Virtual Holonomic Constraints for Euler-Lagrange systems under sampling

Mohamed Elobaid[1,2], Mattia Mattioni[1], Salvatore Monaco[1] and Dorothée Normand-Cyrot[2]

***Abstract*— In this paper, we consider the problem of imposing Virtual Holonomic Constraints to mechanical systems in Euler-Lagrangian form under sampling. An exact solution based on multi-rate sampling of order two over each input channel is described. The results are applied to orbital stabilization of the pendubot with illustrative simulations.**

***Index Terms*— Sampled data control, Feedback linearization, Algebraic/geometric methods.**

I. Introduction

Most control problem rely upon the design of feedback laws asymptotically zeroing a given function of the state as, for instance, orbital or set-point stabilization, motion planning, tracking, path following, to cite a few [1]–[5]. When dealing with mechanical systems, such a function writes $h(q) = 0$, with generalized coordinates q, and is generally referred to as *Virtual Holonomic Constraints* (VHCs, [6]–[9]). Stabilization of VHCs unavoidably requires to asymptotically drive the trajectories of the system onto a sub-manifold associated to the zero-level set of the function. Accordingly, the problem can be recast into a zero-dynamics perspective: setting the function as a dummy output for the dynamics, one must define a feedback making the corresponding zero-dynamics (and the zero-dynamics submanifold) invariant and attractive. With this in mind, it has been proved in [8] that the VHC is stabilizable if and only if the system possesses a well-defined vector relative degree $r = (2, \dots, 2)$ with respect to the associated dummy output function. Accordingly, under suitable hypothesis, the feedback imposing the VHC is the one rendering the corresponding zero-dynamics attractive and invariant (e.g., input-output feedback linearization) while preserving boundedness of the whole system trajectories.

All of this essentially concerns continuous-time systems despite the practical interest to treat mechanical systems under sampling when the control is piecewise constant and the state or output measures are sampled [10]. In this context, it is well known that the relative degree falls to one under single-rate sampling with the rising of an unstable *sampling zero dynamics* making the corresponding sampled-data system non-minimum phase in general [11], [12]. Accordingly, when dealing with stabilization of VHCs, single-rate sampling and, in particular, mere emulation control cannot be employed as the necessary relative degree condition is lost. This motivates the present paper whose contribution stands in providing control strategies ensuring the preservation of VHCs for mechanical systems under sampling. Based on the work in [13], it is shown that VHCs can be imposed under sampling according to a multi-rate device of suitably defined order. The proof is constructive and the feedback is shown to be the solution to a nonlinear implicit equality parameterized by δ, the sampling period, naturally recovering the continuous-time counterpart as $\delta \to 0$. Despite exact forms are hard to be computed in practice, approximate feedback laws computed as approximate solutions to the associated equalities are naturally defined and implemented in practice with notably improved performances with respect to standard emulation controllers [14], [15]. The results are applied to orbital stabilization of the pendubot [8], [16] and based on the VHC associated to the preliminary continuous-time I&I design recently proposed in [17].

The rest of the paper is organized as follows. In Section II, the problem is formally formulated with preliminaries on VHCs for continuous-time mechanical systems and sampled-data dynamics. The main result in Section III with the simulated example of the pendubot in Section IV. Section V concludes the paper with some perspectives.

Notations. $\mathbb{R}$ and $\mathbb{N}$ denote the set of real and natural numbers including 0. For any vector $z \in \mathbb{R}^n$, $\|z\|$ and $z^\top$ define respectively the norm and transpose of z. Given a full rank matrix $B \in \mathbb{R}^{n\times m}$ with $n > m$, $B^\dagger = (B^\top B)^{-1}B^\top$ denotes the pseudoinverse, while $B^\perp$ its orthogonal complement verifying $B^\perp B = 0$. Also, $\ker\{B\}$ denotes the null space of B. Given two matrices of any dimension, $A \otimes B$ denotes the Kronecker product. $\mathbf{1}_n \in \mathbb{R}^n$ is the vector with all unitary entries while 0 denotes the zero matrix of suitable dimensions. $\text{diag}\{a_1, \dots, a_n\} \in \mathbb{R}^{n\times n}$ denotes the diagonal matrix with $a_i \in \mathbb{R}$ the coefficients on the main diagonal for $i = 1, \dots, n$. $x = \text{col}\{a_1, \dots, a_n\} \in \mathbb{R}^{n_1+\dots+n_n}$ denotes the column vector with entries provided by $a_i \in \mathbb{R}^{n_i}$ of suitable dimensions. If $(\mathcal{X}, d)$ is a metric space, $\Gamma \subset \mathcal{X}$ and $x \in \mathcal{X}$, then $\|x\|_\Gamma = \inf_{y\in\Gamma} d(x, y)$ defines the point-to-set distance of x to Γ. I and I_d denote the identity matrix and Identity operator (or function, depending on the context) of suitable dimensions, respectively. Given a twice continuously differentiable function $S(\cdot) : \mathbb{R}^n \to \mathbb{R}$, $\nabla S(\cdot)$ represents its gradient (column) vector while $\nabla^2 S(\cdot)$ is its Hessian matrix. Given a n-dimensional vector field $f(x)$ with $x \in \mathbb{R}^n$,

