbol{\pi}}, \\ \mathrm{H}_1(\mathcal{A}_1) &\xrightarrow[\hookrightarrow]{} \tfrac{\boldsymbol{\mu}}{\boldsymbol{\pi}}\cdot\mathbb{Z}_{\boldsymbol{\pi}}. \end{aligned} \tag{1.303}$$

The homomorphism '$\hookrightarrow$' and its inverse '$\hookleftarrow$', acting on homology classes '$\because$', also act *co*homology classes '$\therefore$' of $\mathrm{H}^1(\mathcal{A}_1)$, with a slight abuse of notation, [9]–§§2.2,3. They are push-forwards and pull-backs of cover transformations, thus they are represented by the deck transformation group $\mathrm{Deck}(\mathcal{A}_1 \overset{f^{-1}}{\hookrightarrow} \mathcal{A}_0)$ defined in §2.3.1.

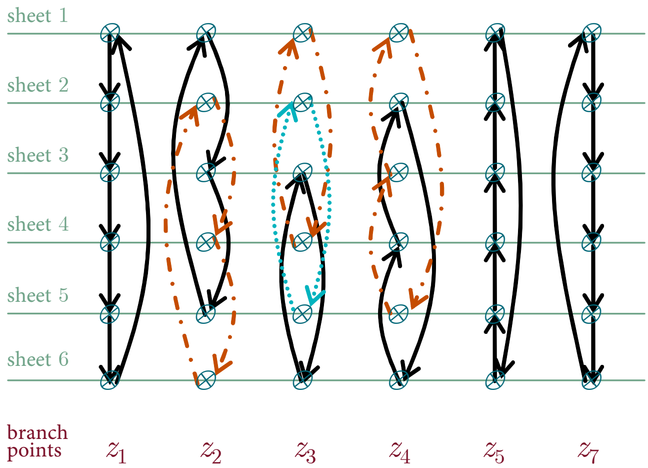


Figure 25 All the possible travels on the 6-sheeted initiator Riemann surface explicitly defined by $f(z)$ branched cover $(z-z_6)\sqrt[6]{(z-z_1)(z-z_2)^2(z-z_3)^3(z-z_4)^4(z-z_5)^5(z-z_7)^7}$ and classified according to the 6 branch points (z_6 is *not* since $\pi_6=1$); its parameters are $p=6$, $d=5040$, $r=7$ and $\mathbf{m}=(1,2,3,4,5,6,7)$, so that $\boldsymbol{\mu}=(1,1,1,2,5,1,7)$ and $\boldsymbol{\pi}=(6,3,2,3,6,1,6)$ featuring different-order branch points.

The additive group $\mathbb{Z}_p(+)$ will be used as a prototype counting-set for $\mathcal{A}_1$'s sheets.

> For example, let $p=3$, $p=4$, $\mathbf{m}=(2,1,4,1)$ and consider a path winding as $\curvearrowright z_2 \curvearrowright z_1 \overset{2}{\curvearrowleft} z_2 \curvearrowright z_4 \curvearrowleft z_2 \curvearrowright z_3 \curvearrowright z_4 \curvearrowleft z_3 \overset{4}{\curvearrowright} z_4$, thus its associated winding vector is $\mathbf{k}=(1,-2,0,6)$. For considerations before it follows that $\frac{\boldsymbol{\mu}}{\boldsymbol{\pi}}=\left(\frac{2}{3},\frac{1}{3},\frac{4}{3},\frac{1}{3}\right)$; thus the path is homologous to a simpler one winding as $\mathbf{k}$, e.g. $\curvearrowright z_1 \overset{2}{\curvearrowleft} z_2 \overset{6}{\curvearrowright} z_4$, or $\curvearrowright z_1 \curvearrowright z_2$, since $\boldsymbol{\pi}=3\cdot\mathbf{1}_4$. Then $[\![\mathbf{k}]\!]=[\![(1,1,0,0)]\!]\in \times^4\mathbb{Z}_3 \equiv \mathbb{Z}_{(3,3,3,3)}$.

> Starting for example on the 1st sheet, the sequence of visited sheets corresponding to loop $\curvearrowright z_1 \curvearrowright z_2$ is $1 \rightsquigarrow 3 \rightsquigarrow 1$, whereas that for loop $\curvearrowright z_1 \overset{2}{\curvearrowleft} z_2 \overset{6}{\curvearrowright} z_4$ is $1 \rightsquigarrow 2 \rightsquigarrow 1 \rightsquigarrow 3 \rightsquigarrow 2 \rightsquigarrow 1 \rightsquigarrow 3 \rightsquigarrow 2 \rightsquigarrow 1$, so in both cases the loop's lift on $\mathcal{A}_1$ is closed (i.e. it ends on the same sheet); furthermore, this property is inherent for any loops' class $[\![\mathbf{k}]\!]$, as will be proven in the next Chapter.

As far as the branch cuts on $\mathcal{A}_0$ are concerned, an arbitrary choice (employing, for example, the conventional cut $]0,\infty[:=\mathbb{R}_-$ for each arithmetic p^{th} root function) consists in inter-linking couples of branch points in B by a finite number of curves such that every branch point of order $\pi\in\{\pi_j,\pi_\infty\}_{1\le j\le r}$ is the endpoint of $\pi-1$ such curves.

3.1.2. Monodromy

By §2.4.5 the monodromy groups of $\mathcal{A}_1$ do not depend on the base point $z\in\mathcal{A}_0$; also recall that, the sheets' permutation group being $\mathbb{S}_p$, the monodromy action is $F:\mathbb{Z}_p\times\pi_1(\mathcal{A}_1)\to\mathbb{Z}_p$, which is unique and computed by formula:

$$s\bullet F([\![\gamma]\!]) = \left(s + [\![\,\mathbf{k}\,]\!]\cdot\mathbf{m}\right)\operatorname{mod}p, \tag{1.304}$$

which can more simply be written, by the associative property of $\mathbb{Z}_p(+)$, as $F([\![\gamma]\!])=[\![\,\mathbf{k}\,]\!]\cdot\mathbf{m}\operatorname{mod}p\equiv\left[\!\!\left[[\![\,\mathbf{k}\,]\!]\cdot\mathbf{m}\right]\!\!\right]_p$.

Consider the surface of §3.1.1's example above and a loop γ of winding vector $\mathbf{k}=(1,-2,0,6)$ starting from the s^{th} sheet (i.e. $s\in\mathbb{Z}_p$) and winding as $[\![(1,1,0,0)]\!]\equiv(1,-2,0,6)\bmod 3$. By (1.304) its lift on $\mathcal{A}_1$ lands on the same sheet where it started from; in fact it is still a closed path:

$$s\bullet F([\![\gamma]\!])\equiv\left(s+\begin{pmatrix}1 & 1 & 0 & 0\end{pmatrix}\cdot\begin{pmatrix}2\\1\\4\\1\end{pmatrix}\right)\bmod 3\equiv(s+3)\bmod 3=s$$

or, simply, $F([\![\gamma]\!])\equiv\big((1,1,0,0)\cdot(2,1,4,1)\big)\bmod 3=0.$

Since the sheets of $\mathcal{A}_1$ are selected by choosing a monodromic (single-valued) branch of f, the action of $\pi_1(\mathcal{A}_1)$ on the z-coordinated sheets (rather then on $\mathbb{Z}_p$) and the homology group $\mathrm{H}_1(\mathcal{A}_1)$ act on the Pontryagin's dual of $\frac{\mu}{\pi}\cdot\mathbb{Z}_\pi$ as the following character depending on the monodromy action $F\in\mathrm{Hom}(\mathbb{S}_p,\pi_1(\mathcal{A}_1))$:

$$\hookrightarrow:\widehat{\tfrac{\mu}{\pi}\cdot\mathbb{Z}_\pi}\cong\tfrac{\mu}{\pi}\cdot\mathbb{Z}_\pi\longrightarrow\mathbb{T},$$

$$[\![\gamma]\!]_{\because}\mapsto[\![\mathbf{k}]\!]\mapsto\hookrightarrow\mathbf{k}\overset{\text{def}}{=}\exp\Big(2\pi\imath F([\![\gamma]\!])\Big):=e^{2\pi\imath\mathbf{k}\cdot\frac{\mu}{\pi}}. \tag{1.305}$$

The diagram (1.303) is thus completed so far as:

$$\begin{array}{ccc}\mathrm{H}_1(\mathcal{A}_0) & \xrightarrow{\text{wind}} & \frac{\mu}{\pi}\cdot\mathbb{Z}_\pi\\ {\scriptstyle f}\downarrow & & \uparrow{\scriptstyle F}\\ \mathrm{H}_1(\mathcal{A}_1) & \xrightarrow[\cdot/[\cdot|\cdot]]{} & \pi_1(\mathcal{A}_1)\end{array} \tag{1.306}$$

and also the epimorphism $\pi_1(\mathcal{A}_1)\rightarrow\mathrm{H}_1(\mathcal{A}_1)$ is to be considered, since two homotopic paths are also homologous, [45]. The explicit expression of '$\hookrightarrow$' on $\mathcal{A}_1$ is otherwise *formally* defined —like (1.266) used as an example of monodromy matrix in §2.4.6— as the complex exponential of the monodromy action:

$$\hookrightarrow::=\exp(2\pi\imath F). \tag{1.307}$$

By using the discrete topology on $\frac{\mu}{\pi}\cdot\mathbb{Z}_\pi$ and from diagram (1.306), the above can be reformulated in the language of Lie algebras, cfr. §1.2.2, in order to be more formal:

Corollary

Under the above assumptions, $\mathrm{Lie}_{\mathbb{C}}\pi_1(\mathcal{A}_1)\leq\mathbb{T}$, whose exponential map is the '↬' homomorphism (1.303), mapping the Lie algebra of (homotopy classes of) loops on a Riemann surface into the unit-circumference subgroup representing the overall rotation in the complex plane cause by local monodromy action.

It has an immediate meaning on the explicit definition (1.291) of $\mathcal{A}_1$ itself: recalling that the k^{th} monodromic branch of an arithmetic p^{th} root $(0\leq k<p)$ is $\underset{k}{\sqrt[p]{z}} := e^{\frac{2\pi\imath k}{p}}\sqrt[p]{|z|}$:

$$f(\gamma(t)) = e^{2\pi\imath F(\gamma(t))}\underset{0}{\sqrt[p]{P(\gamma(t))}} \equiv e^{2\pi\imath\mathbf{k}\cdot\frac{\mu}{\pi}}\sqrt[p]{\left|\gamma(t)\mathbf{1}_r-\mathbf{z}\right|^{\mathbf{m}}}, \qquad (1.308)$$

for any path $\gamma\in\mathrm{C}^0([0,1];\mathcal{A}_0)$. To interpret (1.308) a branch cuts' choice has to be made, because γ may not be a closed curve $\forall t\in[0,1)$: at every "time" t such that γ crosses a branch cut, i.e. its lift $\gamma\uparrow f^{-1}$ changes the sheet it is travelling on $\mathcal{A}_1$, the monodromy action changes —it may in fact be a discontinuous function of $\mathrm{C}^0([0,1];\mathcal{A}_0)$— and a new associated single-valued branch of f is selected.

Note that branch transitions (1.308) corresponding to discontinuities of the monodromy action are even *analytic* because $f\in\mathcal{O}(\mathcal{A}_0)$ and $f^{-1}\in\mathcal{O}(\mathcal{A}_1)$; this is the most intimate link between Riemann surfaces (as complex analytic manifolds — cfr. §2.1.4) and the discontinuous nature of discrete groups acting on them — cfr. §2.4. The computation of the monodromy group follows immediately from §2.4.5:

Monodromy Theorem initiator Riemann surface

The monodromy group of $\mathcal{A}_1$ is the symmetric group of $\frac{\boldsymbol{\mu}}{\boldsymbol{\pi}}\cdot\mathbb{Z}_{\boldsymbol{\pi}}$:

$$\mathrm{MG}\mathcal{A}_1 = \mathrm{Sym}\left(\tfrac{\boldsymbol{\mu}}{\boldsymbol{\pi}}\cdot\mathbb{Z}_{\boldsymbol{\pi}}\right) \leq \mathbb{S}_{\mathrm{LCM}\boldsymbol{\pi}}. \tag{1.309}$$

The most "complete" case is when the monodromy group is *maximal*, i.e. $\mathrm{MG}\mathcal{A}_1 = \mathbb{S}_p$ — there are $p!$ monodromies to every loop on $\mathcal{A}_1$. This happens if and only if $\mathbf{m} = m\mathbf{1}_r$ and $\mathrm{GCD}(m,p)=1$ (and $\mathrm{LCM}\boldsymbol{\pi}=p$) that is P is the m^{th} power of a simple polynomial and m is coprime with p.

<u>Proof</u>: Equation (1.309) is easily obtained by previous reasoning about the behaviour of loops' lifts on $\mathcal{A}_1$ and the properties of the monodromy action defined above. The maximal monodromy group is $\mathbb{S}_p$, which happens if and only if $\frac{\boldsymbol{\mu}}{\boldsymbol{\pi}}\cdot\mathbb{Z}_{\boldsymbol{\pi}} \cong \mathbb{Z}_p \quad \Leftrightarrow \quad \frac{\boldsymbol{\mu}}{\boldsymbol{\pi}} = \frac{m}{p}\mathbf{1}_r$, in fact $\frac{m\mathbf{1}_r}{p}\cdot\mathbb{Z}_{p\mathbf{1}_r} \equiv \frac{m}{p}\mathbb{Z}_{p\mathbf{1}_r} = \mathbb{Z}_{p\mathbf{1}_r} \equiv \bigoplus^r \mathbb{Z}_p \cong \mathbb{Z}_p$. □

Since the branch points of f are present on every sheet (same orders), the only local difference between sheets is represented by their multiplicities' and orders' vectors, $\boldsymbol{\mu}$ and $\boldsymbol{\pi}$; this easily leads to the computability of the solvable monodromy group of $\mathcal{A}_1$, which is done closed-form for special cases:

Corollary

Other relevant cases are when some coordinates of $\frac{\boldsymbol{\mu}}{\boldsymbol{\pi}} \in \mathbb{Q}^r$ are equal or multiple with each other; the monodromy group is then a symmetric *sub*group of $\mathbb{S}_p$ — for example the alternating group $\mathbb{A}_p$ or some diherdral group $\mathbb{D}_{2q}$ with $2q \mid p!$.

<u>Proof</u>: Let $\frac{\mu_i}{\pi_i} = \frac{\mu_j}{\pi_j}$ $\left[\text{or } \frac{\mu_i}{\pi_i} = l\frac{\mu_j}{\pi_j}\right]$ for some $1 \leq i < j \leq r$ [and $l \in \mathbb{N}$], then any homotopy class $[\![\gamma]\!] \in \pi_1(\mathcal{A}_0)$ of winding vector $[\![\mathbf{k}]\!] \equiv [\![(k_j)_{1\leq j\leq r}]\!] \in \mathbb{Z}_{\boldsymbol{\pi}}$ shares the monodromy (i.e. is either called isotopic or **isomonodromic**) with the homotopy class of 1-cycles whose winding vector $[\![(k_1,\ldots,k_{i-1},k_j,k_{i+1},\ldots,k_{j-1},k_i,k_{j+1},\ldots,k_r)]\!] \in \mathbb{Z}_{\boldsymbol{\pi}}$ is

obtained by a simple permutation exchanging the i^{th} with the j^{th} coordinate. For this reason all the permutations in $\mathrm{MG}\mathcal{A}_1$ must include the symmetry between loops around z_i and z_j. This is represented using the fundamental polygon of $\mathcal{A}_0$, [42], [37], by adding a reflection along the normal axis to the i^{th} and j^{th} vertices of the polygon (hence the linking with dihedral groups, cfr. §1.1.1). Due to the arbitrariety of $1 \leq i < j \leq r$, the same reasoning holds whenever more than one couple (i,j) is involved (but not all, otherwise the complete case of previous Theorem holds), which proves the assertion, Q.E.D.. □

Consider the following multi-valued functions f and g.

$$f(z) = \sqrt[5]{z(z^3-1)}, \qquad g(z) = \sqrt[6]{z(z+\imath)(z-1)^2(z-2)^2}.$$

In the former case $p=5$, $d=r=4$, $\mathbf{m}=\mathbf{1}_4$, then $\frac{\boldsymbol{\mu}}{\boldsymbol{\pi}}=\frac{1}{5}\mathbf{1}_4$ and the monodromy group has maximal order $5! \equiv 120$ — it is $\mathbb{S}_5$.

In the latter case $p=d=6$, $r=4$, $\mathbf{m}=\boldsymbol{\mu}=\mathbf{1}_4$ and $\boldsymbol{\pi}=(6,6,3,3)$. Loops like $\overset{2}{\curvearrowright}1\curvearrowright2\curvearrowright0\overset{3}{\curvearrowright}-\imath$ and $\curvearrowright1\overset{2}{\curvearrowright}2\overset{3}{\curvearrowright}0\curvearrowright-\imath$ (obtained exchanging k_1 with k_2, and k_3 with k_4, since $\frac{\mu_1}{\pi_1}=\frac{\mu_2}{\pi_2}$ and $\frac{\mu_3}{\pi_3}=\frac{\mu_4}{\pi_4}$) are isotopic: starting for example from the 1st sheet, the visited sequence is always $1 \rightsquigarrow 3 \rightsquigarrow 5 \rightsquigarrow 1 \rightsquigarrow 2 \rightsquigarrow 3 \rightsquigarrow 4 \rightsquigarrow 5$. No branch points join sheets "2-by-2" so *no direct* sheet transitions are possible between the ones which are exactly "3 sheets far" from each other, like $1 \rightsquigarrow 4$ or $6 \rightsquigarrow 3$. All the permutations $[\![k]\!]_6 \mapsto [\![k+3]\!]_6$ in $\mathbb{S}_6$ $(\forall [\![k]\!]_6 \in \mathbb{Z}_6)$ must be removed from the monodromy group of g, so it has far less than $6! \equiv 720$ elements..

Some multi-valued Julia sets associated to the above prefractal Riemann surfaces are shown in Figure 48 and Figure 49 respectively.

From the algebraic point of view, consider the d symmetric polynomials $\{\mathrm{sy}_n(\mathbf{z})\}_{0 \leq n \leq d}$ defined in §1.5.4; when valuated on the d roots of P (represented as a d-ple $\mathbf{z} \in \mathbb{C}^d$ counted with multiplicities from the r distinct ones for simplicity's sake),

$$P(z) = \sum_{k=0}^{d} (-1)^k \operatorname{sy}_k(\mathbf{z}) z^{d-k}. \tag{1.310}$$

The monodromy can also be represented via a wreath recursion matrix (1.94) on the surface's sheets, $H_1 \in \mathbb{M}_{2\kappa_0, \mathrm{LCM}\boldsymbol{\pi}}(\mathbb{Z}_p)$ and $2\kappa_0 = \operatorname{rk}\pi_1(\mathcal{A}_0)$ by (1.295). In particular, the $(j,s)^{\text{th}}$ element of H_1 for $1 \le j \le 2\kappa_0$ and $0 \le s < \mathrm{LCM}\boldsymbol{\pi}$, is the monodromy action of a path $[\![(\delta_{j,k})_{1\le k\le r}]\!] \in \mathbb{Z}_{\boldsymbol{\pi}}$ winding once around the j^{th} branch point, and based on the s^{th} sheet, that is its ending sheet. By (1.304)–(1.305) the wreath recursion matrix is immediately computed to be:

$$H_1 = \Big(s \bullet F\big([\![(\delta_{j,k})_{j,k}]\!] \big) \Big)_{\substack{1\le j\le r,\\ 0\le s<p}} = \Big([\![s + m_j]\!]_p \Big)_{\substack{1\le j\le r,\\ 0\le s<p}}. \tag{1.311}$$

This matrix completely maps the transition between sheets induced by all the possible homotopy classes of loops in $\mathcal{A}_0$ and is sometimes called in Literature the **connections' matrix** of its associated Riemann surface. Since $\pi_1(\mathcal{A}_1)$ is *abelian* (i.e. all the branch points are present on every sheet), the first column of H is simply $\mathbf{m} \bmod p$, whereas each subsequent column is computed by the previous one by simply adding to $\mathbf{1}_r$ modulo p.

Consider the function $f(z) = \sqrt[p]{z^{m_1}(z+\imath)^{m_2}(z-1)^{m_3}(z+6)^{m_4}}$ $(r=4)$. As first example, let $p=6$, $\mathbf{m}=(1,1,2,3)$, $d=|\mathbf{m}|=7$, $\boldsymbol{\mu}=\mathbf{1}_4$ and $\boldsymbol{\pi}=(6,6,3,2)$. The associated wreath recursion matrix (1.311) is

$$\begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 0 \\ 1 & 2 & 3 & 4 & 5 & 0 \\ 2 & 3 & 4 & 5 & 0 & 1 \\ 3 & 4 & 5 & 0 & 1 & 2 \end{pmatrix} \in \mathbb{M}_{4,6}(\mathbb{Z}_6).$$

Let now $p{=}60$, $\mathbf{m}{=}(30,20,15,12)$, $d{=}|\mathbf{m}|{=}77$, so $\boldsymbol{\pi}{=}(2,3,4,5)$ and $\boldsymbol{\mu}{=}\mathbf{1}_4$. The associated wreath recursion matrix is a left-circulant matrix of $\mathbb{M}_{4,6}(\mathbb{Z}_{60})$ whose first column is $\mathbf{m}$.

3.1.3. Calculus and entropy

Each single-valued (monodromic) branch of $f{\in}\mathcal{O}(\mathbb{C})$ is clearly an entire function with a removable singularity in ∞ and $f(\mathcal{A}_0){=}\mathbb{C}$ because for every point $w{\in}\mathbb{C}$ the (1.292) can be inverted delivering 1 to pd distinct results in $\mathbb{C}$, since the polynomial $w^p{-}P(z)$ has degree d with respect to z. Its logarithmic derivative $\mathrm{D}\log f{\in}\mathcal{K}(\mathbb{C})$ is

$$\mathrm{D}\log f(z) \equiv \frac{f'(z)}{f(z)} = \sum_{j=1}^{r}\frac{\mu_j}{\pi_j}\frac{1}{z-z_j}, \tag{1.312}$$

which is well-known to have only simple poles and a residues' vector equal to $\frac{\boldsymbol{\mu}}{\boldsymbol{\pi}}$, i.e. $\mathrm{Res}_{\mathrm{D}\log f} z_j{=}\mu_j/\pi_j$, $1{\le}j{\le}r$; the common derivative is then

$$f'(z) = f(z)\sum_{j=1}^{r}\frac{\mu_j}{\pi_j}\frac{1}{z-z_j}. \tag{1.313}$$

This means that f is onto (except in its branch points) and there exists an open subset C, with $B{\subset}C{\subset}\mathbb{C}$ which $|f'|{<}1$ holds inside, thus $f|_C$ is a contraction mapping (or, better, each of its monodromic branches is). In the Complex Dynamics' language, C is called the **basin of attraction** of f; its points in C are called attractive points for f, whereas those in $(\mathbb{C}/C)^\circ$ (i.e. the points where $|f'|{>}1$) are called repelling points of f. The remaining points (where $|f'|{=}1$) are called critical points, whereas their pre-images are the *post-critical* points of f and are important for the dynamics of Julia sets associated to f, §2.4.1, [77], [87]. If the set of post-critical points is finite, then is said to be a ***post-critically finite*** function.

Since f is an algebraically-multi-valued function (i.e. it only features algebraic branch points) the most useful type of series associated to its analysis are the Puiseux series introduced in §1.3.4. In particular, the Puiseux series around ∞ in $\mathbb{C}[\![z^{-1}]\!]^{\natural}$ may also be considered to evaluate the behaviour around the possible branch point at the infinity. It is well-known that Puiseux series around a branch point can be treated like Laurent or —in the case of holomorphic functions like *f*'s monodromic branches— power (Taylor) series: their convergence domain is an open circle whose radius of convergence is, in this case, the arithmetic root of the distance between the brach point which the Puiseux expansion is centered at, and the nearest branch point (or singularity) to it. From (1.228) it follows that around $z_j\in B$ for example $\exists N_j\in\mathbb{N}_0$ and $(a_n)_{-N_j\leq j<\infty}\in\ell^1$:

$$f(z)=\sum_{k=-N_j}^{\infty}a_k\sqrt[\pi_j]{(z-z_j)^k}\ ,\qquad \forall z\in\mathbb{B}_{\rho_j}(z_j)\backslash\{z_j\},\tag{1.314}$$

where the right-hand side series belongs to $\mathbb{C}[\![z-z_j]\!]^{\natural}$ and in this case $N_j=0$ $(1\leq j\leq r)$ since *f* has no finite singular polarities, whereas the convergence radius $\rho_j\in\mathbb{R}_+^*$ is:

$$\rho_j=\lim_n\sqrt[n]{|a_n|}\equiv\lim_n\left|\frac{a_{n+1}}{a_n}\right|=\sqrt[\pi_j]{\min_{\substack{1\leq i\leq r,\\ i\neq j}}|z_i-z_j|}\,.$$

The link with differential equations, as introduced in §2.4.6, is immediate since, from (1.270), the Riemann surface $\mathcal{A}_0$ is the underlying manifold of 1st-order linear ODE

$$w'(z)=\sum_{j=1}^{r}\frac{\frac{\mu_j}{\pi_j}w(z)}{z-z_j}\equiv\frac{\frac{\boldsymbol{\mu}}{\boldsymbol{\pi}}}{z\mathbf{1}_r-\mathbf{z}}w(z),\tag{1.315}$$

with $f(z)$ in (1.292) being its fundamental solution. Since the equation is scalar, its monodromy matrices (1.264) are the scalar characters $\exp\left(2\pi\imath\frac{\mu_j}{\pi_j}\right)$, $1\leq j\leq r$, as already pointed out in §3.1.2. The monodromy group is thus also the group generated by the winding vector $[\![\mathbf{k}]\!]\in\mathbb{Z}_{\boldsymbol{\pi}}$ as parameter,

$$\looparrowright\mathbf{k}:=e^{2\pi\imath\mathbf{k}\cdot\frac{\boldsymbol{\mu}}{\boldsymbol{\pi}}}.\tag{1.316}$$

In a shot abuse of notation, the fundamental (generating) monodromy matrix of ODE (1.315) is $\looparrowright \equiv \exp(2\pi i F)$ itself — the monodromy action's exponential (1.307).

In this case the branch points $\{z_j\}_{1\leq j\leq r}$ (plus ∞, if required — cfr. §3.1.1) are *fixed* singularities of the ODE above, as well as simple poles of the dynamical "matrix" (1.267), which by definition coincides with its logarithmic derivative (1.312). Analytic continuation of each of the LCM$\boldsymbol{\pi}$ solutions to each other, cfr. §2.4.1, is thus trivial and done with the aid of the automorphy factor(s) (1.305). In the language of Differential Galois Theory, cfr. §1.1.10, $f(z)$ in (1.292) belongs to an elementary differential extension of $\mathbb{C}\{z\}$, made up of successive simple logarithmic extensions, $\mathbb{C}\left(\sqrt[p]{(z-z_j)^{m_j}}\right), 1\leq j\leq r$, since

$$\mathrm{D}\log\sqrt[p]{(z-z_j)^{m_j}} = \frac{\mu_j/\pi_j}{z-z_j} \in \mathbb{C}\{z\}.$$

Let the following "entropy-like" function,[104] called ***monodromic enrtropy***, be defined with respect to any base $b\in\mathbb{R}_+^*\setminus\{1\}$:

$$\mathfrak{H}_b\colon \mathbb{Q}^n \longrightarrow \mathbb{R}_+$$

$$\boldsymbol{\alpha} \mapsto \mathfrak{H}_b(\boldsymbol{\alpha}) \stackrel{\text{def}}{=} -\frac{\boldsymbol{\alpha}}{|\boldsymbol{\alpha}|}\cdot\log_b\boldsymbol{\alpha}\,; \tag{1.317}$$

whenever $b=e$ (the Napier's constant) let $\mathfrak{H}:\equiv\mathfrak{H}_e$. The monodromic entropy is a variant of the *Shannon*-McMillan entropy (well-known in both Information/Coding and Dynamical Systems' Theory); see [67], [101], [48], [49], [6], and [9]–§6.1 for similar and equivalent definitions in several Mathematics and Physics contexts. For applications to discrete dynamical systems (like cellular automata and LBGK equations), cfr. also [91] and [97]. The monodromic entropy of a Riemann surface like (1.291), formally $\mathfrak{H}_b(f):\equiv\mathfrak{H}_b\left(\frac{\boldsymbol{\mu}}{\boldsymbol{\pi}}\right)$, is of interest here, in particular:

---

[104] The word «entropy» comes from Ancient Greek noun ἐντροπή,-ῆς, meaning «*conversion*», «*confusion*», fig. «*growth*». For multiple (both different and alternative) notions and applications of entropy in pure and applied sciences see also [60], [84], [91], as well as [85] and [104]. The base-b logarithm of a real n-tuplet $\mathbf{x}\equiv(x_j)_{1\leq j\leq n}\in\mathbb{R}^n$ is $\log_b\mathbf{x}:=(\log_b x_j)_{1\leq j\leq n}$.

$$\begin{aligned}\mathfrak{H}_b\left(\tfrac{\boldsymbol{\mu}}{\boldsymbol{\pi}}\right) &= -\frac{p}{|\mathbf{m}|}\left(\frac{\boldsymbol{\mu}}{\boldsymbol{\pi}}\cdot\log_b\frac{\boldsymbol{\mu}}{\boldsymbol{\pi}}\right) \\ &= -\frac{p}{d}\sum_{j=1}^{r}\frac{\mu_j}{\pi_j}\log_b\frac{\mu_j}{\pi_j} \equiv \log_b p - \frac{1}{d}\sum_{j=1}^{r} m_j \log_b m_j \\ &= \log_b p - \frac{\mathbf{m}}{|\mathbf{m}|}\cdot\log_b \mathbf{m}\,.\end{aligned} \tag{1.318}$$

For the two surfaces introduced in §3.1.2's example, one easily finds $\mathfrak{H}(f) = \log 5 \simeq 1.609437$ and $\mathfrak{H}(g) = \frac{1}{3}\log 54 \simeq 1.329661$; in the former case the entropy is the highest possible (in fact the associated monodromy group is maximal), in the latter case it is less than the maximum value $\log 6 \simeq 1.791759$ and, in fact, the monodromy group is smaller than $\mathbb{S}_6$.

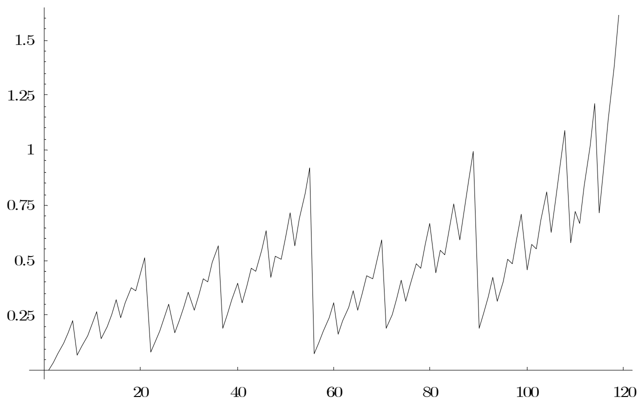


Figure 26 Growth of monodromic entropy $0 \leq \mathfrak{H}(f) \leq \log 5$ versus the monodromy group $\mathrm{MG}\mathcal{A}_1$ complexity, for all the $5!=120$ different (*non*-isomonodromic) Riemann surfaces for $p=r=5$ (thus $p \leq |\mathbf{m}| \leq rp$). Notice the self-similar trend.

Lemma

The monodromic entropy of (1.291), in the case $\max\boldsymbol{\mu}<p$ (i.e. $\max\mathbf{m}\leq p$), attains its extreme values 0 and $\log_b p$ whenever the monodromy group of the ramified cover is either trivial or maximal, cfr. Figure 26

<u>Proof</u>: $\mathfrak{H}_b$ is modeled along the Shannon-McMillan entropy 0, which is well-known to be the only function (modulo a choice of b) to satisfy the following properties (and which a complete list of can be found in [67]):

- It attains the minimum value 0 if and only if $\boldsymbol{\alpha}=\mathbf{1}_r$;
- It attains the maximum value $\log_b p$ if and only if $\boldsymbol{\alpha}\in\mathbb{Z}^r$;
- $\mathfrak{H}_b\left(\frac{\boldsymbol{\mu}}{\boldsymbol{\pi}}\frac{\boldsymbol{\nu}}{\boldsymbol{\varrho}}\right) \equiv \mathfrak{H}_b\left(\frac{\mu_1\nu_1}{\pi_1\varrho_1},\frac{\mu_2\nu_2}{\pi_2\varrho_2},\ldots,\frac{\mu_r\nu_r}{\pi_r\varrho_r}\right) = \mathfrak{H}_b\left(\frac{\boldsymbol{\mu}}{\boldsymbol{\pi}}\right)+\mathfrak{H}_b\left(\frac{\boldsymbol{\nu}}{\boldsymbol{\varrho}}\right), \qquad \forall\frac{\boldsymbol{\nu}}{\boldsymbol{\varrho}}\in\mathbb{Q}^r$
(the *extensive* property of entropy functions).

The topological interpretation of the first two points implies the following *alternative*, thanks to the Monodromy Theorem of §3.1.2:

- It attains the minimum value 0 if and only if the Riemann surface $\mathcal{A}_1$ is one-sheeted (and P is simple), i.e. if and only if $\frac{\mathbf{m}}{p}\equiv\frac{\boldsymbol{\mu}}{\boldsymbol{\pi}}=\mathbf{1}_r$, thus $\mathrm{MG}\mathcal{A}_1=\{\circlearrowleft\}$;
- It attains the maximum value $\log_b p$ if and only if $\boldsymbol{\mu}=\mathbf{1}_r$ and $\boldsymbol{\pi}\neq\mathbf{1}_r$ (thus, necessarily from §3.1.1, $\boldsymbol{\pi}=p\mathbf{1}_r$), that is, all the branch points connect consecutive sheets, then $\mathrm{MG}\mathcal{A}_1\cong\mathbb{S}_p$, Q.E.D.. □

Note that conditions in the previous Lemma are sufficient but not necessary, so the monodromic entropy is just an indicator of 'topologic complexity', *not* a topologic invariant itself. Every time $\mu_j>1$ the sheets ramifying from the j^{th} branch point connect non-consecutive sheets, thus forbidding some sheet transitions: this reflects on the monodromy group lacking some permutations, thus not being maximal any longer. For example, if $p=10$, $\mu_j=3$ and $\pi_j=5$, sheet transitions like $1 \rightsquigarrow 2$ or $5 \rightsquigarrow 3$ are forbidden (at least winding around the j^{th} branch point) whereas those like $1 \rightsquigarrow 7$ and $6 \rightsquigarrow 2$ are allowed.

Equivalent definitions of monodromic entropy may be given, with its extreme values being 0 and $\pm\log_b p$, in any order. Also note that the rightmost term in the central line of (1.318), which is the most important one, is nothing more than the weighted arithmetic average of $\mathbf{m}\in\mathbb{N}^r$, with their logarithms as weights: $\log_b\mathbf{m}=\{\log_b m_j\}_{1\leq j\leq r}$.

3.1.4. Hyperelliptic case

Let us analyze the case where $\mathcal{A}_1$ is hyperelliptic which, from (1.292), is:

$$f(z) = \sqrt{\prod_{j=1}^{d}(z - z_j)} = \sqrt{(z\mathbf{1}_d - \mathbf{z})^{\mathbf{1}_d}}. \tag{1.319}$$

It is well-known that the hyperellipticty (1.295) of $\mathcal{A}_1$ happens whenever $p=2$ and P has simple roots only (i.e. $r=d$), [37], [42]. Then $\mathbf{m}=\boldsymbol{\mu}=\mathbf{1}_d$, $\boldsymbol{\pi}=2\cdot\mathbf{1}_d$, $\pi_\infty=2/\mathrm{GCD}(d,2)$, $|\boldsymbol{\pi}|=2d$, $\mathrm{LCM}\boldsymbol{\pi}=p=2$ and, noting that $\pi_\infty-1=d\bmod 2$,

$$g = g_1 = \frac{d+\pi_\infty-3}{2} = \frac{d+(d\bmod 2)-2}{2} = \left\lceil\frac{d}{2}\right\rceil-1. \tag{1.320}$$

In literature g-genus hyperelliptic surfaces are defined in fact as Riemann surfaces of kind $w^2-P(z)=0$, with P simple polynomial and $\deg P=d=2g+\delta_{0,(d\bmod 2)}+1$, which is equivalent to (1.320), i.e. either $d=2g+1$ (∞ is a branch point) or $d=2g+2$ (∞ *not* a branch point).

A canonical branch cuts' construction is in this case straightforward: let $\gamma_1,\gamma_2,\ldots,\gamma_{g+1}\subset\mathcal{A}_0$ be any distinct, non-intersecting $g+1$ curves such that, let $\sigma\in\mathbb{S}_d$ be a permutation for the d finite branch points, $\gamma_j = {>}z_{\sigma(j)},z_{2\sigma(j)}{<}$ for $0\leq j\leq g$. If ∞ is a branch point then the $(g+1)^{\text{th}}$ branch cut joins ∞ with the permutated d^{th} branch point (i.e. the $(2g+1)^{\text{st}}$ one, which has not been included in the previous cuts' definition): $\gamma_{g+1} = {>}\infty, z_{\sigma(d)}{<}$; otherwise, since $d{=}2g{+}2$, $\gamma_{g+1}{=}{>}z_{\sigma(d-1)}, z_{\sigma(d)}{<}$, i.e. the $(g{+}1)^{\text{th}}$ branch cut joins the last two branch points according to the permutation (∞ not being a branch point). The branch cuts are identical on both sheets of $\mathcal{A}_0$ and every crossing (or, better, looping) any of them leads to the "opposite" sheet (i.e. the sheet where the sign of square root function is the opposite one).

It is well known [42] that, let $\mathrm{H}_1(\mathcal{A}_1){=}\mathrm{span}\{[\![a_k]\!],[\![b_k]\!]\}_{1\le k\le g}$. A common choice for the homology basis is letting 1-cycles $a_k, b_k{\in}\mathrm{Z}_1(\mathcal{A}_1)$ commute as

$$a_i\circ a_j = b_i\circ b_j = 0, \qquad a_i\circ b_j = \delta_{i,j}, \qquad 1\le i,j\le g; \tag{1.321}$$

furthermore these loops wind around g of the above $g{+}1$ branch cuts (since winding once around the $(g{+}1)^{\text{th}}$ one is topologically equivalent to winding once in the opposite direction around all the other ones).

The holomorphic (1,0)-forms $\Lambda^{1,0}(\mathcal{A}_1)$ admit the canonical basis $\mathscr{B}^{1,0}\{\omega_k\}_{1\le k\le g}$, whose local expression is, [37], [42],

$$\omega_k(z) = \frac{z^{k-1}\mathrm{d}z}{w} \equiv \frac{z^{k-1}}{f(z)}\mathrm{d}z = \frac{z^{k-1}\mathrm{d}z}{\sqrt{\prod_{j=1}^{d}(z-z_j)}}, \qquad 1\le k\le g, \tag{1.322}$$

and they provide an homotopy basis for $\pi_1(\mathcal{A}_1)$. Similarly the holomorphic (1,1)-forms $\Lambda^{1,1}(\mathcal{A}_1)$ admit the canonical basis $\mathscr{B}^{1,1}\{\varpi_k\}_{1\le k\le g}$:

$$\varpi_k(z) = \frac{z^{k-1}}{w^2}\mathrm{d}z\wedge\mathrm{d}\overline{z} \equiv \frac{z^{k-1}}{P(z)}\mathrm{d}z\wedge\mathrm{d}\overline{z} = \frac{z^{k-1}\mathrm{d}z\wedge\mathrm{d}\overline{z}}{\prod_{j=1}^{d}(z-z_j)}. \tag{1.323}$$

3.2 Higher iterations

Let us consider now the iterations of f with itself $f^n = \overbrace{f \circ f \circ \cdots \circ f}^{n \text{ times}}$, $\forall n \in \mathbb{N}$, as defined in (1.291) from the multi-valued equations' point of view, that is considering the Riemann surfaces $\mathcal{A}_n$ as defined by explicit equations like

$$z \mapsto w = f^n(z) = \overbrace{\sqrt[p]{P\left(\sqrt[p]{P\left(\cdots \sqrt[p]{P(z)}\right)}\right)}}^{n \text{ times}}. \tag{1.324}$$

$\mathcal{A}_n$ is a covering space of $\mathcal{A}_{n-1}$ via the ramified cover[105] $f^{-1}: \mathcal{A}_n \to \mathcal{A}_{n-1}$, or as a covering space of $\mathcal{A}_{n-2}$ via the ramified cover $f^{-2}: \mathcal{A}_n \to \mathcal{A}_{n-2}$, and so on. In this paragraph the properties of the surfaces of sequence $(\mathcal{A}_n)_{n \in \mathbb{N}_0}$ will be analyzed and inter-related.

3.2.1. Iterated ramifications, foliations and branching

First of all let us suppose that polynomial P is *uncascaded*, i.e. its iterations P^n do not share any roots $\forall n \in \mathbb{N}_0$; otherwise P (and, by extension, f) is said to be ***cascading*** and the corresponding case will be treated in §3.2.3.

Treating f^n as a multi-valued function, it has p^n single-valued branches, that is $\mathcal{A}_n$ has p^n sheets: this is evident by deriving its implicit-equation definition by raising (1.324) at the power of p^n until a polynomial of degree $\max(p^n, d^n)$ in $\mathbb{C}[w,z]$ is obtained.

The topology of $\mathcal{A}_n$, as well as the computation of the branch points of f^n, is retrieved by induction on $n \in \mathbb{N}_0$. Surface $\mathcal{A}_1$ has been extensively analyzed in §3.1 (along with its genus $\kappa_1 \in \mathbb{N}_0$), its branch points referred to as 1st generation. Now consider surface $\mathcal{A}_n$ defined as

$$w_n = f(w_{n-1}) \equiv f^n(w_0), \tag{1.325}$$

---

[105] Or as both a branched cover $f: \mathcal{A}_{n-1} \to \mathcal{A}_n$ and $f^n: \mathcal{A}_0 \to \mathcal{A}_n$, cfr. §2.3.1.

where w_n is a multi-valued complex function with $w_0 :\equiv z$. Clearly (1.325) is a local definition, identical to (1.293), so $\mathcal{A}_n$ locally covers $\mathcal{A}_{n-1}$ like $\mathcal{A}_1$ does with $\mathcal{A}_0$ and the geometries of $(\mathcal{A}_n)_{\mathbb{N}_0}$ are *locally* the same. More specifically, for every single-valued branch of w_{n-1}, i.e. on every sheet of $\mathcal{A}_{n-1}$ —which is hereinafter called a $(n-1)^{\text{th}}$-generation sheet— the map f defines p different branches for multi-valued function w_n, that is p sheets of $\mathcal{A}_n$ —the *n^{th}-generation sheets*— are said to be **foliating** from each sheet of $\mathcal{A}_{n-1}$ via the branched cover f (i.e. the ramified cover f^{-1}).

Globally $\mathcal{A}_n$ is the connected sum of p^{n-k} homeomorphic copies of $\mathcal{A}_k$, each having p^k k^{th}-generation sheets $(1 \leq k \leq n)$, for a total of p^n sheets. The hierarchy induced by these generation-wise “labels” agrees to the surface $\mathcal{A}_k$ which the sheets foliated from. Also, the ramification points of $\mathcal{A}_n$ are classified according to generations: in fact, a ramification point w_n exists for every branch point w_{n-k}, since (1.325) is also written as $w_n = f^k(w_{n-k})$. Such branch points are singular points between k^{th}-generation sheets, are thus called *k^{th}-generation branch points* and are recursively computable: in fact the ramification points of $\mathcal{A}_{n-k}$ are the k^{th}-generation branch points of $\mathcal{A}_n$.

This fact is immediately clear by induction on (1.324): all the branch points of $w_n :\equiv w$ are the branch points of one of the nested arithmetic p^{th} roots (cfr. §1.5.3), whose branch points are in fact solutions to equations $f(w_k) = 0$ and $f(w_k) = \infty$ (for this purpose only, w_k is considered to be single-valued).

> Consider the case $P(z) = z^2 + 1$, $p = 3$ and consider:
>
> $$f^3(z) = \sqrt[3]{1 + \left[\sqrt[3]{1 + \left(\sqrt[3]{1 + z^2}\right)^2}\right]^2}.$$

The parameters of §3.1.1 for f are $d=r=2$, $\boldsymbol{\mu}=\mathbf{1}_2$, $\boldsymbol{\pi}=3\cdot\mathbf{1}_2$ and $(\mu_\infty,\pi_\infty)=(2{,}3)$. The 1st-generation branch points are the branch points of the innermost cubic root (the triple branch points of *f*), that is $\{\pm1{,}\infty\}$. The 2nd-generation branch points are those of the cubic root in the "middle," solutions to

$$1+(1+z^2)^{\frac{2}{3}}=\begin{cases}0\\ \infty\end{cases}\quad\Rightarrow\quad\begin{cases}z^4+2z^2+2=0\\ z=\infty\end{cases}$$

that is $\mp\sqrt{-1\dot{\pm}\imath}$ (where $\dot{\pm}$ means that the two signs are independent with each other, whereas $\pm$ accounts for the square root's two branches); ∞ is also a branch point, which means that it joins *all* the sheets of $\mathcal{A}_2$). 3rd-generation branch points are those of the outermost cubic root, solutions to

$$1+\left[\sqrt[3]{1+\left(\sqrt[3]{1+z^2}\right)^2}\,\right]^2=\begin{cases}0\\ \infty\end{cases}\quad\Rightarrow\quad\begin{cases}(z^2+1)^{\frac{4}{3}}+2(z^2+1)^{\frac{2}{3}}+2=0\\ z=\infty\end{cases}$$

> so the 8 finite solutions are $\mp\sqrt{-1 \dot{\pm} \sqrt{-1 \ddot{\mp} \imath}}$ (where again symbols $\dot{\pm}$ and $\ddot{\mp}$ mean that the three signs for the two square roots' principal branches are independent with each other). ∞ is again a branch point (as expected), so it joins *all* the sheets of $\mathcal{A}_3$ with each other.
>
> It is extremely important to understand the topology of $\mathcal{A}_3$ via its sheets' connections: there are three possible 4-tuplets of 2^{nd}-generation branch points according to the branch of innermost cubic root (according to the 1^{st}-generation sheet they are computed on), whereas there are three possible 8-tuplets of 3^{rd}-generation branch points according to the branch of "middle" cubic root (according to the 2^{nd}-generation sheet they are on).

While the computation of branch points on every iterated cover is trivial (they are the subset of $\{\mathbf{z},\infty\}$ according to §3.1.1), their computation on the *nested* p^n-cover f^{-n}:$\mathcal{A}_n \to \tilde{\mathbb{C}}$ is instead almost impossible for large n's — even when the roots of P are known in closed form, by either synthesis of P, or direct computation.

The most important conclusion is that, inspecting the branch points, f^n is partitioned into hierarchies, and it is evident that every sheet of $\mathcal{A}_n$ may have different branch points: k^{th}-generation branch points are fixed on every k^{th}-generation sheet, but the other branch points depend on the branching from lower- and towards higher-generation sheets; that is the reason why monodromy actions on rooted trees are the best tools to cope with this self-similar branching structure that will be described in §§3.2.1–3.2.6, 3.3.4–3.3.5.

From the previous example it is clear that, as long P is a polynomial (i.e. a function with the trivial behaviour to have one d-ple pole at ∞) and *if* the complex infinity is a branch point, then it is an "any-generation one": all the p^n sheets of $\mathcal{A}_n$ may have different branch points joining different-generation sheets, but they all join together in ∞; more on this subject (and on previous example) will be discussed in §3.2.4.

> Let $f(z)=\sqrt[3]{z(z-\frac{1}{2})}$, with $p=2$, $r=3$, $\pi_\infty=3$, $\boldsymbol{\pi}=3\cdot\mathbf{1}_2$ and $g_1=1$. See **Figure 27** and **Figure 43** for a local, one-sheeted representation of $f^n(z)$ for $1\leq n\leq 3$. In particular.

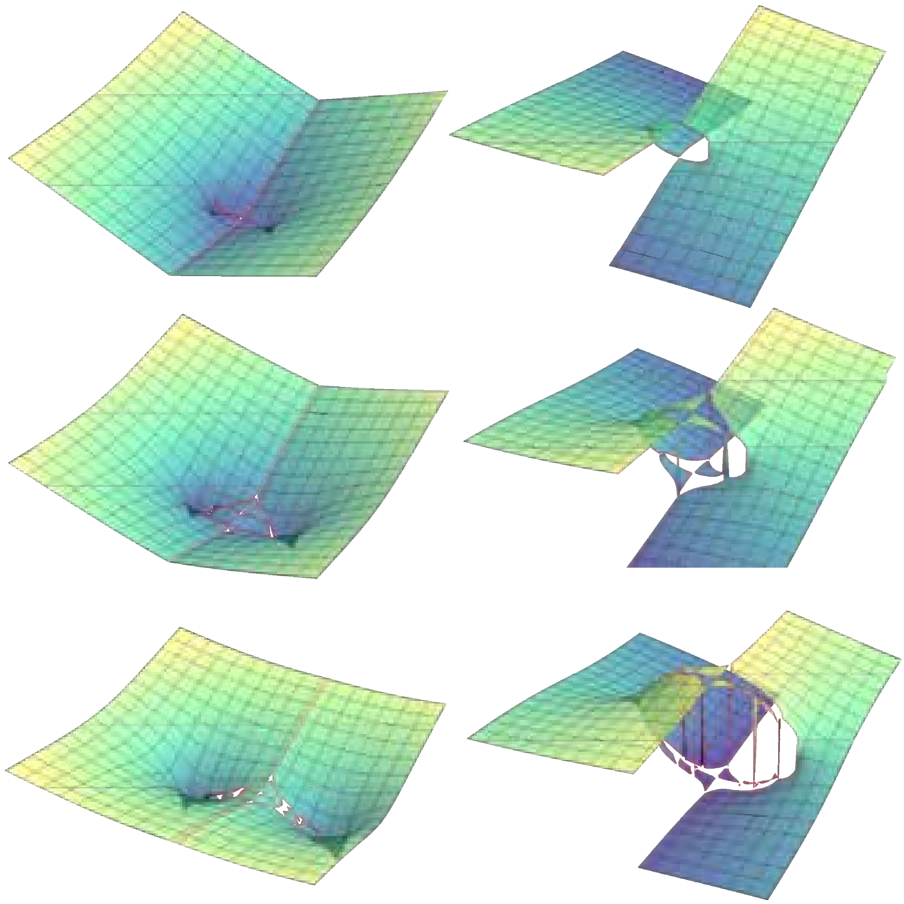


Figure 27 Real (left) and imaginary parts (right) of one single-valued branch of the above $f^n(z)$, with $1\leq n\leq 3$ (top-to-bottom). Sheets' self-similar distribution and branch cuts (in red) are shown.

Theorem

The compact Riemann surface $\mathcal{A}_n$ defined by (1.324) has genus

$$\kappa_n = \frac{(\mathrm{LCM}\,\boldsymbol{\pi})^n - 1}{\mathrm{LCM}\,\boldsymbol{\pi} - 1}(|\boldsymbol{\pi}| + \pi_\infty - 2\,\mathrm{LCM}\,\boldsymbol{\pi} - r + 1), \qquad (1.326)$$

$\forall n \in \mathbb{N}$, i.e. it is homeomorphic to the connected sum of κ_n real projective planes $\mathbb{RP}_{\kappa_n}$. If κ_n is even then it is a g_n-handled torus, with $g_n := \frac{\kappa_n}{2}$.

Proof: Theorem is proved as an application of the Riemann-Hurwitz's Formula (1.205) in §2.3.1 to the ramified p-cover $f^{-1}: \mathcal{A}_n \to \mathcal{A}_{n-1}$, since $\mathrm{b}_{f^{-1}}(z_j) = \pi_j$, $1 \leq j \leq r$, thus leading to the following I-order difference equation ($\kappa_0 = 0$):

$$2 - \kappa_n = (2 - \kappa_{n-1})\mathrm{LCM}\,\boldsymbol{\pi} + (\pi_\infty - 1) + \sum_{j=1}^{r} (\pi_j - 1). \qquad \square$$

When —in most cases— $\mathrm{LCM}\,\boldsymbol{\pi} = \pi_\infty = p$, (1.326) becomes

$$\kappa_n = \frac{p^n - 1}{p - 1}(1 - p)(1 - r) = (p^n - 1)(r - 1) \qquad (1.327)$$

whereas, if $\mathrm{LCM}\,\boldsymbol{\pi} = p$ and $\pi_\infty = 1$ (whenever ∞ is *not* a branch point), it becomes $\kappa_n = (p^n - 1)(r - 2)$, which an orientability condition for arbitrary $\mathcal{A}_n$ can be derived from:

Corollary

The Riemann surface $\mathcal{A}_n$ defined by (1.324), whose genus κ_n is given by (1.327) is *non*-orientable (i.e. it contains a spare real projective plane) if and only if

- both p and r are even, when ∞ is a branch point,
- p is even and r is odd, when ∞ is *not* a branch point,

and its orientability does not depend on the iteration order $n\in\mathbb{N}$.

As a conclusion, this Theorem allows the closed-form computation of the topological genus of any Riemann surface $\mathcal{A}_n$, which is shown to depend on the parameters defined in §3.1.1 alone. For example, the orientable genus of the example's surface is $g_n=(3^n-1)/2$, so the fundamental polygon is homeomorphic to the Sierpiński gasket, cfr. §§1.4.2–1.4.3, Figure 9 , Figure 10 and [9]–§1.8.

The topology of $\mathcal{A}_n$ is then gained for free in analogy with (1.296):

$$\begin{cases} \pi_1(\mathcal{A}_n) \cong \mathbb{Z}_2^{\kappa_n \bmod 2} * \mathop{\ast}^{\left\lfloor \frac{\kappa_n}{2} \right\rfloor} \mathbb{Z} \\ \mathrm{H}^1(\mathcal{A}_n) \cong \mathbb{Z}^{\kappa_n} \end{cases} \tag{1.328}$$

The fundamental group is then isomorphic to $\ast^{g_n}\mathbb{Z}$ in the orientable case $\mathcal{A}_n \asymp \mathbb{T}_{g_n}$, otherwise it is isomorphic to $\left(\ast^{\lfloor\kappa_n/2\rfloor}\mathbb{Z}\right)*\mathbb{Z}_2$ in the non-orientable case $\mathcal{A}_n \asymp \mathbb{RP}\#\mathbb{T}_{\lfloor\kappa_n/2\rfloor} \asymp \mathbb{RP}_{\kappa_n}$.

Let $f(z)=\sqrt[7]{(z-1)(z^7+1)}$, with $p=7$, $r=8$, $\pi_\infty=7$, $\boldsymbol{\pi}=7\cdot\mathbf{1}_8$, $g_n=3(7^n-1)$ and the (simple) branch points being 1 plus the 7th roots of unity. See **Figure 28** and **Figure 47** for a local, one-sheeted representation of $f(z)$ showing its cyclotomic branching structure.

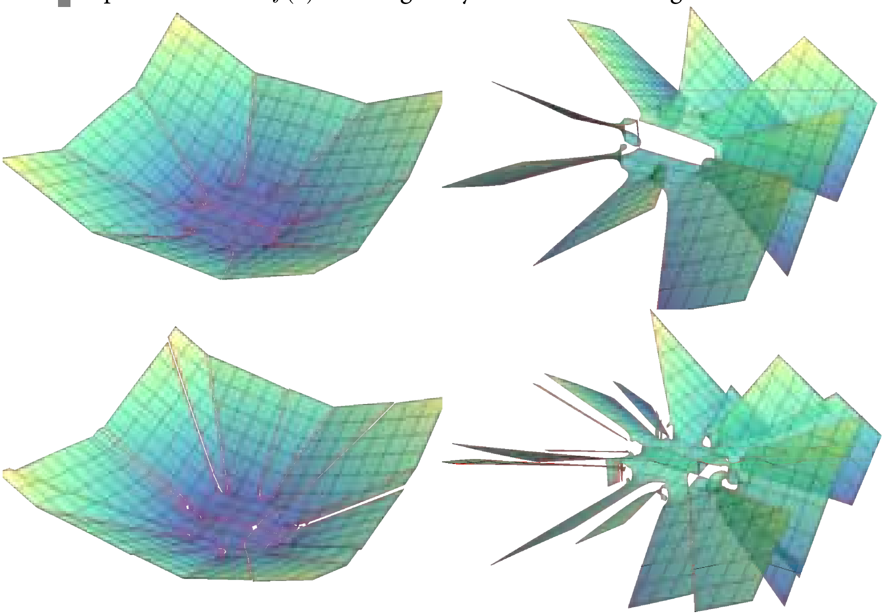


Figure 28 Real (left) and imaginary parts (right) of one single-valued branch of the above $f^n(z)$, for $n=1$ (top) and $n=2$ (bottom). Sheets' self-similar distribution and branch cuts (in red) are shown.

Let now $f(z)=\sqrt{(z+\imath)(z-e)}$, and see Figure 29 and Figure 37 for comparisons between low- and high-order iterations of such surfaces.

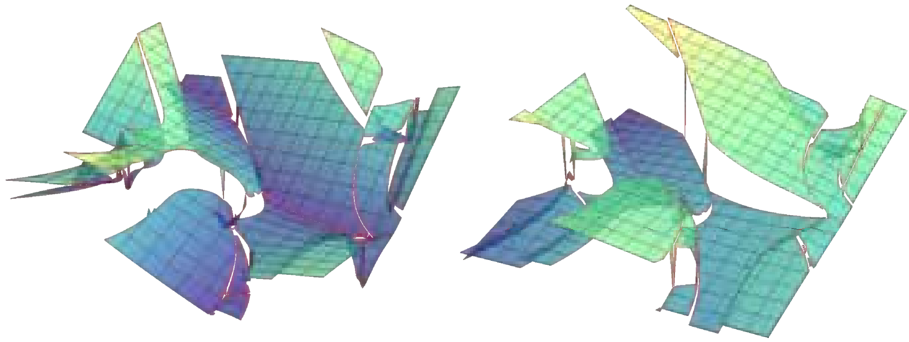


Figure 29 Real (left) and imaginary parts (right) of $f^3(z)$ in the above example. Notice the apparently unordered sheets' distributions along with the branch cuts shown as red curves.

3.2.2. Addressing sheets as rooted trees

It is clear that surfaces $\mathcal{A}_n$ have a finite-order self-similar structure, both in their sheet-branching topology and branch points' distribution. In particular, by using $\mathbb{Z}_p$ as an alphabet for $\mathcal{A}_1$'s sheets, the k^{th}-generation sheets of $\mathcal{A}_n$ are represented by k-characters words of $\mathbb{Z}_p^k$ and the hierarchy structure of different generation sheets is represented by the p-ary rooted[106] tree $\bigsqcup_{k=1}^n \mathbb{Z}_p^k$, as defined in §1.4.1. Let us consider the limit infinite rooted tree $\mathbb{Z}_p^{\mathsf{T}} = \bigsqcup_{k=0}^\infty \mathbb{Z}_p^k$ for simplicity of notation's sake (considerations about the limit $n\to\infty$ will be carried out in the next paragraphs).

