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LOW ENERGY BEHAVIOR IN FEW-PARTICLE QUANTUM SYSTEMS: EFIMOV EFFECT AND ZERO-RANGE INTERACTIONS

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ABSTRACT

We investigate the emergence of a universal behavior in certain few-particle quantum systems at low-energy.

First we consider a system composed by two identical fermions of mass one and a different particle of mass m in dimension three. Under the assumption that the two-particle Hamiltonians composed by one of the fermions and the third particle have a resonance at zero-energy, and for m less than a mass threshold m_*, we prove the occurrence of the Efimov effect, i. e., the existence of an infinite number of three-body bound states accumulating at zero. Then we study three-particle systems with zero-range interactions. In dimension one we give a rigorous definition of the Hamiltonian for three identical bosons and we prove that it is the limit of suitably rescaled regular Hamiltonians. In dimension three we write the expression of the quadratic form associated to the STM extension for a generic three-particle system. Then we focus on a system of three identical bosons proving stability outside the *s-wave* subspace. As a third example of universal behavior in few-particle system we consider a quantum Lorentz gas in dimension three: a particle moving through N obstacles whose positions are independently chosen according to a given common probability density. We assume that the particle interact with each obstacle via a Gross Pitaevskii potential. We prove the convergence, as $N \to \infty$, to a Hamiltonian depending on the common distribution density of the obstacles and such that the only dependence on the interaction potential is through its scattering length.

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CONTENTS

1 INTRODUCTION

In this thesis we discuss the behavior of some quantum few-particle systems with two-body short-range interactions in the limit of sufficiently low-energy. In this regime the wave function is so delocalized that the details of the two-body interactions become irrelevant and the behavior of the system can be characterized using only few physical parameters. This situation is referred to as *universality* of low energy physics, meaning that systems with different microscopic interactions exhibit the same physical behavior.

In the case of three-body systems a strong evidence of this scenario is the so-called *Efimov Effect* discovered by Vitaly Efimov in 1970 (see [28, 29]). He considered three identical bosons interacting via short-range, resonant interactions and proved the existence of an infinite number of three-body bound states of increasing size whose energies E_n accumulate at the scattering threshold. The bound states, known as *Efimov states*, or *trimers*, exhibit universal features whose paradigmatic example is the asymptotic geometrical law satisfied by the eigenvalues E_n :

$$\frac{E_{n+1}}{E_n} \to e^{-\frac{2\pi}{s_0}}, \qquad \text{as } n \to +\infty \tag{1.1}$$

where the parameter $s_0 > 0$ depends only on the mass ratios and, possibly, on the statistics of the particles.

In this context short-range interaction means that the two-body potential $v(r)$ decays faster than $1/r^2$ as r goes to infinity so that it can support, at most, a finite number of bound states. Moreover, a crucial role for the occurrence of the effect is played by the zero-energy resonance condition for the two-body subsystems. From the physical point of view the condition means that the scattering length of the two-body problem is much larger than the effective range of the interaction. We recall that the scattering length a is defined by

$$a = \lim_{|k| \to 0} f(k, k') \tag{1.2}$$

where f is the scattering amplitude.

In mathematical terms the resonance condition is realized sending the scattering length to infinity and keeping the range of the interaction fixed. In particular if the interaction is resonant there are no two-body bound states, hence Efimov trimers are an example of *Borromean bindings*: three-body bound states such that the two-body subsystems are unbounded. Hence, as soon as one of the particle is removed also the others fall apart. The denomination comes from the Borromean rings (Figure 1.1): three interlocking circles organized in such a way that removing one of them the others two remain unlinked.

Another interesting and counterintuitive aspect related to the Efimov effect is that it disappears as long as the interaction is made more attractive because in this way the resonance is destroyed. Indeed, the presence of a zero-energy resonance implies that, even if the interactions are short range, an effective long range attraction appears. According to an intuitive physical picture one can say that in a trimer the attraction between two particles is mediated by the third one, which is moving back and forth between the two.

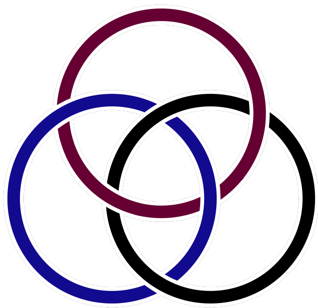

Figure 1.1: Borromean rings

As a consequence of its universality character, the effect can be realized and observed in various physical contexts. It was originally predicted for nuclear systems but, unfortunately, in that context it was very difficult to obtain experimental evidences also due to the obstruction represented by Coulomb interactions. As a result, Efimov's prediction was for a long time considered as a theoretical pathology. The first convincing experimental confirmations has been obtained only forty years later starting from 2006 in the field of ultra-cold quantum gases (we refer the reader to the reviews [15, 57] and references therein). The main advantage in that context is the unique possibility of a precise tuning of the scattering length via the so-called *Feshbach resonance*.

The original Efimov's physical argument is based on the replacement of the two-body potential with a boundary condition, which is essentially equivalent to consider a two-body zero-range interaction (defined below), and on the introduction of hyper-spherical coordinates. If the resonance condition is satisfied, in these coordinates the problem becomes separable and in the equation for the hyper-radius R the long range, attractive effective potential $-(s_0^2+1/4)/R^2$ appears. For small R the behavior R^{-2} is too singular and an extra boundary condition at short distance must be imposed to restore self-adjointness. On the other hand, for large R the decay rate of this effective potential guarantees the existence of an infinite number of eigenvalues.

From a mathematical point of view the first rigorous proof was given by Yafaev in 1974 in [72]. He made rigorous the line of reasoning of [6] studying a symmetrized form of the Faddeev equations for a system of three different particles such that at least two of the two-body subsystems are resonant (i. e., with infinite scattering length) and proving the existence of an infinite number of negative eigenvalues. Later Sobolev (see [67]) used the same techniques to prove the following asymptotic behavior for the number $N(z)$ of eigenvalues smaller than $z<0$:

$$\lim_{z\to 0^-}\frac{N(z)}{|\log|z||}=\frac{s_0}{2\pi}. \tag{1.3}$$

Notice that (1.3) is in agreement with the geometrical law (1.1). In the same year Tamura ([69]) obtained the same result under more general assumptions on the two-body potentials. Other proofs of the effect adopt a variational approach based on a Born-Oppenheiner approximation. In particular we recall the work by Ovchinnikov and Sigal ([58]) where a system of two heavy particles and a light one is considered. See also the physics paper [43] where the case of an explicit separable potential is considered in the case of two heavy and one light particle and the paper by Tamura [68] where the method is generalized to a system of generic masses. For more results on the subject see also [71], [44] and [45].

It is worth mentioning that a mathematical proof of the geometric asymptotic law (1.1) is still lacking (see the conjecture discussed in [4]).

The technique used by [72] can be easily adapted to a system of three identical bosons while for three identical fermions the resonance condition as defined above cannot occur (see Remarks 4.1 and 4.2). On the other hand, resonance can occur looking at the interaction of a fermions and a different particle, for instance a fermion of different specie. In physical applications the case of a mixture of N identical fermions of one species and M identical fermions of another species is particularly relevant and this put in a new perspective the question of presence or absence of the Efimov effect in a system composed of two identical fermions and a different particle.

In this thesis we address this question (see also [9]). We consider the case of a three-particle system composed of two identical fermions of unitary masses and a third particle of mass m assuming the resonance condition on the interactions between one of the fermions and the third particle. We use the same approach as in [67, 72] with the modifications required to take into account the characteristic features of our system. Some peculiar aspects appear in our case. First we find the existence of a threshold m_* on the mass ratio: Efimov effect occurs only for $m < m_*$ while for $m > m_*$ the Hamiltonian has a finite number of eigenvalues. Moreover, considering a partial wave decomposition, the effect takes place only in the subspaces corresponding to the *odd-waves* (contrary to the case of identical bosons or distinguishable particles where the effect appears in the *s-wave* subspace). It is worth mentioning that our result is in agreement with physical results on the subject (see [15, 57] and references therein).

Another interesting regime is obtained keeping the scattering length as fixed and sending to zero the range of the interaction. In this way one obtains the so-called *point interactions*, also known as *contact* or *zero-range* interactions. Quantum mechanical Hamiltonians with zero-range interactions have been introduced around the middle of the last century to describe short-range nuclear forces at low-energy starting with the works of Bethe and Peierls [13], Thomas [70] and, later, Fermi [37]. More recently, interesting applications have been developed also in the physics of ultra-cold atoms, in particular in the unitary limit (see, e. g., [15, 22, 23]). Roughly speaking, the unitary limit is characterized by two-body interactions with zero range and an infinite scattering length.

An N-particle system in $\mathbb{R}^d$, $d = 1, 2, 3$, with zero-range two-body interactions is described by the formal Hamiltonian

$$\mathscr{H}_{\nu}^{N\,\mathrm{body},d} = -\sum_{i=1}^{N} \frac{1}{2m_i}\Delta_{x_i} + \sum_{\substack{i,j=1\\ i<j}}^{N} \nu_{ij}\, \delta(x_i - x_j), \tag{1.4}$$

where $x_i \in \mathbb{R}^d$, $i = 1, \dots, 3$, is the coordinate of the i-th particle, m_i is the corresponding mass, Δ_{x_i} is the Laplacian relative to x_i, and $\nu_{ij} \in \mathbb{R}$ is the strength of the interaction between particles i and j. To simplify the notation we set $\hbar = 1$. The correct definition of the Hamiltonian and the analysis of the stability problem, i.e., the existence of a finite lower bound for the Hamiltonian, have been widely studied both in the physical [15, 19–23, 26, 46] and in the mathematical [3, 5, 24, 25, 27, 36, 42, 50–53, 61] literature.

From a mathematical point of view the first step is to give a rigorous definition of the formal Hamiltonian (1.4) as a self-adjoint operator. The case $N = 2$ reduces, removing the center of mass motion, to the well known case (see [2]) of a single particle interacting with a point interaction placed at the origin. Let us focus on three-particle systems.

In the one dimensional case the problem can be treated using the fact that the interaction term is a small perturbation, in the quadratic form sense, of the free Hamiltonian.

Using this fact, in this thesis we provide a rigorous definition of the Hamiltonian as a self-adjoint operator bounded from below. Moreover, we obtain this Hamiltonian as the limit of Hamiltonians with suitably rescaled interactions potentials. These results will appear in [7].

In dimension two a natural class of Hamiltonians with local zero-range interactions was constructed in [27] and it was also shown that such Hamiltonians are all bounded from below.

In dimension three the analysis is more delicate. A strategy to circumvent the difficulty is to construct a class of operators by analogy with the two-particle case, the so-called *STM extensions* named after Ter-Martirosyan and Skornyakov ([66]). The domains of these operators turn out to be composed by functions with the expected asymptotic behavior prescribed by physical heuristics when two particles come into contact. Unfortunately, in general they do not define a self-adjoint operator. Indeed, for a system of three identical bosons it was shown in [36] that the STM operators are not self-adjoint and all their self-adjoint extensions are unbounded from below owing to the presence of an infinite discrete sequence of bound states whose energy diverge to minus infinity. In [48] this result was generalized to the case of three distinguishable particles with different masses. This kind of instability is known in the literature as the *Thomas effect*.

It should be stressed that the Thomas and Efimov effects are strongly related (we refer the reader to [1, 15, 57] and references therein). On the other hand it is worth noticing that, to our knowledge, a rigorous mathematical investigation of this connection is still lacking (see the conjecture in [4]).

The Thomas effect can be avoided if the Hilbert space is suitably restricted introducing symmetry constraints on the wave function.

Following this line of thought, in this work we prove that, excluding the *s-wave* space, a system of three identical bosons with point interactions is stable (see also [10]). A second possibility is to consider antisymmetry constraint. In fact, a wave function that is antisymmetric under exchange of coordinates of two particles necessarily vanishes at the coincidence points of such two particles, thus making their mutual zero-range interaction ineffective. Therefore, it is reasonable to expect that in a system of two identical fermions plus a different particle the interaction term in the Hamiltonian is less singular, thus making the system stable. Indeed, it has been shown that this is in fact the case for suitable values of the mass ratio (see, e. g., [24, 25, 52, 53, 55]). Remarkably, the optimal mass ratio found in [24] which distinguish between the stable and the unstable case for a system composed by two identical fermions and a different particle is exactly the same obtained in our work [9] characterizing the occurrence of the Efimov effect. This fact enlighten once more the strict connection between the two phenomena.

Finally we discuss a third example of universal behavior in few-particle quantum systems at low energy. Consider a non relativistic quantum particle, whose position is denoted by

$x \in \mathbb{R}^3$, interacting with N obstacles. Suppose that the interaction with the i-th obstacle, placed at y_i, is described by the Gross Pitaevskii potential

$$V_i^N(x) = N^2 V(N(x - y_i)), \tag{1.5}$$

where the unscaled potential V decays to zero at infinity sufficiently fast. Therefore, the Hamiltonian of the particle is

$$H_N = -\Delta + \sum_{i=1}^{N} V_i^N. \tag{1.6}$$

Here we have chosen units such that $\hbar = 1$ and the mass of the particle is $1/2$. Moreover suppose that y_i, $i = 1, \ldots, N$ are independent and identically distributed random variables with density given by the function $W(x)$. We are interested in the limit $N \to \infty$.

We note that for N large the range r_0 of the potential V_i^N is of order N^{-1} while the average distance d among the obstacles is of order $N^{-1/3}$. If the wavelength of the particle λ_p is taken of order 1, we are studying the regime

$$r_0 \ll d \ll \lambda_p,$$

which is the case occurring, for example, in the analysis of scattering of slow neutrons from condensed matter (Neutron Optics). Hence, we are once more considering a low energy limit and we expect to find a universal behavior.

Indeed, in this thesis we prove the convergence of H_N to the limiting Hamiltonian

$$H = -\Delta + 4\pi a W, \tag{1.7}$$

where the only dependence on the interaction potential V is through its scattering length a. More precisely we prove convergence in the strong resolvent sense in probability with respect to the product probability measure $\{W(x)dx\}^{\otimes N}$. Moreover we give the rate of the convergence and characterize the fluctuations (see also [8]).

Remarkably, this result is in accordance with the results known for the more difficult many-body problem of n particle interacting via Gross-Pitaevskii potentials and undergoing Bose-Einstein condensation. Here the limiting behavior is described by the Gross-Pitaevskii equation where again the scattering length appears (see, e. g., [11, 18, 31–33, 62]). On the other hand our simpler Lorentz gas model allows a more explicit analysis of the convergence and of the fluctuations.

The thesis is organized as follows.

Part I is devoted to the study of the occurrence of the Efimov Effect in a system of two identical fermions and a different particle.

In Part II we study three-particle systems with point interactions in dimension one and three. In the one dimensional case we present the results on the rigorous definition of the Hamiltonian and its approximation. As for the three dimensional case, the construction of the quadratic form and the result on stability for three identical bosons outside the *s-wave* is presented together with a short review on the known results in the fermionic case.

In Part III we analyze the Lorentz quantum gas in three dimensions with Gross-Pitaeskii potentials and we characterize the limiting behavior and the fluctuations.

Part I

EFIMOV EFFECT

2 INTRODUCTION

It is well known that a two-particle quantum system in dimension three interacting via a potential vanishing at infinity faster than $1/|x|^2$ has only a finite number of eigenvalues.
One could expect that a similar result holds for a three-particle system. On the contrary, the situation is more involved and not merely dependent on the long range behavior of the potentials. Indeed, a remarkable physical phenomenon known as *Efimov effect* has been discovered by Vitaly Efimov in 1970 ([29], [28]).
Efimov considered particles interacting through short-range potentials such that the two-particle subsystems do not have bound states but exhibit a zero-energy resonance. Under this condition he proved that the three-particle system has an infinite number of eigenvalues accumulating at zero.

Here we present the result obtained in [9]. We study the case of a three-particle system in dimension three composed of two identical fermions with mass one and a different particle with mass m. In the physical literature such a system has been extensively studied (see, e.g., [20], [57] and references therein) and it is known that the Efimov effect can be present with some peculiar features. Indeed, the effect is present only for $m < m_* = (13.607)^{-1}$ and, considering a partial wave decomposition, it takes place only in the subspaces corresponding to the *odd waves* (contrary to the case of identical bosons or distinguishable particles where the effect takes place in *even waves*). Following the approach based on the analysis of the Faddeev equations, we give a mathematical proof of these facts. More precisely, we assume that: a) the two-body potentials are short range, rotationally invariant, non positive; b) the Hamiltonians of all the two-particle subsystems are positive; c) the Hamiltonian of the subsystems composed of the particle with mass m and a fermion has a zero-energy resonance. Then we prove that the Hamiltonian of the three-particle system has a finite number of negative eigenvalues for $m > m_*$ and an infinite number of negative eigenvalues accumulating at zero for $m < m_*$. We also prove the asymptotic behavior (1.3), where the constant at the right hand side, which in our case is denoted by $\mathcal{C}(m)$, depends only on the mass m and it is estimated from below and from above.

We note that, under our assumptions, the interaction potential between the two fermions does not produce zero-energy resonance and therefore it plays no role in the occurrence of the Efimov effect.

The method of the proof follows the line of reasoning of [67], with the modifications required to take into account of the peculiarity of our system. In particular, we formulate the eigenvalue problem $H\psi = z\psi$, $z < 0$, for the three-particle Hamiltonian in terms of symmetrized Faddeev equations $\Psi = \boldsymbol{A}(z)\Psi$, where $\boldsymbol{A}(z)$ is a 2×2 matrix (compact) operator, and we prove that $N(z) = n(1, \boldsymbol{A}(z))$, where the right hand side is the number of eigenvalues of $\boldsymbol{A}(z)$ larger than one. Then the problem is reduced to the study of the asymptotic of $n(1, \boldsymbol{A}(z))$ for $z \to 0^-$. The presence of the zero-energy resonance for the subsystems composed of the particle with mass m and a fermion determines a singular behavior (and a lack of compactness) of $\boldsymbol{A}(z)$ for $z = 0$ and this is the reason for the possible divergence of $n(1, \boldsymbol{A}(z))$ for $z \to 0^-$. Through some successive steps, we single out such singular behavior neglecting operators which, for $z \leq 0$, are compact and continuous in z. At the end we find that the asymptotic for $z \to 0^-$ of $n(1, \boldsymbol{A}(z))$ reduces to the asymptotic for $R \to \infty$ of an operator S_R which has an explicit form. By a direct analysis of such an operator, we conclude the proof of the main result in the two cases $m < m_*$ and $m > m_*$.

