Minimal positive realizations: A survey

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Abstract

This survey aims to present a comprehensive and systematic synthesis of concepts and results on the minimal state space realization problem for positive, linear, time-invariant systems. Positive systems are systems for which the state and the output are always non-negative for any non-negative initial state and input. They are used to model phenomena in which the variables must take non-negative values due to the nature of the underlying physical system. Restricting the state-space realization to positive systems makes the problem extremely different and much more difficult than that for ordinary systems. Indeed, a minimal positive realization may have a dimension even much larger than the order of the transfer function it realizes. Although the problem of finding a finite-dimensional positive state-space realization of a given transfer function has been solved, the characterization of minimality for positive systems is still an open problem. This survey introduces the reader to different aspects of the problem and presents the mathematical approaches used to tackle it as well as some relevant related problems. Moreover, some partial results are presented. Finally, a comprehensive bibliography on positive systems, organized by topics, is provided.

Key words: Positive systems, realization theory, minimal realizations, non-negative matrices.

1 Introduction

Positive systems are dynamical systems in which the state and output variables assume positive (or at least non-negative) values for all times, for any non-negative initial state and non-negative input. This feature makes positive systems an appropriate modeling tool for dynamic phenomena whose describing variables correspond to the quantities or concentrations of any possible type of resource or substance. Consider, for example, the Leontief model used by economists for predicting productions and prices (Leontieff, 1951), the Leslie model used to study age-structured population dynamics (Leslie, 1945), and the compartmental models commonly used in physiology (Jacquez, 1985) and epidemiology (Anderson & May, 1991), just to cite the most known. Moreover, positive systems are commonly used to model stochastic phenomena as well. To get a sense of how relevant they are, it suffices to mention the Markov Chains (Bhat, 1972), the Hidden Markov Models (HMMs) (Rabiner, 1989), and the phase-type distributions (O'Cinneide, 1990).

Positive systems are also remarkably useful in several applications in very different fields of science, ranging from biology and medicine to civil and electronic engineering. HMMs, for example, have been used in some problems in computational biology, such as identifying the genes of an organism from its DNA and classifying proteins into a small number of families (see the references in the survey by Vidyasagar (2011). Moreover, a key issue in modeling genetic regulatory networks is the positivity constraints on the state variables that represent the amount or concentration of some gene products (proteins and RNAs) (de Jong, 2002). Similarly, the design of digital filters using technologies such as optical components (Moslehi, Goodman, Tur, & Shaw, 1984) and Charge Coupling Devices (CCDs) (Gersho & Gopinath, 1979) needs to take into account the positivity constraints on the state variables of the filter that represent light intensity levels and electrical charge quantities, respectively. Finally, positive systems have shown to be a central topic in the design of state-feedback controllers to avoid over and under-shooting since a common way to obtain this behavior is to design a closed-loop system with a non-negative impulse response (Darbha, 2003; Bement & Jayasuriya, 2004). For example, in intelligent vehicle/highway systems, traffic congestion is reduced by managing traffic in platoons, that is, groups of automated vehicles following each other at a very small distance and high-velocity (Varaiya, 1993). It has been shown that safety, that is collision avoidance between cars in the platoon, is guaranteed if and only if each

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controlled vehicle has a non-negative impulse response (Lunze, 2019; Schwab & Lunze, 2022).

The systematic study of positive systems has been initiated by Luenberger (1979) who devoted an entire chapter of his book on dynamic systems to such a class of systems. From that time on, several contributions dealing with the study of classic topics of system theory in the positive systems setting have appeared. For example, reachability and controllability (Bru, Coll, Hernandez, & Sanchez, 1997; Coxson & Shapiro, 1987; Fanti, Maione, & Turchiano, 1989, 1990; Murthy, 1986; Rumchev & James, 1989; Valcher, 1996, 2009; Zaslavsky, 1993; Maeda & Kodama, 1980; Ohta, Maeda, & Kodama, 1984), observability (Back & Astolfi, 2008; Dautrebande & Bastin, 1999; van den Hof, 1998), stability and stabilizability (De Leenheer & Aeyels, 2001, 2002; Fischer, 1997; Hinrichsen & Son, 1998a, 1998b; Hinrichsen & Plischke, 2007; Muratori & Rinaldi, 1991; Roszak & Davison, 2009; Son, 1995; Son & Hinrichsen, 1996), just to cite the main ones. The interested reader is referred to the tutorial paper (Rantzer & Valcher, 2018) for a general overview. The growing interest in positive systems research and its pervasiveness in very different fields of science and technology is also proved by the multidisciplinary symposium "Positive systems: Theory and applications (POSTA)" that was established in 2003 and has now reached its sixth edition (Benvenuti, De Santis, & Farina, 2003; Commault & Marchand, 2006; Bru & Romero-Vivo, 2009; Cacace, Farina, Setola, & Germani, 2016; Lam, Chen, Liu, Zhao, & Zhang, 2018).

One of the most challenging problems investigated in the context of positive systems is certainly the Positive Realization Problem (PRP) (Benvenuti & Farina, 2004), which consists of determining the conditions for a given transfer function to be that of a positive system and a positive realization when these conditions are fulfilled. The PRP is of interest, for example, to the identification of compartmental models and HMMs, and the design of digital filters using optical fibers or CCDs. In more detail, an important problem in the analysis of compartmental systems (Haddad, Chellaboina, & Hui, 2010) is the determination of the internal structure of a compartmental model - specifically, the number of compartments - from the impulse response obtained through an input-output experiment (Benvenuti, De Santis, & Farina, 2002; De Santis & Farina, 2002; Maeda, Kodama, & Kajiya, 1977). An immediate application of this problem in clinical medicine is the determination of the number of organs involved in a given tracer kinetics experiment. Moreover, Picci (1978) showed that the approach to the stochastic realization problem for HMMs, that is the problem of determining an HMM that reproduces the statistics of a given stationary stochastic process, is analogous to that of a finite-dimensional linear system and it is just a complication of the PRP. A solution can then be found by exploiting the technique used to solve the PRP (Anderson, 1999b; Vidyasagar, 2011). Furthermore, the implementation of a given transfer function using optical fibers or CCDs calls for a positive realization of it, since the state variables of the filter may assume only non-negative values (Benvenuti & Farina, 1996, 2001). In general, the PRP is highly relevant in data-driven modeling for all the dynamic phenomena in which one wants the identified system's variables to assume only non-negative values. Indeed, it is not known how to customize the existing identification methods to deal with positive systems. Finally, the positive realization theory has recently been used to investigate the so-called Hessenberg forms of non-negative and Metzler matrices (Grussler & Rantzer, 2022).

Although the problem of determining the existence of a positive realization and computing it has been solved by Anderson, Deistler, Farina, and Benvenuti (1996) and Farina (1996), the characterization of minimality for positive systems is still an open problem. Note that minimality is often a key issue in applications. For example, when implementing a filter, one wishes to reduce space occupation and power consumption, and hence a positive realization with minimal dimension is desirable. Even in the case of HMMs identification, an interesting open problem is that of determining the minimum number of states needed to realize the collection of probabilities (Vidyasagar, 2011). In general, when a positive system has to be identified from its transfer function, its dimension (when a positive realization does exist) may be much larger than the order of the transfer function itself making its use impractical. In this case, identifying the minimum dimension of a positive realization of the transfer function becomes a fundamental issue.

The characterization of minimality for positive systems is inherently different from that for general systems. Indeed, positive systems are defined on cones rather than linear spaces: for example, the reachable set of a positive system, that is the set of all the states that can be reached from the origin using positive inputs, is a cone (Farina & Rinaldi, 2000). Consequently, the well-established methods for general linear systems cannot be used for positive systems. In more detail, when considering linear systems, a minimal state-space realization is both reachable and observable and its dimension corresponds to the order of the corresponding transfer function, and the rank of the Hankel matrix, as well. A minimal positive realization, in contrast, may have a dimension larger than the order of the transfer function of the system, i.e., it may be not jointly reachable and observable, according to the usual definition. In (Benvenuti & Farina, 1999; Nagy & Matolcsi, 2003b) examples are provided in which the minimal dimension of a positive realization of a transfer function is much "larger" than the order of the transfer function itself. Moreover, the non-negative rank of the Hankel matrix [1] is only a lower

[1] The non-negative rank of a non-negative matrix is defined

bound for the dimension of a minimal positive realization, and, as shown in (Benvenuti & Farina, 1998, 2006), there are systems for which the dimension of any minimal positive realization is larger than the non-negative rank of the Hankel matrix.

Conditions for a positive realization to be minimal have been given only for special classes of positive systems such as the tree compartmental systems considered in (Maeda et al., 1977) and the positive reachable systems in (Farina, 1996b). Results for even more specific classes of positive systems are given in (Astolfi & Colaneri, 2004; Halmschlager & Matolcsi, 2005; Nagy & Matolcsi, 2005). Moreover, in (Benvenuti, Farina, Anderson, & Bruyne, 2000; Benvenuti, 2013), the necessary and sufficient conditions for a third-order transfer function with real poles to have a third-order (minimal) positive realization are given.

In general, only lower and upper bounds for the dimension of a minimal positive realization have been given to date. Some of these bounds can be found in (Benvenuti, 2020a, 2020b; Czaja, Jaming, & Matolcsi, 2008; Hadjicostis, 1999; Halmschlager & Matolcsi, 2005; Nagy & Matolcsi, 2003b).

This survey aims to present a comprehensive and systematic synthesis of concepts and results on the minimal positive realizations. Some other reviews on positive systems are available but they present only basic or dated results on minimality. More precisely, some results regarding the stability, the positive realization, and the positive stabilization through state-feedback are summarized in (Rantzer & Valcher, 2018). However, only some very basic results on minimality are there presented. The tutorial paper (Benvenuti & Farina, 2004) is instead devoted to illustrating in detail the results on the PRP but only one section deals with the minimality problem. Finally, the survey paper (Benvenuti & Farina, 2003) aimed to present specifically some results on the minimality problem for positive realizations available at that time. The aim of the current review is then to renew and increase interest in this problem by deepening and integrating the presentations given in these previous review papers taking into account the development of the research and results during the last twenty years. Moreover, some hints on problems related to the characterization of minimality for positive systems, such as the NIEP and the nonnegative rank computation, will be given.

Most of the results on minimality available in literature are related to discrete-time single input-single output (SISO) systems, but a corresponding continuous-time formulation can be easily derived (Anderson et al., 1996; Benvenuti & Farina, 2002; Ohta et al., 1984; van den Hof, 1997b). Therefore, the results for discrete-time SISO systems are presented and discussed throughout this paper, while those for continuos-time systems and multi-input-multi-output (MIMO) systems are provided in dedicated sections.

The paper is organized as follows. In Section 2, the problem is formally stated together with two examples illustrating the main two reasons for a minimal positive realization to possibly have a dimension greater than the order of the corresponding transfer function. Section 3 summarizes some results for low-dimensional cases and special classes of positive systems. The results on the possible location of the eigenvalues of non-negative real matrices are presented in Sections 4 and 5. These results can be exploited to determine lower and upper bounds on the minimum dimension of a positive realization. Such bounds are provided in Section 6. The relation between the non-negative rank of the Hankel matrix and the minimum dimension of a positive realization is deepened in Section 7 and the results for the case of continuos-time systems and MIMO systems are presented in Sections 8 and 9, respectively. A discussion on the possible research directions is addressed in Section 10. The bibliographic section contains references related to the general area of positive systems theory and applications. In particular, it refers to basic results on positive systems theory and their application in different fields (economy, biology, electronics, ...), contributions related to non-negative matrices and the non-negative inverse eigenvalue problem, and results on the existence and minimality of positive realizations.