The sheets of $\mathcal{A}_l$ are 1-to-1 with words of length $l\in\mathbb{N}$ in the free-product additive group $\overset{n}{\ast}\mathbb{Z}_p$ (cfr. §1.1.8, with $1\leq l\leq n$); such words will be indicated as

$$[\![s_1]\!]_{[k_1]}[\![s_2]\!]_{[k_2]}\cdots[\![s_l]\!]_{[k_l]}, \tag{1.329}$$

[106] The tree's root, i.e. the empty word $\varnothing$, represents the punctured, one-sheeted surface $\mathcal{A}_0$.

where $[\![s]\!]_{[h]}\in\mathbb{Z}_p$ represents the h^{th}-generation sheet number s (among the possible p^{n-h}). Every single character $[\![s]\!]_{[h]}$, alone, could *not* represent a specific sheet of $\mathcal{A}_n$: only the most-significant characters (i.e. those on its left if the juxtaposition is read from left to right as usual) allow the identification of sheets. Reduced words are obtained, for example, simplifying every **diphthong** with two adjacent, same-generation sheets, e.g. $[\![a]\!]_{[h]}[\![b]\!]_{[k]}[\![c]\!]_{[k]}\sim[\![a]\!]_{[h]}[\![b+c]\!]_{[k]}$ (if $h\neq k$). A reduced word in $\overset{n}{\ast}\mathbb{Z}_p$ —whose underlying space is $\bigsqcup^n\mathbb{Z}_p$ by definition §1.1.8— is thus a length-n word like:

$$[\![s_1]\!]_{[1]}[\![s_2]\!]_{[2]}\cdots[\![s_n]\!]_{[n]} := s_1 s_2\cdots s_n\,, \tag{1.330}$$

where the right-hand side of (1.330) is unambiguous once it is agreed for it to be a reduced word. Since $s_j\in\mathbb{Z}_p$ for $1\leq j\leq n$, the best way to represent the sheets of $\mathcal{A}_n$ (which is a common choice for regular rooted trees, cfr. Figure 6) is then via *n-digits, base-p natural numbers*, i.e. using the 1-to-1 correspondence

$$s_1 s_2\cdots s_n \;\leftrightarrow\; s=(s_1 s_2\cdots s_n)_p:=\sum_{j=1}^{n} p^{\,n-j}s_j\in\mathbb{N}_0. \tag{1.331}$$

For the monodromy action $F^n{:}\mathbb{Z}_p^{\intercal}\times\pi_1(\mathcal{A}_n)\to\mathbb{Z}_p^{\intercal}$ for loops $\gamma\in Z_1(\mathcal{A}_n)$ (or for paths after a branch cuts' choice on f^n is made) is the homomorphism $F^n\in\mathrm{Hom}(\mathrm{Aut}\mathbb{Z}_p^{\intercal},\pi_1(\mathcal{A}_n))$ such that the same considerations of (1.306) in §§3.1.1–2 hold, i.e.:

$$\begin{array}{ccc} \mathrm{H}_1(\mathcal{A}_0) & & \overset{n}{\ast}\tfrac{\mu}{\pi}\cdot\mathbb{Z}_\pi \\ f^n\big\downarrow & & \big\uparrow F \\ \mathrm{H}_1(\mathcal{A}_n) & \xrightarrow[\cdot/[\cdot|\cdot]]{} & \pi_1(\mathcal{A}_n) \end{array} \tag{1.332}$$

The words in $\overset{n}{\ast}\tfrac{\mu}{\pi}\cdot\mathbb{Z}_\pi$ are algebraic free objects of finite length $L\in\mathbb{N}$ representing the monodromy of γ as

$$_{[s_0]}[\![\mathbf{k}_1]\!]_{[h_1]}\rightsquigarrow[\![\mathbf{k}_2]\!]_{[h_2]}\rightsquigarrow\cdots\rightsquigarrow_{[h_{L-1}]}[\![\mathbf{k}_L]\!]_{[h_L]}, \tag{1.333}$$

where $[\![\mathbf{k}]\!]_{[h]}\in\mathbb{Z}_p$ are the *h^{th}-generation winding numbers* of γ reporting the winding numbers of the loop on h^{th}-generation sheets number s (among the possible p^{n-k}). Every single character $[\![s]\!]_{[h]}$ does *not* represent a specific sheet of $\mathcal{A}_n$: it can represent it only if considered along with its more-significant winding numbers. The word (1.333) has to be read as the monodromy of γ: starting from the basepoint of γ, which is on the s_0^{th} *global* sheet —which, by (1.330), is given by a reduced word like $s_0=\sum_{j=1}^{n}p^{n-j}s_j$ — it winds first $[\![\mathbf{k}_1]\!]$ times around the branch points on the h_1^{th}-generation sheet branching from s_0, then winds $[\![\mathbf{k}_2]\!]$ times around the branch points on the h_2^{th}-generation sheet branching from it, …, and so on until it finally winds $[\![\mathbf{k}_L]\!]$ times around the branch points on the h_L^{th}-generation sheet branching from the penultimate-specified-generation sheet.

Note that $h_1,h_2,\ldots,h_L\in\mathbb{Z}_p$ because the order of visited sheets *does* depend on the monodromy; the word (1.333) can anyway be reduced by both the simplification rules in $\overset{n}{\ast}\frac{\mu}{\pi}\cdot\mathbb{Z}_\pi$ and the associative property in $\mathbb{Z}_p(+)$. For example, let two (or more) winding numbers belong to sheets of same generation (e.g. the h_m^{th} generation — for some $1\leq m\leq L$) be juxtaposed with each other: γ is travelling on a portion of Riemann surface which is, in fact, locally homeomorphic to $\mathcal{A}_1$ on $\mathcal{A}_{h_m}$, then:

$$\cdots\rightsquigarrow[\![\mathbf{k}]\!]_{[h_m]}\rightsquigarrow[\![\mathbf{k}']\!]_{[h_m]}\rightsquigarrow\cdots\quad\sim\quad\cdots\rightsquigarrow[\![\mathbf{k}+\mathbf{k}']\!]_{[h_m]}\rightsquigarrow\cdots. \tag{1.334}$$

This leads to a reduced-word monodromy representation for $\overset{n}{\ast}\frac{\mu}{\pi}\cdot\mathbb{Z}_\pi$ which can be formally represented in closed form like in §1.1.8 and will be introduced in §3.2.6. That sheet addressing on $\mathcal{A}_n$ is exactly the a wreath recursion (1.92) on a finite tree, as defined in §1.4.2 so, by the Wreath Recursion Theorem, a self-similar group acting on $\mathbb{Z}_p^{\mathsf{T}}$ must exist, and will be defined later in §3.2.6.

3.2.3. Cascading phenomenon

First of all let us suppose that, for a fixed $1\leq j\leq r$, $\exists h\in\mathbb{N}\backslash\{1\}$ such that $P^h(z_j)=0$ (and also $P^{h-1}(z_j)\neq 0$ if $h\geq 3$) i.e. the j^{th} root of P reappears as a root of the h^{th}

iterated polynomial P^h. Trivially z_j is also a root of P^{lh}, $\forall l\in\mathbb{N}$. When this happens for $1\leq r'\leq r$ roots of P (even if for different h's) the polynomial itself —so also f— is said to be **cascading** (or **cascaded**) of order $h-1$; in the case of multiple cascading roots, h is the greatest common divisor among the every root's cascating orders, so that P^h is the least-iterated polynomial to have the maximum number of cascading roots. The most trivial case is when P is a homogeneous poynomial ($h=2$ for $z_1=0$), so P^n has all the roots of P, i.e. $P\,|\,P^n$ $\forall n\in\mathbb{N}$ and:

$$\mathrm{mul}_{P^n} z_j \geq \mathrm{mul}_P z_j = m_j, \qquad 1\leq j\leq r. \tag{1.335}$$

The destiny of cascading branch points (i.e. any cascading roots z_j with $\pi_j>1$) is to link different-generation sheets with each other, apparently breaking the branching self-similarity discussed in §3.2, whereas their multiplicities (as roots of P^n) [their orders (as branch points of f^n)] are bound to exponentially increase with n. More exactly, a cascading branch point z_j of order h_j-1 asymptotically behaves like:

$$\begin{cases} \mathrm{mul}_{P^n} z_j = m_j^{n-h_j} \\ \mathrm{ord}_{f^n} z_j = \pi_j^{n-h_j} - 1 \end{cases} \tag{1.336}$$

If $\mathcal{A}_n$ is homogeneous (and, letting $z_1=0$, $\pi_1>1$), both 0 and ∞ are branch points linking all its p^n sheets together. A topological representation of $\mathcal{A}_n$'s branching is still possible, although the rooted p-ary tree is not a tree anymore, bur rather a graph; a description via period matrices —i.e. the graph's incidence matrices— is then preferred.

Also, the monodromy groups of $\mathcal{A}_n$ *may* be solved with more difficulties than the uncascaded case, although some even algebraically "simpler" situations may occurr (cfr. §3.3.4), In general, no closed-form representation like (1.304) is available for "large-n" monodromy actions. Apart from the differences detailed above, the cascaded case is mostly equivalent to the uncascaded one.

3.2.4. Analytical considerations

Let us consider case $r=1$, $\mathrm{GCD}(q,p)=1$: $f(z)=\sqrt[p]{(z-z_0)^q}$, with monodromy group $\mathbb{Z}_p$. Clearly it is a 1-to-p contraction mapping on all of its branches if $|z-z_0|<1$; every single-valued branch maps the whole circle $\mathbb{B}_1(z_0)$ to one of its p open sectors, $\left\{re^{\imath\theta}-z_0\in\mathbb{C}\;\middle|\;r\in[0,1[,\ \theta\in\frac{2\pi}{p}[k,k+1[\right\}$, $0\leq k<p$. The map is still contracting on every sector of centre z_0, radius 1 and angle less than $2\pi\frac{p-1}{p}$, due to the position of the branch cut $\rangle z_0,\infty\langle$ intersecting the circumference $|z-z_0|=1$ in one point.

Analytic continuation of the whole Riemann surface of $f(z)$ makes it contracting on the whole closed circle $\overline{\mathbb{B}_1(z_0)}$, whose interior is also the basin of attraction of $f^n(z)$, $\forall n\in\mathbb{N}$. The limit Riemann surface $\lim_n f^n(z)$ is defined on $\mathbb{B}_1(z_0)$ to be constantly 0.

Let us consider the example of §3.2.1 where the branch points have an interesting algebraic symmetry:

> Let $f(z)=\sqrt[3]{z^2+1}$; working by induction on $n\in\mathbb{N}$, the finite n^{th}-generation branch points are 2^{n+1} and closed-form computable as
>
> $$\mp\underbrace{\sqrt{-1\,\dot{\pm}\sqrt{-1\,\ddot{\mp}\sqrt{-1\cdots\sqrt{-1\,\dddot{\pm}\,\imath}}}}}_{n\ \text{times}}=:\frac{\mathbf{1}_n^{\pm}}{\sqrt{\mathbf{1}_{n-1}\oplus(\dot{\mp}\imath)}}\,.$$

> In this case the symbol $\mathbf{1}_n^{\pm}\in\{-1,1\}^n$ stands for the *generic* n-tuple which attains 2^n different values ± 1 on every coordinate — and overall represents the vertices of the n-dimensional hypercube $[-1,1]^n$. The right-hand side of the expression stands for a finite nested radical (with every complex square root's branch specified by both the initial sign and the n signs of $\mathbf{1}_n^{\pm}$), as defined in §1.5.3.
>
> These n^{th}-generation branch points are thus the solution to algebraic equations of degree 2^{n+1} (see below the properties of nested radicals); in particular, the one branch point with all the positive signs and for $n\to\infty$ tends to the golden mean g, cfr. §1.5.1.

The right-hand side of the expression in the example is a compact way to represent an infinite nested radical, [30], [104]. Both nested radicals and continuous fractions can be employed (as the aforementioned example showed) to write down the numerical relationships between different-generation branch points on Riemann surfaces (which are often computed solving polynomial equations), due to the iterated nature of their generating covers.

Another fact to consider is that, as far as analytical considerations are given, functions $f(z)$ like those used in this section (as well as will be in §3.3 for the meromorphic case) have forward orbits whose single branches are **normal** families (cfr. last part of §2.1.3) in $\mathcal{O}(\tilde{\mathbb{C}})$, since $\{f^n(z)\}_{n\in\mathbb{N}_0}$ admits, at least, a sub-sequence like $(f^{nm})_{n\in\mathbb{N}_0}$ which uniformly converges either to a analytic function or to ∞ on every different connected component (where m is the cascading index of f, i.e. the GCD of the iteration orders wherever cascading occurs).

As far as the more general, limit Riemann surface is concerned, it should satisfy the following fixed-point condition, letting $F(z)=\lim_n f^n(z)$:

$$F(z) = \sqrt[p]{P\left(F(z)\right)} \qquad \Leftrightarrow \qquad F(z)^p - P\left(F(z)\right) = 0. \tag{1.337}$$

Equation (1.337) is the functional analogue to (1.293), where $F(z)$ is equivalent to w; this always admits $\max(n,p)$ constant solutions — the roots of a polynomial whose leading-degree binomial is always $z^p - z^n$. These constants are the values the limit function converges to on disconeected components on various of the infinite sheets. If other non-constant functions $F(z)$ solve (1.337) they are nontrivial functions that need to be checked for further suitability as solutions of $F(z)=\lim_n f^n(z)$ also.

> Let $f(z)=\sqrt[3]{(z-z_1)(z-z_2)}$ and consider the functional solution of (1.337) as $F(z)^3 - F(z)^2 + (z_1+z_2)F(z) - z_1 z_2 \equiv 0$. Actually, three different constant solutions exist:
>
> $$\begin{cases} F_0 = \dfrac{1}{3}\left[1 - \sqrt[3]{\dfrac{2}{\Delta}}(3z_1+3z_2-1) + \sqrt[3]{\dfrac{\Delta}{2}}\right] \\ F_{1,2} = \dfrac{1}{3}\left[1 + (1 \mp i\sqrt{3})\dfrac{3z_1+3z_2-1}{\sqrt[3]{4\Delta}} - \dfrac{1 \pm i\sqrt{3}}{2}\sqrt[3]{\dfrac{\Delta}{2}}\right] \end{cases}$$
>
> where
>
> $$\begin{cases} \Delta = \tilde{\Delta} + \sqrt{\tilde{\Delta}^2 + 4\left[3(z_1+z_2)-1\right]^3} \\ \tilde{\Delta} = 27 z_1 z_2 - 9(z_1+z_2) + 2 \end{cases}$$
>
> Those are the three constat values

3.2.5. IFS setting for Riemann surfaces

It is known that the space of germs of a Riemann surface $\mathcal{R}$ on one of its points $p\in\mathcal{R}$ is one-to-one with the Lie algebra $\mathbb{C}[\![p]\!]^\natural$ of Puiseux series, and that this algebra becomes compact as soon as $\mathcal{R}$ locally is. As observed in the introductory paragraph, maps of kind $z \mapsto \sqrt[p]{(z-z_0)^q}$, become contractions in the space of germs around z_0. Thus the general prefractal initiator is itself a contraction in some sense which has to be defined with the use of the Poincaré metric on $\mathbb{B}_1(0)$

$$\underline{g}_P(z,\overline{z}) :\equiv g_P(z)\frac{\partial}{\partial z}\otimes\frac{\partial}{\partial \overline{z}} \overset{\text{def}}{=} \frac{2}{1-z\overline{z}}\frac{\partial}{\partial z}\otimes\frac{\partial}{\partial \overline{z}}. \qquad (1.338)$$

On $\mathcal{A}_{n+1}$ the metric is defined recursively from the metric on $\mathcal{A}_n$ as:

$$\begin{cases} g_1(w_1)\dfrac{\partial}{\partial w_1}\otimes\dfrac{\partial}{\partial \overline{w}_1} = \dfrac{2|f'(z)|^2}{1+|f(z)|^2}\dfrac{\partial}{\partial z}\otimes\dfrac{\partial}{\partial \overline{z}} \\ g_n(w_n)\dfrac{\partial}{\partial w_{n-1}}\otimes\dfrac{\partial}{\partial \overline{w}_{n-1}} = \dfrac{2|\mathrm{D}f^n(z)|^2}{1+|f^n(z)|^2}\dfrac{\partial}{\partial z}\otimes\dfrac{\partial}{\partial \overline{z}} \end{cases} \tag{1.339}$$

Let $C\subset\mathbb{C}$ be the open set where f is contracting in at least one branch, $|f'|<1$, as was observed in §§3.1.3, 3.2.4: this contains, for example, the intersection of the open sets where each of the mappings $z\mapsto\sqrt[p]{(z-z_j)^{m_j}}$ is simultaneously contracting, and $\{z_j\}_{1\leq j\leq r}\subset C$ always holds. The whole f represents an IFS on $C\subset\mathrm{FJ}(f)$ whose contractions are the r multi-valued maps $z\mapsto\sqrt[p]{(z-z_j)^{m_j}}$ restricted to C and whose attractor is the filled Julia set $\mathrm{FJ}(f)$, as defined in §2.4.1, which is called a "**Riemann** Iterated Function System" (or **R-IFS**). For definitions and properties see §5.1.1, [9]–§1.7, see [26] about IFSs and [77] about Complex Dynamics and Julia sets. The existence of basins of attraction for every pre-fractal Riemann surface $\mathcal{A}_n$ is guaranteed by the Analytic Continuation Theorem applied to every single-valued branch of f^n.

Taking the closure of $\mathcal{O}_0=\overline{C}$ as compact initiator for this IFS, then a sequence of prefractal compact sets $(\mathcal{O}_n)_{n\in\mathbb{N}_0}$ is generated, with $f{:}\mathcal{O}_n\to\mathcal{O}_{n+1}$ being the branched-cover analogue of contraction mapping "w" of §3.1 and $f^{-1}{:}\mathcal{O}_{n+1}\to\mathcal{O}_n$ the partial self-covering as defined in the previous paragraph. $(\mathcal{O}_n)_{n\in\mathbb{N}}$ is also a sub-orbifolds' sequence, where $\mathcal{O}_n\subset\mathcal{A}_n$ topologically and $\mathrm{MG}\mathcal{O}_n\leq\mathrm{MG}\mathcal{A}_n$ algebraically, $\forall n\in\mathbb{N}_0$. From the topologic point of view, this convergence is guaranteed, by definition of sequential compactness, within the multi-valued Fatou set $\mathrm{F}(f)$.

This fact suggests that there always exist a loop $\gamma_n\in\mathrm{C}^0([t_n,t_{n+1}];\mathring{\mathcal{O}}_n)$ based in $\gamma_n(t_n)\in\mathring{\mathcal{O}}_n$, such that its lift $\gamma_n\uparrow f^{-1}$ lies in $\mathring{\mathcal{O}}_{n+1}$; let $F_n\in\mathrm{MG}\mathcal{A}_n$ be the corresponding monodromy action. The paths on the sub-orbifold $\mathcal{O}_n$ introduced in §2.3.2 are described by a 'mixed words' like those in (1.209), where on the endpoints:

$$F_n \bullet \gamma_n(t_{n+1}) = \gamma_{n+1}(t_{n+1}), \qquad \forall n \in \mathbb{N}_0. \tag{1.340}$$

The same fact holds for paths in $(\mathcal{A}_n)_{n\in\mathbb{N}_0}$, although the convergence is not guaranteed everywhere for $n\to\infty$. In the polynomial case, though, even convergence might stand for trivial behaviour since, from §3.2.4, we know that

$$\lim_n f\big|_{\mathcal{O}_n}^n(z) \equiv 0 .$$

Algebraic relations might anyway be non-trivial: in this case *the fractal Riemann surface is just a* continuum *model for a self-similar group* (as will be explained in §3.3.4), despite no interesting standard Calculus may exist on the limit surface.

A proper definition of the limit space of $(\mathcal{A}_n)_{\mathbb{N}_0}$ involves using different topologies: regarding $\mathcal{A}_n$ as topological manifolds alone, the compact-set topology is employed; regarding the symmetries' group on $\mathcal{A}_n$ as a Riemann surface, the topology of the associated *compactified* rooted tree

$$\overset{\infty}{\ast}\mathbb{Z}_p \cong \overline{\mathbb{Z}_p^{\mathsf{T}}}, \tag{1.341}$$

jointly defined in §§1.4.3, 2.4.1, 3.2.2. Under this topology the quotient space is the defined by means of the symbolic homomorphism of monodromy actions (1.332). The equivalence relation '$\sim$' (1.334) is then extended to the asymptotical equivalence relation '$\sim$' (1.104) to account for paths that are asymptotically equivalent under the monodromy action for $n\to\infty$. Then

$$\mathcal{A}_\infty := \overline{\mathbb{Z}_p^{\mathsf{T}}}\big/\sim; \tag{1.342}$$

The limit space of f is naturally the boundary of $\mathcal{A}_\infty$ which, by the IMG-Theorem in §2.3.4, is (homeomorphic to) the quotient space $\overline{\mathbb{Z}_p^{\mathsf{T}}}\big/\mathrm{IMG}\mathcal{A}_\infty$, then leading to the computation of its iterated monodromy group.

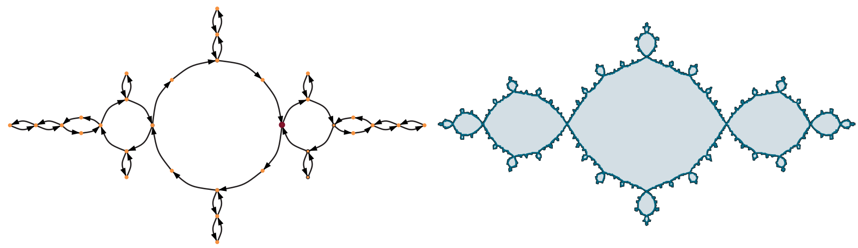


Figure 30 Schreier graph of $\mathcal{J}_{\mathrm{IMG}(z^2+1)}$ (limit set of the ***Basilica* group**: IMG of complex map $z \mapsto z^2+1$) and its Julia set, shown (and further commented) in Figure 36

Considering f as the R-IFS defined above, it is even possible to estimate a lower bound for the Hausdorff dimension of its Julia set $\mathrm{J}(f)$ (also refer to (1.471) for another way to compute it for "standard" IFSs) as:

$$\dim_{\mathrm{H}} \mathrm{J}(f) \geq \inf_{z_0 \in \mathcal{A}_0} \left\{ d \in \mathbb{R}_+^* \;\middle|\; \sum_{j=1}^{r} \frac{\mu_j^d}{\pi_j^d} \left| z_0 - z_j \right|^{d\left(\frac{\mu_j}{\pi_j} - 1\right)} < \infty \right\}. \quad (1.343)$$

Addressing the sheets of $\mathcal{A}_n$ as described in §3.2 turns out to be a symbolic dynamics which is driven by a topologically conjugate IFS (as those which will be used in §0) to the just-introduced R-IFS, and whose contraction ratio is analogue to the contraction coefficient (1.97).

Another interesting way to represent such R-IFS is via the sequence of fundamental polygons of Riemann surfaces $(\mathcal{A}_n)_{n \in \mathbb{N}_0}$, cfr. §§2.3.1, 2.4.2, [37], [42]: from (1.327) it follows that these polygons are $2\kappa_n$-gons with sides identified 2-by-2 (if $\mathcal{A}_n$ is orientable, they are $4g_n$-gons with *couples* of sides identified 2-by-2).

> Let us consider the Siepriński carpet $\check{S}_2$, cfr. §5.1.5, [9]–§1.8, as well as Figure 65 It is well-known that $\check{S}_2$ is not orientable, as well as its prefractals $\check{S}_{2,n}$ (of square initiator, $\forall n\in\mathbb{N}$): they have as many holes as their non-orientable genus $\kappa_n=(8^n-1)/7$. The corresponding fundamental polygon is then a regular (8^n-1)-gon.

From the period matrices' point of view $A_n, B_n\in\mathbb{M}_{g_n}(\mathbb{Z}(\imath))$, cfr. §2.4.4, their structure can be recovered via a block-oriented procedure; let the simplified formula (1.327) be used, where $\kappa_n=(p^n-1)(r-1)\equiv 0 \bmod 2$. Of course the circulation integrals (1.253) that define the Gaussian-integer elements of such matrices are done on more and more complicated multi-valued functions for increasing $n\in\mathbb{N}$ (and require integration on evermore analytically-convoluted paths); while that is an unfeasible task to be solved via Riemann theta functions (1.250), it has enormous difficulties even numerically, because branch cuts, though arbitrary, get progressively closer to each other on certain areas of the complex plane. Anyway, the block-by-block construction algorithm of prefractal period matrices should be further investigated, since it reflects the Riemann surfaces' topological self-similarity. It is suspected that, since their elements depend on the position of different-generation branch points and cuts —which have a self-similar distribution as further investigated throughout §§3.3.3–3.3.5— the dependence of specific matrix elements (or whole sub-matrices) on the generation themselves satisfies a recursion relation, and the whole matrices display some kind of self-similar structure. That would be much in a similar way as the wreath recursion and other self-similar matrices are built — for example the incidence/adjacency ones for fractal networks, simplicial complexes and electric lumped-parameter circuits, [9]–§§5, 7, [10], [12], [16], [17], [18], [20], [53].

3.2.6. Iterated Monodromy Groups

As said in the previous Chapter, Riemann surfaces of the sequence $(\mathcal{A}_n)_{\mathbb{N}_0}$ are all orbifolds, §2.3.4 (with respect to their proper symmetries' group — cfr. §2.4.2), whereas the inclusive sequence of self-covering orbifolds $(\mathcal{O}_n)_{\mathbb{N}_0}$ associated to the R-IFS f (i.e. where it is contracting) is a *prefractal orbifolds*' sequence. The last sequence to introduce is the self-covering $(f^{-n})_{\mathbb{N}}$ (cfr. §2.3.2), which the iterated monodromy group of f is defined thanks to — cfr. §2.4.5, [83]:

$$\mathrm{IMG}f \stackrel{\text{def}}{=} \mathrm{IMG}\mathcal{A}_\infty = \frac{\pi_1(\mathcal{A}_\infty)}{\bigcap\limits_{n\in\mathbb{N}} \mathrm{Ker}\,F^n}, \tag{1.344}$$

the quotienting set $\bigcap_{n\in\mathbb{N}}\mathrm{Ker}F^n$ being the nucleus (1.100) of the iterated monodromy action (whereas, if that intersection is stopped at a finite step $N\in\mathbb{N}$, it would simply be the nucleus of the monodromy action of $\mathcal{A}_N$).

By (1.342) the limit space is $\mathcal{J}_{\mathrm{IMG}f} \cong \partial\mathcal{A}_\infty$ so the topology of $\partial\mathcal{A}_\infty$ (and then of $\mathcal{A}_\infty$) is essentially described by the self-similarity of $\mathcal{J}_{\mathrm{IMG}f}$, i.e. it is symbolically represented by the algebraic symmetries of IMGf, which is a self-similar group (cfr. §1.4.2). To state it in another way, the topology of IMGf is inherited by the forward [backward] orbits of the branched cover f [ramified cover f^{-1}], which are handled like a p-regular rooted tree, as per (1.214):

$$\mathrm{W}_f^{\pm}(z_0) \stackrel{\text{def}}{=} \mathrm{T}_{f^{\mp 1}}(z_0) = \bigsqcup_{n=0}^{\infty} f^{\pm n}(z_0).$$

It is still possible to describe —at least formally, if not numerically— all the possible homotopy classes in $\mathcal{A}_n$ (and, by extension thanks to the asymptotic equivalence, in $\mathcal{A}_\infty$) via a *linear* representation of the monodromy action on the global surface. In analogy with (1.304)–(1.305) let (1.333), i.e.

$$[\![\mathbf{k}_1]\!]_{[h_1]} \rightsquigarrow [\![\mathbf{k}_2]\!]_{[h_2]} \rightsquigarrow [\![\mathbf{k}_3]\!]_{[h_3]} \rightsquigarrow \cdots,$$

be the symbolic dynamics of such a homotopy class $[\![\gamma]\!]\in\pi_1(\mathcal{A}_n)$. The above diagram is, in fact, nothing more than a definition of a generic path on the rooted tree $\mathbb{Z}_p^{\mathsf{T}}$.

Using the base-p representation (1.331) for the sheets, $\forall n\in\mathbb{N}$, the monodromy action $F^n\in\mathrm{Hom}(\mathrm{Aut}\mathbb{Z}_p^{\intercal},\pi_1(\mathcal{A}_n))$ is defined by the *wreath recursion* (§1.4.2):

$$\begin{aligned} F^n(\llbracket\gamma\rrbracket) &\overset{\text{def}}{=} \sum_{j=1}^{n} F(\llbracket\gamma\rrbracket)\, p^{h_j} \\ &= \sum_{j=1}^{n} \Big\llbracket \llbracket \mathbf{k}_j \rrbracket \cdot \mathbf{m} \Big\rrbracket_p p^{h_j}\,. \end{aligned} \tag{1.345}$$

From the computational point of view this does not mean that the monodromy is computable *a priori*: despite the action (1.306) of the corresponding homology group $\mathrm{H}_1(\mathcal{A}_n)$ belongs to the Pontryagin's dual of $\overset{n}{\ast}\frac{\boldsymbol{\mu}}{\boldsymbol{\pi}}\cdot\mathbb{Z}_{\boldsymbol{\pi}}$, there is no reflexive isomorphism with itself, so (1.307) does not hold anymore. That can be seen by directly inspecting multi-valued function $f^n(\gamma(t))$ in order to compute the deck transformation '$\looparrowright$' induced by the lifted path, which is a expressed by a nested radical — cfr. §1.5.3 and §3.2.4's example:

$$\begin{aligned} f^n(\gamma(t)) &= \overbrace{\sqrt[p]{\prod_{j_1=1}^{r}\left\{-z_{j_1}+\sqrt[p]{\prod_{j_2=1}^{r}\left[-z_{j_2}+\cdots\sqrt[p]{\prod_{j_n=1}^{r}(\gamma(t)-z_{j_n})^{m_{j_n}}}\right]^{m_{j_2}}}\right\}^{m_{j_1}}}}^{n\ \text{times}} \\ &::= \underbrace{\left\{\left[\cdots(\gamma(t)\mathbf{1}_r-\mathbf{z})^{\frac{\boldsymbol{\mu}}{\boldsymbol{\pi}}}-\mathbf{z}\right]^{\frac{\boldsymbol{\mu}}{\boldsymbol{\pi}}}-\mathbf{z}\right\}^{\frac{\boldsymbol{\mu}}{\boldsymbol{\pi}}}}_{n\ \text{times}} \end{aligned}$$

$$\Downarrow$$

$$\looparrowright(f^n\uparrow\gamma) =$$

$$e^{2\pi\imath\llbracket\mathbf{k}_n\rrbracket\cdot\frac{\boldsymbol{\mu}}{\boldsymbol{\pi}}}\prod_{j_1=1}^{r}\overbrace{\left|-z_{j_1}+e^{2\pi\imath\llbracket\mathbf{k}_{n-1}\rrbracket\cdot\frac{\boldsymbol{\mu}}{\boldsymbol{\pi}}}\prod_{j_2=1}^{r}\left|-z_{j_2}+\cdots e^{2\pi\imath\llbracket\mathbf{k}_1\rrbracket\cdot\frac{\boldsymbol{\mu}}{\boldsymbol{\pi}}}\prod_{j_n=1}^{r}\left|\gamma(t)-z_{j_n}\right|^{\frac{\mu_{j_n}}{\pi_{j_n}}}\right|^{\frac{\mu_{j_2}}{\pi_{j_2}}}\right|}^{n\ \text{times}}{}^{\frac{\mu_{j_1}}{\pi_{j_1}}}$$

If the sequence of visited sheets' generations is progressive (for example from the s_0^{th} global sheet to a sheet of 1st generation sheet, up to a sheet of a L^{th} generation, $L\leq n$ — or vice versa) then the curve winds $\llbracket\mathbf{k}_1\rrbracket$ times around the s_1^{th} 1st-generation sheet, then it winds $\llbracket\mathbf{k}_2\rrbracket$ times around the s_2^{th} 2nd-generation sheet branching from the global sheet s_0+ps_1, then it winds $[\![\mathbf{k}_3]\!]$ times around the s_3^{th} 3rd-generation sheet branching from the global sheet $s_0+ps_1+p^2s_2$, and so on until it finally winds $[\![\mathbf{k}_L]\!]$ times around the s_L^{th} L^{th}-generation sheet branching from the global sheet $s_0+ps_1+p^2s_2+\ldots+p^Ls_L$. In this representation scheme, each sheet is an element of the bi-monoid $\mathbb{N}[p]$ (*bi*monoid with respect to polynomial addition and multiplication).

It is evident that a p-ary representation of the global sheets of every surface $\mathcal{A}_n$ is also feasible; using negative powers of p instead, the infinite sheets of $\mathcal{A}_\infty$ are mapped into $[0,1]\subset\mathbb{R}$, where each digit represents the branching of the sheet from a specific generation. On such a surface $\mathcal{A}_\infty$ the number $\xi\in[0,1]$ written in base p as

$$\xi=\sum_{n=1}^{\infty}s_n p^{-n}, \tag{1.346}$$

is the sheet recursively foliated from the s_n^{th} n^{th}-generation sheets, $\forall n\in\mathbb{N}_0$. Like happens for fractional real numbers —algebraically represented by asymptotic equivalence relations like those described in (1.104)— some monodromies may be represented by more than one integer digits' sequence. This coincides with the quotient structure of $\mathrm{IMG}f$, where $\mathcal{J}_{\mathrm{IMG}f}$ is the corresponding quotient topological space. Its Riemann sheets are in a 1-to-1 bijection with a subset of $[0,1]$, which can even feature fractal properties, like a Cantor dust or, better, a Julia set, cfr. §§2.3.2, 2.4.1. Every finite-generation sheet is represented by a rational number in $\mathbb{Q}\cap]0,1[$, their "location" within the points of the limit set following the same topology of the embedding $\mathbb{Q}\subset\mathbb{R}$, and unformally represented in Figure 11 (left).

Let $f(z)=\sqrt[3]{z(z^3+1)}$; working by induction on $n\in\mathbb{N}$, the surface $\mathcal{A}_n$ admits $2(2^{n-1}+1)$ branch points, of which 4 of each n^{th} generation. On the n^{th}-generation coordinate system of the complex (multi-valued) points w_n, those 4 branch points have all branching number 3 and are the three cubic roots $\sqrt[3]{-1}=\left\{-1,\ \mp\frac{1}{2}+\imath\frac{\sqrt{3}}{2}\right\}$, plus branch points 0 and ∞ which are common for all coordinates w_n and both have branching number 3^n. By (1.327) $\mathcal{A}_n$ has 3^n sheets and orientable genus $g_n=3^n-1$. On the limit Riemann surface $\mathcal{A}_\infty$ the (finite-generation) sheets are represented by elements of $\mathbb{N}[3]$, such as $s_0+3s_1+9s_2+27s_3+\ldots+3^L s_L$, i.e. by polynomials (where the finite-degree requirement is essential). In the bijection with $[0,1]$ instead, they are represented as rational numbers, i.e. by numbers with either a finite number of ternary digits, like 0.021211_3, or with periodic digit trails, like $0.21002\overline{11021}_3$.

The multi-valued Julia sets associated to this prefractal Riemann surfaces are shown in Figure 45

In particular, what are the elements of $\mathbb{Q}\triangle[0,1]\equiv[0,1]\setminus\mathbb{Q}\equiv\mathbb{I}\cap[0,1]$ bijected to in $\mathcal{A}_\infty$? Basically, they are points added by the topological closure analogue to that of infinite rooted trees in §§1.4.1, 1.4.3, and correspond to the limit set $\mathcal{J}_{\mathrm{IMG}f}\cong\partial\mathcal{A}_\infty$.

The wreath recursion (or connections') matrix $H_n\in\mathbb{M}_{2\kappa_n,p}(\mathbb{Z}_p)$ can be defined as it was done in §1.4.2 with the same considerations done in §3.1.2 for H_1; this time, the non-commutativity of $\pi_1(\mathcal{A}_n)$ implies that H_n loses the nice column-symmetry of (1.311) and, basically, its computability corresponds to the complete topological solution of $\mathcal{A}_n$, because H_n addresses each possible sheet transition on the prefractal Riemann surface.

3.2.7. Differential equations

As it was done in §3.1.3 for the initiatior Riemann surface, $\mathcal{A}_n$ is the underlying surface of the following differential equation.

$$\mathrm{d}w_{n+1} = \sum_{j=1}^{r} \frac{\mu_j/\pi_j}{w_n - z_j}\mathrm{d}w_n = \prod_{k=0}^{n}\sum_{j_k=1}^{r} \frac{\mu_{j_k}/\pi_{j_k}}{w_k - z_{j_k}}\mathrm{d}z, \tag{1.347}$$

$$w(z_0) = w_0 \in \mathbb{C}.$$

It is a *non*linear 1st-order ODE with $w_n(z) = f^n(z)$, $\forall n \in \mathbb{N}_0$. The system is described by means of a nonlinear operator $\tilde{T}$ such that $w'(z) = \tilde{T}w(z)$ and

$$\tilde{T} = \prod_{n}^{0} T \equiv \underbrace{TT \cdot \ldots \cdot T}_{n \text{ times}}$$

and $T = \mathrm{D}\log f(\mathbb{1}) = \sum_{j=1}^{r}\frac{\mu_j/\pi_j}{\mathbb{1} - z_j}$. A *formal* solution to (1.347) is $w_n(z) = e^{\tilde{T}z}w_0$, with:

$$e^{\tilde{T}z} \equiv \exp\left(\prod_{k=0}^{n}\sum_{j=1}^{r}\frac{\mu_j/\pi_j}{f^n(z) - z_j}\right). \tag{1.348}$$

It can also be considered a system of 1st-order *non*linear ODEs with $\mathbf{w} = (w_n, w_{n-1}, \ldots, w_1, w_0)$, $n \in \mathbb{N}_0$. The system is described by means of a *multiplicative* nonlinear triangular operator T_n such that (1.347) becomes $\mathbf{w}'(z) = T_n\mathbf{w}(z)$, and the l^{th} component of vector $\mathbf{w}'(z)$, for $0 \le l \le n$, is the following matrix-product of nonlinear operators. In particular, $T_{l,m} = 0$ if $l \ge m$, otherwise

$$T_{l,m} = \sum_{j_l=1}^{r}\frac{\mu_{j_l}/\pi_{j_l}}{w_m - z_{j_l}}, \qquad \text{if } l < m. \tag{1.349}$$

In either cases, no solutions of the above equations are known, either in closed- or semiclosed-form, but the Inverse Scattering theory could be successfully applied in order to reduce that to a complex Riemann-Hilbert problem for mathematical-physical applications, like those briefly introduced in §5.3.1 and investigated further in [1] and references therein. More on the dynamics of such solutions will be commented in §3.4.

3.3 Prefractal Riemann surfaces: rational case

The analysis of fractal and prefractal Riemann surfaces done in §§3.1–3.2 can be extended in many ways by including wider and wider classes of Riemann surfaces (and complex-dynamic behaviours); the first kind of extension, which is coherent from an algebraic point of view (as far as function fields are concerned) is considering rational functions $P\in\mathbb{C}\{z\}$ instead of polynomials, i.e. considering 'rational' extensions, cfr. §1.1.3. Other generalizations are possible, for example including the iteration of transcendent branched covers like $z\mapsto(z\mathbf{1}_r-\mathbf{z})^{\boldsymbol{\alpha}}$ (where $\boldsymbol{\alpha}\in\mathbb{R}^r\backslash\mathbb{Q}^r$), $z\mapsto\mathbf{m}\cdot\log_{\mathbf{b}}(z\mathbf{1}_r-\mathbf{z})$ (where $\mathbf{m}\in\mathbb{N}^r$ and $\mathbf{b}\in(\mathbb{R}_+\backslash\{1\})^r$) or even hypergeometric functions.

Treating such transcendent cases —more delicate as far as Algebra and convergence is concerned due to the presence of logarithmic branch points at every iteration stage— goes beyond the scopes and applications of this work.

Compositions of different (finite) sets of functions rather than the same function iterated over and over onto itself should also be considered.

3.3.1. Algebro-geometric differences

Let $N,D\in\mathbb{C}[z]/\mathbb{C}$ with $\deg N=d'$, $\deg D=d''$ for some $d',d''\in\mathbb{N}$ such that $d'+d''>1$ and $\mathrm{GCD}(N,D)=1$ (i.e. N and D are two monic, co-prime polynomials) with $1<r\leq d'$ and $1<s\leq d''$ distinct roots $z_1,z_2,\ldots,z_r;\zeta_1,\zeta_2,\ldots,\zeta_s\in\mathbb{C}$ whose multiplicities are $\mathrm{mul}_N z_i=m_i$, $1\leq i\leq r$ and $\mathrm{mul}_D\zeta_j=n_j$, $1\leq j\leq s$. Consider the rational function of $\mathbb{C}\{z\}$.

$$\frac{N(z)}{D(z)} = \frac{\prod_{i=1}^{r}(z-z_i)^{m_i}}{\prod_{j=1}^{s}(z-\zeta_j)^{n_j}} = \frac{(z\mathbf{1}_r-\mathbf{z})^{\mathbf{m}}}{(z\mathbf{1}_s-\boldsymbol{\zeta})^{\mathbf{n}}} \equiv (z\mathbf{1}_r-\mathbf{z})^{\mathbf{m}}(z\mathbf{1}_s-\boldsymbol{\zeta})^{-\mathbf{n}}, \quad (1.350)$$

where, as in (1.292), roots' and multiplicities' vectors $\mathbf{z}=(z_1,z_2,\dots,z_r)\in\mathbb{C}^r$, $\boldsymbol{\zeta}=(\zeta_1,\zeta_2,\dots,\zeta_s)\in\mathbb{C}^s$ and $\mathbf{m}=(m_1,m_2,\dots,m_r)\in\mathbb{N}^r$, $\mathbf{n}=(n_1,n_2,\dots,n_s)\in\mathbb{N}^s$ were used.

Now, fixed two relatively primes $p,q\in\mathbb{N}$ such that $\mathrm{GCD}(p,q)=1$ and $p+q>1$, let $\boldsymbol{\mu}=(\mu_1,\mu_2,\dots,\mu_r),\boldsymbol{\pi}=(\pi_1,\pi_2,\dots,\pi_r)\in\mathbb{N}^r$ and $\boldsymbol{\nu}=(\nu_1,\nu_2,\dots,\nu_s),\boldsymbol{\varphi}=(\varphi_1,\varphi_2,\dots,\varphi_s)\in\mathbb{N}^s$ and:

$$\begin{cases} m_i = \mu_i\,\mathrm{GCD}(m_i,p) \\ p = \pi_i\,\mathrm{GCD}(m_i,p) \end{cases} \quad \begin{cases} n_j = \nu_j\,\mathrm{GCD}(n_j,p) \\ q = \varphi_j\,\mathrm{GCD}(n_j,p) \end{cases} \qquad 1\le i\le r,\quad 1\le j\le s \qquad (1.351)$$

and $\delta',\delta''\in\mathbb{N}$ such that $p=\delta'\mathrm{GCD}(d',p)$ and $q=\delta''\mathrm{GCD}(d'',q)$. Finally consider the complex mapping

$$z \mapsto w = f(z) = \frac{\sqrt[p]{N(z)}}{\sqrt[q]{D(z)}} = \frac{(z\mathbf{1}_r-\mathbf{z})^{\frac{\boldsymbol{\mu}}{\boldsymbol{\pi}}}}{(z\mathbf{1}_s-\boldsymbol{\zeta})^{\frac{\boldsymbol{\nu}}{\boldsymbol{\varphi}}}} \equiv (z\mathbf{1}_r-\mathbf{z})^{\frac{\boldsymbol{\mu}}{\boldsymbol{\pi}}}(z\mathbf{1}_s-\boldsymbol{\zeta})^{-\frac{\boldsymbol{\nu}}{\boldsymbol{\varphi}}} \quad (1.352)$$

with, as usual: $\frac{\boldsymbol{\mu}}{\boldsymbol{\pi}}\equiv p^{-1}\mathbf{m}\in\mathbb{Q}^r$ and $\frac{\boldsymbol{\nu}}{\boldsymbol{\varphi}}\equiv q^{-1}\mathbf{n}\in\mathbb{Q}^s$. The Riemann surface $\mathcal{B}_1$ defined by explicit equation $w=f(z)$ is also defined by implicit equation

$$w^{pq}\,(D(z))^p - (N(z))^q = 0, \qquad (1.353)$$

by elevation at the $(pq)^{\text{th}}$ power and $Q(w,z)=w^{pq}(D(z))^p-(N(z))^q$ is a two-variable Weierstrass polynomial of degree $\max(\max(p,d')+pq,\max(q,d''))$, cfr. §2.1.3.

For simplicity's sake, only the case $p=q$ will be considered (except for a couple of cases shown in Figure 63 , Figure 64), thus (1.352) and (1.353) become

$$z \mapsto w = f(z) = \sqrt[p]{\frac{N(z)}{D(z)}} \tag{1.354}$$

and, respectively,

$$Q(w,z) = w^p D(z) - N(z) = 0, \tag{1.355}$$

with $\boldsymbol{\nu} \equiv p^{-1}\mathbf{n}$ and $\deg Q = \max(d', d''+p)$.

In analogy with the polynomial case, $f^{-1}: \mathcal{B}_1 \to \tilde{\mathbb{C}}$ is a p-cover of the Riemann sphere (puncuted in the polar singularities $\boldsymbol{\zeta}$), whereas letting

$$\mathcal{B}_0 := \tilde{\mathbb{C}} \setminus \{z_1, z_2, \ldots, z_r; \zeta_1, \zeta_2, \ldots, \zeta_s; \infty \mid \pi_1, \pi_2, \ldots, \pi_r; \varrho_1, \varrho_2, \ldots, \varrho_s; |\delta' - \delta''| > 1\}$$

be the initiator punctured sphere, $f^{-1}: \mathcal{B}_1 \to \mathcal{B}_0$ is a ramified p-cover; the branch points are the zeroes and poles of rational function $\frac{N(z)}{D(z)}$ such that $\pi_i, \varrho_j > 1$.

Infinity is a branch point of order $|\delta' - \delta''| - 1$, whereas points of $\mathbf{z}$ are branch points of order $\boldsymbol{\pi} - \mathbf{1}_r$ and points of $\boldsymbol{\zeta}$ branch of order $\boldsymbol{\varrho} - \mathbf{1}_s$. The difference between such *branch zeroes* and *branch poles* is that the lift $\gamma \uparrow f$ on $\mathcal{B}_1$ of any loop $\gamma \in Z_1(\mathcal{B}_0)$ winding $h_i \in \mathbb{Z}$ times around the around the i^{th} branch zero $z_i \in \mathbb{C}$ —as in §3.1.1— "jumps" *up* [down, if $h_i < 0$] by $h_i m_i \bmod p$ sheets, whereas if γ winds $k_j \in \mathbb{Z}$ times around the j^{th} branch pole $\zeta_j \in \mathbb{C}$ then $\gamma \uparrow f$ "jumps" *down* [up, if $k_j < 0$] by $k_j n_j \bmod p$ sheets. If the infinity is a branch point (i.e. if $(\delta' - \delta'') \not\equiv 0 \bmod p$), then it is either a branch zero [branch pole] for f if it is a zero [pole] for $\frac{N(z)}{D(z)}$, that is if $\delta' \lessgtr \delta''$.

Consider

$$f(z) = \sqrt[5]{\frac{z(z^6+1)}{z^7 - 3z^4 + 2z - 1}}, \tag{1.356}$$

whose branching structure is depicted in Figure 31 and Figure 62

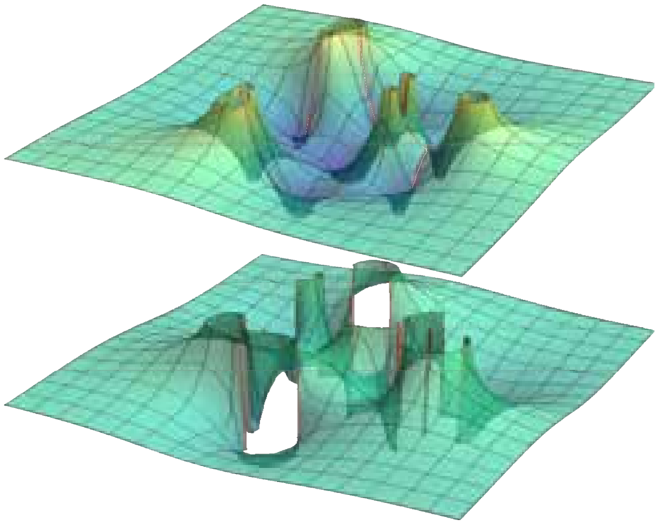


Figure 31 Real (top) and imaginary parts (bottom) of one single-valued branch of (1.356), with the branch cuts represented as red curves.

Both the winding vector around branch zeroes $\mathbf{h}\in\mathbb{Z}^r$ and that around branch poles $\mathbf{k}\in\mathbb{Z}^s$ are important for the Topology of $\mathcal{B}_1$, albeit they have to be considered with opposite signs — as it is evident from the compact formalism on the right-hand side of (1.352).

All the loops $\gamma\in C^0([0,1];\mathcal{B}_0)$ are mapped to a 'double' winding vector $\mathbf{h}\ominus\mathbf{k}::=(\mathbf{h},-\mathbf{k})\in\mathbb{Z}^r\times\mathbb{Z}^s$, so its homotopy class $[\![\gamma\uparrow f]\!]\in\pi_1(\mathcal{B}_1)$ is mapped to a modular vector $[\![\mathbf{h}\ominus\mathbf{k}]\!]\in\mathbb{Z}_{(\pi,\varrho)}$ by (1.299) — the winding map 'wind' acting as

$$\mathrm{Z}_1(\mathcal{B}_1)\xleftrightarrow{\text{wind}}\mathbb{Z}^r\times\mathbb{Z}^s\,, \tag{1.357}$$

is a isomorphism like (1.302), whereas a proper quotient like (1.303) leads to:

$$\begin{tikzcd}
\mathrm{H}_1(\mathcal{B}_0) \arrow[dr, leftrightarrow, "\text{wind}"] & \\
& \frac{\boldsymbol{\mu}}{\boldsymbol{\pi}}\cdot\mathbb{Z}_{\boldsymbol{\pi}}\ominus\frac{\boldsymbol{\nu}}{\boldsymbol{\varrho}}\cdot\mathbb{Z}_{\boldsymbol{\varrho}}\,. \\
\mathrm{H}_1(\mathcal{B}_1) \arrow[ur, "\looparrowright"'] &
\end{tikzcd} \tag{1.358}$$

Almost every additional consideration on the topology of f is identical to the polynomial case: branch poles are equivalent to branch zeroes except for the fact that the winding direction around them is inverted when lifting paths from $\mathcal{B}_0$ to $\mathcal{B}_1$.

> As a last example of prefractal Riemann surfaces, consider now $f(z)=\sqrt[11]{(z+1)^7(z-\imath)^{-5}}$, which is an example of rational type with different arithmetic roots at numerator and denominator, according to (1.352). See **Figure 32** and **Figure 64** for a local, one-sheeted representation of $f(z)$ showing its cyclotomic branching structure.

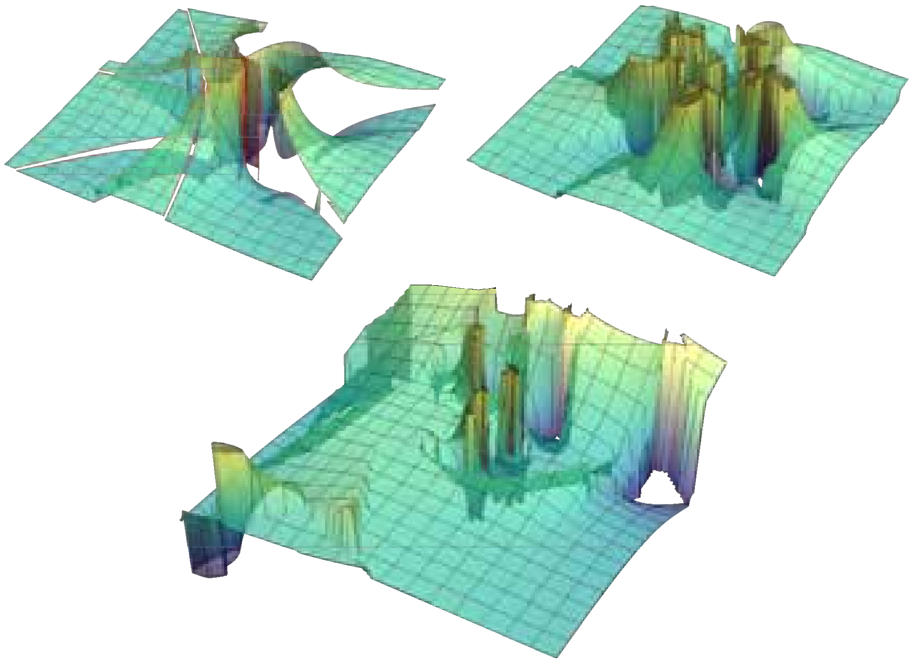


Figure 32 Plot of $\operatorname{Im} f^n(z)$ in the above example, for $n=1,3,4$ (clockwise starting from top-left). Notice the highly chaotic distrubition of sheets and branch cuts.

3.3.2. Pole-Zero cancellations

Higher iterations are treated like in §3.2 for the polynomial initiator case; the exception is that the formalisms are twice as complicated. So $\mathcal{B}_n$ is the Riemann surface defined by explicit equation ($\forall n \in \mathbb{N}$)

$$z \mapsto w = f^n(z) = \underbrace{\sqrt[p]{\frac{N\left(\sqrt[p]{\frac{N\left(\cdots\sqrt[p]{\frac{N(z)}{D(z)}}\right)}{D\left(\cdots\sqrt[p]{\frac{N(z)}{D(z)}}\right)}}\right)}{D\left(\sqrt[p]{\frac{N\left(\cdots\sqrt[p]{\frac{N(z)}{D(z)}}\right)}{D\left(\cdots\sqrt[p]{\frac{N(z)}{D(z)}}\right)}}\right)}}}_{n \text{ times}}. \tag{1.359}$$

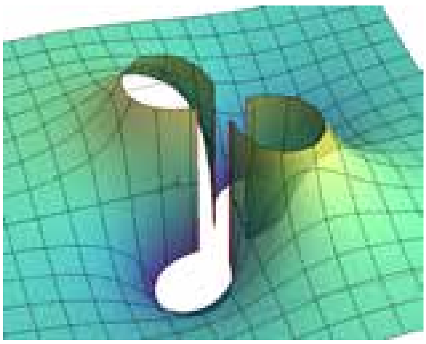
$\mathcal{B}_n$ is the covering space of both $\mathcal{B}_{n-1}$ via the ramified cover $f^{-1}{:}\mathcal{B}_n \to \mathcal{B}_{n-1}$, and so on up of $\mathcal{B}_0$ via the ramified cover $f^{-n}{:}\mathcal{B}_n \to \mathcal{B}_0$. In this paragraph the properties of surfaces of sequence $(\mathcal{B}_n)_{n\in\mathbb{N}_0}$ will be analyzed and inter-related.

The right-hand side of (1.359) resembles a fractal fraction —introduced in §1.5.2— despite arithmetic roots being also present in the self-similar construction of the explicit function.

Since the topology and the (iterated) monodromy are identical to the polynomial case, the two main differences arising in the rational case are generalization to the cascading phenomenon (§3.2.3) described here and convergence considerations summarized in the next paragraph. It is evident since from the second iteration

$$f^2(z) = \sqrt[p]{\frac{N\left(\sqrt[p]{\frac{N(z)}{D(z)}}\right)}{D\left(\sqrt[p]{\frac{N(z)}{D(z)}}\right)}}, \tag{1.360}$$

that, not only cascading[107] may occur on either the iterated numerator and denominator, but also the [2nd-generation branch] zeroes and/or poles of the resulting upper rational function $N\left(\sqrt[p]{\frac{N}{D}}\right)$ may coincide with the [2nd-generation branch] zeroes and/or poles of the lower one $D\left(\sqrt[p]{\frac{N}{D}}\right)$. At higher iterations, this pole-zero cancellation phenomenon —abbreviated to f being "PZ"— may even be more likely to happen. Cascading and "PZ" phenomena may lead to several branch points disappearing, or at least joining different-generation sheets together, all in all deviating from the "linear" scenario presented in the uncascaded polynomial case and easily represented via regular rooted trees.

> Not only the meromorphic multi-valued function f (whose principal argument $\operatorname{Arg} f(z)$ with branch cuts in red is shown below) is "PZ,"
>
> $$f(z) = \sqrt{\frac{z^2+1}{z^2-1}},$$
>
> 
>
> but its iteration operator is also *involutory*, since the 2nd iteration leads back to the trivial covering map $z \mapsto f(z) \mapsto \sqrt{z^2} = z$:
>
> $$f^2(z) = \sqrt{\frac{\frac{z^2+1}{z^2-1}+1}{\frac{z^2+1}{z^2-1}-1}} = \sqrt{\frac{z^2 \cancel{+1} + z^2 \cancel{-1}}{\cancel{z^2}+1-\cancel{z^2}+1}} \equiv \sqrt{z^2} = z.$$

[107] In this case the cascading is the coincidence of *candidate* 1st-generation branch zeroes with 1st-generation branch poles, i.e. critical points of belonging to the lower generation.

> As far as geometry is concerned, f is a branched 2-cover with simple branch zeroes and poles, so $\mathbf{z}=(-\imath,\imath)$, $\boldsymbol{\zeta}=(-1,1)$ and $\frac{\boldsymbol{\mu}}{\boldsymbol{\pi}}=\frac{\boldsymbol{\nu}}{\boldsymbol{\varrho}}=\frac{1}{2}\mathbf{1}_2$. As shown on the picture above, a wise branch cuts' choice is $\rangle -1,1\langle = [-1,1] \subset \mathbb{R}$ and $\rangle \imath,-\imath\langle = \imath(]-\infty,-1]\cup[1,+\infty[) \subset \mathbb{I}$ (e.g. with $\rangle \imath,-\imath\langle$ crossing regular point ∞ along the imaginary axis).

The analysis would have to be done on an almost case-by-case basis, since the associated complex dynamics are so variegated to leave little room for a general treatment (as shown in the rational examples in the Gallery Chapter §3.4.2). It has to be noted, however, that a proper choice of the starting polynomials N and D's roots (for example being incommensurable complex numbers between each other, as in the polynomial case) is not sufficient anymore, because there is a mixing of critical points between numerators and denominators.

3.3.3. Convergence issues

Although the closed-form computability of nested radicals is harder, things are different when it comes down to numerical convergence: as discussed in §3.3.1, not only the complex infinity ∞ may not be a branch point, but it may neither be a singularity for $\lim_n f^n(z)$. It is just handy Calculus stuff to prove that if $d'=d''$ the asymptotic behaviour $z\to\infty$ trivially leads to the $(p^n)^{\text{th}}$ roots of unity, which are finite numbers:

$$\lim_{z\to\infty} f^n(z) = f^n\Big(\lim_{z\to\infty} z\Big) = \underbrace{\sqrt[p]{\sqrt[p]{\ddots\sqrt[p]{1}}}}_{n \text{ times}} \equiv 1^{p^{-n}}. \tag{1.361}$$

If $d'<d''$ then there is just one asymptotic value, $\lim_n f^n(\infty)=0$ and, as 'reciprocal' case $d'>d''$ (which always happens in the polynomial case), the whole limit function is zero outside the corresponding filled Julia set $\mathrm{FJ}(f)$, cfr. §2.4.1.

Since this time the finite branch zeroes and poles are spread and mixed with each other on the Riemann sphere, the filled Julia sets of f correspond to finite (though infinitely multi-valued) images.

3.3.4. Iterated Monodromy and Galois' groups

It is interesting to note that the branching of surfaces $\mathcal{A}_n$ is globally described by the multi-valued function $f^n(z)$, which is locally equivalent to a Puiseux series. The set of *all* the possible branches of f^n which the lifted paths in $\mathcal{A}_n$ may travel on —so that every possible monodromy is accounted for— is described by isomorphic copies of $\mathbb{C}[\![p]\!]^\natural$, where the isomorphisms with respect to local charts/coordinates $p \mapsto z_j$ are

$$\mathbb{C}[\![p]\!]^\natural \cong \mathbb{C}[\![z - z_j]\!]^\natural.$$

This has a nice algebraic interpretation: since [formal] Puiseux series are the algebraic closure of [formal] Laurent series [§1.3.3], then $\mathbb{C}[\![p]\!]^\natural$ contains, in some sense, the *splitting* fields of the series in $\mathbb{C}[\![p]\!]$. The lift $\gamma\uparrow f^{-n}$ of any path γ in the (single-valued) punctured Riemann sphere $\mathcal{A}_0$ exists as solution to implicit equations like $w - f^n(z) = 0$, cfr. §3.2.

Suppose that $p=1$; then $f(z)$ defined according to (1.350) —along with its n^{th} iteration— is a rational function, of numerator and denominator $N_n(z), D_n(z) \in \mathbb{C}[z]$ respectively, and $\mathrm{GCD}(N_n, D_n) = 1$, $\forall n \in \mathbb{N}$. Explicit equation (1.293) may be rewritten in this case as

$$wD_n(z) - N_n(z) = 0, \tag{1.362}$$

where $w \equiv w_{n+1}$ is this time a *single*-valued rational function of $z \in \mathbb{C}$, i.e. $w_{n+1} \in \mathbb{C}\{z\}$.