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[1]Dipartimento di Ingegneria Informatica, Automatica e Gestionale *A. Ruberti* (Sapienza University of Rome); Via Ariosto 25, 00185 Rome, Italy { mohamed.elobaid, mattia.mattioni, salvatore.monaco}@uniroma1.it.

[2]Laboratoire de Signaux et Systèmes (L2S, CNRS Université Paris-Saclay); Centrale Supelec 3, Rue Joliot Curie, 91192, Gif-sur-Yvette, France dorothee.normand-cyrot @centralesupelec.fr

$\mathrm{L}_f = \sum_{i=1}^{n} \frac{\partial}{\partial x_i}$ denotes the Lie derivative operator, and recursively, $L_f^i = \mathrm{L}_f \circ \mathrm{L}_f^{i-1}$ with $\mathrm{L}_f^0 = I_d$. For $\delta > 0$, $e^{\delta \mathrm{L}_f} = I_d + \sum_{i>0} \frac{\delta^i}{i!} \mathrm{L}_f^i$ denotes the Lie exponential. Given a smooth function $H : \mathbb{R}^n \to \mathbb{R}$ by the Exchange Theorem $H(e^{\delta \mathrm{L}_f} x) = e^{\delta \mathrm{L}_f} H(x) = H(x) + \sum_{i>0} \frac{\delta^i}{i!} \mathrm{L}_f^i H(x)$. A function $R(x, \delta) = O(\delta^p)$ is said of order δ^p, $p \geq 1$ if whenever it is defined it can be written as $R(x, \delta) = \delta^{p-1} \tilde{R}(x, \delta)$ and there exist a function $\theta \in \mathcal{K}_\infty$ and $\delta^* > 0$ s. t. $\forall \delta \leq \delta^*$, $|\tilde{R}(x, \delta)| \leq \theta(\delta)$.

II. Preliminaries and Problem Statement

In the sequel, we consider Euler-Lagrange systems of the form

$$D(q)\ddot{q} + C(q, \dot{q})\dot{q} + \nabla P(q) = B(q)u \tag{1}$$

with n-dimensional generalized coordinates $q \in \mathcal{Q}$, $\mathcal{Q} \subset R^n$ the configuration space, input torque $u \in \mathbb{R}^m$ with $m = n - 1$, $D(q) = D^\top(q) \succ 0$ the generalized inertia matrix, $C(q, \dot{q})\dot{q}$ representing the Coriolis and centrifugal forces, $P(q)$ the potential energy function, $B(q)$ of rank $m = n - 1$ and

$$L(q, \dot{q}) = \frac{1}{2} \dot{q}^\top D(q) \dot{q} + P(q)$$

being the Lagrangian verifying

$$\dot{L}(q, \dot{q}) = \dot{q}^\top B^\top(q) u.$$

A. VHCs for EL systems in continuous time

As formalized in [8], a virtual holonomic constraint (VHC) for a mechanical system (1) is a relation of the form $h(q) = 0$ which can be made invariant under feedback. In this sense, the following formal definition is recalled.

Definition 2.1: A virtual holonomic constraint is a relation $h(q) = 0$ where $h : \mathcal{Q} \to \mathbb{R}^m$ is smooth, $\mathrm{rank}(\mathrm{d}h) = m$ for all $q \in h^{-1}(0)$ and the constraint manifold

$$\Gamma = \{(q, \dot{q}) : h(q) = 0,\ \mathrm{d}h(q)\dot{q} = 0\} \tag{2}$$

is controlled invariant. A VHC is stabilizable if there exists a smooth feedback $u(q, \dot{q})$ that asymptotically stabilizes[1] Γ.

In the following, it is assumed that $m = n - 1$ so that the configuration variable can be regrouped, with a slight abuse of notation, as $q = \mathrm{col}\{q_a, q_u\} \in \mathbb{R}^{n-1} \times \mathbb{R}$ so that the VHC can be described in parametric form as

$$q_a = \varphi(q_u), \quad h(q) = q_a - \varphi(q_u)$$

with hence $h^{-1}(0)$ a closed curve. The definition of regular VHC is recalled below as fundamental to characterize asymptotically stabilizable VHS [8][2]

Definition 2.2 ([8]): Consider a smooth relation $h : \mathcal{Q} \to \mathbb{R}^m$ and $\mathrm{rank}(\mathrm{d}h) = m$ for all $q \in h^{-1}(0)$. The relation $h(q) = 0$ is said to be a *regular* VHC of order $m > 0$ for the system (1) if it possesses relative degree $\{2, \ldots, 2\}$ everywhere on the constrained manifold (2); i.e., the matrix

$$U(q) = \mathrm{d}h(q) D^{-1}(q) B(q) \tag{3}$$

has full rank for all $q \in h^{-1}(0)$.