2 On minimality of positive realizations

A discrete-time, time-invariant, SISO linear system of the form

$$\begin{aligned} x_{k+1} &= Ax_k + bu_k \\ y_k &= cx_k, \end{aligned} \tag{1}$$

is said to be a *positive system* if the state and output sequences x_k and y_k are non-negative at any time for any non-negative input sequence u_k and non-negative initial state x_0. In particular, positive systems exhibit a non-negative impulse response

$$h_k = cA^{k-1}b, \qquad k \geq 1. \tag{2}$$

As shown in (Farina & Rinaldi, 2000; Luenberger, 1979), the non-negativity constraint on the sequences x_k and y_k is equivalent to the non-negativity constraint on the entries of the system's matrices A, b, and c. By definition, The PRP consists then of the following question: given a system with a non-negative impulse response h_k, find necessary and sufficient conditions for the existence of a

as the smallest inner size of a factorization of the matrix as the product of two non-negative matrices (Cohen & Rothblum, 1993).

positive system realizing it, that is of non-negative real matrices A, b, and c satisfying (2) or, equivalently,

$$H(z) = \sum_{k\geq 0} h_k z^{-k} = c(zI - A)^{-1}b,$$

where $H(z)$ is the transfer function of the system. The PRP has been solved by Anderson et al. (1996) and Farina (1996) using a geometric approach. This approach is based on the result presented by Ohta et al. (1984) and Farina and Benvenuti (1995) for continuous and discrete-time systems, respectively. The existence of a positive realization is cast in terms of the existence of a polyhedral cone with certain properties. In more detail, a strictly proper rational transfer function $H(z)$ with a minimal (i.e., jointly reachable and observable) realization $\{A, b, c\}$ of dimension n has a positive realization if and only if there exists an A-invariant polyhedral proper cone $\mathcal{K}$ such that

$$\mathcal{R} \subseteq \mathcal{K} \subseteq \mathcal{O} \tag{3}$$

where [2]

$$\mathcal{R} = cone\left(\begin{bmatrix} b & Ab & A^2b & \ldots \end{bmatrix}\right)$$

and

$$\mathcal{O} = \{x \in \mathbb{R}^n | cA^{k-1}x \geq 0, k = 1, 2, \ldots\}$$

are the reachability and observability cone, respectively. Knowledge of a cone $\mathcal{K}$ immediately yields a positive realization of a dimension equal to the number of the extreme rays of the cone. The necessary and sufficient conditions are awkward and the interested reader is referred to the tutorial paper (Benvenuti & Farina, 2004) for a detailed description of the solution. The following are "simple to state" sufficient conditions for the existence of a positive realization [3] (Benvenuti & Farina, 2004):

Theorem 1 *Let $H(z)$ be a strictly proper rational transfer function with a non-negative impulse response. Then $H(z)$ has a positive realization of some finite dimension if every pole p_i of the transfer function has the property that $p_i/|p_i|$ is a root of unity.* ■

The problem of determining the minimum dimension of a positive realization of a given transfer function is still an open problem. In general, the minimum dimension may be larger than the order of the transfer function, that is a positive realization may not be jointly reachable and observable. As far as is known, this may depend mainly on two different reasons.

1) The first reason relies on the fact that the poles of the transfer function are a subset of the eigenvalues of the matrix A of any of its positive realizations. Hence, the poles must be a subset of the eigenvalues of a non-negative real matrix. As it will be clear hereinafter, the non-negativity constraints on the entries of a matrix impose some limitations on the location of its eigenvalues and, roughly speaking, these limitations are weaker as the dimension of the matrix increases. As a consequence, just the location of the poles may require the dimension of a positive realization to be larger than a minimum threshold. The following example illustrates this point:

Example 1 Consider a transfer function with three simple real poles equal to 1, -0.8, and -0.4, and with a corresponding impulse response that is non-negative. Any positive realization of such a transfer function has necessarily a dimension not smaller than four since there does not exist a three-dimensional non-negative real matrix having those poles as eigenvalues. Indeed, the sum of the eigenvalues of a non-negative real matrix is equal to its trace and hence must be non-negative while the sum of the three given poles is not. On the other hand, there exist non-negative real matrices of size four having the three poles among their four eigenvalues, as, for example, the following one:

$$A = \begin{pmatrix} 0.1 & 0.7 & 0 & 0.2 \\ 0.7 & 0.1 & 0.2 & 0 \\ 0 & 0.2 & 0.1 & 0.7 \\ 0.2 & 0 & 0.7 & 0.1 \end{pmatrix}$$

■

This kind of mechanism takes into consideration only the poles of the transfer function and allows to determine a lower bound on the minimum dimension of a positive realization, if any. To this aim, however, it is necessary to know the admissible locations for the eigenvalues of non-negative real matrices. The complete characterization of the spectra of non-negative real matrices is a long-standing problem initiated at the beginning of the 20th-century with the celebrated theorem by Perron and Frobenius (Frobenius, 1912; Perron, 1907). This problem has proved to be very difficult and it is still unsolved. Some results on this topic, together with their application to the minimality problem for positive realizations, will be presented in Sections 4 and 5.

2) The second reason for which a minimal positive realization may possibly have a dimension larger than the order of the transfer function it realizes is related not only to the non-negativity of the matrix A but also to

[2] The notation $cone(M)$ indicates the closed convex cone consisting of all non-negative linear combinations of the columns of the matrix M.

[3] The case of transfer functions with not all the poles equal to zero is considered. In fact, as proved in (Benvenuti & Farina, 2004), a strictly proper rational transfer function of order n with a non-negative impulse response and with all the poles equal to zero has a positive realization of order n.

that of the vectors b and c. The next example illustrates this point:

Example 2 Consider the system with the transfer function

$$H(z) = \frac{1}{z-1} + 4 \cdot \frac{z+1}{z^2 + 1/4}$$

whose corresponding impulse response

$$h_k = 1 + 8 \cdot \left(\frac{1}{2}\right)^k \cdot \left(\sin\frac{k\pi}{2} - 2\cos\frac{k\pi}{2}\right)$$

has positive samples apart from h_3 and h_4 that are equal to zero. The poles of the transfer function are equal to 1 and $\pm\frac{i}{2}$, and there exist non-negative real matrices of size three having such poles as eigenvalues, as, for example, the following one:

$$A = \frac{1}{6}\begin{pmatrix} 2 & 2+\sqrt{3} & 2-\sqrt{3} \\ 2-\sqrt{3} & 2 & 2+\sqrt{3} \\ 2+\sqrt{3} & 2-\sqrt{3} & 2 \end{pmatrix}.$$

Nevertheless, there does not exist a positive realization of dimension three of the given transfer function. To prove this, assume the opposite. In this case the similarity transformation T between the positive realization and a minimal realization $\{A, b, c\}$ defines a cone $\mathcal{K} = cone(T)$ with three extreme rays. This cone should be A-invariant and should satisfy conditions (3). Since $cA^2b = h_3 = 0$ and $cA^3b = h_4 = 0$, the following hold:

$$\begin{array}{ccc} c(b) > 0 & c(Ab) > 0 & c(A^2b) = 0 \\ cA(b) > 0 & cA(Ab) = 0 & cA(A^2b) = 0 \\ cA^2(b) = 0 & cA^2(Ab) = 0 & cA^2(A^2b) > 0 \\ cA^3(b) = 0 & cA^3(Ab) > 0 & cA^3(A^2b) > 0 \\ cA^k(b) > 0 & cA^k(Ab) > 0 & cA^k(A^2b) > 0 \end{array}$$

for $k = 4, 5, \ldots$. Consequently, the three vectors b, Ab, and A^2b of the reachability cone $\mathcal{R}$ lie on different extreme rays of the observability cone $\mathcal{O}$. The cones $\mathcal{R}$ and $\mathcal{O}$ are depicted in Figure 1. Then, from conditions (3), these three vectors are necessarily extreme rays of $\mathcal{K}$, that is

$$\mathcal{K} = cone\left(b, Ab, A^2b\right).$$

Since $A^3b \notin \mathcal{K}$, then $\mathcal{K}$ is not A-invariant, thus arriving at a contradiction. Therefore, the following fourth-order positive realization

$$A = \begin{pmatrix} 0 & 0 & 0 & 1/4 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 3/4 \\ 0 & 0 & 1 & 0 \end{pmatrix}, \quad b = \begin{pmatrix} 1 \\ 0 \\ 0 \\ 0 \end{pmatrix}, \quad c^T = \begin{pmatrix} 5 \\ 5 \\ 0 \\ 0 \end{pmatrix}$$

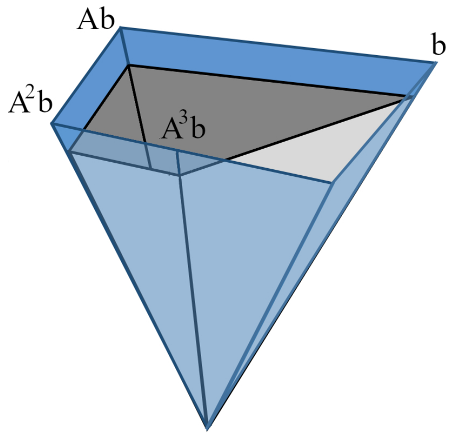

Figure 1. The containment relation between the reachability cone $\mathcal{R} = cone\left(b, Ab, A^2b, A^3b\right)$ (gray) and the observability cone $\mathcal{O}$ (blue) for the system considered in Example 2.

is minimal as a positive system. This property remains true for all the third-order transfer functions with an impulse response that is always positive apart from $h_3 = h_4 = 0$. For example, the same result has been proved for a transfer function with three positive real poles in (Benvenuti & Farina, 1999). This last example has been generalized by Nagy and Matolcsi (2003b) using a digraph approach by considering non-negative impulse responses for which there exists a time instant at which the impulse response is equal to zero and becomes strictly positive from that time on. ■

The first reason, which relies only on the location in the complex plane of the poles of the transfer function, has been studied by exploiting the results on the eigenvalues of non-negative real matrices. The results of these studies consist mainly in lower and upper bounds on the minimum dimension of a positive realization. They are illustrated in Section 6. As expected, the second reason has proved to be harder to analyze and only partial results are available. To date, no general method to determine the minimum dimension of a positive realization exists. As a consequence, the available results mainly consist of evaluating the circumstances under which a given transfer function of order n has a positive realization of dimension n, which is then minimal. Since even this problem proved to be very difficult, only seemingly small results have been obtained for $n \leq 3$ or by considering particular classes of transfer functions and systems. These results are presented in the following section.