Let Σ_n to be the splitting field of the polynomial $N_n - wD_n \in \mathbb{C}\{w\}[z]$, which is both a Weierstrass polynomial in the unknowns (w,z) and a polynomial in the unknown z with coefficients in the field $\mathbb{C}\{w\}$ of rational functions in the variable w_{n+1}, since

$$w_{n+1} = \frac{N_n(z)}{D_n(z)} = \frac{N(w_n)}{D(w_n)} \tag{1.363}$$

(with $N_0 = N$ and $D_0 = D$), thus $\Sigma_n = \mathbb{C}\{w\}(\ker(N_n - wD_n))$ is an *algebraic* extension field. Now let the associated Galois' group $\mathrm{Gal}(\Sigma_n/\mathbb{C}\{w\})$ be considered, which was proven to be isomorphic to the monodromy group of the branched covering $f^n : \tilde{\mathbb{C}} \to \tilde{\mathbb{C}}$, i.e. to the permutations' group of the sets $f^{-n}(w_0)$ induced by the action of the fundamental group $\pi_1(\mathcal{B}_n, w_0)$ (which does not depend on the starting point $w_0 \in \mathcal{B}_n$); also see §§1.1.10, 1.5.4, [45], [79]:

$$\mathrm{Gal}\frac{\Sigma_n}{\mathbb{C}\{w\}} \cong \mathrm{MG}f^n. \tag{1.364}$$

Theorem

If $f = N_0/D_0 \in \mathbb{C}\{z\}$ is a post-critically finite rational function (cfr. §2.4.1) as defined above, then its profinite IMG is isomorphic to the corresponding Galois' group,

$$\overline{\mathrm{IMG}} f \cong \mathrm{Gal}\left(\Sigma/\mathbb{C}\{z\}\right), \tag{1.365}$$

where

$$\Sigma = \bigcup_{n=1}^{\infty} \Sigma_n = \lim_n \Sigma_n.$$

More information on the Galois' groups and their applications to the solution of polynomial equations can be found in §§1.1.10, 1.5.4, as long as in [86] and [71]. Trivially the splitting fields in the sequence $(\Sigma_n)_{n \in \mathbb{N}_0}$, with $\Sigma_0 :\equiv \mathbb{C}\{w\}$, are inclusively mutual extension fields of the rational functions of w, that is

$$\Sigma_n \leq \Sigma_{n+1}, \qquad \forall n \in \mathbb{N}_0, \tag{1.366}$$

and since $\mathrm{Gal}(\Sigma_n/\mathbb{C}\{w\}) = \mathrm{Aut}(\Sigma_n/\mathbb{C}\{w\})$, this is the group of automorphic functions fixing any rational functions of w and then, by induction on $n \in \mathbb{N}_0$ since

$w=f^n(z)$, fixing $\mathbb{C}\{z\}$ too. The Galois' group (1.365) is a profinite group for which all the results about Infinite Galois Theory introduced in §1.2.3 hold.

From the mutual inclusion of field extensions Σ_{n+1}/Σ_n and given that each splitting field is a simple extension of the previous one (Σ_{n+1} is obtained adjoining to Σ_n the $(n+1)^{\text{th}}$-generation critical points of $f(w_n)$), the roots of polynomials $N_n(z)-w_{n+1}D_n(z)$ are expressed employing nested radicals.

Now let the (multi-valued) case $p>1$ be considered. It is important to note that $f(z)$ is a "quasi-algebraic" transcendental function, since it the p^{th} arithmetic root of either a polynomial P (§3.2) or a rational function N/D (§3.3) with its branch points (characterizing both the fundamental and the monodromy groups of $\mathcal{A}_n$ or $\mathcal{B}_n$, respectively) corresponding to the critical points of the above functions (respectively, the roots of P and the zeroes/poles of N/D). This means that the extension fields considered for this result essentially provide *algebraic*, not transcendental extensions.

Let the "polynomial" covers' case ($D(z)\equiv 1$) be considered for simplicity's sake from now on. As stated throughout §§0–2.4.2, 2.4.5, and in particular thanks to diagram (1.259), built as per §1.1.10, with $w_n=f^n(z)$ as usual. Suppose neither cascading nor "PZ" phenomena are present (the latter already excluded since only the case treated in §§3.1–3.2 is considered).

Lemma

The monodromy group of any non-"PZ" prefractal Riemann surface $\mathcal{B}_n$ contains the monodromy groups of all the previous-order prefractal surfaces as *normal* subgroups: $\mathrm{MG}\mathcal{B}_n \lhd \mathrm{MG}\mathcal{B}_{n+1}\ \ \forall n\in\mathbb{N}$:

$$\operatorname{ord}\mathrm{MG}\mathcal{B}_n=(\operatorname{ord}\mathrm{MG}\mathcal{B}_1)^n. \tag{1.367}$$

<u>Proof</u>: First of all, consider the mutual isomorphism of each Galois group in the sequence $(G_n)_{n\in\mathbb{N}}=\left(\mathrm{Gal}\left(\mathbb{C}\{w_n\}/\mathbb{C}\{w_{n-1}\}\right)\right)_{\mathbb{N}}$ with each other which, by (1.259), is the monodromy group of the initiator surface, i.e. $G_1\cong\mathrm{MG}\mathcal{B}_1$ and, $\forall n\in\mathbb{N}_0$:

$$\operatorname{Gal}\frac{\mathbb{C}\{w_{n+1}\}}{\mathbb{C}\{w_n\}} \cong \operatorname{Gal}\frac{\mathbb{C}\{f(z)\}}{\mathbb{C}\{z\}} \equiv \mathrm{MG}_f\mathbb{C}\{z\} \cong \mathrm{MG}\mathcal{B}_1. \qquad (1.368)$$

The quotient groups G_{n+1}/G_n are all trivial: this is understandable, since, by induction on $n\in\mathbb{N}$, all the automorphisms fixing $f^n(z)$, also fix $f^{n-1}(z)$, noting that simply letting $w_1=z$ implies that everything invariant under $f^n(z)$ is also invariant for $f^{n-1}(w_1)$ (where $f^0(z)\equiv z$ and $f(w_1)\equiv f^2(z)$). Topologically, this is consistent with the fact that the m^{th}-generation sheets of $\mathcal{B}_m$ locally foliate p^n times from the n^{th}-generation sheets of $\mathcal{B}_{n+m}$, $\forall m\in\mathbb{N}$. This is a Galois Theory analogue to the wreath recursion property analyzed in §3.2. The above considerations imply that the fields $\mathbb{C}\{w_n\}=\mathbb{C}\{w_n,w_{n-1},\dots,w_1\}$ are all *transcendental* p-radical extension fields of $\mathbb{C}\{z\}$; furthermore, from the field extensions' chain rule (cfr. §1.1.3) and (1.259):

$$\begin{aligned}\operatorname{ord}\mathrm{MG}\mathcal{B}_n &= \Big[\mathbb{C}\{w_n\}:\mathbb{C}\{z\}\Big] = \prod_{k=1}^{n}\Big[\mathbb{C}\{w_k\}:\mathbb{C}\{w_{k-1}\}\Big]\\ &= \Big[\mathbb{C}\{f(z)\}:\mathbb{C}\{z\}\Big]^n \equiv (\operatorname{ord}\mathrm{MG}\mathcal{B}_1)^n,\end{aligned}$$

$$\Downarrow$$

$$\mathrm{MG}\mathcal{B}_n \lesssim \mathrm{MG}\mathcal{B}_{n+1}, \qquad \forall n\in\mathbb{N}.$$

Let $[\![\gamma]\!]\in\pi_1(\mathcal{B}_n)$ be (the homology class of) a loop on $\mathcal{B}_n$ and consider its lift $\gamma\uparrow f^{-1}$ on $\mathcal{B}_{n+1}$ along with the corresponding monodromy action $F_1\in\mathrm{MG}f^{-n}(\mathcal{B}_{n+1})\cong\mathrm{MG}\mathcal{B}_1$. The path $\gamma\uparrow f^{-1}$ will in general "land" on a different $(n+1)^{\text{th}}$-generation sheet of local complex coordinate w_{n+1} but, on it, all the lower-generation branch points are located in the same position than the starting $(n+1)^{\text{th}}$-generation sheet because $w_{n+1}=f(w_n)$ and w_n is the local coordinate of $\gamma(t)$ itself. For example the n^{th}-generation branch points are always $w_n\in\{z_j,\infty \mid \pi_j,\pi_\infty>1\}_{1\le j\le r}$, both on the starting and the ending sheets. This numerically means that the starting and ending points of $\gamma\uparrow f^{-1}$ differ by an integer multiple of $e^{2\pi\imath/p}$, i.e. by $\looparrowright(\gamma\uparrow f^{-1})=\exp\Big(2\pi\imath F\Big([\![\gamma\uparrow f^{-1}]\!]\Big)\Big)=e^{2\pi\imath\mathbf{k}_{n+1}\cdot\frac{\boldsymbol{\mu}}{\boldsymbol{\pi}}}$, where $[\![\mathbf{k}_{n+1}]\!]\in\mathbb{Z}_{\boldsymbol{\pi}}$ is the $(n+1)^{\text{th}}$-generation winding vector, as shown in (1.308). The path can be reverted back to the original loop by chosing the inverse homology class in $\pi_1(\mathcal{B}_{n+1})/\pi_1(\mathcal{B}_n)$, i.e. considering the inverse monodromy action $F_1^{-1} \in \mathrm{MG} f^{-n}(\mathcal{B}_{n+1})$. This proves that $\mathrm{MG}\mathcal{B}_n$ is not only a subgroup of $\mathrm{MG}\mathcal{B}_{n+1}$, but rather a *normal* subgroup i.e., $\forall n \in \mathbb{N}$ and $\forall F_{n+1} \in \mathrm{MG}\mathcal{B}_{n+1}$ there exist a homotopy class $\exists [\![\gamma]\!] \in \pi_1(\mathcal{B}_n)$ such that F_{n+1} is decomposed as $F_{n+1} = F^n [\![\gamma]\!] \circ F_1$, which implies

$$F_1^{-1} \circ \mathrm{MG}\mathcal{B}_n \circ F_1 = \mathrm{MG}\mathcal{B}_n \qquad \Rightarrow \qquad \mathrm{MG}\mathcal{B}_n \lhd \mathrm{MG}\mathcal{B}_{n+1}. \qquad \square$$

It is interesting to note that (1.367) holds even if $\mathcal{B}_n$ is cascading, cfr. §3.2.3, because this phenomenon does not prevent branch point(s) from belonging to a specific generation: it simply makes it/them connect different-generation sheets altogether, but preserving individual branching orders, because in (1.336) trivially $\pi_j \,|\, \pi_j^{n-h_j}$.

The following result is a transcendental generalization of Theorem (1.365), which holds for prefractal Riemann surfaces considered throughout this Chapter, rather than only for those generated by iterating post-critically finite rational functions. Consider also that to be a "Structure Theorem," for both $\mathcal{A}_n$ and $\mathcal{B}_n$, $\forall n \in \mathbb{N} \cup \{\infty\}$, because many algebraic and topological aspects are unveiled in its proof. Particularly check out for the properties given by numbered equations below, since they provide, up to now, the most general and most important results on such (pre-)fractal Riemann surfaces.

Fractal Riemann Theorem

If f is is non-"PZ" the monodromy group of $\mathcal{B}_n$ is solvable if and only if $\mathrm{MG}\mathcal{B}_1$ is solvable itself, and:

$$\mathrm{MG}\mathcal{B}_{m+n} / \mathrm{MG}\mathcal{B}_m \cong \mathrm{MG}\mathcal{B}_n, \qquad \forall m,n \in \mathbb{N}. \qquad (1.369)$$

The profinite IMG of the limit fractal surface $\mathcal{B}_\infty$, that is $\overline{\mathrm{IMG}} f$, is the inverse limit of the (finite) monodromy groups' sequence, i.e.:

$$\overline{\mathrm{IMG}} f = \varprojlim_n \mathrm{MG}\mathcal{B}_n \, . \qquad (1.370)$$

<u>Proof</u>: This one is a natural completion of the previous Lemma, stating that for any prefractal Riemann surface $\mathcal{B}_n$ a finite sequence of mutually normal monodromy groups exists:

$$\{\circlearrowleft\} \lhd \mathrm{MG}\mathcal{B}_1 \lhd \mathrm{MG}\mathcal{B}_2 \lhd \ldots \lhd \mathrm{MG}\mathcal{B}_n.$$

Now consider the groups $\mathrm{MG}\mathcal{B}_{n+1}/\mathrm{MG}\mathcal{B}_n$, $\forall n \in \mathbb{N}$, whose order is $\mathrm{ord}\,\mathrm{MG}\mathcal{B}_1$ and, despite being equipotent groups, could not be isomorphic to each other. The same terminology in the previous Lemma's proof and the "language" introduced in §§3.1.2, 3.2.1, 3.3.1 will be used throughout: formalization of the travels on such Riemann surfaces by means of rooted trees and different-generation winding vectors is stated as in (1.333).

Let $[\![\gamma]\!] \in \pi_1(\mathcal{B}_n)$ be (the homology classes of) a loop, and consider its monodromy action $F^n[\![\gamma]\!] \in \mathrm{MG}\mathcal{B}_n$, cfr. (1.344), together with its lift $\gamma \uparrow f^{-1}$ on $\mathcal{B}_{n+1}$ and with its monodromy action: $F^{n+1} \in \mathrm{MG}\mathcal{B}_{n+1}$. By the isomonodomicity condition, the base points of $\gamma \uparrow f^{-1}$ and $\gamma' \uparrow f^{-1}$ may be on different $(n+1)^{\mathrm{th}}$-generation sheets but lie on homeomorphic lower-generation sub-sheets foliated from their respective two base sheets. That is equivalent to fix one branch of the outermost arithmetic p^{th} root in f^{n+1} such that $\mathbf{k}_{n+1} \in \mathbb{Z}^r$ is its overall $(n+1)^{\mathrm{th}}$-generation winding vector, as shown in (1.308). Alternatively, this is the cover transformation of $\mathrm{Deck}\mathcal{B}_{n+1}$ represented by $\looparrowright[\![\mathbf{k}_{n+1}]\!] = e^{2\pi\imath \mathbf{k}_{n+1} \cdot \frac{\boldsymbol{\mu}}{\boldsymbol{\pi}}}$ on the single-valued branch of $f(\gamma(t))$ and $f(\gamma'(t))$. As it was done at the end of the previous Lemma's proof, and $\forall F_{n+1} \in \mathrm{MG}\mathcal{B}_{n+1}$ there exists a homotopy class $\exists [\![\gamma]\!] \in \pi_1(\mathcal{B}_n)$ such that $F_{n+1} = F^n[\![\gamma]\!] \circ F_1$, i.e. $\looparrowright[\![\gamma \uparrow f^{-1}]\!] = d$, which means that $\mathrm{MG}\mathcal{B}_n$ is a direct factor of $\mathrm{MG}\mathcal{B}_{n+1}$ and, cfr. §1.1.4,

$$\mathrm{MG}\mathcal{B}_{n+1} = \mathrm{MG}\mathcal{B}_1 \rtimes \mathrm{MG}\mathcal{B}_n$$

$$\Downarrow$$

$$\mathrm{MG}\mathcal{B}_{n+1}/\mathrm{MG}\mathcal{B}_n \cong \mathrm{MG}\mathcal{B}_1. \tag{1.371}$$

The orthogonal decomposition above —which holds, $\forall n\in\mathbb{N}$, only for the outermost level (1.371)— means that the *highest*-generation winding vector $[\![\mathbf{k}_{n+1}]\!]\in\mathbb{Z}_\pi$ can always be recovered from symbolic dynamics sequences (1.333) because it only depends on the outermost arithmetic p^{th} root's branch — i.e. by (1.304) and (1.307):

$$\begin{aligned}\looparrowright(F_{n+1}) &\equiv \looparrowright(F^n[\![\gamma]\!]\circ F_1) = \looparrowright(F_1)\looparrowright(F^n[\![\gamma]\!]) \\ &= e^{2\pi\imath\mathbf{k}_{n+1}\cdot\frac{\boldsymbol{\mu}}{\pi}}\looparrowright(F^n[\![\gamma]\!]).\end{aligned} \tag{1.372}$$

Such a symbolic dynamics is well represented on the limit surface $\mathcal{B}_\infty$ by the infinite free-product group $\overset{\infty}{\ast}\mathbb{Z}_\pi\cong\overline{\mathbb{Z}_p^{\mathsf{T}}}$, defined similarly to (1.341) and whose elements are juxtapositions of kind $[\![\mathbf{k}_1]\!]_{[h_1]}\rightsquigarrow[\![\mathbf{k}_2]\!]_{[h_2]}\rightsquigarrow[\![\mathbf{k}_3]\!]_{[h_3]}\rightsquigarrow\cdots$, together with the usual asymptotic equivalences (1.105).

Instead, as shown in §3.2, it would be impossible to characterize the "finer" topology of $\gamma\uparrow f^{-1}$ and $\gamma'\uparrow f^{-1}$ only via lower-generation winding vectors, that is because such symbolic dynamics (1.333) would not be homology-invariant.

Furthermore (1.371) implies, by induction, (1.369). Now $\mathrm{MG}\mathcal{B}_{n+1}$ is solvable if and only if both $\mathrm{MG}\mathcal{B}_n$ and $\mathrm{MG}\mathcal{B}_{n+1}/\mathrm{MG}\mathcal{B}_n\cong\mathrm{MG}\mathcal{B}_1$ are. Iterating this consideration, it follows that $\mathrm{MG}\mathcal{B}_n$ is then solvable if and only if both $\mathrm{MG}\mathcal{B}_{n-1}$ and $\mathrm{MG}\mathcal{B}_n/\mathrm{MG}\mathcal{B}_{n-1}$ are solvable, with the latter group being again isomorphic to $\mathrm{MG}\mathcal{B}_1$. By inverse induction on $n\in\mathbb{N}$ then, it follows from (1.371) that $\mathrm{MG}\mathcal{B}_n$ is solvable if and only if $\mathrm{MG}\mathcal{B}_1$ is, Q.E.D. for the first part of the Theorem.

Let $(\mathrm{MG}\mathcal{B}_n)_{n\in\mathbb{N}}=\left(\mathrm{Gal}_{\mathbb{C}\{z\}}\mathbb{C}\{w_n\}\right)_{\mathbb{N}}$ be the inclusive monodromy groups' sequence, which is an inverse [direct] system, as defined in §1.1.7, where $\{\phi_m^n\}_{m,n\in\mathbb{N},\ m\leq n}$ is defined as the homomorphisms' semigroup whose elements are just push-forwards of brached covers f_*^{n-m} [of ramified covers f_*^{m-n}] on the respective monodromy groups (cfr. note [81]). So $(\mathrm{MG}\mathcal{B}_n)_{\mathbb{N}}$ admits a profinite group as inverse limit and, at the same time, it also generates a sequence of $(\mathbb{C}\{w_n\})_{\mathbb{N}}$ of intermediate field extensions $\mathbb{C}\{w_n\}/\mathbb{C}\{z\}$.

On the other hand, $\mathrm{MG}\mathcal{B}_n \cong \mathrm{Gal}(\mathbb{C}\{w_n\}/\mathbb{C}\{z\})$ is the Galois group associated to the finite extension field $\mathbb{C}\{w_n\}$ which, by (1.44), has degree $[\mathbb{C}\{w_n\}:\mathbb{C}\{z\}] = \mathrm{ord}\,\mathrm{MG}\mathcal{B}_n$. It is also true that any monodromy action $F_{n+1} \in \mathrm{MG}\mathcal{B}_{n+1}$ induces a Galois cover $\looparrowright \in \mathrm{Deck}(\mathcal{B}_{n+1} \xrightarrow{f^{-1}} \mathcal{B}_n)$, because $\mathrm{MG}\mathcal{B}_n \lhd \mathrm{MG}\mathcal{B}_{n+1}$ and also from (1.204):

$$w, w' \in \mathcal{B}_{n+1},\ f^{-1}(w) = f^{-1}(w') \qquad \Rightarrow \qquad \looparrowright(w) = e^{2\pi\imath F_{n+1}} w = w';$$

then, by Theorem (1.234) and since $f(F[\![\gamma]\!]) = f(F)[\![\gamma]\!]$, $\mathrm{MG}\mathcal{B}_m \lhd \mathrm{MG}\mathcal{B}_n$ implies:

$$\mathrm{Deck}(\mathcal{B}_n \xrightarrow{f^{m-n}} \mathcal{B}_m) \cong \mathrm{Gal}\,\frac{\mathcal{K}(\mathcal{B}_n)}{\mathcal{K}(\mathcal{B}_m)} = \mathrm{Gal}\,\frac{\mathbb{C}\{w_n\}}{\mathbb{C}\{w_m\}} \cong \mathrm{MG}\mathcal{B}_{n-m}, \tag{1.373}$$

for any $\forall m, n \in \mathbb{N}$ such that $m < n$. The geometric structure of these changing of field extensions can be easily computed by the Fundamental Theorem of Galois Theory; in fact by (1.44) and (1.373), it is simply:

$$\Big[\mathbb{C}[w_n, w_{n-1}, \dots, w_1, z] : \mathbb{C}[z]\Big] \le \prod_{k=1}^{n} \Big[\mathbb{C}\{w_n\}[z] : \mathbb{C}[z]\Big];$$

$$\mathrm{ord}\,\mathrm{Gal}\,\frac{\mathbb{C}\{w_n\}}{\mathbb{C}\{w_m\}} = \Big[\mathbb{C}\{w_n\} : \mathbb{C}\{w_m\}\Big] = (\mathrm{ord}\,\mathrm{MG}\mathcal{B}_1)^{n-m}. \tag{1.374}$$

Let us now build the inverse limit group $\varprojlim_n \mathrm{MG}\mathcal{B}_n$. Since the monodromy groups $\mathrm{MG}\mathcal{B}_n$ act freely on (paths in) $\mathcal{B}_n$ and the covers f^{-1} are regular, then by the Monodromy Theorem (1.212) those groups are isomorphic to fibres' symmetric groups $\mathrm{Sym}\, f(w_n)$, $\forall w_n \in \mathcal{B}_n$, and are quotients of homotopy classes in $\pi_1(\mathcal{B}_n)$ which are invariant by homotopy classes in $\pi_1(\mathcal{B}_{n+1})$. In other words, the monodromy groups algebraically relate all the possible, homotopically different lifts $\gamma \uparrow f^{-1}$ of equivalent paths $[\![\gamma]\!] \in \pi_1(\mathcal{B}_n)$, i.e.:

$$\mathrm{MG}\mathcal{B}_n \cong \frac{\pi_1(\mathcal{B}_n)}{f_*^{-1}\pi_1(\mathcal{B}_{n+1})}, \tag{1.375}$$

where the "denominator" group is the monodromy action's kernel, $\mathrm{Ker}F_1$.

In this case $\mathrm{MG}\mathcal{B}_n$ in (1.371) acts as both a stabilizer (1.23), or nucleus (1.100), since $f_*^{-1}\pi_1(\mathcal{B}_{n+1})$ is the topological closure of the preimages of lifts $[\![\gamma\uparrow f^{-1}]\!]\in\pi_1(\mathcal{B}_{n+1})$.

Iteration of the previous formula recalling that $\mathrm{MG}\mathcal{B}_n$ contains isomorphic copies of every lower-order monodromy groups as *normal* subgroups, and observing that the f_*^{n-m}'s are equivalent to the push-forwarded homeomorphisms of f^{n-m} definining the monodromy groups' inverse system, a simple iteration of (1.375) along different-order monodromy groups leads to, for any $m<n$:

$$\mathrm{MG}\mathcal{B}_m \cong \frac{\pi_1(\mathcal{B}_m)}{f_*^{m-n}\pi_1(\mathcal{B}_n)} \qquad \Leftrightarrow \qquad \mathrm{MG}\mathcal{B}_n \cong \frac{\pi_1(\mathcal{B}_n)}{f_*^{n-m}\pi_1(\mathcal{B}_m)}$$

$$\Downarrow$$

$$\mathrm{MG}\mathcal{B}_n \cong \frac{\pi_1(\mathcal{B}_n)}{\bigcap_{m=1}^{n} \operatorname{Ker} F^m}\,.$$

This re-definition is possible by induction because $f^{-1}:\mathcal{B}_1\to\mathcal{B}_0$ can be considered a *partial self*-covering. Since the inverse limit group is always profinite, taking the "algebraic" limit $n\leftarrow\infty$ on the left-hand side forces the use of a compact-closed topology for the topological limit $n\to\infty$ on the right-hand side. As shown in §§2.3.4, 3.2.6 and [82], such topology is inherited by the asymptotic equivalences represented by the forward orbits (1.214) of "regular" points[108] $z\in\mathcal{B}_0$:

$$\mathrm{W}_f^+(z)\equiv \mathrm{T}_{f^{-1}}(z)=\bigsqcup_{n=0}^{\infty} f^n(z).$$

It is now evident that, for the limit surface, the above construction of an inverse-limit monodromy group coincides with the definition (1.213) of the profinite IMG:

[108] If f is either cascading and "PZ" such points must be carefully in order to avoid trivial or periodic orbits, as is well known in Complex Dynamics, cfr. [77], [86].

$$\varprojlim_n \mathrm{MG}\,\mathcal{B}_n = \frac{\varprojlim_n \pi_1(\mathcal{B}_n)}{\bigcap\limits_{n\in\mathbb{N}} \mathrm{Ker}\, F^n} = \overline{\mathrm{IMG}}\,\mathcal{B}_\infty\,,$$

Q.E.D. for the last part of the Theorem. □

From (1.309) for the "polynomial cover case" remember that $\mathrm{MG}\mathcal{A}_1 = \mathrm{Sym}\left(\frac{\mu}{\pi}\cdot\mathbb{Z}_\pi\right) \leq \mathbb{S}_p$. The solvability of such group, as just stated, has deep consequences on the Galois Theory of all the other prefractal Riemann surfaces $\mathcal{A}_n$.

The main two results on fractal Riemann surfaces are contained in the above Theorem: the solvability of any finite-order monodromy group $\mathrm{MG}\mathcal{B}_n$ is equivalent to the solvability of $\mathcal{B}_1$'s monodromy group, and the asymptotic equivalence between monodromy groups and profinite IMGs — (1.370). Proving those results paves the way to many other speculations on such topological spaces, especially many connections between their self-similar topology and the Galois Theory. From here to the end of the Chapter, several of these aspects will be treated.

First of all a solvable monodromy group $\mathrm{MG}\mathcal{B}_1$ implies that all the groups of $(\mathrm{MG}\mathcal{B}_n)_{\mathbb{N}} = \left(\mathrm{Gal}(\mathbb{C}\{w_n\}/\mathbb{C}\{z\})\right)_{\mathbb{N}}$, being solvable, are also amenable, cfr. §1.2.4.

Secondly, since the nucleus of the profinite IMG is defined as the forward orbits' closure (1.225) of regular, non-periodic points $z \in \mathrm{J}(f)$, cfr. §2.4.1, it follows that the geometry of $\overline{\mathrm{IMG}}\,\mathcal{B}_\infty$ and that of the Julia set $\mathrm{J}(f)$ are strictly correlated. In particular, since the closure of IMG's nucleus is isomorphic to the Julia set itself, i.e.

$$\mathrm{IMG}\mathcal{B}_\infty \equiv \frac{\pi_1(\mathcal{B}_\infty)}{\mathcal{N}_{\mathrm{IMG}\mathcal{B}_\infty}} \qquad \Rightarrow \qquad \overline{\mathcal{N}_{\mathrm{IMG}\mathcal{B}_\infty}} \asymp \mathrm{J}(f)\,.$$

Points in the (multi-valued) Julia set $\mathrm{J}(f)$ thus represent homotopy classes of iteratively isomonodromic paths on $\mathcal{B}_\infty$, cfr. §2.3.4, and are at the same time their base-points (since the points in $\mathcal{B}_\infty$ are equivalence classes of asymptotically-equivalent points under the iterated, multi-valued map $z \mapsto f(z) \mapsto f^2(z) \mapsto \cdots$). In particular, the accumulation points in $\mathcal{D}\mathrm{J}(f)$ are associated with the accumulation points in the *profinite* IMG of $\mathcal{B}_\infty$. Points in the $\mathcal{B}_\infty/\mathrm{IMG}\,\mathcal{B}_\infty$ (base points for non-isomonodromic paths) correspond instead, by the IMG-Theorem, to the boundary points, $\mathcal{J}_{\mathrm{IMG}f} = \partial\mathcal{B}_\infty$. The nucleus $\mathcal{N}_{\mathrm{IMG}f}$ is, by definition (1.100), the minimal set of (iterated) monodromy actions which *definitely* fix the homology classes on $\mathcal{A}_\infty$. All the other points are associated to (homotopy classes of) finite-order isomonodromic paths, i.e. to actions in a (finite-order) monodromy group $\mathrm{MG}\mathcal{B}_n$, for some $n \in \mathbb{N}$.

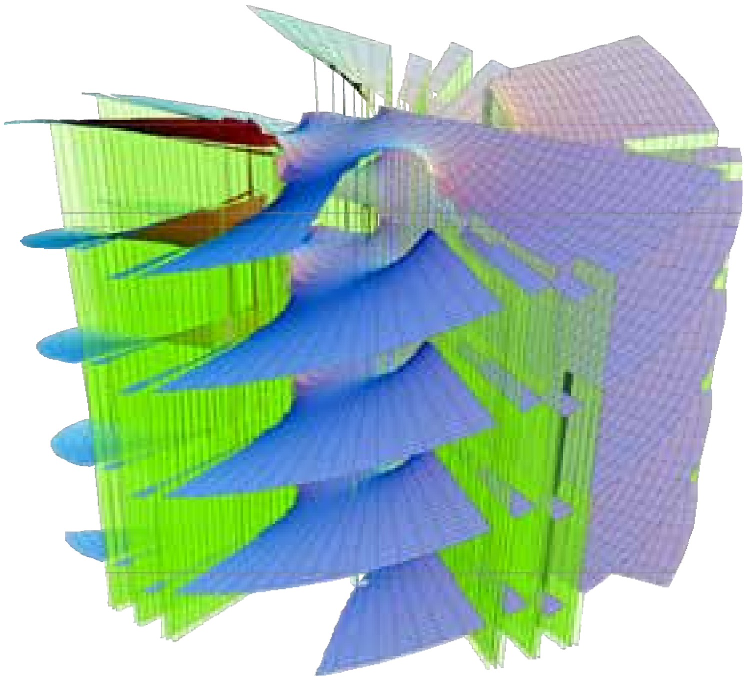


Figure 33 The above picture shows the real part of a prefractal Riemann surface of ‘cyclotomic type’ like those of Figure 24 and Figure 28 with the difference that several branches are shown together with some interconnections between same- and different-generation sheets highlighted in green.

For further considerations on the structure of the IMG, cfr. [27], [83]; for analysis of the different dynamical behaviour associated to different points within (single-valued) Julia sets, cfr. [77], [26] and [39].

Furthermore, by the Wreath Recursion Theorem of §1.4.2, the self-similar action of IMGf on $\mathcal{B}_\infty$ and, in general, on any monodromy group of $\mathcal{B}_n$, $\forall n\in\mathbb{N}$, is given by the wreath recursions $\mathrm{IMG}f\cong\mathbb{S}_p\wr\mathrm{IMG}f$ and $\mathrm{IMG}f\cong\mathrm{MG}\mathcal{B}_n\wr\mathrm{IMG}f$, where the p elements which the (iterated) monodromy group(s) acts on are the p sheets of each generation; this fact has two main consequences.

First of all, the wreath recursion matrix (1.311) can always be built for $\mathcal{B}_n$ as a block-matrix (in general rectangular) H_n, whose blocks are composed in a self-similar way; usually the matrix is pseudo-diagonal, with the pseudo-diagonal block all equal to the wreath recursion matrix H_1 of the initiator surface $\mathcal{B}_1$, whereas sparse, off-diagonal-block elements are present to join the connection between different-generation sheets. These self-similar matrices are then built using formulas like,

$$H_n = V_n + \bigoplus^{p} H_{n-1}, \qquad \text{with } V_n \text{ sparse, off-diagonal increasing-order matrices,}$$

which are well-known methods in fractal networks' theory, and were used by the author as incidence [adjacency] matrices of fractal graphs [simplicial complexes], cfr. [9]–§§5, 7, [10], [12], [16], [17], [18], [20], [53].

Secondly, at this point $\mathcal{B}_\infty$ will always be paired with the discrete dynamical system $(\mathcal{J}_{\mathrm{IMG}f};f^{-1})$, cfr. §1.4.2, where f^{-1} exactly behaves like a shift map on $\partial\mathcal{B}_\infty$: in fact, since (1.341)–(1.342), $\mathcal{B}_\infty=\overline{\mathbb{Z}_p^{\mathsf{T}}}\big/\!\sim$ whereas $\mathcal{J}_{\mathrm{IMG}f}$ is its boundary. In particular, $\partial\mathcal{B}_\infty$ is invariant (but *not* fixed) by f, simply because, with a slight abuse of notation:

$$f^{-1}\Big({}_{[s_0]}[\![\mathbf{k}_1]\!]_{[s_1]}\rightsquigarrow[\![\mathbf{k}_2]\!]_{[s_2]}\rightsquigarrow[\![\mathbf{k}_3]\!]_{[s_3]}\rightsquigarrow\cdots\Big)={}_{[s_1]}[\![\mathbf{k}_2]\!]_{[s_2]}\rightsquigarrow[\![\mathbf{k}_3]\!]_{[s_3]}\rightsquigarrow[\![\mathbf{k}_4]\!]_{[s_4]}\rightsquigarrow\cdots.$$

The interior set $\partial\mathcal{B}_\infty$ is instead either "nilpotent" or periodic with respect to the shift map f^{-1} (more on this subject in §3.4). Wreath recursion $\mathrm{IMG}f\cong\mathbb{S}_p\wr\mathrm{IMG}f$ is the algebraic equivalence to the invariance of under both the *forward* and *backward* shift maps $f;^{\pm 1}$ and to the complete invariance of the Julia and Fatou sets $\mathrm{J}(f)$ and $\mathrm{F}(f)$ under the same maps $f^{\pm 1}$ — cfr. §2.4.1. In particular, the following result is true:

Fractal Monodromy Theorem

The limit set for a post-critically finite, branched self-cover $f:\mathcal{B}_0 \to \mathcal{B}_1$ is homeomorphic to the multi-valued Julia set of f, i.e.:

$$\mathrm{J}(f) \asymp \mathcal{J}_{\mathrm{IMG}f} = \partial\mathcal{B}_\infty. \tag{1.376}$$

This manifold comparison between the boundary and interior of $\mathcal{B}_\infty$, between multi-valued Julia and Fatou sets, between finite rooted trees and limit sets (i.e. infinite rooted trees) is analogue to a numbering system in one or more dimensions (the latter also called a "number **tiling**"): for more information on this particular subject in Number Theory, cfr. [82]. Also cfr. Figure 30 where an example of the homeomorphism (1.376) is shown, where the Basilica group $\mathrm{IMG}(z^2+1)$ is defined and is visually compared with an approximation of its Julia set $\mathrm{J}(z^2+1)$.

3.3.5. Iterated Monodromy and Puiseux series

This paragraph is mainly devoted to the applications of the conjuction between the the Infinite Galois Theory introduced in §1.2.3 and in [79], the Puiseux's Theorem (1.87) the Monodromy Theorem for Riemann surfaces stated in §2.4.5, the Fractal Monodromy Theorem (1.376) and the Fractal Riemann Theorem proved in §3.3.4.

From isomorphism (1.374) it follows that the fields $F_{n,m} := \mathbb{C}\{w_n\}/\mathbb{C}\{w_m\}$, for $\forall m,n \in \mathbb{N}_0$ with $m < n$, apart from being p-radical, $(\operatorname{ord}\mathrm{MG}\mathcal{B}_{n-m})^{\text{th}}$-degree Galois field extensions, are also isomorphic, by the Theorem of §0, to the splitting fields of some monic (Weierstrass) polynomials $P_{n,m} \in \mathbb{C}\{w_n\}[z]$ of degree $(\operatorname{ord}\mathrm{MG}\mathcal{B}_1)^{n-m}_{;}$ cfr. (1.233). In particular:

- Fields $F_n := F_{n,0} \equiv \mathbb{C}\{w_n\}/\mathbb{C}\{z\}$ are extensions of degree $(\operatorname{ord}\mathrm{MG}\mathcal{B}_1)^n$.
- Fields $F_{n+1,n} \equiv \mathbb{C}\{w_{n+1}\}/\mathbb{C}\{w_n\} \cong \mathrm{Gal}_f\mathbb{C}\{z\}$ are isomorphic $(\operatorname{ord}\mathrm{MG}\mathcal{B}_1)^{\text{th}}$-degree extensions, with $\mathrm{Gal}F_n = G_n$, $\forall n \in \mathbb{N}$, by (1.368) and the corresponding Lemma.
- The Weierstrass polynomials $P_n(w,z) := P_{n,0}(w,z)$ implicitly define the prefractal Riemann surfaces $\mathcal{B}_n$, whereas the associated Laurent series $g_n \in \mathbb{C}[\![z]\!]$ such that, by (1.87), $P_n(g_n(z), z^{p^n}) \equiv 0$, and the Puiseux series $g_n(\sqrt[p]{z})$ are their local multi-valued explicit definitions;
- Polynomials $P_{n,m}(w,z)$ satisfy the following identity, for any $m < n$:

$$P_{n,m} \equiv P_{m-n,0}.$$

Three *triangular* sequences (i.e. indexed by integer couples $m,n \in \mathbb{N}_0$ with $m < n$) were thus defined: that of the field extensions $(F_{n,m})_{m<n}$, that of their Galois groups $(\mathrm{Gal}F_{n,m})_{m<n}$, and that of the Weierstrass polynomials $(P_{n,m}(w,z))_{m<n}$ which are solved by certain Puiseux series in $\mathcal{B}_n$.

The first explicit definitions of $\mathcal{A}_n$ and $\mathcal{B}_n$ —which are given in (1.324) and (1.359), respectively— imply that a gross numerical (though severely multi-valued) estimate of their points is done, at least numerically, by computing the nested radicals of a certain polynomial or rational function. The most important objects —or obstacles, according to computational purposes— in this process are the branch points (both branch zeroes and poles in the case of $\mathcal{B}_n$'s) and the arbitrary choice of the associated branch cuts. Even when the roots of either $P(z)$ in (1.324) or $N(z)$ and $D(z)$ in (1.359) are exactly assigned, higher-generations branch points could be found neither in closed nor semi-closed forms; that is a problem which a partial solvability condition was given to in §3.3.4: the monodromy groups' solvability implies, in fact, the computability "by radicals" of any-order branch points, other than the possibility to compute all the homotopically-equivalent paths on the surfaces.

In fact, we can say that the branch points of generation n and below for $\mathcal{B}_n$ belong to the p-radical extension field $\mathbb{C}\{w_n\}\equiv\mathbb{C}\{w_n,w_{n-1},\ldots,w_1,z\}\equiv\mathbb{C}\{f^n(z)\}$, since $w_n=f^{n-m}(w_m)$ for $m<n$. The proper extension in order to consider n^{th}-generation *ramification* points only is instead the field F_n, although it is a field of rational functions in the transcendental variable w_{n-1}, which gives the trivial solution $\{\mathbf{z},\infty\}\subset\mathcal{B}_{n-1}$, *not* the branch points for $z\in\mathcal{B}_0$. Closed-form expressions for points in the pre-fractal surface $\mathcal{B}_n$ take then the form of nested radicals like (1.117):

$$\sqrt[p]{\frac{\mathbf{b}(\zeta)}{\mathbf{a}(\zeta)}}=\overbrace{\sqrt[p]{a_1(\zeta)+b_1(\zeta)\sqrt[p]{a_2(\zeta)+b_2(\zeta)\sqrt[p]{\cdots+b_{n-1}(\zeta)\sqrt[p]{a_n(\zeta)+b_n(\zeta)}}}}}^{n\ \text{times}}\,,$$

which is an element of splitting fields of the form $\mathbb{C}\{\zeta\}[z]/P_n$.

Another big problem, in this case, is finding a simpler expression for such nested radicals, which is often possible, especially when they come from a radical extension rather than being randomly generated. This problem, which has not been applied to prefractal Riemann surfaces yet, is also called radicals' "***denesting***", cfr. [28] and references therein. In the "rational cover case" of §3.3.1, such radicals may be combine with, or be denested into prefractal fractions like (1.114).

If the monodromy of the initiator surface is solvable (then also $\mathrm{MG}\mathcal{B}_n$ are solvable $\forall n\in\mathbb{N}$) a Puiseux series $\phi_\zeta\in\mathbb{C}\{\zeta\}[\![z-\zeta]\!]^\natural$ is always available around any n^{th}-generation branch point $\zeta\in\mathcal{B}_0$ ($\zeta\neq\infty$, for simplicty's sake), which converges to $f^n(z)$ inside an annulus around that point (the annulus turning into a disc if ζ is a branch zero rather than a branch pole — in which case $N_\zeta=\mathrm{ord}_{\phi_\zeta}\zeta=0$):

$$\phi_\zeta(z)=\sum_{k=-N_\zeta}^{\infty}a_k(\zeta)(z-\zeta)^{p^{-n}k}. \tag{1.377}$$

In the above expression, note that $\phi_\zeta \in \mathbb{C}\{\zeta\}[\![z-\zeta]\!]^\natural$ is a Laurent series with respect to variable $\sqrt[p^n]{z-\zeta}$, that because $\mathbb{C}\{w_n\}$ requires in general nested radicals up to depth n which become arithmetic $(p^n)^{\text{th}}$ roots when developed using Puiseux series. That is also consistent with the fact that (except for a few cases where LCM$\boldsymbol{\pi} < p$) the prefractal surface $\mathcal{B}_n$ has p^n sheets by (1.232)–(1.233). Of course, the ability to compute local Puiseux series expansions for such complicated surfaces rigidly depends on the solvability of their monodromy groups and is, in general, quite a difficult task to be accomplished in closed form for higher enough iteration orders n, cfr. [100].

As far as the branch points' multiplicity is concerned, also remember that, if neither cascading nor "PZ" phenomenon are present, each branch point has a finite and fixed order within each $\mathcal{B}_n$. In the cascading case, instead, the cascading branch points have exponentially increasing orders and, in the limit fractal surface $\mathcal{B}_\infty$, they become logarithmic points, where infinite sheets glue together. If "PZ" phenomenon is present instead, some branch points might either disappear, have their order decreased, and eventually turn from branch zeroes to branch poles (or vice versa). Although such facts deeply influence the monodromy actions, almost all the considerations done in previous paragraph remain unvaried.

3.4 Riemann Chaos

In this paragraph the reader is supposed to be familiar with formalism and concepts of the geometric Theory of Dynamical Systems, like those exposed in [5]–[8], [37] and [77]. Let $\omega \in \Lambda^{1,0}(\mathcal{B}_\infty)$ be the phase (1,0)-form of a meromorphic dynamical system whose phase space is a fractal Riemann manifold like $\mathcal{B}_\infty$ (or $\mathcal{A}_\infty$, or their prefractals $\mathcal{A}_n$ and $\mathcal{B}_n$). If the dynamical system is 1st-order and time-periodic, the corresponding ordinary differential equation (system) obeys the ***Folquet*'s Theorem** and its *linearized* Poincaré map is nothing more that the monodromy action on the symmetries' group of the Poincaré section (which is a Riemann sub-manifold), [7]. Many other non-time-periodic differential systems are globally described by means of their monodromy 'operator', which inter-relates all the system's solutions via the algebraic symmetries featured when a solution develops (in many senses, Puiseux series' scenario included, cfr. §3.3.5) into another — [69].

3.4.1. Algebraic interpretation of chaos

Let now the logarithmic derivative of multi-valued function $f^n(z)$ be examined; for $n=2$ it can be computed, in the polynomial case for simplicity's sake, from (1.312) as

$$\begin{aligned}
\mathrm{D}\log f^2(z) &\equiv \frac{\mathrm{D}f\left(f(z)\right)}{f\left(f(z)\right)} = \frac{f'\left(f(z)\right)}{f\left(f(z)\right)}f'(z) \\
&= f'(z)\sum_{j=1}^{r}\frac{\mu_j}{\pi_j}\frac{1}{f(z)-z_j} = \sum_{j=1}^{r}\frac{\mu_j}{\pi_j}\frac{1}{\frac{1}{\mathrm{D}\log f(z)}-\frac{z_j}{f'(z)}} \\
&= \left(\sum_{j=1}^{r}\frac{\mu_j}{\pi_j}\frac{1}{1-\frac{z_j}{f(z)}}\right)\mathrm{D}\log f(z) \\
&= \sum_{i,j=1}^{r}\frac{\mu_i\mu_j}{\pi_i\pi_j}\frac{f(z)}{(z-z_i)(f(z)-z_j)};
\end{aligned} \tag{1.378}$$

then by induction over $n\in\mathbb{N}$:

$$\begin{aligned}
\mathrm{D}\log f^n(z) &= \left(\sum_{j=1}^{r}\frac{\mu_j}{\pi_j}\frac{1}{1-\frac{z_j}{f^{n-1}(z)}}\right)\mathrm{D}\log f^{n-1}(z) \\
&= \frac{1}{z}\sum_{i_0,i_1,\ldots,i_{n-1}=1}^{r}\prod_{k=0}^{n-1}\frac{\mu_{i_k}}{\pi_{i_k}}\frac{f^k(z)}{f^k(z)-z_{i_k}}.
\end{aligned} \tag{1.379}$$

These formulas are useful to manipulate the expression in §3.2.7 and treating a possible (ordinary) differential equation whose underlying surface is $\mathcal{A}_n$.

The name ***Riemann Chaos*** was coined by the author in order to embrace the effects of a generic smooth curve $\gamma\in\mathrm{C}^\infty([0,1];\mathcal{B}_n)$ travelling on a (pre-)fractal Riemann surface $\mathcal{B}_n$ (as well as on one like $\mathcal{A}_n$) for a fixed $n \in \mathbb{N} \cup \{\infty\}$. First of all, if the curve is represented as a discrete "path" (for example, numerically on either a triangulation or another meshed version of $\mathcal{B}_n$), this means that, for high enough n's, the path must be carefully chosen such that it does not pass by any branch point or singularity in local coordinate z; then, the curve will certainly cross one or more branch cuts which, though arbitrary, become more and more dense as n increases, especially in some regions which will be the real "chaotic" regions of $\mathbb{C}$. When such a sheet transition happens (which can be represented by some symbolic dynamics on a rooted tree, cfr. §3.2.2) the curve generally undertakes different kinds of dynamics (which *never* depends on the arbitrary branch cuts' choice), being now roughly classified with respect to that curve.

- Regular motion— It crosses a finite number of branch cuts which asymptotically increases, at most, like a negative power of the iteration order n, i.e. $\exists \chi \in]1, +\infty[$ such that the crosses behave like $\mathrm{O}(n^{-\chi})$. Then the number of branch-cut crosses stays finite for any finite- or infinite-length paths.

- Irregular (weakly chaotic) motion— It crosses a finite number of branch cuts which logarithmically increases like $\mathrm{O}(\log n)$. The number of branch-cut crosses stays finite for any finite path lengths but is eventually infinite for infinite path lengths for $n \to \infty$ (i.e. for the fractal limit surface only).

- Chaotic motion— It crosses a finite number of branch cuts which asymptotically increases like some positive power $\chi \in \mathbb{R}_+$ of n, i.e. like $\mathrm{O}(n^{\chi})$. It crosses a *countably* infinite number of branch cuts for any arbitrarily *finite*-length paths for $n \to \infty$ — that is where Riemann Chaos onsets.

- Singular motion— It crosses a finite number of branch cuts which increases at least like $\mathrm{O}(e^n)$. It then crosses an *uncountably* infinite number of branch cuts of the limit surface (when $n \to \infty$) for any arbitrarily *finite*-length paths. This behaviour is termed "singular" because it poses serious convergence problems and, usually, the Riemann surface can no longer be analytically continued into those regions as a functions' limit from definitions like those given throughout §3.

The most interesting case is the third one — chaotic motion. When that happens, trajectories starting arbitrarily close on the same sheet exponentially diverge in the sense that they "get lost" among different sheets. This is the "sensitive dependence on initial conditions" which is the main feature of deterministic chaos, [105], [84] and is associated to non-negative (and, usually, strictly positive) Lyapunov exponents (1.224).

In the end, the study of such Riemann surfaces has left not a few open problems, which will be sketched here:

- Is there a way to compute an (approximate) Lyapunov exponent for a trajectory on an arbitrary $\mathcal{B}_n$?

- Is it possible to estimate the sign of the above exponents only from a few data about the Riemann surface's initiator (for example, from its branch points' position and order, like was done for the monodromic entropy in §3.1.3) ?

- Is it possible to classify the fractal Riemann surfaces $\mathcal{A}_n$ and $\mathcal{B}_n$ by building a phase space whose coordinates work like a sort of *parameter space* for all such possible multi-valued functions? Like the **Mandelbrot set** is a parameter space for every family of complex (single-valued) maps and their Julia sets [39], [77], is it possible to estimate *a priori* the surfaces' asymptotic behaviour by either the algebraic, geometric or topologic features of a manifold depending —for example— on a vector on such a phase space, like for example $(\mathbf{z},\boldsymbol{\zeta};p,q;(\mathbf{m},\mathbf{n}))\in\mathbb{C}^2\times(\mathbb{N}\backslash\{1\})^2\times\mathbb{Z}^{r+s}$?

- Are surfaces like either $\mathcal{A}_n$ and $\mathcal{B}_n$ (and, especially, $\mathcal{A}_\infty$ and $\mathcal{B}_\infty$) the underlying Riemann manifolds for some (quasi-)*integrable* physical system?

- Is there a way to estimate *a priori* the cascading and "PZ" phenomena from the algebraic knowledge of $P(z)$ or $(N(z),D(z))$?

3.4.2. Gallery of of Riemann surfaces and Julia sets

This paragraph presents a brief gallery of the Riemann surfaces analyzed throughout the Chapter. It is mainly composed of two-dimensional images because the entangled branching structures are harder to discern and be interpreted on a static three-dimensional (3D) picture. Since the surfaces are *complex* 1-manifolds the projections on real spaces $\mathbb{R}^2$ and $\mathbb{R}^3$ are to be commented first.

Two-dimensional images are the greyscale plots of the multi-valued functions' *principal* arguments $\mathrm{Arg} f^n(z)$ versus the portion of a "realized" complex interval $z\in[z_1,z_2]$ — i.e. for $z=x+\imath y$ with $x\in[\mathrm{Re}z_1,\mathrm{Re}z_2]$ and $y\in[\mathrm{Im}z_1,\mathrm{Im}z_2]$. The brightness of the picture's dots (dpi resolution) goes *linearly* from minimum ($-\pi$, corresponding to **black**) to maximum principal argument (π, corresponding to white).

One-dimensional brightness' discontinuities on adjacent dots (white-to-black wrap curves) indicate ***both*** regular 360° phase lines and the position of *branch cut*s along the line, which were all computed by fixing the standard branch cut $\rangle 0,\infty\langle$ for the arithmetic p^{th} root function to be the negative real half-line $\mathbb{R}_-$. Branch cuts indicate where a sheet of $\mathcal{A}_n$ (or $\mathcal{B}_n$) is glued to another sheet, whereas the cuts' endpoints indicate the presence of a branch point; multiple branch points are the intersection of two or more branch cuts. Unfortunately there is no visual way to distinguish, on these plots, the phase lines from the conventional branch cuts.

The associated captions just specify the essential parameters to the plots:

- The explicit formula of $f(z)$.
- The iteration order(s) n; when more than one is listed as $n = n_1, n_2, \ldots, n_r$, illustrations are ordered left-to-right, top-to-bottom.
- The windowing interval $[z_1, z_2]$; if the latter is centre-symmetric of type $(1+\imath)[-\alpha, \alpha]$, then just the window's half-side $\alpha \in \mathbb{R}_+$.

Figures made up of multiple images have different parameters sequentially indicated in the caption and always referred from left to right and from top to bottom. If the modulus of the function is also displayed, to show for example the distribution of (branch) zeroes and poles, it is specifically written in the caption; then $|f^n(z)|$ is plotted to the left, whereas $\operatorname{Arg} f^n(z)$ is plotted to the right.

Despite the images not showing how different sheets are glued together (or the topology of different-generation sheet branching), they are effective in depicting the convergence ratio of Julia and filled Julia sets; the former sets, in particular, are the underlying spaces of the iterated monodromy groups themselves, so these representations provide an insight into the monodromy of different curves travelling on such Riemann surfaces.

First of all, in order to acquaint the reader with these plotting conventions, let the standard (single-valued) Julia set for some quadratic maps $z \mapsto z^2 + c$, $c \in \mathbb{C}$, be considered, [26], [39], [77].

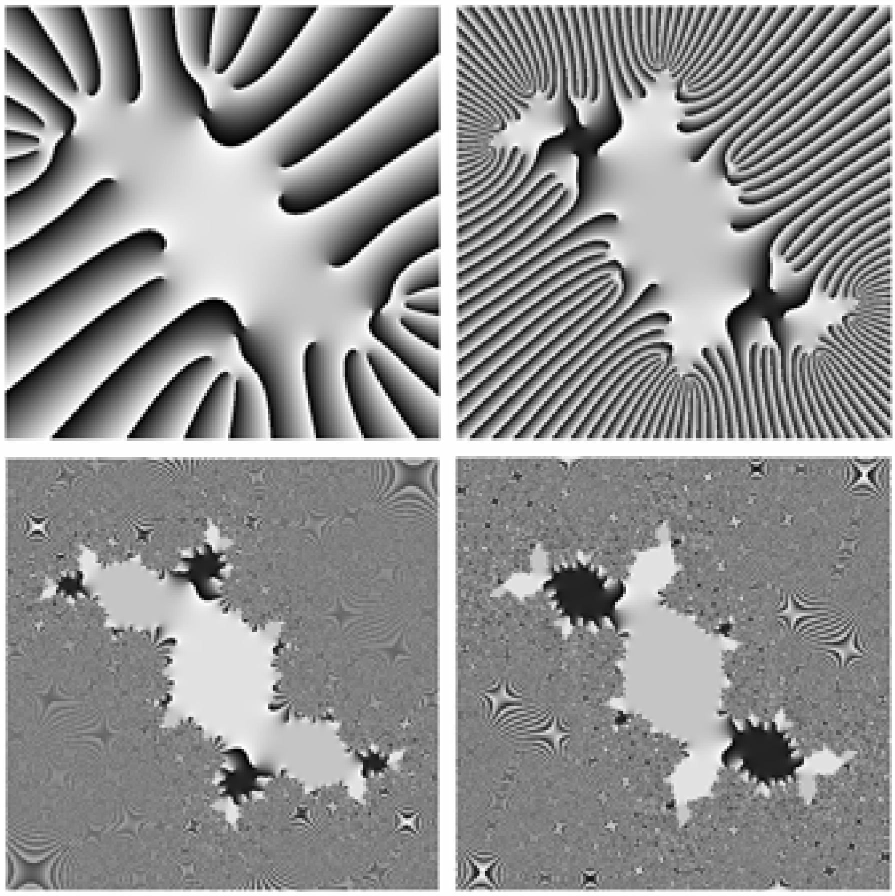


Figure 34 $f(z)=z^2+c$ for $c=0.1-0.75\imath$, $n=5,7,11,13$ and $\alpha=1.5$. The "Douady rabbit" Julia set is the boundary between "tame" and "wild" behaviour of the branch cuts of f^{-1}.

The plot of the argument functions shows many "spurious" branch cuts (which do not exist since the iterated functions are always even-degree polynomials in $\mathbb{C}[z]$).

In particular, Figure 19 is tied with Figure 36 because they show two representations of algebraically equivalent sets (by the IMG-Theorem in §2.3.4): the Basilica group $\mathrm{IMG}(z^2+1)$ and the "Basilica" Julia set $\mathrm{J}(z^2+1)$; from the topological point of view in fact, $\mathcal{J}_{\mathrm{IMG}(z^2+1)}\asymp \mathrm{J}(z^2+1)$.

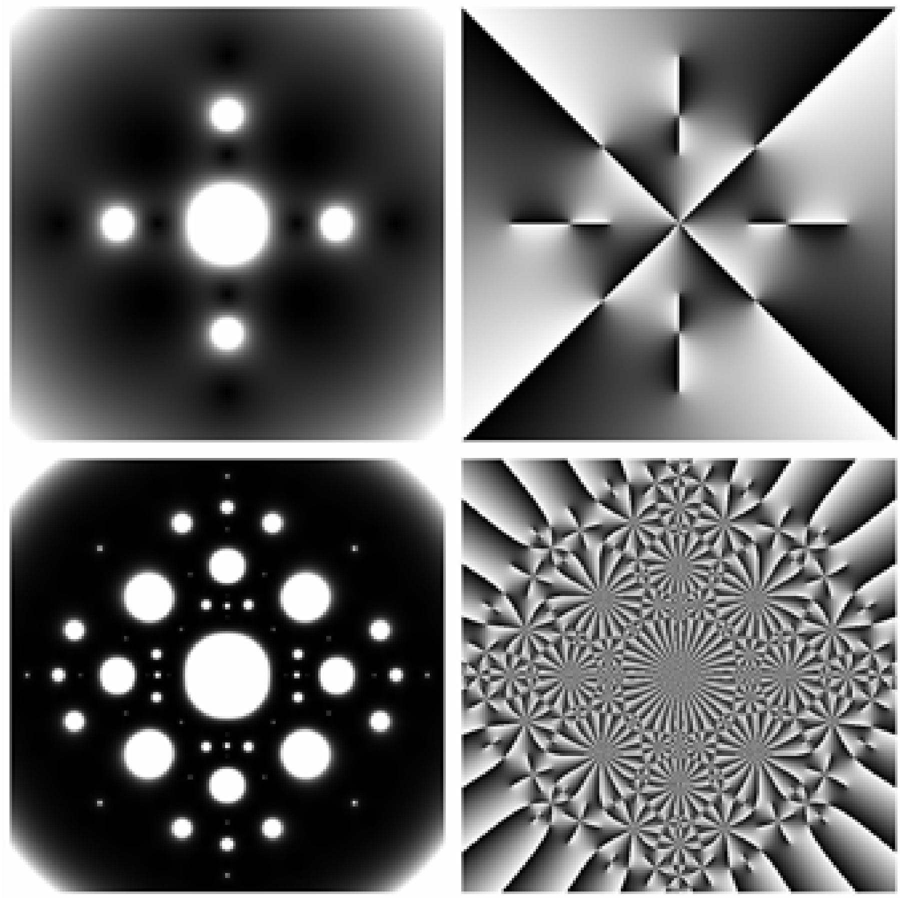


Figure 35 $f(z)=z^2+16z^{-2}$ for $n=2,5$ and $\alpha=1$; the "Sierpiński carpet" Julia set.

Figure 35 is instead associated to a rational Julia set, since $f(z)=z^2+16z^{-2}\in\mathbb{C}\{z\}$ so polar singularities come into play and, along zeroes, reveal a self-similar distribution; Such discrete pattern of (zeroes and) poles is in fact a Cantor dust (cfr. note [80]), whose fractal geometry is deduced by counting the singularities along with their orders, which in Figure 35 are "proportional" to the sizes of the white circles.

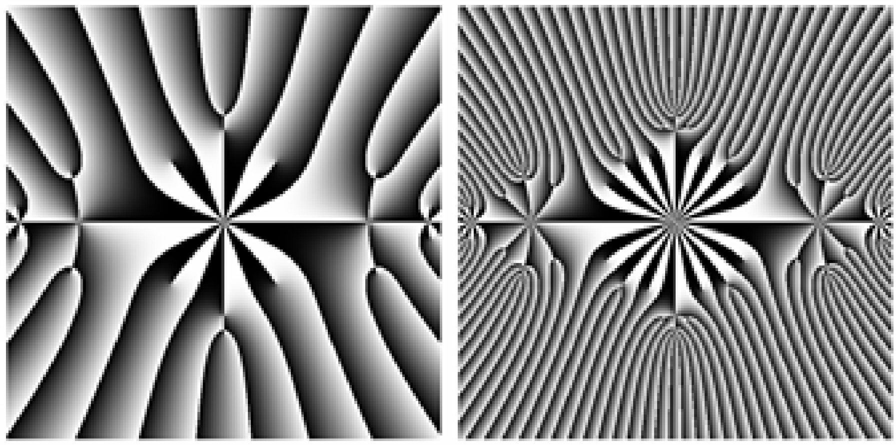


Figure 36 $f(z)=z^2+c$ for $c=1$, $n=5,7$ and $\alpha=1.5$. The "Basilica" Julia set is the boundary between the tame and wild behaviour of the branch cuts of $f_\cdot^{-1}$

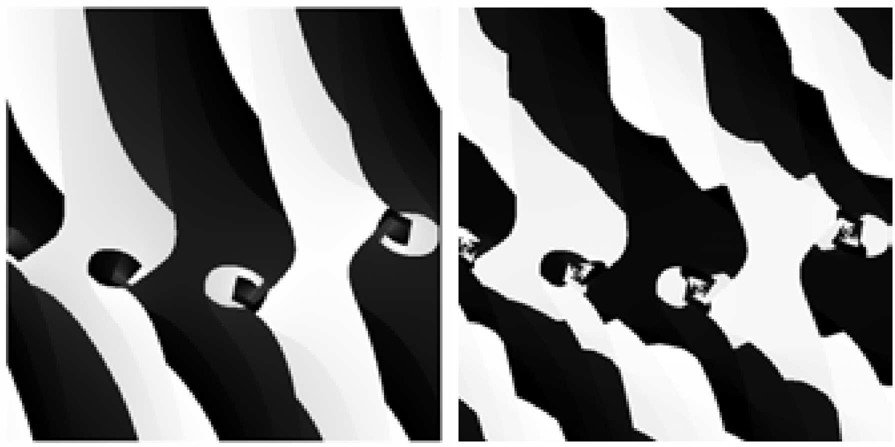


Figure 37 $\sqrt{(z+\imath)(z-e)}$ for $n=7,30$ and $\alpha=3$.