By Definition 2.1, the constraint manifold (2) is the zero-dynamics manifold corresponding to the output $e = h(q)$ so that the reduced dynamics coincides with the zero-dynamics. As a consequence, Γ is asymptotically stabilized via feedback linearization under mild hypotheses on the maps $h(q), \mathrm{d}h(q)\dot{q}$ as recalled in the result below.

Proposition 2.1: Let $h(q) = 0$ be a regular VHC of order $n - 1$ for (1) with constraint manifold Γ in (2). Let

$$H(q, \dot{q}) = \begin{pmatrix} h(q) \\ \mathrm{d}h(q)\dot{q} \end{pmatrix} \tag{4}$$

and assume there exist functions $\alpha_1, \alpha_2 \in \mathcal{K}$ such that

$$\alpha_1(\|(q, \dot{q})\|_\Gamma) \leq \|H(q, \dot{q})\| \leq \alpha_2(\|(q, \dot{q})\|_\Gamma).$$

Then, the input-output linearizing controller

$$\begin{aligned} u(q, \dot{q}) =& U^{-1}(q) \Big(\mathrm{d}h(q) D^{-1}(q) \big(C(q, \dot{q})\dot{q} + \nabla P(q) \big) \\ & - \mathcal{H}(q, \dot{q}) - \kappa_p e - \kappa_d \dot{e} \Big) \end{aligned} \tag{5}$$

with decoupling matrix $U(q)$ as in (3), $h(q) = \mathrm{col}\{h_1(q), \ldots, h_{n-1}(q)\}$, $\mathcal{H}(q, \dot{q}) = \mathrm{col}\{\dot{q}^\top \nabla^2 h_1(q) \dot{q}, \ldots, \dot{q}^\top \nabla^2 h_{n-1}(q) \dot{q}\}$, makes Γ in (2) asymptotically stable for all $\kappa_p, \kappa_d \in \mathbb{R}^{(n-1)\times(n-1)}$ rendering the matrix below Hurwitz

$$A(\kappa_p, \kappa_d) = \begin{pmatrix} 0 & I_{n-1} \\ -\kappa_p & -\kappa_d \end{pmatrix}. \tag{6}$$

B. Problem statement and motivations

From the result recalled in the previous section, imposing VHC generally corresponds to making Γ the zero-dynamics manifold of the dynamics, with a stable zero dynamics. Accordingly, a VHS can be imposed under a continuous-time control if it is regular in the sense of Definition 2.2. However, what does it occur if the control is of sampled-data type, that is a piecewise constant signal based on sampled measures of the configuration variables $q, \dot{q}$? More in detail, denoting by $\delta > 0$ the sampling period, we address the following problem.

VHCs under digital control. Consider the mechanical dynamics (1) and a regular VHC $h(q) = 0$ in the sense of Definition 2.2 with constraint manifold Γ in (2). Let $\delta > 0$ be the sampling period and $q_k := q(k\delta)$, $\dot{q}_k = \dot{q}(k\delta)$ for all $k \geq 0$. Design, if any, a piecewise constant control $u_k = u^\delta(q_k, \dot{q}_k)$, enforcing the VHC $h(q) = 0$, while asymptotically stabilizing the constraint manifold Γ. □

More in detail, setting $u(t) = u_k$ for $t \in [k\delta, (k+1)\delta)$, $x = \mathrm{col}\{q, \dot{q}\}$ and $x_k = x(k\delta)$ for all $k \geq 0$, (1) is described by the so-called sampled-data equivalent model [18]

$$x_{k+1} = F^\delta(x_k, u_k) \tag{7}$$

[1] In the sense of [9, Definition 3]

[2] Necessary and sufficient conditions for the relation $h(q) = 0$ to be a regular VHC are given in [8].

with

$$F^{\delta}(x,u) = e^{\delta \mathrm{L}_{f(x)+G(x)u}} x = x + \sum_{i>0} \frac{\delta^i}{i!} \mathrm{L}^i_{f(x)+G(x)u} x$$

$$f(x) = \begin{pmatrix} \dot{q} \\ -D^{-1}(q)\big(C(q,\dot{q})\dot{q} + \nabla P(q)\big) \end{pmatrix}$$

$$G(x) = \begin{pmatrix} 0 \\ D^{-1}(q)B(q) \end{pmatrix}.$$

It is convenient to rewrite (7) in block-component-wise as

$$\begin{aligned} q_{k+1} =& F^{\delta}_q(q_k, \dot{q}_k, u_k) \\ \dot{q}_{k+1} =& F^{\delta}_{\dot{q}}(q_k, \dot{q}_k, u_k) \end{aligned}$$

with, by definition

$$F^{\delta}_q(q, \dot{q}, u) = q + \int_0^{\delta} F^{s}_{\dot{q}}(q, \dot{q}, u)\mathrm{d}s.$$

Accordingly, the definition of relative degree for discrete-time systems is recalled here below from [19].