3 Special results

In the next subsection, some results on the positive minimality problem are given by restricting to the case of transfer functions up to the third-order. To this end, note that the non-negativity of the impulse response h_k implies some limitations on the location of the poles of

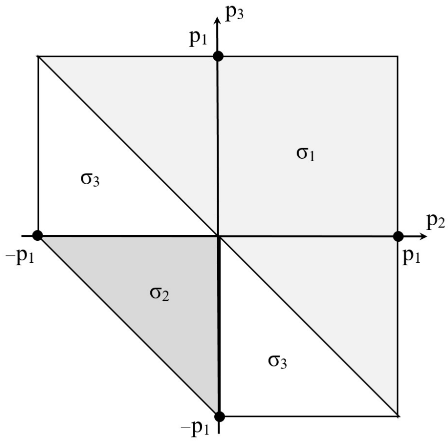

Figure 2. Planar section of the sets σ_1, σ_2, σ_3, and of the set $\sigma = \sigma_1 \cup \sigma_2 \cup \sigma_3$.

the corresponding transfer function $H(z)$. In particular, non-negativity in the long term implies that one of the dominant poles of $H(z)$, i.e., the poles of maximum modulus, is positive real and has the maximal multiplicity among all the dominant poles.

3.1 Low dimensional results

The case of transfer functions of degree one or two has been solved by Ohta et al. (1984). In these cases, the non-negativity of the impulse response is the necessary and sufficient condition for the existence of a (minimal) positive realization of a dimension equal to the order of the transfer function itself.

Consider then the case of a third-order transfer function with distinct real poles, i.e.,

$$H(z) = \frac{r_1}{z - p_1} + \frac{r_2}{z - p_2} + \frac{r_3}{z - p_3} \tag{4}$$

with $r_1, r_2, r_3 \neq 0$. As stated above, when looking for a (minimal) three-dimensional positive realization of $H(z)$, there is no loss of generality in assuming one of the poles of maximum modulus, say p_1, to be positive real and the sum of the poles to be non-negative. These are effectively the necessary and sufficient conditions for the three poles to be the spectrum of a non-negative real matrix of dimension three (see Section 5.2). The set σ defined by these conditions is a polyhedral cone and can be divided into three subsets as follows:

(1) $\sigma_1 = \{\{p_1, p_2, p_3\} \in \sigma \mid p_2 + p_3 \geq 0\}$
(2) $\sigma_2 = \{\{p_1, p_2, p_3\} \in \sigma \mid (p_2 + p_3 < 0) \wedge (p_2 p_3 \geq 0)\}$
(3) $\sigma_3 = \{\{p_1, p_2, p_3\} \in \sigma \mid (p_2 + p_3 < 0) \wedge (p_2 p_3 < 0)\}$

The subsets σ_i are solid and pointed cones and their sections with the plane $p_1 = const$ are shown in Figure 2.

The next two theorems provide necessary and sufficient conditions for a transfer function $H(z)$ as in (4) with poles in the sets σ_1 or σ_2 to have a (minimal) positive realization of dimension three. These conditions are expressed in terms of lower bounds for the first three samples of the impulse response and therefore are very easy to check. Moreover, the a priori knowledge about non-negativity of the whole impulse response is not required so that there is no need to check such condition before applying the theorems. Indeed, the satisfaction of the conditions of the theorems implies the non-negativity of the whole impulse response. These results are presented in (Benvenuti, 2013) and are an extension of a previous result for transfer functions with distinct real positive poles by Benvenuti et al. (2000).

Theorem 2 *Let $H(z)$ be a strictly proper rational third-order transfer function as in (4) with distinct real poles $\{p_1, p_2, p_3\} \in \sigma_1$. Then, $H(z)$ has a third-order positive realization if and only if the following conditions hold:*

(1) $h_1 \geq 0$
(2) $h_2 \geq \theta_{h_2}(h_1)$
(3) $h_3 \geq \theta_{h_3}(h_1, h_2)$

where

$$\theta_{h_2}(h_1) = \begin{cases} K h_1, & \text{if } p_1^2 + p_2^2 + p_3^2 < 2(p_1 p_2 + p_1 p_3 + p_2 p_3) \\ 0, & \text{otherwise} \end{cases}$$

$$\theta_{h_3}(h_1, h_2) = \begin{cases} (p_2 + p_3) h_2 - p_2 p_3 h_1, & \text{if } \dfrac{h_2}{h_1} \geq \min(p_2, p_3) \\ \dfrac{h_2^2}{h_1} & \text{otherwise} \end{cases}$$

with

$$K = \frac{p_1 + p_2 + p_3 - 2\sqrt{(p_2 - p_3)^2 + (p_1 - p_2)(p_1 - p_3)}}{3}$$

■

Theorem 3 *Let $H(z)$ be a strictly proper rational third-order transfer function as in (4) with distinct real poles $\{p_1, p_2, p_3\} \in \sigma_2$. Then, $H(z)$ has a third–order positive realization if and only if the following conditions hold:*

(1) $h_1 \geq 0$
(2) $h_2 \geq 0$
(3) $h_3 \geq 0$ ■

The case of transfer functions with poles lying in the set σ_3 is still open but sufficient conditions for the existence of a (minimal) third-order positive realization are given in (Benvenuti, 2013). The case of third-order transfer functions with a pair of complex poles is open as well. However, some results can be found in (Benvenuti, 2020b; Sun, Yu, Yu, & Wang, 2006; Wang, Wang, Yu, & Liu, 2004). In particular, some sufficient conditions for the existence of a (minimal) third-order positive realization are given in (Sun et al., 2006; Wang et al., 2004) while an upper bound on the dimension of a positive realization is provided in (Benvenuti, 2020b) (see Theorem 11 hereinafter).

3.2 Classes of systems with a minimal positive realization

Some results of the minimality problem for transfer functions of order greater than three have been obtained by considering particular classes of transfer function or by restricting the set of realizing positive systems.

The case of discrete-time positive linear systems having the non-negative orthant reachable from the origin in a finite time interval with non-negative inputs is considered by Farina (1996b). In this case, the minimum dimension of a positive realization may be larger than the order of the transfer function and can be obtained using a simple iterative procedure and by solving a set of linear equalities with non-negativity constraints.

The class of discrete-time positive systems having a diagonal matrix A is considered by Astolfi and Colaneri (2004). The necessary and sufficient conditions for a transfer function to have such kind of positive realization are that it is minimum phase and with non-negative real and interlaced poles and zeros. In this case, the transfer function describes a system composed of the parallel interconnection of one-dimensional subsystems with distinct poles. Moreover, the number of subsystems is equal to the order of the transfer function, and hence the positive realization results to have minimal dimension. Further results, presented in (Astolfi & Colaneri, 2004), are related to transfer functions of the form

$$H(z) = \frac{1}{(z-p_1)(z-p_2)\cdots(z-p_n)},$$

with n non-negative real distinct poles. In this case, the transfer function admits a (minimal) positive realization of dimension n. The same result holds if the transfer function is asymptotically stable, has only one zero ζ such that

$$0 \le \zeta < \min_{i=1,\ldots,n} p_i$$

and has a positive high frequency gain.

Halmschlager and Matolcsi (2005) consider the class of transfer functions with non-negative real distinct poles of the form:

$$H(z) = \frac{r_1}{z-p_1} + \frac{r_2}{z-p_2} + \ldots + \frac{r_n}{z-p_n}$$

where $0 \le p_i < p_1$ and $r_i < 0$, for $i = 2, \ldots, n$. The necessary and sufficient condition for $H(z)$ to have a (minimal) positive realization of dimension n is that

$$r_1 + r_2 + \ldots + r_n \ge 0$$

This result has been extended in (Nagy & Matolcsi, 2005) to the case of transfer functions with non-negative real distinct multiple poles. In particular, the class of transfer functions of the form

$$H(z) = \frac{r_1}{z-p_1} + \sum_{i=2}^{h} \sum_{k=1}^{l_i} \frac{r_{i,k}}{(z-p_i)^k}$$

with $r_1 > 0$, and $0 \le p_i < p_1$ for $i = 2, \ldots, h$, is considered. If r_1 is sufficiently large, then $H(z)$ has a (minimal) positive realization of dimension equal to the order $n = 1 + l_2 + l_3 + \ldots + l_n$ of the transfer function.

4 Dominant poles do matter

In this section, some results related to the eigenvalue locations for non-negative real matrices are presented. Since the matrix A of a positive system is non-negative, these results can be used to characterize the eigenvalue locations for positive systems and, as discussed in Section 2, may give reasons for a minimal positive realization to be not jointly reachable and observable. As noted above, the non-negativity of the impulse response h_k implies one of the dominant poles of the corresponding transfer function $H(z)$ to be positive real. Moreover, as shown in (Anderson, 1997), if the transfer function has a positive realization, its poles of maximum modulus must be a subset of the allowed eigenvalues of maximum modulus of a non-negative real matrix. Finally, the positive real dominant pole must coincide with the spectral radius $\rho(A)$ of the matrix A of any minimal positive realization, (Anderson et al., 1996). Then, a lower bound on the dimension of a minimal positive realization of a transfer function, if any, is given by the minimum size of a non-negative real matrix A having the poles of the transfer function in its spectrum and with a spectral radius $\rho(A)$ equal to the positive real dominant pole of the transfer function itself.

The first result on the eigenvalues of non-negative real matrices is due to Perron and Frobenius (Frobenius, 1912; Perron, 1907). It concerns non-negative irreducible matrices, that is matrices that cannot be written in an upper-triangular block form by simultaneous row/column permutations. This result mainly defines some properties of the dominant eigenvalues and the corresponding eigenvectors of the matrix. The following formulation disregards the properties of the eigenvectors:

Theorem 4 *The dominant eigenvalues of an irreducible non-negative real matrix A of size n are all the roots of*

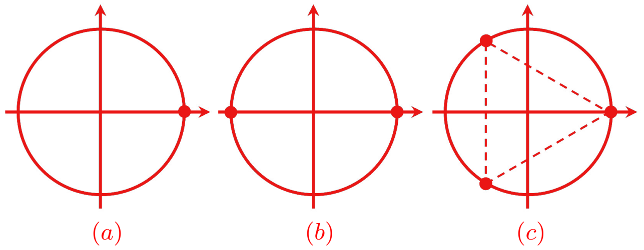

Figure 3. Admissible configurations of the dominant eigenvalues (black dots) of an irreducible non-negative real matrix of size three. All the eigenvalues are simple roots of the characteristic polynomial.