The prefractal Riemann surface shown in Figure 37 —as many others below— is not symmetric with respect to the real axis $\mathbb{R}$ because it has trivially non-real roots. The principal argument plot is very interesting in fact, since (1.326), it is an even-sheeted Riemann surface, with its sheets glued by "diagonal-like" branch cuts: every continuous sections of them is equivalent to a side of its fundamental polygon.

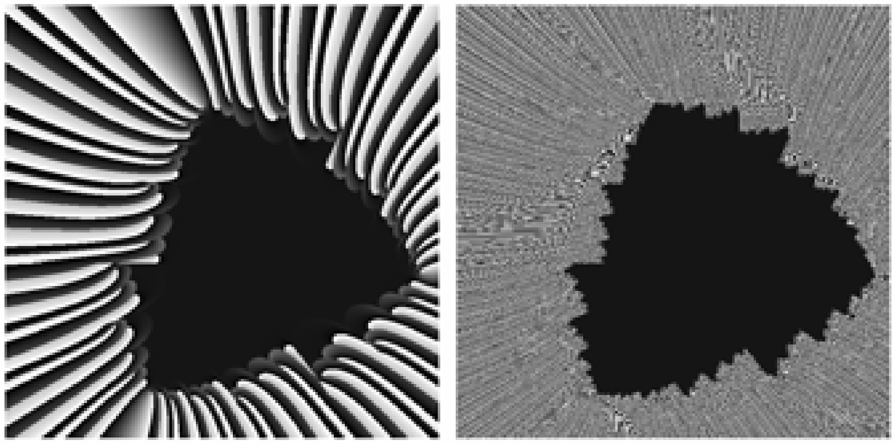


Figure 38 $\sqrt{z(z-1)(z+\imath)}$ for n=6,12 and α=2. The boundary of the "trivial" region (in black) is a typical example of a "***dragon* curve**" Julia set.

The "dragon" curves shown in Figure 38 and Figure 39 as the boundaries of the "tame" regions (in darker colours), where only a finite number of branch cuts is present for every finite-length section of the dragon curves, are well-known to be both perfect fractal-curve Julia sets and natural boundaries for functions either inside or outside of them, cfr. §§2.4.1, 5.3.1.

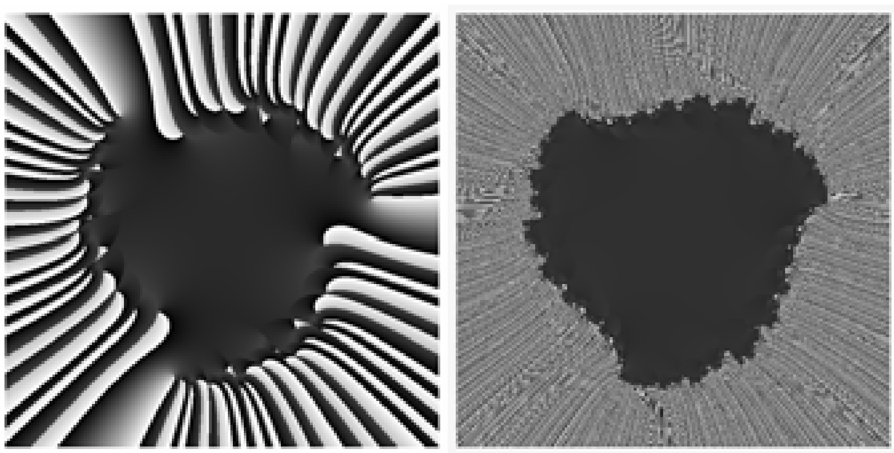


Figure 39 $\sqrt{z^3+0.2\imath}$ for $n=6,12$ and $\alpha=1.5$. Another "dragon curve" Julia set example as boundary of the "wild ∞ branch cuts"' region.

Riemann Chaos phenomenon is also evident at sight by observing the density of branch cuts on the multi-valued escaping sets (i.e. the complements to filled Julia sets, cfr. §§2.4.1, 0, 3.3.3), where branch cuts become more and more dense, always showing self-similar bifurcations over increasing-generation branch points. Note also that some discontinuities in the pictures below are "*fake*" branch cuts, i.e. they are simply normal wrap in the complex argument $\mathrm{Arg}\, f^n(z)$. It is handy complex numerics to prove that, for increasing iteration orders $n \gg 1$ and for *almost all* the possible algebraic functions of kind (1.292) —i.e. in their parameters' space as defined in §3.4.1— the number of $\mathcal{A}_n$'s branch cuts asymptotically increases as fast as the number of curves such that $\arg f^n(z) \equiv 0 \bmod 2\pi$, so the plots in this paragraph are significant to show the branch cuts' distribution to an extent which is neither higher nor lower as $n \in \mathbb{N}$ increases.

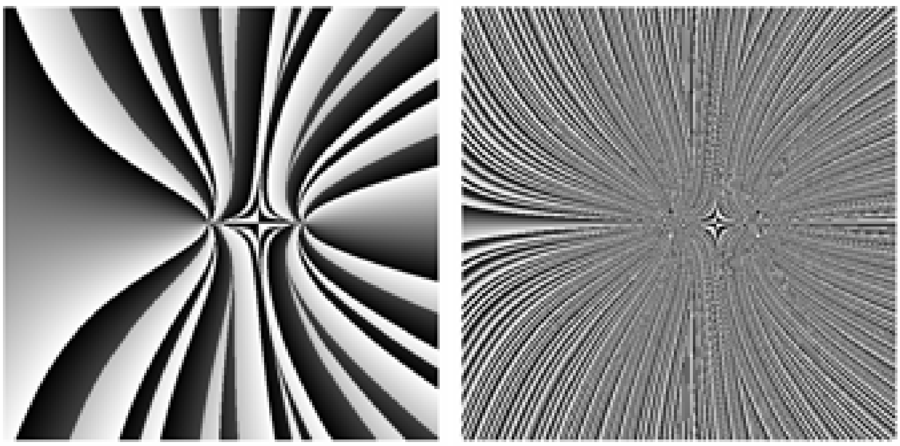


Figure 40 $\sqrt{z(z-1)(z+6)}$ for $n=5, 11$ and $\alpha=3$.

The top-left image of Figure 41 has a gray circle within, which is a logarithmic *lens space* (cfr. [59]) ingenuously used to improve the dot's contrast and show the existence of evident branch cuts even for the modulus of the function, which is actually $\log|f^9(z)|$ within the circle.

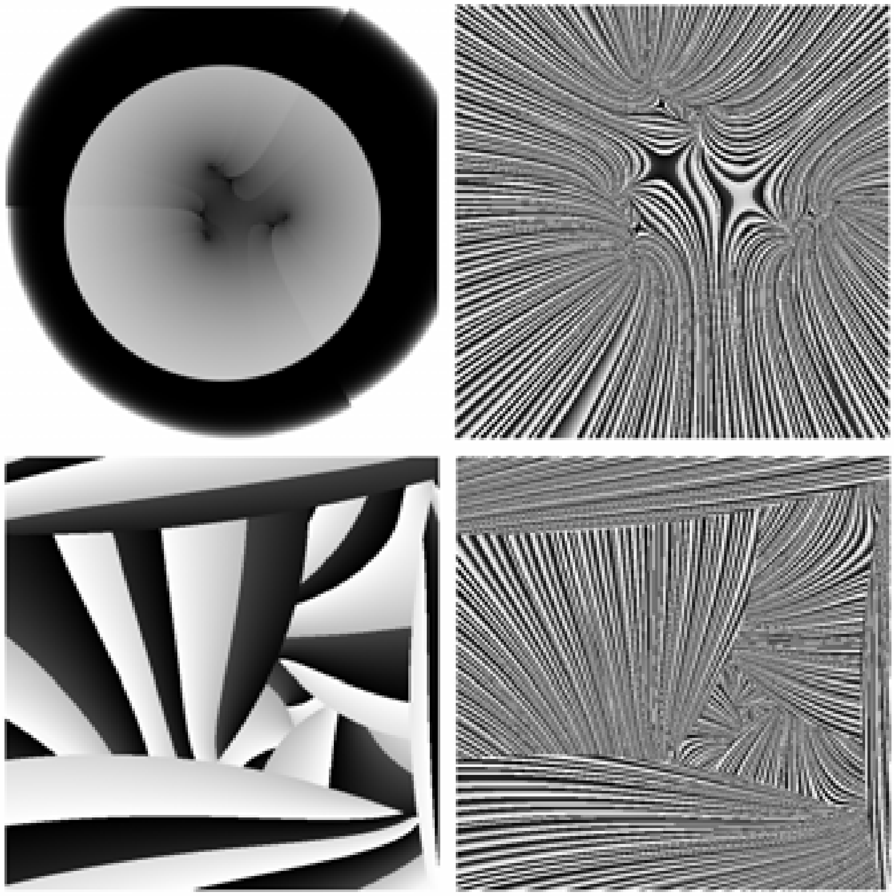


Figure 41 $\sqrt{(z+1)(z-\pi)(z-\imath e)}$. Top-left: $|f^9(z)|$ with $\log|f^9(z)|$ plotted within the gray circle with $\alpha=12$. Top-right: $\mathrm{Arg} f^9(z)$ with $\alpha=6$. Bottom: Detail of the region $[-0.45+3.05\imath, -0.35+3.15\imath]$ for $n=9$ (left) and $n=16$ (right).

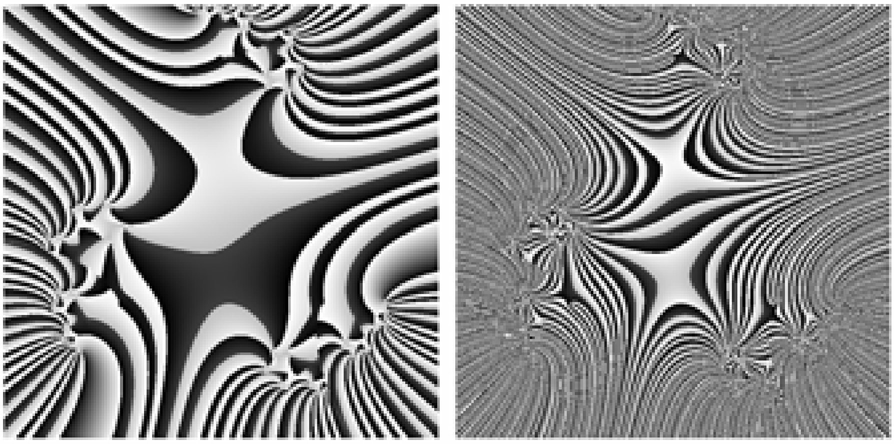


Figure 42 $\sqrt{z^3+0.48z+0.928\imath}$ for $n{=}7,11$, $\alpha{=}1.5$.

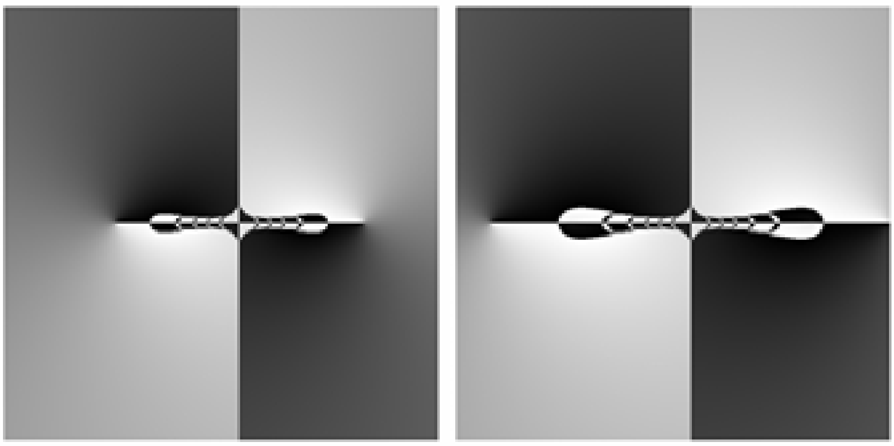


Figure 43 $\sqrt[3]{z(z-0.5)}$ for $n{=}5,7$, $\alpha{=}3$.

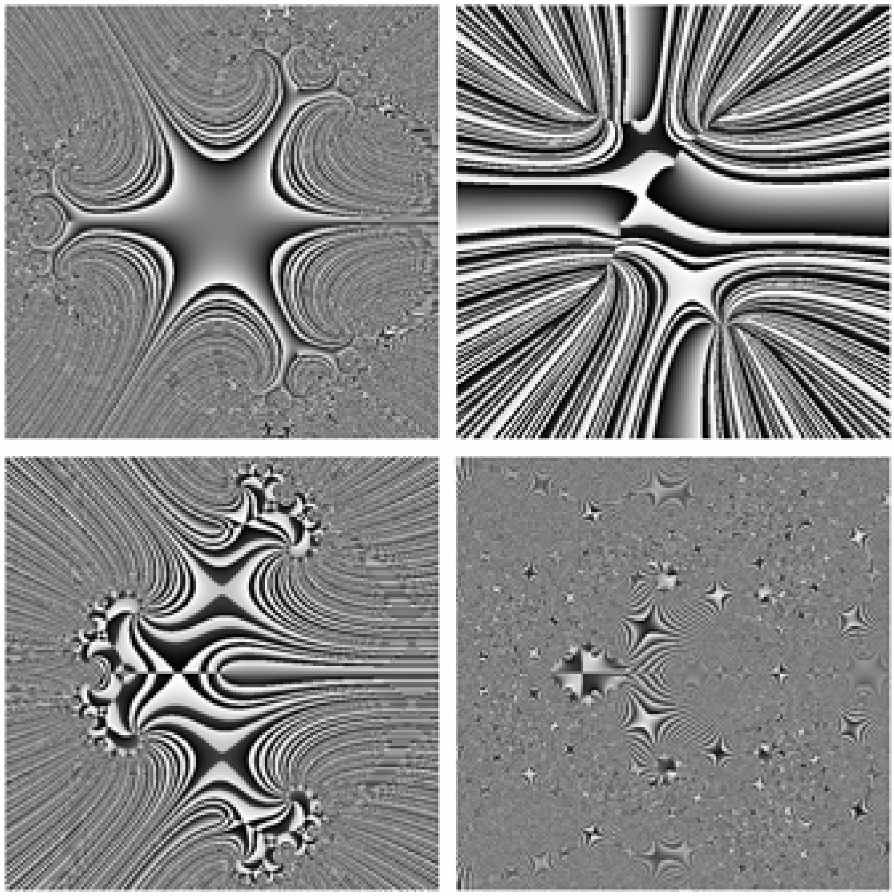


Figure 44 $\sqrt{z^3+1}$, $\sqrt[3]{(z+1)(z-\imath)(z^2+\pi+\imath e)}$, $\sqrt{z(z^2+1)}$ and $\sqrt{z(z^3+1)}$, all for n=13 and α=1.5.

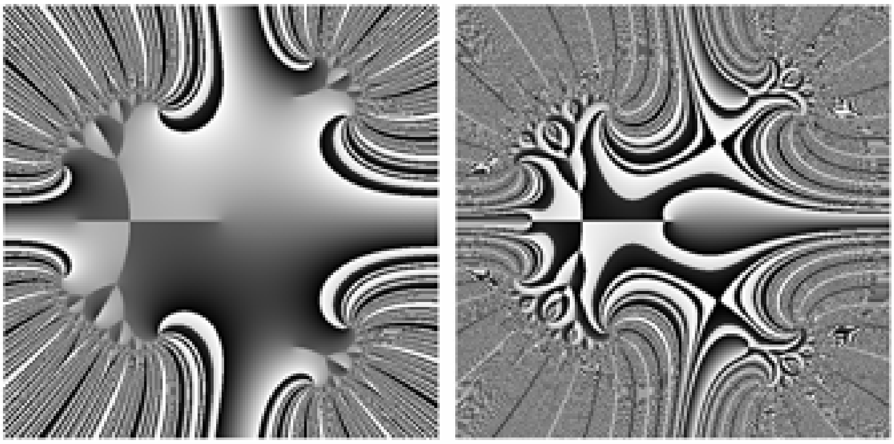


Figure 45 $\sqrt[3]{z(z^3+1)}$ for $n=9,13$ and $\alpha=1$.

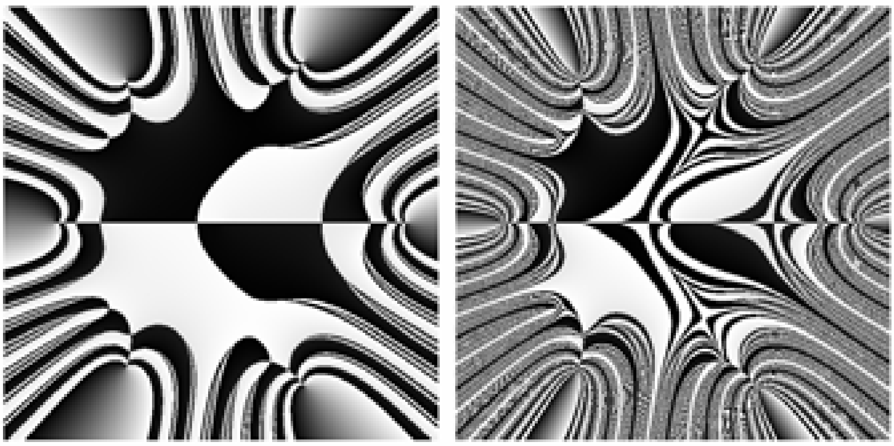


Figure 46 $\sqrt[5]{z(z^5-1)}$ for $n=6,12$ and $\alpha=1.5$.

The prefractal Riemann surfaces of "polynomial" type (§3.2), as well as those of "rational" type (§3.3) depicted in these figures are extremely tied to the algebraic symmetries of the equations initiator defining them. One fact which is evident "at sight" by looking at the images is that for a function of type either $f(z)=\sqrt[p]{P(z)}$ or $f(z)=\sqrt[p]{N(z)/D(z)}$, the branch cuts' distribution is mainly sensitive to both p, $\deg P$, $\deg N$ and $\deg D$, and usually locally shows a *local* "cyclotomic" self-similarity, which is at the base of the topologic and algebraic facts described in §3.3.4.

For example the three Riemann surfaces depicted in Figure 47 and Figure 51 (left) show a cyclotomic similarity of orders $\deg P = 8$ and $\deg P = 3$ with respect to the origin (and to its pre-images), whereas a cascade of sheets' foliation is present, progressing with a p-fold symmetry; that is evident since the "wilder" branch cuts' distributions, either between couples of higher-order branch cuts —Figure 47 with 7^n-fold (left) an a 8^n-fold (right) self-foliation— or around higher-order branch points — Figure 51 (left) with 13 branch cuts departing from every branch points.

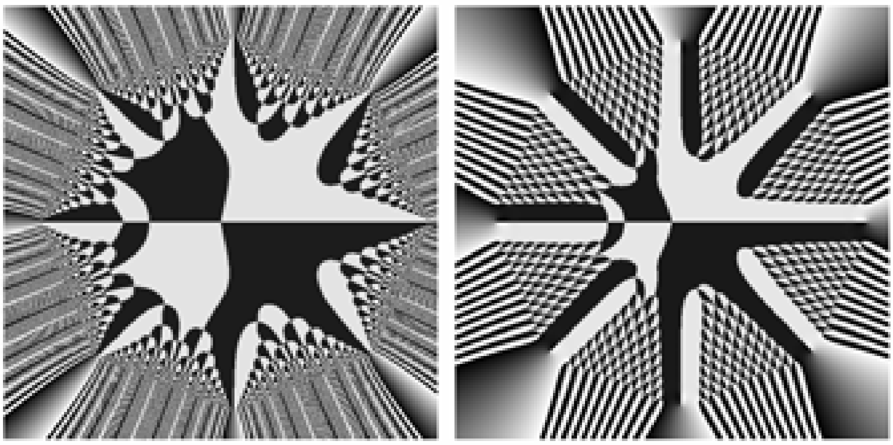


Figure 47 $\sqrt[7]{(z-1)(z^7+1)}$, $\sqrt[8]{(z-1)(z^7-1)}$ for $n=11$ and $\alpha=3$.

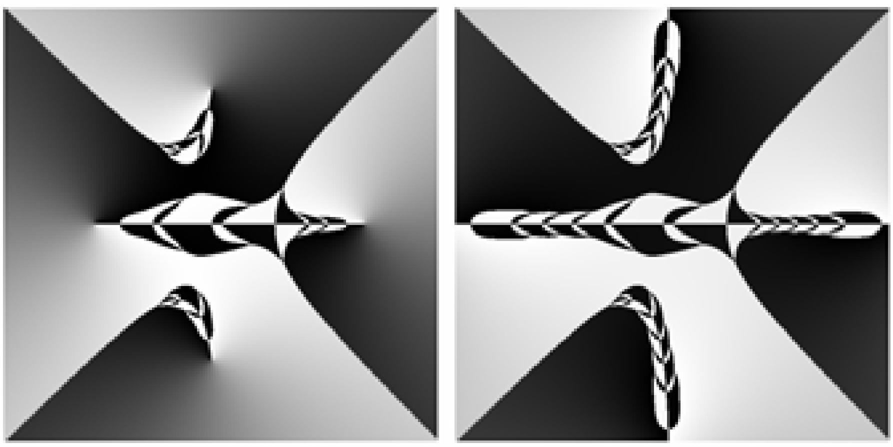


Figure 48 $\sqrt[5]{z(z^3-1)}$ for $n=4,7$ and $\alpha=2.5$.

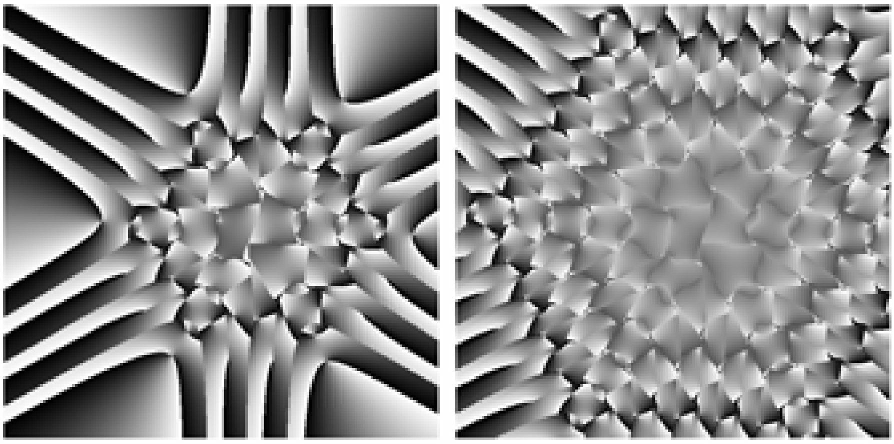


Figure 49 $\sqrt[6]{z(z+\imath)(z-1)^2(z-2)^2}$ for $n{=}4{,}7$ and $\alpha{=}6$.

It is still possible to observe beautiful and almost “humanly regular” symmetties, like that of crystals, for example for the multi-valued Julia sets in Figure 49 where the branch cuts easily follow a 6-fold cyclotomic symmetry at the “outside” the Julia set (in this case towards ∞), and a wonderful *hexagonal* symmetry at the “inside” of it (although such distinction would be topologically indifferent, since the behaviour is homogeneous local charting as $z \mapsto z^{-1}$).

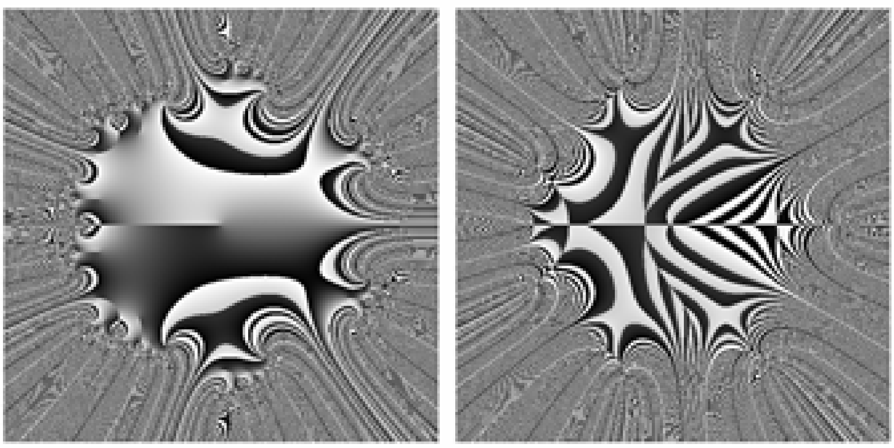


Figure 50 $\sqrt[5]{z(z^6 \pm 1)}$ for $n{=}11$ and $\alpha{=}1.5$.

Figure 51 (right) shows a transcendental case, which was not really analytically treated due to the presence of *logarithmic* branch points and thus an infinite number of sheets at any iteration orders. Despite the analytical difficulties in treating surfaces of this kind, their multi-valued Julia set seem not to be too different (at least *locally* — on each single-valued sheet) from algebraic cases. This fact suggests that the result analyzed throughout the whole §3 could be extended even to iterations of transcendental functions, in order to incorporate even wider classes of Riemann Chaos, which is infinite-sheeted even for finitely-iterated functions.

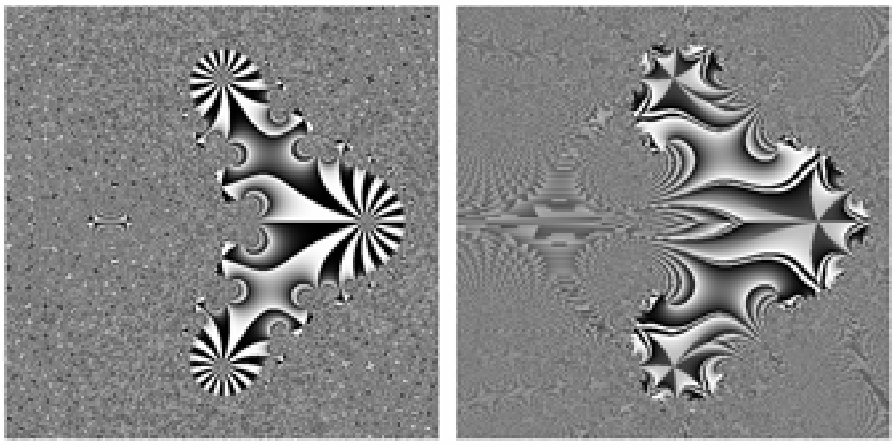


Figure 51 $\sqrt[p]{(z-1)(z^2+1)}$ for $n=3$ and $\alpha=1.5$. Left: algebraic case $p=13$. Right: *transcendental* case $p=\pi$.

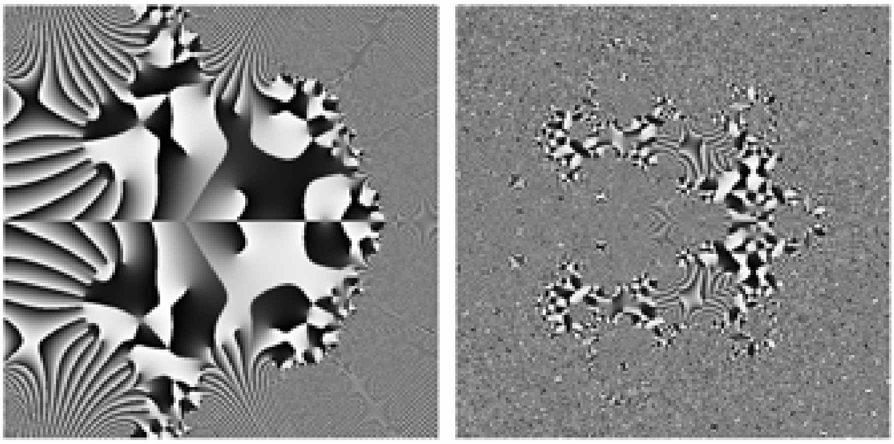


Figure 52 $\sqrt{(z-1)(z^7-1)}$ for $n{=}7{,}11$ in $[-0.5-\imath,\, 1.5+\imath]$.

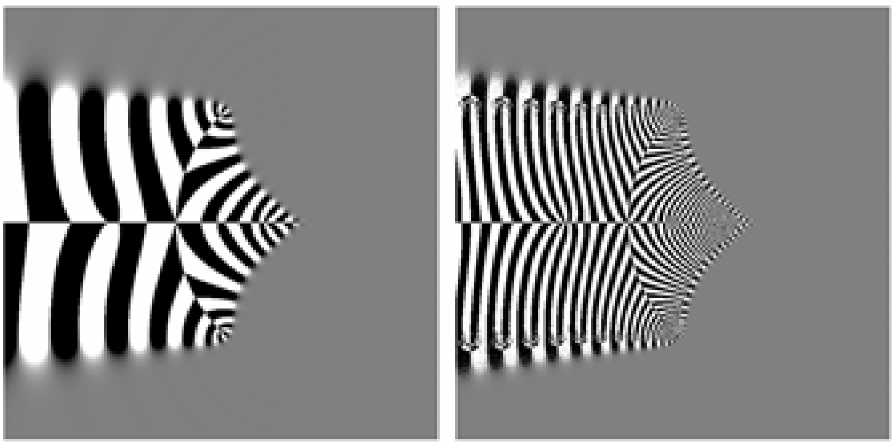


Figure 53 $\sqrt{\dfrac{z(z+1)}{z-1}}$ for $n{=}7{,}8$ and $\alpha{=}2$.

All in all, the scenariœs depicted in this gallery show how varied and topologically non-trivial can be the behaviour of fractal Riemann surfaces defined starting from so simple functions (such as polynomials or rational functions). However, and apart from the algebraic solvability case (§§3.3.4–3.3.5), the numerical difficulties in following such convoluted paths, without "missing" even a branch cut, thus travelling by more and more different sheets, is ultimately the sign of deterministic chaos which was sought for in §3.4.1.

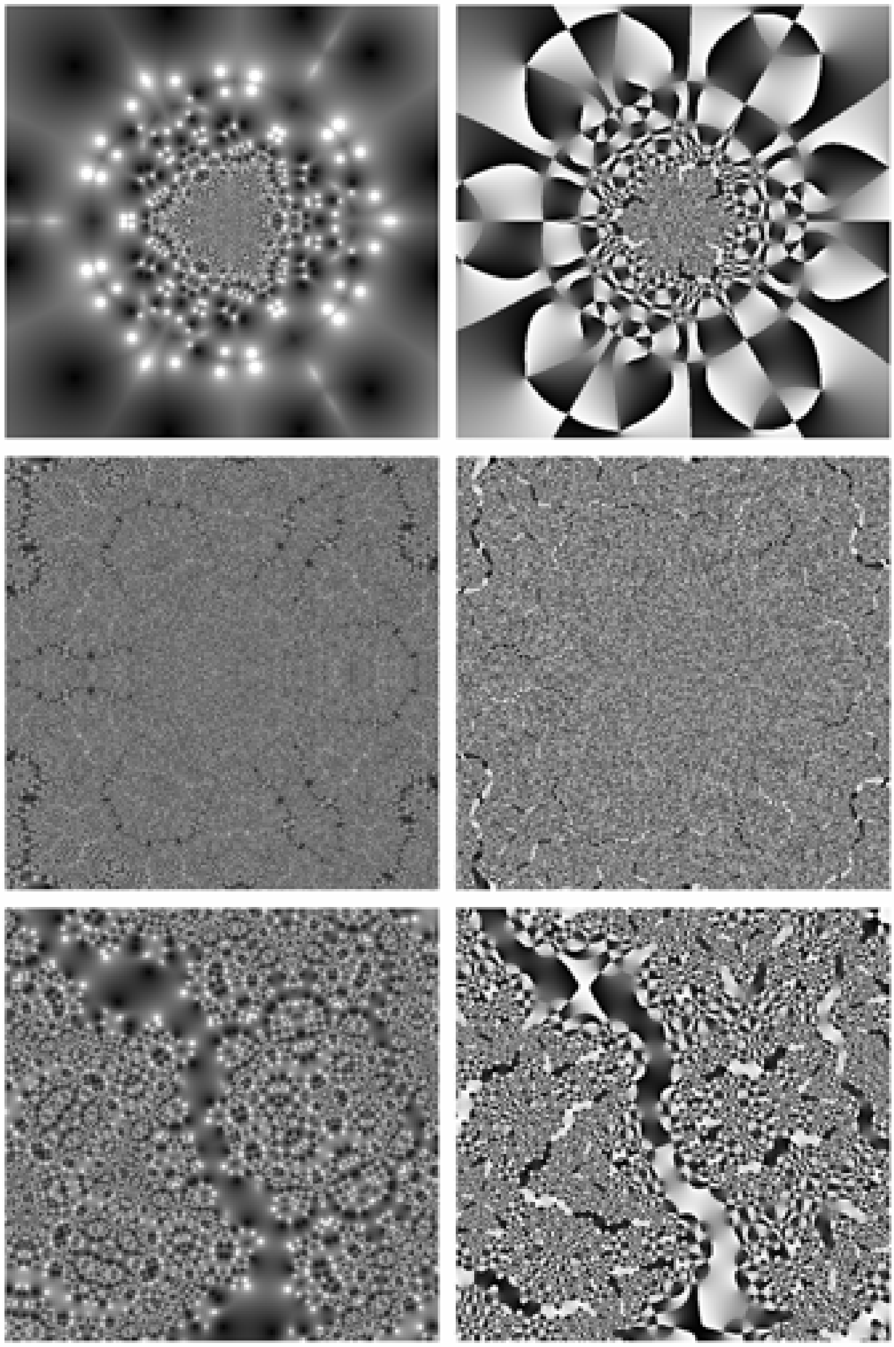


Figure 54 $\sqrt{(z-\frac{1}{10})(z+2)(z^5-1)^{-1}}$ (modulus and argument) for $n=13,23$ (top and middle) and $\alpha=30$. Bottom: detail of “tame rivers” for $n=23$ and $z\in(1+\imath)[20,30]$.

The iteration of functions whose critical points are prevalently imaginary, cfr. Figure 55 , features some interesting covering behaviour which is analoguous to another pretty “chaotic” phenomenon: **intermittency**. It is evident at sight that the Julia set in this case becomes disconnected in a curious way: sheets fold around branch points in a way such that each sheet is “rotated” with an “angle” which is greater than that of the previous sheet, so many of them actually become “invisible” from the first sheet (although they are reachable if trajectories sufficiently close to the branch point are chosen).

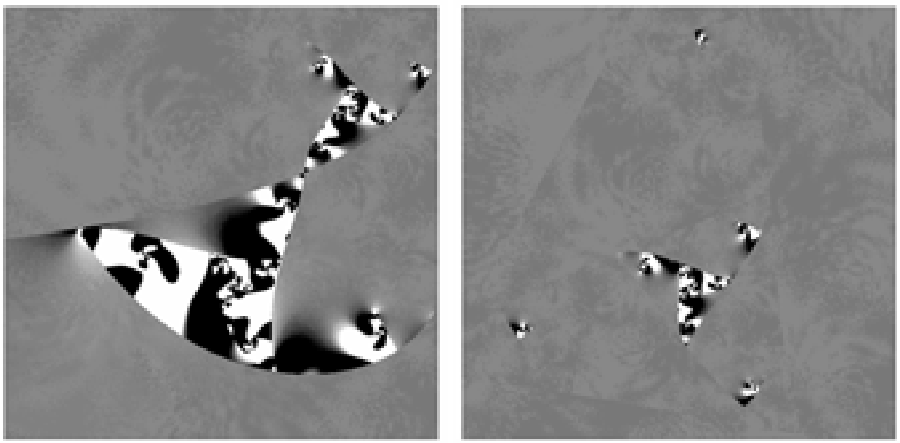


Figure 55 $\sqrt{(z+\imath)^2(z-1)^{-1}}$ Left: $n=17$ for $z\in[-2.25-3\imath, 1+0.25\imath]$. Right: $n=29$, detail for $z\in[-0.57-1.77\imath, -0.46-1.88\imath]$.

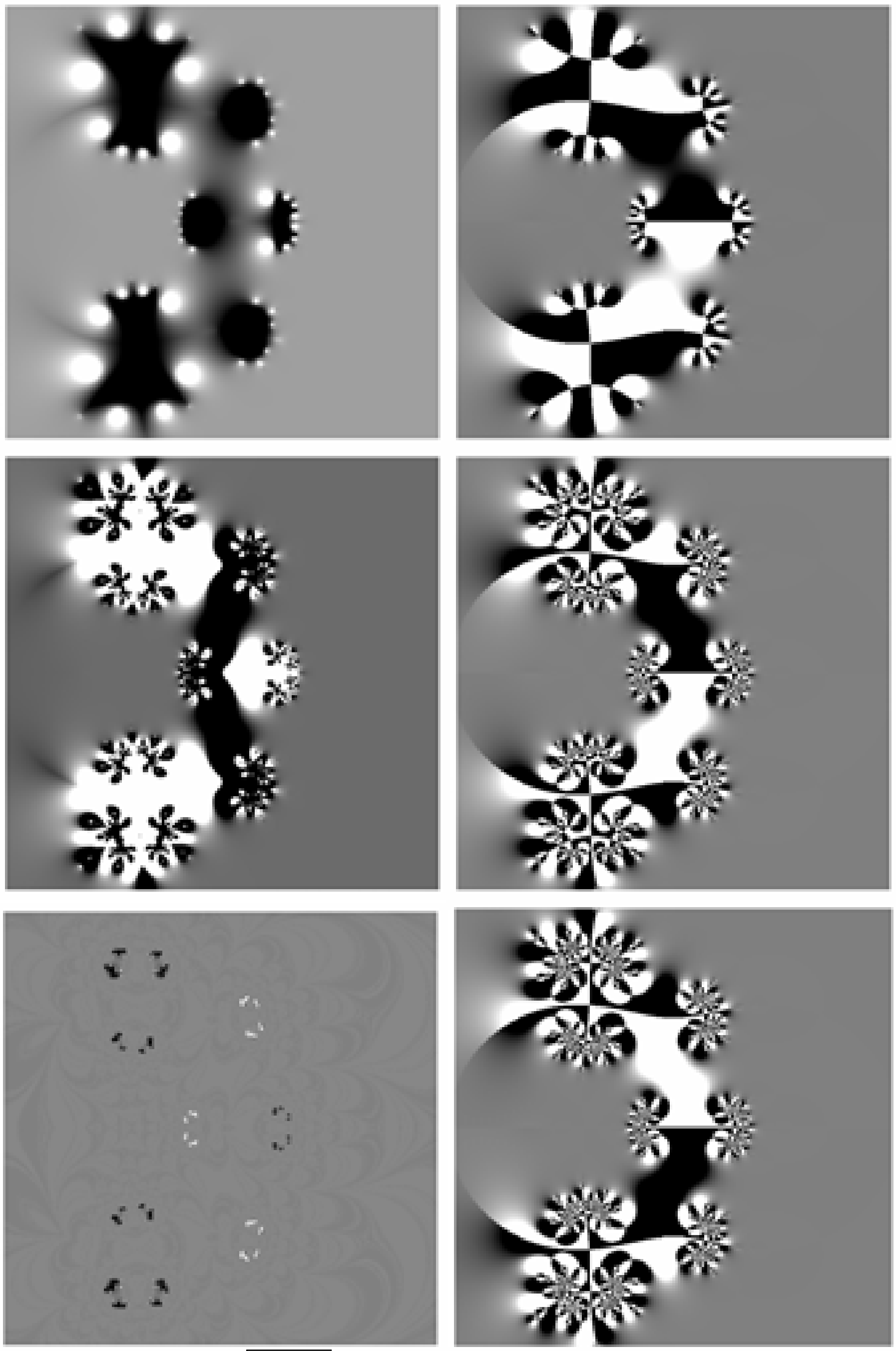


Figure 56 $\sqrt{\dfrac{z(z^2+1)}{z^2-1}}$ for n=7,13,32 and α=2.

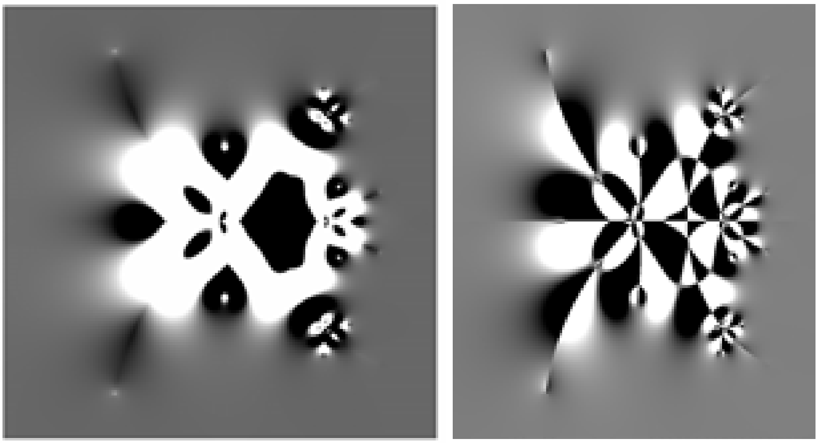


Figure 57 $\sqrt{\dfrac{z^2+1}{z-1}}$ in $[-3-2\imath, 1+2\imath]$ for $n=11$.

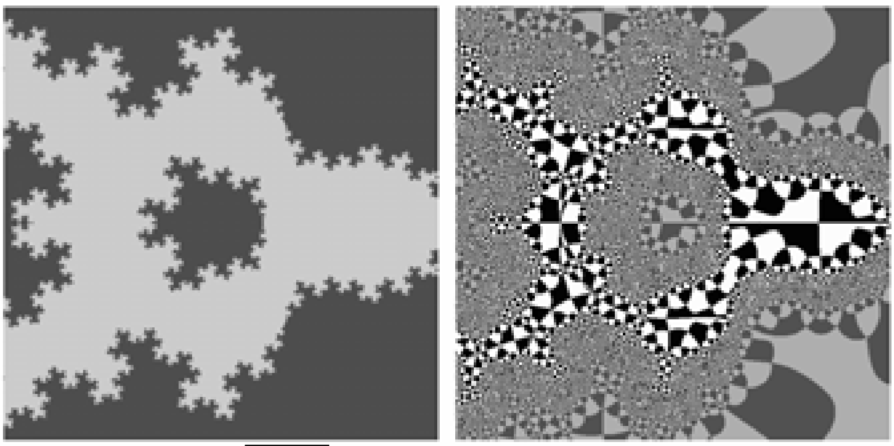


Figure 58 $\sqrt[3]{\dfrac{z^3+1}{z(z^3-1)}}$ for $n=32$, in $2[-1.6-\imath, 0.6+\imath]$.

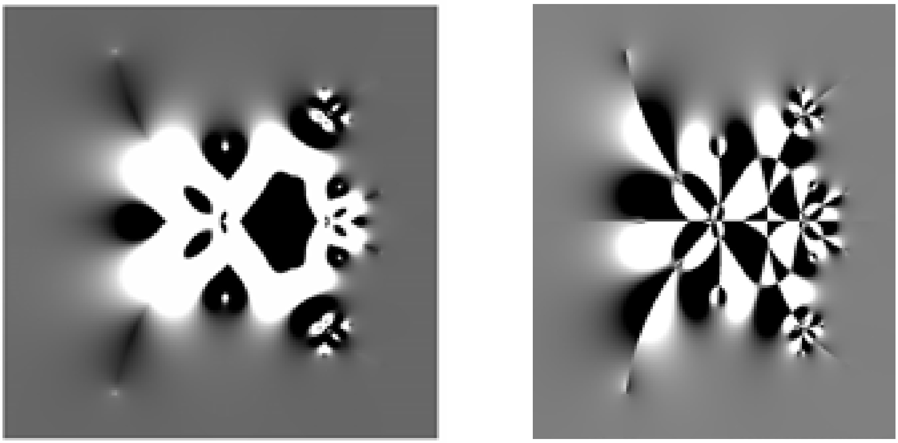


Figure 59 $\sqrt{\dfrac{z^2+1}{z-1}}$ in $[-3-2\imath, 1+2\imath]$ for $n=11$.

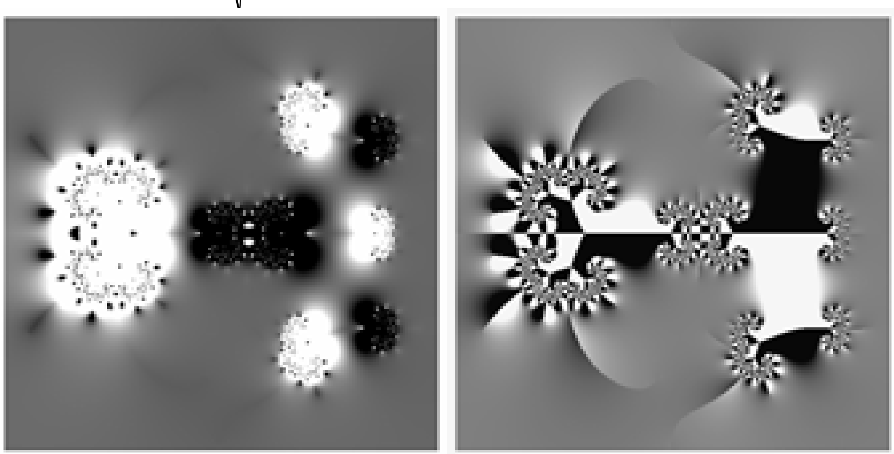


Figure 60 $\sqrt[5]{\dfrac{z^5+1}{z(z^5-1)}}$ for $n=23$, in $2[-1.6-\imath, 0.6+\imath]$.

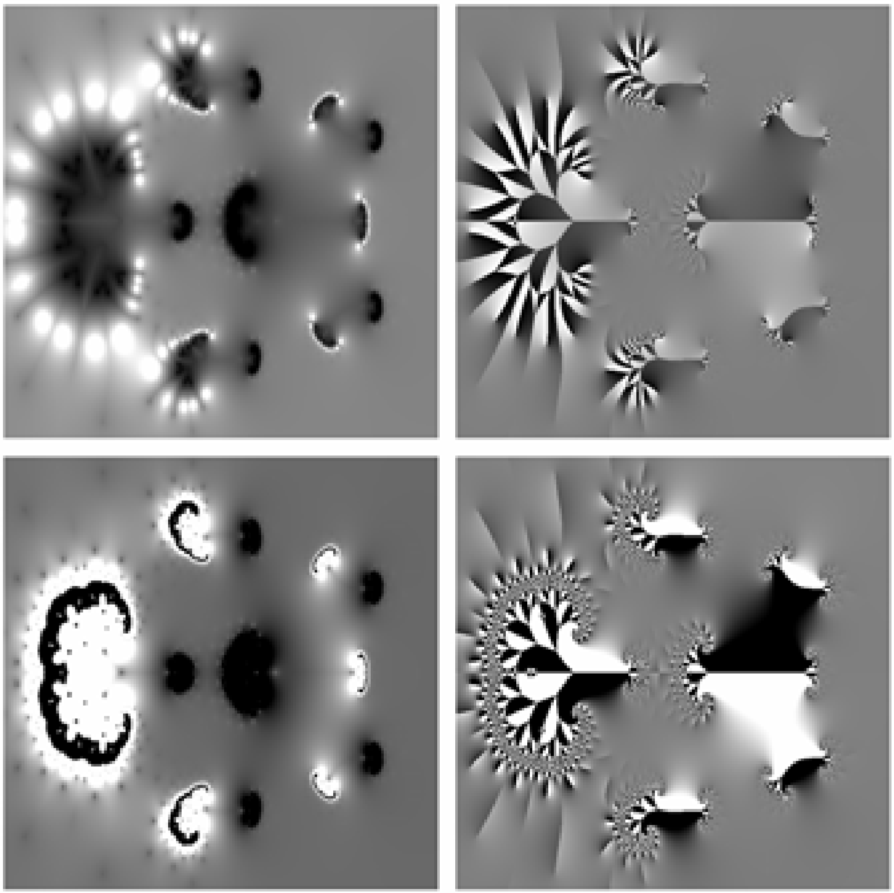


Figure 61 $\sqrt[5]{\dfrac{z(z^5+1)}{z^5-1}}$ for $n=11, 19, 23$ and $\alpha=1.5$.

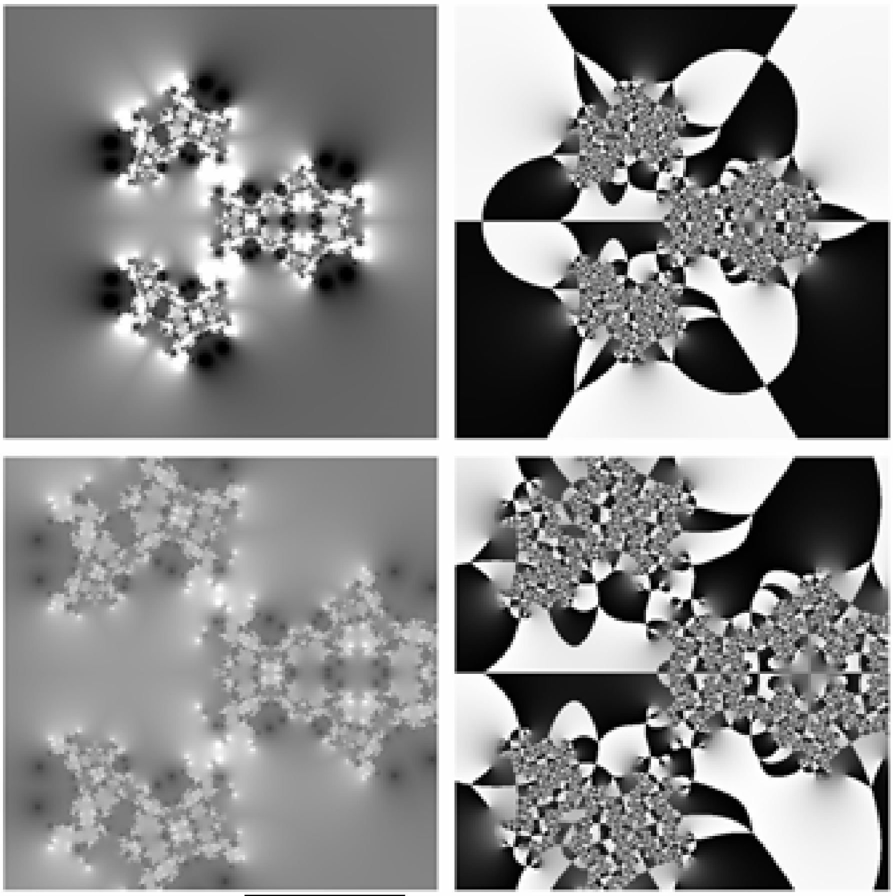


Figure 62 $\sqrt[5]{\dfrac{z(z^6+1)}{z^7-3z^4+2z-1}}$ for $n=11$, $\alpha=2.5, 1.5$.

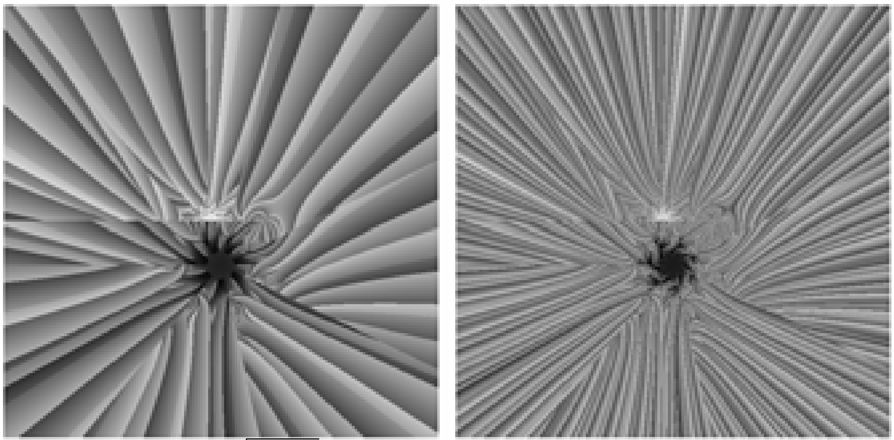


Figure 63 $\dfrac{\sqrt[3]{(z+\imath)^7}}{\sqrt{z-1}}$ for n=4,7 and α=5.

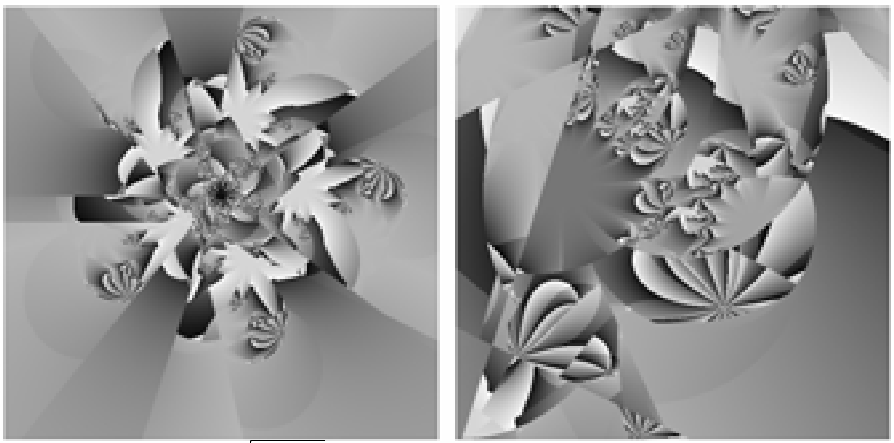


Figure 64 $\dfrac{\sqrt[11]{(z+1)^7}}{\sqrt{(z-\imath)^5}}$ for n=7 and α=8, 0.5.

4 Electromagnetics

> *πολλὰ δ’ἐϋπλοκάμου πολιῆς ἁλὸς ἐν πελάγεσσι*
> *θεσσάμενοι γλυκερὸν νόστον.*
>
> …Over n’over the white sea, amongst foam-curls,
> Grievin’ for a homecomin͜ the sweet’st͡.
>
> Archilochus, fragment №12 (Diehl)

> **3.** *Ἕνεκα γὰρ σώματος καὶ χρόνος καὶ τὸπος καὶ φύσις.* **[…]** *ἐπεί τοί γε τὸ σῶμα ἐδεῖτο τόπου,* **[…]** *εἶναι ἄευ ἁρμονίας.* **4.** *Ἕνεκα τοὶνυν τοῦ σώματος ἐστιν ὁ τόπος·* **[…] 5.** *Ἀσώματος τοὶνυν ὁ τόπος καὶ ὁ χρόνος καὶ ἡ φυσικὴ κίνησις.*
>
> 3. It is for the body that time and space and growth exist: […] No bodies could exist lacking any [covering] space, and […] be formed without harmony. 4. It is therefore due to body that space exists. […] 5. The space is thus incorporeal, as are time and natural movement.
>
> *Corpus Hermeticus*, Stobæus’ extracts, XVI

4.1 Exterior formulation

A formulation of Electromagnetics using the exterior algebra of differential forms is useful for formal reasons, above all when advanced geometric methods are to be used during the analysis of fields: from the computational point of view though, traditional and exterior formulations are completely equivalent.

Differential forms, mathematically introduced in [9]–§3.2, [36] and [88], as well as in §2.2.1, and applied to Electromagnetics in [9]–§4.4, [25] and [102]–[103], feature several pros and cons as far as representing a proper field theory is concerned:

- ▲ Exterior Algebra is more self-consistent and self-contained than Calculus, as far as analysis on vector fields is concerned. For example, Stoke's and Green's Theorems are natural Corollaries of one algebraic property of any homologic theory.
- ▲ There are just two (instead of three) 1st order differential operators with simple associative and (anti-)commutative rules, so less confustion than dealing with curl, gradient and divergence operators.
- ▲ Differential forms carry a greater physical coherence about the quantities they represent (instead of just scalar, vector and sometimes tensor fields): the integer order of a form carries information of the 'dimension' of the real medium required to directly, physically 'observe' that given quantity (at least in principle). In any $(3+1)$-dimensional Classical Field Theory, in fact:
    - 0-forms represent quantities measured ***on*** (or *between*) *points*, like scalar potentials;
    - 1-forms represent quantities measured ***along*** *paths*, like trajectories (speed, acceleration, …), forces, and linear densities;
    - 2-forms represent quantities measured ***across*** *surfaces*, like fluxes, velocities and surface densities;
    - 3-forms represent quantities measured ***within*** *volumes*, like volumetric densities (e.g. density of mass, charge, energy).
- ▼ Differential Geometry and Algebraic Topology are more advanced topics than Calculus on $\mathbb{R}^2$ or $\mathbb{R}^3$ and need a deeper mastering of exterior algebra,

topology, cohomology and operator theory, otherwise typical mistakes are on the way of reading or writing formulæ.

- Despite several field theories are naturally taugh in the exterior algebra's language even in basic courses (e.g. General Relativity and sometimes Quantum Mechanics), most other field theories are not widely known in this language, so expoloiting it for wide research interchange between several institutions used to other formalisms inevitably leads to another obstacle in communications.

4.1.1. Maxwell's equations

Quite general assumptions will be made regarding the nature of media involved in the setting of Maxwell's equation in the exterior algebra's language, but not the most general possible, to which other references are [9]–§4.4, [25] and [102]–[103]. As Electrodynamics of moving bodies is a relativistic field theory (converging to classical one in the slow-moving bodies' limit), the spacetime is the Minkowski 4-manifold $\mathbb{R}^4_3$, whose measure element is the 4-form $\mathrm{d}\mathbf{r}\wedge c\mathrm{d}t$. $\mathrm{d}\mathbf{r}$ is the volume element 3-form (i.e. $\mathrm{d}x\wedge\mathrm{d}y\wedge\mathrm{d}z$ in Cartesian coordinates) and $c:=\sqrt{\varepsilon_0\mu_0}^{-1}\simeq 2.99792458\cdot 10^8$ m/s the speed of light in vacuum.[109]

When the presence of different media is considered and they are neither still, homogeneous, isotropic nor lossless, $\mathrm{d}\mathbf{r}\wedge c\mathrm{d}t$ is replaced with different volume elements according to the electric-like or magnetic-like quantities to be measured: to this end, permittivity and permeability are defined as two different *metrics* in the spacetime, and two different Hodge operators $\star_\mathrm{e}$ and $\star_\mathrm{m}$ are defined (instead of the Hodge star operator $\star$ depending on the Minkowski metric, cfr. §2.1.2) for constitutive equations between the electric and magnetic fields' 1-forms

[109] $\varepsilon_0 \simeq 2.8542$ F/m and $\mu_0 \simeq 4\pi{\cdot}10^7$ H/m are the electric **permettivity** and **permeability** in vacuum, respectively.

$E,H\in\Lambda^1(\mathbb{R}^4_3)$ and their induction 2-forms: electric displacement and magnetic flux density $D,B\in\Lambda^2(\mathbb{R}^4_3)$:

$$\begin{cases} D = \star_{\mathrm{e}} E \\ B = \star_{\mathrm{m}} H \end{cases} \tag{1.380}$$

For example, in the case of a still, stationary, heterogeneous and anisotropic medium, the first of (1.380) can be rewritten in just the space Cartesian coordinates (x,y,z) as:

$$\begin{aligned} D \;=\; & \left(\varepsilon_{1,1}E_1 + \varepsilon_{1,2}E_2 + \varepsilon_{1,3}E_3\right)\mathrm{d}y\wedge\mathrm{d}z \\ + \; & \left(\varepsilon_{2,1}E_1 + \varepsilon_{2,2}E_2 + \varepsilon_{2,3}E_3\right)\mathrm{d}z\wedge\mathrm{d}x \\ + \; & \left(\varepsilon_{3,1}E_1 + \varepsilon_{3,2}E_2 + \varepsilon_{3,3}E_3\right)\mathrm{d}x\wedge\mathrm{d}y, \end{aligned} \tag{1.381}$$

with $E = E_1\mathrm{d}x + E_2\mathrm{d}y + E_3\mathrm{d}z$, $D = D_1\mathrm{d}y\wedge\mathrm{d}z + D_2\mathrm{d}z\wedge\mathrm{d}x + D_3\mathrm{d}x\wedge\mathrm{d}y$ and the functions $(\varepsilon_{i,j})_{1\leq i\leq j\leq 3}$ are the components of the symmetric, positive-definite permittivity 2-tensor $\underline{\varepsilon}(\mathbf{r})\equiv\underline{\varepsilon}(\mathbf{r})^{\mathsf{T}}$. This is the differential-form analogue of vector equation $\mathbf{D} = \underline{\varepsilon}:\mathbf{E}$ in the standard formalism.

Also note that, in the case of still media, the Minkowski spacetime (i.e. a pseudo-Euclidean metric), orthogonally changes into a flat (Euclidean) one: the Galilean metric $\mathrm{d}\mathbf{r}+\mathrm{d}t$. In this case the electrodynamic behaviour can be simply set into the "space" submanifold of metric $\mathrm{d}\mathbf{r}$ and the electromagnetic quantities' differential forms all lie in spaces $\Lambda^p(\mathbb{R}^3)$, $0\leq p\leq 3$.