Definition 2.3 (Discrete-time vector relative degree): A discrete-time system

$$\begin{aligned} x_{k+1} =& F(x_k, u_k) \\ y^1_k =& h_1(x_k), \dots, y^m_k = h_m(x_k) \end{aligned}$$

with $x \in \mathbb{R}^n$, $u \in \mathbb{R}^m$, $y^i \in \mathbb{R}$ for $i = 1, \dots, m$, is said to possess vector relative degree $r = (r_1, \dots, r_m) \in \mathbb{R}^m$ at $x^{\circ} \in \mathbb{R}^n$ if the following holds:

- for all $i = 1, \dots, m$, $\ell = 1, \dots, r_i - 1$
  $$\frac{\partial}{\partial u} h_i(F^{\ell-1}_0(F(x,u)) = 0, \ \ \frac{\partial}{\partial u} h_i(F^{r_i-1}_0(F(x,u)) \neq 0;$$
- the decoupling matrix
  $$U_d(x,u) = \begin{pmatrix} \frac{\partial}{\partial u} h_1(F^{r_1-1}_0(F(x,u))) \\ \vdots \\ \frac{\partial}{\partial u} h_m(F^{r_m-1}_0(F(x,u))) \end{pmatrix}$$
  is non-singular at $x = x^{\circ}$, $u = 0$.

It is well-known that under sampling the relative degree is not preserved [12]. As a matter of fact, let us consider the output map $e = h(q)$ with $h(q) = 0$ a regular VHC in the sense of Definition 2.2. By Definition 2.3, one computes for the sampled-data model (7) the discrete-time decoupling matrix

$$\frac{1}{\delta^2} U_d(x,u) = \mathrm{d}h(q)D^{-1}(q)B(q) + O(\delta)$$

that is non-singular in Γ by non-singularity of the continuous-time one in (3). Thus, $h(q)$ is no longer a regular VHC for (7) as it possesses a discrete vector relative degree $(1, \dots, 1) \in \mathbb{R}^{n-1}$ with respect to the output $e = h(q)$ with a zero-dynamics sub-manifold $\mathcal{Z}^{\delta}$ of dimension n and corresponding to a generally unstable reduced dynamics (due to the rise of the so-called sampled-data zero dynamics [12]). In addition, $h(q) = 0$ is not a VHC for (7) in the weaker sense of Definition 2.1 as control invariance of Γ is lost under sampling and thus its stabilizability via piecewise constant control. As a particular case, it is naturally deduced that standard emulation of the continuous-time feedback (5) via sampling and hold devices (i.e., setting $u_k = u(q_k, \dot{q}_k)$) fails into imposing $h(q) = 0$.

Remark 2.1: The characterization of the properties and structure of the reduced dynamics (i.e., the scalar dynamics governing (1) over Γ) under sampling is not addressed here. As a matter of fact, such a characterization relies upon sampled-data Lagrangian structures as particular classes of discrete-time Hamiltonian structures as proposed in [20].

III. Main result

The proposed solution, relying upon the results in [13], ensures that regularity of the VHC is preserved under multi-rate sampling of order $(2, \dots, 2) \in \mathbb{R}^{n-1}$ (i.e., of the same order 2 over each input channel). Namely, we assume the control piecewise constant over the sub-interval of the sampling period of length $\bar{\delta} = \frac{\delta}{2}$; namely, $u^i_k = u(k\delta + (i-1)\bar{\delta}) \in \mathbb{R}^{n-1}$ for $i = 1, 2$ with the multi-rate equivalent model of (1)

$$x_{k+1} = F^{\bar{\delta}}_2(x_k, \underline{u}_k) \tag{8}$$

with $\underline{u} = \mathrm{col}\{u^1, u^2\} \in \mathbb{R}^{2(n-1)}$ and

$$\begin{aligned} F^{\bar{\delta}}_2(x, \underline{u}) =& F^{\delta}(\cdot, u^2) \circ F^{\bar{\delta}}(x, u^1) \\ =& e^{\delta \mathrm{L}_{f+Gu^1}} \circ e^{\delta \mathrm{L}_{f+Gu^2}} x. \end{aligned}$$

At this point, the following main result can be proved.