$\lambda^k - \rho(A)^k = 0$ *for some positive value* $k \leq n$*. Each of the dominant eigenvalues is a simple root of the characteristic polynomial and, in particular, one of them is positive real. Moreover, the spectrum of the matrix is invariant under the rotation of the complex plane by the angle* $2\pi/k$*.* ■

Figure 3 shows the admissible configurations of the dominant eigenvalues of an irreducible non-negative real matrix of size three. These configurations correspond, for example, to the following matrices with unitary spectral radius:

$$(a): \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1/4 & 3/4 & 0 \end{pmatrix}, (b): \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1/2 & 1/2 & 0 \end{pmatrix}, (c): \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{pmatrix}.$$

The Perron–Frobenius theorem does not apply directly to general non-negative real matrices. Nevertheless, any non-negative real matrix A may be written in an upper-triangular block form where each block is a square non-negative matrix that is either irreducible or zero. Hence, the spectrum of A is just the union of the spectra of the blocks. Therefore, the following limitations on the location of the dominant eigenvalues hold:

Theorem 5 *The dominant eigenvalues of a non-negative real matrix* A *of size* n *are all the roots of* $\lambda^{k_i} - \rho(A)^{k_i} = 0$ *for some positive values* $k_i \leq n$ *such that* $\sum_i k_i \leq n$*. In particular, if* A *is not nilpotent, one of the dominant eigenvalues is positive real and has maximal multiplicity among all the dominant eigenvalues.* ■

For example, a non-negative real matrix of size three may have more admissible configurations for the dominant eigenvalues than those depicted in Figure 3. These additional configurations refer to the case in which the matrix is reducible and are depicted in Figure 4. In this case, the dominant eigenvalues can be multiple roots of the characteristic polynomial. The configurations correspond, for example, to the following matrices with unitary spectral radius:

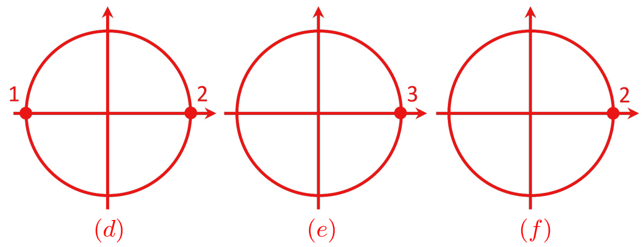

Figure 4. Further admissible configurations of the dominant eigenvalues (black dots) of a non-negative real matrix of size three. Numbers indicate the multiplicity of the eigenvalues as roots of the characteristic polynomial.

$$(d): \left(\begin{array}{cc|c} 0 & 1 & * \\ 1 & 0 & * \\ \hline 0 & 0 & 1 \end{array}\right), \quad (e): \left(\begin{array}{c|c|c} 1 & * & * \\ \hline 0 & 1 & * \\ \hline 0 & 0 & 1 \end{array}\right)$$

$$(f): \left(\begin{array}{cc|c} 0 & 1 & * \\ 1/2 & 1/2 & * \\ \hline 0 & 0 & 1 \end{array}\right) \text{ or } \left(\begin{array}{c|c|c} 1 & * & * \\ \hline 0 & 1 & * \\ \hline 0 & 0 & 1/2 \end{array}\right)$$

where the asterisk denotes a non-negative entry. Note that the dominant eigenvalues of a reducible non-negative real matrix may have the configurations (a) and (b) of Figure 3 as well, but not the configuration (c).

From Theorem 5, it follows that if a transfer function has a positive realization, then its dominant poles are among the r-th roots of ρ^r for some positive integer r, where ρ is the modulus of the dominant poles.

The limitation on the location of the dominant eigenvalues may "raise" the dimension of minimal positive realizations as illustrated in the following example (Förster & Nagy, 1998):

Example 3 Consider a transfer function with three simple poles equal to 1 and $e^{\pm 2\pi i/q}$, and with a corresponding impulse response that is non-negative. From Theorem 4 it follows that any non-negative matrix having these poles as eigenvalues must have also as eigenvalues all the q-th roots of unity. Hence, any minimal positive realization of such a transfer function has a dimension not lesser than q. ■

Based on the above results, it is possible to determine the minimum size of a non-negative real matrix A having all the dominant poles of a given transfer function among its dominant eigenvalues. To this end, denote by $\mathcal{R}_k$ the set of all the k–th roots of unity and by $\mathcal{P}_k$ that of all

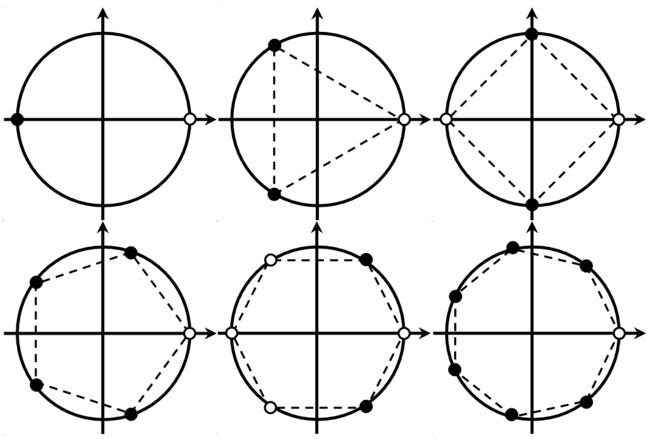

Figure 5. The set $\mathcal{R}_k$ of all the k-th roots of unity (both black and white dots) and the set $\mathcal{P}_k$ of all the primitive ones (black dots only) for $k = 2, \ldots, 7$.

the primitive ones [4] . These sets are depicted in Figure 5 for $k = 2, \ldots, 7$.

The next theorem is given in (Benvenuti, 2020c):

Theorem 6 *Let $H(z)$ be a strictly proper rational transfer function with a non-negative impulse response. Let $\mathcal{D}$ be the set of its dominant poles $p_1, \ldots, p_n$ with corresponding multiplicity $m(p_1), \ldots, m(p_n)$. Denote by ρ their modulus and assume that they are among the r-th roots of ρ^r for some positive integer r. Then the minimum size $N_{\mathcal{D}}$ of a non-negative real matrix with a spectral radius equal to ρ and having all the dominant poles among its dominant eigenvalues is equal to* [5] *:*

$$N_{\mathcal{D}} = \sum_{h|r} h \cdot m_{\mathcal{R}_h}$$

where $m_{\mathcal{R}_h}$ is recursively defined as follows:

- *if $h = r$, then $m_{\mathcal{R}_r} = m_{\mathcal{P}_r}$;*
- *if h is a proper divisor of r, then*

$$m_{\mathcal{R}_h} = \max\{0, m_{\mathcal{P}_h} - \sum_{\substack{h|k|r \\ k \neq h}} m_{\mathcal{R}_k}\}$$

with

$$m_{\mathcal{P}_h} = \begin{cases} 0, & \text{if } \nexists p_i \in \mathcal{D} : p_i/\rho \in \mathcal{P}_h \\ \max\{m(p_i) : p_i \in \mathcal{D} \text{ and } p_i/\rho \in \mathcal{P}_h\}, & \text{otherwise} \end{cases}$$

[4] A k–th root of unity z is said to be primitive if it is not an h–th root of unity for some smaller h, that is if $z^k = 1$ and $z^h \neq 1$ for $h = 1, 2, \ldots, k-1$.

[5] As usual, the notation $h|k$ means that h goes through all the positive divisors of k, including 1 and k. Similarly, the notation $h|k|r$ means that k goes through all the positive divisors of r that are multiples of h.

■

The following examples illustrate the result given in Theorem 6:

Example 4 Consider a transfer function with (dominant) simple poles equal to 1, -1, and $e^{\pm 2\pi i/3}$. The location of the poles in the complex plane is depicted in the left picture of Figure 6. The minimum size of an irreducible non-negative real matrix having all the given poles among its dominant eigenvalue is six. But when considering reducible matrices, this size reduces to five, as, for example,

$$A = \begin{pmatrix} C_2 & 0 \\ 0 & C_3 \end{pmatrix}$$

where C_h is the basic circulant matrix of dimension h. In fact, in this case, $r = 6$, $m_{\mathcal{P}_6} = 0$, and $m_{\mathcal{P}_3} = m_{\mathcal{P}_2} = m_{\mathcal{P}_1} = 1$. Hence,

$$\begin{aligned} m_{\mathcal{R}_6} &= m_{\mathcal{P}_6} = 0; \\ m_{\mathcal{R}_3} &= \max\{0, m_{\mathcal{P}_3} - m_{\mathcal{R}_6}\} = 1; \\ m_{\mathcal{R}_2} &= \max\{0, m_{\mathcal{P}_2} - m_{\mathcal{R}_6}\} = 1; \\ m_{\mathcal{R}_1} &= \max\{0, m_{\mathcal{P}_1} - m_{\mathcal{R}_2} - m_{\mathcal{R}_3} - m_{\mathcal{R}_6}\} = 0. \end{aligned}$$

Consequently, a non-negative real matrix with unitary spectral radius and having all the given poles as dominant eigenvalues must have a size at least equal to 5. ■

A more elaborated example is the following:

Example 5 Consider a transfer function with (dominant) poles equal to 1 with multiplicity 6, $e^{\pm 2\pi i/3}$ with multiplicity 3, $\pm i$ with multiplicity 2, and -1, $e^{\pm \pi i/3}$, $e^{\pm \pi i/4}$ with multiplicity 1. The location of the poles in the complex plane, together with their multiplicities, is depicted in the right picture of Figure 6. To compute the size of the matrix A provided by Theorem 6 note that, in this case, $r = 24$ and $m_{\mathcal{P}_{24}} = m_{\mathcal{P}_{12}} = 0$, $m_{\mathcal{P}_8} = m_{\mathcal{P}_6} = 1$, $m_{\mathcal{P}_4} = 2$, $m_{\mathcal{P}_3} = 3$, $m_{\mathcal{P}_2} = 1$, and $m_{\mathcal{P}_1} = 6$. Hence, the values of the $m_{\mathcal{R}_h}$'s can be recursively computed, for $h|24$, as follows:

$$\begin{aligned} m_{\mathcal{R}_{24}} &= m_{\mathcal{P}_{24}} = 0; \\ m_{\mathcal{R}_{12}} &= \max\{0, m_{\mathcal{P}_{12}} - m_{\mathcal{R}_{24}}\} = 0; \\ m_{\mathcal{R}_8} &= \max\{0, m_{\mathcal{P}_8} - m_{\mathcal{R}_{24}}\} = 1; \\ m_{\mathcal{R}_6} &= \max\{0, m_{\mathcal{P}_6} - m_{\mathcal{R}_{12}} - m_{\mathcal{R}_{24}}\} = 1; \\ m_{\mathcal{R}_4} &= \max\{0, m_{\mathcal{P}_4} - m_{\mathcal{R}_8} - m_{\mathcal{R}_{12}} - m_{\mathcal{R}_{24}}\} = 1; \\ m_{\mathcal{R}_3} &= \max\{0, m_{\mathcal{P}_3} - m_{\mathcal{R}_6} - m_{\mathcal{R}_{12}} - m_{\mathcal{R}_{24}}\} = 2; \\ m_{\mathcal{R}_2} &= \max\{0, m_{\mathcal{P}_2} - m_{\mathcal{R}_4} - m_{\mathcal{R}_6} - m_{\mathcal{R}_8} - m_{\mathcal{R}_{12}} \\ &\qquad - m_{\mathcal{R}_{24}}\} = 0; \\ m_{\mathcal{R}_1} &= \max\{0, m_{\mathcal{P}_1} - m_{\mathcal{R}_2} - m_{\mathcal{R}_3} - m_{\mathcal{R}_4} - m_{\mathcal{R}_6} - \\ &\qquad m_{\mathcal{R}_8} - m_{\mathcal{R}_{12}} - m_{\mathcal{R}_{24}}\} = 1; \end{aligned}$$

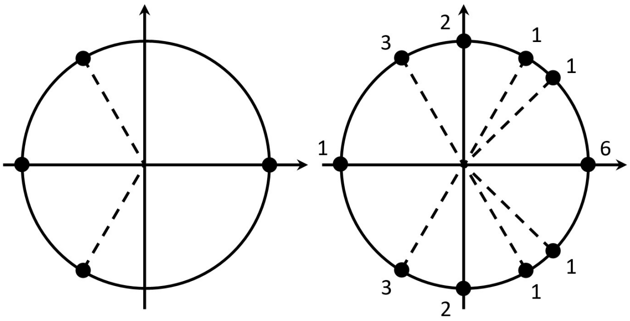

Figure 6. Pole locations for the transfer functions in Examples 4 and 5.