When the medium is isotropic (but in general not stationary) all the permittivity and permeability components are all equal so, $\star_{\mathrm{e}}\equiv\varepsilon(\mathbf{r},\mathbf{r}';t,t')\star$ and $\star_{\mathrm{m}}\equiv\mu(\mathbf{r},\mathbf{r}';t,t')\star$; if the medium is stationary, then $\varepsilon(\mathbf{r},\mathbf{r}';t,t')\equiv\varepsilon(\mathbf{r}-\mathbf{r}',\mathbf{0};t-t',0)$, i.e. permittivity [and permeability as well] only depend from a source-point in the space-time and is indicated as $\varepsilon(\mathbf{r},t)$ [$\mu(\mathbf{r},t)$]. If the medium is also homogeneous, $\varepsilon(\mathbf{r})\equiv\varepsilon=\varepsilon_0\varepsilon_{\mathrm{r}}$ and $\mu(\mathbf{r})\equiv\mu=\mu_0\mu_{\mathrm{r}}$.

If, on the contrary, the medium has losses (or is dispersive in the spacetime) and/or is moving, (1.380) are convolutions of kind $D = \circledast_{\mathrm{e}} E$ and $B = \circledast_{\mathrm{m}} H$, where the effect of operator $\circledast$ on the general term $\varepsilon_{i,j} E_j$ in (1.381) is the convolution

$$\varepsilon_{i,j} \circledast E_j = \int_{\mathbf{R}_3^4} \varepsilon_{i,j}(\mathbf{r},t) E_j(\mathbf{r}-\mathbf{r}', t-t')\mathrm{d}\mathbf{r}'\mathrm{d}t'. \tag{1.382}$$

Under the light of this, the **Maxwell's equations** can be easily rewritten 'at sight' as:

$$\begin{cases} \mathrm{d}D = \varrho \\ \mathrm{d}B = \varrho_{\mathrm{m}} \end{cases} \qquad \begin{cases} \mathrm{d}E + \dot{B} = -J_{\mathrm{m}} \\ \mathrm{d}H - \dot{D} = J \end{cases} \tag{1.383}$$

where the over-dot indicates ordinary differentiation respect to time t, whereas '$_{\mathrm{m}}$' subscript denotes the magnetic "charge" and current densities added for renormalization (in this case, algebraic symmetry) of the equations. Uniqueness and existence Theorem for the solution to (1.383) is subordinate to the usual set of initial and boundary conditions [50]. By the Maxwell's I equation, the electric field's 1-form E [the electric displacement's 2-form D] must be exact, i.e $E \in \mathrm{B}^1(\Omega)$ [$D \in \mathrm{B}^2(\Omega)$] whereas, by the Maxwell's II equation, the magnetic field's 1-form H [the magnetic flux density's 2-form B] must be closed, i.e $H \in \mathrm{Z}^1(\Omega)$ [$B \in \mathrm{Z}^2(\Omega)$]: these cohomology facts will be further investigated in §4.1.3.

Continuity equations for the electric and magnetic currents[110] are

$$\begin{cases} \mathrm{d}J + \dot{\varrho} = 0 \\ \mathrm{d}J_{\mathrm{m}} + \dot{\varrho}_{\mathrm{m}} = 0 \end{cases} \tag{1.384}$$

whereas the electric and magnetic energy densities 3-forms are defined as

$$\begin{cases} v_{\mathrm{e}} := \dfrac{1}{2} E \wedge D = u_{\mathrm{e}} \mathrm{d}\mathbf{r} \\ v_{\mathrm{m}} := \dfrac{1}{2} H \wedge B = u_{\mathrm{m}} \mathrm{d}\mathbf{r} \end{cases} \tag{1.385}$$

[110] The electric [magnetic] charge 3-form is written as $\varrho = \rho\,\mathrm{d}\mathbf{r}$ [$\varrho_{\mathrm{m}} = \rho\,\mathrm{d}\mathbf{r}$].

The **Poynting** 2-**form**, i.e. the flux of electromagnetic radiated power density, is:

$$\Pi := E \wedge H.$$

Poynting's Theorem — Minkowski space–time domain

On every Lebesgue-measurable domain $\Omega \subseteq \mathbb{R}^3$:

$$\oint_{+\partial\Omega} \Pi + \frac{\partial}{\partial t}\int_{\Omega} (v_{\mathrm{e}} + v_{\mathrm{m}}) + \int_{\Omega} \left(E \wedge J + H \wedge J_{\mathrm{m}} \right) = 0. \quad (1.386)$$

$E \wedge J + H \wedge J_{\mathrm{m}}$ is the 3-form of the electromagnetic power density dissipated inside Ω due to matter-radiation interactions (e.g. phonon-photon inelastic collisions macroscopically generating, among others, the Joule effect [33]). The I-term surface integral —the Poynting flux— represents the electromagnetic power *radiated* from Ω outward.

It's worth nothing to note that the numerical value of electromagnetic energy densities (1.385) can also be obtained by computing either the canonical scalar product '$\langle\cdot|\cdot\rangle$' induced by the natural Hodge-$\star$ between the electric [magnetic] field and its induction field, or the field's norm '$\|\cdot\|_{\mathrm{e}}$' [or '$\|\cdot\|_{\mathrm{m}}$'] with respect to the inner product inherited by the electric [magnetic] Hodge operator, i.e. $u_{\mathrm{e}} = \star v_{\mathrm{e}}$ [$u_{\mathrm{m}} = \star v_{\mathrm{m}}$]:

$$\begin{aligned} u_{\mathrm{e}} &= \frac{1}{2}\langle E\,|\,D\rangle = \frac{1}{2}\sum_{i=1}^{3} E_i D_i = \frac{1}{2}\sum_{i,j=1}^{3} \mu_{i,j} E_i E_j \\ &= \frac{1}{2}\|E\|_{\mathrm{e}}^2 = \frac{1}{2}\langle E\,|\,E\rangle_{\mathrm{e}}\,; \end{aligned} \quad (1.387)$$

$$\begin{aligned} u_{\mathrm{m}} &= \frac{1}{2}\langle H\,|\,B\rangle = \frac{1}{2}\sum_{i=1}^{3} H_i B_i = \frac{1}{2}\sum_{i,j=1}^{3} \mu_{i,j} H_i H_j \\ &= \frac{1}{2}\|H\|_{\mathrm{m}}^2 = \frac{1}{2}\langle H\,|\,H\rangle_{\mathrm{m}}\,. \end{aligned} \quad (1.388)$$

All these facts strongly suggest that there is more than just a physical symmetry which will be investigated throughout §4.3.

It is well known that, supposing linear, static, lossless, and homogeneous (but, in general, *an*isotropic) media —and the absence of sources for simplicity's sake— the Maxwell's equations are equivalent to the following couple of wave equations

$$\begin{cases} \Delta E + \star_{\mathrm{m}}\star_{\mathrm{e}}\ddot{E} = 0 \\ \Delta H + \star_{\mathrm{m}}\star_{\mathrm{e}}\ddot{H} = 0 \end{cases} \tag{1.389}$$

where in this case the Laplace operator is $\Delta := \star\mathrm{d}\star\mathrm{d} + \mathrm{d}\star\mathrm{d}\star$. The operator '$\star_{\mathrm{m}}\star_{\mathrm{e}}$' is positive-definite and its inverse square-root operator $\sqrt{\star_{\mathrm{m}}\star_{\mathrm{e}}}^{-1}$, called the phase velocity operator, is well-defined with its three eigenvalues being the phase velocities along the metric's normal vector fields (i.e. the tangent bundle's basis). If the involved media are also isotropic and since $\star\star \equiv \star^2 \equiv \mathbb{1}$, then $c = 1/\sqrt{\varepsilon\mu}$ and (1.389) become:

$$\begin{cases} \Delta E + \dfrac{1}{c^2}\dfrac{\partial^2 E}{\partial t^2} = 0 \\ \Delta H + \dfrac{1}{c^2}\dfrac{\partial^2 H}{\partial t^2} = 0 \end{cases} \tag{1.390}$$

As far as a general formulation of electromagnetic problems is concerned, all the considered physical fields are of temporary nature, in the sense that they may be thought to persist —and having persisted— for a finite time, though of arbitrary length; for this reason, they behave like compact-supported *energy* signals. The physical quantities in the real space-time also have to be *smooth*, in the sense that they should be modelled by C^∞-class functions, although it is extremely more easy to consider *occasional* (i.e. almost nowhere) discontinuities in correspondence, for example, of medium interfaces or sudden components' and feeds' state changes (e.g. current/voltage AC/DC generators being either switched ON/OFF, or nonlinear devices changing their functioning regime).[111] For this reason, the most general and suitable function spaces for electromagnetic problems are the real-valued, square-integrable

[111] For example electronic components such as diodes and transistors, whose 1st- and 2nd-order approximated models are described by picewise-differentiable functions.

Lebesgue space $\mathrm{L}^2_{\mathbb{R}}(\mathbb{R}^4_3)$ and its weakly-differentiable counterparts — the real-valued Hilbert-Sobolev spaces $H^k_{\mathbb{R}}(\mathbb{R}^4_3)$, where usually $k=2$ as just II-order derivatives are needed for the linear PDEs derived from (1.383), [8].

When electromagnetic equations are set into such "weak" formulation, boundary conditions on a two-dimensional domains $\mathcal{S}\subset\mathbb{R}^3$ (e.g. quantities like surface densities or field components on $\mathcal{S}$) are imposed to belong to the Hardy *trace*-spaces $H^{k-\frac{1}{2}}_{\mathbb{R}}(\mathcal{S})$ since $\mathcal{S}$ is always a set of zero "three-dimensional" measure.

As far as the time-dependence is concerned (especially in the case of static media, so that space-time becomes Galileian: $\mathbb{R}^3\oplus\mathbb{R}$), functions are also integrable and compactly supported in $\mathrm{L}^1_{\mathrm{loc}}(\mathbb{R})$ with respect to time t, in order for the Laplace transform introduced in §1.2.5 to be applicable, if necessary.

4.1.2. Electromagnetics in the frequency domain

From final considerations in the previous Chapter it is clear that the electromagnetic fields can always be Fourier-transformed with respect to any space-time metric or Laplace-transformed with respect to time.

Let $\phi\in\mathrm{L}^p_2(\mathbb{R}^3)\otimes\mathrm{L}^2_{\mathrm{loc}}(\mathbb{R})$ be a square-integrable p-form $(0\leq p\leq 3)$ with respect to space coordinates $\mathbf{r}\in\mathbb{R}^3$, simultaneously locally-square-integrable function of time $t\in\mathbb{R}$, representing some physical quantity in the Galileian space-time $\mathbb{R}^3\oplus\mathbb{R}$. The Pontryagin's characters for the space-time Fourier transform (cfr. §1.2.4) is

$$\exp\left[\imath(\boldsymbol{\kappa}\cdot\mathbf{r}-\omega t)\right],\qquad \forall(\mathbf{r},t),(\boldsymbol{\kappa},\omega)\in\mathbb{R}^3\oplus\mathbb{R},$$

which can be trivially factored into the product of the space-like character $e^{\imath\boldsymbol{\kappa}\cdot\mathbf{r}}$ and the time-like character $e^{-\imath\omega t}$ (which confirms direct-sum decomposition of Pontryagin's dual spaces: $\widehat{\mathbb{R}^3\oplus\mathbb{R}}=\widehat{\mathbb{R}^3}\oplus\widehat{\mathbb{R}}=\mathbb{R}^4$). Conventionally, Fourier transforms respect to time coordinates $t\hat{\leftrightarrow}\omega$ will be indicated as $\hat{\phi}$, whereas whose with respect to *both* space and time coordinates ($\mathbf{r}\hat{\leftrightarrow}\boldsymbol{\kappa}$ too) as $\hat{\hat{\phi}}$.

For most uses in applied Electromagnetism such transforms provide a useful physical interpretation: the value of $\hat{\phi}$ at a given angular velocity $\omega = \pm 2\pi\nu \in \mathbb{R}$ represents the monochromatic field component (also called a plane-wave spectrum) at electromagnetic (photonic) frequency ν. Equations (1.380)–(1.384),(1.390) are rewritten as:

$$\begin{cases} \hat{D} = \star_{\mathrm{e}} \hat{E} \\ \hat{B} = \star_{\mathrm{m}} \hat{H} \end{cases} \tag{1.391}$$

$$\hat{\varepsilon}_{i,j} \circledast \hat{E}_j = \int_{\mathbb{R}^3} \hat{\varepsilon}_{i,j}(\mathbf{r},\omega)\hat{E}_j(\mathbf{r}-\mathbf{r}',\omega)\mathrm{d}\mathbf{r}' \tag{1.392}$$

$$\begin{cases} \mathrm{d}\hat{D} = \hat{\varrho} \\ \mathrm{d}\hat{B} = \hat{\varrho}_{\mathrm{m}} \end{cases} \qquad \begin{cases} \mathrm{d}\hat{E} + \imath\omega\hat{B} = -\hat{J}_{\mathrm{m}} \\ \mathrm{d}\hat{H} - \imath\omega\hat{D} = \hat{J} \end{cases} \tag{1.393}$$

$$\begin{cases} \mathrm{d}\hat{J} + \imath\omega\hat{\varrho} = 0 \\ \mathrm{d}\hat{J}_{\mathrm{m}} + \imath\omega\hat{\varrho}_{\mathrm{m}} = 0 \end{cases} \tag{1.394}$$

$$\begin{cases} \Delta\hat{E} - \omega^2 \star_{\mathrm{m}} \star_{\mathrm{e}} \hat{E} = -\imath\omega \star_{\mathrm{m}} \hat{J}_{\mathrm{i}} - \star\mathrm{d}\star\hat{J}_{\mathrm{m}} \\ \Delta\hat{H} - \omega^2 \star_{\mathrm{m}} \star_{\mathrm{e}} \hat{H} = -\imath\omega \star_{\mathrm{e}} \hat{J}_{\mathrm{i}} + \star\mathrm{d}\star\hat{J}_{\mathrm{i}} \end{cases} \tag{1.395}$$

The value of $\hat{\hat{\phi}}$ at a given "**wave vector**" $\boldsymbol{\kappa} \equiv (\kappa_1, \kappa_2, \kappa_3) \in \mathbb{R}^3$ (and velocity $\omega \in \mathbb{R}$) represents the component of the field with spatial periodicity given by $\boldsymbol{\kappa}$, i.e. the component which in a spectral decomposition produce a periodic field over the lattice $\mathbb{R}^3 / \boldsymbol{\kappa}\mathbb{R}^3$ (whose elementary period-cell is the parallelogram of side lengths κ_1, κ_2 and κ_3) — all at (photonic) frequency ν.

The Poynting form and the electromagnetic energy densities (1.385) are naturally computed in the frequency domain using formula (1.396) instead of (1.387)–(1.388), where scalar product is replaced by the hermitian product:

$$\hat{\Pi} := \frac{1}{2}\hat{E}\wedge\overline{\hat{H}}, \qquad \begin{cases} \hat{v}_{\mathrm{e}} := \frac{1}{4}\hat{E}\wedge\overline{\hat{D}} = \hat{u}_{\mathrm{e}}\mathrm{d}\mathbf{r} \\ \hat{v}_{\mathrm{m}} := \frac{1}{4}\hat{H}\wedge\overline{\hat{B}} = \hat{u}_{\mathrm{m}}\mathrm{d}\mathbf{r} \end{cases} \tag{1.396}$$

Inner products in the frequency domain are slightly different due to both the rescaling factor ½ and the use of complex-conjugated forms on the right (the product is in fact *hermitian* rather than simply symmetric): this is 'physically' consistent with the fact that, in case of an electromagnetic monochromatic regime at velocity ω (which also means considering *permanent regime* conditions only), the inverse transform of the generic form $\hat{\phi}$ is simply $\phi(\mathbf{r},t)\equiv 2\,\mathrm{Re}(e^{\imath\omega t}\phi(\mathbf{r}))$.

In the case of a homogeneous and isotropic medium, a wave number $\kappa(\omega) := \omega\sqrt{\varepsilon(\omega)\mu(\omega)}$ can be defined such that (1.395) become Helmholtz's equations:

$$\begin{cases} \Delta\hat{E} + \kappa^2\hat{E} = 0 \\ \Delta\hat{H} + \kappa^2\hat{H} = 0 \end{cases} \tag{1.397}$$

We also want to stress that, from the quantum-mechanical point of view [33]–[34], for any single (ideal) plane-wave of wave vector $\boldsymbol{\kappa}$ —i.e. for any single photon of energy[112] $h\nu$— the momentum $\hbar\boldsymbol{\kappa}$ is the semi-classically canonically-conjugate coordinate to the photon's position vector $\mathbf{r}$ (although in Quantum Mechanics they are probabilistic compact self-adjoint operators rather than deterministic vectors).

In the case of linear, static, lossless homogeneous and isotropic media and supposing no magnetic-current sources are present ($J_{\mathrm{m}}\equiv 0$) one has in general to account for impressed electric currents due to real sources/feeds (J_{i} and ϱ_{i}), i.e. replacing J with $J+J_{\mathrm{i}}$ (and ϱ with $\varrho+\varrho_{\mathrm{i}}$) in (1.383). Currents represented by J are due to the media's conducibility, which is another Hodge operator $\star_{\mathrm{c}}$ such that $J = \star_{\mathrm{c}}E$, like per (1.380); in the case of isotropic media $\star_{\mathrm{s}}\equiv\sigma(\mathbf{r},\mathbf{r}';t,t')\star$ and the same considerations done in

[112] $h \simeq 6.626184\cdot 10^{-34}$ J·s is the Planck's constant, whereas $\hbar := h/2\pi$.

§4.1.1 for both permittivity and permeability hold: in the simplest case of linear, still, stationary, homogeneous and isotropic media $\star_{\mathrm{c}} \equiv \sigma\star$.

Passing to the frequency domain (and supposing non-time-stationary media, so that their electromagnetic characteristics vary with the photonic frequency) the **equivalent** dielectric constant $\hat{\varepsilon}_{\mathrm{c}}(\omega) \overset{\mathrm{def}}{=} \hat{\varepsilon}(\omega) - \imath\omega^{-1}\hat{\sigma}(\omega)$ is defined; Maxwell's III and IV equations are rewritten incorporating constitutive equations (1.391) as

$$\begin{cases} \mathrm{d}\hat{E} + \imath\omega\hat{\mu}\star\hat{H} = 0 \\ \mathrm{d}\hat{H} - \imath\omega\hat{\varepsilon}_{\mathrm{c}}\star\hat{E} = \hat{J}_{\mathrm{i}} \end{cases} \tag{1.398}$$

which are equivalent to (1.393), whereas wave equations (1.395) are derived from them.

The Pontryagin's duality '$\widehat{\leftrightarrow}$' respect to space and time metrics is reflected by the algebra of their derivations, since differential operators get replaced by the following algebraic operators[113]

$$\left.\begin{matrix} \mathrm{d} \\ \dfrac{\partial}{\partial t} \end{matrix}\right\} \quad \widehat{\longleftrightarrow} \quad \begin{cases} \boldsymbol{\kappa}\mathbb{1}\wedge\mathrm{d}\boldsymbol{\kappa} \\ \imath\omega\mathbb{1} \end{cases} \tag{1.399}$$

with the first operator written in Cartesian coordinates:

$$\mathrm{d} = \frac{\partial}{\partial x}\mathrm{d}x\wedge\mathbb{1} + \frac{\partial}{\partial y}\mathrm{d}y\wedge\mathbb{1} + \frac{\partial}{\partial z}\mathrm{d}z\wedge\mathbb{1} \quad \widehat{\longleftrightarrow} \quad \imath\sum_{j=1}^{3}\kappa_j\mathbb{1}\wedge\mathrm{d}\kappa_j. \tag{1.400}$$

In this case, (1.382), (1.383), (1.390) become:

$$\hat{\hat{\varepsilon}}_{i,j}(\boldsymbol{\kappa},\omega)\hat{\hat{E}}_j(\boldsymbol{\kappa},\omega) \tag{1.401}$$

[113] Formalisms $\mathbf{p}\wedge\mathbf{q} := \sum_j p_j \wedge q_j$ and $\mathbf{pq} := \sum_j p_j q_j$ for two differential forms' n-tuplets or multivectors $\mathbf{p} \equiv (p_1, p_2, \ldots, p_n)$ and $\mathbf{q} \equiv (q_1, q_2, \ldots, q_n)$ are used, which are common in Classical Mechanics [8]).

$$\begin{cases} \boldsymbol{\kappa}\hat{\hat{D}}\wedge\mathrm{d}\boldsymbol{\kappa} = \hat{\hat{\varrho}} \\ \boldsymbol{\kappa}\hat{\hat{B}}\wedge\mathrm{d}\boldsymbol{\kappa} = \hat{\hat{\varrho}}_{\mathrm{m}} \end{cases} \qquad \begin{cases} \boldsymbol{\kappa}\hat{\hat{E}}\wedge\mathrm{d}\boldsymbol{\kappa} + \imath\omega\hat{\hat{B}} = -\hat{\hat{J}}_{\mathrm{m}} \\ \boldsymbol{\kappa}\hat{\hat{H}}\wedge\mathrm{d}\boldsymbol{\kappa} - \imath\omega\hat{\hat{D}} = \hat{\hat{J}} \end{cases} \tag{1.402}$$

$$\begin{cases} \left(\|\boldsymbol{\kappa}\|^2 - \kappa^2\right)\hat{\hat{E}} = 0 \\ \left(\|\boldsymbol{\kappa}\|^2 - \kappa^2\right)\hat{\hat{H}} = 0 \end{cases} \tag{1.403}$$

From (1.403) it is clear that the "full-spectral" electromagnetic waves are described by the kernel of the above linear operator, i.e. $\mathrm{Ker}(\boldsymbol{\kappa}\otimes\boldsymbol{\kappa} - \kappa^2\mathbb{1})$, i.e. by the eigenspaces of $\boldsymbol{\kappa}\otimes\boldsymbol{\kappa}$ (which is trivially hermitian and positive-definite, but in general non-compact). More on this subject can be found in §4.2.

Another way to solve Maxwell's equations is to resort to a spectral representation in the spatial coordinates only. In this case, let $\overleftrightarrow{\phi}$ be the corresponding Fourier transform

$$\overleftrightarrow{\phi}(\boldsymbol{\kappa}, t) := \int_{\mathbf{R}^3} e^{\imath\boldsymbol{\kappa}\cdot\mathbf{r}}\phi(\mathbf{r}, t)\mathrm{d}\mathbf{r},$$

then (1.383), or (1.402), become:

$$\begin{cases} \boldsymbol{\kappa}\overleftrightarrow{D}\wedge\mathrm{d}\boldsymbol{\kappa} = \overleftrightarrow{\varrho} \\ \boldsymbol{\kappa}\overleftrightarrow{B}\wedge\mathrm{d}\boldsymbol{\kappa} = \overleftrightarrow{\varrho}_{\mathrm{m}} \end{cases} \qquad \begin{cases} \boldsymbol{\kappa}\overleftrightarrow{E}\wedge\mathrm{d}\boldsymbol{\kappa} + \dot{\overleftrightarrow{B}} = -\overleftrightarrow{J}_{\mathrm{m}} \\ \boldsymbol{\kappa}\overleftrightarrow{H}\wedge\mathrm{d}\boldsymbol{\kappa} - \dot{\overleftrightarrow{D}} = \overleftrightarrow{J} \end{cases}$$

Latter formalism is "less practical" for several reasons, since most analysis and synthesis problems in Applied Electromagnetism are related with the time-frequency characteristics of radiating structures at and around certain operating frequencies and bandwidths. Some numerical methods in Electromagnetism rely instead on the time-domain behaviour. Furthermore this formalism is better suited for certain problems to be solved via differential equations, which can be treated and solved by analytically continuing the solution with respect to a complexified time variable $\tau\in\mathbb{C}$ such that $\mathrm{Re}\,\tau = t$, thus resorting to a (system of) ODE(s) on Riemann surfaces, like those introduced in §2.4.6. These problems will be treated in §5.3.1, where the fractal Riemann surfaces studied throughout §3 will briefly turn back in.

4.1.3. Electrodynamic potentials

Considering the cohomologic features of Maxwell's equations (1.398) in a Lebesgue-measurable three-dimensional domain $\Omega\subseteq\mathbb{R}^3$, let

$$\begin{cases} \hat{H} = \star \mathrm{d}\hat{A} \\ \hat{E} = -\mathrm{d}V - \imath\omega\hat{\mu}\hat{A} \end{cases} \tag{1.404}$$

with above differential forms $A\in\mathrm{L}_2^1(\Omega)\otimes\mathrm{L}^2(\mathbb{R})$ and $V\in\mathrm{L}_2^0(\Omega)\otimes\mathrm{L}^2(\mathbb{R})\cong\mathrm{L}^2(\Omega\times\mathbb{R})$. Different choices are possible for the electrodynamic potentials (A,V) since $\forall\phi\in\mathrm{L}^2(\Omega\times\mathbb{R})$ the couple $(A+\mathrm{d}\phi,V+\phi)$ still satisfies (1.404) for the same electromagnetic field (E,H), so the space of equivalence classes of electrodynamic potentials $([\![A]\!],[\![V]\!])$ is a topological invariant for the electromagnetic problem called —in the Field Theory of physicists— a *gauge*; the gauge is basically a choice of cohomology classes, since[114] $[\![A]\!]\in\mathrm{H}^1(\Omega\times\mathbb{R})$ and $[\![V]\!]\in\mathrm{H}^0(\Omega\times\mathbb{R})$. The arbitrary choice of a gauge field ϕ allows to impose an arbitrary functional relation between 1-form A an 0-form V such that the number of equations in computing the electromagnetic field is lowered: in applied Electromagnetics a useful choice is the one of the ***Lorenz* gauge**

$$\star\mathrm{d}\star\hat{A} + \imath\omega\hat{\varepsilon}_{\mathrm{c}}\hat{V} = 0, \tag{1.405}$$

thanks to which, substituting (1.404) into (1.398), the Helmholtz's equation is obtained:

$$\begin{cases} \Delta A + \kappa^2 A = -J_{\mathrm{i}} \\ A\big|_{\partial\Omega} = A_0 \end{cases} \qquad \begin{cases} A\in H_2^1(\Omega) \\ A_0\in H_2^{\frac{1}{2}}(\partial\Omega) \end{cases} \tag{1.406}$$

[114] By DeRham's Theorem —whose complexified version is (1.199)— the 2nd- and 3rd-order Cohomology spaces are isomorphic to the 1st- and 0th-order ones, respectively.

or, in case the media inside Ω are either heterogeneous and anisotropic,

$$\Delta A + \omega^2 \star_{\mathrm{m}} \star_{\mathrm{e}} A = -J_{\mathrm{i}}. \tag{1.407}$$

The electric and magnetic field's 1-forms are computed back from (1.406) as:

$$\begin{cases} \hat{E} = \dfrac{\mathrm{d}\star\mathrm{d}\star A}{\imath\omega\hat{\varepsilon}_{\mathrm{c}}} - \imath\omega\hat{\mu}\hat{A} \\ \hat{H} = \star\mathrm{d}\hat{A} \end{cases} \tag{1.408}$$

Whenever static fields are considered ($\omega = 0$), Electrostatics and Magnetostatics become uncoupled problems (as well as the former couple of Maxwell's equations becomes independent on the latter couple); this also corresponds in (1.407) to the $\kappa \equiv 0$ case which the Helmholtz's equations become a Laplace equation for:

$$\Delta A = 0. \tag{1.409}$$

The cohomology space for the electrodynamic potentials is thus the space of harmonic differential 1-forms in Ω, $[\![A]\!] \in \mathrm{H}^1(\Omega)$ and there are $\beta_1(\Omega)$ independent solutions for both the electrostatic and magnetostatic field, as many as the genus g of Ω; this fact agrees with $\dim_{\mathbb{Z}} \mathrm{H}^1(\Omega) = 2g$.

By the Hodge Decomposition Theorem (real case, [9]–§3.7) the electromagnetic 1-forms are both subject to an orthogonal decomposition such that each of these two fields depends on three 'potential-like' forms: for the electric field's form E, they are the 'scalar' potential V, the 'vector potential' A and an harmonic 1-form's cohomology class $[\![\phi]\!] \in \mathrm{H}^1(\Omega)$ depending on the domain's topology (and which, in the electrostatic case (1.409) is the only left by such a decomposition).

4.2 Green's function theory for Electromagnetics

The Hemlholtz's equation, for example (1.407) for the electrodynamic potential, is solved in Functional Analysis by computing the resolvent of the Helmholtz's operator '$\Delta+\omega^2\star_{\mathrm{m}}\star_{\mathrm{e}}$', which is a normal operator and is also both self-adjoint and non-positive definite if the media inside Ω are lossless (i.e. $\star_{\mathrm{c}}=\star_{\mathrm{e}}$ or $\varepsilon_{\mathrm{c}}(\omega)\equiv\varepsilon(\omega)$) and whose inverse is compact if Ω is bounded. Considering a monochromatic regime (i.e. $\omega\in\mathbb{R}_+$ fixed) and $\hat{A},\hat{J}_{\mathrm{i}}\in H_2^p(\Omega)$ ($p=1,2$ respectively), the resolvent operator is formally $G_\lambda:=(\Delta+\omega^2\star_{\mathrm{m}}\star_{\mathrm{e}}-\lambda\mathbb{1})^{-1}$ such that

$$G_0\hat{J}_{\mathrm{i}} = \hat{A}. \tag{1.410}$$

In the next Chapter a few facts about the p-Laplacian Δ_p of a $(d+1)$-dimensional classical field theory, which the Laplace operator of the electromagnetic problem constitutes the $(d,p)=(3,1)$ case for, will be summarized and generalized to a II-order differential operator. This background theory will be applied to derive the general properties of the resolvent in both the discrete (guided waves) and continuous spectrum (electromagnetic radiation) cases. For further references throughout the whole Chapter, see [47], [58], [63] [43], [63], [9]–§§B.4, 5 and [35] where not indicated otherwise.

4.2.1. General properties and the discrete spectrum

Let $\Omega\subset\mathbb{R}^d$ be a sufficiently regular bounded (i.e. compact) d-dimensional domain and $F\in\mathfrak{n}_{\mathbb{C}}(H_2^p(\Omega))$ be a linear normal operator with compact inverse and with particularly 'well-behaved' boundary conditions (e.g. $H_2^p(\Omega)$ is separable so that F is orthogonally diagonalizable in $\mathbb{C}$); let us eventually consider the self-adjoint and sign-definite case (e.g. F orthogonally diagonalizable in $\mathbb{R}_\pm$) as a special variant — motivated by electromagnetic assumptions in §4.1.1).

This means that F has a discrete (sign-definite, e.g. positive) spectrum and finite-dimensional eigenspaces: $\mathrm{Spec}F\equiv\mathrm{spec}F=\{\lambda_n\}_{n\in\mathbb{N}_0}$, $\lim_n\lambda_n=\pm\infty$.

By the Spectral Theorem of compact normal operators:

$$H_2^p(\Omega) = \overline{\bigoplus_{n=0}^{\infty} \mathrm{Ker}(F - \lambda_n \mathbb{1})}, \tag{1.411}$$

that is $H_2^p(\Omega)$ admits a complete orthogonal basis of p-eigenforms of F, relative to complex eigenvalues $\{\lambda_n\}_{\mathbb{N}_0}$. Let us consider the operator

$$G_\lambda : H_2^p(\Omega) \to H_2^p(\Omega)\,, \qquad \forall \lambda \in \mathbb{C}:$$

$$F\varphi = \eta \qquad \Leftrightarrow \qquad G_0\eta = \varphi. \tag{1.412}$$

From tensor Algebra [9]–§A.4 we know that G_λ is a 2-tensor of $H_2^p(\Omega)^{\otimes 2}_{,}$ then it can be written as a "**double** p-form," respect to any conformal metric $\mathbf{x} \equiv (x_1, x_2, \ldots, x_d) \in \Omega$ (cfr. [9]–§3.3):

$$\underline{G}_\lambda(\mathbf{x}, \mathbf{x}'; \lambda) = \sum_{\substack{1 \le i_1 < i_2 < \ldots < i_p \le d, \\ 1 \le j_1 < j_2 < \ldots < j_p \le d}} G^{i_1, i_2, \ldots, i_p}_{j_1, j_2, \ldots, j_p}(\mathbf{x}, \mathbf{x}'; \lambda) \bigwedge_{h,k=1}^{p} (\mathrm{d}x_{i_h} \otimes \mathrm{d}x'_{j_k}) \tag{1.413}$$

(where $\mathbf{x}' \in \Omega$ is an isomorphic copy of the same metric). By the Steinberg's Theorem, [58], the $\binom{d}{p}^2$ coefficients of (1.413) have II-order weak derivatives with respect to both coordinate systems in Ω^2 and are meromorphic functions of $\lambda \in \mathbb{C}$, that is $G^{i_1, i_2, \ldots, i_p}_{j_1, j_2, \ldots, j_p} \in H^2(\Omega^2) \otimes \mathcal{K}(\mathbb{C})$ for $1 \le i_1, j_1 < i_2, j_2 < \ldots < i_p, j_p \le d$. In particular G_λ is a meromorphic tensor with respect to λ, whose polar singularities are the eigenvalues of F and whose n^{th} holomorphic derivatives, $\forall n \in \mathbb{N}$, are

$$\frac{\mathrm{d}^n G_\lambda}{\mathrm{d}\lambda^n} = n!\, G_\lambda^{n+1} = n!(F - \lambda \mathbb{1})^{-n-1}. \tag{1.414}$$

Recalling that the inverse of a linear operator $T \in \mathfrak{GL}_{\mathbb{C}}(\mathbf{B})$ on a Banach space $\mathbf{B}_{\mathbb{C}}$ can be written as the following power series of T's iterates

$$T^{-1} = \sum_{n=0}^{\infty} T^n \in \mathbb{C}[\![\mathfrak{GL}(\mathbf{B})]\!],$$

any time that $T\in\mathbb{B}_1(\mathbb{0})$ (i.e. $\|T\|<1$, with $\mathbb{0}$ being the null operator) then (1.414) provides the operator-analogue of MacLaurin series expansion for F (provided $0\notin\operatorname{spec}F$):

$$F = \sum_{n=0}^{\infty}\frac{1}{n!}\frac{\mathrm{d}^n G_\lambda}{\mathrm{d}\lambda^n}\bigg|_{\lambda=0}. \tag{1.415}$$

Such a resolvent's representation is called a ***Green*'s function** for the Helmholtz's equation/problem[115]

$$G_\lambda\eta = \varphi \tag{1.416}$$

and its functional setting is:

$$G_\lambda = (F-\lambda\mathbb{1})^{-1} \in H_2^p(\Omega)^{\otimes 2}\otimes\mathcal{K}(\mathbb{C}). \tag{1.417}$$

Another definition of the Green's function is via the resolvent itself:

$$FG_\lambda = \mathbb{1}_p^{\otimes 2}\delta(\mathbf{x}-\mathbf{x}'), \qquad \forall(\mathbf{x},\mathbf{x}')\in\bar{\Omega}\times\mathring{\Omega} \tag{1.418}$$

where $\mathbb{1}_p^{\otimes 2}\in H_2^p(\Omega^2)$ is the **identity** double p-form

$$\mathbb{1}_p^{\otimes 2} \overset{\text{def}}{=} \sum_{\substack{1\le i_1<i_2<\ldots<i_p\le d,\\ 1\le j_1<j_2<\ldots<j_p\le d}} \bigwedge_{h,k=1}^{p}(\mathrm{d}x_{i_h}\otimes\mathrm{d}x'_{j_k}) ::= \mathrm{d}\mathbf{x}\otimes\mathrm{d}\mathbf{x}' \tag{1.419}$$

and $\delta\in C_0^\infty(\Omega)^*$ is the ***Dirac*'s delta** distribution, cfr. §2.2.2, (1.151), [58], [68].

In this context the solution $\varphi\in H_2^p(\Omega)$ to the problem —the image of $\mathcal{F}[J_\mathrm{i}]$ under the resolvent in the electromagnetic case (1.410)— is computed as the induced L_2^p-product of the Green's function with the known term η (with respect to $\mathbf{x}'$'s metric):

$$G_\lambda\eta = \langle \underline{G}_\lambda \,|\, \overline{\eta}\rangle_{\mathrm{L}^2(\Omega)} = \int_\Omega \underline{G}_\lambda\wedge\star\eta \in H_2^p(\Omega). \tag{1.420}$$

115 Always remember that operators acting on functional spaces are always considered **together** with their boundary (intial) conditions, otherwise they are associated to ill-posed problems.

In the self-adjoint case $\mathrm{Spec}F\subset\mathbb{R}$, the reality of the eigenforms lets G_λ have the following hermitian symmetry property, $\forall\mathbf{x}\in\Omega$, $\forall\mathbf{x}'\in\mathring{\Omega}$ and $\forall\lambda\in\mathbb{C}\backslash\mathrm{spec}F$:

$$G_\lambda(\mathbf{x},\mathbf{x}') \equiv \overline{G_\lambda(\mathbf{x}',\mathbf{x})} \equiv \overline{G_{\overline{\lambda}}(\mathbf{x},\mathbf{x}')}. \tag{1.421}$$

If $\eta\in\Lambda^p(\Omega)$ (i.e. it is a *smooth* differential p-form), then G_λ is bounded and almost-everywhere Frechét-differentiable in $\mathring{\Omega}\times\mathring{\Omega}$, i.e. $\underline{G}_\lambda\in\Lambda^p(\Omega)^{\otimes 2}\otimes\mathcal{K}(\mathbb{C})$, with its discontinuities given by the line $\{(\mathbf{x},\mathbf{x}')\in\Omega\times\mathring{\Omega}\mid\mathbf{x}=\mathbf{x}'\}$.

If Ω is bounded, let $\mathcal{B}\{\phi_n\}_{\mathbb{N}_0}$ be a complete orthonormal basis of p-eigenforms by (1.411), then $\mathcal{B}^{\otimes 2}\{\phi_m\otimes\phi_n\}_{m,n\in\mathbb{N}_0}$ is a complete **bi**-orthonormal basis for $H_2^p(\Omega)^{\otimes 2}$ and the Green's function is developable into its *Neumann* series $\underline{G}_\lambda\in H_2^p(\Omega)^{\otimes 2}[\![\lambda]\!]$, which is basically a Laurent series of double p-forms and least order -1:

$$\underline{G}_\lambda(\mathbf{x},\mathbf{x}') = \sum_{n=0}^{\infty}\frac{\phi_n(\mathbf{x}^{\leqslant})\otimes\overline{\phi_n(\mathbf{x}^{>})}}{\lambda_n-\lambda}. \tag{1.422}$$

Map $(\mathbf{x},\mathbf{x}')\mapsto(\mathbf{x}^{\leqslant},\mathbf{x}^{>})$ is continuous, almost-everywhere differentiable, with

$$(\mathbf{x}^{\leqslant},\mathbf{x}^{>}) \stackrel{\text{def}}{\equiv} \begin{cases}(\mathbf{x},\mathbf{x}'), & \mathbf{x}\leq\mathbf{x}' \quad (x_j\leq x_j',\ 1\leq j\leq d)\\ (\mathbf{x}',\mathbf{x}), & \mathbf{x}>\mathbf{x}' \quad (x_j>x_j',\ 1\leq j\leq d)\end{cases} \tag{1.423}$$

G_λ has simple poles in $\{\lambda_n\}_{\mathbb{N}_0}$, accumulating at the cluster point ∞ (cfr. §2.4.1), with $\mathrm{Res}_{\underline{G}_\lambda}\lambda_n=\phi_n\otimes\overline{\phi_n}\ \ \forall n\in\mathbb{N}_0$ and

$$\sum_{n=0}^{\infty}\phi_n(\mathbf{x})\otimes\overline{\phi_n(\mathbf{x}')} = \mathbb{1}_p^{\otimes 2}\delta(\mathbf{x}-\mathbf{x}'), \tag{1.424}$$

(which holds in the distribution's sense since, in general, $\delta\notin\mathrm{L}^2(\Omega)^\dagger$), is a consequence of the Residue Theorem (1.222) and is called a completeness relation for $\mathcal{B}^{\otimes 2}$.

Consider the II-order differential operator $L = -\frac{\mathrm{d}^2}{\mathrm{d}x^2} \in \mathsf{H}\left(H^2([0,a])\right)$ and the following Sturm-Liouville problem [9]–§B.5:

$$\begin{cases} u''(x) + \lambda u(x) = 0 \\ u(0) = u(a) = u'(0) = u'(a) = 0 \end{cases}$$

The operator is self-adjoint and positive-definite, with a discrete spectrum $\operatorname{spec} L = \left\{\frac{n^2\pi^2}{a^2}\right\}_{n\in\mathbb{N}_0}$ of simple eigenvalues, corresponding to the following complete basis of L^2-orthonormal eigenfunctions:

$$u_n(x) = \sqrt{\frac{2-\delta_{n,0}}{a}} \cos\frac{n\pi x}{a}, \qquad \forall n \in \mathbb{N}_0.$$

The Green's function is thus written as the Neumann series (1.422)

$$G(x,\xi;\lambda) = \frac{1}{a}\sum_{n=0}^{\infty} \frac{(2-\delta_{n,0})\cos\frac{n\pi x}{a}\cos\frac{n\pi\xi}{a}}{\lambda - \left(\frac{n\pi}{a}\right)^2},$$

along with the spectral representation (1.424):

$$\frac{1}{a}\sum_{n=0}^{\infty}(2-\delta_{n,0})\cos\frac{n\pi x}{a}\cos\frac{n\pi\xi}{a} = \delta(x-\xi).$$

Cases where the operator is *not* normal can be considered for particular domains and boundary conditions: this goes beyond our scopes but need it say that in these cases the Laplace operator —despite not being diagonalizable— still admits an infinite-dimensional spectral decomposition analogue to Jordan decomposition (by means of *generalized* eigenspaces) [9]–§§B.4, C.2. A general property of Green's functions for non-normal operators is that their polar singularities still coincide with the eigenvalues of F and their order as poles of G_λ is their index as eigenvalues of F, i.e.:

$$\operatorname{ord}_{G_\lambda}\lambda = \operatorname{idx}_F \lambda, \qquad \forall \lambda \in \operatorname{spec} F. \tag{1.425}$$

Thus Green's functions have a Neumann series (1.422) with higher-order negative terms, i.e. more "wide" formal Laurent series $\underline{G}_\lambda \in H_2^p(\Omega)^{\otimes 2}[\![\lambda]\!]$, cfr. §1.3.3.

Consider the II-order differential operator $L = -\frac{\mathrm{d}^2}{\mathrm{d}x^2} \in \mathrm{H}\left(H^2([0,1])\right)$ and the following Sturm-Liouville problem [9]–§B.5:

$$\begin{cases} u''(x) + \lambda u(x) = 0 \\ u(0) = 0, \quad u'(1) = cu(1) \end{cases}$$

with $c = \sqrt{\lambda_0}\cot\sqrt{\lambda_0}$ and $(\lambda_n)_{n\in\mathbb{N}_0}$ being the increasing sequence of non-zero solutions to non-linear equation $\sin 2\sqrt{\lambda} = 2\sqrt{\lambda}$ (thus λ_0 being the *least* one). Operator L is self-adjoint and positive-definite, with discrete spectrum $\{\lambda_n\}_{n\in\mathbb{N}_0}$; the Green's function is then[116]

$$G(x,\xi;\lambda) = \frac{\sin\left(\sqrt{\lambda}x^{\leqslant}\right)}{\sqrt{\lambda}} \frac{\sqrt{\lambda}\cos\left[\sqrt{\lambda}(1-x^{>})\right] - c\sin\left[\sqrt{\lambda}(1-x^{>})\right]}{\sqrt{\lambda}\cos\sqrt{\lambda} - c\sin\sqrt{\lambda}},$$

whose poles are in fact $\{\lambda_n\}_{n\in\mathbb{N}_0}$. In particular, λ_0 is a *double* pole, whereas λ_n are *simple* poles $\forall n\in\mathbb{N}$. Residues' estimation leads to:

$$\mathrm{Res}_G\lambda_0 = \lim_{\lambda\to\lambda_0}(\lambda-\lambda_0)^2\frac{\partial G}{\partial\lambda} = 2\cot\sqrt{\lambda_0}x^{>}\sin\left(\sqrt{\lambda_0}x^{\leqslant}\right)\cos\left(\sqrt{\lambda_0}x^{>}\right)$$

$$\mathrm{Res}_G\lambda_n = \lim_{\lambda\to\lambda_n}(\lambda-\lambda_n)G = 4\frac{\sqrt{\lambda_0}\sin\left(\sqrt{\lambda_n}x^{\leqslant}\right)\cos\left(\sqrt{\lambda_n}x^{>}\right)}{2\sqrt{\lambda_0} - \sin 2\sqrt{\lambda_0}}.$$

To the eigenvalues λ_n, the associated eigenfunctions are $\left\{\sin\sqrt{\lambda_n}x\right\}_{n\in\mathbb{N}_0}$ (with *geometric* multiplicities 1), whereas the function $x\cos\sqrt{\lambda_0}x$ is a 2nd-rank *generalized* eigenfunction, i.e.:

$$\begin{cases} \mathrm{mul}_L\lambda_n = \mathrm{idx}_L\lambda_n = \mathrm{ord}_G\lambda_n = 1 + \delta_{n,0} \\ \mathrm{gmul}_L\lambda_n = 1 \end{cases} \quad \forall n\in\mathbb{N}_0;$$

$$\begin{cases} (L - \lambda_0\mathbb{1})^2(x\cos\sqrt{\lambda_0}x) \equiv 0 \\ (L - \lambda_0\mathbb{1})(x\cos\sqrt{\lambda_0}x) = 2\sqrt{\lambda_0}\sin\sqrt{\lambda_0}x. \end{cases}$$

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116 As per (1.423), $(x^{\leqslant}, x^{>}) :\equiv (x,\xi)$ if $x \leq \xi$, otherwise $(x^{\leqslant}, x^{>}) :\equiv (\xi, x)$.

This means that $H^2([0,1])$ can be orthogonally decomposed with respect to the complete basis composed by the above 1st- and 2nd-rank generalized eigenfunctions as [66]

$$H^2([0,1]) = \mathrm{Ker}(L-\lambda_0 \mathbb{1})^2 \oplus \overline{\bigoplus_{n=0}^{\infty} \mathrm{Ker}(L-\lambda_n \mathbb{1})}$$

and, with respect to such basis, L's matrix-elements representation is:

$$[L] = J_{\lambda_0,2} \oplus \bigoplus_{n=1}^{\infty} J_{\lambda_n,1} \equiv \left(\begin{array}{cc|c|c|c} \lambda_0 & 1 & & & \\ 0 & \lambda_0 & & & \\ \hline & & \lambda_1 & & \\ \hline & & & \lambda_2 & \\ \hline & & & & \lambda_3 \\ & & & & \ddots \end{array}\right).$$

4.2.2. Continuous spectrum and multi-valuedness

If Ω is *not* bounded, F also has a *continuous* spectrum (but in general no residual spectrum) i.e. $\mathfrak{spec} F = \mathrm{Spec} F \setminus \mathrm{spec} F = K \subset \mathbb{C}$.

From the Functional Analysis point of view (1.411) and (1.424) do not hold since $H_2^p(\Omega)$ is *not* separable anymore, [74]. From the Complex Analysis point of view G_λ is meromorphic on a compact, positive-genus (possibly orientable) Riemann surface $\mathcal{S}$ rather than on $\mathbb{C}$: it still features regular polar singularities coinciding with the discrete spectrum $\mathrm{spec} F$ (if $\mathrm{spec} F \neq \varnothing$) but everything has to be properly reformulated and $\mathcal{S}$ has to be correctly defined for the Green's functions' technique to be useful in $H_2^p(\Omega)^{\otimes 2} \otimes \mathcal{K}(\mathcal{S})$. A detailed review of the matter would extend far beyond the scope of this text (also because there are still many open problems and radically-different ways to approach them), so a brief excursus of the main analytic-geometric properties will be given.

Let μ be the measure used for Ω (the d-dimensional Lebesgue measure was supposed up to now); it can be proven that there exist two refinement measures [three if there is a residual spectrum too] $\mu_{\aleph}$ and $\mu_{\mathfrak{c}}$ [and $\mu_{\mathfrak{r}}$] such that for every μ-measurable subset $E\subseteq\Omega$:

- $\mu_{\aleph}$ is a **discontinuous** point measure with support the discrete spectrum, i.e. $\mathrm{supp}\mu_{\aleph}=\mathrm{spec}F$, and
$$\mu_{\aleph}(E)=\mathrm{Card}\{\lambda\in\mathrm{spec}F \mid \lambda\in E\};$$
- $\mu_{\mathfrak{c}}$ is **absolutely continuous** with support the continuous spectrum, i.e. $\mathrm{supp}\mu_{\mathfrak{c}}=\mathfrak{spec}F$ and $\mu_{\mathfrak{c}}\in\mathrm{C}^0(K;\mathbb{R}_+)$;
- [$\mu_{\mathfrak{r}}$ is **singular** with support the residual spectrum, [68];]
- $\mu=\mu_{\aleph}\oplus\mu_{\mathfrak{c}}\oplus\mu_{\mathfrak{r}}$, i.e. for every μ-measurable subset $E\subseteq\Omega$, $\mu(E)=\mu_{\aleph}(E)+\mu_{\mathfrak{c}}(E)+\mu_{\mathfrak{r}}(E)$;
- $\mathrm{L}^2(\Omega,\mu)\cong\mathrm{L}^2(\Omega,\mu_{\aleph})\oplus\mathrm{L}^2(\Omega,\mu_{\mathfrak{c}})^{\dagger}\oplus\mathrm{L}^2(\Omega,\mu_{\mathfrak{r}})^{\dagger}$;

Since we are dealing with II-order differential operators F, $\dim K\leq 1$ i.e. the continuum spectrum $K\subset\tilde{\mathbb{C}}$ is either a totally disjoint set (a "**dust**" set if $\mathrm{Card}K=\mathfrak{c}$) or the union of $l\in\mathbb{N}$ complex curves in the Riemann sphere: $K=\gamma_1\cup\gamma_2\cup\ldots\cup\gamma_l$, which can be interpreted as branch cuts $\rangle\upsilon_{2j-1},\upsilon_{2j}\langle\ =\gamma_j$ ($1\leq j\leq l$ and distinct points $\upsilon_0,\upsilon_1,\ldots,\upsilon_{2l}\in\tilde{\mathbb{C}}$) for a Riemann surface. Despite usual interpretations in dealing with Riemann surfaces, where the choice of branch cuts is quite arbitrary in order to build a suitable branched covering and perform Calculus on it, in this case of physical relevance, it is the Riemann surface which is defined in order to incorporate the "imposed" branch cuts. The mathematical tools used for the rest of the paragraph were introduced in §2.2.2 and §2.4, as well as in [1] and [42].

In order to define $\mathcal{S}$ let the branched covering $\upsilon:\mathcal{S}\to\tilde{\mathbb{C}}$ be introduced whose fibers are chosen by partitioning $\tilde{\mathbb{C}}$ and using K as their boundaries (the branch cuts, in fact). Different physical constraints imposed on the spectral parameter λ (see below and §4.2.3) for "incorporating" the point spectrum $\mathrm{spec}F$ on some or all of the sheets of $\mathcal{S}$, which may lead to υ having from extremely simple to quite "exotic" multi-valued-function structure (like for example those described in §3).

The use of a Riemann surface is mandatory since, in general, the eigenvalues of F — despite being poles for the resolvent operator— may feature different contribution to the evaluation of (1.416) according not only to their relative position respect to the continuous spectrum, but also on the spectral decomposition of the known term η in (1.416), which involves a special contouring over the loops on $\mathcal{S}$.

It can be proven that a p-form[117] $\psi\in H^p_{2,\mathrm{loc}}(\Omega,\mu)\otimes H^2(\mathcal{S},\mu_{\mathfrak{c}})$ exists such that

$$\psi(\mathbf{x},\lambda_n) \equiv \phi_n(\mathbf{x}), \qquad \forall n\in\mathbb{N}_0 \tag{1.426}$$

and that an orthogonal decomposition analogue to (1.411) and for every loop $\gamma\in\tilde{\mathbb{C}}\backslash\mathrm{Spec}F$ a completeness relation analogue to (1.424), exist:

$$H^p_2(\Omega) = \overline{\bigoplus_{\lambda\in\mathrm{Spec}F} \mathrm{Ker}(F-\lambda\mathbb{1})}; \tag{1.427}$$

$$\begin{aligned}\int_{+\gamma\uparrow\upsilon} \psi(\mathbf{x}^{\lessgtr},\upsilon)\otimes\overline{\psi(\mathbf{x}^{>},\upsilon)}\mathrm{d}\upsilon &= \oint_{+\gamma} \psi(\mathbf{x}^{\lessgtr};\upsilon(z))\otimes\overline{\psi(\mathbf{x}^{>};\upsilon(z))}\upsilon'(z)\mathrm{d}z \\ &= \mathbb{1}_p^{\otimes 2}\delta(\mathbf{x}-\mathbf{x}').\end{aligned} \tag{1.428}$$

It is important to note that (1.428) always holds in the distributions' sense, since ψ is *not* a p-form of class $H^2_{\cdot}$ Furthermore, the **exact** 1-form $\mathrm{d}\upsilon\equiv\upsilon'\mathrm{d}z\in \mathrm{B}^1(\mathcal{S})$, restricted to $\mathfrak{spec}F$, constitutes the measure element for the continuous spectrum defined above, i.e.:

[117] In literature ψ is equivalent to an *un*conutable family of functions $\mathcal{I}\{\psi_\upsilon\}_{\upsilon\in\mathcal{S}}$ called ***improper eigenfunctions*** (eigenforms in the exterior case $p>0$) since they don't belong to operator's domain. The topological closure in (1.427) only stands for the discrete spectrum, since the operator's domain is not separable anymore, so it cannot be obtained as the closure of a countable set union alone.

$$\upsilon\big|_K \equiv \mu_{\mathfrak{c}}\,. \tag{1.429}$$

The point measure $\mu_{\aleph}$ instead, thanks to (1.426), is responsible for generating the contributions of polar singularities $\{\lambda_n\}_{\mathbb{N}_0}$ as an adjoint completeness relation to (1.428) and (1.429); using the complex Dirac's delta (1.151) $\delta \in \mathrm{C}_0^\infty(\mathbb{C})^*$ one can formally write:

$$\mu_{\aleph}(z) = \sum_{n\in\mathbb{N}_0} \delta(z-\lambda_n). \tag{1.430}$$

Of course ψ is a meromorphic p-form of $\mathcal{S}$, whereas its composition by the branched covering, $\psi\circ\upsilon$, is a multi-valued p-form in the Riemann sphere, so both integrals of (1.428) make sense and provide a quite interesting completeness relation.

Using (1.426)–(1.430) it can be proven that the Green's function is the tensor

$$\begin{aligned}
\underline{G}_\lambda(\mathbf{x},\mathbf{x}') &= \int_{\upsilon(K)} \frac{\psi(\mathbf{x}^{\lessgtr},\upsilon)\otimes\overline{\psi(\mathbf{x}^{>},\upsilon)}}{\lambda-\upsilon}\,\mathrm{d}\upsilon \\
&= \int_{\operatorname{spec} F} \frac{\psi(\mathbf{x}^{\lessgtr},\upsilon)\otimes\overline{\psi(\mathbf{x}^{>},\upsilon)}}{\lambda-\upsilon}\,\mathrm{d}\mu_{\aleph} + \int_{\mathfrak{spec}\,F} \frac{\psi(\mathbf{x}^{\lessgtr},\upsilon)\otimes\overline{\psi(\mathbf{x}^{>},\upsilon)}}{\lambda-\upsilon}\,\mathrm{d}\mu_{\mathfrak{c}} \\
&\equiv \sum_{n=0}^{\infty} \frac{\phi_n(\mathbf{x}^{\lessgtr})\otimes\overline{\phi_n(\mathbf{x}^{>})}}{\lambda-\lambda_n} + \int_K \frac{\psi(\mathbf{x}^{\lessgtr},\upsilon(z))\otimes\overline{\psi(\mathbf{x}^{>},\upsilon(z))}}{\lambda-\upsilon(z)}\,\mathrm{d}\mu_{\mathfrak{c}}(z).
\end{aligned}$$

Under this formalism, the Green's function can be interpreted as the ***generating function*** of the solutions to problem (1.416), whereas the monodromy groups $\mathrm{MG}_z\mathcal{S}$ (cfr. §2.4.5 and [69]) describe the symmetries between any subsets of different solutions as algebraic properties relating the topology of $\mathcal{S}$ with the geometry of the complete spectrum of operator F.

A similar example to that of §4.2.1 is shown here for the simultaneous occurrence of discrete and continuous spectra. Consider the Sturm-Liouville problem for $L \in H^2([0,1])$ given by ($\alpha \in \mathbb{C}^*$):

$$\begin{cases} u''(x) + \lambda u(x) = 0 \\ u(0) = \alpha u(1) \\ u'(0) = \alpha u'(1) \end{cases}$$

The operator L is not normal and has both a *discrete* spectrum $\operatorname{spec} L = \{n^2\pi^2\}_{n \in \mathbb{N}_0}$ made by simple eigenvalues, and a *continuous* spectrum $\mathfrak{spec}\, L = \{\alpha\}$ (which is an isolated set, due to non-normality). The associated L^2-orthonormal eigenfunctions, *not* being a complete basis for $\mathrm{L}^2([0,1])$, are

$$u_n(x) = \sqrt{\frac{2}{n^2\pi^2 + \alpha^2}} \Big(n\pi \cos(n\pi x) + \alpha \sin(n\pi x) \Big), \qquad \forall n \in \mathbb{N};$$

whereas the “improper” eigenfunction (1.426) associated to α is

$$u_\alpha(x) = \sqrt{\frac{2\alpha}{e^{2\alpha} - 1}}\, e^{\alpha x}.$$

Of course $\operatorname{Spec} L$ and $\{u_\alpha(x)\} \cup \{u_n(x)\}_{n \in \mathbb{N}}$ are respectively the complex-conjugated complete spectrum and eigenfunctions of the adjoint operator $L^\dagger$ due to duality isomorphism $H^2([0,1])^\dagger \cong H^2([0,1])$.

The Green’s function is thus written as stated just above:

$$G(x, \xi; \lambda) =$$

$$2\left[\frac{\alpha e^{\alpha(x+\xi)}}{(e^{2\alpha} - 1)(\alpha - \lambda)} + \sum_{n=0}^{\infty} \frac{\big(n\pi\cos(n\pi x) + \alpha\sin(n\pi x)\big)\big(n\pi\cos(n\pi\xi) + \alpha\sin(n\pi\xi)\big)}{\big(n^2\pi^2 + \alpha^2\big)\big(n^2\pi^2 - \lambda\big)} \right].$$

The multi-valuedness of the Green's function with respect to the spectral parameter v (it is *not* multi-valued with respect to λ) is due to the analytic continuation outside K in order for the Green's function to be a meromorphic tensor with respect to λ.

The non-arbitrary choice of branch cuts is due to physical reasons instead: the physical meaning of $\lambda=\kappa^2$ in Electromagnetics as the square of the propagation constant κ requires, for example, $\mathrm{Im}\kappa\geq 0$ for modes attenuating towards ∞, so a branch choice for the square-root function (called **proper** Riemann sheet) $\kappa=\sqrt{\lambda}$ and its branch cut $\rangle 0,\infty\langle$ (usually equal to either $\mathbb{R}_\pm$ or $\mathbb{I}_\pm$) is imposed. Choosing the other (**improper**) Riemann sheet means accounting for mode being amplified as long as they get farther away (towards ∞), which is *almost* always non-physical (cfr. "leaky" waves on §4.2.5), because that would not respect the principle of energy conservation.

4.2.3. Electrodynamic Green's function

If Ω is bounded and sufficiently smooth (and for particularly 'well-behaved' boundary conditions) Δ_p is a normal operator whose inverse is also compact. If Ω is filled with lossless media then $\Delta_p\in\mathfrak{sym}_\leq H_2^p(\Omega)$, i.e. it is also self-adjoint, non-positive definite[118] and with compact inverse.

From a distributional point of view, (1.416) in the Electromagnetics case is equal to

$$G_{\kappa^2}\hat{A} = \hat{J}_\mathrm{i}.$$

Let us consider, for simplicity's sake, the Hemholtz's operator '$\Delta+\kappa^2\mathbb{1}$' (no Hodge operators are involved and $\Delta:\equiv\Delta_1$); i.e. the presence of a bounded static, homogeneous and isotropic media. It is compact and normal since media losses, i.e. $\Delta+\kappa^2\mathbb{1}\in\mathfrak{nC}(H_2^1(\Omega))$; but only the lossless case will be considered for simplicity's

[118] Even in the "lossy" case the discrete spectrum may have positive real part if and only if Ω contains ***active*** media, i.e. materials generating electromagnetic power instead of absorbing or scattering it (e.g. by the effect of chemical and/or nuclear reactions, like in plasma media).

sake. In this case, since $\operatorname{spec}\Delta \subset \mathbb{R}_-$, the eigenvalues of the 1-Laplacian are $\operatorname{spec}\Delta = \{-\kappa_n^2\}_{n\in\mathbb{N}_0}$ with $0 \le \kappa_0 < \ldots < \kappa_n < \kappa_{n+1} < \ldots$, $\forall n \in \mathbb{N}$. In Electromagnetics the spectral parameter $\lambda \in \mathbb{C}$ represents the square of the propagation wave-number, $\lambda \equiv \kappa^2$, so the resolvent operator and its Green's function for the Helmholtz's equation are written as depending on κ, which may lead to some confusion to this and next paragraph's equations.