3 NOTATION AND MAIN RESULT

We consider a quantum system in dimension $d = 3$ composed by two identical fermions of unitary mass and a different particle of mass $m_3 = m$. Let $x_1, x_2 \in \mathbb{R}^3$ denote the coordinates of the fermions and $x_3 \in \mathbb{R}^3$ the coordinate of the third particle. The Hilbert space is then $L^2_{\text{asym}}(\mathbb{R}^9) = \{\psi \in L^2(\mathbb{R}^9) \text{ s. t. } \psi(x_2, x_1, x_3) = -\psi(x_1, x_2, x_3)\}$ and due to the symmetry constraint, we suppose the pairwise interactions v_α to be spherically symmetric and $v_{23} = v_{31} := v$.

Proceeding as in Appendix A we reduce our analysis to the following Hamiltonian in momentum space in the Jacobi coordinates (k_α, p_α)

$$H = H_0 + \sum_{\alpha'} V_{\alpha'} = \frac{k_\alpha^2}{2\mu_\alpha} + \frac{p_\alpha^2}{2n_\alpha} + \sum_{\alpha'} V_{\alpha'}$$

where the operators V_α and the new masses μ_α, n_α have been defined in (A.4) and (A.2) respectively. In particular under our assumptions

$$\mu_{12} = \frac{1}{2} \qquad \mu_{23} = \mu_{31} = \mu = \frac{m}{m+1}$$
$$n_{12} = \frac{2m}{m+2} \qquad n_{23} = n_{31} = n = \frac{m+1}{m+2}.$$

Taking into account of the definitions given in (A.3), the symmetry constraint reduces to $\psi(k_{23}, p_{23}) = -\psi(-k_{31}, p_{31})$ or equivalently $\psi(k_{12}, p_{12}) = -\psi(-k_{12}, p_{12})$. Therefore, choosing for instance the coordinates (k_{23}, p_{23}), the Hilbert space of the system is

$$\mathcal{H}^{\text{asym}} = \left\{ \psi \in L^2(\mathbb{R}^6) \,|\, \psi(k_{23}, p_{23}) = -\psi\left(\frac{k_{23}}{m+1} - \frac{m(m+2)}{(m+1)^2} p_{23}, -k_{23} - \frac{p_{23}}{m+1} \right) \right\}. \tag{3.1}$$

As pointed out in Appendix A we can adopt as coordinates any pair $(p_\alpha, p_{\alpha'})$, in particular we will often use in the study of this system the coordinates (p_{23}, p_{31}). Equation (A.6) yields

$$H_0(p_{23}, p_{31}) = \frac{p_{23}^2}{2\mu} + \frac{p_{31}^2}{2\mu} + \frac{p_{23} \cdot p_{31}}{m}. \tag{3.2}$$

By definition the exchange of the fermionic coordinates corresponds to the exchange of p_{23} and p_{31}, thus, using the coordinates (p_{23}, p_{31}), the symmetry condition becomes $\psi(p_{23}, p_{31}) = -\psi(p_{31}, p_{23})$. For notational convenience we describe the symmetry by an operator P acting on $L^2(\mathbb{R}^6)$ so that $\psi \in \mathcal{H}^{\text{asym}}$ if and only if $P\psi = \psi$. In the coordinates (k_{23}, p_{23}) P acts as

$$(P\psi)(k_{23}, p_{23}) = -\psi\left(\frac{k_{23}}{m+1} - \frac{m(m+2)}{(m+1)^2} p_{23}, -k_{23} - \frac{p_{23}}{m+1} \right)$$

while in the coordinates (p_{23}, p_{31}) and (k_{12}, p_{12}) we have

$$(P\psi)(p_{23}, p_{31}) = -\psi(p_{31}, p_{23}) \qquad (P\psi)(k_{12}, p_{12}) = -\psi(-k_{12}, p_{12}).$$

Equations (3.2) immediately yields that P commutes with H_0. Furthermore by the spherical symmetry of the potentials it follows that P also commutes with V_{12} and it satisfies $PV_{31} =$

$V_{23}P$. Indeed by Equation (A.4) and spherical symmetry of v_{12}, one obtains $PV_{12} = V_{12}P$. While using Equation (A.4), (A.5) and the spherical symmetry of v we can write

$$(V_{23}\psi)(p_{23}, p_{31}) = \frac{1}{(2\pi)^{3/2}} \int dp\, \widehat{v}(p - p_{31})\psi(p_{23}, p)$$
$$(V_{31}\psi)(p_{23}, p_{31}) = \frac{1}{(2\pi)^{3/2}} \int dp\, \widehat{v}(p - p_{23})\psi(p, p_{31})$$

which implies $PV_{23} = V_{31}P$ and $PV_{31} = V_{23}P$.

Let us introduce the two-particle subsystems of our three-particle system, described by the following Hamiltonians in $L^2(\mathbb{R}^3)$

$$h_\alpha = -\frac{1}{2\mu_\alpha}\Delta_x + v_\alpha(x). \tag{3.3}$$

In order to formulate our main result we introduce below our assumptions on the two-body potentials, on the zero-energy properties of the two-particle subsystems and on the mass ratio.

Concerning the two-body potentials v_α and the Hamiltonians h_α, we assume that the following conditions hold for any pair α

(A_1) $|v_\alpha(x)| \leq \dfrac{C}{(1+|x|)^b}$, with $b > 3$

(A_2) $v_\alpha(x) = v_\alpha(|x|)$

(A_3) $v_\alpha \leq 0$

(A_4) $h_\alpha \geq 0$

A further important assumption is the presence of a zero-energy resonance for the two-particle Hamiltonian $h_{23} = h_{31}$. Here we recall the definition of a zero-energy resonance, while further comments and some useful results will be given in the next section.

Let us consider a Hamiltonian in $L^2(\mathbb{R}^3)$

$$h = -\frac{1}{2\mathfrak{m}}\Delta_x + u(x) \tag{3.4}$$

with $\mathfrak{m} > 0$ and u a generic potential satisfying (A_1) and (A_4). We denote by g_0 the integral operator with kernel

$$g_0(x, x') = \frac{\mathfrak{m}}{2\pi|x - x'|}\,.$$

Definition 3.1. Zero is a (simple) resonance for h if there exists $\varphi \in L^2(\mathbb{R}^3)$ such that

$$\tilde{u} g_0 \mathfrak{u}\, \varphi = -\varphi, \qquad (\mathfrak{u}, \varphi) \neq 0, \tag{3.5}$$

where $\mathfrak{u}(x) = |u|^{1/2}(x)$ and $\tilde{u}(x) = \mathrm{sgn}(u(x))|u|^{1/2}(x)$.

Remark 3.1. We note that if zero is a resonance for h then the scattering length a, defined by Equation (1.2), is infinite. The scattering amplitude $f(k, k')$ is given by

$$f(k, k') = -\frac{\mathfrak{m}}{2\pi} \int dx\, e^{-ik\cdot x}\, u(x)\xi(x, k')|_{|k|=|k'|} \tag{3.6}$$

where ξ is the solution of the Lippman-Schwinger equation:

$$\xi(x,k') = e^{ik'\cdot x} - \frac{\mathfrak{m}}{2\pi}\int dy\, \frac{e^{-i|k||x-y|}}{|x-y|}u(y)\xi(y,k').$$

The equation for ξ yields

$$\tilde{u}\xi(\cdot,k') = (1+b_{k'})^{-1}\tilde{u}\varepsilon_{k'} \tag{3.7}$$

where we set $\varepsilon_{k'}(x) = e^{ik'\cdot x}$ and $b_{k'}$ denotes the operator acting as

$$b_{k'}\phi(x) = \frac{\mathfrak{m}}{2\pi}\int dy\, \tilde{u}(x)\frac{e^{-i|k'||x-y|}}{|x-y|}\mathrm{u}(y)\phi(y).$$

Inserting now Equation (3.7) in Equation (3.6) we get

$$f(k,k') = \frac{\mathfrak{m}}{2\pi}\int dx\, e^{-ikx}\,\mathrm{u}(x)\left[(1+b_{k'})^{-1}\tilde{u}\varepsilon_{k'}\right](x).$$

Using now Lemma 1.2.4 (case II) in [2] we get that if h has a resonance at zero energy then

$$(1+b_{k'})^{-1} \simeq -\frac{2\pi}{i|k'|\,|(\mathrm{u},\varphi)|^2\,\mathfrak{m}}(\mathrm{sgn}(u)\varphi,\cdot)\varphi \quad \text{as } |k'|\to 0$$

which implies

$$f(k,k') \simeq \frac{(\varphi,\mathrm{u}\,\varepsilon_{k'})}{i|k'||(\mathrm{u},\varphi)|^2}\overline{(\varphi,\mathrm{u}\,\varepsilon_k)}.$$

From the above equation, recalling Equation (1.2), we immediately get $a=+\infty$.
Let us notice that if on the contrary zero is neither an eigenvalue nor a resonance for h then $(1+b_{k'})^{-1}$ converges to $(1-\tilde{u}g_0\mathrm{u})^{-1}$ which is a bounded operator. Hence in this case one has

$$a = \frac{1}{4\pi}\int dx\, \mathrm{u}(x)[(1+\tilde{u}g_0\mathrm{u})^{-1}\tilde{u}](x) = \frac{1}{4\pi}(\mathrm{u},(1+\tilde{u}g_0\mathrm{u})^{-1}\tilde{u}).$$

In particular a is finite and can be equivalently rewritten as $a=\frac{1}{4\pi}(u,\phi_0)$ where ϕ_0 is the solution of $(-\frac{1}{2\mathfrak{m}}\Delta_x + u)\phi_0 = 0$.

Let us comment on the above assumptions. We remark that if each v_α satisfies condition (A_1) then, via Kato-Rellich Theorem (see, for instance, [64]), we have self-adjointness and lower boundedness of H on the same domain of H_0.

Moreover, using the HVZ Theorem (see [64]), we know that the essential spectrum of H is of the form $[l,+\infty)$, where l is the lowest point of the spectra of the operators h_α describing the two-particle subsystems. Thus the assumption (A_4) implies that on the left of zero the spectrum of H consists of isolated eigenvalues with finite multiplicities.

The assumptions (A_2), (A_3) are introduced to simplify the analysis. Note that under the assumption (A_3) the first equation in (3.5) reduces to

$$|u|^{1/2}g_0|u|^{1/2}\varphi = \varphi.$$

Moreover (A_2) prevents the presence of zero-energy resonance in the subsystem composed by the identical fermions (see Remark 4.2 at the end of the next section).

As we already pointed out in the introduction, a peculiar aspect of the Efimov effect in our fermionic system is that it only occurs for a certain range of values of the mass ratio $(0,m_*)$, which can be defined as follows. Let $\Lambda(m)$ be the following function of the mass of the third particle

$$\Lambda(m) = \frac{2(m+1)^2}{\pi}\left(\frac{1}{\sqrt{m(m+2)}} - \arcsin\left(\frac{1}{m+1}\right)\right). \tag{3.8}$$

It is easy to check that $\Lambda(m)$ is decreasing and

$$\lim_{m\to 0}\Lambda(m) = +\infty, \qquad \lim_{m\to+\infty}\Lambda(m) = 0$$

Thus the following definition makes sense

Definition 3.2. The critical mass m_* is the unique solution of the equation $\Lambda(m) = 1$.

Note that for $m < m_*$ ($m > m_*$) we have $\Lambda(m) > 1$ ($\Lambda(m) < 1$). Moreover, we stress that m_* is the same mass threshold obtained in the study of the corresponding system with point interactions, see Section 9.3 (for more detailed results we refer the reader to [42], [24], [53]).

We are now ready to formulate our main result.

Theorem 3.1. *Let us assume that conditions (A_1)–(A_4) hold for any pair α and that the two-particle Hamiltonian $h_{23} = h_{31}$ has a zero-energy resonance. Then the following holds.*

i) For $m < m_*$ there exists a positive constant $\mathcal{C}(m)$ such that

$$\lim_{z\to 0^-}\frac{N(z)}{|\log|z||} = \mathcal{C}(m). \tag{3.9}$$

Moreover, $\mathcal{C}(m)$ satisfies

$$\mathcal{C}_1(m) \le \mathcal{C}(m) \le \mathcal{C}_2(m) \tag{3.10}$$

where $\mathcal{C}_1(m)$ is the unique positive solution of the equation $F_m(x) = 1$, with

$$F_m(x) := \frac{m+1}{\sqrt{m(m+2)}}\int_0^1 dy\, y\frac{\sinh\left(\frac{2\pi}{3}x\arcsin\left(\frac{y}{m+1}\right)\right)}{\sinh\left(\frac{\pi^2}{3}x\right)\cos\left(\arcsin\left(\frac{y}{m+1}\right)\right)},$$

and

$$\mathcal{C}_2(m) = \frac{1}{4\pi\beta(m)}\log\left(\sqrt{\frac{2\pi}{3}}\alpha(m)\right)\left(l_0^2(m) + 3l_0(m) + 2\right) \tag{3.11}$$

with

$$\alpha(m) := \frac{(m+1)^{3/2}}{2\sqrt{\pi}\sqrt{m(m+2)}}\log^{1/2}\left(1+\frac{2}{m}\right), \quad \beta(m) := \frac{\pi}{2} - \arcsin\left(\frac{1}{m+1}\right) \tag{3.12}$$

and $l_0(m)$ is the largest odd integer smaller than $\pi\alpha(m)^2 - 1/2$.

ii) For $m > m_*$ the number of negative eigenvalues of H is finite.

4 PRELIMINARY RESULTS

4.1 TWO-PARTICLE SUBSYSTEMS

Here we recall some properties of the two-body Hamiltonian operator h which expression in the position space is given by (3.4), under the hypothesis that the assumptions (A_1)–(A_4) are satisfied. In particular, we are interested in the low energy behavior connected with the presence of zero-energy resonance (see Definition 3.1).
We first observe that if φ is a square integrable solution of $|u|^{1/2}g_0|u|^{1/2}\varphi = \varphi$ then $\psi = g_0|u|^{1/2}\varphi$ satisfies $h\psi = 0$ in the sense of distributions. Furthermore, ψ is an eigenfunction of h with eigenvalue zero if and only if $(|u|^{1/2}, \varphi) = 0$ (see, e.g., Section 1 in [69]). On the other hand, if zero is a resonance for h then there exists ψ solution of $h\psi = 0$ in the sense of distribution with $\psi \in L^2_{\text{loc}}(\mathbb{R}^3)$ but $\psi \notin L^2(\mathbb{R}^3)$.

Remark 4.1. The spherical symmetry of the potential u implies that zero-energy resonance can only occur in *s-wave* subspace. Indeed, if φ belongs to the subspace with angular momentum $l \geq 1$ then obviously $(|u|^{1/2}, \varphi) = 0$.

Let us introduce the resolvent of h

$$r(z) = (h - z)^{-1}$$

which is a bounded operator in $L^2(\mathbb{R}^3)$ for any $z < 0$ due to condition (A_4), the free resolvent

$$r_0(z) = \Big(-\frac{1}{2\mathrm{m}}\Delta - z\Big)^{-1}$$

(note that $r(0) = g_0$) and the operator

$$w(z) = I + |u|^{1/2}r(z)|u|^{1/2}.$$

Using the resolvent identity $r(z) = r_0(z) - r(z)\,u\,r_0(z)$ one verifies that

$$w(z) = (I - |u|^{1/2}r_0(z)|u|^{1/2})^{-1}$$

The following lemma describes the behavior of the operator $w(z)$ if 1 is not an eigenvalue of $|u|^{1/2}g_0|u|^{1/2}$.

Lemma 4.1. *Let us suppose that* 1 *is not an eigenvalue of the operator* $|u|^{1/2}g_0|u|^{1/2}$. *Then* $w(z)$ *is continuous in* $z \leq 0$.

Let us assume that h has a zero-energy resonance. We fix the following normalization condition on the eigenvector φ

$$(|u|^{1/2}, \varphi) = (2\pi)^{3/2}(\widehat{\varphi|u|^{1/2}})(0) = 2^{1/4}\pi^{1/2}\mathrm{m}^{-3/4} \tag{4.1}$$

and we characterize the behavior of the operator $w(z)$ when $z \to 0$.

Lemma 4.2. *Suppose that h has a zero-energy resonance. If $z < 0$ is small enough and $\delta < min\{1, b-3\}$ then*

$$w(z) = \frac{(\cdot,\varphi)\varphi}{|z|^{1/2}} + |z|^{-\frac{1-\delta}{2}} w^{(\delta)}(z)$$

where the operator $w^{(\delta)}(z)$ is continuous in $z \leq 0$. Moreover

$$(w(z))^{1/2} = \frac{(\cdot,\varphi)\varphi}{||\varphi|||z|^{1/4}} + |z|^{-\frac{1-\delta}{4}} \tilde{w}^{(\delta)}(z)$$

where the operator $\tilde{w}^{(\delta)}(z)$ is continuous in $z \leq 0$.

For the proof of the previous lemmas we refer the reader to [67].

Remark 4.2. Let us consider the Hamiltonians of the two-body subsystems defined in (3.3). We note that h_{12}, due to the symmetry constraint, acts on the space

$$L^2_{\text{asym}}(\mathbb{R}^3) = \{\psi \in L^2(\mathbb{R}^3) \,|\, \psi(-x) = -\psi(x)\}.$$

Hence, by Remark 4.1, it cannot exhibit a zero-energy resonance because its domain does not include s-wave functions. Moreover, using the assumptions (A_2) and (A_4), h_{12} cannot have zero as an eigenvalue (see e. g.[73]). Hence we conclude that 1 is not an eigenvalue for the operator $|v_{12}|^{1/2} g_0 |v_{12}|^{1/2}$. Then we can apply Lemma 4.1 and, as we will see, this implies that the potential between the two fermions does not play any role in the occurrence of the Efimov effect. On the other hand, as we have already stressed, the presence of a zero-energy resonance for $h_{23} = h_{31}$ is a crucial ingredient of the proof.