Consequently, a non-negative real matrix with unitary spectral radius and having all the given poles as dominant eigenvalues must have a size at least equal to

$$N_{\mathcal{D}} = \sum_{h|24} h \cdot m_{\mathcal{R}_h} = 25$$

Such a matrix is, for example, the following one:

$$A = \begin{pmatrix} C_8 & 0 & 0 & 0 & 0 & 0 \\ 0 & C_6 & 0 & 0 & 0 & 0 \\ 0 & 0 & C_4 & 0 & 0 & 0 \\ 0 & 0 & 0 & C_3 & 0 & 0 \\ 0 & 0 & 0 & 0 & C_3 & 0 \\ 0 & 0 & 0 & 0 & 0 & C_1 \end{pmatrix}.$$

■

5 Non–dominant poles matter too

It is interesting to note that also the non-dominant poles of the transfer function may play a role in increasing the size of the matrix A, for example, due to the rotational symmetry of the spectrum defined in the Perron–Frobenius theorem:

Example 6 Consider a transfer function with three simple poles equal to 1, −1, and −0.5, and with a corresponding impulse response that is non-negative. Also in this case, any minimal positive realization of such a transfer function has a dimension not smaller than four. In fact, from Theorem 4, the spectrum of any non-negative real matrix having these poles as dominant eigenvalues must be invariant under the rotation of the complex plane by the angle π and, consequently, must also contain an eigenvalue equal to 0.5 [6]. ■

[6] This result, in this case, can be obtained also by using the same argument of Example 1.

The limitations imposed by the non-negativity of a matrix on its whole spectrum have been completely characterized only for matrices of small dimensions. A partial characterization is given by the solution of the so-called Stochastic Inverse Single Eigenvalue Problem, that is the determination of which individual complex numbers occur in the spectra of non-negative real matrices. The complete characterization is instead known as the Nonnegative Inverse Eigenvalue Problem (NIEP) and consists of determining which lists of n complex numbers occur as the eigenvalues of some n-dimensional nonnegative real matrix. A solution to this problem is known only for $n \leq 4$, although there have been important results for several variations of the problem that consider special classes of matrices or lists of numbers. For example, the doubly stochastic NIEP (DS-NIEP) restricts the question to matrices with both row and column unitary sums, while the real NIEP (R-NIEP) to real spectra. Some results related to these problems are presented hereinafter.

5.1 The Stochastic Inverse Single Eigenvalue Problem

Kolmogorov (1937) posed the problem of characterizing the subset of the complex plane, denoted by Θ_n, consisting of the individual eigenvalues of all n-dimensional stochastic matrices. Such a characterization implies that of the region $\Theta_n(\rho)$, consisting of the eigenvalues of n-dimensional non-negative real matrices with a spectral radius equal to ρ. In fact, as shown in (Minc, 1987),

$$\Theta_n(\rho) = \rho \cdot \Theta_n = \{\rho z : z \in \Theta_n\}.$$

The sets Θ_n were characterized by Karpelevič (1951) after a partial result by Dmitriev and Dynkin (1946). The Karpelevič result is unwieldy, but simplifications were given by Đoković (1990) and Ito (1997). Moreover, recently, Johnson and Paparella (2017) provided parameterized stochastic matrices realizing the borders of these regions. The interested reader may refer to (Mirsky, 1963) for references to the earlier literature on this topic and Chapter XIII of (Gantmacher, 1959). The following theorem (Ito, 1997) is the characterization of the sets Θ_n:

Theorem 7 *The region Θ_n is symmetric relative to the real axis, is included in the disc $|z| \leq 1$, and intersects the circle $|z| = 1$ at points $e^{2\pi i a/b}$, where a and b run over the relatively prime integers satisfying $0 \leq a \leq b \leq n$. The boundary of Θ_n consists of these points and of curvilinear arcs connecting them in circular order. Let the endpoints of an arc be $e^{2\pi i a_1/b_1}$ and $e^{2\pi i a_2/b_2}$ $(b_1 \leq b_2)$. Each of these arcs is given by the following parametric equation:*

$$z^{b_2}(z^{b_1} - s)^{[n/b_1]} = (1 - s)^{[n/b_1]} z^{b_1\,[n/b_1]}$$

where the real parameter s runs over the interval $0 \leq s \leq 1$ and $[x]$ denotes the nearest integer to x. ■

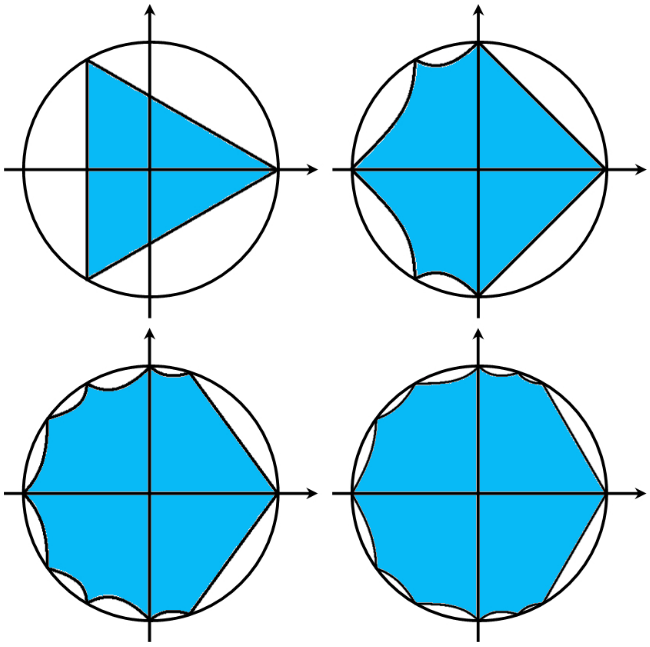

Figure 7. The Karpelevič regions Θ_3 (upper left), Θ_4 (upper right), Θ_5 (lower left), and Θ_6 (lower right).

For the sake of illustration, the regions Θ_3, Θ_4, Θ_5, and Θ_6 are depicted in Figure 7.

The following example shows how these regions may play a role in increasing the minimum dimension of a positive realization:

Example 7 Consider a transfer function with three simple poles equal to 1 and $\pm 0.9\,i$, and with a corresponding impulse response that is non-negative. As one can easily check from Figure 7, the poles $\pm 0.9\,i$ lie outside Θ_3, and hence any minimal positive realization of such a transfer function has a dimension not smaller than four. Note that, in this case, the sum of the poles is positive and the dominant pole is unique so that no symmetry of the dominant spectrum is required by the Perron–Frobenius theorem. ■

As the example makes clear, the location of the poles of the transfer function must be consistent with the Karpelevič regions and this allows to define the following basal lower bound for the minimum dimension of a positive realization (Benvenuti & Farina, 2004):

Theorem 8 *Let $H(z)$ be a strictly proper rational transfer function of order n with non-negative impulse response and n poles $p_1, \ldots, p_n$. Denote by ρ the modulus of the dominant poles of the transfer function. Then, the minimum dimension of a positive realization of $H(z)$ is not less than $\max\{n, N\}$ where N is the minimal value such that $p_i \in \Theta_N(\rho)$ for any $i = 1, \ldots, n$.* ■

5.2 The Non-negative Inverse Eigenvalue Problem

The NIEP is the problem of characterizing all possible spectra of non-negative real matrices, that is, determining the necessary and sufficient conditions for a given list of n complex numbers, $\lambda_1, \ldots, \lambda_n$, to be the spectrum of a non-negative real matrix A of size n. This problem was formulated in its present form by Suleĭmanova (1949) and a complete solution is known only for matrices of dimension $n \leq 4$. Some necessary conditions on the numbers of the list can be derived by the Perron-Frobenius theorem and by the fact that the matrix is real and non-negative [7] :

1) the list has to be closed under complex conjugation;
2) it must contain a positive real number greater than or equal to the modulus of any other number of the list;
3) the k-th moment s_k of the list, defined as

$$s_k = \sum_{i=1}^{n} \lambda_i^k,$$

must be non-negative for all $k \geq 1$.

Further necessary conditions are more subtle. It is worth citing those noticed independently by Johnson (1981) and Loewy and London (1978) that follow from the fact that every power of a non-negative matrix is non-negative and that positive diagonal entries must contribute to positive diagonal entries in powers:

$$s_k^m \leq n^{m-1} s_{km}, \qquad k, m = 1, 2, \ldots$$

It is easy to prove that, in some cases, some necessary conditions result to be also sufficient. This is the case of conditions 1 and 2 for lists of two numbers, and conditions 2 and 3 with $k = 1$, for lists of three or four real numbers (Perfect, 1952; Loewy & London, 1978). The case of lists of three numbers (not all real) was solved by Loewy and London (1978). The geometric interpretation of this result corresponds to the result of Dmitriev and Dynkin (1946), that is the region $\Theta_3(\rho)$.

The solution to the case $n = 4$ for a list of not all real numbers is presented by Torre Mayo, Abril-Raymundo, Alarcia-Estévez, Marijuán, and Pisonero (2007) using inequalities on the coefficients of the characteristic polynomial of the matrix. The conditions are very unwieldy, but their geometrical representation was recently given by Benvenuti (2014). This representation provides the sets $\mathcal{N}(\lambda)$ of possible locations in the complex plane of the pair of complex numbers that together with 1 and λ, with $-1 \leq \lambda \leq 1$, are the spectrum of some four-dimensional

[7] In what follows the trivial case of a list of n zeros, which is the spectrum of a nilpotent non-negative real matrix of dimension n, is not considered

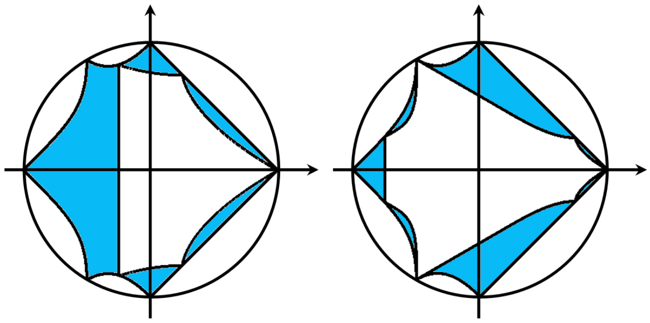

Figure 8. The sets $\mathcal{N}(-0.5)$ (left) and $\mathcal{N}(0.5)$ (right) and the set Θ_4 (grey)

non-negative matrix with unitary spectral radius. For the sake of illustration, the sets $\mathcal{N}(-0.5)$ and $\mathcal{N}(0.5)$ are depicted in Figure 8. By definition $\mathcal{N}(\lambda) \subset \Theta_4$ for all $\lambda \in [-1, 1]$ and

$$\bigcup_{\lambda \in [1,1]} \mathcal{N}(\lambda) = \Theta_4$$

The available solutions of the NIEP provide further insight into the minimality problem for positive realizations. The following example illustrates this point:

Example 8 Consider a transfer function with four simple poles equal to 1, 0.5, and $\pm 0.7\,i$, and with a corresponding impulse response that is non-negative. Although the sum of the poles is non-negative and all of them lie in the Karpelevič region Θ_4, any minimal positive realization of such a transfer function has a dimension not smaller than five. In fact, as Figure 8 makes clear, the pair of complex poles $\pm 0.7\,i$ lie outside the region $\mathcal{N}(0.5)$, and hence there does not exist a non-negative real matrix of size four having these four poles as eigenvalues. On the contrary, if the poles were equal to 1, -0.5, and $\pm 0.7\,i$ one could not exclude the existence of a positive realization of dimension four, being the pair of complex poles $\pm 0.7\,i$ inside the region $\mathcal{N}(-0.5)$. Indeed, a non-negative real matrix of size four with the given eigenvalues is, for example, the following:

$$A = \begin{pmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 0.245 & 0.245 & 0.01 & 0.5 \end{pmatrix}$$

■

Several sufficient conditions together with a map of inclusion or independence relations between them can be found in (Marijuan, Pisonero, & Soto, 2007, 2017). The interested reader may find more details and references on the NIEP in the survey paper (Johnson, Marijuán, Paparella, & Pisonero, 2018).