The zero eigenvalue $\kappa_0 = 0$ exists if and only if $\beta_1(\Omega) + \beta_2(\Omega) > 0$, i.e. Ω contains one or more cavities such that resonating non-zero static fields may exist inside, of either **electric** type (Ω multiply *superficially* connected, i.e. $\beta_2(\Omega) > 0$) and **magnetic** type (Ω multiply *linearly* connected, i.e. $\beta_1(\Omega) > 0$).

Let the standard physical-space position vector $\mathbf{r} \in \mathbb{R}^3$ be used and let $\mathbb{1}^{\otimes 2} :\equiv \mathbb{1}_1^{\otimes 2}$ be the identity double 1-form $\mathbb{1}^{\otimes 2} := \mathrm{d}\mathbf{r} \otimes \mathrm{d}\mathbf{r}' :\equiv \mathrm{d}x \otimes \mathrm{d}x' + \mathrm{d}y \otimes \mathrm{d}y' + \mathrm{d}z \otimes \mathrm{d}z'$.

The free-space radiation problem (where $\mathbb{R}^3$ is filled with a static, lossless, homogeneous and isotropic medium like vacuum) leads to the well-known Green's function for the electromagnetic problem, whose tensor is of scalar type:

$$\underline{G}_{\kappa^2}(\mathbf{r}, \mathbf{r}') = g_0(\mathbf{r} - \mathbf{r}', \omega)\, \mathbb{1}^{\otimes 2},$$

$$g_0(\mathbf{r}, \omega) = \frac{e^{-\imath\kappa(\omega)\|\mathbf{r}\|}}{4\pi\,\|\mathbf{r}\|}, \qquad \kappa(\omega) = \omega\sqrt{\varepsilon\mu} \equiv \frac{\omega}{v_{\mathrm{ph}}}. \tag{1.431}$$

Given any impressed electric current density 2-form $J_{\mathrm{i}} \in H_2^2(\Omega)$, the electrodynamic potential is computed via (1.420) which, thanks to the scalar nature of the Green's function (1.431), resorts to a convolution with respect to source coordinates $\mathbf{r}' \in \mathring{\Omega}$:

$$\begin{aligned} \hat{A}(\mathbf{r}, \omega) &= \left\langle \underline{G}_{\kappa^2} \,\middle|\, \overline{\hat{J}_{\mathrm{i}}} \right\rangle_{\mathrm{L}^2(\Omega)} = \int_{\Omega} g_0(\mathbf{r} - \mathbf{r}', \omega)\, \mathbb{1}^{\otimes 2} \wedge \star \hat{J}_{\mathrm{i}}(\mathbf{r}', \omega) \\ &= \frac{1}{4\pi} \int_{\Omega} \frac{e^{-\imath\kappa(\omega)\|\mathbf{r}-\mathbf{r}'\|}}{\|\mathbf{r} - \mathbf{r}'\|} \star \hat{J}_{\mathrm{i}}(\mathbf{r}', \omega) = g_0 \mathbb{1}^{\otimes 2} \circledast \hat{J}_{\mathrm{i}}\,. \end{aligned} \tag{1.432}$$

The resolvent for the Helmholtz's equation is sometimes computed using the operator Fourier transform $\mathrm{L}_2^p(\Omega) \to \mathrm{L}_2^p\big(\mathbb{R}^d, \frac{\mu_{\mathrm{L}}(\boldsymbol{\kappa})}{\|\boldsymbol{\kappa}\|^2-\kappa^2}\big)^{\dagger}$:

$$\mathcal{F}\Big[\Delta - \kappa^2 \mathbb{1}\Big] = \boldsymbol{\kappa}\otimes\boldsymbol{\kappa} - \kappa^2\mathbb{1} \in \partial\mathbb{B}_1(0) \subset H_2^p(\Omega)^{\dagger}, \tag{1.433}$$

which is not of tensor type any more, because the Fourier transform of any n^{th}-order linear differential operator is a n^{th}-degree algebraic operator in some extension field, i.e. $\mathcal{F}\Big[\big(\Delta - \kappa^2\mathbb{1}\big)^{-1}\Big] \in \mathbb{C}\{\hat{\hat{J}}_{\mathrm{i}}\}$, cfr. §§1.1.3, 1.2.4.

If the space-time spectral domain is employed, then (1.432) turns into a multiplication of the spectral current density 2-form by the spectral Green's function $\hat{G}_{\kappa^2} = \big(\boldsymbol{\kappa}\otimes\boldsymbol{\kappa} - \kappa^2\mathbb{1}\big)^{-1}$, whereas the singularities' evaluation is "pulled back" into the inverse Fourier transform's computation, which is a contract product:

$$\begin{aligned}\hat{\hat{A}}(\boldsymbol{\kappa},\omega) &= \big(\boldsymbol{\kappa}\otimes\boldsymbol{\kappa} - \kappa^2\mathbb{1}\big)^{-1}\hat{\hat{J}}_{\mathrm{i}}(\boldsymbol{\kappa},\omega) \\ &= \underline{\hat{G}}_{\kappa^2} :\star \hat{\hat{J}}(\boldsymbol{\kappa},\omega).\end{aligned} \tag{1.434}$$

In a free-space problem, (1.431) is used (1.434) becomes:

$$\hat{\hat{A}}(\boldsymbol{\kappa},\omega) = \hat{g}_0(\boldsymbol{\kappa},\omega)\star\hat{\hat{J}}_{\mathrm{i}}(\boldsymbol{\kappa},\omega) = \frac{\star\hat{\hat{J}}_{\mathrm{i}}(\boldsymbol{\kappa},\omega)}{\|\boldsymbol{\kappa}\|^2 - \kappa(\omega)^2}\,. \tag{1.435}$$

The spectral Green's function thus is simply $\hat{g}_0(\boldsymbol{\kappa},\omega) = \big(\|\boldsymbol{\kappa}\|^2 - \kappa(\omega)^2\big)^{-1}$, which can be inverse-Fourier transformed and, employing the fact that it is a homogeneous function (i.e. it only depends on the norm of $\boldsymbol{\kappa}$), such computation can be simplified down to a scalar integral; by letting $q = \|\boldsymbol{\kappa}\|$ and $R = \|\mathbf{r}\|$, the Fourier integral is solved by the Residue Theorem (1.222), with the real axis as integration path with holes around the simple poles $\pm\kappa$, to obtain (1.431):

$$
\begin{aligned}
g_0(\mathbf{r},\omega) &= \mathcal{F}^{-1}[\,\hat{g}_0\,](\mathbf{r}) = \frac{1}{8\pi^3}\int\limits_{\mathbb{R}^3}\frac{e^{\imath\boldsymbol{\kappa}\cdot\mathbf{r}}\mathrm{d}\boldsymbol{\kappa}}{\|\boldsymbol{\kappa}\|^2-\kappa^2(\omega)} \\
&= \frac{4\pi}{R}\mathcal{F}^{-1}[\,q\hat{g}_0(q,\omega)\sin(qR)\delta_{-1}(q)\,](R) \\
&= \frac{1}{2\pi^2 R}\int\limits_0^{\infty}\frac{q\sin(qR)}{q^2-\kappa^2}\mathrm{d}q = \frac{1}{8\imath\pi^2 R}\int\limits_{-\infty}^{+\infty}\frac{q\left(e^{\imath qR}-e^{-\imath qR}\right)}{(q+\kappa)(q-\kappa)}\mathrm{d}q \\
&= \frac{1}{4\pi R}\left(\operatorname{Res}_{\frac{qe^{\imath qR}}{q^2-\kappa^2}}(\pm\kappa)-\operatorname{Res}_{\frac{qe^{-\imath qR}}{q^2-\kappa^2}}(\pm\kappa)\right) = \frac{e^{-\imath\kappa R}}{4\pi R}\,.
\end{aligned}
$$

Equation (1.434) is the explicit formula for computing the spectral domain Green's function (1.433) but, as said above, the singularity and multi-valuedness' problems analyzed in §4.2.1 turn up again when inverse-transforming and imposing physical constraints with respect to spectral coordinates $\boldsymbol{\kappa}$ rather than $\|\boldsymbol{\kappa}\|^2$ (cfr. §5.1.7).

In this case $\hat{G}_{\kappa^2}$ becomes a multi-valued tensor and the inverse transform

$$
\hat{A}(\mathbf{r},\omega) = \frac{1}{8\pi^3}\oint\limits_{+\gamma}\hat{\underline{G}}_{\kappa^2}(\boldsymbol{\kappa},\omega){:}\star\hat{J}_{\mathrm{i}}(\boldsymbol{\kappa},\omega) \tag{1.436}
$$

is to be computed by analytic continuation on complexified paths $\boldsymbol{\kappa}\in\gamma\subset\mathbb{C}^3$ different than $\boldsymbol{\kappa}\in\mathbb{R}^3$; the non-uniqueness of the continuation (component by component: κ_1, κ_2 and κ_3), leads to integration on the Riemann surface $\mathcal{S}$ defined in §2.4.1. By the Hartog's Theorem in fact (cfr. §2.1.3) the zeroes and singularities of a complex function of several complex variables cannot be isolated points. That is why the circulation around singular paths in (1.436) is associated to the singular points of $\hat{G}_{\kappa^2}$ with respect to the scalar spectral parameter κ —be they poles and/or branch points— cfr. §4.2.5. The physical interpretation of the singularities is usually possible at component-wise integration time, on each inverse-Fourier-transform integral [43]: dealing with inverse Fourier transforms with respect to single wavenumbers κ_j, that means dealing with isolated singularities one at a time (i.e. neglecting the singular curves which are interpreted as the presence of a continuous spectrum).

4.2.4. Electromagnetic Green's function

Since many applications in Electromagnetics require that both electric $J_{\mathrm{E}}{:}{\equiv}J_{\mathrm{i}}$ and (equivalent) magnetic impressed current densities $J_{\mathrm{M}}{:}{\equiv}J_{\mathrm{m}}$ are involved, and since electromagnetic fields (E,H) are directly useful for synthesis problems, it is easier to compute the Green's function for the coupled Maxwell's equations (1.393) —i.e. for the uncoupled Helmholtz's equations (1.397)— than those for the electrodynamic potential A, which (E,H) are computed from by means of (1.404), [35] and [24].

Many practical problems require in fact numerical approximations of the electromagnetic field equations — for example via the Method of Moments (**MoM**), bondary/volume elements' method (all variants of the Galerkin's method), or discrete Fourier transform (**DFT**) methods, like the FDTD. Such approximate solutions are "discrete" in either space and time: for this reason the computation of a discrete version of A would inevitably introduce additional approximations for the determination of discretized versions of either E and H via (1.404), which definitely involves continuum differential operators.

For all these reasons Maxwell's equations (1.393) need to be directly "inverted" to obtain a set of 4 resolvent operators (called electric-electric, electric-magnetic, magnetic-electric and magnetic-magnetic Green's functions respectively):

$$\begin{cases} \mathrm{d}\hat{E} + \imath\omega\star_{\mathrm{m}}\hat{H} = -\hat{J}_{\mathrm{M}} \\ \mathrm{d}\hat{H} - \imath\omega\star_{\mathrm{c}}\hat{E} = \hat{J}_{\mathrm{E}} \end{cases} \quad\Leftrightarrow\quad \begin{pmatrix} \imath\omega\star_{\mathrm{c}} & -\mathrm{d} \\ \mathrm{d} & \imath\omega\star_{\mathrm{m}} \end{pmatrix}\begin{pmatrix} \hat{E} \\ \hat{H} \end{pmatrix} = -\begin{pmatrix} \hat{J}_{\mathrm{E}} \\ \hat{J}_{\mathrm{M}} \end{pmatrix} \tag{1.437}$$

$$\Updownarrow$$

$$-G\begin{pmatrix} \hat{J}_{\mathrm{E}} \\ \hat{J}_{\mathrm{M}} \end{pmatrix} \equiv -\begin{pmatrix} G_{\mathrm{EE}} & G_{\mathrm{EM}} \\ G_{\mathrm{ME}} & G_{\mathrm{MM}} \end{pmatrix}\begin{pmatrix} \hat{J}_{\mathrm{E}} \\ \hat{J}_{\mathrm{M}} \end{pmatrix} = \begin{pmatrix} \hat{E} \\ \hat{H} \end{pmatrix}, \tag{1.438}$$

where the operator matrix $G{=}(G_{hk})_{h,k\in\{\mathrm{E,M}\}}$ is computable as a matrix of double 1-forms acting as exterior L_2^2 products as in (1.420) and (1.432), $\forall h,k\in\{\mathrm{E,M}\}$:

$$G_{hk}\hat{J}_k = \left\langle \underline{G}_{hk} \middle| \bar{\hat{J}}_k \right\rangle_{\mathrm{L}^2(\Omega)} = \int_\Omega \underline{G}_{hk}(\mathbf{r},\mathbf{r}';\omega)\wedge\star\hat{J}_k(\mathbf{r}',\omega). \tag{1.439}$$

The Green's tensors too have to satisfy both Maxwell-like (coupled) and Helmholtz-like (decoupled) equations — all differential and Hodge operators acting on $\mathbf{r}\in\Omega$:

$$\begin{cases} \mathrm{d}\,G_{\mathrm{EE}} + \imath\omega\star_{\mathrm{m}} G_{\mathrm{ME}} = 0 \\ \mathrm{d}\,G_{\mathrm{EM}} + \imath\omega\star_{\mathrm{m}} G_{\mathrm{MM}} = \mathbb{1}^{\otimes 2}\delta(\mathbf{r}-\mathbf{r}') \\ -\mathrm{d}\,G_{\mathrm{MM}} + \imath\omega\star_{\mathrm{c}} G_{\mathrm{EM}} = 0 \\ -\mathrm{d}\,G_{\mathrm{ME}} + \imath\omega\star_{\mathrm{c}} G_{\mathrm{EE}} = \mathbb{1}^{\otimes 2}\delta(\mathbf{r}-\mathbf{r}') \end{cases} \tag{1.440}$$

$$\Updownarrow$$

$$\begin{cases} \Delta\,G_{\mathrm{EM}} - \omega^2\star_{\mathrm{m}}\star_{\mathrm{e}} G_{\mathrm{EM}} = \mathrm{d}(\mathbb{1}^{\otimes 2}\delta) \\ \Delta\,G_{\mathrm{ME}} - \omega^2\star_{\mathrm{m}}\star_{\mathrm{e}} G_{\mathrm{ME}} = -\mathrm{d}(\mathbb{1}^{\otimes 2}\delta) \end{cases} \tag{1.441}$$

with boundary conditions $n_\Omega\wedge\underline{G}_{hh}\big|_{\partial\Omega}\equiv 0$ (where $n_\Omega\in\Lambda^2(\partial\Omega)$ is the outward-oriented normal 2-form to $\partial\Omega$), $\forall h\in\{\mathrm{E},\mathrm{M}\}$. Equations (1.440) are equivalent to (1.393), as well as their decoupled versions (1.441) are equivalent to (1.395).

For antenna problems where an impressed electric field E^{i} excites a (ideally) perfectly conducting antenna $A\subset\Omega$ the equations (1.438) are further simplified: no impressed magnetic currents are generated ($J_{\mathrm{M}}\equiv 0$ so just the electric-electric Green's function $\underline{G}_{\mathrm{EE}}$ is needed), $n_A\wedge E\big|_{\partial A}\equiv 0$, and usually the incident field is known as a "far-field" **approximation** (due to $|\boldsymbol{\kappa}\cdot\mathbf{r}|\gg 1$). Equation (1.439) becomes the well-known Electric Field Integral Equation[119] (EFIE), [35]:

$$n_A\wedge\left(\hat{E}^{\mathrm{i}} + \oint_{+\partial A} \underline{G}_{\mathrm{EE}}\wedge\star\hat{J}_{\mathrm{i}}\right) = 0. \tag{1.442}$$

The dual equation for the magnetic field (MFIE) can be easily formulated by duality from (1.443).

[119] In many practical problems $\Omega\backslash A$ is filled by a "well-behaved" material (e.g. air), so $\underline{G}_{\mathrm{EE}}\equiv g_0\mathbb{1}^{\otimes 2}$.

4.2.5. Singularities' classification

Physical interpretation and classification of singularities and branch points of the Green's function is the key point to qualitatively understand the electromagnetic phenomena of the given problem. Yet, a further insight into such phenomena goes beyond the scope of this work. Currently, there is still not a general, systematic classification of such special points for either guided or free-space propagation.

In §4.2.3 it was observed that the singularities' interpretation is possible when the inverse Fourier transforms are computed component-wise (also in order to get rid of a continuous spectra = continuous singularities by the Hartog's Theorem); the way such integrals are approached heavily depends on the geometric symmetries of the domain (i.e. on the associated coordinates' system): the most easy classifiable singularities arise in fact in strongly symmetric domains, like stratified media which infinitely extend along one or two dimensions (ideal models for optic fibres, patch antennas, microstrip lines and other high-frequency electronic paths): these geometries are in fact axial- or plane-symmetric, respectively.

Following instead is a quite "rough" classification of the spectral Green's function's properties, for each type of special, isolated point. Remember that this classification has only to be taken as a general layout, because several electromagnetic problems may (or may not) behave in the manner discussed below. More insight on the singularities' classification problem, especially to specific domain and electromagnetic phenomena, is found in [35] and [43], whereas its classification oriented to the Inverse Scattering method is found in [1]:

- Polar singularities which are *not* branch points (i.e. regular poles) are associated to either **guided modes** of [partially] closed structures such as waveguides and resonators [slabs and microstrips], or to smooth geometric (periodic) boundaries.

- When the Green's function is multi-valued the presence of regular poles only on some sheets means that a continuous-spectrum excitation may or may not excite the corresponding mode. This just depends on the integration path in (1.436): an impressed current exciting a particular mode corresponds to an integration path travelling on a specific sheet of the Riemann surface which that mode's pole exists on and is winded around (such that its corresponding residue must be accounted for). Since the wavenumber κ is a function (1.431) of ω, that corresponds to the well-known **dispertion relation** which defines each mode's ***cut-off frequencies*** and excitation degree. The dispertion equation's complexity depends on the geometrical symmetries and on the physical-chemical properties of the media. More on that in the special case of "fractal" geometries/materials will be studied in §5.3.

- Branch point polar singularities (i.e. branch poles) are the most interesting ones: they are related to the **radiating modes** of an open or partially open structure (e.g. linear and patch antennas, as well as microstrips): the branch cuts of the Green's function are defined (between any two such poles) by the dispertion equation, i.e. to the associated continuous spectrum of the electromagnetic radiation. Integration paths avoiding such cuts often involve winding around different complex poles from time to time. The solutions associated to them are invariant by monodromy of the integration path and, by the Residue Theorem, so are the branch cuts. They are a effective in 'cumulating' certain subsets of singularities which —globally— define other modes, and whose classification depends on the quadrant which the poles' propagation constants lie in.

- Usually, branch poles in $\mathbb{R}$ represent **surface waves** (propagating "along" the interface between different media), whereas poles lying in $\mathbb{C}\backslash\mathbb{R}$ represent **space waves**; both can be of *proper* (attenuation for $\|\mathbf{r}\|\to\infty$) or *improper* (propagation for $\|\mathbf{r}\|\to\infty$) type. Among these four types of "continuous-spectrum modes," proper surface-waves and improper space-waves (the latter called ***leaky* waves**) are associated 1-to-1 to branch poles, called "***surface* poles**" or "***leaky* poles**," respectively. For more details, cfr. [70] and [98]. The reason why branch poles represent such modes is due to the singular nature of the Green's function at the interface between the closed structure (e.g. the dielectric slab or in between its stratified layers) and the open space, where radiation takes the place to free oscillations (cfr. note [121]).

- Regular algebraic branch points located in the numerator of the Green's function (not necessarily on its zeroes) are usually related to geometric singularities of the domain, such as wedges, pin-points or other discontinuities: a path winding around them leads to multiple solutions accounting for different scattering angles of the electromagnetic fields on either sides of the discontinuity [99].

- Logarithmic branch-point singularities may arise — cfr. [76]: they are usually for $\|\boldsymbol{\kappa}\|^2=\kappa(\omega)^2$, i.e. *at* the cut-off frequencies, or at other physical 'criticalities' of the model. For this reason, they are the worst singularities to deal with, since they cannot be neglected, yet they pose serious convergence problems for most numerical integration techniques. Asymptotic evaluation of such integrals is thus needed in order to separate the logarithmic singularity's contribution from other (polar or regular) contributions.

- Essential singularities correspond to the model's geometric limits, whenever the domain boundaries are reached (e.g. for $\mathbf{r}=\mathbf{r}'$, cfr. (1.432)), i.e. to the singular nature of the radiation kernel. Such singulariries may overlap with logarithmic branch points and the same above problematics arise.

It is also important to stress that, since $\kappa(\omega)$ is a function of the frequency, such points usually move in $\mathbb{C}$ as the sources' frequency changes: different singular points may split [join together] at certain frequencies into [higher-] lower-order points, and beginning (or stopping) to "travel" on $\mathbb{R}$ — in such sense, they are movable singularities, as defined in §2.4.6. The propagation characteristics and efficiency of electromagnetic structures —such as a optic fibre, a stratified PCB (printed circuit board) or microstrips— may change (either gradually or abruptly), for example, from being an efficient waveguide to being an efficient directive antenna — and it is clear that only one of the above requirements should be fulfilled at a time within each of the structure's operating bandwidths for it to be effective; cfr. [24] and [70].

4.3 Complexified Electromagnetics

Let us consider the so called "symmetry principle" of Electrodynamics: «the physics of an electromagnetic system is invariant by orthogonal transformations of any couples of an electric and a magnetic state variables — let E and H». This means that, using the language of group actions introduced in §0, *Electrodynamics is invariant by the 1st unitary group*[120] $\mathrm{U}(1)\cong\mathrm{SO}(2)$. By the sentence's ansatz, $\forall Z\in\mathrm{SO}(2)$ the formal electromagnetic system of state variables $\begin{pmatrix}E\\H\end{pmatrix}$ is equivalent to system $Z\begin{pmatrix}E\\H\end{pmatrix}$.

---

[120] Like many other matrix Lie groups, orthogonal [unitary] ones are defined on abstract rings [for $\mathbb{C}$ and the **hypercomplex** bodies, e.g. quaternions $\mathbb{H}$, octonions $\mathbb{O}$, etc.], but they are all isomorphic to real [complex] ones; for this reason they are often written as $\mathrm{SO}(n):\cong\mathrm{SO}_n(\mathbb{R})$, $\mathrm{U}(n):\cong\mathrm{U}_n(\mathbb{C})$ and $\mathrm{SU}(n):\cong\mathrm{SU}_n(\mathbb{C})$, in Group Theory and Mathematical Physics, $\forall n\in\mathbb{N}_0$, [63] and [8].

4.3.1. Introduction

Since $\mathrm{SO}_2(\mathbb{R})(\cdot) \cong \mathrm{U}_1(\mathbb{C})(\cdot) \cong \mathbb{T}(\cdot) \cong (\mathbb{R}/\mathbb{Z})(+)$ acts on $\mathbb{R}^2$ by a rotation around the origin $\mathbf{0} \equiv (0,0)$, its representation is by the 1-parameter group of plane-rotation matrices:

$$Z(\zeta) = \begin{pmatrix} \cos\zeta & -\sin\zeta \\ \sin\zeta & \cos\zeta \end{pmatrix}, \qquad \forall \zeta \in \mathbb{R}/2\pi\mathbb{Z};$$

if E and H are respectively an electric field and its associated magnetic field components, then Z is an **impedance matrix** (and the trigonometric functions of ζ have the physical dimension of resistances). In practice almost any choice of $\zeta \in [0, 2\pi[$ may result in "non-physical" electromagnetic quantities' couples but —when inter-related with each other— they become *automorphic* to the physical ones. An easy interpretation of such a strong **gauge symmety** is evident for $\zeta \in \{0, \pi/2\}$: in those cases in fact $Z|_{\{0,\frac{\pi}{2}\}} = \{I_2, \left(\begin{smallmatrix} 0 & -1 \\ 1 & 0 \end{smallmatrix}\right)\} \cong \mathbb{Z}_2$. $\zeta = 0$ is the trivial automorphism, leaving all quantities unchanged; $\zeta = \pi/2$ exchanges all the "electric"-type quantities of the system with the associated "magnetic"-type ones, whereas magnetic quantities are exchanged with their associated electric ones with **opposite** signs.

This fact is evident at sight in the Maxwell's equations (1.383)–(1.385) and explains why "non-physical" magnetic densities like the ϱ_{m} and J_{m} may (and *need*) be added to restore the symmetry. Due to this $\mathrm{SO}(2)$ gauge, the I and II Maxwell's equations can be formally exchanged with each other, as well as the III and IV one, by the imaginary automorphisms $\boldsymbol{\imath}_p \in \mathfrak{GL}_{\mathbb{R}}\Lambda_p(\mathbb{R}^4_3)$, for $0 \le p \le 3$, with $\boldsymbol{\imath}_p^2 = -\mathbb{1}$:

$$\begin{array}{ccc} E \xrightarrow{\imath_1^*} H \xrightarrow{\imath_1^*} -E & \qquad & J \xrightarrow{\imath_2^*} J_{\mathrm{m}} \xrightarrow{\imath_2^*} -J \\ D \xrightarrow{\imath_2^*} B \xrightarrow{\imath_2^*} -D & \qquad & \varrho \xrightarrow{\imath_3^*} \varrho_{\mathrm{m}} \xrightarrow{\imath_3^*} -\varrho \end{array} \tag{1.443}$$

The electromagnetic field possesses indeed an almost-complex structure, as defined in §2.1.1: loosely speaking, it is sufficient to extend for a moment the physical three-dimensional space where Electromagnetics takes place —and where both the classical electric vector field $\mathbf{E}$ and the magnetic vector field $\mathbf{H}$ "live"— to a *hexa-dimensional* space whose "points" are the 6 independent components of both vector fields, ordered —for example with respect to the 3 orthogonal coordinates of each one— as $(\mathbf{E},\mathbf{H}) \equiv (E,H)^* \equiv (E_1,E_2,E_3,H_1,H_2,H_3) \in \mathbb{R}^6$.

From the mathematical point of view this just means considering the electric and magnetic vector fields as belonging to a 6-dimensional fibre bundle of the Minkowski space-time manifold, instead to a 3-dimensional one. Since the Maxwell's equations are coupled equations for both fields, the degree of the variety is again lowered by such relations —as it is well known to any 2nd-year Physics student— down to 3.

Even in Classical Mechanics the electric and magnetic fields components are canonically conjugate physical quantities, as long as many other couples: the electrodynamic potentials $(\mathbf{A}, V)$, the electric tensions and currents $(\mathbf{V},\mathbf{I})$ in a lumped-parameter electric network [9]–§D.4. They are, thus, like the momentum-position $(\mathbf{p},\mathbf{q})$ or the action-angle $(\mathbf{I},\phi)$ coordinate couples for any Hamiltonian system, whose phase space is in fact even-dimensional — cfr. [8], [33], [37], [49], [91] and §2.1.

4.3.2. The Maxwell's bundle

Firtst of all $\Lambda_{1,1}(\mathcal{M}) \cong \mathsf{T}^{\mathbb{C}}\mathcal{M} \equiv \mathsf{T}^{1,0}\mathcal{M} \oplus \mathsf{T}^{0,1}\mathcal{M}$ for every almost-complex manifold, so basically its complexified vector fields can be identified with its $(1,1)$-vectors. If $\mathcal{M}$ is finite-dimensional, this association holds for its complex differential 1-forms too, by vector space duality: $\Lambda^1_{\mathbb{C}}(\mathcal{M}) \equiv \Lambda^{1,0}(\mathcal{M}) \oplus \Lambda^{0,1}(\mathcal{M}) \equiv \Lambda^1(\mathsf{T}^{1,0}\mathcal{M}) \otimes \Lambda^1(\mathsf{T}^{0,1}\mathcal{M})$, cfr. §2.2.1.

For simplicity's sake the hypothesis of electromagnatically-linear and stationary matter will be assumed inside $\Omega \subseteq \mathbb{R}^3$ and the complexified physical quantities to be introduced will be named after the well-known tensor formulation for Electrodynamics [9]–§4.3. The **Maxwell's p-bundle**, for $0 \leq p \leq 3$, is the complexified cotangent bundle of p-forms for $\Lambda^{p,p}(\Omega)=\Lambda^p(\mathsf{T}^{1,0}\mathcal{M})\otimes\Lambda^p(\mathsf{T}^{0,1}\mathcal{M})$ with respect to $\boldsymbol{i}_p$ (i.e. using sets of isomorphic coordinates for electric-type and magnetic-type quantities).

Let $\mathcal{B}^*\{\mathrm{d}x_1,\mathrm{d}x_2,\mathrm{d}x_3\}$ be an orthogonal cotangent basis for §4.1.1's real differential forms (e.g. the Cartesian one: $(x_1,x_2,x_3)\equiv\mathbf{r}\equiv(x,y,z)$) and $\mathcal{B}'^*\{\mathrm{d}y_1,\mathrm{d}y_2,\mathrm{d}y_3\}$ another cotangent basis isomorphic to the former one; throughout this paragraph a 'prime' symbol ' $'$ ' will denote that a differential form or operator refers to the primed coordinates' system; The associated complexified cotangent basis $\mathcal{B}^\dagger_{\mathbb{C}}\{\mathrm{d}z_1,\mathrm{d}z_2,\mathrm{d}z_3,\mathrm{d}\overline{z}_1,\mathrm{d}\overline{z}_2,\mathrm{d}\overline{z}_3\}$ is then used, where "electric"-type and "magnetic"-type quantities are functions of $\mathbf{z}=(z_1,z_2,z_3)$ and $\overline{\mathbf{z}}=(\overline{z}_1,\overline{z}_2,\overline{z}_3)$, respectively. It is important to stress the fact that functions of either $\mathbf{z}$ and $\overline{\mathbf{z}}$ are not necessarily *holomorphic* functions of such variables (i.e. the *real* electromagnetic fields' and impressed currents' expressions cannot be usually analytically continued to obtain the complexified ones), that is because the Maxwell's (or wave, or Helmholtz's) equations are *not* of Cauchy-Riemann type (1.142) —except in the static cases, when they become Laplace's equations, thus requiring harmonicity— cfr. §2.1.3.

The **electromagnetic** 1-form and the electromagnetic **induction** 2-form, as well as the electromagnetic **current density** 2-form and the electromagnetic **charge** density 3-form are defined, by analogy to the electromagnetic tensors' definition in [9]–§4.3, as the following complex differential forms: $\boldsymbol{e}\in\Lambda^{1,0}(\Omega)\oplus\Lambda^{0,1}(\Omega)<\Lambda^1_{\mathbb{C}}(\Omega)$, $\boldsymbol{\gamma},\boldsymbol{j}\in\Lambda^{2,0}(\Omega)\oplus\Lambda^{0,2}(\Omega)$ and $\boldsymbol{p}\in\Lambda^{3,0}(\Omega)\oplus\Lambda^{0,3}(\Omega)$, defined taking (1.130) into account; furthermore, $\boldsymbol{e}$ is also fully computed here for clarity's sake:

$$\boldsymbol{e}(\mathbf{z},\overline{\mathbf{z}}) = \sum_{k=1}^{3}\Big(E_k(\mathbf{z})\mathrm{d}z_k + \imath H_k(\overline{\mathbf{z}})\mathrm{d}\overline{z}_k \Big);$$

$$\begin{aligned}\boldsymbol{\gamma}(\mathbf{z},\overline{\mathbf{z}}) &= \frac{1}{2}\sum_{\substack{j,h,k=1,\\ j\neq h\neq k}}^{3}\Big(D_j(\mathbf{z})\mathrm{d}z_h\wedge\mathrm{d}z_k + \imath B_j(\overline{\mathbf{z}})\mathrm{d}\overline{z}_h\wedge\mathrm{d}\overline{z}_k \Big) = \\ &= \quad D_1\mathrm{d}z_2\wedge\mathrm{d}z_3 + D_2\mathrm{d}z_3\wedge\mathrm{d}z_1 + D_3\mathrm{d}z_1\wedge\mathrm{d}z_2 + \\ &\quad + \imath\Big(B_1\mathrm{d}\overline{z}_2\wedge\mathrm{d}\overline{z}_3 + B_2\mathrm{d}\overline{z}_3\wedge\mathrm{d}\overline{z}_1 + B_3\mathrm{d}\overline{z}_1\wedge\mathrm{d}\overline{z}_2 \Big);\end{aligned}$$

$$\boldsymbol{p}(\mathbf{z},\overline{\mathbf{z}}) = \rho(\mathbf{z})\mathrm{d}z_1\wedge\mathrm{d}z_2\wedge\mathrm{d}z_3 + \imath\rho_{\mathrm{m}}(\overline{\mathbf{z}})\mathrm{d}\overline{z}_1\wedge\mathrm{d}\overline{z}_2\wedge\mathrm{d}\overline{z}_3.$$

Hereinafter such differential forms might be rewritten as the following (formal) abbreviated forms:

$$\begin{cases} \boldsymbol{e} = \mathbf{E}\mathrm{d}\mathbf{z} + \imath\mathbf{H}\mathrm{d}\overline{\mathbf{z}} \\ \boldsymbol{\gamma} = \mathbf{D}\mathrm{d}\mathbf{z}^2 + \imath\mathbf{B}\mathrm{d}\overline{\mathbf{z}}^2 \\ \boldsymbol{j} = \mathbf{J}\mathrm{d}\mathbf{z}^2 + \imath\mathbf{J}_{\mathrm{m}}\mathrm{d}\overline{\mathbf{z}}^2 \\ \boldsymbol{p} = \rho\mathrm{d}\mathbf{z}^3 + \imath\rho_{\mathrm{m}}\mathrm{d}\overline{\mathbf{z}}^3 \end{cases} \tag{1.444}$$

If the gauge of §4.3.1 is chosen to be $\zeta=0$, all the complexified electromagnetic quantities are physically meaningful for $\mathbf{z}=\overline{\mathbf{z}}$, i.e. when both coordinate systems are **real**-valued and equal to each other (becoming $\mathbf{r}\equiv\mathbf{r}'$). For any other gauge choice — that is $\forall\zeta\in\mathbb{R}^*/2\pi\mathbb{Z}$— the physical coherence of Complexified Electromagnetics obeys to other I-order equations of type $P(\mathbf{z},\overline{\mathbf{z}})=0$, where $P\in\mathbb{C}^1[\mathbf{z},\overline{\mathbf{z}}]/\mathbb{C}$ is also a 1st-degree trigonometric polynomial with respect to ζ.

The gauge symmetry discussed before is represented by pseudo-invariances $\boldsymbol{i}_p=\imath\mathbb{1}$ and $\boldsymbol{i}_p^*\equiv\imath$, i.e. $\boldsymbol{i}_1\boldsymbol{e}=\imath\boldsymbol{e}$, $\boldsymbol{i}_2\boldsymbol{\gamma}=\imath\boldsymbol{\gamma}$, $\boldsymbol{i}_2\boldsymbol{j}=\imath\boldsymbol{j}$ and $\boldsymbol{i}_3\boldsymbol{p}=\imath\boldsymbol{p}$ with respect to (1.443), which simply act on the Maxwell's bundle, as the subgroup $\mathcal{Z}\big|_{\{0,\pi/2\}}\cong\mathbb{Z}_2$ whose orbit (as a quotient group by the stabilizers of $\mathrm{SO}_2(\mathbb{R})$, cfr. §1.1.5) is isomorphic to $\mathbb{Z}_4$ (each element corresponding to a rotation by $\pi/2$ radiants in $\mathrm{SO}(2)$).

As happens in the real case and by (1.133) to the real permittivity and permeability Hodge-$\star$ operators, a unique real fundamental form $\varpi_{\mathrm{EM}}\in\Lambda^{1,1}(\Omega)\cap\Lambda^2(\Omega)$ acts with respect to the complex tangent basis $\mathcal{B}_{\mathbb{C}}\left\{\frac{\partial}{\partial z_1},\frac{\partial}{\partial z_2},\frac{\partial}{\partial z_3},\frac{\partial}{\partial \overline{z}_1},\frac{\partial}{\partial \overline{z}_2},\frac{\partial}{\partial \overline{z}_3}\right\}$ defined in §2.2.1:

$$\langle Z,W\,|\,\varpi_{\mathrm{EM}}\rangle := -\langle Z\,|\,\boldsymbol{\imath}_1 W\rangle \tag{1.445}$$

an hermitian product

$$\langle Z|W\rangle_{\mathrm{EM}} \overset{\text{def}}{=} \langle \mathrm{Re}\,Z\,|\,\mathrm{Re}\,W\rangle - \imath\langle Z;W\,|\,\varpi_{\mathrm{EM}}\rangle \tag{1.446}$$

is also defined $\forall Z,W\in\mathsf{T}\Omega$ from (1.385)–(1.388). Of course the Kähler form ϖ_{EM} is always defined as (1.133),

$$\begin{aligned}\varpi_{\mathrm{EM}} &= \frac{\imath}{2}(\star_{\mathrm{e}}+\boldsymbol{\imath}\star_{\mathrm{m}})\mathrm{d}\mathbf{z}\wedge\mathrm{d}\overline{\mathbf{z}}\\ &= \frac{\imath}{2}\sum_{h,k=1}^{3}\left(\varepsilon_{h,k}+\mu_{h,k}\boldsymbol{\imath}\right)\mathrm{d}z_h\wedge\mathrm{d}\overline{z}_k,\end{aligned} \tag{1.447}$$

where the matrix $H=\left(\varepsilon_{h,k}+\imath\mu_{h,k}\right)_{1\le h,k\le 3}$ is almost-everywhere hermitian and positive-definite. Then there usually exist a ***conformal*** (curvilinear) coordinate system $\boldsymbol{\varrho}\equiv(\varrho_1,\varrho_2,\varrho_3)\in\mathbb{C}^3$ respect to which the Kähler form is a *diagonal*, positive-definite, i.e. $H\sim\mathrm{diag}(\lambda_1,\lambda_2,\lambda_3)$, with $\lambda_1,\lambda_2,\lambda_3>0$.

The constitutive equations (1.380) are complexified into one (plus its inverse) by means of the associated Lefshetz operators

$$\begin{cases} L\boldsymbol{e}=\boldsymbol{\gamma}\\ \Lambda\boldsymbol{\gamma}=\boldsymbol{e}\end{cases} \tag{1.448}$$

The Poynting 2-form is $\boldsymbol{\pi}:=\boldsymbol{e}\wedge\boldsymbol{\imath}\boldsymbol{e}$, that is:

$$\begin{aligned}\boldsymbol{\pi} &= \left(\mathbf{E}(\mathbf{z})\mathrm{d}\mathbf{z}+\imath\mathbf{H}(\overline{\mathbf{z}})\mathrm{d}\overline{\mathbf{z}}\right)\wedge\left(\imath\mathbf{E}(\overline{\mathbf{z}})\mathrm{d}\overline{\mathbf{z}}-\mathbf{H}(\mathbf{z})\mathrm{d}\mathbf{z}\right)\\ &= \imath\mathbf{E}(\mathbf{z})\otimes\mathbf{E}(\overline{\mathbf{z}})\mathrm{d}\mathbf{z}\wedge\mathrm{d}\overline{\mathbf{z}}-\mathbf{E}(\mathbf{z})\otimes\mathbf{H}(\mathbf{z})\mathrm{d}\mathbf{z}^2-\mathbf{E}(\overline{\mathbf{z}})\otimes\mathbf{H}(\overline{\mathbf{z}})\mathrm{d}\overline{\mathbf{z}}^2+\\ &\quad+\imath\mathbf{H}(\overline{\mathbf{z}})\otimes\mathbf{H}(\mathbf{z})\mathrm{d}\mathbf{z}\wedge\mathrm{d}\overline{\mathbf{z}}.\end{aligned} \tag{1.449}$$

The energy density 3-form is $\boldsymbol{u}:=\frac{1}{2}\boldsymbol{e}\wedge\boldsymbol{\gamma}$, that is:

$$
\begin{aligned}
\mathscr{u} &= \frac{1}{2}\Big(\mathbf{E}(\mathbf{z})\mathrm{d}\mathbf{z}+\imath\mathbf{H}(\overline{\mathbf{z}})\mathrm{d}\overline{\mathbf{z}}\Big)\wedge\Big(\mathbf{D}(\mathbf{z})\mathrm{d}\mathbf{z}^2+\imath\mathbf{B}(\overline{\mathbf{z}})\mathrm{d}\overline{\mathbf{z}}^2\Big) = \\
&= \frac{1}{2}\Big(\mathbf{E}(\mathbf{z})\otimes\mathbf{D}(\mathbf{z})\mathrm{d}\mathbf{z}^3+\imath\mathbf{E}(\mathbf{z})\otimes\mathbf{B}(\overline{\mathbf{z}})\mathrm{d}\mathbf{z}\wedge\mathrm{d}\overline{\mathbf{z}}^2+ \\
&\quad +\imath\mathbf{H}(\overline{\mathbf{z}})\otimes\mathbf{D}(\mathbf{z})\mathrm{d}\overline{\mathbf{z}}\wedge\mathrm{d}\mathbf{z}^2-\mathbf{H}(\overline{\mathbf{z}})\otimes\mathbf{B}(\overline{\mathbf{z}})\mathrm{d}\overline{\mathbf{z}}^3\Big) \\
&\equiv \Big(\upsilon_{\mathrm{e}}(\mathbf{z})+\upsilon_{\mathrm{m}}(\overline{\mathbf{z}})\Big)+\imath\Big(\upsilon_{\mathrm{em}}(\mathbf{z},\overline{\mathbf{z}})-\upsilon_{\mathrm{me}}(\mathbf{z},\overline{\mathbf{z}})\Big).
\end{aligned}
\tag{1.450}
$$

where $\upsilon_{\mathrm{e}}\in\Lambda^{3,0}(\Omega)$ and $\upsilon_{\mathrm{m}}\in\Lambda^{0,3}(\Omega)$ are the *real* electric and magnetic energy density 3-forms defined in (1.385),

$$
\begin{cases}
\upsilon_{\mathrm{e}}(\mathbf{z}) = u_{\mathrm{e}}(\mathbf{z})\mathrm{d}\mathbf{z} \equiv u_{\mathrm{e}}(z_1,z_2,z_3)\mathrm{d}z_1\wedge\mathrm{d}z_2\wedge\mathrm{d}z_3 \\
\upsilon_{\mathrm{m}}(\overline{\mathbf{z}}) = u_{\mathrm{m}}(\overline{\mathbf{z}})\mathrm{d}\overline{\mathbf{z}} \equiv u_{\mathrm{m}}(\overline{z}_1,\overline{z}_2,\overline{z}_3)\mathrm{d}\overline{z}_1\wedge\mathrm{d}\overline{z}_2\wedge\mathrm{d}\overline{z}_3
\end{cases}
$$

expressed by means of hermitian products and Lefshetz operators, cfr. (1.139) too:

$$
\begin{aligned}
u_{\mathrm{e}}+u_{\mathrm{m}} &= \frac{1}{2}\langle\boldsymbol{e}\,|\,\boldsymbol{e}\rangle_{\mathrm{EM}}=\frac{1}{2}\langle\boldsymbol{e}\,|\,\boldsymbol{\gamma}\rangle_{\mathbb{C}}=\frac{1}{2}\langle\boldsymbol{e},\Lambda\boldsymbol{\gamma}\,|\,\varpi_{\mathrm{EM}}\rangle \\
&= \frac{1}{2}\langle\boldsymbol{e}\,|\,\Lambda\boldsymbol{\gamma}\rangle_{\mathrm{EM}}\equiv\frac{1}{2}\langle L\boldsymbol{e}\,|\,\boldsymbol{\gamma}\rangle_{\mathrm{EM}}\,.
\end{aligned}
\tag{1.451}
$$

The 3-form $\upsilon_{\mathrm{em}}-\upsilon_{\mathrm{me}}\in\Lambda^{2,1}(\Omega)\oplus\Lambda^{1,2}(\Omega)$ is the complex mutual induction's energy density (although its components, having different physical dimensions, are not directly 'usable').

The gauge symmetry exposed in §4.3.1 is reformulated into the Maxwell's bundle observing that I and III equations use the holomorphic differential in $\mathsf{T}^{1,0}\Omega$, whereas II and IV equations use the antiholomorphic differential on $\mathsf{T}^{0,1}\Omega$, cfr. (1.172):

$$
\begin{cases}
\partial D = \varrho \\
\overline{\partial} B = \varrho_{\mathrm{m}}
\end{cases}
\qquad
\begin{cases}
\partial E+\dot{B} = -J_{\mathrm{m}} \\
\overline{\partial} H-\dot{D} = J
\end{cases}
$$

with differential operators' exchange leading to $\begin{cases}\overline{\partial} D = 0 \\ \partial B = 0\end{cases}$ and $\begin{cases}\overline{\partial} E = 0 \\ \partial H = 0\end{cases}$.

Using (1.444) and the differential's decomposition $\mathrm{d}=\partial+\overline{\partial}$, the complexified Maxwell's equations (1.383) are:

$$\begin{cases} \mathrm{d}\gamma = p \\ \mathrm{d}e - i(\dot{\gamma} + j) = 0 \end{cases} \tag{1.452}$$

Poynting's Theorem — Maxwell's bundle – time domain

Forms (1.449)–(1.451) satisfy the following complexified version of (1.386) on every analytical domain $\Omega \subseteq \mathbb{C}^6$:

$$\oint_{+\partial\Omega} \pi + \int_{\Omega} (\dot{u} + e \wedge i j) = 0. \tag{1.453}$$

If (1.453) is computed on any 3-dimensional subdomain $\Omega' = \{ (\mathbf{z}, \overline{\mathbf{z}}) \in \Omega \mid \mathbf{z} = \overline{\mathbf{z}} \} \subset \Omega$ (thus $\Omega' \subseteq \mathbb{R}^3$ by isomorphism) then the physical meaning of this balance equation is the electromagnetic energy's conservation principle, as in the Minkowski space-time case.

Combining (1.452) with (1.448) and the continuity equation $\mathrm{d}j + \dot{p} = 0$ leads to the complexified wave equation for the electromagnetic field — in the case of homogeneous, lossless media and the absence of impressed currents, for simplicity's sake. First of all let the 1st constitutive equation of (1.448) be combined with II equation of (1.452):

$$\mathrm{d}e - i(L\dot{e} + j) = 0\,; \tag{1.454}$$

let the I equation of (1.452) be differentiated with respect to time and the continuity equation be put inside: then there exists a closed 2-form $\exists \phi \in Z^2_{\mathbb{C}}(\Omega)$ such that:

$$\mathrm{d}\dot{\gamma} = -\,\mathrm{d}j \qquad \Rightarrow \qquad \dot{\gamma} = -\,j + \phi\,. \tag{1.455}$$

Apply the codifferential 'δ' to (1.454), using $[\delta \,|\, i] = 0$ and $\delta L = \Lambda \mathrm{d}$ by (1.141), so that

$$\delta \mathrm{d}e - i\delta(L\dot{e} + j) = \Delta e - \mathrm{d}\delta e - i(\Lambda \mathrm{d}\dot{e} + \delta j) = 0\,; \tag{1.456}$$

Let (1.454) be mixed back again into left-hand side of (1.456) to get

$$\Delta\boldsymbol{e}-\mathrm{d}\delta\boldsymbol{e}-\boldsymbol{i}\Lambda\Big[\boldsymbol{i}(L\ddot{\boldsymbol{e}}+\dot{\boldsymbol{j}})\Big]-\boldsymbol{i}\delta\boldsymbol{j}=\Delta\boldsymbol{e}-\mathrm{d}\delta\boldsymbol{e}+\Lambda L\ddot{\boldsymbol{e}}+\Lambda\dot{\boldsymbol{j}}-\boldsymbol{i}\delta\boldsymbol{j}\,,$$

then rewrite the equation:

$$\Delta\boldsymbol{e}+\Lambda L\ddot{\boldsymbol{e}}=\mathrm{d}\delta\boldsymbol{e}-\Lambda\dot{\boldsymbol{j}}+\boldsymbol{i}\delta\boldsymbol{j}. \tag{1.457}$$

Finally, put the 2nd constitutive equation of (1.448) into the right-hand side of (1.457) such that $\mathrm{d}\delta\boldsymbol{e}=\mathrm{d}\delta\Lambda\boldsymbol{\gamma}=\mathrm{d}L\mathrm{d}\boldsymbol{\gamma}$; making use of (1.455) and considering the electromagnetic problem to "start" at a past time $t_0\in\mathbb{R}$ (i.e. initial conditions for $\boldsymbol{j}$ at time t_0 are known), then the **complexified** wave equation is obtained by integration:

$$\Delta\boldsymbol{e}+\Lambda L\ddot{\boldsymbol{e}}\,=\,\boldsymbol{i}\delta\boldsymbol{j}-\Lambda\dot{\boldsymbol{j}}+\mathrm{d}\delta\Lambda\int_{t_0}^{t}\boldsymbol{j}\mathrm{d}\tau. \tag{1.458}$$

The complexified wave operator is then $\Delta+\Lambda L\frac{\mathrm{d}^2}{\mathrm{d}t^2}$ which, for electromagnetically-linear, stationary, lossless, homogeneous and anisotropic media, becomes $\Delta+c_0^{-2}\frac{\mathrm{d}^2}{\mathrm{d}t^2}$, the complex analogue to the D'Alambert's operator.

Explicit reading of formula (1.458) —in the absence of impressed currents and in a linear, homogeneous, isotropic and lossless medium for simplicity's sake— on a conformal coordinate system $(\mathbf{z},\overline{\mathbf{z}})\equiv(z_1,z_2,z_3;\overline{z}_1,\overline{z}_2,\overline{z}_3)$ leads to a system of more "familiar at sight" II-order elliptic PDEs:

$$\Delta\boldsymbol{e}+\Lambda L\ddot{\boldsymbol{e}}\,=\,0$$

$$\Updownarrow$$

$$\begin{aligned}
&\frac{\partial^2\boldsymbol{e}}{\partial z_1\partial\overline{z}_1}+\frac{\partial^2\boldsymbol{e}}{\partial z_2\partial\overline{z}_2}+\frac{\partial^2\boldsymbol{e}}{\partial z_3\partial\overline{z}_3}+\frac{\imath}{2c^2}\frac{\partial^2\boldsymbol{e}}{\partial t^2}=\\
&=\frac{\partial^2\boldsymbol{e}}{\partial\overline{z}_1^2}+\frac{\partial^2\boldsymbol{e}}{\partial\overline{z}_2^2}+\frac{\partial^2\boldsymbol{e}}{\partial\overline{z}_3^2}-\frac{\imath}{c^2}\frac{\partial^2\boldsymbol{e}}{\partial t^2}=\\
&=\frac{\partial^2\boldsymbol{e}}{\partial z_1^2}+\frac{\partial^2\boldsymbol{e}}{\partial z_2^2}+\frac{\partial^2\boldsymbol{e}}{\partial z_3^2}-\frac{\imath}{c^2}\frac{\partial^2\boldsymbol{e}}{\partial t^2}=0.
\end{aligned} \tag{1.459}$$

4.3.3. Cohomology

The Hodge Decomposition Theorem, §2.2.4, (1.196), can be successfully applied to the Maxwell's bundle, to infer from (1.452) the proper spaces where each component belongs and the influences between the domain's topology and its Dolbeault cohomology (particularly in the case of Ω compact, i.e. for guided waves phenomena). Similar application of DeRham's cohomology to the *real* exterior formulation of Electromagnetics (§4.1.1) was carried out in [9]–§§3.7, 4.4.

For example, if *no* magnetic currents/charges are present (e.g. a gauge where $J_{\mathrm{m}} \equiv 0$ and $\varrho_{\mathrm{m}} \equiv 0$ is chosen) then γ is a $\overline{\partial}$-closed, ∂-exact form, i.e. $\gamma \in \mathrm{H}^0(\Omega)$; vice versa, if there are no electric sources (i.e. $J \equiv 0$ and $\varrho \equiv 0$) γ is a ∂-closed, $\overline{\partial}$-exact form. If there are only *static* sources, the electromagnetic problem splits, as usually, in two independent ones — **Electrostatics** and **Magnetostatics**. For Electrostatics (with Ω bounded), there exist $V \in \mathrm{H}^0(\Omega) \cong \mathrm{H}^0_{\mathbb{C}}(\Omega)$, $A \in \Lambda^{2,0}(\Omega)$ and $[\![\varphi]\!] \in \mathrm{H}^{1,0}(\Omega)$ such that

$$\Lambda^{1,0}(\Omega) \ni \boldsymbol{e} = \partial V + \partial^\dagger A + \varphi; \tag{1.460}$$

this is the *physical* proof of Helmholtz's Decomposition (1.196) in the static case, where V and A are the electrostatic potentials, whereas there are as many *harmonic* 1-forms ϕ as many times Ω' is multiply superficially connected. For Magnetostatics instead, $\boldsymbol{e}$ features the "dual" Hodge Decomposition as depending on a mangetostatic potentials and on $[\![\psi]\!] \in \mathrm{H}^{0,1}(\Omega)$, where there are as many ***co**harmonic* 1-forms ψ as many times Ω' is multiply linearly connected, i.e. the number of holes inside Ω' (or handles, cfr. §2.4.2) which is the 1st Betti number of Ω' or the $(0,1)$th Hodge number of Ω, i.e. $\beta_1(\Omega') = \mathrm{h}_{0,1}(\Omega)$ by (1.197).

At last, in the trivial case of homogeneous charge distribution, $\boldsymbol{e}$ becomes a d-closed form and by the $\partial\overline{\partial}$-Lemma of §2.2.4 it is also a $\partial\overline{\partial}$-exact form, i.e. $\exists [\![V]\!] \in \mathrm{H}^{0,0}(\Omega)$ such that (Poincaré equation) $\boldsymbol{e} = \partial\overline{\partial}^\dagger \star \boldsymbol{p}$.

The Hodge's Decomposition Theorem holds also for full electrodynamic problems on bounded domains (with all the complex forms depending on time t), but the Dolbeault cohomology spaces are always finite-dimensional and depending on the (real) domain Homology. It is well-knon in fact that such resonating modes (also called stationary waves, free oscillations,[121] or TEM modes for guided-waves' transverse propagation on the cross-section [50]) are the solutions to the Laplace's equation (i.e. the "static" waves' equation) for the electromagnetic field:

$$\Delta \boldsymbol{e} = 0\,; \tag{1.461}$$

From the algebraic point of view, this means that static modes are the eigenvectors of the Laplace's operator associated to eigenvalue $0 \in \mathrm{spec}\Delta$ (if any). From the geometrical point of view this means that $\boldsymbol{e} \in \mathrm{H}^1_{\mathbb{C}}(\Omega) = \mathrm{H}^{1,0}(\Omega) \oplus \mathrm{H}^{0,1}(\Omega)$ and there are possibly $\mathrm{h}_{1,0}(\Omega) + \mathrm{h}_{0,1}(\Omega)$ stationary solutions: $\mathrm{h}_{1,0}(\Omega)$ of them, of "electric type" —since $\mathrm{Re}\boldsymbol{e} \equiv E$ by (1.444)— due to superficial connection (i.e. the presence of "bubbles" inside Ω'); $\mathrm{h}_{0,1}(\Omega)$ of them, of "magnetic type" —since $\mathrm{Im}\boldsymbol{e} \equiv H$ with respect to $\overline{\mathbf{z}}$ — due to linear connection (i.e. the presence of handles 'across' Ω'). An example of such electro-magnetostatic modes on a fractal resonator with hull-like symmetry (like the Sierpiński and Menger sponges) is found in §5.1.6, as well as in [15].

As a last remark the harmonic forms like ϕ and ψ are trivially both coharmonic (since Ω is compact Kähler), holomorphic (since holomorphicity and harmonicity coincide by complexification, cfr. §2.2.3) and thus also anti-holomorphic.

Since $\mathrm{H}^1_{\mathbb{C}}(\Omega) = \mathrm{Z}^1_{\mathbb{C}}(\Omega)/\mathrm{B}^1_{\mathbb{C}}(\Omega)$ has an orbifold structure the paths over it are continuous homotopies between cohomology classes; then it is possible to represent $\mathrm{Z}^1_{\mathbb{C}}(\Omega)$ as a ramified covering [multi-valued function] whose fibres [sheets] are elements of the cohomology classes $[\![\varphi]\!] + [\![\psi]\!] \equiv [\![\varphi + \psi]\!] \in \mathrm{H}^{0,1}(\Omega) \oplus \mathrm{H}^{0,1}(\Omega)$. As long

[121] The name 'free oscillations' because they are non-trivial solutions to homogeneous differential equations. From the physical point of view such oscillations are the macroscopic states due to the vaccum states of Quantum Mechanics (i.e. ground state's matter-radiation interaction), [34].

as a path $\gamma\subset\Omega$ is followed along with the complexified coordinates $(\mathbf{z},\overline{\mathbf{z}})\in\gamma$, different and different harmonic differentials are travelled upon; as ramification points of the Kähler manifold Ω are 'crossed' (analogue to loops' contouring in the 1-dimensional case of Riemann surfaces, cfr. §2.4.2) different fibres [sheets] are visited, so the cohomology changes and different resonating modes of the Maxwell's equations must be taken into account when evaluating the full complexified radiation problem.

4.3.4. Example: the cubic resonator

As a sample application to the Maxwell's bundle consider a cubic cavity $[0,1]^3\subset\mathbb{R}^3$, with perfectly conducting boundaries and filled with an electromagnetically-linear, stationary, lossless, homogeneous and isotropic medium whose permittivity and permeability constants are $\varepsilon,\mu\in\mathbb{R}_+$, respectively: the fundamental form (1.447) is

$$\varpi_{\text{EM}}=(\varepsilon+\mu\boldsymbol{\imath})\star\ .$$

From the two isomorphic Cartesian coordinate systems $\mathbf{x}\equiv(x_1,x_2,x_3)$ and $\mathbf{y}\equiv(y_1,y_2,y_3)$ the complexified domain with its conformal metric is the complex hull $\mathbf{z}\equiv(z_1,z_2,z_3)\in(1+\imath)[0,1]^3\subset\mathbb{C}^3$. By solving (1.461) with appropriate boundary conditions the free oscillations of the electromagnetic resonator $\boldsymbol{e}_{l,m,n}\in\operatorname{Ker}\Delta<\Lambda^1_{\mathbb{C}}((1+\imath)[0,1]^3)$ are found to be the following complex trigonometric functions, $\forall l,m,n\in\mathbb{N}_0$:

$$\begin{aligned}\boldsymbol{e}_{l,m,n}(\mathbf{z},\overline{\mathbf{z}}) = {}&\\ &\sin(l\pi z_1)\cos(m\pi z_2)\cos(n\pi z_3)\mathrm{d}z_1+\cos(l\pi z_1)\sin(m\pi z_2)\cos(n\pi z_3)\mathrm{d}z_2\ +\\ &+\cos(l\pi z_1)\cos(m\pi z_2)\sin(n\pi z_3)\mathrm{d}z_3+\imath\cos(l\pi\overline{z}_1)\sin(m\pi\overline{z}_2)\sin(n\pi\overline{z}_3)\mathrm{d}\overline{z}_1\ +\\ &+\imath\sin(l\pi\overline{z}_1)\cos(m\pi\overline{z}_2)\sin(n\pi\overline{z}_3)\mathrm{d}\overline{z}_2+\imath\sin(l\pi\overline{z}_1)\sin(m\pi\overline{z}_2)\cos(n\pi\overline{z}_3)\mathrm{d}\overline{z}_3.\end{aligned}$$

Of course the eigenforms $\boldsymbol{e}_{l,m,n}\in\operatorname{Ker}\Delta=\mathrm{H}^{1,1}((1+\imath)[0,1]^3)$ are holomorphic because they are harmonic; on a fully electrodynamic problem —like those solved starting from the homogeneous problem's eigenfunctions via the Green's function methods shown in §§4.2.3–4.2.4— the electromagnetic field is not represented anymore by holomorphic functions but rather by complex-differentiable, as discussed in §4.3.3.