4.2 FADDEEV EQUATIONS FOR THE EIGENVALUES

In the derivation of the Faddeev equation for the eigenvalues given in Appendix A we have not assumed any symmetry property of our system and therefore it is valid for a generic three-particle system. In order to take into account of the fermionic symmetry we proceed as follows.

Recalling the definition of the operator P and its properties (see Chapter 3) and the definition of η_α given in (A.9) we obtain $P\eta_{12} = \eta_{12}$, $\eta_{31} = P\eta_{23}$ and the system (A.10) reduces to

$$\begin{cases} \eta_{23} = -R_{23}(z)V_{23}(P\eta_{23} + \eta_{12}) \\ \\ \eta_{12} = -R_{12}(z)V_{12}(I + P)\eta_{23} \end{cases} \tag{4.2}$$

where $\eta_{23} \in L^2(\mathbb{R}^6)$ and $\eta_{12} \in \mathcal{H}$. Consequently, in our fermionic system the solution of the eigenvalue equation (A.7) reads $\psi = \eta_{12} + (I + P)\,\eta_{23}$.

In this paper we find convenient to use a symmetrized form of the Faddeev equations similar to the one used in [67]. In order to derive such equations, we first introduce the following bounded and positive operators on $L^2(\mathbb{R}^6)$ for $z < 0$

$$W_\alpha(z) = I + |V_\alpha|^{1/2} R_\alpha(z) |V_\alpha|^{1/2}.$$

Using the resolvent identity $R_\alpha(z) = R_0(z) - R_0(z)V_\alpha R_\alpha(z)$, we find

$$W_\alpha(z) = \left(I - |V_\alpha|^{1/2} R_0(z) |V_\alpha|^{1/2}\right)^{-1} \tag{4.3}$$

Moreover, we define the resolvent of h_α

$$r_\alpha(z) = (h_\alpha - z)^{-1}$$

which is a bounded operator in $L^2(\mathbb{R}^3)$ for any $z < 0$, the bounded and positive operator

$$w_\alpha(z) = I + |v_\alpha|^{1/2} r_\alpha(z) |v_\alpha|^{1/2}$$

and the Fourier transform $\mathcal{F}_\alpha$ with respect to x_α

$$(\mathcal{F}_\alpha f)(k_\alpha, p_\alpha) = \frac{1}{(2\pi)^{3/2}} \int dx_\alpha \, e^{-ik_\alpha \cdot x_\alpha} \, f(x_\alpha, p_\alpha).$$

Then one verifies that

$$W_\alpha(z) = \mathcal{F}_\alpha w_\alpha \left(z - \frac{p_\alpha^2}{2n_\alpha}\right) \mathcal{F}_\alpha^*. \tag{4.4}$$

Let us reconsider the first equation in (4.2). Using the resolvent identity and taking into account that $V_\alpha = -|V_\alpha|$, we have

$$\begin{aligned} \eta_{23} &= -\big(R_0(z) - R_0(z)V_{23}R_{23}(z)\big)V_{23}(P\eta_{23} + \eta_{12}) \\ &= \big(R_0(z) + R_0(z)|V_{23}|R_{23}(z)\big)|V_{23}|(P\eta_{23} + \eta_{12}) \\ &= R_0(z)|V_{23}|^{1/2} W_{23}(z)^{1/2} W_{23}(z)^{1/2} |V_{23}|^{1/2} (P\eta_{23} + \eta_{12}) \end{aligned} \tag{4.5}$$

Analogously, the second equation in (4.2) can be rewritten as

$$\eta_{12} = R_0(z)|V_{12}|^{1/2} W_{12}(z)^{1/2} W_{12}(z)^{1/2} |V_{12}|^{1/2} (I + P)\eta_{23}. \tag{4.6}$$

Now we apply P to both sides of (4.5) and sum it with (4.6). On the resulting equation we apply $W_{23}^{1/2}|V_{23}|^{1/2}$. Denoting

$$\begin{aligned}\psi_{23} &= W_{23}(z)^{1/2}|V_{23}|^{1/2}(P\eta_{23}+\eta_{12})\\ \psi_{12} &= W_{12}(z)^{1/2}|V_{12}|^{1/2}(I+P)\eta_{23}\end{aligned}$$

we find

$$\psi_{23} = W_{23}(z)^{1/2}|V_{23}|^{1/2}R_0(z)\left(P|V_{23}|^{1/2}W_{23}(z)^{1/2}\psi_{23} + |V_{12}|^{1/2}W_{12}(z)^{1/2}\psi_{12}\right).$$

On the other hand, applying $W_{12}(z)^{1/2}|V_{12}|^{1/2}(I+P)$ to both sides of (4.5) we find

$$\psi_{12} = W_{12}(z)^{1/2}|V_{12}|^{1/2}R_0(z)(I+P)|V_{23}|^{1/2}W_{23}(z)^{1/2}\psi_{23}.$$

Hence we have the following symmetrized form of the Faddeev equations for our model of two identical fermions and a different particle

$$\Psi = \boldsymbol{A}(z)\Psi$$

where $\Psi = (\psi_{23}, \psi_{12})$ and $\boldsymbol{A}(z)$ is a 2×2 matrix operator acting on the space

$$\mathcal{K} = L^2(\mathbb{R}^6)\times\mathcal{H}^{\text{asym}}$$

defined by

$$\boldsymbol{A}(z) = \boldsymbol{W}(z)^{1/2}\boldsymbol{U}(z)\boldsymbol{W}(z)^{1/2},$$

with

$$\boldsymbol{W}(z)^{1/2} = \text{diag}\{W_{23}(z)^{1/2}, W_{12}(z)^{1/2}\}$$

and

$$\boldsymbol{U}(z) = \begin{pmatrix} |V_{23}|^{1/2}R_0(z)P|V_{23}|^{1/2} & |V_{23}|^{1/2}R_0(z)|V_{12}|^{1/2} \\ & \\ |V_{12}|^{1/2}R_0(z)(I+P)|V_{23}|^{1/2} & 0 \end{pmatrix}.$$

The advantage of such symmetrized form of the equations is the fact that $\boldsymbol{A}(z)$ is compact for $z<0$.

Theorem 4.1. *For $z<0$ the operator $\boldsymbol{A}(z)$ is compact and it is continuous in z.*

The proof of Theorem 4.1 goes exactly as that of Theorem 4.1 in [67] and it is omitted.

It turns out that the number of eigenvalues of H smaller than $z<0$ equals the number of eigenvalues of $\boldsymbol{A}(z)$ larger than 1. In order to prove this fact it is useful to introduce the following definition.

Definition 4.1. Let B be a self-adjoint operator on the Hilbert space $\mathfrak{h}$ and let $\lambda\in\mathbb{R}$. We set

$$n(\lambda, B) = \sup_{\mathfrak{h}_B(\lambda)} \dim \mathfrak{h}_B(\lambda)$$

where $\mathfrak{h}_B(\lambda)$ denotes any subspace of $\mathcal{D}(B)$ such that if f belongs to $\mathfrak{h}_B(\lambda)$ then $(Bf,f) > \lambda||f||^2$.

We stress that if the spectrum of the operator B on the right of λ is purely discrete then $n(\lambda, B)$ coincides with the number of eigenvalues (with multiplicities) on the right of λ. Thus in particular

$$N(z) = n(-z, -H).$$

Due to the compactness of $\boldsymbol{A}(z)$ stated in Theorem 4.1, we also have that $n(1, \boldsymbol{A}(z))$ equals the number of eigenvalues of $\boldsymbol{A}(z)$ larger than 1.

In the next Theorem we prove a "Birman-Schwinger Principle" for our three-particle system, which is crucial for our analysis.

Theorem 4.2. *For $z < 0$ we have*

$$N(z) = n(1, \boldsymbol{A}(z)).$$

Proof. We adapt the proof of Theorem 3.1 in [67] to our case. First we show that

$$N(z) = n\big(1, R_0(z)^{1/2}|V|R_0(z)^{1/2}\big) \tag{4.7}$$

where $|V| = |V_{23}| + P|V_{23}| + |V_{12}|$. Indeed, let $x \in \mathfrak{h}_{-H}(-z)$ then

$$((H_0 - z)x, x) < (|V|x, x).$$

Setting $y = (H_0 - z)^{1/2}x$ and using self-adjointness of $(H_0 - z)^{1/2}$ and $R_0(z)^{1/2}$ we have

$$(R_0(z)^{1/2}|V|R_0(z)^{1/2}y, y) > (y, y)$$

that is $y \in \mathfrak{h}_{R_0(z)^{1/2}|V|R_0(z)^{1/2}}(1)$. This implies $n(1, R_0(z)^{1/2}|V|R_0(z)^{1/2}) \geq N(z)$. Reversing the argument we get the opposite inequality.

Next we introduce the matrix operator $\boldsymbol{L}(z)$ on $\mathcal{H}^{\text{asym}} \otimes \mathcal{H}^{\text{asym}}$ defined by

$$\boldsymbol{L}(z) = \frac{1}{2}\begin{pmatrix} R_0(z)^{1/2}|V|R_0(z)^{1/2} & R_0(z)^{1/2}|V|R_0(z)^{1/2} \\ \\ R_0(z)^{1/2}|V|R_0(z)^{1/2} & R_0(z)^{1/2}|V|R_0(z)^{1/2} \end{pmatrix}$$

and we note that

$$n\big(1, R_0(z)^{1/2}|V|R_0(z)^{1/2}\big) = n(1, \boldsymbol{L}(z)). \tag{4.8}$$

The operator $\boldsymbol{L}(z)$ can be written as

$$\boldsymbol{L}(z) = \boldsymbol{S}(z)\boldsymbol{S}^*(z)$$

where $\boldsymbol{S}(z) : \mathcal{K} \to \mathcal{H}^{\text{asym}} \otimes \mathcal{H}^{\text{asym}}$ is defined by

$$\boldsymbol{S}(z) = \begin{pmatrix} \frac{1}{2}R_0(z)^{1/2}(I+T)|V_{23}|^{1/2} & \frac{1}{\sqrt{2}}R_0(z)^{1/2}|V_{12}|^{1/2} \\ \\ \frac{1}{2}R_0(z)^{1/2}(I+T)|V_{23}|^{1/2} & \frac{1}{\sqrt{2}}R_0(z)^{1/2}|V_{12}|^{1/2} \end{pmatrix}$$

and its adjoint $\boldsymbol{S}^*(z)$ is given by

$$\boldsymbol{S}^*(z) = \begin{pmatrix} |V_{23}|^{1/2}R_0(z)^{1/2} & |V_{23}|^{1/2}R_0(z)^{1/2} \\ \\ \frac{1}{\sqrt{2}}|V_{12}|^{1/2}R_0(z)^{1/2} & \frac{1}{\sqrt{2}}|V_{12}|^{1/2}R_0(z)^{1/2} \end{pmatrix}$$

where we have used $T = I$ on the space $\mathcal{H}^{\text{asym}}$. Since $n(\lambda, BB^*) = n(\lambda, B^*B)$ for any bounded operator B (see e. g., Lemma 4.2 in [67]), we have

$$n(1, \boldsymbol{L}(z)) = n(1, \boldsymbol{S}^*(z)\boldsymbol{S}(z)) \tag{4.9}$$

where $\boldsymbol{S}^*(z)\boldsymbol{S}(z)$ is an operator on $\mathcal{K}$ explicitly given by

$$\boldsymbol{S}^*(z)\boldsymbol{S}(z) = \begin{pmatrix} |V_{23}|^{1/2}R_0(z)(I+P)|V_{23}|^{1/2} & \sqrt{2}|V_{23}|^{1/2}R_0(z)|V_{12}|^{1/2} \\ \\ \frac{\sqrt{2}}{2}|V_{12}|^{1/2}R_0(z)(I+P)|V_{23}|^{1/2} & |V_{12}|^{1/2}R_0(z)|V_{12}|^{1/2} \end{pmatrix}.$$

Let us decompose the above operator as follows

$$\boldsymbol{S}^*(z)\boldsymbol{S}(z) = \boldsymbol{D}_1(z) + \boldsymbol{D}_2(z)$$

where

$$\boldsymbol{D}_1(z) = \begin{pmatrix} |V_{23}|^{1/2}R_0(z)|V_{23}|^{1/2} & 0 \\ \\ 0 & |V_{12}|^{1/2}R_0(z)|V_{12}|^{1/2} \end{pmatrix}$$

and

$$\boldsymbol{D}_2(z) = \begin{pmatrix} |V_{23}|^{1/2}R_0(z)T|V_{23}|^{1/2} & \sqrt{2}|V_{23}|^{1/2}R_0(z)|V_{12}|^{1/2} \\ \\ \frac{\sqrt{2}}{2}|V_{12}|^{1/2}R_0(z)(I+T)|V_{23}|^{1/2} & 0 \end{pmatrix}.$$

Moreover, let us define

$$\widetilde{\boldsymbol{A}}(z) = (I - \boldsymbol{D}_1(z))^{-1/2}\boldsymbol{D}_2(z)(I - \boldsymbol{D}_1(z))^{-1/2} \tag{4.10}$$

and note that, from the definition of W_α given in Equation (4.3), it follows $(I - \boldsymbol{D}_1(z))^{-1/2} = \boldsymbol{W}(z)^{1/2}$. Let us prove that

$$n(1, \boldsymbol{S}^*(z)\boldsymbol{S}(z)) := n(1, \boldsymbol{D}_1(z) + \boldsymbol{D}_2(z)) = n(1, \widetilde{\boldsymbol{A}}(z)). \tag{4.11}$$

Assume $x \in \mathfrak{h}_{\tilde{\boldsymbol{L}}(z)}(1)$, i.e.,

$$((\boldsymbol{D}_1(z) + \boldsymbol{D}_2(z))x, x) > (x, x)$$

then

$$(\boldsymbol{D}_2(z)x, x) > ((I - \boldsymbol{D}_1(z))x, x).$$

Defining $y = (I - \boldsymbol{D}_1(z))^{1/2}x$ and using (4.10) we get

$$(\widetilde{\boldsymbol{A}}(z)y, y) > (y, y)$$

which means $y \in \mathfrak{h}_{\widetilde{\boldsymbol{A}}(z)}(1)$. This proves $n(1, \boldsymbol{S}^*(z)\boldsymbol{S}(z)) \le n(1, \widetilde{\boldsymbol{A}}(z))$. To get the opposite inequality it is sufficient to reverse the argument.

We also note that for $z < 0$ the operator $\widetilde{\boldsymbol{A}}(z)$ is compact and it is continuous in z.

Finally, by a direct computation one verifies that

$$\widetilde{\boldsymbol{A}}(z) = \begin{pmatrix} 1 & 0 \\ 0 & \frac{\sqrt{2}}{2} \end{pmatrix} \boldsymbol{A}(z) \begin{pmatrix} 1 & 0 \\ 0 & \sqrt{2} \end{pmatrix}.$$

This implies that $\widetilde{A}(z)$ and $A(z)$ have the same eigenvalues and if $\widetilde{\Psi} = (\psi_{23}, \psi_{12})$ is an eigenfunction of $\widetilde{A}(z)$ then $\Psi = (\psi_{23}, \sqrt{2}\psi_{12})$ is an eigenfunction of $A(z)$ with the same eigenvalue.

$$n(1, \widetilde{A}(z)) = n(1, A(z)) \tag{4.12}$$

Taking into account of (4.7), (4.8), (4.9), (4.11), (4.12), we conclude the proof. □

We conclude this section describing the behavior of the operators $W_\alpha(z)$ when $z < 0$ is small.
Let us introduce the multiplication operator in $L^2(\mathbb{R}^6)$

$$(\Gamma_\alpha(z)f)(k_\alpha, p_\alpha) = \gamma\left(\frac{p_\alpha^2}{2n_\alpha} - z\right) f(k_\alpha, p_\alpha) \tag{4.13}$$

where $\gamma \in C^\infty(\mathbb{R}_+)$ is such that $\gamma(t) > 0$ for all t, $\gamma(t) = t$ if $t \leq 1$ and $\gamma(t) = 1$ if $t \geq 2$.

Moreover, for the resonant pair 23, we define the operator in $L^2(\mathbb{R}^6)$

$$(\Pi_{23}f)(k_{23}, p_{23}) = \frac{1}{\|\varphi\|}\hat{\varphi}(k_{23}) \int dk\, f(k, p_{23})\overline{\hat{\varphi}(k)} \tag{4.14}$$

where φ is the eigenfunction of $|v|^{1/2}g_0|v|^{1/2}$ with eigenvalue 1 (see Definition 3.1).

Using the relation (4.4) and Lemma 4.2, we find

$$W_{23}(z)^{1/2} = \Gamma_{23}(z)^{-1/4}\,\Pi_{23} + \Gamma_{23}(z)^{-\frac{1-\delta}{4}}\,\widetilde{W}_{23}^{(\delta)}(z) \tag{4.15}$$

where $\delta < \min\{1, b-3\}$ and $\widetilde{W}_{23}^{(\delta)}(z)$ is continuous in $z \leq 0$.
On the other hand, Lemma 4.1 implies that $W_{12}(z)^{1/2}$ is continuous in $z \leq 0$.