Theorem 3.1: Let $h(q) = 0$ be a regular VHC of order $n-1$ for (1) under the hypotheses of Proposition 2.1 and constraint manifold Γ in (2). Then, $h(q) = 0$ is a stabilizable regular VHC of order $n-1$ under multi-rate digital control of order 2; equivalently, it is a regular VHC of order $n-1$ for the sampled-data equivalent model (8).

Proof: For showing the result, one must show that the extended output (4) possesses vector relative degree $(1, 1, \dots, 1, 1) \in \mathbb{R}^{2(n-1)}$ everywhere on the constraint manifold Γ in (2). By Definition 2.3, one gets that the discrete-time decoupling matrix associated to (8) with the extended output (4) is given by

$$U^{\delta}_d(q,u) = \Delta \otimes \mathrm{d}h(q)D^{-1}(q)B(q) + O(\delta)$$

with

$$\Delta = \begin{pmatrix} \frac{3}{2} & \frac{1}{2} \\ 1 & 1 \end{pmatrix}.$$

The matrix above is invertible everywhere on Γ because $U(q)$ in (3) is such by assumption and Δ is non singular. ∎

In the next result, the sampled-data control law enforcing the VHC $h(q) = 0$ is proved to exist in a constructive way starting from the continuous-time solution.

Proposition 3.1: Let $h(q) = 0$ be a regular VHC of order $n-1$ for (1) under the hypotheses of Proposition 2.1 and constraint manifold Γ in (2). Let (8) be the sampled-data equivalent model of order 2 with extended output $H(x) = H(q, \dot{q})$ in (4). Then, the following holds true.

(i) The implicit equality

$$H(F_2^{\bar{\delta}}(x,\underline{u})) = A^{\delta}(\kappa_p,\kappa_d)H(x) \tag{9}$$

with, for $A(\kappa_p,\kappa_d)$ as in (6),

$$A^{\delta}(\kappa_p,\kappa_d) = e^{\delta A(\kappa_p,\kappa_d)} \tag{10}$$

admits a unique solution $\underline{u} = \underline{u}^{\bar{\delta}}(x)$ in the form of a series expansion in powers of δ around the continuous-time one in (5); namely, one gets

$$\underline{u}^{\bar{\delta}}(x) = \mathbf{1}_2 \otimes u(x) + \sum_{\ell>0} \frac{\bar{\delta}^{\ell}}{(\ell+1)!}\underline{u}_{\ell}(x). \tag{11}$$

(ii) The feedback $\underline{u} = \underline{u}^{\bar{\delta}}(x)$ solution to (9) enforces the VHC $h(q) = 0$ that is, it makes Γ in (2) asymptotically stable.

Proof: The feedback solution to (9) is the one ensuring input-output linearization with respect to the extended mapping $H(x)$. Accordingly, the proof of (i) follows from Theorem 3.1 and by the Implicit Function Theorem along the lines of [13]. As far as (ii) is concerned, by construction of the matrix (6), $A^{\delta}(\kappa_p,\kappa_d)$ in (10) is asymptotically stable (in the discrete-time sense[3]) and $H(x_k) \to 0$ as $k \to \infty$ ensuring that the trajectories asymptotically converge to Γ so getting the result. ■

Remark 3.1: The constraining feedback is given by the solution of the implicit equality (9) ensuring I/O linearization under sampling with, moreover, output matching of the continuous-time output trajectories under the feedback (5). We underline that such a choice is made to allow, as developed in the next section, comparison with respect to the nominal continuous-time behavior. More general assignments of the output linear dynamics are possible for the closed-loop sampled-data equivalent model. In general, one can compute the feedback so to assign a desired LTI discrete-time equivalent dynamics of the form

$$H(F_2^{\bar{\delta}}(x,\underline{u})) = (A^{\delta} + B^{\delta}F^{\delta})H(x)$$

with

$$A_0 = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, \quad B_0 = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$$

$$A^{\delta} = e^{A_0\delta} \otimes I_{n-1}, \quad B_0^{\delta} = \int_0^{\delta} e^{A_0 s}\mathrm{d}s B_0$$

$$B^{\delta} = \begin{pmatrix} A^{\frac{\delta}{2}}B_0^{\frac{\delta}{2}} & B_0^{\frac{\delta}{2}} \end{pmatrix} \otimes I_{n-1}$$

and F^{δ} any feedback gain ensuring asymptotic stability of the closed loop.

Remark 3.2: The (invariant and attractive) zero-dynamics (i.e., the reduced dynamics over Γ) of the closed-loop sampled-data system under the feedback solution to (9) preserves the same stability and boundedness properties as the continuous-time counterpart, at least in first approximation [12]. Accordingly, nothing can be concluded on the possible preservation of the sampled-data Lagrangian structure.

[3] i.e., all the eigenvalues are in the open unit circle.