6 Bounds on the dimension

As illustrated in the previous sections, even considering only the limitations imposed by the non-negativity of the matrix A of a positive realization, determining its minimum dimension seems to be a very hard task. Indeed, this problem results to be a variant of the NIEP (Benvenuti, 2020c) that is a long-standing, hard, and sought-after problem, perhaps the hardest in matrix analysis (Johnson et al., 2018).

A lower bound to the dimension of a minimal positive realization was proposed by Hadjicostis (1999). The key idea is that of determining the minimum number of eigenvalues that the matrix A must have, beyond the poles of the transfer function, to fulfill the necessary condition 3) of the NIEP for $k = 1$. This bound has been recently refined by Benvenuti (2020a) by adding the condition on the minimum number of dominant eigenvalues given in Theorem 6:

Theorem 9 *Let $H(z)$ be a strictly proper rational transfer function with a non-negative impulse response and having n poles $p_1, \dots, p_n$, with corresponding multiplicity $m(p_1), \dots, m(p_n)$. Denote by ρ the modulus of the dominant poles of the transfer function and assume they are among the r-th roots of ρ^r for some positive integer r. Then, the minimum dimension of a positive realization of $H(z)$, if any, is not less than*

$$N_{\mathcal{D}} + \sum_{|p_i|<\rho} m(p_i) + \zeta$$

with

$$\zeta = \left\lceil - \sum_{|p_i|<\rho} p_i/\rho \cdot m(p_i) - m_{\mathcal{R}_1} \right\rceil$$

if

$$\sum_{|p_i|<\rho} p_i \cdot m(p_i) + \rho \cdot m_{\mathcal{R}_1} < 0,$$

and $\zeta = 0$ otherwise, and where $N_{\mathcal{D}}$ and $m_{\mathcal{R}_1}$ are those defined in Theorem 6. ■

The following example illustrates the result of the theorem:

Example 9 Consider the strictly proper rational transfer function

$$H(z) = \frac{1}{z-1} + \frac{0.1}{z+1} + \frac{0.2z+0.1}{z^2+z+1} + \frac{0.4}{z-0.3} - \frac{1}{z-0.1} - \frac{0.3}{z-0.2} + \frac{0.1}{z+0.5} + \frac{-0.1z+0.1}{z^2+1.141z+0.36}$$

whose impulse response is non-negative. The dominant poles of $H(z)$ are equal to 1, -1 and $e^{\pm 2\pi i/3}$. Consequently, as shown in Example 4, $N_{\mathcal{D}} = 5$ and $m_{\mathcal{R}_1} = 0$. Moreover, $\zeta = 2$ given that

$$\sum_{|p_i|<1} p_i \cdot m(p_i) = 0.3+0.1+0.2-0.5-2\cdot 0.5706 = -1.0412.$$

Hence, any minimal positive realization of $H(z)$ has a dimension not less than

$$N_{\mathcal{D}} + \sum_{|p_i|<\rho} m(p_i) + \zeta = 5 + 6 + 2 = 13.$$

■

Another lower bound is given in (Nagy & Matolcsi, 2003b) for a very special class of transfer functions, i.e., those for which there exists a time instant at which the non-negative impulse response is equal to zero and is strictly positive from that instant onward. This bound is obtained by using digraph techniques and extends a previous result by Benvenuti and Farina (1999) by using a geometric approach. An improvement on this result is given by Czaja et al. (2008).

Bounding from above the dimension of a minimal positive realization of a given transfer function is a more difficult task than defining a lower bound. Indeed, while a lower bound may be determined by taking into account only the non-negativity constraints on the entries of the matrix A of the realization, the definition of an upper bound requires the construction of a positive realization and hence must consider also the non-negativity constraints on the vectors b and c.

An upper bound on the dimension of a minimal realization of a transfer function with only real and simple poles was provided by Hadjicostis (1999). This result was improved by Halmschlager and Matolcsi (2005) as follows:

Theorem 10 *Let $H(z)$ be a strictly proper rational transfer function of the form:*

$$H(z) = \frac{r_1}{z-\rho} + \sum_{j=1}^{N_1+N_2+N_3} \frac{r_j}{z-p_j}$$

with a simple dominant real pole of modulus ρ, N_1 non-negative real and simple non-dominant poles with positive residue, N_2 non-negative real and simple non-dominant poles with negative residue, and N_3 negative real and simple non-dominant poles. Assume that $H(z)$ has a non-negative impulse response. Then $H(z)$ has a minimal positive realization of dimension not greater than

$$N + 1 + N_1 + N_2 + 2N_3$$

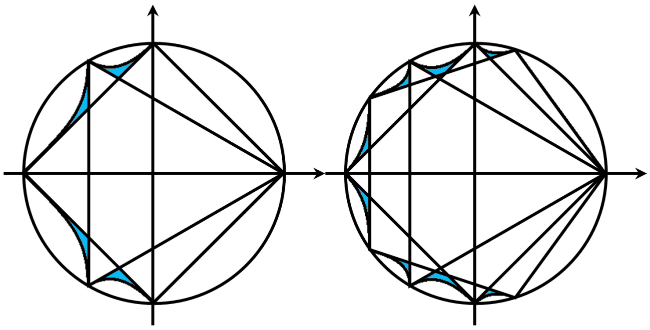

Figure 9. The Karpelevič regions Θ_4 (left) and Θ_5 (right) and the corresponding approximations $\widetilde{\Theta}_4 = \Pi_3 \cup \Pi_4$ and $\widetilde{\Theta}_5 = \Pi_3 \cup \Pi_4 \cup \Pi_5$.

where N denotes the smallest positive integer such that

$$\sum_{j=N_1+1}^{N_1+N_2+N_3} |r_j| \cdot |p_j|^N \le r_1 \cdot \rho^N$$

■

The key idea is that of decomposing the impulse response into the sum of appropriate non-negative sequences for which a positive realization of "reduced" dimension is known and then combine these realizations. The next theorem (Benvenuti, 2020b) is used to deal with the pairs of complex poles. In this theorem, the Karpelevič regions Θ_n are approximated from below through the polygons Π_k, i.e., regular polygons with k edges, having one vertex in the point $(1,0)$ and center in the origin of the complex plane, as follows:

$$\Theta_n \supset \widetilde{\Theta}_n = \bigcup_{k=3}^{n} \Pi_k.$$

Note that $\widetilde{\Theta}_3 = int(\Theta_3)$ while the accuracy of the approximation for $k > 3$ is shown, for example, in Figure 9 for $n = 4$ and 5.

Theorem 11 *Let*

$$H(z) = \frac{r_1}{z-p_1} + \frac{r_{11}z + r_{12}}{z^2 - (2\sigma\cos\theta)\,z + \sigma^2}$$

be a third-order strictly proper rational transfer function with a pair of complex poles $\rho e^{\pm i\theta}$, with $0 < \sigma \le p_1$ and $0 < \theta < \pi$. Denote by m a positive integer such that the complex poles of $H(z)$ satisfy $\sigma/p_1 \in \Pi_m$. If

$$\xi_1 = \frac{\sqrt{r_{11}^2\sigma^2 + r_{12}^2 + 2r_{11}r_{12}\sigma\cos\theta}}{\sigma\sin\theta} \le r_1,$$

then $H(z)$ has a positive realization of dimension m. ■

The following theorem, which is a restatement of a result provided in (Benvenuti, 2020b), makes use of the previous result and Theorem 6 to extend the result of Theorem 10 to the case of transfer functions with possibly complex poles. It provides an upper bound on the dimension of a minimal positive realization of a given transfer function with simple poles:

Theorem 12 *Let $H(z)$ be a strictly proper rational transfer function of the form:*

$$H(z) = \frac{r_1}{z-\rho} + \frac{r_1'}{z+\rho} + \sum_{k=1}^{N_1} \frac{r_{k1}z + r_{k2}}{z^2 - (2\rho\cos\theta_k)\,z + \rho^2} + \sum_{j=1}^{N_2+N_3+N_4} \frac{r_j}{z-p_j} + \sum_{h=N_1+1}^{N_1+N_5} \frac{r_{h1}z + r_{h2}}{z^2 - (2\sigma_h\cos\theta_h)\,z + \sigma_h^2}$$

with $2+2N_1$ (or $1+2N_1$ if $r_1'=0$) simple dominant poles of modulus ρ, N_2 non-negative real and simple non-dominant poles with positive residue, N_3 non-negative real and simple non-dominant poles with negative residue, N_4 negative real and simple non-dominant poles, N_5 pairs of complex and simple non-dominant poles. Assume that $H(z)$ has a non-negative impulse response and its dominant poles are among the r-th roots of ρ^r for some positive integer r. If

$$|r_1'| + \sum_{k=1}^{N_1} \xi_k < r_1, \tag{5}$$

then $H(z)$ has a minimal positive realization of dimension not greater than

$$N_{\mathcal{D}} + N + 1 + N_2 + N_3 + 2N_4 + \sum_{N_1+1}^{N_1+N_5} m_h$$

where $N_{\mathcal{D}}$ is that defined in Theorem 6, ξ_k is that defined in Theorem 11, m_h denotes the minimum value of m for which the non-dominant complex poles $\sigma_h e^{\pm i\theta_h}$ belong to the set $\widetilde{\Theta}_m(\rho)$, and N denotes the smallest positive integer such that

$$\sum_{j=N_2+1}^{N_2+N_3+N_4} |r_j|\cdot|p_j|^N + \sum_{h=N_1+1}^{N_1+N_5} \sigma_h^N \xi_h \le \left(r_1 - |r_1'| - \sum_{k=1}^{N_1}\xi_k\right)\cdot\rho^N. \tag{6}$$

■

The following example illustrates the result of Theorem 5:

Example 10 Consider the transfer function in Example 9. For this transfer function, the following relations hold: $N_1 = N_2 = 1$, $N_3 = 2$, $N_4 = 1$, and $N_5 = 1$. The dominant poles of $H(z)$ have unitary modulus and are equal to 1, -1, and $e^{\pm i2\pi i/3}$. Consequently, as shown in Example 4, $N_{\mathcal{D}} = 5$. The non-dominant complex poles of $H(z)$, which are equal to $0.6e^{\pm 9\pi i/10}$, lie in the set $\widetilde{\Theta}_4$ but not in the set $\widetilde{\Theta}_3$. This is made clear in Figure 10 where the poles of $H(z)$ are shown together with the sets Π_3 and Π_4. Consequently $m_2 = 4$. Condition (5) of Theorem 12 holds and condition (6) holds true for $N = 2$. Hence, the transfer function has a positive realization of dimension equal to 17. Such a positive realization can be found in (Benvenuti, 2020b). Taking into account the result in Example 9, the dimension of any minimal positive realization of $H(z)$ is between 13 and 17. ■

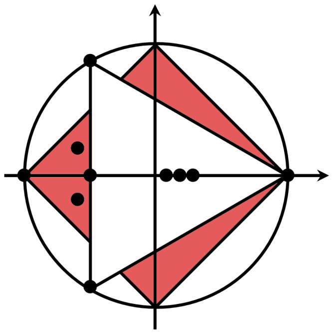

Figure 10. The location of the poles of the transfer function considered in Examples 9 and 10.