5 LEADING TERM OF $\mathbf{A}(z)$ FOR $z \to 0^-$

The proof of our main result expressed in Theorem 3.1 requires, via Theorem 4.2, an asymptotic analysis of $n(1, \boldsymbol{A}(z))$ for $z \to 0^-$. From Theorem 4.1, we know that for $z < 0$ the operator $\boldsymbol{A}(z)$ is compact but there is a lack of compactness for $z = 0$ and this is the reason why we find that $N(z)$ diverges for $z \to 0^-$. In this section we shall prove various intermediate results where, at each step, we single out the leading term of $\boldsymbol{A}(z))$ for $z \to 0^-$, neglecting operators which are compact and continuous in z for $z \le 0$. At the end, we shall obtain the following integral operator acting in $L^2((0,R) \times \mathbb{S}^2, dr \otimes d\Omega)$

$$(S_R f)(r,\omega) = \int_0^R d\rho \int_{\mathbb{S}^2} d\Omega(\zeta) S(r-\rho, \omega\cdot\zeta) f(\rho,\zeta) \tag{5.1}$$

where

$$S(x,y) = -b(m)\,\frac{1}{\cosh x + \frac{y}{m+1}}\,, \qquad x \in \mathbb{R}, \quad y \in [-1,1] \tag{5.2}$$

$$R = R(z) = \frac{1}{2}|\log|z||\,, \tag{5.3}$$

$$b(m) = \frac{1}{4\pi^2}\,\frac{m+1}{\sqrt{m(m+2)}}\,. \tag{5.4}$$

In section 6 we shall prove that the asymptotic behavior of $n(1, \boldsymbol{A}(z))$ for $z \to 0^-$ coincides with that of $n(1, S_R)$ for $R \to +\infty$.

As a first step, we show that the terms in $\boldsymbol{A}(z)$ depending on the interaction between the two fermions give a compact contribution which can be neglected.

Lemma 5.1. *For $z \le 0$ the operator $\boldsymbol{A}(z) - \boldsymbol{A}_0(z)$ is compact and it is continuous in z, where*

$$\boldsymbol{A}_0(z) = \begin{pmatrix} A_0(z) & 0 \\ 0 & 0 \end{pmatrix}$$

and

$$A_0(z) = \Pi_{23}\Gamma_{23}^{-1/4}(z)|V_{23}|^{1/2}R_0(z)T|V_{23}|^{1/2}\Gamma_{23}^{-1/4}(z)\Pi_{23}. \tag{5.5}$$

Proof. Let us introduce the operators

$$\boldsymbol{\Gamma}(z) = \mathrm{diag}\{\Gamma_{23}(z), \Gamma_{12}(z)\}\,,$$
$$\widetilde{\boldsymbol{W}}^{(\delta)}(z) = \mathrm{diag}\{\widetilde{W}_{23}^{(\delta)}(z), \widetilde{W}_{12}^{(\delta)}(z)\}\,,$$
$$\boldsymbol{\Pi} = \begin{pmatrix} \Pi_{23} & 0 \\ 0 & 0 \end{pmatrix}$$

where $\delta < \min\{1, b-3\}$, $\widetilde{W}_{12}^{(\delta)}(z) = \Gamma_{12}(z)^{\frac{1-\delta}{4}} W_{12}(z)$ is continuous in $z \le 0$ and the other terms have been defined in (4.13), (4.14), (4.15). Using the above notation we write

$$\boldsymbol{A}(z) = \boldsymbol{A}_0(z) + \boldsymbol{R}(z)$$

where

$$\boldsymbol{R}(z) = \boldsymbol{\Pi}\,\boldsymbol{U}^{(1/4,(1-\delta)/4)}(z)\widetilde{\boldsymbol{W}}^{(\delta)}(z) + \widetilde{\boldsymbol{W}}^{(\delta)}(z)\boldsymbol{U}^{((1-\delta)/4,(1-\delta)/4)}(z)\widetilde{\boldsymbol{W}}^{(\delta)}(z) + \widetilde{\boldsymbol{W}}^{(\delta)}(z)\boldsymbol{U}^{((1-\delta)/4,1/4)}(z)\boldsymbol{\Pi}$$

and

$$\boldsymbol{U}^{(\mu,\nu)}(z) = \boldsymbol{\Gamma}(z)^{-\mu}\boldsymbol{U}(z)\boldsymbol{\Gamma}(z)^{-\nu}, \qquad 0 \le \mu,\nu \le \frac{1}{4}, \quad \mu+\nu < \frac{1}{2}.$$

By Lemma 4.4 in [67], for $z \le 0$ the operator $\boldsymbol{U}^{(\mu,\nu)}(z)$ is compact and it is continuous in z. Since $\widetilde{W}_\alpha^{(\delta)}(z)$, $z \le 0$, and Π_{23} are bounded, we conclude that for $z \le 0$ the operator $\boldsymbol{R}(z)$ is compact and it is continuous in z. □

In the next step we reduce the problem to the analysis of an operator in $L^2(\mathbb{R}^3)$. Such an operator is better analyzed using the coordinates (p_{23}, p_{31}).

Lemma 5.2. *For $\lambda > 0$ and $z < 0$ we have*

$$n(\lambda, \boldsymbol{A}_0(z)) = n(\lambda, \mathcal{B}(z))$$

where $\mathcal{B}(z)$ is the integral operator in $L^2(\mathbb{R}^3)$ with kernel

$$\mathcal{B}(p,q;z) = -\frac{\widehat{|v|^{1/2}\varphi}\left(p+\frac{q}{m+1}\right)\overline{\widehat{|v|^{1/2}\varphi}\left(q+\frac{p}{m+1}\right)}}{\gamma\left(\frac{p^2}{2n}-z\right)^{1/4}(H_0-z)\,\gamma\left(\frac{q^2}{2n}-z\right)^{1/4}}.$$

Proof. We first observe that $n(\lambda, \boldsymbol{A}_0(z)) = n(\lambda, A_0(z))$. Moreover, $A_0(z)$ is compact for $z < 0$ and then $n(\lambda, A_0(z))$ is the number of its eigenvalues larger than λ. By (5.5), recalling (4.14) and (4.13) we can write explicitly the eigenvalue equation for A_0 corresponding to the eigenvalue $\widetilde{\lambda} > \lambda > 0$

$$\begin{aligned}\widetilde{\lambda}\psi(p_1,p_2) = &-\frac{1}{(2\pi)^{3/2}||\varphi||^2}\,\hat{\varphi}\left(p_2+\frac{p_1}{m+1}\right)\int dp\,\overline{\hat{\varphi}\left(p+\frac{p_1}{m+1}\right)}\\ &\times\int dp_2'\frac{\widehat{|v|^{1/2}}(p-p_2')\widehat{|v|^{1/2}\varphi}\left(p_1+\frac{p_2'}{m+1}\right)}{\gamma\left(\frac{p_1^2}{2n}-z\right)^{1/4}(H_0(p_1,p_2')-z)\gamma\left(\frac{p_2'}{2n}-z\right)^{1/4}}\\ &\times\int dq\,\overline{\hat{\varphi}\left(q++\frac{p_2'}{m+1}\right)}\,\psi(p_2',q).\end{aligned} \tag{5.6}$$

Let us define

$$\xi(p) = \int dq\,\overline{\hat{\varphi}\left(q+\frac{p}{m+1}\right)}\psi(p,q).$$

Then $\xi \in L^2(\mathbb{R}^3)$ and, by (5.6), it satisfies the equation

$$\begin{aligned}\widetilde{\lambda}\xi(p_1) &= -\frac{1}{(2\pi)^{3/2}}\int dp_2\,\overline{\hat{\varphi}\left(p_2+\frac{p_1}{m+1}\right)}\\ &\quad\times\int dp_2'\frac{\widehat{|v|^{1/2}}(p_2-p_2')\widehat{|v|^{1/2}\varphi}\left(p_1+\frac{p_2'}{m+1}\right)}{\gamma\left(\frac{p_1^2}{2n}-z\right)^{1/4}(H_0(p_1,p_2')-z)\gamma\left(\frac{p_2'^2}{2n}-z\right)^{1/4}}\xi(p_2')\\ &= -\int dq\,\frac{\widehat{|v|^{1/2}\varphi}\left(p_1+\frac{q}{m+1}\right)\overline{\widehat{|v|^{1/2}\varphi}\left(q+\frac{p_1}{m+1}\right)}}{\gamma\left(\frac{p_1^2}{2n}-z\right)^{1/4}(H_0(p_1,q)-z)\gamma\left(\frac{q^2}{2n}-z\right)^{1/4}}\xi(q)\end{aligned}$$

that is $\mathcal{B}(z)\xi = \widetilde{\lambda}\xi$. On the other hand, if $\xi \in L^2(\mathbb{R}^3)$ is such that $\mathcal{B}(z)\xi = \widetilde{\lambda}\xi$ then

$$\psi(p_1,p_2) = -\frac{1}{(2\pi)^{3/2}}\int dq \, \frac{\widehat{|v|^{1/2}}(p_2-q)\widehat{|v|^{1/2}\varphi}\left(p_1+\frac{q}{m+1}\right)}{\gamma\left(\frac{p_1^2}{2n}-z\right)^{1/4}(H_0(p_1,q)-z)\gamma\left(\frac{q^2}{2n}-z\right)^{1/4}}\,\xi(q)$$

satisfies the equation $A_0(z)\psi = \widetilde{\lambda}\psi$ and therefore the Lemma is proved. □

The lack of compactness of $\mathcal{B}(z)$ for $z=0$ is clearly due to the behavior of its integral kernel near the origin. Indeed, in the following Lemma we show that the difference of $\mathcal{B}(z)$ with an operator whose kernel is different from zero only in a ball of radius one is compact and continuous in $z \le 0$. Denoted by χ_a the characteristic function of the ball of radius $a>0$, we have

Lemma 5.3. *For $z \le 0$ the operator $\mathcal{B}(z) - \widetilde{\mathcal{B}}(z)$ is compact and it is continuous in z, where $\widetilde{\mathcal{B}}(z)$ is the operator in $L^2(\mathbb{R}^3)$ with kernel*

$$\widetilde{\mathcal{B}}(p,q;z) = -\frac{1}{c(m)}\frac{\chi_1(p)\chi_1(q)}{\left(\frac{p^2}{2n}-z\right)^{1/4}(H_0(p,q)-z)\left(\frac{q^2}{2n}-z\right)^{1/4}}$$

and $c(m) = 2^{5/2}\pi^2\left(\frac{m}{m+1}\right)^{3/2}$. Moreover

$$n(\lambda,\widetilde{\mathcal{B}}(z)) = n(\lambda,\mathcal{B}_0(z)) \tag{5.7}$$

where $\mathcal{B}_0(z)$ is the integral operator in $L^2(\mathbb{R}^3)$ with kernel

$$\mathcal{B}_0(p,q;z) = -\frac{1}{c(m)}\frac{\chi_{|z|^{-1/2}}(p)\chi_{|z|^{-1/2}}(q)}{\left(\frac{p^2}{2n}+1\right)^{1/4}(H_0(p,q)+1)\left(\frac{q^2}{2n}+1\right)^{1/4}}.$$

Proof. The proof is divided into three steps. We first introduce the operator $\mathcal{E}_0(z)$ with integral kernel

$$\mathcal{E}_0(p,q;z) = -\chi_1(p)\frac{\widehat{|v|^{1/2}\varphi}\left(p+\frac{q}{m+1}\right)\overline{\widehat{|v|^{1/2}\varphi}\left(q+\frac{p}{m+1}\right)}}{\gamma\left(\frac{p^2}{2n}-z\right)^{1/4}(H_0(p,q)-z)\gamma\left(\frac{q^2}{2n}-z\right)^{1/4}}\chi_1(q)$$

and note that the integral kernel of the difference $\mathcal{B}(z) - \mathcal{E}_0(z)$ can be written as

$$\begin{aligned}(\mathrm{I}+\mathrm{II}+\mathrm{III})(p,q;z) &= (1-\chi_1(p))\frac{\widehat{|v|^{1/2}\varphi}\left(p+\frac{q}{m+1}\right)\overline{\widehat{|v|^{1/2}\varphi}\left(q+\frac{p}{m+1}\right)}}{\gamma\left(\frac{p^2}{2n}-z\right)^{1/4}(H_0(p,q)-z)\gamma\left(\frac{q^2}{2n}-z\right)^{1/4}}\chi_1(q)\\
&+\chi_1(p)\frac{\widehat{|v|^{1/2}\varphi}\left(p+\frac{q}{m+1}\right)\overline{\widehat{|v|^{1/2}\varphi}\left(q+\frac{p}{m+1}\right)}}{\gamma\left(\frac{p^2}{2n}-z\right)^{1/4}(H_0(p,q)-z)\gamma\left(\frac{q^2}{2n}-z\right)^{1/4}}(1-\chi_1(q))\\
&+(1-\chi_1(p))\frac{\widehat{|v|^{1/2}\varphi}\left(p+\frac{q}{m+1}\right)\overline{\widehat{|v|^{1/2}\varphi}\left(q+\frac{p}{m+1}\right)}}{\gamma\left(\frac{p^2}{2n}-z\right)^{1/4}(H_0(p,q)-z)\gamma\left(\frac{q^2}{2n}-z\right)^{1/4}}(1-\chi_1(q)).\end{aligned}$$

Let us consider $\mathrm{I}(p,q;z)$. We note that $|v|^{1/2}\varphi \in L^1(\mathbb{R}^3)$ which implies $\widehat{|v|^{1/2}\varphi} \le c$. Moreover, $H_0(p,q) - z \ge cp^2$ and therefore we obtain

$$\mathrm{I}(p,q;z) \le c\,\frac{(1-\chi_1(p))\,\chi_1(q)}{p^2\,q^{1/2}}$$

which is a square integrable function. Analogously, one can prove square integrability of $\mathrm{II}(p,q;z)$. In order to estimate $\mathrm{III}(p,q;z)$, we note that $\widehat{|v|^{1/2}\varphi} \in L^2(\mathbb{R}^3)$ and therefore

$$\mathrm{III}(p,q;z) \le c(1-\chi_1(p))\frac{\widehat{|v|^{1/2}\varphi}\,(p+\frac{q}{m+1})}{q^2}(1-\chi_1(q))$$

is square integrable. Hence we conclude that for $z \le 0$ the operator $\mathcal{B}(z) - \mathcal{E}_0(z)$ is Hilbert-Schmidt and it is continuous in z. Now we consider the operator $\mathcal{E}_1(z)$ with integral kernel

$$\mathcal{E}_1(p,q;z) = -\frac{1}{c(m)}\,\frac{\chi_1(p)\chi_1(q)}{\gamma\left(\frac{p^2}{2n}-z\right)^{1/4}(H_0(p,q)-z)\,\gamma\left(\frac{q^2}{2n}-z\right)^{1/4}}\,.$$

We note that

$$||\widehat{|v|^{1/2}\varphi}(k) - \widehat{|v|^{1/2}\varphi}(0)| \le c\,|k|^{\nu}\,, \qquad 0<\nu<\frac{b-3}{2}. \tag{5.8}$$

Indeed, using $|e^{-ik\cdot x}-1| \le c\,|k|^{\nu}|x|^{\nu}$, we have

$$\begin{aligned} ||\widehat{|v|^{1/2}\varphi}(k) - \widehat{|v|^{1/2}\varphi}(0)| &\le c\,|k|^{\nu}\int dx\,|x|^{\nu}|v|^{1/2}(x)|\varphi|(x) \\ &\le c\,\|\varphi\|\,|k|^{\nu}\left(\int dx\,|x|^{2\nu}|v|(x)\right)^{1/2} \end{aligned}$$

and the last integral is finite by assumption (A_1). Moreover, using Young's inequality, we get

$$H_0(p,q) - z = \frac{p^2}{2\mu} + \frac{q^2}{2\mu} + \frac{p\cdot q}{m} \ge c\left(\frac{(p^{2\kappa})^{1/\kappa}}{\kappa} + \frac{(q^{2\kappa'})^{1/\kappa'}}{\kappa'}\right) \ge c\,p^{2\kappa}q^{2\kappa'} \tag{5.9}$$

for any $\kappa,\kappa' > 0$ such that $\kappa+\kappa'=1$.

By (5.8), (5.9) and the normalization condition (4.1), we find

$$\begin{aligned} |\mathcal{E}_0(p,q;z) - \mathcal{E}_1(p,q;z)| \le& c\,\chi_1(p)\frac{|p|^{\nu}+|q|^{\nu}}{\left(\frac{p^2}{2n}-z\right)^{1/4}(H_0(p,q)-z)\left(\frac{q^2}{2n}-z\right)^{1/4}}\chi_1(q) \\ \le& c\,\frac{\chi_1(p)\chi_1(q)}{|p|^{-\nu+2\kappa+1/2}|q|^{2\kappa'+1/2}} + c\,\frac{\chi_1(p)\chi_1(q)}{|q|^{-\nu+2\kappa+1/2}|p|^{2\kappa'+1/2}} \end{aligned}$$

which is square integrable choosing $\kappa \in \left(\frac{1}{2},\frac{1+\delta}{2}\right)$.
In order to obtain the operator $\widetilde{\mathcal{B}}(z)$ from $\mathcal{E}_1(z)$, it remains to replace $\gamma\left(\frac{p^2}{2n}-z\right)$ and $\gamma\left(\frac{q^2}{2n}-z\right)$ with $\frac{p^2}{2n}-z$ and $\frac{q^2}{2n}-z$, respectively. One can easily see that the difference $\mathcal{E}_1(z) - \widetilde{\mathcal{B}}(z)$ is compact up to $z=0$. This concludes the proof that for $z \le 0$ the operator $\mathcal{B}(z) - \widetilde{\mathcal{B}}(z)$ is compact and it is continuous in z.

In order to prove (5.7) it is sufficient to observe that $\widetilde{\mathcal{B}}(z)$ is unitarily equivalent to $\mathcal{B}_0(z)$ via the unitary operator U_z defined by $U_z\xi(p) = |z|^{3/4}\xi(|z|^{1/2}p)$. □

In the following lemma we finally arrive at the operator S_R defined in (5.1), (5.2).