Although closed forms to (9) are hard to compute, all terms of the series expansion (5) can be deduced via an iterative and constructive procedure: first, one substitutes (11) into (9) and then equates all terms with the same power of δ; each term $\underline{u}_{\ell}(x)$ is the solution to a linear equality depending on x and the previous terms $\underline{u}_{\ell-1}(x),\ldots,\underline{u}_0(x)$. For the first term, one gets

$$\underline{u}_1(x) = \frac{1}{3}\begin{pmatrix} 1 \\ 5 \end{pmatrix} \otimes \dot{u}(x), \quad \dot{u}(x) = \mathrm{L}_{f+Gu(x)}u(x).$$

Accordingly, only controllers computed as truncations of the series expansion (11) at all desired order $\beta \geq 0$ are implementable in practice; namely, the β^{th}-order approximate feedback law is defined by

$$\underline{u}_{[\beta]}^{\bar{\delta}}(x) = \mathbf{1}_2 \otimes u(x) + \sum_{\ell=1}^{\beta} \frac{\bar{\delta}^{\ell}}{(\ell+1)!}\underline{u}_{\ell}(x) \tag{12}$$

with $\bar{\delta} = \frac{\delta}{2}$ so recovering for $\beta = 0$ the usual emulation-based control [15]. The stabilizing properties under such feedback laws are guaranteed only in a practical sense with $h(q)$ converging to a ball containing the origin with radius in $O(\delta^{\beta+1})$ [18, Theorem 5.1].

IV. Digital orbital stabilization of the Pendubot

Consider the dynamics of a pendulum robot (Pendubot) in Figure 1 in the form (1) with $n = 2$ and [8], [17], [21]

$$D(q) := \begin{pmatrix} d_{uu}(q_u) & d_{ua}(q_u) \\ d_{ua}(q_u) & d_{aa}(q_u) \end{pmatrix}, \quad C(q,\dot{q})\dot{q} = \begin{pmatrix} c_u(q,\dot{q}) \\ c_a(q,\dot{q}) \end{pmatrix}\dot{q}$$

$$\nabla P(q) = \begin{pmatrix} \nabla P_a(q) \\ \nabla P_u(q) \end{pmatrix}$$

where

$$\begin{aligned}
d_{uu}(q_u) =& m_2\ell_{c2}^2 + I_2 \\
d_{ua}(q_u) =& m_2\ell_{c2}^2 + I_2 + m_2\ell_1\ell_{c2}\cos q_u \\
d_{au}(q_u) =& m_2\ell_{c2}^2 + I_2 + m_2\ell_1\ell_{c2}\cos q_u \\
d_{aa}(q_u) =& m_1\ell_{c1}^2 + m_2\ell_1^2 + I_1 + m_2\ell_{c2}^2 + I_2 \\
&+ 2m_2\ell_1\ell_{c2}\cos q_u \\
c_u(q,\dot{q})\dot{q} =& (m_2\ell_1\ell_{c2}\sin q_u)\dot{q}_a \\
c_a(q,\dot{q})\dot{q} =& (m_2\ell_1\ell_{c2}\sin q_u)\dot{q}_a - 2(m_2\ell_1\ell_{c2}\sin q_u)\dot{q}_u \\
\nabla P_a(q) =& m_2\ell_{c2}g_r\sin(q_a+q_u) \\
\nabla P_u(q) =& (m_1\ell_{c1} + m_2\ell_{c2})g_r\sin q_a \\
&+ m_2\ell_{c2}g_r\sin(q_a+q_u)
\end{aligned}$$

$m_i, I_i, \ell_i, \ell_{ci}, g_r$ being the mass, inertia, length and length to center of mass for link $i = 1,2$ and the gravity constant respectively.

Following [8], [17], the design goal stands in the generation of non-trivial stable oscillations of the underactuated link under a sampled-data feedback. This corresponds to stabilizing the regular VHC

$$h(q) = q_a + \kappa q_u \tag{13}$$

with $\kappa \in \mathbb{R}$ [17] and, equivalently, the set

$$\Gamma = \{(q,\dot{q}) : q_a - \kappa q_u = 0, \dot{q}_a - \kappa\dot{q}_u = 0\}. \tag{14}$$