Whenever $\mathbf{z}=\overline{\mathbf{z}}=\mathbf{x}$ the well-known trigonometric eigenvectors for either the electric and magnetic fields' 1-forms $E,H \in \Lambda^1([0,1]^3)$ are obtained, since $\mathscr{e}_{l,m,n} \equiv E_{l,m,n}+\imath H_{l,m,n}$, cfr. [15] and [50]:

$$
\begin{aligned}
&\mathscr{e}_{l,m,n}(x_1,x_2,x_3) = \\
&\qquad \Big(\sin(l\pi x_1)\cos(m\pi x_2)\cos(n\pi x_3)+\imath\cos(l\pi y_1)\sin(m\pi y_2)\sin(n\pi x_3)\Big)\,\mathrm{d}x_1 \;+ \\
&\quad + \Big(\cos(l\pi x_1)\sin(m\pi x_2)\cos(n\pi x_3)+\imath\sin(l\pi x_1)\cos(m\pi x_2)\sin(n\pi x_3)\Big)\,\mathrm{d}x_2 \;+ \\
&\quad + \Big(\cos(l\pi x_1)\cos(m\pi x_2)\sin(n\pi x_3)+\imath\sin(l\pi x_1)\sin(m\pi x_2)\cos(n\pi x_3)\Big)\,\mathrm{d}x_3 \;.
\end{aligned}
$$

4.3.5. Connection with Topological Calculus

Is is worth to note that a discrete theory for such field equations (both real and complexified) already exists, naturally employing the simplicial cohomology groups $\mathrm{H}^p(\Sigma;\mathbb{R})$ or $\mathrm{H}^p(\Sigma;\mathbb{C})$, where Σ is any simplicial complex triangulating Ω (in the real and complexified case, respectively, where $\dim\Sigma=2n$). This theory is called **Topological Calculus** and the author has been investigating its pairing to some Classical Field Theories formulated in the Exterior Algebra's language, such as Electrodynamics (in both time- and frequency-domain), Fluid-Dynamics, [84], [97], [91], and Magnetohydrodynamics (MHD), [6]. In this field of research several results were obtained, particularly when the electromagnetic properties of complicated geometries are concerned. In the case of IFS-generated fractals for example, whose geometry is self-similar, the algebraic relations used to generate a discrete triangulation are easily translated to algebraic presentations (in a fashion like was done in §1.4.2 for self-similar group actions) and the discrete-space electromagnetic fields feature the same self-similarity symmetries of the domain which simplicial Maxwell's (or Helmholtz's) equations are formulated in. For other references on discrete homology-based Topological Field Theories, cfr. [52] and [65].

The mathematical details of Topological Calculus, as well as its applications to Electrodynamics and Electromagnetism, can be found in [9]–§§3, 4, [10], [12], [20].

4.4 Appendix –2D complexified Electromagnetism

As commonly practiced in several works about elliptic operators, it is more convenient to turn a two-dimensional, real, elliptic PDE [system] in a complex ODE [system] where the real variables' vector $\mathbf{r}=(x,y)\in\Omega\subseteq\mathbb{R}^2$ is complexified in the 'trivial' way by setting $z:\equiv x+\imath y\in\mathbb{C}$ (or, equivalently, $\mathbf{r}:\equiv(\mathrm{Re}z,\mathrm{Im}z)$ — cfr. (1.131)). In this case the real elliptic operator L becomes a II-order differential operator $L_\mathbb{C}$ on the complexified domain $\Omega_\mathbb{C}\subseteq\mathbb{C}$, here shown as general, homogeneous scalar (one ODE) case with Dirichlet's boundary conditions:

$$\begin{cases} L_\mathbb{C} u = 0 \\ u\big|_{\partial\Omega_\mathbb{C}} = 0 \end{cases} \qquad u\in\mathcal{K}(\Omega_\mathbb{C}). \tag{1.462}$$

The complexified equation (and the corresponding operator) is said to be integrable if and only if there exist a map $\mathrm{Ker}L_\mathbb{C}\to\mathrm{Z}^{1,1}(\Omega_\mathbb{C})$ such that $u\mapsto\varpi\in\mathrm{Z}^{1,1}(\Omega_\mathbb{C})$ (which, by the $\partial\overline{\partial}$-Lemma of §2.2.4 is also exact in many ways) is a solution to (1.462) which

$$\oint_{+\partial\Omega_\mathbb{C}}\varpi = 0. \tag{1.463}$$

For example, a two-dimensional Helmholtz's equation like (1.397) in a homogeneous medium (of free-space wave-vector $\kappa_0\in\mathbb{R}$), by §2.4.1 and (1.459), becomes

$$\Delta u - \kappa_0^2 u \equiv \frac{\partial^2 u}{\partial z\partial\overline{z}} - \kappa_0^2 u(z,\overline{z}) = 0, \tag{1.464}$$

and the associated 1-dimensional d-closed forms' subspace is

$$\mathrm{Z}^{1,0}(\Omega_\mathbb{C}) = \left\{\varpi_\kappa(z,\overline{z}) = e^{-\imath\kappa z+\imath\frac{\kappa_0^2}{\kappa}\overline{z}}\left(\frac{\partial u}{\partial z}\mathrm{d}z + \imath\frac{\kappa_0^2}{\kappa}u(z,\overline{z})\mathrm{d}\overline{z}\right)\right\}_{\kappa\in\mathbb{R}}$$

$$\Downarrow$$

$$\mathrm{d}\varpi_\kappa = e^{-\imath\kappa z+\imath\frac{\kappa_0^2}{\kappa}\overline{z}}\left(\frac{\partial^2 u}{\partial z\partial\overline{z}} - \kappa_0^2 u(z,\overline{z})\right)\mathrm{d}z\wedge\mathrm{d}\overline{z} \equiv 0, \tag{1.465}$$

if and only if $u\in\mathcal{K}(\Omega_{\mathbb{C}})$ is a solution to (1.462)–(1.464) and a Riemann surface $\mathcal{R}$ is defined such that (1.465) holds $\forall\kappa\in\mathcal{R}$ as a spectral parameter.

Applications of this method to electromagnetic scattering problems from wedges, half-strips, half-planes and cones can be found in the works of Y.A. Antipov: [4] and the bibliography therein. More mathematical details in order to re-formulate such elemectromagnetic problems as Riemann-Hilbert problems can be found in §§5.3.1–5.3.2.

5 Fractal Electrodynamics

Keith Jarrett, *The Köln Concert–Part IIc*
©1975 ECM Records
(live 24.1.1975; #1–4 re-arranged excerpt)

Ἡσίοδος οἴεται καὶ περιόδοις τισὶ χρόνων γίγνεσθαι τοῖς δαίμοσι τὰς τελευάς· λέγει γὰρ·

> *ἐννέα τοι ζώει γενεὰς λακέρυζα κορώνη // ἀνδρῶν ἡβώντων· ἔλαφος δέ τε τετρακόρωνος· // τρεῖς δ᾽ ἐλάφους ὁ κόραξ γηράσκεται· αὐτὰρ ὁ φοῖνιξ // ἐνέα τοὺς κόρακας· δέκα δ᾽ ἡμεῖς τοὺς φοίνικας* […].

Τοῦτον τὸν χρόνον εἰς πολὺ ὗηθος ἀριθμοῦ συνάγουσιν οἱ μὴ καλῶς δεχόμενοι τὴν γενεάν. Ἔστι γὰρ ἐνιαυτός· ὥστε γίγνεσθαι τὸ σύμπαν ἐννακισχίλια ἔτη καὶ ἑπτακόσια καὶ εἴκοσι τῆς τῶν δαιμόνων ζωῆς, ἔλαττον μ※ν οὐ νομίζουσιν οἱ πολλοὶ τῶν μαθηματικῶν […].

Hesiod believes that after certain periods of time even dæmons have an end […]:

> 9 generations liveth the noisy Crow of lusty men: 4 times the crow the Stag.
> 3 stags outlive the Raven; the Phoenix 9 times the raven; 10 times We outlive…

This space is calculated at a vast extent of time by people incorrectly understanding the word 'generation' (for it really means a year), so that they maketh the total duration of ye dæmons' life to be 9,720 years. Most mathematicians assign a shorter duration […].

Plutarch, *On the Cessation of Oracles*, XI

5.1 Electromagnetics on prefractal geometries

In this paragraph the Green's function for electromagnetic structures featuring some kind of geometric self-similarity is analyzed. After a brief review of the most important 'fractal-generating algorithm' is given, and a generalized analysis of such Green's functions is approached, [14], [21], a couple of cases are immediately treated: the Sierpiński sponge resonator [15] and the Sierpiński carpet patch antenna [13], [16], [53].

5.1.1. Introductory Iterated Function Systems

Attention will be given first to fractals generated via ***Iterated Function Systems*** (IFS), whose mathematical definitions and properties are left to [26], [39] and [9]–§1.7.

Recall that an IFS $\{S;w_1,w_2,\ldots,w_p\}$ on a metric space (S,d) is a collection of contraction mappings $w_1,w_2,\ldots,w_p\colon S\rightarrow S$. If $\wp_{©}(S)$ is the subset of compact sets of S then —no matter whether S is complete or not— $\wp_{©}(S)$ is a complete metric space with respect to the Hausdorff metric, $\forall A,B\in\wp_{©}(S)$:

$$\mathrm{d}_{©}(A,B)\overset{\text{def}}{=}\max\left\{\sup_{x\in A}\mathrm{d}(x,B),\sup_{y\in B}\mathrm{d}(y,A)\right\}. \qquad (1.466)$$

The following mapping $w\colon\wp_{©}(S)\rightarrow\wp_{©}(S)$

$$E\ \mapsto\ w(E):=\bigcap_{j=1}^{p}w_j(E) \qquad (1.467)$$

is contracting on $(\wp_{©}(S),\mathrm{d}_{©})$, thus it has one fixed point $E_\infty\in\wp_{©}(S)$ (i.e. a unique compact subset $E_\infty\subset S$), called the IFS' ***attractor***: whatever is the starting compact $E_0\in\wp_{©}(S)$ then, a sequence $(E_n)_{n\in\mathbb{N}}\in\wp_{©}(S)^{\mathbb{N}}$ is generated, with $E_{n+1}:=w(E_n)$, such that (with respect to the Hausdorff metric in $\wp_{©}(S)$ — i.e. to metric 'd' in S):

$$E_\infty=\lim_n E_n=\lim_n w^n(E_0)=\bigcap_{n\in\mathbb{N}}w^n(S). \qquad (1.468)$$

It is important to note that, despite E_∞ is fixed and depends only on the IFS, all the finite-step sets E_n —called for this reason **pre**-fractals— heavily depend on the choice of starting compact set E_0, also called the initiator (set).

An IFS $\{S; w_1, w_2, \ldots, w_p\}$ is said to be *just-touching* if $w_i(E)^\circ \cap w_j(E)^\circ = \varnothing$, $1 \le i < j \le p, \forall E \subseteq S$ (in general $\bigcap_{j=1}^{p} w_j(E) \neq \varnothing$, otherwise the IFS is called sparse).

If w_1, w_2, …, w_p are similarity mappings then the IFS may be also called an ***I**terated **S**imilarity **S**ystem* (**ISS**). In particular, when $S = \mathbb{R}^d$ the similarities (whose contraction ratios are $c_1, c_2, \ldots, c_p \in]0,1[$, respectively) are written as affine transformations

$$\mathbf{w}_j(\mathbf{x}) = A_j \mathbf{x} + \mathbf{b}_j\,, \qquad 1 \le j \le p, \tag{1.469}$$

where $\mathbf{w}_j \equiv (w_{j,k})_{1 \le k \le d}$, $A_j \equiv (A_{j;h,k})_{1 \le h,k \le d} \in \mathbb{M}_d(\mathbb{R})$ and $\mathbf{b}_j \equiv (b_{j,k})_{1 \le k \le n} \in \mathbb{R}^d$ and of course $c_j = \max|\mathrm{spec} A_j| < 1$ (i.e. each contraction ratio is the dominant eigenvalue's modulus of the respective matrix A_j). Furthermore, considering the contractions' composition like $\mathbf{w}_i \circ \mathbf{w}_j$ it is clear that since $\|\mathbf{w}_j(\mathbf{x}) - \mathbf{w}_j(\mathbf{y})\| \le c_j \|\mathbf{x} - \mathbf{y}\|$ $\forall \mathbf{x}, \mathbf{y} \in \mathbb{R}^d$, then $\|\mathbf{w}_i(\mathbf{w}_j(\mathbf{x})) - \mathbf{w}_i(\mathbf{w}_j(\mathbf{x}))\| \le c_i c_j \|\mathbf{x} - \mathbf{y}\|$ with:

$$c_i c_j \;=\; \max|\mathrm{spec}\, A_i| \cdot \max|\mathrm{spec}\, A_j|. \tag{1.470}$$

As shown in (1.96), the group $\mathrm{GL}_n(\mathbb{R}) \ltimes \mathbb{R}^n$ acts self-similarly on the topological subspace of prefractal iterations of the ISS (1.467): the symbolic dynamics of such a ISS, such that $\mathbf{i} \equiv (i_1, i_2, \ldots, i_L) \mapsto (\mathbf{w}_{i_1} \circ \mathbf{w}_{i_2} \circ \cdots \circ \mathbf{w}_{i_L})(E)$ allows to factor every finite-length addressing of prefractal subsets of the attractor as an infinite number of syllables (multi-indices $\mathbf{i} \in \{1, 2, \ldots, p\}^L$) of length L, which the group above acts upon.

The different notions of 'fractal dimensions' often numerically agree to the same values for that class of fractal sets generated as attractors to IFSs; if the IFS is just-touching the Hausdorff and box-counting dimensions '$\dim_{\mathrm{H}}$' and '$\dim_{\mathrm{B}}$' are equivalent [9]–§§1.4, 5 and are the *smallest* positive real solution to the following non-linear equation, [26]:

$$\sum_{j=1}^{p} c_j^{\dim_{\mathrm{H}} E_\infty} = 1. \tag{1.471}$$

It is well-known that, if the IFS is just-touching and composed of affinities with the same contraction ratio $c \in]0,1[$, then (1.471) has the exact solution

$$\dim_{\mathrm{H}} E_\infty = \frac{\log p}{\log c^{-1}} = \rho(\mathrm{GL}_n(\mathbb{R}) \ltimes \mathbb{R}^n, \mathbb{R}^n), \tag{1.472}$$

which (for the right-hand side above) in the case of ISSs, also equals the growth degree (1.101) of the group $\mathrm{GL}_n(\mathbb{R}) \ltimes \mathbb{R}^n$ acting on prefractal sets in $\wp_{©}(\mathbb{R}^n)$ and the contraction coefficient (1.97) of such action is exactly $\rho(\mathrm{GL}_n(\mathbb{R}) \ltimes \mathbb{R}^n, \mathbb{R}^n) = c^{-1}$ (where $\mathbb{R}^n$ can be replaced with any infinite metric space, even *not* complete thanks to the Haudorff metric).

5.1.2. Diaperiodic modes

Suppose $E_\infty \subset E_0 \subset \mathbb{R}^d$ are bounded domains, filled with an electromagnetically-linear, stationary, lossless, homogeneous and isotropic medium and that $(E_N)_{N\in\mathbb{N}}$ is the inclusive sequence of prefractals of a just-touching IFS $\{\mathbb{R}^d; \mathbf{w}_1, \mathbf{w}_2, \ldots, \mathbf{w}_p\}$ whose initiator is E_0 and whose attractor is E_∞. Now let Δ_N be Laplace operator on the generic E_N, with respect to the generic 0-forms (functions) on E_n and with Dirichlet's boundary conditions for simplicity's sake. Let also $G_N \in H^2(E_N)^{\otimes 2}[\![\lambda]\!]$ be the Neumann series of the Green's function for the corresponding Helmholtz's operator on E_N — cfr. §§4.1.2, 4.2.1.

Let $\mathcal{B}\{\varphi_{N,n}\}_{n\in\mathbb{N}_0}$ be an complete orthonormal basis of eigenfunctions for $-\Delta$ on E_N and $\{\lambda_{N,n}\}_{n\in\mathbb{N}_0}$ the corresponding eigenvalues, counted along with their multiplicities; with these positions the Neumann series for G_N can be written as:

$$G_N(\mathbf{x}, \mathbf{x}'; \lambda) = \sum_{n=0}^{\infty} \frac{\varphi_{N,n}(\mathbf{x})\varphi_{N,n}(\mathbf{x}')}{\lambda_{N,n} - \lambda}. \tag{1.473}$$

First of all recall that the eigenvalue $\lambda_0 = 0 \in \operatorname{spec}\Delta$ is present if and only if internal resonances (or free oscillations) are possible inside E_N; this corresponds to the case of E_N multiply connected and the finite-dimensionality of $\operatorname{Ker}\Delta$ implies that a finite number of electromagnetic (static) modes can exist in domains with non-trivial homology: that is the case of TEM modes for waveguides with multiply connected cross-sections (which will be discussed further in §§5.1.5–5.1.6). The part of the Green's function accounting for such "**cohomologic**" modes is just singular for $\lambda = 0$ and can be neglected a priori as:

$$\begin{aligned} G_N(\mathbf{x},\mathbf{x}';\lambda) &= \frac{g_N^0(\mathbf{x},\mathbf{x}')}{\lambda} + g_N(\mathbf{x},\mathbf{x}';\lambda) \\ &\equiv \frac{1}{\lambda}\sum_{k=1}^{\beta_N} \psi_{N,n}(\mathbf{x})\psi_{N,n}(\mathbf{x}') + \sum_{n=1}^{\infty} \frac{\varphi_{N,n}(\mathbf{x})\varphi_{N,n}(\mathbf{x}')}{\lambda_{N,n} - \lambda}, \end{aligned} \tag{1.474}$$

where $\operatorname{Ker}\Delta = \operatorname{span}\{\psi_{N,k}\}_{1\le k\le \beta}$ is a cohomology basis for E_N — which is to be homeomorphic to a β_N-fold torus, where β_N is its relevant Betti number, cfr. §§2.3.1, 4.3.3:

$$g_N^0(\mathbf{x},\mathbf{x}') = \sum_{k=1}^{\beta_N} \psi_{N,n}(\mathbf{x})\psi_{N,n}(\mathbf{x}'). \tag{1.475}$$

Such modes will not be considered for now, because they are responsible for static modes, whereas the Green's function decomposition we are working towards is mainly useful for full-wave electromagnetic problems. So we are left with a meromorphic function g_N which is regular in 0.

Now it is time to use the self-similarity or, at least, the recursive similarity of just-touching prefractal sets. The $(N+1)^{\text{th}}$-step prefractal E_{N+1}, can be partitioned in p distinct sub-domains, images of E_N under the p different contraction mappings, and joined with each other by at most their mutual boundaries, i.e.:

$$E_{N+1} = \mathbf{w}(E_N) = \bigcup_{j=1}^{p} \mathbf{w}_j(E_N). \tag{1.476}$$

For any fixed $N\in\mathbb{N}_0$ and $1\leq j\leq p$, let[122]

$$\varphi_{N+1,j,n}(\mathbf{x}) := \varphi_{N,n}(\mathbf{w}_j(\mathbf{x}))\chi_{\mathbf{w}_j(E_N)}(\mathbf{x});$$

this function 'replicates' the pattern of the eigenfunction $\varphi_{N,n}$, contracted on the subdomain $\mathbf{w}_j(E_N)$ and being zero outside. In particular $\varphi_{N+1,j,n}$, complying with Dirichlet's boundary conditions on $\partial\mathbf{w}_1(E_N)\cup\partial\mathbf{w}_2(E_N)\cup\ldots\cup\partial\mathbf{w}_p(E_N)$. Globally on $\partial\mathbf{w}(E_N)\equiv\partial E_{N+1}$, is a solution of the Helmholtz's equation on E_{N+1}, so

$$\mathrm{span}\{\varphi_{N,j,n}\}_{\substack{n\in\mathbb{N},\\ 1\leq j\leq p}} \leq \mathrm{span}\{\varphi_{N,n}\}_{n\in\mathbb{N}}\,.$$

The eigenfunctions of this subspace are associated to special resonating modes called ***diaperiodic** modes* of the cavity (from the Ancient Greek preposition διά + gen., meaning «*across*», so 'dia-periodic' roughly means «across the periods»). A basis (in general *not* complete) of this vector space is represented by eigenfunctions $\varphi_{N+1,1,n}+\varphi_{N+1,2,n}+\ldots+\varphi_{N+1,p,n}$ (i.e. the sum of the same 'elementary' diaperiodic mode associated to each of the p subdomains).

All the non-diaperiodic modes, for example the solutions to the Helmholt'z equation that do *not* vanish on the subdomains' inner boundaries $\partial\mathbf{w}_1(E_N)\cup\partial\mathbf{w}_2(E_N)\cup\ldots$ $\ldots\cup\partial\mathbf{w}_p(E_N)\setminus\partial E_{N+1}$, are called ***interconnective** modes*, because they do not trivially depend on the eigenfunctions of the initiator, but rather depend on the interconnections between the prefractal sets with each other. The eigenfunctions associated to interconnective modes for the N^{th}-step prefractal are indicated as $\{\varphi_{N,0,n}\}_{n\in\mathbb{N}}$.

All in all, the spectrum of the Laplace operator on E_N has been orthogonally decomposed as:

$$\mathrm{span}\{\varphi_{N,n}\}_{n\in\mathbb{N}_0} = \mathrm{span}\{\psi_{N,k}\}_{1\leq k\leq\beta_N} \oplus \mathrm{span}\{\varphi_{N,0,n}\}_{n\in\mathbb{N}} \oplus \bigoplus_{j=1}^{p} \mathrm{span}\{\varphi_{N,j,n}\}_{n\in\mathbb{N}}\,,$$

---

[122] The characteristic function of $E\neq\varnothing$ is $\chi_E\colon E\to\{0,1\}$ such that $\chi_E(x)=1$ if $x\in E$ and 0 if $x\notin E$.

with:

$$g_N(\mathbf{x},\mathbf{x}';\lambda) = g_N^{\text{cnn}}(\mathbf{x},\mathbf{x}';\lambda) + g_N^{\text{dia}}(\mathbf{x},\mathbf{x}';\lambda)$$
$$\equiv \sum_{n=1}^{\infty} \frac{\varphi_{N,0,n}(\mathbf{x})\varphi_{N,0,n}(\mathbf{x}')}{\lambda_{N,0,n}-\lambda} + \sum_{j=1}^{p}\sum_{n=1}^{\infty} \frac{\varphi_{N,j,n}(\mathbf{x})\varphi_{N,j,n}(\mathbf{x}')}{\lambda_{N,j,n}-\lambda};$$

$$\left\|\varphi_{N,j,n}\right\|_{H^2(E_N)} = c_j \left\|\varphi_{N-1,n}\right\|_{H^2(E_{N-1})}, \qquad 1\leq j\leq p. \tag{1.477}$$

The eigenvalues associated to a specific mode (either cohomologic, diaperiodic or interconnective) are labelled with the same indices, so for example the eigenvalue associated to a diaperiodic mode on the j^{th} subdomain $\varphi_{N,j,n}$ is $\lambda_{N,j,n}\in\mathbb{R}_+$. Since the eigenvalues are counted along with their multiplicities, they are not necessarily distinct values, i.e. $\lambda_{N,n}\leq\lambda_{N,n+1}$ and $\lambda_{N,j,n}\leq\lambda_{N,j,n+1}$ (where an equal sign means that $\text{mul}_{-\Delta}\lambda_{N,n}>1$ or $\text{mul}_{-\Delta}\lambda_{N,j,n}>1$, respectively, $\forall n\in\mathbb{N}$ and $1\leq j\leq p$. In this way the association between eigenvalues and eigenfunctions is 1-to-1 because, being diagonalizable by the Spectral Theorem of compact self-adjoint operators [9]–§B.4, [68], the geometric multiplicities are all equal to the algebraic ones, and to every (possibly multiple) eigenvalue, listed in the sequence as many times as its algebraic multiplicity, one eigenfunction is exactly associated the same number of times.

5.1.3. Self-similar renormalization of the Green's function

For any general IFS the diaperiodic eigenvalues $1\leq j\leq p$ get rescaled with N as

$$\lambda_{N,j,n} \geq c_j^{-1}\lambda_{N-1,n} \tag{1.478}$$

due to the similarities' property (1.470) and (1.477), with the equality holding if and only if $\mathbf{w}_j$ is a similarity (i.e. a combination of translations, rotations, reflections and homotheties): in the case of an ISS a 'self-similar' renormalization of the Green's function is available.

Using the generic decomposition (1.474) and (1.477) for the Green's function, g_N^0/λ accounts for cohomologic modes, g_N^{cnn} for interconnective modes and g_N^{dia} for diaperiodic modes. By the similarities of the ISS and the fact that $\lambda_{N,j,n}= c_j^{-1}\lambda_{N-1,n}$:

$$\begin{aligned} g_{N+1}^{\mathrm{dia}}(\mathbf{x},\mathbf{x}';\lambda) &= \sum_{j=1}^{p}\sum_{n=1}^{\infty}\frac{\varphi_{N+1,j,n}(\mathbf{x})\varphi_{N+1,j,n}(\mathbf{x}')}{\lambda_{N+1,j,n}-\lambda} \\ &= \sum_{j=1}^{p}\sum_{n=1}^{\infty}\frac{\varphi_{N,n}(\mathbf{w}_j(\mathbf{x}))\varphi_{N,n}(\mathbf{w}_j(\mathbf{x}'))}{c_j^{-1}\lambda_{N,n}-\lambda} \\ &= \sum_{j=1}^{p}c_j\sum_{n=1}^{\infty}\frac{\varphi_{N,n}(\mathbf{w}_j(\mathbf{x}))\varphi_{N,n}(\mathbf{w}_j(\mathbf{x}'))}{\lambda_{N,n}-c_j\lambda} \\ &= \sum_{j=1}^{p}c_j g_N(\mathbf{w}_j(\mathbf{x}),\mathbf{w}_j(\mathbf{x}');c_j\lambda). \end{aligned} \tag{1.479}$$

Recursively applying this formula allows to recover the diaperiodic part of the Green's function on prefractals of any order $N\in\mathbb{N}$ as just depending on the Green's function on the initiator set E_0. Since the limit set (and, up to a certain approximation, any sufficiently high-order prefractal) is geometrically (almost) independent from the initiator, this is a powerful tool to estimate the spectrum of diaperiodic resonating modes within a (pre-)fractal structure.

Of course (1.479) only holds for the diaperiodic part g_N^{dia} of the Green's function so it seems that only a part of it can be easily recovered exploiting the self-similarity, but this can be enough for certain practical considerations below and in §5.2.

5.1.4. Practical considerations

First of all the Helmholtz's equation (as well as most differential equations) on the limit fractal set E_∞ may not be correctly defined since the fractal set is, from the analytic point of view, too irregular and low-dimensional to be domain to continuous models (e.g. *no* disks exist inside the Sierpiński gasket and carpet, as well as *no* straight lines do inside a Cantor set, cfr. Figure 15 and example of §2.3.2 — no matter how small they are). Furthermore there are the size limitation of the physical model considered: below a certain microscopic scale, for example, the Classic Electrodynamics equations must be replaced with semi-classical or genuinely quantum model; even below those Planck length-scales, many other phenomena have to be taken into account, and so on.

Anyway the practical application of electromagnetic problems on fractal domains has been addressed as crucial for the production of fractal antennas (cfr. Bibliography in [9], [53]), as well as other integrated microwave components, where 'fractal' means in this case just 'prefractal of a high enough order'. It can be proven, in fact, that the electromagnetic behaviour of fields irradiated from or propagating within such a prefractal geometry asymptotically relax with the iteration order N, so the key point is to estimate such an N. Under this light the electromagnetic behaviour of prefractal objects is more important that the properties on their attractors.

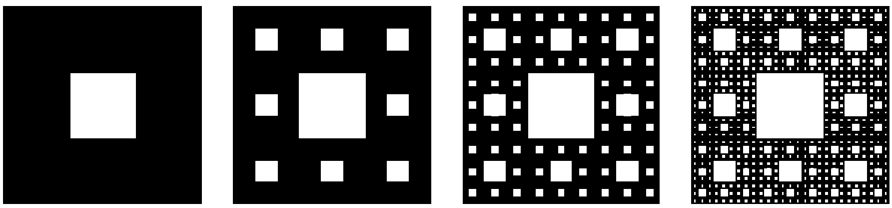


Figure 65 The first four prefractals of the Sierpiński carpet $\check{S}_2$ built from a square initiatior $[0,1]^2$.

5.1.5. The Sierpiński carpet waveguide

The Sierpiński n-**sponge** $\check{S}_n \subset [0,1]^n$ is the attractor to the ISS of $\mathbb{R}^n$ made by $3^n - 1$ similarities (of contraction ratio 1/3) described in [9]–§1.8, [15] and its Hausdorff dimension is

$$\dim_{\mathrm{H}} \check{S}_n = \log_3(3^n - 1), \tag{1.480}$$

For $n=2$ the fractal set $\check{S}_2$ represented in Figure 1 is the well-known Sierpiński **carpet**. If the unit square $[0,1]^2$ itself is chosen as initiator then the N^{th}-step prefractal

$\check{S}_{2,N}$ has $(8^N-1)/7$ holes (but it is not a closed surface, so this is not properly a surface genus).

Consider a cylindrical waveguide whose cross-section is $\check{S}_{2,N}$; let its boundaries be perfectly conducting and its interior filled with an electromagnetically-linear, stationary, lossless, homogeneous and isotropic medium (of permittivity ε and permeability μ). It is well known that the electromagnetic propagation inside this waveguide is subordinated by a countable infinity of both TE and TM modes, whereas $(8^N-1)/7$ TEM modes are also present; these last modes correspond to the cohomologic modes defined in §5.1.2 and are associated to the simple pole in 0 for the two-dimensional Green's function of the transverse Helmholtz's equation, as decomposed in §5.1.3. From the analytical point of view these modes can be found by solving the following scalar Laplace's equation:

$$\begin{cases} \Delta V = 0 \\ \mathrm{d}V\big|_{\partial \check{S}_{2,N}} = 0 \end{cases} \tag{1.481}$$

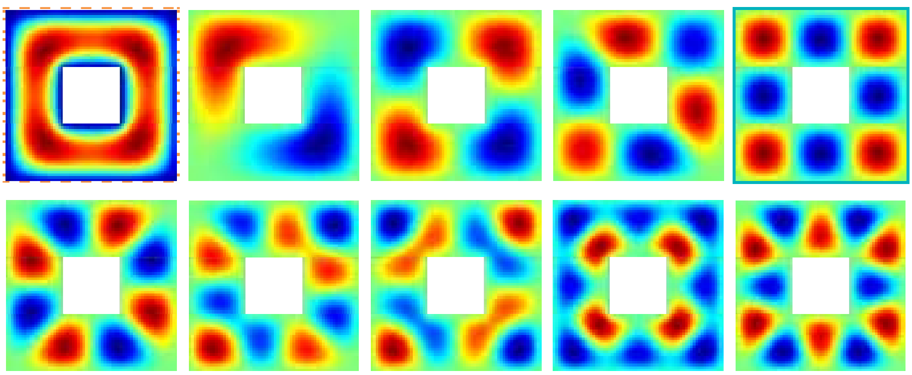


Figure 66 Sample of Laplace eigenmodes for the Sierpiński carpet's first prefractal $\check{S}_{2,1}$, ordered by increasing eigenvalues. Note the one **cohomologic** mode (top-left) and a diaperiodic mode (top-right).

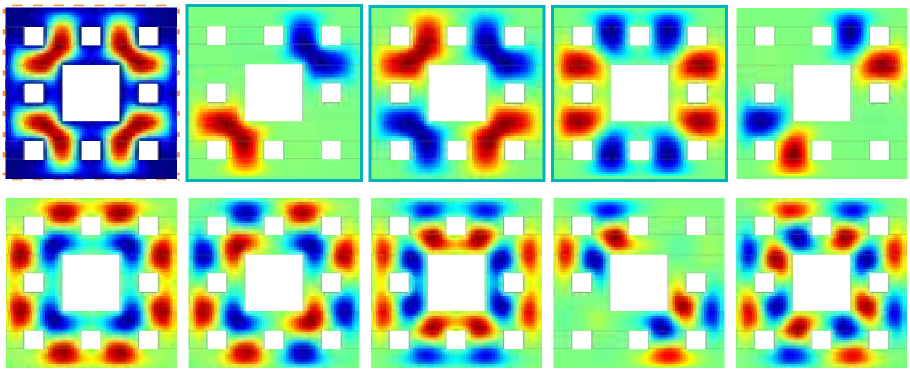


Figure 67 Sample of Laplace eigenmodes for the Sierpiński carpet's second prefractal $\check{S}_{2,2}$, ordered by increasing eigenvalue. Note one of the 9 **cohomologic** modes (top-left).

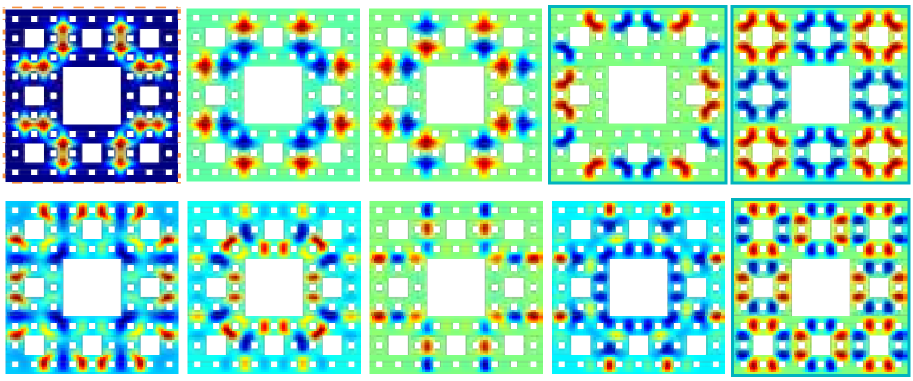


Figure 68 Sample of Laplace eigenmodes for the Sierpiński carpet's third prefractal $\check{S}_{2,3}$, ordered by increasing eigenvalues. Note one of the 73 **cohomologic** modes (top-left) and some **diaperiodic** modes, (4,1), (5,1), (5,2), respectively similar to modes (2,1), (3,1), (4,1) of Figure 67 .

i.e. by imposing the scalar electrostatic potential to be constant on each of the $(8^N-1)/7$ components of $\partial\check{S}_{2,N}$; for example, imposing a **normalized** potential to attain value 0 on all the square curves except for one, which is set to constant value 1. Figure 66 to Figure 70 show some of these potentials for 1^{st}- through 5^{th}-step prefractal cross-sections.

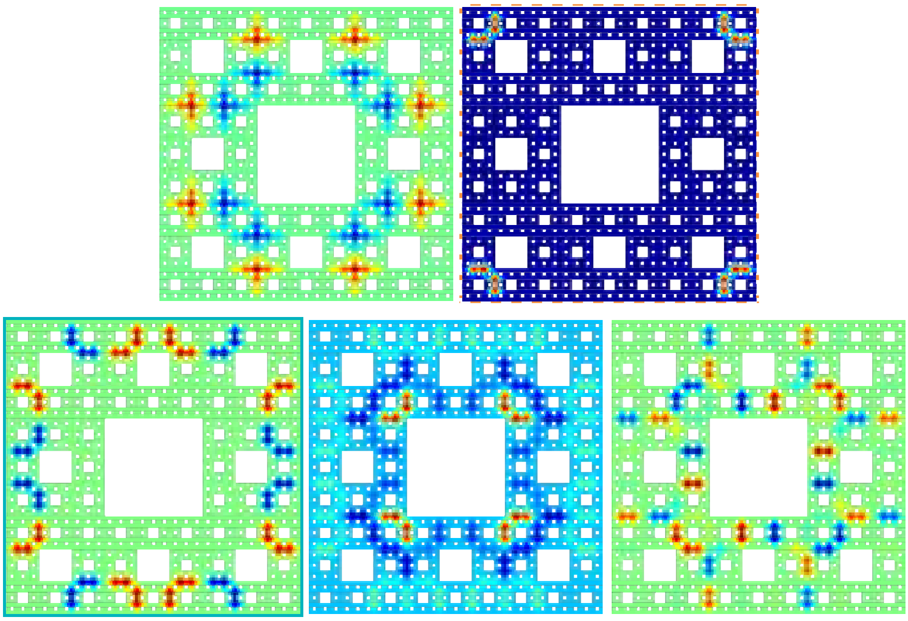


Figure 69 Sample of Laplace eigenmodes for the Sierpiński carpet's fourth prefractal $\check{S}_{2,4}$, ordered by increasing eigenvalues. Note two of the 585 **cohomologic** modes, (1,1), (3,1), and a **diaperiodic** mode, (2,1), respectively self-similar to the mode (4,1) of Figure 68 and to mode (3,1) of Figure 67 .

Cylindrical separation for potential $A\in\Lambda^1(\check{S}_{2,N})$ is employed (as in [50]) such that, in Cartesian coordinates $\{x,y,z\}$, let $A(x,y,z)\mathrm{d}z = T(x,y)L(z)\mathrm{d}z$.

Diaperiodic $\mathrm{TE}^{\mathrm{dia}}_{m,n}$ and $\mathrm{TM}^{\mathrm{dia}}_{m,n}$ modes, normalized in $\mathrm{L}^2(\check{S}_{2,N})$, are easily computed in closed-form employing considerations done in §5.1.3 and found to be ($\forall m,n\in\mathbb{N}$ for TM modes, $\forall m,n\in\mathbb{N}_0$ with $m+n>0$ for TE modes):

$$\begin{cases} T^{\mathrm{TE}^{\mathrm{dia}}}_{N;m,n}(x,y) = 2^{1-\frac{N}{2}}3^N\cos\left(3^N m\pi x\right)\cos\left(3^N n\pi y\right) \\ T^{\mathrm{TM}^{\mathrm{dia}}}_{N;m,n}(x,y) = 2^{1-\frac{N}{2}}3^N\sin\left(3^N m\pi x\right)\sin\left(3^N n\pi y\right) \end{cases} \tag{1.482}$$

where the associated free oscillations (in the transverse direction of the wave-guide) are associated to wave-numers $\kappa_{N;m,n} = 2\pi\nu_{N;m,n}\sqrt{\varepsilon\mu}$, with their frequencies being

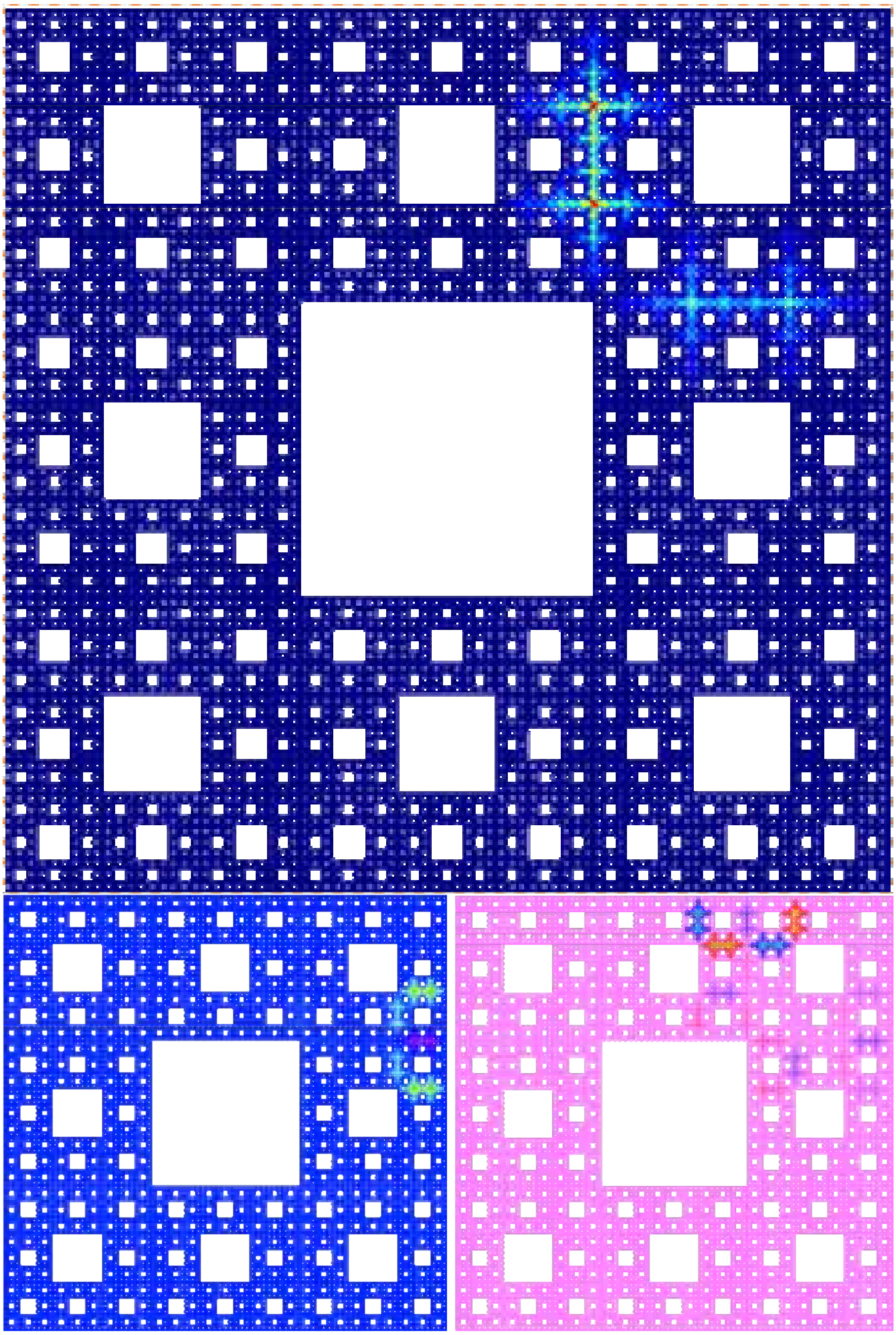


Figure 70 (*previous page*) Sample of Laplace eigenmodes for the Sierpiński carpet's fifth prefractal $\check{S}_{2,5}$, ordered by increasing eigenvalues. Note one of the 4681 **coho-mologic** modes (top).

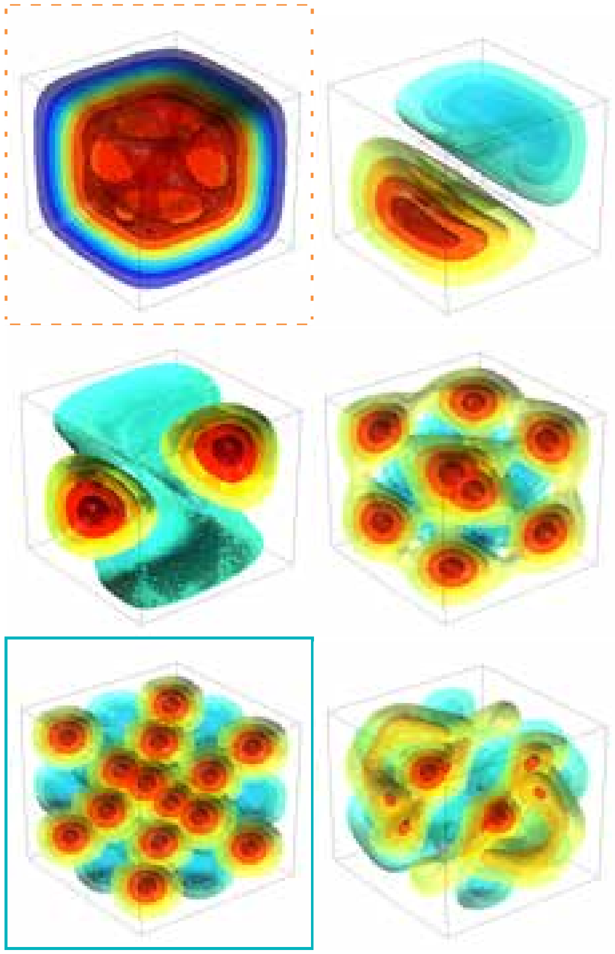
$$\nu_{N;m,n}=\frac{3^N}{2\sqrt{\varepsilon\mu}}\sqrt{m^2+n^2}\qquad\forall m,n\in\mathbb{N}_0. \tag{1.483}$$

Since we are interested in smooth solutions, the decomposition in diaperiodic modes may not be the most physically interesting, so it is better to solve the electromagnetic numerically and found a localized basis of eigenfunctions at increasing frequencies. Figure 66 to Figure 69 report a graphical representation of such 'hybrid' modes. Please read the caption to distinguish interconnective and diaperiodic modes, which may also be individually addressed within as '(*m*,*n*)', there denoting the picture at the *m*th row and *n*th column of the figure.

Again, the relation between diaperiodic modes along with increasing iteration steps is emphasised in the above Figures, where a threaded coloured background is used to easily highlight **cohomologic** modes, as well as **diaperiodic** modes' self-similarity across iterations (hence, again, the use of the Greek suffix «διά-» in the word '*dia*periodic').

5.1.6. The Sierpiński and Menger sponge resonators

Similar considerations lead to the closed-form computation of electromagnetic free oscillations for a resonator whose geometry is that of the Sierpiński 3-sponge $\check{S}_3$ (or, better, of its Nth-step prefractal $\check{S}_{3,N}$), with the electric and magnetic field's 1-forms:



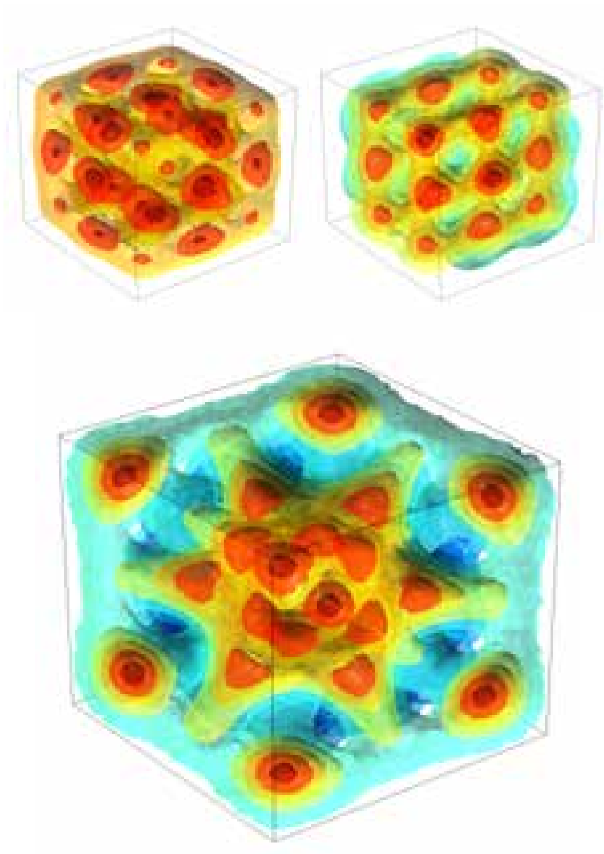
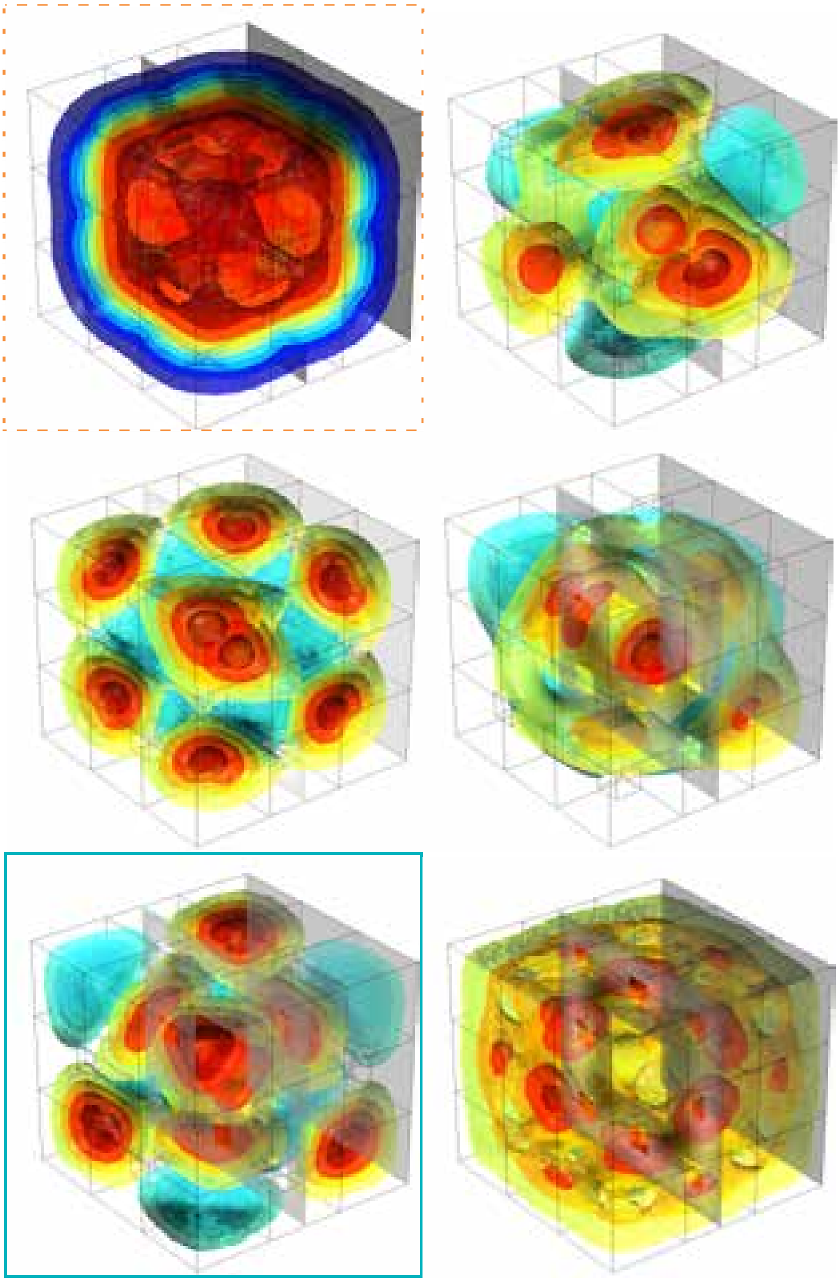


Figure 71 Samples of Laplace eigenmodes for the Sierpiński sponge's first prefractal $\check{S}_{3,1}$, ordered by increasing eigenvalues. Note the one **cohomologic** mode (top-left) and the **diaperiodic** mode (1,3).

$$\begin{aligned} E_{N;l,m,n}(x,y,z) \propto\ & \sin(3^N l\pi x)\cos(3^N m\pi y)\cos(3^N n\pi z)\mathrm{d}x + \\ & + \cos(3^N l\pi x)\sin(3^N m\pi y)\cos(3^N n\pi z)\mathrm{d}y + \\ & + \cos(3^N l\pi x)\cos(3^N m\pi y)\sin(3^N n\pi z)\mathrm{d}z\ ; \end{aligned}$$

$$\begin{aligned} H_{N;l,m,n}(x,y,z) \propto\ & \cos(3^N l\pi x)\sin(3^N m\pi y)\sin(3^N n\pi z)\mathrm{d}x + \\ & + \sin(3^N l\pi x)\cos(3^N m\pi y)\sin(3^N n\pi z)\mathrm{d}y + \\ & + \sin(3^N l\pi x)\sin(3^N m\pi y)\cos(3^N n\pi z)\mathrm{d}z\ . \end{aligned}$$



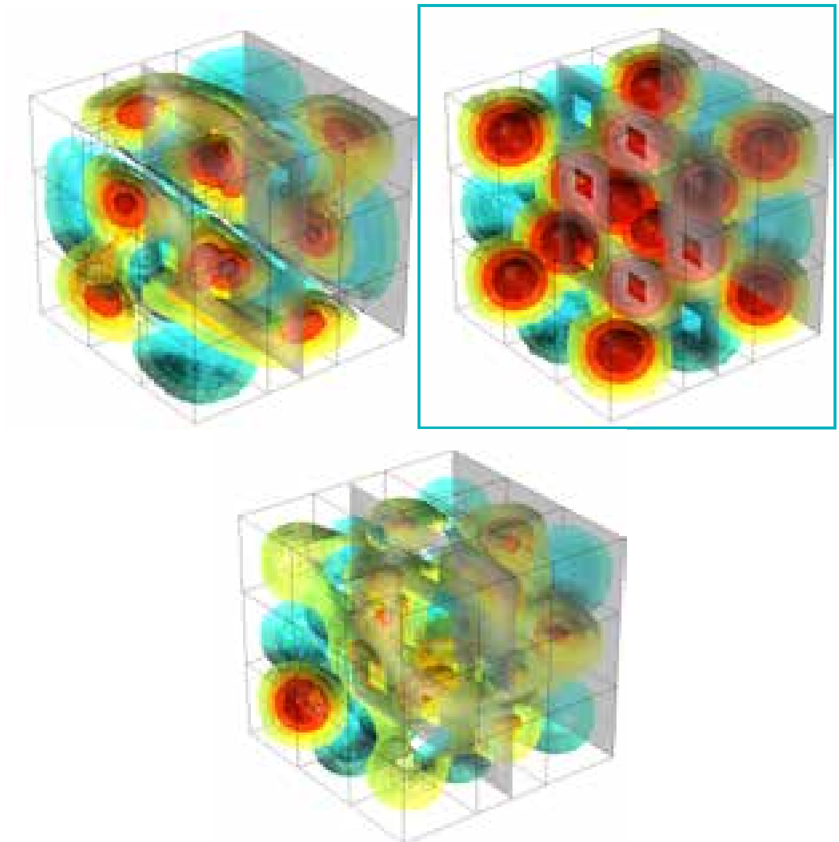


Figure 72 Sample of Laplace eigenmodes for the Sierpiński sponge's second prefractal $\check{S}_{3,2}$, ordered by increasing eigenvalues. Note one of the 27 **cohomologic** modes (top-left) and some **diaperiodic** modes: (1,3) and (2,4).

The associated eigenfrequencies, in analogy with (1.483), show how diaperiodic eigenvalues feature a self-similar, ***log**-periodic* distribution on the real line (also cfr. Figure 71 and Figure 72 for the eigenvalues on Sierpiński sponge's first two prefractals of cubic initiator), as both theoretically expected —since considerations in §§5.1.2–5.1.3— and experimentally measured on radiation patterns of prefractal IFS antennas and resonators, [15].

The Sierpiński sponge's N^{th}-step prefractal has exactly $(26^N-1)/25$ "bubbles" within, so it is a simply linearly connected, but $\frac{26^N-1}{25}$-times **superficially** multiply connected domain: it features exactly the same number of linearly independent electrostatic modes but *no* magnetostatic fields exist.

The *Menger sponge* (always cfr. [9]–§1.8) is a "topologically-dual" structure to the Sierpiński 3-sponge: it is in fact built starting from a similar ISS of $\mathbb{R}^3$ made by 20 similarities (of contraction ratio 1/3), exactly 20 of the Sieprinski sponges' 26 ones. The topological difference is that the Menger sponge's prefractals (of initiator the unit cube) are superficially simply connected, but **linearly** multiply connected: such prefractals are in fact homeomorphic to g_N-fold solid tori, with

$$\begin{cases} g_N = 5^N 4^{N-1} - \dfrac{2^{3N}3+24}{7} \\ g_0 = 0 \end{cases}$$

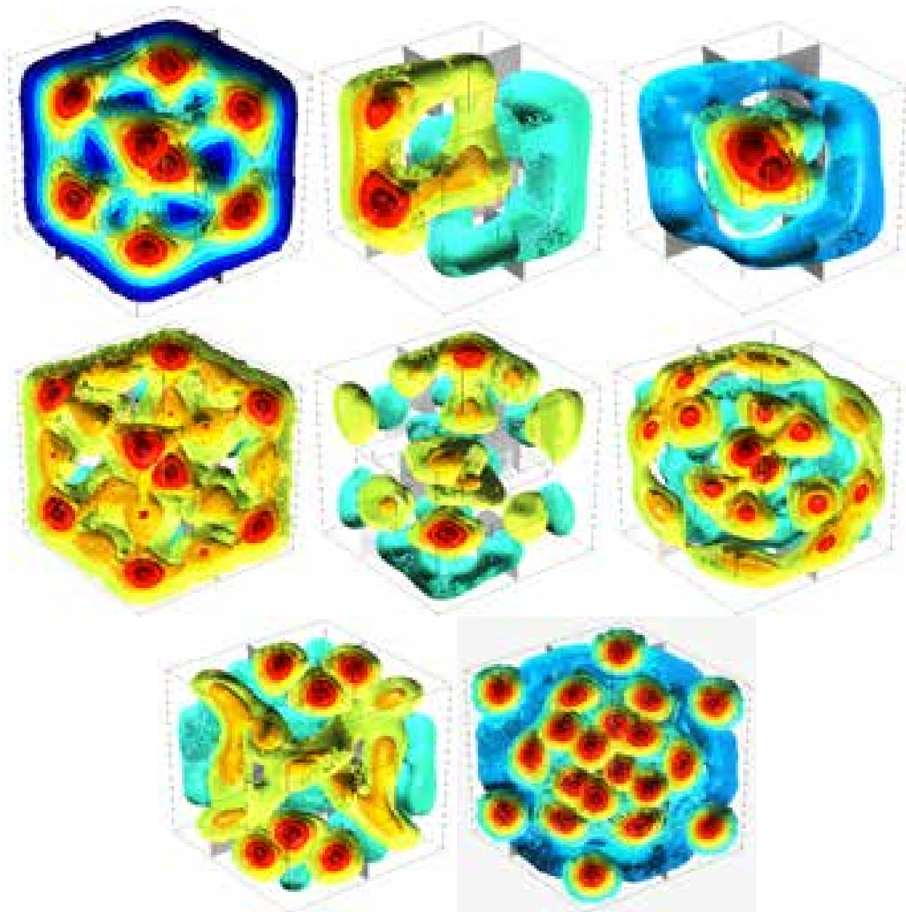


Figure 73 Sample of Laplace eigenmodes for the Menger sponge's first prefractal, ordered by increasing eigenvalues. Note one of the $g_1 = 5$ **cohomologic** modes (top-left).

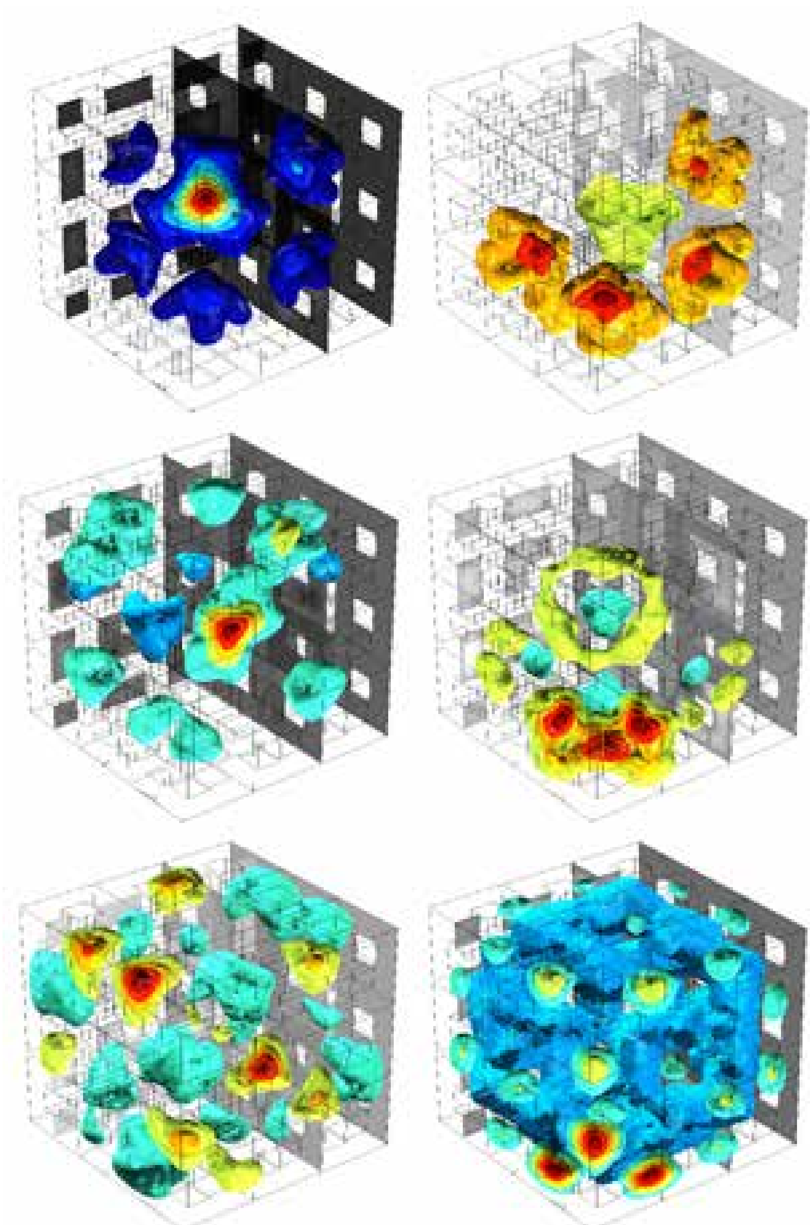


Figure 74 Sample of Laplace eigenmodes for the Menger sponge's second prefractal, ordered by increasing eigenvalues. Note one of the $g_2 = 76$ **cohomologic** modes (top-left).

The free oscillations on a Menger sponge's prefractals include exactly g_N linearly independent magnetostatic fields, whereas *no* electrostatic fields exist. Figure 73 and Figure 74 depict some of these modes for the first two prefractals of cubic initiator.