Lemma 5.4. *For $z \le 0$ the operator $\mathcal{B}_0(z) - \mathcal{S}(z)$ is compact and it is continuous in z, where $\mathcal{S}(z)$ is the integral operator in $L^2(\mathbb{R}^3)$ with kernel*

$$\mathcal{S}(p,q;z) = -\frac{(2n)^{1/2}}{c(m)} \frac{\big(\chi_{|z|^{-1/2}} - \chi_1\big)(p)\big(\chi_{|z|^{-1/2}} - \chi_1\big)(q)}{|p|^{1/2}\left(\frac{p^2}{2\mu} + \frac{q^2}{2\mu} + \frac{p\cdot q}{m}\right)|q|^{1/2}}.$$

Moreover

$$n(\lambda, \mathcal{S}(z)) = n(\lambda, S_R).$$

Proof. The first point is easy to check. Let us prove the second statement. Using the unitary operator $M : L^2(\mathbb{R}^3) \to L^2(\mathbb{R} \times \mathbb{S}^2, dr \otimes d\Omega)$ defined by

$$(M\xi)(r,\omega) = e^{3r/2}\xi(e^r,\omega) \tag{5.10}$$

we see that $\mathcal{S}(z)$ is unitarily equivalent to the operator on $L^2(\mathbb{R} \times \mathbb{S}^2, dr \otimes d\Omega)$ with integral kernel

$$-\,b(m)\frac{\chi_{(0,R)}(x)\chi_{(0,R)}(x')}{\cosh(x-x') + \frac{\zeta\cdot\omega}{m+1}}. \tag{5.11}$$

where $\chi_{(a,b)}$ is the characteristic function of the interval (a,b). Indeed,

$$\begin{aligned}
(M\mathcal{S}(z)\xi)(r,\omega) &= -\frac{(m+1)^2}{4\pi^2\sqrt{m^3(m+2)}}e^{3r/2}\int_0^{+\infty} d\rho\,\rho^{3/2}\int_{\mathbb{S}^2} d\Omega(\zeta)\,\xi(\rho,\zeta) \\
&\quad\times \frac{\Big(\chi_{|z|^{-1/2}}(e^r,\omega) - \chi_1(e^r,\omega)\Big)\Big(\chi_{|z|^{-1/2}}(\rho,\zeta) - \chi_1(\rho,\zeta)\Big)}{e^{r/2}\left(\dfrac{e^{2r}}{2\mu} + \dfrac{\rho^2}{2\mu} + e^r\rho\dfrac{\omega\cdot\zeta}{m}\right)} \\
&= -\frac{(m+1)^2}{4\pi^2\sqrt{m^3(m+2)}}\,e^r\int_{-\infty}^{+\infty} dx\, e^{5x/2}\int_{\mathbb{S}^2} d\Omega(\zeta) \\
&\quad\times \frac{\chi_{(0,R)}(r)\chi_{(0,R)}(x)}{\dfrac{e^{2r}}{2\mu} + \dfrac{e^{2x}}{2\mu} + \dfrac{\omega\cdot\zeta}{m}e^{r+x}}\,\xi(e^x,\zeta) \\
&= -\frac{(m+1)^2}{4\pi^2\sqrt{m^3(m+2)}}\,e^r\int_{-\infty}^{+\infty} dx\, e^{x}\int_{\mathbb{S}^2} d\Omega(\zeta) \\
&\quad\times \frac{\chi_{(0,R)}(r)\chi_{(0,R)}(x)}{\dfrac{e^{r+x}}{\mu}\left(\dfrac{e^{r-x}+e^{x-r}}{2} + \dfrac{\omega\cdot\zeta}{m+1}\right)}\,e^{3x/2}\,\xi(e^x,\zeta) \\
&= -b(m)\int_{-\infty}^{+\infty} dx\int_{\mathbb{S}^2} d\Omega(\zeta)\,\frac{\chi_{(0,R)}(r)\chi_{(0,R)}(x)}{\cosh(r-x) + \dfrac{\omega\cdot\zeta}{m+1}}\,(M\xi)(x,\zeta).
\end{aligned}$$

Since the operator with the kernel given by (5.11) can be considered as an operator on $L^2((0,R)\times\mathbb{S}^2, d\rho\otimes d\Omega)$, we find S_R. □

6 PROOF OF THEOREM 3.1

In this section we give the proof of Theorem 3.1. Taking into account of Theorem 4.2, the result for $m < m_*$ is obtained in two steps. We first show that

$$\frac{n(1, S_R)}{2R} \tag{6.1}$$

converges for $R \to \infty$ (see Proposition 6.1 below). Then we prove that

$$\frac{n(1, \boldsymbol{A}(z))}{|\log|z||}$$

converges to the same limit for $z \to 0^-$ (see Proposition 6.2 below).

In order to study the asymptotic behavior of (6.1) for $R \to \infty$, it is convenient to decompose the integral kernel of S_R in spherical harmonics. Indeed, denoted by P_l the Legendre polynomial of order l and by Y_l^ν the spherical harmonic of order l, ν, we write

$$S(x, \omega \cdot \zeta) = \sum_{l=0}^{+\infty} \frac{2l+1}{2} P_l(\omega \cdot \zeta) \int_{-1}^{1} dy\, P_l(y) S(x, y)$$

and, using the addition formula

$$P_l(\omega \cdot \zeta) = \frac{4\pi}{2l+1} \sum_{\nu=-l}^{l} \overline{Y_l^\nu(\zeta)} Y_l^\nu(\omega) \quad \forall \omega, \zeta \in \mathbb{S}^2, \tag{6.2}$$

we find (see (5.1), (5.2))

$$\begin{aligned}
(S_R f)(r, \omega) &= 2\pi \sum_{l=0}^{+\infty} \sum_{\nu=-l}^{l} Y_l^\nu(\omega) \int_0^R d\rho \int_{-1}^{1} dy\, S(r-\rho, y) P_l(y) \int_{\mathbb{S}^2} d\Omega(\zeta)\, f(\rho, \zeta) \overline{Y_l^\nu(\zeta)} \\
&= -2\pi b(m) \sum_{l=0}^{+\infty} \sum_{\nu=-l}^{l} Y_l^\nu(\omega) \int_0^R d\rho\, f_{l\nu}(\rho) \int_{-1}^{1} dy\, \frac{P_l(y)}{\cosh(r-\rho) + \frac{y}{m+1}} \\
&:= \sum_{l=0}^{+\infty} \sum_{\nu=-l}^{l} (S_R^{(l)} f_{l\nu})(\rho) Y_l^\nu(\omega)
\end{aligned}$$

where

$$f_{l\nu}(\rho) = \int_{\mathbb{S}^2} d\Omega(\zeta)\, f(\rho, \zeta) \overline{Y_l^\nu(\zeta)} \tag{6.3}$$

and $S_R^{(l)}$ is the integral operator in $L^2((0, R))$, for any $R > 0$, with kernel $S_m^{(l)}(x)$ independent from R defined in Appendix B. In particular, this decomposition implies

$$n(\lambda, S_R) = \sum_{l=0}^{+\infty} (2l+1) n(\lambda, S_R^{(l)}). \tag{6.4}$$

We are now in position to characterize the asymptotic of (6.1).

Proposition 6.1. *For any $\lambda > 0$ we have*

$$\lim_{R\to+\infty} \frac{n(\lambda, S_R)}{2R} = \sum_{l=0}^{+\infty} \frac{2l+1}{4\pi} \left| \left\{ k \in \mathbb{R} \,|\, \widehat{S}_m^{(l)}(k) > \frac{\lambda}{\sqrt{2\pi}} \right\} \right| \tag{6.5}$$

and the limit is continuous in $\lambda > 0$.

Proof. As noted in Appendix B, the kernel of the operator $S_R^{(l)}$ is an even function and satisfies the estimate

$$|S_m^{(l)}(x)| \leq \frac{c}{\cosh x - \frac{1}{m+1}}\,. \tag{6.6}$$

By (6.6) we have $S_m^{(l)} \in L^1(\mathbb{R}) \cap L^\infty(\mathbb{R})$ and $e^{\varepsilon|x|} S_m^{(l)} \in L^2(\mathbb{R})$ for any $\varepsilon \in [0,1)$. These properties imply that $|\widehat{S}_m^{(l)}(k)| \to 0$ for $|k| \to \infty$, by Riemann-Lebesgue Theorem, and that $\widehat{S}_m^{(l)}(k)$ has an analytic continuation to a neighborhood of the real axis (see, e.g., Theorem IX.13 of [63]). Then, for any $\lambda > 0$ the set $\mathfrak{M}(\lambda) = \left\{ k \in \mathbb{R} \,|\, \widehat{S}_m^{(l)}(k) = \frac{\lambda}{\sqrt{2\pi}} \right\}$ consists of a finite number of points and, in particular, it has measure zero. Thus, the hypotheses of Lemma 4.6 in [67] are satisfied and we have

$$\begin{aligned} \lim_{R\to+\infty} \frac{n(\lambda, S_R^{(l)})}{R} &= \frac{1}{2\pi} \int_{-\infty}^{+\infty} dk\, \chi_{\left(\frac{\lambda}{\sqrt{2\pi}}, +\infty\right)}(\widehat{S}_m^{(l)}(k)) \\ &= \frac{1}{2\pi} \left| \left\{ k \in \mathbb{R} \,|\, \widehat{S}_m^{(l)}(k) > \frac{\lambda}{\sqrt{2\pi}} \right\} \right| \end{aligned}$$

where $|A|$ denotes the Lebesgue measure of the set A. Taking into account of (6.4), we obtain (6.5). Concerning the continuity of the limit, we observe that $\lim_{\lambda'\to\lambda} \chi_{(\lambda',+\infty)}(\widehat{S}_m^{(l)}(k)) = \chi_{(\lambda,+\infty)}(\widehat{S}_m^{(l)}(k))$ for any k such that $\widehat{S}_m^{(l)}(k) \neq \lambda$. Using the Dominated Convergence Theorem we conclude the proof. □

For the proof of the last step we make repeatedly use of the following technical Lemma (for the proof see Lemma 4.9 in [67]).

Lemma 6.1. *Let $B(z) = B_0(z) + K(z)$, where for $z < 0$ ($z \leq 0$) the operator $B_0(z)$ ($K(z)$) is compact and continuous in z. Suppose that for a function f such that $f(z) \to 0$ when $z \to 0^-$ there exists the limit*

$$\lim_{z\to 0^-} f(z) n(\lambda, B_0(z)) = l(\lambda)$$

and $l(\lambda)$ is continuous in $\lambda > 0$. Then the following holds

$$\lim_{z\to 0^-} f(z) n(\lambda, B(z)) = l(\lambda).$$

Then we have

Proposition 6.2.

$$\lim_{R\to+\infty} \frac{n(1, S_R)}{2R} = \lim_{z\to 0^-} \frac{n(1, \boldsymbol{A}(z))}{|\log|z||}\,.$$

Proof. The proof is obtained using Proposition 6.1, Lemma 6.1 and Lemmas 5.1, 5.2, 5.3, 5.4. □

Let us prove our main result in the case $m < m_*$.

Proof of Theorem 3.1 (case $m < m_*$). By Theorem 4.2 and Propositions 6.1, 6.2 we find that the limit relation (3.9) holds with

$$\mathcal{C}(m) := \sum_{\substack{l=1,\\ l \text{ odd}}}^{+\infty} \frac{2l+1}{2\pi} \left| \left\{ k \in [0,+\infty) \,|\, \widehat{S}_m^{(l)}(k) > \frac{1}{\sqrt{2\pi}} \right\} \right| \tag{6.7}$$

where we have used the parity of $\widehat{S}_m^{(l)}(k)$ and the sum is only on l odd due to the property (B.3). It remains to show that $\mathcal{C}(m)$ is finite and strictly positive.

Let us prove the upper bound for $\mathcal{C}(m)$. We first look for an estimate of $\widehat{S}_m^{(l)}(k)$ for l odd and $k \geq 0$. By Cauchy-Schwarz inequality we have

$$\begin{aligned}\widehat{S}_m^{(l)}(k) &\leq \frac{m+1}{\sqrt{2\pi}\sqrt{m(m+2)}} \int_0^1 dy\, |P_l(y)| \frac{\sinh\left(k \arcsin\left(\frac{y}{m+1}\right)\right)}{\sinh\left(k\frac{\pi}{2}\right)\cos\left(\arcsin\left(\frac{y}{m+1}\right)\right)} \\ &\leq \frac{(m+1)^{3/2}}{\sqrt{2\pi}\sqrt{m(m+2)}} \frac{1}{\sqrt{2l+1}} \left[\int_0^{z_0} dz \frac{\sinh^2(kz)}{\sinh^2\left(k\frac{\pi}{2}\right)\cos z} \right]^{1/2}\end{aligned}$$

where $z_0 = \arcsin\left(\frac{1}{m+1}\right)$. Using the estimate $\dfrac{\sinh(kz)}{\sinh(k\pi/2)} \leq e^{-k(\pi/2 - z_0)}$ for $z \in (0, z_0)$, we find

$$\begin{aligned}\widehat{S}_m^{(l)}(k) &\leq \frac{(m+1)^{3/2}}{\sqrt{2\pi}\sqrt{m(m+2)}} \frac{1}{\sqrt{2l+1}} e^{-k(\pi/2-z_0)} \left[\int_0^{z_0} dz \frac{1}{\cos z} \right]^{1/2} \\ &\leq \frac{(m+1)^{3/2}}{\sqrt{2\pi}\sqrt{m(m+2)}} \frac{1}{\sqrt{2l+1}} e^{-k(\pi/2-z_0)} \left[\log\left(\tan\left(\frac{\pi}{4} + \frac{z_0}{2} \right) \right) \right]^{1/2} \\ &\leq \frac{(m+1)^{3/2}}{\sqrt{2\pi}\sqrt{m(m+2)}} \frac{1}{\sqrt{2l+1}} e^{-k(\pi/2-z_0)} \left[\log\left(\sqrt{1+\frac{2}{m}} \right) \right]^{1/2}\end{aligned}$$

where in the last step we have used the elementary formula $\tan\frac{x}{2} = \frac{\sin x}{1+\cos x}$ and the definition of z_0. Taking into account of the definition of $\alpha(m)$ and $\beta(m)$ given in (3.12), we have shown

$$\widehat{S}_m^{(l)}(k) \leq \frac{\alpha(m)}{\sqrt{2l+1}} e^{-\beta(m)k}.$$

By Equation (6.7) and the above inequality we obtain

$$\mathcal{C}(m) \leq \sum_{\substack{l=1,\\ l \text{ odd}}}^{+\infty} \frac{2l+1}{2\pi} \left| \left\{ k \in [0,+\infty) \,|\, \frac{\alpha(m)}{\sqrt{2l+1}} e^{-\beta(m)k} > \frac{1}{\sqrt{2\pi}} \right\} \right|. \tag{6.8}$$

The measure of the set in the r.h.s. of (6.8) is different from zero only if

$$\frac{\alpha(m)}{\sqrt{2l+1}} > \frac{1}{\sqrt{2\pi}}$$

i. e., only if $l \leq l_0(m)$, where $l_0(m)$ is the largest odd integer smaller than $\pi\alpha(m)^2 - \frac{1}{2}$. Therefore we have

$$\mathcal{C}(m) \leq \sum_{\substack{l=1,\\ l \text{ odd}}}^{l_0(m)} \frac{2l+1}{2\pi} \left| \left\{ k \in [0,+\infty) \,|\, \frac{\alpha(m)}{\sqrt{2l+1}} e^{-\beta(m)k} > \frac{1}{\sqrt{2\pi}} \right\} \right|.$$

For any $l \leq l_0(m)$, let

$$K_l(m) := \frac{1}{\beta(m)} \log \frac{\sqrt{2\pi}\,\alpha(m)}{\sqrt{2l+1}}$$

be the unique positive solution of the equation $\frac{\alpha(m)}{\sqrt{2l+1}} e^{-\beta(m)k} = \frac{1}{\sqrt{2\pi}}$. Then

$$\begin{aligned} \mathcal{C}(m) &\leq \sum_{\substack{l=1,\\ l \text{ odd}}}^{l_0(m)} \frac{2l+1}{2\pi\beta(m)} \log\left(\frac{\sqrt{2\pi}\,\alpha(m)}{\sqrt{2l+1}}\right) \\ &\leq \frac{1}{2\pi\beta(m)} \log\left(\frac{\sqrt{2\pi}\,\alpha(m)}{\sqrt{3}}\right) \sum_{\substack{l=1,\\ l \text{ odd}}}^{l_0(m)} (2l+1) \\ &= \frac{1}{4\pi\beta(m)} \log\left(\frac{\sqrt{2\pi}\,\alpha(m)}{\sqrt{3}}\right) (l_0(m)^2 + 3l_0(m) + 2) \end{aligned}$$

and this concludes the proof of (3.11).

Let us prove the lower bound for $\mathcal{C}(m)$. By (6.7) we immediately get

$$\mathcal{C}(m) \geq \frac{3}{2\pi} \left| \left\{ k \in [0,+\infty) \mid \widehat{S}_m^{(1)}(k) > \frac{1}{\sqrt{2\pi}} \right\} \right|. \tag{6.9}$$

Using Equation (B.5) and Definition 3.2, we find that

$$\widehat{S}_m^{(1)}(0) = \frac{\Lambda(m)}{\sqrt{2\pi}} > \frac{1}{\sqrt{2\pi}} \quad \text{for } m < m_*$$

and this implies strictly positivity of the right hand side of (6.9). Furthermore, by monotonicity of $\widehat{S}_m^{(1)}(k)$, stated in Lemma B.1, we have

$$\left| \left\{ k \in [0,\infty) | \widehat{S}_m^{(1)}(k) > \frac{1}{\sqrt{2\pi}} \right\} \right| = k_1(m) \tag{6.10}$$

where $k_1(m)$ is the unique positive solution of the equation $\widehat{S}_m^{(1)}(k) = \frac{1}{\sqrt{2\pi}}$. From (6.9) and (6.10) the lower bound (3.10) follows.

□

Remark 6.1. Let us check that $l_0(m) \geq 1$ for $m < m_*$. Recall that $l_0(m)$ is the largest odd integer smaller than $g(m) := \pi\alpha(m)^2 - 1/2$. It is easy to see that the function g is decreasing, $g(m) \to +\infty$ for $m \to 0$ and $g(m) \to 0$ for $m \to +\infty$. Then the assertion follows if one observes that $g(m_*) > 1$ (indeed, $g(m_*) \simeq 6.65$).