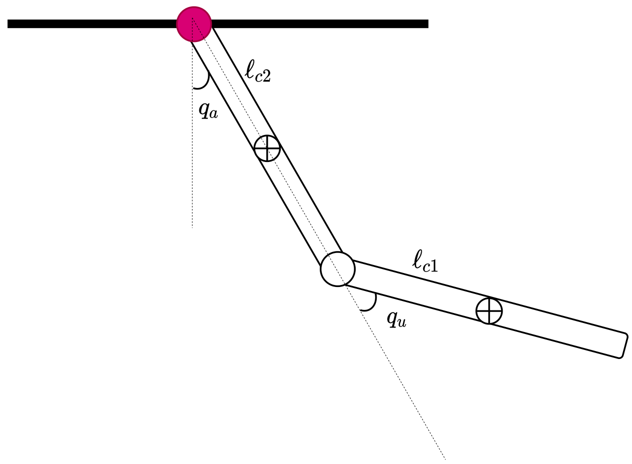

Fig. 1. The pendubot robot

Following Proposition 2.1, the stabilizing continuous-time control is provided by (5) which is specified as

$$u = c_a(q,\dot q)\dot q + \nabla P_a(q) + \gamma_1(q_u)(c_u(q,\dot q)\dot q + \nabla P_u(q) - \gamma_2(q_u)\Big(\kappa_p h(q) + \kappa_d \dot h(q) - c_u(q,\dot q)\dot q + \nabla P_u(q)\Big)$$

with

$$\gamma_1(q_u) = \frac{d_{au}(q_u)}{d_{uu}(q_u)}$$
$$\gamma_2(q_u) = \frac{d_{aa}(q_u)}{(m_2\ell_{c2}^2 + I_2)(d_{aa}(q_u) - \kappa d_{au}(q_u))}.$$

As far as the sampled-data solution is concerned, following Proposition 3.1 the problem is solved under multi-rate control of order $r = 2$ with $\underline{u}_k = (u_k^1 \ u_k^2)^\top$, $u_k^i = u(k\delta + (i-1)\bar\delta$, $\bar\delta = \frac{\delta}{2}$ and $i = 1,2$ as in Section III. Applying Proposition 3.1, the feedback takes the form of the series expansion (11) with the first term specified by

$$\begin{aligned}\underline{u}_1(q,\dot q) &= \begin{pmatrix}\frac{1}{3} & \frac{5}{3}\end{pmatrix}^\top \dot u(q,\dot q)\\ \dot u(q,\dot q) &= \dot c_a(q,\dot q)\dot q + c_a(q,\dot q)\ddot q + \nabla^2 P_a(q)\dot q\\ &+ (\dot\gamma_1(q_u) + \dot\gamma_2(q_u))c_u(q,\dot q)\dot q\\ &+ (\gamma_1(q_u) + \gamma_2(q_u))\dot c_u(q,\dot q)\dot q\\ &+ (\gamma_1(q_u) + \gamma_2(q_u))c_u(q,\dot q)\ddot q\\ &+ (\gamma_1(q_u) - \gamma_2(q_u))\nabla^2 P_u(q)\dot q\\ &+ (\dot\gamma_1(q_u) - \dot\gamma_2(q_u))\nabla P_u(q)\\ &- (\gamma_1(q_u) - \gamma_2(q_u))\Big(\kappa_p \dot h(q) + \kappa_d \ddot h(q)\Big)\\ &- (\dot\gamma_1(q_u) - \dot\gamma_2(q_u))\Big(\kappa_p h(q) + \kappa_d \dot h(q)\Big)\end{aligned}$$

We are now in position to compare the stabilization properties of the obtained sampled-data feedback, approximated at the first order, against that of emulation, i.e. when considering the 0^{th}order approximation in (12).

Simulations

To validate the approach proposed, simulations of the controlled pendubot are performed with the parameters reported in the table below.

m_1 $[kg]$	0.2	m_2 $[kg]$	0.052
I_1 $[kgm^2]$	3.38×10^{-1}	I_2 $[kgm^2]$	1.17×10^{-1}
ℓ_1 $[m]$	0.2	ℓ_2 $[m]$	0.28
ℓ_{c1} $[m]$	0.13	ℓ_{c2} $[m]$	0.15

In all cases, we fix the parameter $\kappa = -1$ in (13) so comparing similar situations to those reported in [17]. Additionally, the stabilizing continuous-time gains in (5) are fixed at $\kappa_p = 1$, $\kappa_d = \sqrt{3}$. In Figure 2 we show the performance of the (1^{st} order approximate) sampled-data multi-rate stabilizing controller for two different sets of initial conditions when compared to the ideal continuous-time solution. The dashed lines corresponds to the multi-rate solution while the continuous line is the ideal continuous-time solution. Both are compared staring from the configuration $q_0 = (\frac{\pi}{6} \ \frac{\pi}{1.5})^\top$ with zero velocities, as well as $q_0 = (\frac{\pi}{3} \ \frac{\pi}{1.5})^\top$ with zero velocities. The sampled-data control follows closely the ideal continuous-time behaviour even for larger sampling period $\delta = 0.3$.