5.1.7. Radiation pattern from prefractal apertures and patches

This application simply uses basic Fourier transform properties to compute the Green's function for a fractal-aperture or fractal-patch antenna, [13] and [19]. Using the electric-electric Green's function of §4.2.4, suppose that an electromagnetic radiation on the space $\mathbb{R}^3$ (or the positive half-space $\mathbb{R}^3_+$) is to be evaluated (practically on its "far-field" region) as produced by an patch antenna placed on the xy plane (with respect to the usual Cartesian coordinates' system). Let the positive half-space be filled with an electromagnetically "well-behaved" medium (e.g. air or vacuum) of free propagation constant $\kappa_0(\omega)=\omega\sqrt{\varepsilon\mu}$, whereas the aperture itself is the N^{th}-step prefractal plane domain E_N of a just-touching ɪss subject to definitions in §5.1.1.

The (impressed) surface density current on the patch is the 2-form $J\in\Lambda^2(\mathbb{R}^2)$, whose compact support is E_N (i.e. $\operatorname{supp} J=E_N\subset\mathbb{R}^2$): in this case, using (1.439), one finds

$$\hat{E}(\mathbf{r},\omega) = \left\langle \underline{G}_{\mathrm{EE}} \,\middle|\, \overline{\hat{J}_k} \right\rangle_{\mathrm{L}^2(E_N)} = \int_{E_N} \underline{G}_{\mathrm{EE}}(\mathbf{r},\mathbf{r}';\omega)\wedge\star\hat{J}(\mathbf{r}',\omega) \tag{1.484}$$

or, in the spectral domain by (1.434):

$$\hat{\hat{E}}(\boldsymbol{\kappa},\omega) = \underline{\hat{G}}_{\mathrm{EE}}:\star\hat{\hat{J}}(\boldsymbol{\kappa},\omega). \tag{1.485}$$

Since the medium in free space is well-behaved, the Green's function is of scalar type, i.e. by (1.431),

$$\underline{G}_{\mathrm{EE}}=g_0\mathbb{1}^{\otimes 2}, \qquad g_0(\mathbf{r},\omega) = \frac{e^{-\imath\kappa(\omega)\|\mathbf{r}\|}}{4\pi\|\mathbf{r}\|},$$

It is also well-known that, in "far-field" approximation, i.e. for $|\boldsymbol{\kappa}\cdot\mathbf{r}| \gg 1$, the electric field 1-form $E^{\mathrm{far}} \in \Lambda^1(2\pi\mathbb{T}^2)$ is *proportional* to the Fourier transform of the planar electric current density, evaluated on the spherical-coordinated wave vector

$$\boldsymbol{\kappa}(\varphi,\theta) := \kappa(\omega)\begin{pmatrix}\sin\theta\cos\varphi\\ \sin\theta\sin\varphi\\ \cos\theta\end{pmatrix}, \qquad \forall(\varphi,\theta)\in 2\pi\mathbb{T}^2; \tag{1.486}$$

the far-field radiation pattern is then computed as:

$$\hat{E}^{\mathrm{far}}(\varphi,\theta;\omega) \propto \hat{\chi}_{E_N}(\boldsymbol{\kappa}(\varphi,\theta))\star\hat{\hat{J}}(\boldsymbol{\kappa}(\varphi,\theta),\omega), \tag{1.487}$$

where $\hat{\chi}_{E_N}$ is the Fourier transform of E_N's characteristic function (cfr. note [122]). The computation of integral (1.484) is also performed, implicitly, by means of such function: since trivially $J \equiv J\chi_{E_N}$, the convolution integral can be performed on the whole aperture/patch plane by adding the characteristic function inside:

$$\hat{E}(\mathbf{r},\omega) = \int_{\mathbb{R}^2} g_0(\mathbf{r}-\mathbf{r}')\chi_{E_N}(\mathbf{r}')\star\hat{J}(\mathbf{r}',\omega).$$

Fourer-transforming it, the spectral-domain electric field is the product of the spectral-domain impressed current density by $\hat{\chi}_{E_N}$, i.e. by (1.435):

$$\hat{\hat{E}}(\boldsymbol{\kappa},\omega) = \frac{\hat{\chi}_{E_N}(\boldsymbol{\kappa})\star\hat{\hat{J}}(\boldsymbol{\kappa},\omega)}{\|\boldsymbol{\kappa}\|^2 - \kappa(\omega)^2}. \tag{1.488}$$

This has some easy computability features due to the self-similarity properties of domain E_N, particularly when the contraction mappings are affine or similarity transformations as (1.469), and the ISS is just-touching, in fact

$$\chi_{E_{N+1}}(\mathbf{x}) = \sum_{j=1}^{p}\chi_{\mathbf{w}_j(E_N)}(\mathbf{x}),$$

and

$$\hat{\chi}_{\mathrm{w}_j(E_N)}(\boldsymbol{\kappa}) = \int_{\mathbf{R}^n} e^{\imath\boldsymbol{\kappa}\cdot\mathbf{r}}\chi_{\mathbf{w}_j(E_N)}(\mathbf{r})\mathrm{d}\mathbf{r} = e^{-\imath\boldsymbol{\kappa}\cdot\mathbf{b}_j} A_j^{-1}\hat{\chi}_{\mathrm{w}_j(E_N)}\left(A_j^{-1}\boldsymbol{\kappa}\right). \quad (1.489)$$

Putting (1.489) into (1.488) or (1.487) immediately allows for a direct computation of the (far-field) electric field radiated by such a prefractal aperture/patch antenna. It is crucial to note, however, that the hardest computational efforts are often just delayed to the Fourier anti-transformation phase, since the function to be reverted back to the space-time domain usually features different kinds of singularities and is often multi-valued too.

If the aperture is the N^{th}-step prefractal of a Sierpiński carpet $\check{S}_{2,N}$, as described in §1.4.2 (cfr. Figure 9) then the its characteristic function can be written as a sum of rectangular pulses[123]

$$\chi_{\check{S}_{2,N+1}}(\mathbf{r}) = \mathrm{rect}_{\mathbf{1}_n}\mathbf{r} - \sum_{k=1}^{N+1} \sum_{\mathbf{c}_k\in\frac{(2\mathbb{Z}+1)^n}{2\cdot 3^k}\cap\check{S}_{2,N}} \mathrm{rect}_{3^{-k}\mathbf{1}_n}(\mathbf{r}-\mathbf{c}_k)$$

$$\Downarrow$$

$$\hat{\chi}_{\check{S}_{2,N+1}}(\boldsymbol{\kappa}) = \mathrm{sinc}\frac{\boldsymbol{\kappa}}{2} - \sum_{k=1}^{N+1} 3^{-nk}\,\mathrm{sinc}\frac{\boldsymbol{\kappa}}{2{\cdot}3^k} \sum_{\mathbf{c}_k\in(2\mathbb{Z}+1)^n\cap\check{S}_{2,N}} e^{\imath\boldsymbol{\kappa}\cdot\mathbf{c}_k}$$

Such functions can also be recursively computed as depending on the previous-step functions, as:

$$\chi_{\check{S}_{2,N+1}}(\mathbf{r}) = \sum_{\mathbf{h}\in\{-\frac{1}{2},0,\frac{1}{2}\}^n\setminus\{\mathbf{0}_n\}} \chi_{\check{S}_{2,N}}\Big(3(\mathbf{r}+\mathbf{h})\Big)$$

$$\Downarrow$$

$$\hat{\chi}_{\check{S}_{2,N+1}}(\boldsymbol{\kappa}) = \frac{1}{3}\hat{\chi}_{\check{S}_{2,N}}(3^{-1}\boldsymbol{\kappa}) \sum_{\mathbf{h}\in\{-\frac{1}{2},0,\frac{1}{2}\}^n\setminus\{\mathbf{0}_n\}} e^{-\imath\mathbf{h}\cdot\boldsymbol{\kappa}}$$

---

[123] The **rect**angular pulse of width $\mathbf{b}\in\mathbb{R}_+^n$ is $\mathrm{rect}_{\mathbf{b}}\mathbf{x} := 1$ iff $\mathbf{x}\in$ ½$[\![-\mathbf{b},\mathbf{b}]\!]$, and 0 if $\mathbf{x}\notin$ ½$[\![-\mathbf{b},\mathbf{b}]\!]$.

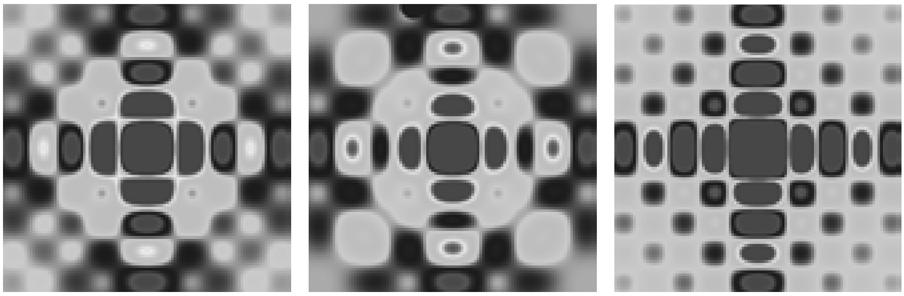


Figure 75 Prefractal aperture radiation pattern in the 2D $\boldsymbol{\kappa}$-space of a Sierpiński carpet patch antenna $\check{S}_{2,N}$, for $1 \le N \le 3$.

From the mathematical point of view, $\hat{\chi}_{\mathrm{w}_j(E_N)}(\boldsymbol{\kappa})\mathrm{d}\boldsymbol{\kappa}$ is the new measure element respect to which integrals like (1.487) have to be computed. This formal approach has more general consequences: when dealing with physical-mathematical problems on fractal sets, it is convenient to resort to specially "crafted" measures obtained as "weighted" Lebesgue measures — until as the limit $N \to \infty$ is reached. This allows for transporting almost all the physical properties and equations into the "prefractal domain" without any formal difficulty.

Figure 75 shows the Fourier transform of a Sierpiński carpet's aperture function, which ideally corresponds to the radiation pattern of a Sierpiński carpet patch antenna with an (equally ideal) constant feeding, when evaluated according to (1.486).

5.2 Can you hear the fractal dimension of a drum?

Mathematician Marc Kac in a 1966 essay titled *Can you hear the shape of a drum?*, as well as in later papers [64], wondered whether it were possible for a 'blind' —but mathematically skilled— listener, to figure out the geometrical shape of a plucked drumhead by listening to its whole acoustic spectrum alone. Though quite a strange question, "hearing the shape of a drum" has become more of a symbolic action —

almost a watermark— for those scientists willing to tackle with problems of a new discipline, which spawn and grew up since: the Spectral Geometry.

What was referred to by Kac as a *drum* is nothing but a bounded plane domain $\Omega \subset \mathbb{R}^2$ (at first glance also thought to be simply connected too) to which a **pressure** function $u \in C^\infty(\Omega)$ is associated, satisfying the wave equation $\nabla^2 u(\mathbf{r},t) + \ddot{u}(\mathbf{r},t) = 0$ with either Dirichlet $u\big|_{\partial\Omega} = 0$ or Neumann $(\operatorname{grad} u)\cdot\mathbf{n}\big|_{\partial\Omega} = 0$ boundary conditions.

This means considering all the free oscillations (cfr. §4.1.2) of an elastic membrane whose border $\partial\Omega$ is either fixed to a (semi-)rigid support (typical of drum instruments, like timpani, toms, congas, etc.), or left floating (i.e. constraints-less — typical of percussions in the cymbals' family). The spectrum of this wave operator is discrete and —due to geometric symmetries of Ω— often degenerate, i.e. partially composed by multiple eigenvalues, with different eigenmodes associated to each. Hearing the shape of a drum seems to be *a priori* an impossible feat because of both the elasticity limits of real drums and the bandwidth limitation of human hearing (the whole spectrum of infinite resonating frequencies would be impossible to be produced by the drum, as to be perceived by human hearing).

But Marc Kac's question is noteworthy for at least two reasons: first of all —to remain on the 'musical side' of the question— anyone even only barely trained to music listening could make the timbres of a bass drum, a xylophone, a triangle, or a güiro out of each other [104]. After all, most western percussions, tuned to be enharmonic with the equally-tempered scale, employ circular membranes (from timpani to cymbals, to drums and toms, and so on). Second, it is well-known that the mathematical model of a plucked membrane can be extended to higher-dimensional domains (even not simply connected) and is equivalent to many other problems in Mathematical Physics, including the well-known Electromagnetic wave equations (1.389). The problem of inferring any information about the domain's geometry from its spectral distribution is important for several applications and is, in fact, a key question of classical Spectral Geometry, the branch studying the correlations between a (Riemann) manifold $\mathcal{M}$ and the spectrum of its (scalar) Laplace operator $\nabla \in \mathfrak{sym}^{-}(C^{\infty}(\mathcal{M}))$, cfr. §4.2.

5.2.1. Hearing the *shape* of a drum

Marc Kac's question was left open for almost three decades, with mathematicians occasionally providing counter-examples of higher-dimensional cases, but it was only in 1992 that C. Gordon, D. Webb and S. Wolpert [55]–[56] definitely disproved the two-dimensional case of 'drums' by showing a couple of isometric, concave polygons with identical Laplace spectra, but different shapes. Figure 76 shows the two different drums (two rows for each geometry) along with their first 8 eigenmodes (left to right, top to bottom) associated to the same 8 eigenvalues, which are approximately reported —as they are located for each eigenmode in the Figure below— in the following table.

10.1515	14.6220	20.7022	26.1502
28.9923	36.8371	42.3879	46.1655

A more general isospectrality statement was also proven by using the following:

Bérard's Theorem

Let $G(\circ)$ be a topological finite group (cfr. §1.2) with $H,K<G$ and $U\in\mathfrak{L}(\mathrm{L}^2(G/H),\mathrm{L}^2(G/K))$ a *unitary* operator (i.e. $\|U\|=1$). If $\mathbf{H}$ is a separable Hilbert space which G acts unitarily on (i.e. $\|g\bullet\mathbf{v}\|=\|\mathbf{v}\|$ $\forall\mathbf{v}\in\mathbf{H}$, $\forall g\in G$), then U induces an isometry $\mathrm{stab}_H\mathbf{H}\rightarrow\mathrm{stab}_K\mathbf{H}\leq\mathbf{H}$.

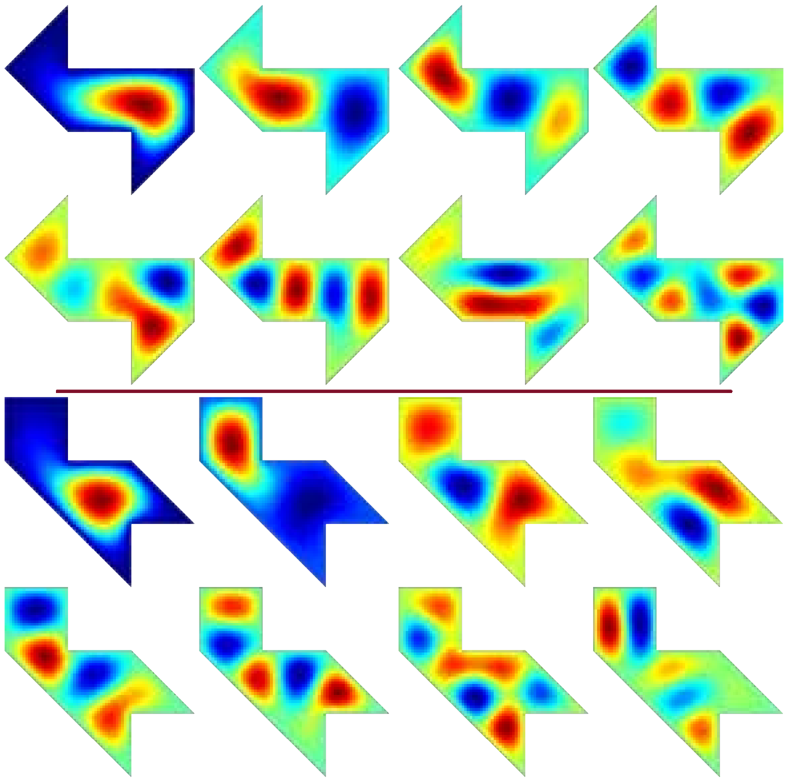


Figure 76 Two isospectral domains ('drums'), along with their first eigenmodes, by increasing eigenvalues, left-to-right, top-to-bottom per drum.

The reason why the concave polygons constructed to disprove Kac's conjecture have the particular shapes of Figure 76 is that the earliest techniques developed for isospectrality problems rely on triangulated manifolds, where the spectral properties are synthesized by properly composing the polygon as the underlying manifold of a simplicial complex. For more details about simplicial complexes and Algebraic Topology in general, cfr. [81] and [9].

Sunada's Theorem

Let $\mathcal{M}/G$ be an orbifold (cfr. §2.3.2) where $\mathcal{M}$ is a compact Riemann manifold and $G(\circ)$ is a finite group acting by isometries. If $H,K<G$ are mutually conjugate[124] subgroups acting freely on $\mathcal{M}$ then $\mathcal{M}/H$ and $\mathcal{M}/K$ are ***isospectral* orbifolds**.

<u>Proof</u>: It is sufficient to apply Bérard's Theorem to Hilbert space $\mathbf{H}=\mathrm{L}^2(\mathcal{M})$, since $\mathrm{stab}_H\mathrm{L}^2(\mathcal{M})\cong\mathrm{L}^2(\mathcal{M}/H)$ and $\mathrm{stab}_K\mathrm{L}^2(\mathcal{M})\cong\mathrm{L}^2(\mathcal{M}/K)$. □

The eigenvalue-**counting** function $\mathbb{N}_L:\mathbb{R}\to\mathbb{N}_0$ of a $\mathbb{R}$-definite compact operator $L\in\mathfrak{L}\mathbf{C}(\mathbf{V})$, where $\mathbf{V}$ is any real or complex linear topological space [9]–§B.1 and $\mathrm{spec}L\subset\mathbb{R}$, is $\mathbb{N}_L(\lambda):=\mathrm{Card}\{\mu\leq\lambda\mid\mu\in\mathrm{spec}L\}$ and 'Card' also counts the eigenvalues along with their multiplicities.

Weyl's Formula

Let $\mathcal{M}$ be a (real) Lebesgue-measurable Riemann n-manifold (respect to μ_{L}^n, the n-dimensional Lebesgue measure's lift on $\mathcal{M}$) and let $\Delta:\equiv\nabla^2$ be its scalar Laplace's operator (with Dirichlet's boundary conditions if $\partial\mathcal{M}\neq\varnothing$ and the boundary is supposed to be μ_{L}^{n-1}-measurable). Let $\mathbb{N}_{-\Delta}:\mathbb{R}_+\to\mathbb{N}_0$ be the eigenvalue-counting function of $-\Delta$, then:

$$\mu_{\mathrm{L}}^n(\mathcal{M})=\frac{\pi^{n+\frac{1}{2}}}{\left\lfloor\frac{n}{2}\right\rfloor!}\lim_{\lambda\to+\infty}\frac{\mathbb{N}_{-\Delta}(\lambda)}{\sqrt{\lambda^n}}. \tag{1.490}$$

---

[124] H and K are *almost* conjugate if $\exists f\in\mathrm{Hom}(H,K)$ such that $h^{f(h)}=h$, $\forall h\in H$ — cfr. § o .

> If $\partial\mathcal{M}\neq\varnothing$ is a smooth $(n-1)$-manifold, then the higher-order asymptotic behaviour of $\mathbb{N}_{-\Delta}$ carries information on the measure of the manifold's boundary too, i.e., for $\lambda\sim+\infty$,
>
> $$\mathbb{N}_{-\Delta}(\lambda)\sim\frac{c_{\mu_{\mathrm{H}}^{n}}}{(2\pi)^{n}}\mu^{n}(\mathcal{M})\sqrt{\lambda^{n}}+c_{\mu_{\mathrm{H}}^{n-1}}\mu^{n-1}(\partial\mathcal{M})\sqrt{\lambda^{n-1}}.$$

The coefficient $\lfloor\frac{n}{2}\rfloor!(\frac{2}{\sqrt{\pi}})^{n}=\mu_{\mathrm{L}}^{n}(\mathbb{B}_{1}^{n}(\mathbf{0}))=:c_{\mu_{\mathrm{H}}^{n}}$, cfr. [39] and [9]–§1.4 (only depending on manifold's dimension n), was introduced in §2.1.2 and essentially maps Lebesgue measures to Hausdorff measures; nonetheless, there is still not a real Theorem proving that the Weyl's formula also holds for sets of non-integer Hausdorff dimension (for example fractal manifolds), but (1.490) show that one can certainly "hear the area of a drum" (as well as its perimeter) !

Since the inverse problem of guessing a manifold's geometry from its laplacian spectrum was solved modulo the manifold's measure itself, only classes of isometric Riemann manifolds are considered, and further divided into equivalence classes whose elements are *isospectral* manifolds, and whose laplacian spectrum (either Neumann's or Dirichlet's) is the same — counting multiplicities in case of degeneracy.

The ***partition*** *function* of a compact Riemann n-manifold $\mathcal{M}$ is the following $\mathcal{Z}_{\mathcal{M}}\in\mathrm{C}^{\omega}(\mathsf{T}\mathcal{M}^{\otimes 2})\otimes\mathcal{K}(U)$, with[125] $\mathbb{R}_{+}\subset U\subseteq\mathbb{C}$.

$$\mathcal{Z}_{\mathcal{M}}(\underline{g},\tau)\overset{\text{def}}{=}\sum_{\lambda_{G}\in\operatorname{spec}\Delta}e^{\tau\lambda_{G}},\tag{1.491}$$

where $G\in\mathrm{sym}_{n}^{+}(\mathbb{R})$ is the symmetric, positive-definite matrix associated to its metric $\underline{g}\in\mathsf{T}\mathcal{M}^{\otimes 2}$ which both Δ and $\mathcal{Z}_{\mathcal{M}}$ depend on. The partition function is related to profound connections with both Spectral Geometry, [55]–[56], [90] (and references therein), Hamiltonian dynamical systems, Classical and Statistical Mechanics, [8], [23], [34], [49], [60], [84], [97], [105], and Number Theory, [2], [71], [72], [78]; the

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[125] Usually, it is defined for $t\in\mathbb{R}_{+}$, since $\operatorname{spec}\Delta\subset\mathbb{R}_{-}$, then analytically continued in $U\subseteq\mathbb{C}$.

latter reference stands since partition functions are strictly connected with the famed and "wild" Riemann zeta function $\zeta(s) := \sum_{n\in\mathbb{N}} n^{-s}$ (cfr. [2]), for which reason functions like $\mathcal{Z}_{\mathcal{M}}$ are sometimes also referred to as "**zeta** functions".

The partition function is often evaluated with respect to a "real" time variable $\tau \equiv t \in \mathbb{R}$, which represents an evolution parameter, and sometimes is even continued outside its convergence domain U where it is just considered to be a formal power series (§1.3.2 — that being a reason why it is called a 'parititon' function). In this way, both its geometrical (zeroes, singularities, branch points) and analytical properties (derivatives, natural boundaries, stationary points, etc.) reveal something about the geometry and topology of the underlying Riemann manifold, such as its measure (volume/area — as well as that of its boundary $\partial\mathcal{M}$), curvature, genus, and so on. Since partition functions would deviate the treatise from the present work, they are just here defined, whereas references to the aforementioned Bibliography items are suggested.

As an example to the previously mentioned properties of $\mathcal{Z}_{\mathcal{M}}$, let us derive an asymptotic evaluation for $t \to 0^+_{,}$ which is similar[126] to Weyl's (1.490) for the eigenvalue-counting function $\mathbb{N}_{-\Delta}$:

$$\lim_{t\ll 1} \mathcal{Z}_{\mathcal{M}}(\underline{g}, t) \approx \frac{1}{\sqrt{(4\pi t)^n}} \sum_{k=0}^{\infty} a_k t^k, \qquad \begin{cases} a_0 = \mu_{\underline{g}}(\mathcal{M}) \\ a_1 = \frac{1}{6} K_{\underline{g}}(\mathcal{M}) \\ \vdots \end{cases}$$

5.2.2. Spectral dimension

Considering successes and failures summarized above, and willing to extend some Spctral Geometry results to Fractal Geometry, one would naturally ask: if not much of geometrical information about the shape of a (possibly fractal) drum can be inferred from its (Laplacian) spectrum, then how much about the 'fractal

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[126] $K_{\underline{g}}(\mathcal{M})$ is the mean scalar curvature of $\mathcal{M}$ with respect to the metric $\underline{g}$, cfr. [36], [88]

dimension'? Is it possible to recover at least some quantitative information about the complexity or self-similarity of a drum (for plane as well as for higher-dimensional fractal manifolds)?

The concept that will be used throughout the rest of the Chapter is that of spectral dimension. Although several authors suggest different, often non-equivalent definitions of "spectral dimension," the following one was invented specifically for this scope; further details can be found in [14] and [21].

Let $\mathcal{M}$ be a bounded real Riemann n-manifold and Δ its scalar Laplace's operator (with Dirichlet's boundary conditions if $\partial\mathcal{M}\neq\varnothing$); then $\mathrm{spec}\,\Delta=\{-\lambda_n\}_{n\in\mathbb{N}}\subset\mathbb{R}_-$ is its negative-definite spectrum (which is only discrete), as in §4.2.1. The ***spectral dimension*** of $\mathcal{M}$ is:

$$\dim_\Delta\mathcal{M}\overset{\text{def}}{=}\frac{\displaystyle\sum_{\lambda\in\mathrm{spec}\,\Delta\setminus\{0\}}\log\mathrm{mul}_\Delta\lambda}{\displaystyle\sum_{\lambda\in\mathrm{spec}\,\Delta\setminus\{0\}}\log|\lambda|}. \tag{1.492}$$

It is easy to prove that the spectral dimension is well-defined and $\dim_\Delta\mathcal{M}\in[0,+\infty[$ for every compact finite-dimensional Riemann manifold; furthermore, (1.492) is well-posed (and has the same numerical value) if Neumann's boundary conditions (or 'mixed' conditions of both) are imposed instead. Since $\mathrm{spec}\Delta$ accumulates at $-\infty$, $\dim_\Delta\mathcal{M}=0$ whenever the degeneracies of Δ's spectrum do not sufficiently increase with the (absolute value of the) eigenvalues themselves; this may trivially happen if $\mathcal{M}$ admits only simple eigenvalues, or at most only a finite number of multiple eigenvalues (versus the countable infinity of its whole spectrum).

What is the geometrical meaning of spectral dimension? It has already been inferred in §5.2.1, (1.490) that eigenvalues' degeneration for the Laplace's operator (even not associated to Riemann manifolds, but also graphs for example) indicates the presence of geometric symmetries in the domain: the greater algebraic multiplicities, the higher is the symmetry's "degree." Also recall that higher is an eigenvalue (in absolute value), the shorter are the "wavelengths" associated to its eigenfunction(s) which —if interpreted as waves for the wave equation relative to Laplace's equation— can be considered to be "scanning" the manifold with a geometric "precision" proportional to the eigenvalues' magnitude. The algebraic multiplicities represent the degree of symmetries at shorter length-scales, proportionally with the eigenvalues themselves.

What happens on a self-similar fractal? First of all a proper concept of spectral dimension must be introduced because fractal sets can be properly covered with atlases to become topological manifolds, but seldom show a sufficiently regular geometry as to become differentiable (or even smooth) manifolds.

Lemma

Let $\{\mathbb{R}^n;\mathbf{w}_1,\mathbf{w}_2,\ldots,\mathbf{w}_p\}$ be a just-touching ISS, then the spectral dimension of its attractor $E_\infty\subset\mathbb{R}^n$ is defined as the limit (with respect to the Hausdorff metric of $\mathbb{R}^n$) of the spectral dimensions of its prefractals $(E_n)_\mathbb{N}$, i.e.:

$$\dim_\Delta E_\infty := \lim_n \dim_\Delta E_n. \tag{1.493}$$

<u>Proof</u>: This is a Lemma because the well-posedness of such a definition has to be proven, i.e. that $\dim_\Delta E_\infty$ does not depend on the chosen initiator $E_0\subset\mathbb{R}^n$ (and thus on the prefractals' sequence).

From now on the formalism and identification of diaperiodic modes will be carried out as was done in §5.1.2, with the only difference that the non-zero eigenvalues are no longer counted along with their multiplicities, so $\lambda_{N,n}<\lambda_{N,n+1}$ and $\lambda_{N,j,n}<\lambda_{N,j,n+1}$, respectively, $\forall n\in\mathbb{N}$ and $1\leq j\leq p$. On the N^{th}-step prefractal $E_N=\mathbf{w}^N(E_0)\equiv\mathbf{w}(E_{N-1})$, let the self-similar spectral decomposition described in §5.1.3 be used: $\mathrm{spec}(-\Delta)=\{\lambda_{N,j,n}\}_{0\leq j\leq p,n\in\mathbb{N}}=\{\lambda_{N,n}\}_{n\in\mathbb{N}}$.

Now let us consider that some $1\leq q\leq p$ of the similarity transformations may have the same contraction ratios, for example there will be p_1 similarities with contraction ratio c_1, p_2 similarities with contraction ratio c_2, … and p_q similarities with contraction ratio c_q (where, of course, $p_1+p_2+\ldots+p_q=p$).

If $q>1$ then the algebraic multiplicities of diaperiodic modes get rescaled as

$$\mathrm{mul}_{-\Delta}\lambda_{N,j,n}=p_j\,\mathrm{mul}_{-\Delta}\lambda_{N-1,n}$$

(where subscript '$_{-\Delta}$' will be hereinafter suppressed since it is unambiguous ad far as this proof is concerned). Combining definitions (1.492) and (1.493), one has:

$$\dim_\Delta E_\infty = \lim_N \frac{\sum_{n=1}^{\infty}\log\mathrm{mul}\lambda_{N,n}}{\sum_{n=1}^{\infty}\log\lambda_{N,n}}. \tag{1.494}$$

The general term in the numerator of (1.494) can be thus evaluated as:

$$\begin{aligned}
\log\mathrm{mul}\lambda_{N,n} &= \sum_{j=0}^{q}\log\mathrm{mul}\lambda_{N,j,n}\\
&= \log\mathrm{mul}\lambda_{N,0,n}+\sum_{j=1}^{q}\log\left(p_j\mathrm{mul}\lambda_{N-1,n}\right)\\
&= \log\mathrm{mul}\lambda_{N,0,n}+q\sum_{j=0}^{q}\log\mathrm{mul}\lambda_{N-1,j,n}+\sum_{j=1}^{p}\log p_j\\
&= \log\mathrm{mul}\lambda_{N,0,n}+q\log\mathrm{mul}\lambda_{N-1,0,n}+\\
&\quad+q\sum_{j=1}^{q}\log\left(p_j\mathrm{mul}\lambda_{N-2,n}\right)+\sum_{j=1}^{p}\log p_j\\
&= \log\mathrm{mul}\lambda_{N,0,n}+q\log\mathrm{mul}\lambda_{N-1,0,n}+\\
&\quad+q^2\log\mathrm{mul}\lambda_{N-2,n}+(1+q)\sum_{j=1}^{p}\log p_j\\
&\vdots\\
&= q^N\log\mathrm{mul}\lambda_{0,n}+\sum_{k=0}^{N-1}q^k\left(\log\mathrm{mul}\lambda_{N-k,0,n}+\sum_{j=1}^{q}\log p_j\right).
\end{aligned}$$

The general term in the denominator of (1.494) is also evaluated in a similar way:

$$\begin{aligned}
\sum_{j=0}^{p} \log \mathrm{mul}\lambda_{N,n} &= \log \mathrm{mul}\lambda_{N,0,n} + \sum_{j=1}^{q} \log \frac{\lambda_{N-1,n}}{c_j} \\
&= \log \mathrm{mul}\lambda_{N,0,n} + q\sum_{j=0}^{q} \log \lambda_{N-1,j,n} - \sum_{j=1}^{q} \log c_j \\
&= \log \mathrm{mul}\lambda_{N,0,n} + q\log \lambda_{N-1,0,n} + q^2 \log \lambda_{N-2,0,n} + \\
&\quad -(1+q)\sum_{j=1}^{q} \log c_j \\
&\vdots \\
&= q^N \log \lambda_{0,n} + \sum_{k=1}^{N}\left(q^{N-k} \log \lambda_{k,p(p+1)+n} - q^{k-q} \sum_{j=1}^{q} \log c_j \right).
\end{aligned}$$

Numerator and denominator are then combined together for the computation of the N^{th}-step prefractal E_N's spectral dimension:

$$\dim_\Delta E_N = \\ = \frac{\left(\sum_{k=0}^{N-1} q^k\right)\left(\sum_{j=1}^{q} \log p_j\right)\sum^{\infty} 1 + \sum_{n=1}^{\infty}\left(q^N \log \mathrm{mul}\lambda_{0,n} + \sum_{k=0}^{N-1} q^k \log \mathrm{mul}\lambda_{N-k,0,n}\right)}{\left(\sum_{k=0}^{N-1} q^k\right)\left(\sum_{j=1}^{q} \log \frac{1}{c_j}\right)\sum^{\infty} 1 + \sum_{n=1}^{\infty}\left(q^N \log \lambda_{0,n} + \sum_{k=0}^{N-1} q^{N-k} \log \lambda_{k,p(p+1)+n}\right)};$$

the rightmost sums '$\sum_k$' at both numerator and denominator, diverging as $\mathrm{O}(q^{N-1})$, can be neglected for the asymptotic evaluation in the limit $N \to \infty$, due to the remaining sums diverging as $\mathrm{O}(q^N)$ and being asymptotically evaluated further as

$$\dim_\Delta E_N \approx \frac{\left(\frac{q^N-1}{q-1}\sum_{j=1}^{q} \log p_j\right)\sum^{\infty} 1 + \sum_{n=1}^{\infty} q^N \log \mathrm{mul}\lambda_{0,n}}{\left(\frac{q^N-1}{q-1}\sum_{j=1}^{q} \log \frac{1}{c_j}\right)\sum^{\infty} 1 + \sum_{n=1}^{\infty} q^N \log \lambda_{0,n}} < \infty \quad (1.495)$$

At this stage it can already be seen that the prefractals' spectral dimension only depends on the initiator's spectrum (since only the eigenvalues $\{\lambda_{0,n}\}_{\mathbb{N}}$ are involved), as well as the ISS' contraction ratios.

As $N \to \infty$ (1.495) does not asymptotically depend on the Laplace's eigenvalues of initiator E_0 anymore, so the Lemma is definitely proven for $q > 1$.

Case $q = 1$ (p similarities all with the same contraction ratio $c \in]0,1[$) is simpler, and even allows for a well-known closed-form computation for the spectral dimension, in fact similar asymptotic evaluation leading to (1.495) ends in:

$$\dim_\Delta E_N \approx \frac{\frac{N(N+1)}{2} \log p_j \sum^{\infty} 1 + q^N \sum_{n=1}^{\infty} \log \mathrm{mul}\lambda_{0,n}}{\frac{N(N+1)}{2} \log \frac{1}{c} \sum^{\infty} 1 + q^N \sum_{n=1}^{\infty} \log \lambda_{0,n}} < \infty \quad (1.496)$$

As $N \to \infty$ the (1.496) is now composed of two diverging sums at both numerator and denominator, and

$$\begin{aligned} \dim_\Delta E_\infty &\approx \cancel{\lim_{N}} \frac{\frac{N(N+1)}{2} \log p \cancel{\sum^{\infty}} 1 + \cancel{q^N \sum_{n=1}^{\infty} \log \mathrm{mul}\lambda_{0,n}}}{\frac{N(N+1)}{2} \log \frac{1}{c} \cancel{\sum^{\infty}} 1 + \cancel{q^N \sum_{n=1}^{\infty} \log \lambda_{0,n}}} \\ &\approx \frac{\log p}{\log \frac{1}{c}} \equiv -\frac{\log p}{\log c} = \dim_{\mathrm{B}} E_\infty, \end{aligned} \quad (1.497)$$

where '$\dim_{\mathrm{B}}$' is the box-counting dimension (cfr. [26], [49], [6]), which has the simple expression (1.497) for IFSs with equal contraction ratios. In this case $\dim_\Delta E_\infty$ only depends on p and c, as in (1.472). Lemma is proved for $q = 1$ also. □

A computation example of spectral dimension is done on the Sierpiński carpet $\check{S}_2$ as it was defined in [9]–§1.8, considering its diaperiodic modes only, as computed in §5.1.5.

$$\begin{aligned}
\dim_\Delta \check{S}_2 &= \lim_N \dim_\Delta \check{S}_{2,N} \approx \\
&\approx \lim_N \frac{\displaystyle\sum_{\substack{m,n=0,\\ m+n>0}}^{\infty} \log(2{\cdot}8^N)}{\displaystyle\sum_{\substack{m,n=0,\\ m+n>0}}^{\infty} \log \kappa_{N;m,n}} = \lim_N \frac{\displaystyle\sum_{\substack{m,n=0,\\ m+n>0}}^{\infty} \log 2^{3N+1}}{\displaystyle\sum_{\substack{m,n=0,\\ m+n>0}}^{\infty} \log\left(3^N \pi \sqrt{m^2+n^2}\right)} \\
&\approx \lim_N \frac{3N \log 2 \sum^{\infty} 1}{(N\log 3 + \log\pi)\sum^{\infty} 1 + \cancel{\frac{1}{2}\displaystyle\sum_{\substack{m,n=0,\\ m+n>0}}^{\infty} \log(m^2+n^2)}} \\
&\approx \lim_N \frac{N \log 2^3}{N\log 3 + \log \pi} \\
&= \frac{\log 8}{\log 3} \simeq 1.892789261 = \dim_{\mathrm{B}} \check{S}_2 .
\end{aligned}$$

5.2.3. One can hear the *fractal dimension* of a drum

Recall that for any subset $E \subseteq S$ in a metric space (S,d), the counting function $\mathbb{N}_\delta(E)$, $\forall \delta \in \mathbb{R}_+$, is the number of sets of diameter at most δ whose union covers[127] E, i.e.:

$$\mathbb{N}_\delta(E) \overset{\text{def}}{=} \inf \operatorname{Card} \left\{ \{U_i\}_{i\in I} \;\middle|\; E = \bigcup_{i\in I} U_i, \ \operatorname{diam} U_i \leq \delta \right\}.$$

The numerical procedures employed to compute this counting function always involve restricting the covering sets to be of a specific geometry (for example using only balls or poly-intervals): this produces different results but it can be proven that the following definition of box-counting dimension does instead *not* depend on that,

[127] Covering sets $\{U_i\}_{i\in I}$ used in this chapter are not to be confounded with coverings of §2.3.1.

and the limit below —when it does exist— is equivalent to both supremum and infimum limits, ‘ $\overline{\lim}_{\delta\to 0^+}$ ’ and ‘ $\underline{\lim}_{\delta\to 0^+}$ ’, respectively — cfr. [39]:

$$\dim_{\mathrm{B}} E \stackrel{\text{def}}{=} -\lim_{\delta\to 0^+} \frac{\log \mathbb{N}_\delta(E)}{\log \delta}. \tag{1.498}$$

This equivalence means that the above limit is invariant by the kind of sets used to cover E because, roughly, as their supremum size δ tends to 0, it asymptotically tends to be covered with the same number of “infinitesimal” sets: this power-law is a feature of most fractal sets — self-similar ones in particular.

It is noteworthy that in the case of contraction mappings sharing the same contraction ratio $c\in]0,1[$, (1.471) is closed-form computed to be the rightmost-hand side of (1.497). This fact is not a pure coincidence, in fact the problem of estimating the fractal dimension of an ISS’ attractor (i.e. the problem of “hearing the fractal dimension of a drum”) is completely solved by generalizing this fact.

‘Fractal Drum’ Theorem

The fractal (box-counting) dimension of a just-touching ISS’ attractor is **equal** to its spectral dimension, as per (1.493).

Proof: Let $\{\mathbb{R}^n;\mathbf{w}_1,\mathbf{w}_2,\ldots,\mathbf{w}_p\}$ be the ISS of the previous Lemma, whereas the initiator is set to be a closed hypercube $B_0=[\![\mathbf{0}_n,\delta_0\mathbf{1}_n]\!]:\equiv \times^n[0,\delta_0+1]$, of side $\delta_0\in\mathbb{R}_+$ and let $(B_N)_{\mathbb{N}}$ be associated prefractals’ sequence (i.e. $B_N=\mathbf{w}^N(B_0)$, $\forall N\in\mathbb{N}$). Each B_N is made of just-touching hypercubes of supremum diameter $\delta_N\sqrt{n}$ and the corresponding sequence $(\delta_N)_{\mathbb{N}}$ is infinitesimal, i.e. $\lim_N\delta_N=0$.

The spectrum of the scalar Laplace’s operator on such an hypercube is then (at most)

$$\operatorname{spec}\Delta = \left\{-\frac{\pi\|\mathbf{m}\|}{\delta_N}\right\}_{\mathbf{m}\in\mathbb{N}_0^n}, \qquad \operatorname{mul}_\Delta\left(-\frac{\pi\|\mathbf{m}\|}{\delta_N}\right)\sim \mathrm{O}(n!),$$

with the asymptotic evaluation of the algebraic multiplicities due to the n-dimensional cubic symmetry. Then it is sufficient to note that this spectrum constitutes the diaperiodic (self-similar) spectrum of compact sets B_N and that, by spectral dimension's definition in the previous Lemma (1.493), only such modes are significant for $N \to \infty$, then (letting $m = \|\mathbf{m}\|$ for every degenerate eigenvalue):

$$\dim_\Delta B_N = \frac{\sum_{m=1}^{\infty} \log \mathrm{mul}_{-\Delta} \dfrac{\pi m}{\delta_N}}{\sum_{m=1}^{\infty} \log \dfrac{\pi m}{\delta_N}}.$$

Also note that $(B_N)_{\mathbb{N}}$ is a suitable *coarse graining* sequence to evaluate the box-counting dimension of the attractor E_∞ (cfr. [39]) and that, by a basic Calculus theorem, its computation as a continuous limit $\delta \to 0^+$ is equivalent to that of a discrete (sequential) limit such as $\delta_N \to 0$. Then:

$$\begin{aligned}
\dim_\Delta E_\infty \equiv \lim_N \dim_\Delta B_N = \lim_N \frac{\sum_{m=1}^{\infty} \log \mathrm{mul}_{-\Delta} \dfrac{\pi m}{\delta_N}}{\sum_{m=1}^{\infty} \log \dfrac{\pi m}{\delta_N}} \\
&\approx \lim_N \frac{\log \mathbb{N}_{\delta_N}(B_N) + \log d!}{\log \pi - \log(\delta_N \sqrt{n})} \approx -\lim_N \frac{\log \mathbb{N}_{\delta_N}(B_N)}{\log \delta_N} \\
&\approx -\lim_N \frac{\log \mathbb{N}_{\delta_N}(E_\infty)}{\log \delta_N} = -\lim_{\delta \to 0^+} \frac{\log \mathbb{N}_{\delta}(E_\infty)}{\log \delta} \\
&\equiv \dim_{\mathrm{B}} E_\infty.
\end{aligned}$$

This proves the 'Fractal Drum' Theorem and positively answers «yes, one can» to the question entitling §5.2, Q.E.D.. □

5.2.4. Electromagnetic applications

The 'Fractal Drum' Theorem may be useful in the case whenever an estimation of a regularity or complexity of an unknown object able to scatter resonate and scatter electromagnetic radiation on a discrete spectrum. Suppose that an unknown dielectric object (possibly dangerous) is buried beneath or beyond a "problematic" or unknown (or, simply, undestroyable) obstacle. The obstacle might be a terrain, a (underwater) relic, a weapon, a mined field, as well as vascular or neoplasic tissue — whose real nature, or danger level, could or could not be known a priori.

Particularly in the last case, the need to estimate the complexity of a dielectric medium without explicitly gain mechanical access to it and/or remove it (e.g. via wireless telecommunications, risky mine-sweeping actions, more or less invasive surgeries, etc.) may be crucial.

Since dielectric objects are believed to act as resonators, but their lack of perfectly conducting boundaries allows to free oscillations —once properly excited— to be partially radiated outside; a special antenna probing the object at different, precise frequencies may receive its emitted spectrum and, from the magnitude of each peak, guess a sort of degeneracy for each positive-response frequency (i.e. estimating the algebraic multiplicities of its eigenfrequencies).

Of course, as remarked at the begginning of the paragraph, only by recovering the *whole* exact spectrum (with its multiplicities) the computation of the object's spectral (and, hence, the fractal) dimension might be possible; but since, at least on self-similar objects, the eigenvalues tend to distribute along a self-similar curve, the computation of a high (though finite) number of frequencies might be enough to infer an approximated fractal dimension.

Correlation between the fractal dimension and human tissues' pathologies, like chronic liver disease or neoplasies, can be found in [85], [96].

5.3 Fractal Riemann surfaces in Electromagnetism

This paragraph briefly reviews the use of (pre-fractal) Riemann surfaces introduced throughout §3 for applications in Electromagnetism. As sketched in §4.2, electromagnetic integro-differential problems in the spectral domain generally involve the evaluation of the inverse Fourier transform with respect to scalar or vector wave-numbers which depend on either the real-valued velocity ω (or $\nu \equiv \omega/2\pi$) or the complex-valued Laplace variable $s \in \mathbb{C}$ (whose imaginary part is $\mathrm{Im}\, s = \omega$ — cfr. §1.2.5). In other cases, as for scattering problems related to the "electromagnetic **leakage**" phenomenon (cfr. §4.2.5), using complex-valued variables for singular integrals is a way to compactly treat, in a discrete way, some scattering phenomena which would otherwise be classically described by far more complicated operators — using "leaky poles" instead of a continuum spectrum (which is usually a natural boundary for compex functions to be integrated).

Even in the former, real-valued case, such Fourier integrals are often singular, so the [real-parameters] functions are [analytically continued along a complex domain and] evaluated on complex paths avoiding singularities and branch points [generated by analytic continuation] by techniques of Numerical Analysis (e.g. the steepest-descent path) —refer to [43], [47], [58], [99], as well as papers [4], [24], [70], [76], [98]— which rely on the Residue and the Mondoromy Theorems discussed throughout §2.

Another "dual" approach, introduced at the end of §4.1.2, is to use the Fourier transform with respect to space variables only, whereas the time-variable is left untransformed. The resulting equations are thus (systems of) ODE(s) with respect to the real time $t \in \mathbb{R}$. Complexifying the time to a complex-valued variable $\tau \in \mathbb{C}$ such that $\mathrm{Re}\, \tau \equiv t$ is a "dual" procedure to Fourier-transforming t into a velocity $\omega \in \mathbb{R}$ and then analytically continuing it along the Laplace variable $s \in \mathbb{C}$ such that $\mathrm{Im}\, s \equiv \omega$, according to a simple 'square' commutative diagram.

Finally, in other electromagnetic problems, resorting to complex-valued wave-numbers is not even a numerical choice any more, but a mandatory way to deal with losses inside the media involved.

Whatever is the case among those exposed above, many differential problems in Electromagnetism (as well as in other branches of Mathematical Physics and applied sciences, [1], [6], [7], [47], [92]) often require complexified integro-differential equations, that is ordinary differential equations on Riemann surfaces (or partial differential equations on Kähler manifolds for vector wave-numbers, i.e. for infinite-dimensional function spaces), to be solved.

5.3.1. Inverse Scattering and Riemann-Hilbert problems

What is now commonly referred to as ***Inverse Scattering Transform*** (IST) is an ensemble of mathematical methods inherited from several branches of Mathematics (like Functional Analysis, Complex Analysis, Harmonic Analysis, Algebraic Geometry) and Mathematical Physics (from Electromagnetism to Signals' theory; from many-body problems, to Solitons' theory) in order to solve both ordinary and partial *nonlinear* differential equations —or at least redirect them to equivalent, better-known, (less partially) solved ones— under an as much general as possible framework. Many of these methods are nonlinear generalizations of Classical Physics' inverse scattering method, which is the inverse problem to reconstruct the "input data" of a physical system from the sole knowledge of "output data," when the physical system itself is not completely known, or is only known "directly" via non-invertible functions (like wave operators). In the linear case, this method is based on the inverse Fourier, Laplace or Hilbert transforms (cfr. §1.2), whose integrals physically represent the convolution of the known output data by the **transfer function** of the system, which is the Green's function of the direct-problem operator (cfr. §4.2).

Stated in this way, such methodology is more affine to control systems in Signals' Theory (which may be usable, for example, to break the encryption code of a crypted message, [44]), but the method borrows the name from its historical first-application field: the inverse problems in Electromagnetism regarding the reconstruction of the incident electric field scattered from one or more (static or moving) obstacles, starting from the sole knowledge of the scattered field (known as "scattering data"), and a few properties of the scattering media, like for example its reflection and transmission coefficients, the scattering matrix (or S-matrix) $S(\boldsymbol{\kappa},\omega)$, or some equivalent to theirs (like impedance or admittances' matrices, [9]–§D, [16], [13], [50], [63]).

This argument deviates too much form the scope of this work; here only a particular case will be treated, where the problem is transformed, via the IST, into a Riemann-Hilbert problem, cfr. §2.4.7, which is a special framework where singularities and branch points of complicated Riemann surfaces (like those analyzed in §3) play a fundamental role, whereas their limit set (§§1.4.3, 3.2) represents a natural boundary (§2.4.1).

Let $\boldsymbol{\jmath}\equiv(\jmath_1,\jmath_2,\ldots,\jmath_n)\in\mathbb{R}^n$ with $\jmath_1>\jmath_2>\ldots>\jmath_n$ (a more general setting would include $\boldsymbol{\jmath}\in\mathbb{C}^n$), let $J=\operatorname{diag}\boldsymbol{\jmath}$ and $Q\equiv(q_{i,j})_{1\leq i,j\leq n}\in\mathbb{M}_n(\mathrm{L}^1_{\mathrm{loc}}(\mathbb{C}))$ with zero diagonal (i.e. $q_{i,i}=0$ for $1\leq i\leq n$). Consider then the following system of matrix linear ODEs in the undetermined matrix function $\Psi_\kappa(z)\equiv\Psi(z,\kappa)$ depending on a complex parameter $\kappa\in\mathbb{C}$ and defined —at least at the beginning— on the complex plane $\mathbb{C}$ as starting Riemann surface, with dynamical matrix $A_\kappa(z)=\imath\kappa J+Q(z)$:

$$\mathrm{d}\Psi_\kappa = (\imath\kappa\boldsymbol{\jmath}^\intercal\Psi_\kappa + Q\Psi_\kappa)\mathrm{d}z. \tag{1.499}$$

If $\boldsymbol{\jmath}\in\mathbb{C}^n$, then Ψ becomes meromorphic in the whole complex plane minus a finite number of rays emanating from a branch point $\kappa_0\in\mathbb{C}$ (possibily $\kappa_0=0$) and the above equations becomes formally more complex (and so $\boldsymbol{\jmath}$, for simplicity's sake). Of course the dynamical matrix could also feature a more complicated dependency on (z,κ), but such formalism is general enough for most physical-mathematical applications.

Let the alternative function $M_\kappa(z):=e^{-\imath\kappa zJ}\Psi_\kappa(z)$ be defined, such that the above ODE is equivalent to a Lax-pair equation

$$\frac{\mathrm{d}M_\kappa}{\mathrm{d}z} = \imath\kappa[J\,|\,M_\kappa] + QM_\kappa, \tag{1.500}$$

where also $[J|M_\kappa] = (\mathrm{adj}\,J)M_\kappa$. It was proven that $\exists M \in \mathbb{M}_n(\mathcal{O}(\mathbb{C})) \otimes \mathcal{K}(\mathbb{C}\backslash\mathbb{R})$, such that $M_\kappa \in \mathbb{M}_n(\mathrm{L}^\infty(\mathbb{R}))$, $\forall \kappa \in \mathbb{C}\backslash\mathbb{R}$, if $M(x,\kappa)$ is valued on the real axis $z = x \in \mathbb{R}$ and, furthermore, $\lim\limits_{\kappa\to\infty} M(z,\kappa) = I_n$. In this case, let[128] $k \equiv \mathrm{Re}\,\kappa$ and

$$M_\pm(z,k) = \lim_{\mathrm{Im}\,\kappa\to 0^\pm} M(z,\kappa),$$

be the values of matrix function M when the spectral parameter becomes real (k) being approached from above or below the real axis, respectively. They can then be analytically continued along the upper or lower half-planes $\mathbb{C}_\pm$, respectively as $M_\pm(z,\kappa)$, being inter-related as $M_+(z,\kappa) = M_-(z,\kappa)(I_n + e^{\imath\kappa zJ}\Gamma_\kappa(z)\,e^{-\imath\kappa zJ})$. The multi-valued matrix function $\Gamma \in \mathbb{M}_n(\mathcal{O}(\mathbb{C})) \otimes \mathcal{K}(\mathbb{C})$ is called the reflection coefficient, whereas $S_\kappa(z) :\equiv S(z,\kappa) \stackrel{\mathrm{def}}{=} e^{\imath\kappa zJ}\Gamma_\kappa(z)e^{-\imath\kappa zJ}$ is best known as the **scattering matrix** of the RH-problem, so $M_-(z,\kappa) = M_+(z,\kappa)(I_n + S_\kappa(z))^{-1}_\cdot$

Furthermore, for any other solution $M'(x,k)$, there exists an automorphy factor $\exists \Phi \in \mathbb{M}_n(\mathrm{C}^\infty(\mathbb{R}))$ depending on the real spectral parameter, such that, cfr. §2.4.6:

$$M'(x,k) = M(x,k)e^{\imath kxJ}\Phi(k)e^{-\imath kxJ}. \tag{1.501}$$

The scattering matrix M_κ, for $\kappa \gtrless 0$, is proven to be the convolution of the general solution M_κ by a Green's function of the differential operator (1.499), i.e.:

$$M_\kappa(z) = I_n + \underline{G}_\kappa \circledast (Q(z)M_\kappa(z)) \tag{1.502}$$

where, for $z = x \in \mathbb{R}$, it is still not defined; in this case, with similar formalism as the matrix functions defined above, let $[\underline{G}_\kappa] = G_\kappa \equiv G(x,\kappa) \equiv (G_\kappa^{i,j})_{1\le i,j\le n}$. Since the components of the Green's function satisfy $\dfrac{\partial G_\kappa^{i,j}}{\partial x} - \imath\kappa(\jmath_i - \jmath_j)G_\kappa^{i,j}(x) = \delta(x)$, then:[129]

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[128] $M(z,\kappa)$ is holomorphic in z, meromorphic in κ everywhere but on $\kappa \in \mathbb{R}$ (where it attains different limited values $M_\pm(z,\mathrm{Re}\kappa)$ for $\mathrm{Im}\kappa \to 0^\pm$), where $M(x,\kappa)$ is a limited function $\forall x \in \mathbb{R}$.

[129] Also note that the diagonal part of $G_\kappa(x)$ is equal to $\imath I_n$, since $G_\kappa^{h,h}(x) = \mathcal{L}^{-1}[s^{-1}] = \imath$.

$$G_\kappa^{i,j}(x) = \mathcal{L}^{-1}\left[\frac{1}{s-(\jmath_i-\jmath_j)\,\mathrm{Re}\,\kappa-\imath(\mathrm{sign}\,\kappa)\,\mathrm{sign}(\jmath_i-\jmath_j)\,\mathrm{Im}\,\kappa}\right](x)$$
$$= \pm e^{\imath\kappa x(\jmath_i-\jmath_j)}\delta_{-1}(\pm x).$$

(where the last formula's $\pm$ signs depend on whether $i \gtrless j$ — and another sign variation is to be added when it is valued for $\mathrm{Im}\kappa \to 0^{\pm}$). That is the Green's function defined as an inverse Laplace transform (whose nautral boundary —the Laplace abscissa— is $\mathbb{R}$ rather than $\mathbb{I}$ as was defined in §1.2.5). In the end, (1.502) is equivalent to the following integral equation, which *is* the Inverse Scattering Transform (IST) introduced at the beginning of the paragraph:

$$M_\kappa(x) = I_n + \int_{-\infty}^{+\infty} G_\kappa(x-\xi)Q(\xi)M_\kappa(\xi)\mathrm{d}\xi\,.$$

Also notice the similarity of the above equations to those in §4.2, which are indeed physically equivalent to them if the IST is applied to electromagnetic scattering problems.

A general treatment to either the Inverse Scattering theory and the scattering problems specitic to Electromagnetism highly deviate from the scope of this work; for this reason we suggest the reader refers to textbooks [1] and [58], [43], respectively, and Bibliography therein.

5.3.2. Electromagnetic Scattering from fractal surfaces

Maxwell's III and IV equations in the time domain and the spectral-space domain $(\boldsymbol{\kappa},t)\in\mathbb{R}^3\oplus\mathbb{R}$, in the hypothesis of linear, stationary, homogeneous, isotropic, non-dispersive media (constant permittivity and permeability) are:

$$\begin{cases} \dfrac{\partial \overleftrightarrow{B}}{\partial t} = -\boldsymbol{\kappa}\overleftrightarrow{E}\wedge\mathrm{d}\boldsymbol{\kappa} - \overleftrightarrow{J}_{\mathrm{m}} \\ \dfrac{\partial \overleftrightarrow{D}}{\partial t} = \boldsymbol{\kappa}\overleftrightarrow{H}\wedge\mathrm{d}\boldsymbol{\kappa} - \overleftrightarrow{J} \end{cases} \qquad \begin{cases} \overleftrightarrow{D} = \varepsilon\star\overleftrightarrow{E} \\ \overleftrightarrow{B} = \mu\star\overleftrightarrow{H} \end{cases}$$

Written in matrix form with respect, for example, to the complex induction fields $(\overleftrightarrow{D}, \overleftrightarrow{B})$, they take the form of the following linear Hamiltonian system, [1], [8]:

$$\frac{\mathrm{d}}{\mathrm{d}t}\begin{pmatrix}\overleftrightarrow{D}\\ \overleftrightarrow{B}\end{pmatrix} = \boldsymbol{\kappa}\begin{pmatrix}0 & \mu^{-1}\\ -\varepsilon^{-1} & 0\end{pmatrix}\star\begin{pmatrix}\overleftrightarrow{D}\\ \overleftrightarrow{B}\end{pmatrix}\wedge\mathrm{d}\boldsymbol{\kappa} - \begin{pmatrix}\overleftrightarrow{J}\\ \overleftrightarrow{J}_{\mathrm{m}}\end{pmatrix}, \tag{1.503}$$

which can be easily reformulated within the formalism of (1.499) so that, applying the IST in some complex domain of the spectral parameter, analytically continued to a complex manifold with $\boldsymbol{\kappa}\in\mathbb{C}^3$, (1.503) becomes in the end a vector Riemann-Hilbert problem; in this case, the natural boundary of integrated (Green's) functions corresponds to separation interfaces in the scattering medium, where discontinuities of both fields and potentials ("**single**-layer" and "**double**-layer" potentials, [47], [58]) are present. Fortunately, since Maxwell's equations (1.383) are intrinsically *linear* (especially when the media are electromagnetically linear and homogeneous) the IST for electromagnetic inverse scattering easily reverts to an inverse Fourier transform like those in §4.2, so the methods detailed in that paragraph, long-time well-known to physicists and electronic engineers, [35], [50], [58], [63], [70], [76], [99], can be perfectly applied.

When the RH-problem associated to the IST of an electromagnetic scattering problem obeys to a functional ODE equivalent to those described in §§3.2.7, 3.4.1 (which happens when the geometry of the scattering medium is self-similar or, more generally, *fractal* — as long as its inhomogeneities), things get horribly worse from the numerical point of view… Yet, amazingly interesting things they are, as this works hopes to inspire.

In this case the scattering phenomenon becomes "chaotic" in the sense that incident plane-waves with only *slight* different phase vectors and frequencies within the same incident wave packet get scattered with arbitrarily large deviations in their corresponding scattering parameters. The inverse scattering becomes numerically difficult, not only because of the convergence problems related to the "ordinary" singularities of the Green's functions (as observed in §4.2), but also *intrinsically* unpredictable because of the *deterministic* (numerical) ***chaos*** of the scattering data, i.e. the Riemann Chaos of the fractal Riemann surface present within the scattering matrices themselves.

For this reason, not only the inverse problem is difficult, but also the direct one, because scattered wave from a "spectrally pure" plane wave is subject to the above "deterministic chaos effect" and is hard to be computed with prescripted precision, which means an arbitrarily large numerical error on the scattering data.

As commented in §3.4, the chaoticity of the surface is described by its (profinite) IMG, which unfortunately is solvable only in a few cases. When algebraic solvability is present, cfr. §3.3.4, integration paths of the IST, which correspond to trajectories on arbitrarily high-order surfaces of type $\mathcal{A}_n$ (§3.2) or $\mathcal{B}_n$ (§3.3), are explicitly computed by using both local Puiseux series' expansions and nested radicals, §3.3.5.

Using surfaces of the latter kind with equal numerator and denominator degrees is physically consistent because, as shown in §3.3.3, the explicit expression of the Riemann surface converges almost everywhere.

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Acknowledgements

```
~$ sudo init 0
Password: _
```

GNU/Linux **sh** console terminal

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