Remark 6.2. The lower bound of $\mathcal{C}(m)$ can be improved. Recall that $m_* := m_*^{(1)}$ is defined as the solution of $\sqrt{2\pi}\,\widehat{S}_m^{(1)}(0) = \Lambda(m) = 1$. By analogy, for each l odd we can define a critical mass $m_*^{(l)}$ as the solution of $\sqrt{2\pi}\,\widehat{S}_m^{(l)}(0) = 1$. By (B.2), $m_*^{(l)}$ solves the equation

$$\frac{2(m+1)}{\pi\sqrt{m(m+2)}} \int_0^1 dy\, P_l(y) \frac{\arcsin\left(\frac{y}{m+1}\right)}{\cos\left(\arcsin\left(\frac{y}{m+1}\right)\right)} = 1.$$

One can show (see [25, Appendix A]) that $m_*^{(l)}$ is uniquely defined, it is decreasing in l and $\sqrt{2\pi}\,\widehat{S}_m^{(l)}(0) > 1$ $\left(\sqrt{2\pi}\,\widehat{S}_m^{(l)}(0) < 1\right)$ if $m < m_*^{(l)}$ ($m > m_*^{(l)}$). Hence, if $m_*^{(L+2)} < m < m_*^{(L)}$ for some L then in (6.7) each term with $l \leq L$ is certainly nonzero and this implies

$$\mathcal{C}(m) \geq \sum_{\substack{l=1\\ l \text{ odd}}}^{L} \frac{2l+1}{2\pi} \left| \left\{ k \in [0,+\infty) | \widehat{S}^{(l)}(k) > \frac{1}{\sqrt{2\pi}} \right\} \right|.$$

Obviously, for $L=1$ we have only the first term and the inequality above reduces to (6.9).

Remark 6.3. It is worth noticing that, by Definition 3.2, we have $\sqrt{2\pi}\,\widehat{S}_m^{(1)}(0)<1$ for $m>m_*$ which, via (6.7), (B.3), (B.4) and (B.5), implies $\mathcal{C}(m)=\lim_{z\to 0^-}\dfrac{N(z)}{|\log|z||}=0$. In fact, we conclude this section showing that for $m>m_*$ the discrete spectrum of H is finite, i.e., there is no Efimov effect.

Proof of Theorem 3.1 (case $m>m_*$). For $z<0$ and $\varepsilon\in(0,1)$ the following inequality holds

$$\begin{aligned} n(1,\boldsymbol{A}(z)) \leq & n(1-\varepsilon,\mathcal{S}(z))+n\left(\frac{\varepsilon}{3},\boldsymbol{A}(z)-\boldsymbol{A}_0(z)\right)+n\left(\frac{\varepsilon}{3},\mathcal{B}(z)-\tilde{\mathcal{B}}(z)\right)\\ &+n\left(\frac{\varepsilon}{3},\mathcal{B}_0(z)-\mathcal{S}(z)\right) \end{aligned} \tag{6.11}$$

where the operators $\boldsymbol{A}_0(z),\mathcal{B}(z),\tilde{\mathcal{B}}(z),\mathcal{S}(z)$ have been defined in the previous section. Such an inequality is a direct consequence of Lemmas 5.1, 5.2, 5.3, 5.4 and of the following technical result (see, e.g., [14]): if A,B are compact self-adjoint operators and $\lambda_i>0$, $i=1,2$ then

$$n(\lambda_1+\lambda_2,A+B)\leq n(\lambda_1,A)+n(\lambda_2,B).$$

Note that for $z\leq 0$ the operators appearing in the last three terms of (6.11) are compact and continuous in z. Therefore, the last three terms of (6.11) remain finite for $z\to 0^-$. Let us consider the first term in the right hand side of (6.11), i.e., $n(1-\varepsilon,\mathcal{S}(z))$.

We notice that

$$(\mathcal{S}(z)\xi,\xi)=(\mathcal{S}\xi_z,\xi_z)=(SM\xi_z,M\xi_z) \tag{6.12}$$

where $\xi_z(p)=(\chi_{|z|^{-1/2}}-\chi_1)(p)\xi(p)$, $\mathcal{S}$ is the operator in $L^2(\mathbb{R}^3)$ defined by

$$(\mathcal{S}f)(p)=-\frac{1}{4\pi^2}\frac{(m+1)^2}{\sqrt{m^3(m+2)}}\int dq\,\frac{f(q)}{|p|^{1/2}\left(\frac{p^2}{2\mu}+\frac{q^2}{2\mu}+\frac{p\cdot q}{m}\right)|q|^{1/2}},$$

M is the unitary operator defined in (5.10) and S acts on $L^2(\mathbb{R}_+\times\mathbb{S}^2,dr\otimes d\Omega)$ as follows

$$(Sf)(r,\omega)=-\frac{1}{4\pi^2}\frac{m+1}{\sqrt{m(m+2)}}\int_{-\infty}^{+\infty}d\rho\int_{\mathbb{S}^2}d\Omega(\zeta)\frac{f(\rho,\zeta)}{\cosh(r-\rho)+\frac{\omega\cdot\zeta}{m+1}}.$$

Let us estimate (Sf,f). By (6.3) and (B.1) we get

$$\begin{aligned}(Sf,f)=&-\frac{1}{2\pi}\frac{m+1}{\sqrt{m(m+2)}}\sum_{l=0}^{+\infty}\sum_{\nu=-l}^{l}\int_{-\infty}^{+\infty}dr\,\overline{f_{l\nu}(r)}\int_{-\infty}^{+\infty}d\rho\,f_{l\nu}(\rho)\\ &\times\int_{-1}^{1}dy\,\frac{P_l(y)}{\cosh(r-\rho)+\frac{y}{m+1}}\\ =&\sum_{l=0}^{+\infty}\sum_{\nu=-l}^{l}\sqrt{2\pi}(\hat{f}_{l\nu},\hat{f}_{l\nu}\widehat{S}_m^{(l)})\\ =&\sqrt{2\pi}\sum_{l=0}^{+\infty}\sum_{\nu=-l}^{l}\int_{-\infty}^{+\infty}dk\,|\hat{f}_{l\nu}(k)|^2\widehat{S}_m^{(l)}(k).\end{aligned}$$

Taking into account of (B.3), (B.4) and (B.5) we deduce

$$
\begin{aligned}
(Sf, f) &\leq \sqrt{2\pi} \sum_{\substack{l=1 \\ l \text{ odd}}}^{+\infty} \sum_{\nu=-l}^{l} \int_{-\infty}^{+\infty} dk \, |\hat{f}_{l\nu}(k)|^2 \widehat{S}_m^{(1)}(k) \\
&\leq \Lambda(m) \sum_{\substack{l=1 \\ l \text{ odd}}}^{+\infty} \sum_{\nu=-l}^{l} ||f_{l\nu}||^2 \\
&\leq \Lambda(m) ||f||^2 .
\end{aligned}
\tag{6.13}
$$

By (6.12) and (6.13) we have

$$
(\mathcal{S}(z)\xi_z, \xi_z) \leq \Lambda(m) \|M\xi_z\|^2 \leq \Lambda(m) \|\xi\|^2
$$

where $0 < \Lambda(m) < 1$ for $m > m_*$ (see Definition 3.2). Therefore, the operator $\mathcal{S}(z)$ does not have eigenvalues larger than $\Lambda(m)$, i.e.,

$$
n(1 - \varepsilon, \mathcal{S}(z)) = 0 \tag{6.14}
$$

for any $\varepsilon \in (0, 1 - \Lambda(m))$. By Theorem 4.2 and (6.11), (6.14) we find

$$
N(z) \leq n\left(\frac{\varepsilon}{3}, \boldsymbol{A}(z) - \boldsymbol{A}_0(z)\right) + n\left(\frac{\varepsilon}{3}, \mathcal{B}(z) - \tilde{\mathcal{B}}(z)\right) + n\left(\frac{\varepsilon}{3}, \mathcal{B}_0(z) - \mathcal{S}(z)\right) \tag{6.15}
$$

for any $z < 0$ and $\varepsilon \in (0, 1 - \Lambda(m))$. Taking the limit $z \to 0^-$ and using compactness and continuity in $z \leq 0$ of the operators in the r.h.s. of (6.15), we obtain the finiteness of the number of negative eigenvalues of H. □

Part II

POINT INTERACTIONS

7 INTRODUCTION

The quantum mechanical three-body problem with pairwise, local zero-range interactions is a subject of considerable interest both for physical applications and for its peculiar mathematical structure. Such a system is described by the formal Hamiltonian $\mathscr{H}_{\mathrm{v}}^{3\,\mathrm{body},d}$ defined by (1.4).

The first step toward a rigorous analysis of the system is to define the formal Hamiltonian (1.4) as a self-adjoint operator. We first notice that, in any reasonable definition, the interaction term of the Hamiltonian must be non trivial only on the hyperplanes $\{x_i = x_j\}$ where the coordinates of two particles coincide. It is therefore natural to consider the operator acting as the free Hamiltonian restricted to a domain of smooth functions vanishing in the neighborhood of each hyperplane $\{x_i = x_j\}$. Such operator is symmetric but not self-adjoint and one (trivial) self-adjoint extension is obviously the free Hamiltonian. Then we define a Hamiltonian for a system of three quantum particles in $\mathbb{R}^d$ with a two-body, zero-range interaction as a non trivial self-adjoint extension of it. As a consequence of the definition, any such Hamiltonian acts as the free Hamiltonian outside the hyperplanes $\cup_{i<j}\{x_i = x_j\}$ and it is characterized by a specific boundary condition satisfied by the wave function at each hyperplane $\{x_i = x_j\}$. The second, and more involved, step is to give an explicit description of these extensions. The most frequently used techniques are Krein's theory of self-adjoint extensions, use of the boundary triplets and approximation by regularized Hamiltonians, in the sense of the limit of the resolvent or of the quadratic form. The result is strongly dependent on dimension.

In Chapter 8 the one dimensional case is considered. Here the problem is relatively simpler due to the fact that the interaction term is a small perturbation of the free Hamiltonian in the sense of quadratic forms as proved in Section 8.1, where the rigorous definition of $\mathscr{H}_{\mathrm{v}}^{3\,\mathrm{body},1}$ is then given. In this one dimensional case it is also possible to obtain the Hamiltonian as the limit of regular interaction rescaled as in the case of a fixed point interaction: $\frac{1}{\varepsilon}v\left(\frac{x_\alpha}{\varepsilon}\right)$. This convergence result is proved in Section 8.2. These results will appear in [7].

In dimension two a natural class of Hamiltonians with local zero-range interactions was constructed in [27] and it was also shown that such Hamiltonians are all bounded from below. We don't consider this case further.

In dimension three the analysis is more delicate and is addressed in Chapter 9. A strategy to circumvent the difficulty is to select a class of extensions based on the analogy with the two-particle case. In this case (see, e.g., [4]) one is reduced to the study of a fixed point interaction in the relative coordinate and the entire class of self-adjoint extensions can be explicitly constructed. One can show that the domain $D(h_\nu)$ of each Hamiltonian h_ν in the center of mass reference frame consists of functions $\psi \in L^2(\mathbb{R}^3) \cap H^2(\mathbb{R}^3 \setminus \{0\})$ such that

$$\psi(x) = \frac{q}{|x|} + r + o(1)\,, \qquad \text{with } r = \nu\, q\,, \tag{7.1}$$

for $|x| \to 0$, where $q \in \mathbb{C}$, $\nu \in \mathbb{R}$ is a parameter proportional to the inverse of the scattering length, and x is the relative coordinate. The relation $r = \nu q$ in (7.1) should be understood as the generalized boundary condition satisfied at the origin by all the elements in the domain. Moreover, by definition h_ν satisfies

$$(h_\nu \psi)(x) = -\frac{1}{2\mu}(\Delta\psi)(x) \qquad \forall\, x \neq 0 \tag{7.2}$$

where μ denotes the reduced mass of the two-body problem. By analogy, in the three-particle case the generalized boundary conditions we ask to be satisfied by the generic element ψ in the domain of $\mathscr{H}_{\mathrm{v}}^{3\,\mathrm{body},d}$ are the following

$$\psi(x_1, x_2, x_3) = \frac{Q^{(\ell)}(r_{ij}, x_\ell)}{|x_i - x_j|} + R_{ij}(r_{ij}, x_\ell) + o(1) \qquad \text{with } R_{ij} = \mathrm{v}_\alpha Q^{(\ell)} \tag{7.3}$$

as $|x_i - x_j| \to 0$. In (7.3) $Q^{(\ell)}$ is a suitable function defined on the hyperplane $\{x_i = x_j\}$ and $r_{ij} = \frac{m_i x_i + m_j x_j}{m_i + m_j}$. Furthermore we ask

$$\mathscr{H}_{\mathrm{v}}^{3\,\mathrm{body},d}\psi = -\sum_{i=1}^{3} \frac{1}{2m_i}\Delta_{x_i}\psi \qquad \text{if } |x_i - x_j| \neq 0\ \forall i \neq j. \tag{7.4}$$

In Section 9.1 the quadratic form corresponding to the operator defined by Equations (7.3) and (7.4) is constructed. Unfortunately, the operator selected in this way, known as *STM extensions*, could be not self-adjoint and their self-adjoint extensions could be unbounded from below (see [36, 48]) exhibiting the occurrence of the Thomas effect. If one resctricts the Hilbert space by imposing some symmetry constraints the Thomas effect can be prevented. In our quadratic form approach this means that the associated quadratic form is closed and bounded from below. In particular, for the three-bosons system the instability is proven to occur only in the *s-wave* subspace. Hence, the operator restricted on the orthogonal complement of this subspace is stable, see Section 9.2. Another system which is stable is the one composed by two identical fermions and a third particle under certain conditions on the masses (see Section 9.3 below and [24, 25] for further details).

8 ONE DIMENSION

8.1 RIGOROUS DEFINITION OF THE HAMILTONIAN

This section is devoted to the rigorous definition of the following formal Hamiltonian in dimension $d = 1$ as presented in [7]. The first step toward a rigorous definition of $\mathscr{H}_{\mathrm{v}}^{3\text{body},1}$ is to remove the center of mass motion. Proceeding as in Appendix A we are left with the formal Hamiltonian in $L^2(\mathbb{R}^2)$

$$\mathscr{H}_{\mathrm{v}}^{1d} = H_0 + \sum_{\alpha} \nu_\alpha \delta(x_\alpha)$$

where H_0 is defined in Equation (A.1). As in the simpler case of a single particle interacting with a fixed point in dimension one we shall use the theory of quadratic form and in particular the KLMN Theorem in order to give a precise definition of $\mathscr{H}_{\mathrm{v}}^{1d}$ (see [2], [65]).

The coincidence plane of a pair of particles, because of the removal of the center of mass, is represented by a line. We denote by π_α the coincidence line of the particles in the pair α, i.e., in the Jacobi coordinates (x_α, y_α), π_α is identified by $x_\alpha = 0$. Furthermore we set $\Pi = \cup_\alpha \pi_\alpha$. The hyperplanes π_α identify six regions Γ_r, $r = 1, \ldots 6$. For the sake of clarity we write explicitly the definition of Γ_r in the coordinates (x_{23}, y_{23}). Obviously we could have equivalently used any other pair of Jacobi coordinates.

$$\Gamma_1 = \left\{ \begin{array}{l} (x_{23}, y_{23}) \colon x_{23} \geq 0, \\ y_{23} < \dfrac{-m_3}{m_2 + m_3} x_{23} \end{array} \right\}, \qquad \Gamma_2 = \left\{ \begin{array}{c} (x_{23}, y_{23}) \colon x_{23} \geq 0, \\ \dfrac{-m_3}{m_2 + m_3} x_{23} < y_{23} < \dfrac{m_2}{m_2 + m_3} x_{23} \end{array} \right\},$$

$$\Gamma_3 = \left\{ \begin{array}{l} (x_{23}, y_{23}) \colon x_{23} \geq 0, \\ y_{23} > \dfrac{m_2}{m_2 + m_3} x_{23} \end{array} \right\}, \qquad \Gamma_4 = \Big\{ (x_{23}, y_1) \colon (-x_{23}, -y_1) \in \Gamma_1 \Big\},$$

$$\Gamma_5 = \Big\{ (x_{23}, y_{23}) \colon (-x_{23}, -y_{23}) \in \Gamma_2 \Big\}, \quad \Gamma_6 = \Big\{ (x_{23}, y_3) \colon (-x_{23}, -y_{23}) \in \Gamma_3 \Big\}.$$

For any function $\psi \in H^s(\mathbb{R}^2)$ with $s > 1/2$ we denote by $\psi|_{\pi_\alpha}$ its trace on the hyperplane π_α and we recall that the map $\psi \to \psi|_{\pi_\alpha}$ extends to a continuous one from $H^s(\mathbb{R}^2)$ to $H^{s-1/2}(\mathbb{R}^1)$ for any $s > 1/2$.

We start by defining the sesquilinear form associated to the free Hamiltonian H_0. In any pair of Jacobi coordinate (x_α, y_α) we have

$$\mathcal{B}_{\text{free}}(\varphi, \psi) := \int dx_\alpha \, dy_\alpha \left[\frac{1}{2\mu_\alpha} \overline{\partial_{x_\alpha} \varphi} \, \partial_{x_\alpha} \psi + \frac{1}{2n_\alpha} \overline{\partial_{y_\alpha} \varphi} \, \partial_{y_\alpha} \psi \right],$$
$$D(\mathcal{B}_{\text{free}}) := H^1(\mathbb{R}^2) \times H^1(\mathbb{R}^2).$$

With a slight abuse of notation, we shall denote by the same letter the associated quadratic form $\mathcal{B}_{\text{free}}(\psi) := \mathcal{B}_{\text{free}}(\psi, \psi)$ with domain $D(\mathcal{B}_{\text{free}}) := H^1(\mathbb{R}^2)$.