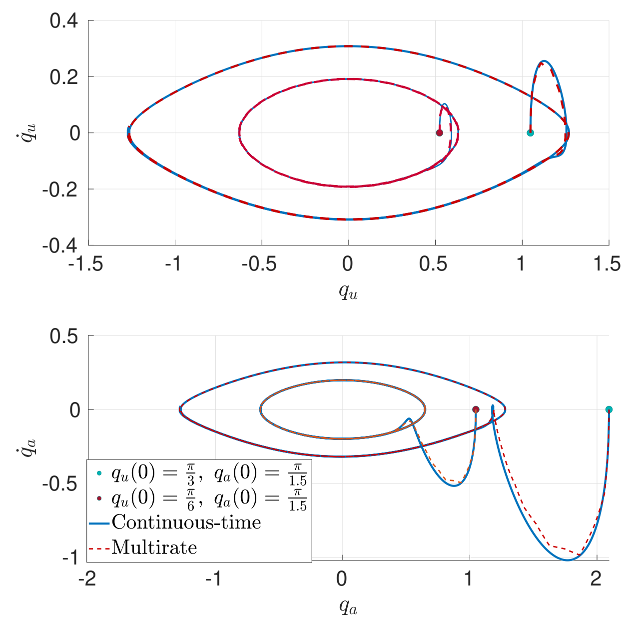

Fig. 2. Sampled-data stabilization for different initial conditions and $\delta = 0.5$ under multi-rate approximate control of order $\beta = 1$.

Figures 3 and 4 highlight the significant benefits obtained from the additive terms introduced by the proposed approximate sampled-data control when compared to standard emulation for increasing values of δ. Also, it is worth to underline that the control effort required by the multi-rate approximate control is comparable with respect to the continuous-time one and much better than emulation. Further simulations have been perormed also considering different control objectives as, for instance, swing-up stabilization[4].

[4]More animated cases at `https://youtu.be/YqGJnm1oNo0`

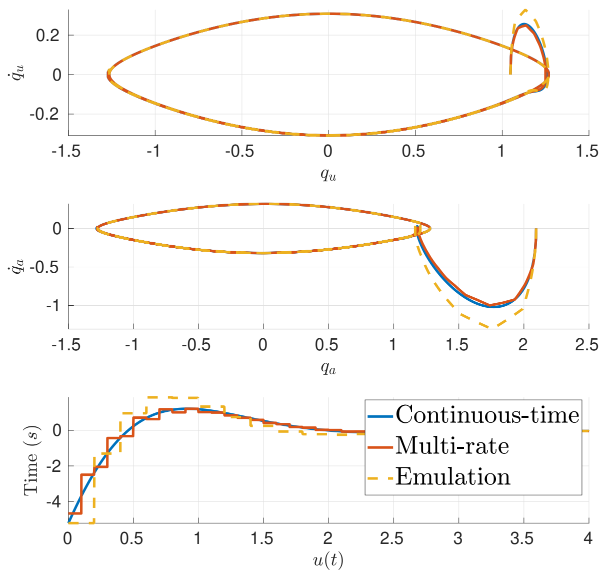

Fig. 3. Simulations for the pendubot when $\delta = 0.2$ seconds

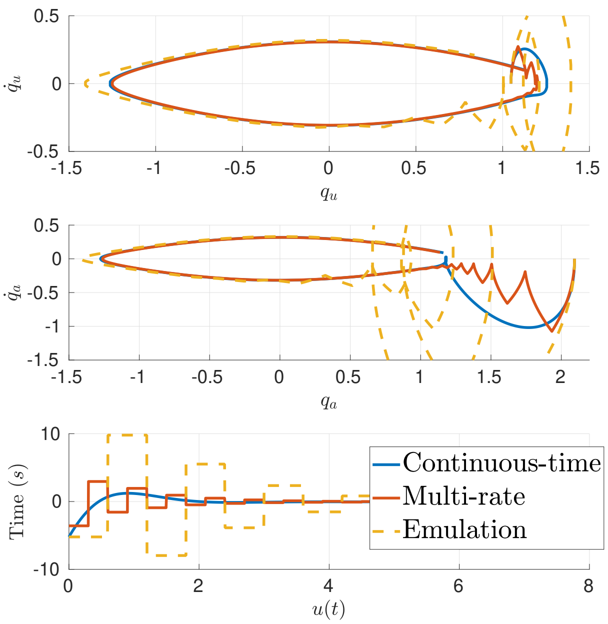

Fig. 4. Simulations for the pendubot when $\delta = 0.6$ seconds

V. Conclusions and Perspectives

In this paper, it is shown that multi-rate sampling allows to impose VHCs to mechanical systems in Euler-Lagrangian form. In particular, the order of the multi-rate must be equal to two over each input channel in order to guarantee invariance of the corresponding surface. Future perspectives concern the problem of preserving the Euler-Lagrangian structure of the residual dynamics (the associated zero-dynamics) possibly exploiting redundant multi-rate control. Also, the application of those methods to deal with control of multi-agent autonomous systems is under investigation [22].

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