7 Hankel matrix factorization

A well-known result from system theory states that the rank of the semi-infinite Hankel matrix H corresponding to a given impulse response is the minimal dimension of a state-space realization of the impulse response itself (Ho & Kalman, 1965). This rank corresponds to the minimal inner size of a factorization of the Hankel matrix. Since the impulse response h_k of a positive system is non-negative, then the Hankel matrix H has non-negative entries. Moreover, given a minimal positive realization $\{A, b, c\}$ of dimension n, the Hankel matrix can be factorized as the product of two non-negative matrices R and S, with inner size n, as follows:

$$H = \begin{pmatrix} c^Tb & c^TAb & \dots \\ c^TAb & c^TA^2b & \dots \\ c^TA^2b & \ddots & \\ \vdots & & \ddots \end{pmatrix} =$$

$$\begin{pmatrix} c^T \\ c^T A \\ c^T A^2 \\ \vdots \end{pmatrix} \begin{pmatrix} b & Ab & \dots \end{pmatrix} = RS$$

Hence, the Hankel matrix has a non-negative rank (Cohen & Rothblum, 1993) equal to n. It is then interesting to study whether the converse is true, that is, if the non-negative rank of the Hankel matrix is, in general, equal to the minimum dimension of a positive realization of the impulse response. The following example (Benvenuti & Farina, 1998) gives a negative answer to this question:

Example 11 Consider the system

$$x_{k+1} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & -1 \\ 0 & 1 & 0 \end{pmatrix} x_k + \begin{pmatrix} 8 \\ 1 \\ 1 \end{pmatrix} u_k$$
$$y_k = \begin{pmatrix} 1 & 1 & 1 \end{pmatrix} x_k,$$

whose impulse response is cyclic and non-negative because it is equal to $h_{1+4i} = 10$, $h_{2+4i} = 8$, $h_{3+4i} = 6$, $h_{4+4i} = 8$ for $i = 0, 1, \dots$. The matrix A of the system has eigenvalues equal to 1 and $\pm i$ so that a positive realization of dimension three does not exist (see Example 3). Nevertheless, a non-negative factorization of the Hankel matrix having an inner size equal to three does exist. In fact, the Hankel matrix of the system can be written as:

$$H = \begin{pmatrix} \widetilde{H} & \widetilde{H} & \dots \\ \widetilde{H} & \ddots & \\ \vdots & & \ddots \end{pmatrix}, \quad \widetilde{H} = \begin{pmatrix} 10 & 8 & 6 & 8 \\ 8 & 6 & 8 & 10 \\ 6 & 8 & 10 & 8 \\ 8 & 10 & 8 & 6 \end{pmatrix}.$$

Moreover, the matrix $\widetilde{H}$ can be factorized as

$$\widetilde{H} = \begin{pmatrix} 5 & 3 & 3 \\ 4 & 2 & 6 \\ 3 & 5 & 5 \\ 4 & 6 & 2 \end{pmatrix} \begin{pmatrix} 2 & 1 & 0 & 1 \\ 0 & 1 & 1 & 0 \\ 0 & 0 & 1 & 1 \end{pmatrix} = PQ,$$

and consequently the following is a non-negative factorization of H with an inner size equal to three:

$$H = \begin{pmatrix} P \\ P \\ \vdots \end{pmatrix} \begin{pmatrix} Q & Q & \dots \end{pmatrix}$$

■

A sufficient condition for a given positive system to be minimal, expressed in terms of the non-negative rank of the associated Hankel matrix, is given by van den Hof (1997b):

Theorem 13 *Given a MIMO positive linear system $\{A, B, C\}$ of dimension n, if there exist p, $q \geq 1$ such that the non-negative rank of the Hankel matrix $H(p, q)$ is equal to n then $\{A, B, C\}$ is a minimal positive linear system.* ■

Moreover, a reformulation of minimality of a positive realization in terms of a non-negative factorization of the Hankel matrix is provided by van den Hof (1996) by adding a further constraint on such a factorization, but the condition involves an infinite test. To date, in this context, a procedure to evaluate the minimum order of a positive realization is not available. Moreover, Vavasis (2009) proved that the non-negative rank r of a non-negative matrix H of dimension $p \times q$ is NP-hard to compute, i.e., an exact algorithm that runs in time polynomial in p, q, and r does not exist. Arora, Ge, Kannan, and Moitra (2012) gave an exact algorithm for deciding if the non-negative rank is at most r which is doubly exponential in r but runs in a polynomial–time algorithm for any fixed r. This result was recently improved by (Moitra, 2016) who provided an algorithm that runs in singly exponential time as a function of r.

8 The case of MIMO systems

It is well known that a strictly proper rational transfer function matrix $H(z)$ has a non-negative realization if and only if each one of its entries has a non-negative realization (Förster & Nagy, 2000). Moreover, even in the MIMO case, the existence of a positive realization can be cast in terms of the existence of an A-invariant polyhedral cone with properties (3), where, in this case,

$$\mathcal{R} = cone\left(\begin{bmatrix} B & AB & A^2B & \dots \end{bmatrix}\right),$$

$$\mathcal{O} = \{x \in \mathbb{R}^n | CA^{k-1}x \geq 0, \ k = 1, 2, \dots\},$$

and $\{A, B, C\}$ denotes a minimal (i.e., jointly reachable and observable) realization of $H(z)$. Even in this case, knowledge of a cone $\mathcal{K}$ immediately yields a positive realization of a dimension equal to the number of the extreme rays of the cone.

The minimality of positive realization in the MIMO case was studied in particular by Förster and Nagy (1998). They provided the following example showing that, at least in the MIMO case, the dimension of any minimal positive realization not only may be larger than the dimension of any minimal realization but it may also be smaller than the minimum of the number of extreme rays of all the A-invariant polyhedral cones satisfying conditions (3).

Example 12 Consider the strictly proper rational transfer function matrix

$$H(z)=\begin{pmatrix} \frac{z^2}{z^3-1} & \frac{1}{z^3-1} & \frac{z}{z^3-1} & \frac{z^2}{z^3-1} & \frac{1}{z^3-1} & \frac{z}{z^3-1} \\ \frac{z}{z^3-1} & \frac{z^2}{z^3-1} & \frac{1}{z^3-1} & \frac{z}{z^3-1} & \frac{z^2}{z^3-1} & \frac{1}{z^3-1} \\ \frac{1}{z^3-1} & \frac{z}{z^3-1} & \frac{z^2}{z^3-1} & \frac{1}{z^3-1} & \frac{z}{z^3-1} & \frac{z^2}{z^3-1} \\ 0 & 0 & 0 & \frac{1}{z-1} & \frac{1}{z-1} & \frac{1}{z-1} \\ \frac{1}{z-1} & \frac{1}{z-1} & \frac{1}{z-1} & 0 & 0 & 0 \end{pmatrix}$$

and its minimal realization:

$$A=\begin{pmatrix} 0&0&1&0\\ 1&0&0&0\\ 0&1&0&0\\ 0&0&0&1 \end{pmatrix}, B=\begin{pmatrix} 1&0&0&1&0&0\\ 0&1&0&0&1&0\\ 0&0&1&0&0&1\\ 0&0&0&1&1&1 \end{pmatrix}, C=\begin{pmatrix} 1&0&0&0\\ 0&1&0&0\\ 0&0&1&0\\ 0&0&0&1\\ 1&1&1&-1 \end{pmatrix}$$

The reachability cone, which by definition is A-invariant, is equal to $\mathcal{R}=cone\,(B)$ and has six extreme rays. Moreover, the observability cone is defined as:

$$\mathcal{O}=\{x\in\mathbb{R}^4 | x_1+x_2+x_3-x_4\geq 0, x_i\geq 0, i=1,\ldots,4\},$$

The edges of this cone can be determined using the technique described in (Goldman & Tucker, 1957) and this results in $\mathcal{O}=\mathcal{R}$. As a consequence, the cone $\mathcal{K}=\mathcal{O}=\mathcal{R}$ is the single A-invariant cone satisfying conditions (3) and a positive realization of dimension 6 can be easily computed:

$$A=\begin{pmatrix} 0&0&1&0&0&0\\ 1&0&0&0&0&0\\ 0&1&0&0&0&0\\ 0&0&0&0&0&1\\ 0&0&0&1&0&0\\ 0&0&0&0&1&0 \end{pmatrix}, B=I_6, C=\begin{pmatrix} 1&0&0&1&0&0\\ 0&1&0&0&1&0\\ 0&0&1&0&0&1\\ 0&0&0&1&1&1\\ 1&1&1&0&0&0 \end{pmatrix}.$$

On the other hand, the following is a fifth-order positive realization of $H(z)$:

$$A=\begin{pmatrix} 0&0&1&0&0\\ 1&0&0&0&0\\ 0&1&0&0&0\\ 0&0&0&1&0\\ 0&0&0&0&1 \end{pmatrix}, B=\begin{pmatrix} 1&0&0&1&0&0\\ 0&1&0&0&1&0\\ 0&0&1&0&0&1\\ 0&0&0&1&1&1\\ 1&1&1&0&0&0 \end{pmatrix}, C=I_5$$

Further, this realization is minimal as a positive one. In fact, if there was a fourth-order positive realization, the similarity transformation matrix T between such a positive realization and the minimal realization would define an A-invariant polyhedral proper cone $\mathcal{K}=cone(T)$ having four extreme rays and satisfying conditions (3), which is impossible by the above result. ■

This example suffices to indicate that the geometric approach is not an appropriate tool to address the minimal positive realization problem for MIMO systems, i.e., minimizing the number of extreme rays of an A-invariant polyhedral cone satisfying conditions (3), does not guarantee the minimality of the corresponding positive realization. Nevertheless, this approach has been used by Matolcsi and Nagy (2004) to compute an upper bound on the dimension of a minimal positive realization of a strictly proper rational transfer function matrix as a function of the number of the extreme rays of the polyhedral cones associated with the positive realizations of the entries of the transfer function matrix.

Finally, also a lower bound on the dimension of a minimal positive realization for a MIMO system can be provided. To this aim, it is first necessary to note that the spectral radius of the matrix A of any minimal positive realization of a given transfer function matrix $H(z)$ must coincide with the positive real dominant pole of the transfer function matrix itself[8]. Then, Theorem 9 can be used to provide a lower bound to a minimal positive realization taking into consideration that, in the MIMO case, the multiplicity of a pole is defined as its multiplicity as a root in the characteristic polynomial of a minimal realization of $H(z)$, which is the least common multiple of the denominators of all possible minors in $H(z)$.