On the other hand, since the potential $\nu_\alpha \delta(x_\alpha)$ is supported by the line π_α, it seems natural to associate to the formal operator $\mathscr{H}_\nu^{1\text{d}}$ the sesquilinear form $\mathcal{B}_\nu$ defined as follows

$$\mathcal{B}_\nu(\varphi, \psi) := \mathcal{B}_{\text{free}}(\varphi, \psi) + \sum_\alpha \nu_\alpha \left(\varphi|_{\pi_\alpha}, \psi|_{\pi_\alpha}\right)_{L^2(\pi_\alpha)}, \quad D(\mathcal{B}_\nu) := H^1(\mathbb{R}^2) \times H^1(\mathbb{R}^2).$$

As before we denote by the same letter the corresponding quadratic form $\mathcal{B}_\nu(\psi) := \mathcal{B}_\nu(\psi, \psi)$ with domain $D(\mathcal{B}_\nu) := H^1(\mathbb{R}^2)$. The bound

$$\|\psi|_{\pi_\alpha}\|^2_{L^2(\pi_\alpha)} \le \|\partial_{x_\alpha}\psi\|_{L^2(\mathbb{R}^2)} \|\psi\|_{L^2(\mathbb{R}^2)} \le \varepsilon \|\partial_{x_\alpha}\psi\|^2_{L^2(\mathbb{R}^2)} + \frac{1}{\varepsilon} \|\psi\|^2_{L^2(\mathbb{R}^2)}, \tag{8.1}$$

which holds true for all $\varepsilon > 0$, immediately yields

$$\sum_\alpha |\nu_\alpha| \, \|\psi|_{\pi_\alpha}\|^2_{L^2(\pi_\alpha)} \le a\, \mathcal{B}_{\text{free}}(\psi) + b\, \|\psi\|^2_{L^2(\mathbb{R}^2)}$$

for some $0 < a < 1$ and $b > 0$. Hence by the KLMN Theorem the form $\mathcal{B}_\nu$ is closed, semi-bounded, and defines a self-adjoint operator bounded from below (see also [16]). To prove (8.1) note that

$$\|\psi|_{\pi_\alpha}\|^2_{L^2(\pi_\alpha)} = \int_{-\infty}^{\infty} dy_\ell \, |\psi(0, y_\ell)|^2 = \int_{-\infty}^{\infty} dy_\alpha \int_{-\infty}^{0} dx_\alpha \, \partial_{x_\alpha} |\psi(x_\alpha, y_\alpha)|^2$$

then calculate the derivative in the r.h.s. and apply the elementary inequality $ab \le 1/2(a^2 + b^2)$.

Hence we have found a rigorous definition for the formal expression $\mathscr{H}_\nu^{1\text{d}}$ and it remains to write its explicit expression. This is the content of the following theorem. Let us introduce the notation $[\partial_{x_\alpha}\psi]_{\pi_\alpha}$ to denote the jump of the normal derivative of the function ψ across the plane π_α, i.e.,

$$[\partial_{x_\alpha}\psi]_{\pi_\alpha} \equiv [\partial_{x_\alpha}\psi]_{\pi_\alpha}(y_\alpha) := \lim_{\varepsilon \to 0^+} \left(\partial_{x_\alpha}\psi(\varepsilon, y_\alpha) - \partial_{x_\alpha}\psi(-\varepsilon, y_\alpha)\right).$$

Theorem 8.1. *The self-adjoint operator associated to the closed and semi-bounded quadratic form $\mathcal{B}_\nu$ is*

$$D(H_\nu) := \left\{\psi \in H^2(\mathbb{R}^2 \backslash \Pi) \cap H^1(\mathbb{R}^2) |\ [\partial_{x_\alpha}\psi]_{\pi_\alpha} = 2\mu_\alpha \nu_\alpha \psi|_{\pi_\alpha} \ \forall\, \alpha\right\} \tag{8.2}$$

$$H_\nu \psi = H_0 \psi \qquad \text{on } \mathbb{R}^2 \backslash \Pi. \tag{8.3}$$

Proof. According to the general theory (see [63]) the operator associated to $\mathcal{B}_\nu$ is defined by

$$D(H_\nu) := \{\psi \in D(\mathcal{B}_\nu) |\, \exists f \in L^2(\mathbb{R}^2) \text{ s.t. } \mathcal{B}_\nu(\varphi, \psi) = (\varphi, f)\ \forall \varphi \in D(\mathcal{B}_\nu)\}$$

$$H_\nu \psi = f.$$

Let $\varphi \in C_0^\infty(\Gamma_1) \subset D(\mathcal{B}_\nu)$ and $\psi \in D(H_\nu)$. Then

$$(\varphi, f) = \mathcal{B}_\nu(\varphi, \psi) = \mathcal{B}_{\text{free}}(\varphi, \psi),$$

hence $\psi \in H^2(\Gamma_1)$ and $H_0\psi = f$ on Γ_1. Repeating the argument for Γ_r, $r = 2, \ldots, 6$ we conclude that $\psi \in D(H_\nu)$ implies $\psi \in H^2(\mathbb{R}^2 \backslash \Pi)$ and $f = H_0\psi$ on $\mathbb{R}^2 \backslash \Pi$. This proves (8.3).

It remains to show the validity of the boundary condition $[\partial_{x_\alpha}\psi]_{\pi_\alpha} = 2\mu_\alpha \nu_\alpha \psi|_{\pi_\alpha}\ \forall\alpha$. To this end we consider $\varphi \in C_0^\infty(\Gamma_6 \cup \Gamma_1) \in D(\mathcal{B}_\nu)$ and ψ as before. Using for definiteness the coordinates (x_{23}, y_{23}), we have

$$(\varphi, f) = \mathcal{B}_\nu(\varphi,\psi) = \mathcal{B}_{\text{free}}(\varphi,\psi) + \nu_{23}\int_{-\infty}^{0} dy_{23}\, \overline{\varphi|_{\pi_{23}}(y_{23})}\psi|_{\pi_{23}}(y_{23}). \tag{8.4}$$

Let $\pi_{23,\varepsilon} = \{(x_{23}, y_1) : |x_{23}| < \varepsilon, y_1 < 0\}$, $\Gamma_1^\varepsilon = \Gamma_1 \setminus \pi_{23,\varepsilon}$, $\Gamma_6^\varepsilon = \Gamma_6 \setminus \pi_{23,\varepsilon}$. Then, by (8.3) it follows

$$(\varphi, f) = \lim_{\varepsilon\to 0}\left[\int_{\Gamma_6^\varepsilon} \overline{\varphi}\, H_0\psi + \int_{\Gamma_1^\varepsilon} \overline{\varphi}\, H_0\psi\right]. \tag{8.5}$$

On the other hand, integrating by parts we have

$$\begin{aligned}\mathcal{B}_{\text{free}}(\varphi,\psi) = \lim_{\varepsilon\to 0}\Bigg[&\int_{\Gamma_6^\varepsilon} \overline{\varphi}\, H_0\psi + \frac{1}{2\mu_{23}}\int_{-\infty}^{0} dy_1\, \overline{\varphi(-\varepsilon, y_{23})}\, \partial_{x_{23}}\psi(-\varepsilon, y_{23}) \\ &+ \int_{\Gamma_1^\varepsilon} \overline{\varphi}\, H_0\psi - \frac{1}{2\mu_{23}}\int_{-\infty}^{0} dy_{23}\, \overline{\varphi(\varepsilon, y_{23})}\, \partial_{x_{23}}\psi(\varepsilon, y_{23})\Bigg].\end{aligned} \tag{8.6}$$

By (8.4), (8.5), (8.6) we conclude

$$\int_{-\infty}^{0} dy_{23}\, \overline{\varphi|_{\pi_{23}}(y_{23})}\left[\nu_{23}\psi|_{\pi_{23}}(y_{23}) + \frac{1}{2\mu_{23}}\lim_{\varepsilon\to 0}\Big(\partial_{x_{23}}\psi(-\varepsilon, y_{23}) - \partial_{x_{23}}\psi(\varepsilon, y_{23})\Big)\right] = 0$$

For any φ in $C_0^\infty(\Gamma_6 \cup \Gamma_1)$. Hence

$$[\partial_{x_\alpha}\psi]_{\pi_{23}} = 2\mu_{23}\nu_{23}\, \psi|_{\pi_{23}}$$

on $\pi_{23}^- = \{(x_{23}, y_{23}) : x_{23} = 0,\, y_{23} < 0\}$. Repeating the argument for each $\Gamma_i \cup \Gamma_j$, $i, j = 1, \ldots, 6$, $i < j$ we conclude the proof. □

In the following we find a different expression for the operator H_ν. First let us introduce the "charges" $\boldsymbol{\xi} = (\xi^{(1)}, \xi^{(2)}, \xi^{(3)})$ where $\xi^{(\ell)} \in H^{1/2}(\mathbb{R}^3)$ and the "potential" for $\lambda > 0$

$$G^\lambda \boldsymbol{\xi} = \sum_\alpha G_\alpha^\lambda \xi^{(\ell)}$$

where

$$\widehat{G_\alpha^\lambda \xi^{(\ell)}}(k_\alpha, p_\alpha) = \frac{\widehat{\xi}^{(\ell)}(p_\alpha)}{\frac{k_\alpha^2}{2\mu_\alpha} + \frac{p_\alpha^2}{2n_\alpha} + \lambda}.$$

Then we rewrite the definition of the operator H_ν as follows

Lemma 8.1.

$$\begin{aligned} D(H_\nu) = \{\psi \in L^2(\mathbb{R}^2) : \psi = \phi^\lambda + G^\lambda\boldsymbol{\xi},\ \phi^\lambda \in H^2(\mathbb{R}^2), \\ \xi^{(\ell)} \in H^{1/2}(\mathbb{R})\,\forall\ell,\ \phi^\lambda\big|_{\pi_\alpha} = -(\Gamma^\lambda\boldsymbol{\xi})_\alpha\,\forall\alpha\}\end{aligned} \tag{8.7}$$

$$(H_\nu + \lambda)\psi = (H_0 + \lambda)\phi^\lambda \tag{8.8}$$

where

$$\widehat{(\Gamma^\lambda\boldsymbol{\xi})}_\alpha(p_\alpha) = \left(\frac{\sqrt{2\pi}}{\nu_\alpha} + \frac{\sqrt{\pi\mu_\alpha}}{\sqrt{\frac{p_\alpha^2}{2n_\alpha} + \lambda}}\right)\widehat{\xi}^{(\ell)}(p_\alpha) + \sum_{\alpha'\neq\alpha}\frac{1}{\sqrt{2\pi}}\int dp_{\alpha'}\frac{\widehat{\xi}^{(\ell')}(p_{\alpha'})}{H_0(p_\alpha, p_{\alpha'}) + \lambda}.$$

Proof. First we note that $\psi = \phi^\lambda + G^\lambda \boldsymbol{\xi}$ in the domain defined by the right hand side of (8.7) belongs to $H^2(\mathbb{R}^2 \backslash \Pi) \cap H^1(\mathbb{R}^2)$. Furthermore from

$$\left[\partial_{x_\alpha} G^\lambda \boldsymbol{\xi}\right]_{\pi_\alpha} = \left[\partial_{x_\alpha} G^\lambda_\alpha \xi^{(\ell)}\right]_{\pi_\alpha} = -2\sqrt{2\pi}\mu_\alpha \xi^{(\ell)}$$

and the boundary condition $\phi^\lambda\big|_{\pi_\alpha} = -(\Gamma^\lambda \boldsymbol{\xi})_\alpha$ we deduce $[\partial_{x_\alpha}\psi]_{\pi_\alpha} = 2\mu_\alpha \nu_\alpha \psi|_{\pi_\alpha}$. This conclude the proof of (8.7). Finally one can easily show that

$$(H_0 + \lambda) G^\lambda \boldsymbol{\xi} = 0 \quad \text{on } \mathbb{R}^2 \backslash \Pi$$

which, recalling (8.3), immediately implies (8.8) . □

Remark 8.1. From the above lemma it follows immediately that for $\lambda > 0$ large enough the resolvent of H_ν acts in the following way

$$(H_\nu + \lambda)^{-1} f = R_0(-\lambda) f + G^\lambda \boldsymbol{\xi}$$

where, defining

$$\tau_\alpha(\lambda) = \frac{1}{2\pi} \frac{\nu_\alpha}{1 + \frac{\nu_\alpha \sqrt{2\mu_\alpha}}{2\sqrt{\lambda}}}, \tag{8.9}$$

$\widehat{\xi}^{(1)}$ solves the equation

$$\widehat{\xi}^{(1)}(p_{23}) = -\int dp_{31} \frac{\tau_{23}(\lambda + \frac{p_{23}^2}{2n_{23}})}{H_0(p_{23}, p_{31}) + \lambda} \widehat{f}(p_{31}, p_{23})$$
$$- \int dp_{31} \frac{\tau_{23}(\lambda + \frac{p_{23}^2}{2n_{23}})}{H_0(p_{23}, p_{31}) + \lambda} \widehat{\xi}^{(2)}(p_{31}) - \int dp_{12} \frac{\tau_{23}(\lambda + \frac{p_{23}^2}{2n_{23}})}{H_0(p_{23}, p_{12}) + \lambda} \widehat{\xi}^{(3)}(p_{12}).$$

The equations for $\widehat{\xi}^{(2)}, \widehat{\xi}^{(3)}$ are obtained by cyclic permutation of the indices.

Remark 8.2. For any $\nu_\alpha \in \mathbb{R}$. There exist $\lambda_0 > 0$ such that, for all $\lambda > \lambda_0$ one has

$$|\tau_\alpha(\lambda)| \leq \frac{|\nu_\alpha|}{\pi}.$$

To see that this is indeed the case: if $\nu_\alpha \geq 0$ one has $\tau_\alpha(\lambda) \leq \frac{\nu_\alpha}{2\pi}$ for all $\lambda > 0$; if $\nu_\alpha < 0$ take $\lambda_0 = 2\mu_\alpha \nu_\alpha^2$.

8.2 APPROXIMATION BY REGULAR POTENTIALS: THREE IDENTICAL BOSONS

In this section we consider a system of three identical bosons of unitary mass in dimension one interacting through contact. Under these assumptions $\mu_\alpha = \mu = 1/2$, $n_\alpha = n = 2/3$ for any α and the Hilbert space is

$$\mathcal{H}^{\text{sym}} = \{\psi \in L^2(\mathbb{R}^2) : \psi(p_\alpha, p_\beta) = \psi(p_\beta, p_\alpha) = \psi(-p_\alpha - p_\beta, p_\alpha)\}.$$

As in Chapter 3 it is useful to introduce operators to describe symmetry. Let $P_{i \to j}$ be the operator which exchanges particles i, j. For instance in the coordinates (p_{23}, p_{31}) one has

$$P_{1\to 2}\psi(p_{31}, p_{23}) = \psi(p_{23}, p_{31}),$$
$$P_{2\to 3}\psi(p_{31}, p_{23}) = \psi(-p_{31} - p_{23}, p_{23}),$$
$$P_{1\to 3}\psi(p_{31}, p_{23}) = \psi(p_{31}, -p_{31} - p_{23}).$$

Then $\psi \in \mathcal{H}^{\text{sym}}$ if and only if $\psi = P_{i\to j}\psi$ for any i, j.

Moreover, using the symmetry assumptions, one easily conclude $\nu_\alpha = \nu$ and $\xi^{(\ell)} = \xi$ for any α. Hence the Hamiltonian H_ν of the system (see Lemma 8.1) reduced to

$$\begin{gathered} D(H_\nu) = \{\psi \in \mathcal{H}^{\text{sym}} : \psi = \phi^\lambda + \sum_\alpha G^\lambda_\alpha \xi,\ \phi^\lambda \in \mathcal{H}^{\text{sym}} \cap H^2(\mathbb{R}^2), \\ \xi \in H^{1/2}(\mathbb{R})\,,\ \sqrt{2\pi}\int dp'_\alpha\, \widehat{\phi}^\lambda(p_{\alpha'}, p_\alpha) = -\widehat{(\Gamma^\lambda \xi)}(p_\alpha)\ \forall \alpha, \alpha' : \alpha \neq \alpha'\} \\ (H_\nu + \lambda)\psi = (H_0 + \lambda)\phi^\lambda \end{gathered}$$

where

$$\widehat{(\Gamma^\lambda \xi)}(p_\alpha) = \left(\frac{\sqrt{2\pi}}{\nu} + \frac{\sqrt{\pi\mu}}{\sqrt{\frac{p_\alpha^2}{2n} + \lambda}}\right)\widehat{\xi}(p_\alpha) + \sum_{\alpha' \neq \alpha} \frac{1}{\sqrt{2\pi}}\int dp_{\alpha'} \frac{\widehat{\xi}(p_{\alpha'})}{H_0(p_\alpha, p_{\alpha'}) + \lambda}.$$

Furthermore the resolvent operator studied in Remark 8.1 under our assumptions acts as follows:

$$(H_\nu + \lambda)^{-1} f(p_{31}, p_{23}) = \frac{\widehat{f}(p_{31}, p_{23})}{H_0(p_{31}, p_{23}) + \lambda} + \frac{\widehat{\xi}(p_{23}) + \widehat{\xi}(p_{31}) + \widehat{\xi}(-p_{23} - p_{31})}{H_0(p_{31}, p_{23}) + \lambda} \tag{8.10}$$

where $\widehat{\xi}$ solves the equation

$$\widehat{\xi}(p) = -\int dq' \frac{\tau(\lambda + \frac{3}{4}p^2)}{H_0(p, q') + \lambda}\widehat{f}(q', p) - 2\int dq' \frac{\tau(\lambda + \frac{3}{4}p^2)\widehat{\xi}(q')}{H_0(p, q') + \lambda}. \tag{8.11}$$

Here $\tau(\lambda) = \tau_\alpha(\lambda) = \frac{1}{2\pi}\frac{\nu}{1+\nu/2\sqrt{\lambda}}$ for any α, $\tau_\alpha(\lambda)$ being defined in Equation (8.9). Our goal is to obtain the Hamiltonian H_ν as the limit of Hamiltonians with regular rescaled potentials. To this end let us consider a system of three identical bosons of unitary mass interacting through two-body potentials rescaled as follows: $v_{\alpha,\varepsilon}(x_\alpha) = \frac{1}{\varepsilon} v\left(\frac{x_\alpha}{\varepsilon}\right)$ for any α and for some spherically symmetric function v. Then the Hamiltonian of the system in the Fourier space is

$$H_\varepsilon = H_0 + \sum_\alpha V_{\alpha,\varepsilon}$$

where

$$V_{\alpha,\varepsilon}\psi(k_\alpha, p_\alpha) = \frac{1}{\sqrt{2\pi}}\int dk\, \widehat{v}(\varepsilon(k - k_\alpha))\psi(k, p_\alpha).$$

We are interested in the limit of H_ε as ε goes to 0. In particular we want to show that H_ε converges to H_ν in strong resolvent sense where $\nu = \int dx\, v(x) = \sqrt{2\pi}\,\widehat{v}(0)$.