9 Special results for continuous-time systems

As shown in (Anderson et al., 1996; Benvenuti & Farina, 2002; Ohta et al., 1984; van den Hof, 1997b), the results on the PRP for the discrete-time case have in general a corresponding continuous-time formulation. Nevertheless, this is not always true when the results on minimality are considered. This is mainly due to the fact that a continuous-time positive system is characterized by a matrix A which is a Metzler matrix A and not, in general, non-negative matrix (Farina & Rinaldi, 2000). As a consequence, all the results on the minimality for discrete-time positive systems that specifically rely on some properties of non-negative matrices cannot be directly extended to the continuous-time case. In particular, the dominant eigenvale of a Metzler matrix, that is the eigenvalue with maximal real part, is unique (possibly multiple) and real (Berman & Plemmons, 1994) so

[8] This result easily follows as an extension of Theorem 3.2 in (Anderson et al., 1996)

that all the considerations about dominant eigenvalue locations, underlying the lower bounds on a positive realization for a discrete-time system, cannot be extended to the continuous-time case.

Similarly to the discrete-time case, some results for the continuous-time case have been obtained for transfer functions of third order or by considering particular classes of transfer functions and systems.

The first result on minimality for continuous-time systems is due to Maeda et al. (1977) and is related to compartmental systems. In more detail, the class of compartmental systems having a tree-compartmental matrix A, that is a matrix whose corresponding non-oriented graph has no closed path, is considered. Two compartmental systems of such a class that are widely used in the field of clinical medicine are, for example, the catenary and the mammillary systems. The authors provide the necessary and sufficient conditions for a transfer function to have a compartmental realization with a tree-compartmental matrix and show that, in this case, the minimal dimension of the positive realization equals the order of the transfer function itself.

In (Benvenuti & Farina, 2002), the equivalent of Theorem 2 for compartmental systems is provided, i.e., the necessary and sufficient conditions for a third-order transfer function $H(s)$ with positive distinct real poles to have a (minimal) positive realization of dimension three. Further results are provided in (Astolfi & Colaneri, 2004) and are the counterpart of those described in Section 3. For example, it is there shown that a system with a transfer function of the form

$$H(s) = \frac{1}{(s-p_1)(s-p_2)\cdots(s-p_n)}$$

with real distinct poles, admits a (minimal) positive realization of dimension n. Moreover, the same result holds if the transfer function is asymptotically stable, has only one zero ζ such that

$$\zeta > \min_{i=1,\ldots,n} p_i$$

and has a positive high frequency gain.

10 Discussion

The issues discussed in this survey paper show that the characterization of minimality for positive realizations proved to be a very difficult problem in which the tools seem not yet clearly defined or, at least, limited. The results are only partial or seemingly weak and new results, or improvements over what is already available, do not appear as often. To underline the difficulty of the problem, it suffices to note that the related problems such as the NIEP (Johnson et al., 2018) and the nonnegative rank computation (Cohen & Rothblum, 1993) are among the most prominent and hard problems in matrix analysis and have proved to be NP-hard (Borobia & Canogar, 2017; Vavasis, 2009). Hence, the solution to the minimality problem for positive systems appears to be still far away.

A major issue is to understand what kind of mathematical "tool" is more suited to tackle it. In fact, the geometric approach has proved to be a fundamental tool for determining the existence of a positive realization but, with regards to minimality, it has led exclusively to some results for the case of third-order transfer functions with real poles (see Theorems 2 and 3). Moreover, in the case of MIMO systems, it has been shown (see Example 12) that finding a cone with the minimum number of extreme rays does not guarantee the minimality of the corresponding positive realization. Nevertheless, this has not been proved nor disproved yet, in the case of SISO systems. Hence, a decisive step to make the geometric approach useful in solving the minimality problem in the case of SISO systems is that of determining if, at least in this case, the dimension of any minimal positive realization is equal to the minimum of the number of extreme rays of the A-invariant polyhedral cones satisfying conditions (3).

Another interesting geometric interpretation of the positive realization problem is given in (Picci, van den Hof, & van Schuppen, 1998; van den Hof, 1997a, 1996, 1997b). This interpretation refers to cones defined from the impulse response h_k instead of the transfer function $H(z)$. The main features of this approach is that the minimality problem is reformulated directly for MIMO systems and in terms of the non-negative factorization of the Hankel matrix. In particular, sufficient conditions for a positive realization to be minimal are given in terms of the non-negative rank of the Hankel matrix (see Theorem 13). However, these conditions are not necessary, as shown in Example 11. Further, as shown in Section 7 such a result is to date mainly theoretical since it involves an infinite test. It could become more interesting as the results in (rank-revealing/exact) non-negative matrix factorization algorithms increase.

A promising approach is the graph-theoretical one used by Nagy and Matolcsi (2003b) to provide a lower bound on the minimum dimension of a positive realization of a given transfer function. Even if the case there considered is related to a very special class of transfer functions, the approach proved to be useful in different problems related to positive systems and hence it is certainly worth investigating the possibilities of such an approach for the study and understanding of the minimality problem in the general framework. In more detail, it proved to be useful in revealing reachability and controllability properties of positive linear systems (Bru, Caccetta, Romero, Rumchev, & Sanchez, 2020; Caccetta & Rumchev, 2000). Moreover, it has been used to find the last available solutions to the NIEP, which, as shown in this survey, is strictly related to the minimality problem of positive systems. Indeed, the case $n = 4$ of the NIEP is solved by Torre Mayo et al. (2007) by introducing some graph-theoretical tools to study the coefficients of the characteristic polynomials of non-negative matrices, being they closely related to the cyclic structure of the weighted digraph with adjacency matrix A. Furthermore, when $n = 5$, a solution to the NIEP for matrices with trace zero is provided by Laffey and Meehan (1999) using again graph-theoretical tools and the Newton's identities.

Another important issue related to minimality is the improvement of the bounds on the minimal dimension of a positive realization. This improvement may occur in several directions:

- find "tighter" bounds;
- extend the class of transfer functions for which the bounds can be computed;
- extend some results to the case of MIMO systems.

Tighter lower bounds could be obtained by exploiting the results of the NIEP. To date, only the conditions on the location of the dominant eigenvalues and that on the non-negativity of the sum of all the eigenvalues have been considered (see Theorem 9). On the other hand, the class of transfer functions for which the upper bounds are defined can be extended by considering the case of multiple poles. Some results in this direction are presented in (Kim, 2012; Nagy & Matolcsi, 2005) where the case of transfer functions with non-negative multiple poles is considered.

There are many other arguments related to the PRP and the problem of minimality. Two of them, together with the corresponding results, deserve to be illustrated in more detail.

10.1 Checking non-negativity of the impulse response

Not only the non-negativity of the impulse response is a necessary condition for the existence of a positive realization of the systems, but it is also a required feature in many applications such as congestion control systems, machine tool axis control, trajectory following in robotics or level control. In fact, this property corresponds, in the linear case, to a non-overshooting (or monotone non-decreasing) step response. Checking the non-negativity of the impulse response showed to be a very hard problem since necessary and sufficient conditions for a general SISO linear system to have this property are still not available. To date, an algorithm to perform this check from the coefficients of the corresponding transfer function is not yet known and, in general, it is not possible to know how many values of the impulse response must be checked to be sure that the impulse response is non-negative for all times (Blondel & Portier, 2002; Farina & Rinaldi, 2000). The existence of a positive realization is a sufficient condition for the impulse response to be non-negative but this may require computing an arbitrarily large number of extreme generators. Moreover, not all the systems with a non-negative impulse response possess a positive realization. However, in the case of third-order transfer functions with real poles, it is possible to infer the non-negativity of the whole impulse response from only the three values that it assumes for $k = 1, 2, 3$ (see Theorems 2 and 3). Recently, Grussler and Rantzer (2021) proposed another sufficient condition that requires the definition of a second-order cone avoiding that of any special state-space realization.

Some sufficient conditions, as well as necessary conditions, are available in terms of zero–pole patterns (Jayasuriya & Franchek, 1991; Leon de la Barra & Salazar, 2002; Liu & Bauer, 2008, 2009). In more detail, sufficient conditions for a transfer function with only real poles and zeros to have a non-negative impulse response are given in (Liu & Bauer, 2008). These conditions have been extended to the case of complex conjugate zeros and poles in (Liu & Bauer, 2009) resulting in an interesting geometric pole-zero pattern, i.e., poles and zeros evenly distributed on different concentric circles centered at the origin. Simpler conditions, involving only the poles of the transfer function, can be given by assuming the non-negativity of the first values of the impulse response. These conditions follow from the result of Benvenuti and Farina (2017) [9], as shown in (Förster & Nagy, 2000).

Theorem 14 *Let $H(z)$ be a strictly proper rational transfer function of order n and assume that the values h_k of the corresponding impulse response are non-negative for $k = 1, \dots, n$. The whole impulse response is non-negative if the following conditions are satisfied: either $H(z)$ has no positive poles, or else*

1) it has a positive real dominant pole ρ,
2) the dominant poles are among the r-th roots of ρ^r for some positive integer r,
3) all the dominant poles are simple,
4) taking the minimal value of r, no non-dominant pole has an argument that is an integer multiple of $2\pi/r$.

■

[9] This result is the correct formulation of a theorem proposed by Roitman and Rubinstein (1992) on the characterization of linear recursions which imply a linear recursion with non-negative coefficients.

10.2 Finding approximate realizations

When a positive realization does not exist or its minimal dimension is much larger than the order n of the corresponding transfer function, it has been recently proposed to consider approximated models with reduced-order by introducing some suitable approximating criteria. Basically, two general criteria can be considered: Either determining positive realizations whose transfer function is as close as possible to that to be realized or determining non-positive realization with an internal behavior "similar" to that of a positive system.

The first approach was recently proposed by Sato and Takeda (2020). In particular, the authors propose some algorithms to compute the nearest n-dimensional positive system, that is, for example, the positive system whose eigenvalues are as close as possible to the poles of the transfer function to be realized.

The second approach considers the so-called quasi-positive realizations (Guidorzi, 2014, 2016) or eventually positive realizations (Altafini, 2016). The quasi-positive realizations are, in general, non-positive but assure non-negativity of the state and output sequences at any time for any non-negative input sequence but only for all the initial states that are reachable with a non-negative input sequence. In particular, the Markov canonical realization (Farina & Rinaldi, 2000) is, by construction, an n dimensional quasi-positive realization of a transfer function of order n with a non-negative impulse response.
The eventually positive realizations are, in general, non-positive as well. They assure that the state sequence is eventually non-negative, i.e., it may fail to be non-negative, but only transiently. In more detail, in (Altafini, 2016) an algorithm is given that, when terminates successfully, provides an n-dimensional eventually positive realization of a transfer function of order n with a non-negative impulse response and with one single dominant pole. While some of the properties of positive systems are retained, analysis of and certificates for eventually positive systems are much harder than for positive systems. Nevertheless, the theoretical development of this class of systems can offer simpler analysis and control tools (Sootla, 2019).

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