Remark 8.3. Notice that on $\mathcal{H}^{\text{sym}}$ one has $P_{i\to j}H_0 = H_0$. Moreover one can easily show $P_{\ell\to\ell'}V_{\alpha,\varepsilon} = V_{\alpha',\varepsilon}$ for $\ell' \neq \ell$ and finally $P_{i\to j}V_{\alpha,\varepsilon} = V_{\alpha,\varepsilon}$ if $i, j \in \alpha$. Hence in particular $P_{i\to j}H_\varepsilon = H_\varepsilon$ for any i, j.

Our aim is to prove the following theorem.

Theorem 8.2. *Let us assume $v \in L^1(\mathbb{R}, (1 + |x|^2)^{1/2})$, $v(x) = v(|x|)$ and $\nu = \sqrt{2\pi}\,\widehat{v}(0)$. Then there exists $\lambda_0 > 0$ such that for $\lambda > \lambda_0$ one has*

$$\lim_{\varepsilon\to 0} \|(H_\varepsilon + \lambda)^{-1} f - (H_\nu + \lambda)^{-1} f\| = 0 \quad \forall f \in \mathcal{H}^{\text{sym}}.$$

Remark 8.4. Note that by the assumption $v \in L^1(\mathbb{R})$ the interaction term in H_ε is a small perturbation of the free Hamiltonian and KLMN Theorem applies (see [2]). Hence H_ε is a well defined self-adjoint operator bounded from below. In particular, for $\lambda > 0$ large enough, the resolvent $(H_\varepsilon + \lambda)^{-1}$ is a bounded operator.

Remark 8.5. From the assumptions on the potential v it follows $\sup_k |\widehat{v}(k)| \leq C$. Furthermore for any $a \in (0, 2)$

$$\sup_k \frac{|\widehat{v}(k) - \widehat{v}(0)|^2}{k^a} \leq C.$$

To check that this is indeed the case, it is enough to recall that $|e^{ix} - 1| \leq |x|$, hence

$$|\widehat{v}(k) - \widehat{v}(0)| = \frac{1}{\sqrt{2\pi}} \left| \int dx (e^{ixk} - 1) v(x) \right| \leq |k| \frac{1}{\sqrt{2\pi}} \int dx |x| |v(x)|.$$

The proof of Theorem 8.2 is based on the analysis of the Faddeev Equations for the resolvent presented in Section A.3. First we obtain the explicit expression of the equations under our assumptions. By Equation (A.11) we know that

$$R_\varepsilon(-\lambda) f := (H_\varepsilon + \lambda)^{-1} f = R_0(-\lambda) f + \sum_{\ell=1}^{3} R_0(-\lambda) \rho_\varepsilon^{(\ell)} \tag{8.12}$$

where $\rho_\varepsilon^{(\ell)} = -R_0(-\lambda) R^{(\ell)}(-\lambda) f = -V_\alpha R_\varepsilon(-\lambda) f$. Noticing that $f \in \mathcal{H}^{\mathrm{sym}}$ and using the commutation properties of $P_{i \to j}$ seen before we easily conclude

$$\begin{aligned} P_{\ell \to \ell'} \rho_\varepsilon^{(\ell)} &= \rho_\varepsilon^{(\ell')}, \quad \ell \neq \ell' \\ P_{i \to j} \rho_\varepsilon^{(\ell)} &= \rho_\varepsilon^{(\ell)}, \quad i, j \in \alpha. \end{aligned} \tag{8.13}$$

Hence $\widehat{\rho}_\varepsilon(p_{31}, p_{23}) := \widehat{\rho}_\varepsilon^{(1)}(p_{31}, p_{23})$ satisfies the equation

$$\begin{aligned} \widehat{\rho}_\varepsilon(p_{31}, p_{23}) = &- \int dp'_{31} \frac{t_\varepsilon(\lambda + \frac{3}{4} p_{23}^2; -\frac{1}{2} p_{23} - p_{31}, -\frac{1}{2} p_{23} - p'_{31})}{H_0(p_{23}, p'_{31}) + \lambda} \widehat{f}(p'_{31}, p_{23}) \\ &- \int dp'_{31} \frac{t_\varepsilon(\lambda + \frac{3}{4} p_{23}^2; -\frac{1}{2} p_{23} - p_{31}, -\frac{1}{2} p_{23} - p'_{31})}{H_0(p_{23}, p'_{31}) + \lambda} \widehat{\rho}_\varepsilon(p_{23}, p'_{31}) \\ &- \int dp'_{31} \frac{t_\varepsilon(\lambda + \frac{3}{4} p_{23}^2; \frac{1}{2} p_{23} + p_{31}, \frac{1}{2} p_{23} + p'_{31})}{H_0(p_{23}, p'_{31}) + \lambda} \widehat{\rho}_\varepsilon(p'_{31}, -p_{23} - p'_{31}) \end{aligned}$$

where $t_\varepsilon(\lambda; k, k,') := t_{23,\varepsilon}(\lambda; k, k')$ has been defined in Remark A.1. Next in the last integral we use the symmetry property $\widehat{\rho}_\varepsilon(p'_{31}, -p_{23} - p'_{31}) = \widehat{\rho}_\varepsilon(p_{23}, -p_{23} - p'_{31})$ and the change of variable $p'_{12} = -p_{23} - p'_{31}$ obtaining

$$\begin{aligned} \widehat{\rho}_\varepsilon(q, p) = &- \int dq' \frac{t_\varepsilon(\lambda + \frac{3}{4} p^2; -\frac{1}{2} p - q, -\frac{1}{2} p - q')}{H_0(p, q') + \lambda} \widehat{f}(q', p) \\ &- \int dq' \left[\frac{t_\varepsilon(\lambda + \frac{3}{4} p^2; -\frac{1}{2} p - q, -\frac{1}{2} p - q') + t_\varepsilon(\lambda + \frac{3}{4} p^2; \frac{1}{2} p + q, -\frac{1}{2} p - q')}{H_0(p, q') + \lambda} \right] \widehat{\rho}_\varepsilon(p, q'). \end{aligned} \tag{8.14}$$

From $\widehat{\rho}_\varepsilon$, using Equations (8.12) and (8.13), we reconstruct the action of the resolvent $R_\varepsilon(-\lambda)$ as follows:

$$\begin{aligned} R_\varepsilon(-\lambda) f(p_{31}, p_{23}) = &\frac{\widehat{f}(p_{31}, p_{23})}{H_0(p_{31}, p_{23}) + \lambda} \\ &+ \frac{\widehat{\rho}_\varepsilon(p_{31}, p_{23}) + \widehat{\rho}_\varepsilon(p_{23}, p_{31}) + \widehat{\rho}_\varepsilon(p_{31}, -p_{31} - p_{23})}{H_0(p_{31}, p_{23}) + \lambda}. \end{aligned} \tag{8.15}$$

Remark 8.6. We stress the analogy between Equations (8.15),(8.14) and Equations (8.10), (8.11) respectively. In fact, if the interactions are of the form $v_\alpha = \nu\delta(\delta,\cdot)$ then the kernel of the operator $t_\alpha(\lambda)$ defined by Equation (A.16) reduces to

$$t_\alpha(\lambda, k, k') = \frac{1}{2\pi}\frac{\nu}{1+\nu/(2\sqrt{\lambda})} = \tau(\lambda).$$

Since t_α does not depend on k, k' the r.h.s. of the Faddeev Equation (8.14) does not depend on q, hence, $\widehat{\rho}^{(\ell)} = \widehat{\xi}^{(\ell)}$ does not depend on q either. Then the Faddeev Equations reduce to Equation (8.11).

By Equations (8.15),(8.14) and Remark 8.1 it follows that to prove Theorem 8.2 is enough to show the following result.

Theorem 8.3. *Under the same assumptions of Theorem 8.2 there exists λ_0 such that for $\lambda > \lambda_0$ and any $a \in (0,2)$ one has*

$$\left\| \frac{\widehat{\rho}_\varepsilon - \widehat{\xi}}{p^2+q^2+pq+\lambda} \right\| \le \varepsilon^{\frac{a}{2}} \frac{C_a}{\lambda^{\frac{3}{2}-\frac{a}{4}}} \|f\|.$$

The proof of Theorem 8.3 is divided into some steps. First we prove some bound on the kernel t_ε and its difference with τ.

Lemma 8.2. *Under the same assumptions of Theorem 8.2 there exists $\lambda_0 > 0$ such that, for all $\lambda > \lambda_0$, one has*

$$\sup_{k,k'} |t_\varepsilon(\lambda; k, k')| \le C. \tag{8.16}$$

Moreover for any $a \in (0,2)$

$$\sup_{k,k'} \frac{|t_\varepsilon(\lambda; k, k') - \tau(\lambda)|^2}{|k|^a + |k'|^a + 1} \le C\varepsilon^a. \tag{8.17}$$

Remark 8.7. In the proof of the above lemma and in the following we will often use the trivial inequality

$$\frac{1}{p^2+q^2+p\cdot q+\lambda} \le \frac{2}{p^2+2+2\lambda}$$

and the identity

$$\int ds\, \frac{1}{(s^2+\eta)^b} = \frac{C_b}{\eta^{b-1/2}}, \qquad \eta > 0,\ b > 1/2.$$

Proof of Lemma 8.2. Under our assumptions Equation (A.17) reduces to

$$t_\varepsilon(\lambda; k, k') = \frac{\widehat{v}(\varepsilon(k-k'))}{\sqrt{2\pi}} - \frac{1}{\sqrt{2\pi}} \int dp\, \frac{\widehat{v}(\varepsilon(k-p))}{p^2+\lambda} t_\varepsilon(\lambda; p, k'). \tag{8.18}$$

By using Remarks 8.5 and 8.7 one infers,

$$|t_\varepsilon(\lambda; k, k')| \le C + \frac{C}{\sqrt{\lambda}} \sup_p |t_\varepsilon(\lambda; p, k')|.$$

By taking the supremum over k and k', and up to choosing λ large enough, the latter bound implies (8.16).
To prove the bound (8.17), we start by noting that the function $\tau(\lambda)$ satisfies the identity

$$\tau(\lambda) = \frac{\widehat{v}(0)}{\sqrt{2\pi}} - \frac{1}{\sqrt{2\pi}} \int dp\, \frac{\widehat{v}(0)}{p^2+\lambda} \tau(\lambda) \tag{8.19}$$

where we have used $\nu = \sqrt{2\pi}\,\widehat{v}(0)$. By taking the difference of Equations (8.18) and (8.19) one has

$$\begin{aligned} t_\varepsilon(\lambda;k,k') - \tau(\lambda) = & \frac{\widehat{v}(\varepsilon(k-k')) - \widehat{v}(0)}{\sqrt{2\pi}} - \frac{1}{\sqrt{2\pi}} \int dp\, \frac{\widehat{v}(\varepsilon(k-p)) - \widehat{v}(0)}{p^2+\lambda} \tau(\lambda) \\ & - \frac{1}{\sqrt{2\pi}} \int dp\, \frac{\widehat{v}(\varepsilon(k-p))}{p^2+\lambda} (t_\varepsilon(\lambda;p,k') - \tau(\lambda)). \end{aligned} \tag{8.20}$$

We shall prove that

$$\sup_{k,k'} \frac{|t_\varepsilon(\lambda;k,k') - \tau(\lambda)|}{|k|^{\frac{a}{2}} + |k'|^{\frac{a}{2}} + 1} \le C\varepsilon^{\frac{a}{2}}, \tag{8.21}$$

which is equivalent to Bound (8.17).
By using Remark 8.5 in Equation (8.20), one has

$$\begin{aligned} \frac{|t_\varepsilon(\lambda;k,k') - \tau(\lambda)|}{|k|^{\frac{a}{2}} + |k'|^{\frac{a}{2}} + 1} \le & C\varepsilon^{\frac{a}{2}} + \frac{C\varepsilon^{\frac{a}{2}}|\tau(\lambda)|}{|k|^{\frac{a}{2}} + |k'|^{\frac{a}{2}} + 1} \int dp\, \frac{|k|^{\frac{a}{2}} + |p|^{\frac{a}{2}}}{p^2+\lambda} \\ & + \frac{C}{|k|^{\frac{a}{2}} + |k'|^{\frac{a}{2}} + 1} \int dp\, \frac{|p|^{\frac{a}{2}} + |k'|^{\frac{a}{2}} + 1}{p^2+\lambda} \frac{|t_\varepsilon(\lambda;p,k') - \tau(\lambda)|}{|p|^{\frac{a}{2}} + |k'|^{\frac{a}{2}} + 1}. \end{aligned}$$

We note that

$$\frac{1}{|k|^{\frac{a}{2}} + |k'|^{\frac{a}{2}} + 1} \int dp\, \frac{|p|^{\frac{a}{2}} + |k|^{\frac{a}{2}} + |k'|^{\frac{a}{2}} + 1}{p^2+\lambda} \le \frac{C}{\lambda^{\frac{1}{2}}} + \frac{C_a}{\lambda^{\frac{1}{2}-\frac{a}{4}}} \qquad a \in (0,2).$$

Hence, for $\lambda > 1$, one has

$$\sup_{k,k'} \frac{|t_\varepsilon(\lambda;k,k') - \tau(\lambda)|}{|k|^{\frac{a}{2}} + |k'|^{\frac{a}{2}} + 1} \le C\varepsilon^{\frac{a}{2}} \left(1 + \frac{C_a|\tau(\lambda)|}{\lambda^{\frac{1}{2}-\frac{a}{4}}}\right) + \frac{C_a}{\lambda^{\frac{1}{2}-\frac{a}{4}}} \sup_{p,k'} \frac{|t_\varepsilon(\lambda;p,k') - \tau(\lambda)|}{|p|^{\frac{a}{2}} + |k'|^{\frac{a}{2}} + 1}.$$

The latter bound implies the bound (8.21), by Remark 8.2 and up to taking λ large enough. □

Lemma 8.3. *Let $\delta = (3-a)/2$. Then, there exists λ_0 such that for $\lambda > \lambda_0$ one has*

$$\sup_q \int dp\, \frac{1}{(q^2+p^2+2\lambda)^{\frac{3}{2}-\delta}} \left|\widehat{\rho}_\varepsilon(q,p) - \widehat{\xi}(p)\right|^2 \le \varepsilon^a \frac{C_a}{\lambda^{\frac{3}{2}}} \|f\|^2.$$

Proof. We start by noting that

$$\begin{aligned} \sup_q \int dp\, \frac{|\widehat{\rho}_\varepsilon(q,p) - \widehat{\xi}(p)|^2}{(p^2+q^2+2\lambda)^{3/2-\delta}} \le & C \sup_q \int dp\, \frac{1}{(p^2+q^2+2\lambda)^{3/2-\delta}} \\ & [|F(q,p)|^2 + |X_+(q,p)|^2 + |X_-(q,p)|^2 + |Y_+(q,p)|^2 + |Y_-(q,p)|^2] \end{aligned} \tag{8.22}$$

where

$$F(q,p) = -\int dq'\, \frac{t_\varepsilon(\lambda + \frac{3}{4}p^2; q + \frac{p}{2}, q' + \frac{p}{2}) - \tau(\lambda + \frac{3}{4}p^2)}{q'^2 + p^2 + q'p + \lambda} \widehat{f}(q',p);$$

$$X_\pm(q,p) = -\int dq'\, \frac{t_\varepsilon(\lambda + \frac{3}{4}p^2; q + \frac{p}{2}, \pm q' \pm \frac{p}{2}) - \tau(\lambda + \frac{3}{4}p^2)}{q'^2 + p^2 + q'p + \lambda} \widehat{\xi}\,(q')\,;$$

and

$$Y_\pm(q,p) = -\int dq'\, \frac{t_\varepsilon(\lambda + \frac{3}{4}p^2; q + \frac{p}{2}, \pm q' \pm \frac{p}{2})}{q'^2 + p^2 + q'p + \lambda} \left(\widehat{\rho}_\varepsilon\,(p,q') - \widehat{\xi}(q')\right).$$

Next we want to estimate all the terms appearing in the r.h.s. of Equation (8.22). In particular, we want to show

$$\sup_q \int dp \, \frac{1}{(q^2+p^2+2\lambda)^{\frac{3}{2}-\delta}} \, |F(q,p)|^2 \leq \varepsilon^a \frac{C_a}{\lambda^{\frac{3}{2}}} \|f\|^2, \tag{8.23}$$

$$\sup_q \int dp \, \frac{1}{(q^2+p^2+2\lambda)^{\frac{3}{2}-\delta}} \, |X_\pm(q,p)|^2 \leq \varepsilon^a \frac{C_a}{\lambda^{\frac{5}{2}}} \|f\|^2, \tag{8.24}$$

$$\begin{aligned} \sup_q \int dp \, &\frac{1}{(q^2+p^2+2\lambda)^{\frac{3}{2}-\delta}} \, |Y_\pm(q,p)|^2 \\ &\leq \frac{C_\delta}{\lambda} \sup_q \int dp \, \frac{1}{(q^2+p^2+2\lambda)^{\frac{3}{2}-\delta}} \left| \widehat{\rho}_\varepsilon\,(q,p) - \widehat{\xi}(p) \right|^2. \end{aligned} \tag{8.25}$$

We start with Bound (8.25).

$$\begin{aligned} &\sup_q \int dp \, \frac{1}{(q^2+p^2+2\lambda)^{\frac{3}{2}-\delta}} \, |Y_\pm(q,p)|^2 \\ &= \sup_q \int dp \frac{1}{(q^2+p^2+2\lambda)^{\frac{3}{2}-\delta}} \left| \int dq' \, \frac{t_\varepsilon(\lambda+\frac{3}{4}p^2; q+\frac{p}{2}, \pm q' \pm \frac{p}{2})}{q'^2+p^2+q'p+\lambda} \left(\widehat{\rho}_\varepsilon\,(p,q') - \widehat{\xi}(q') \right) \right|^2 \\ &\leq C_\delta \sup_q \int dp \, \frac{1}{(q^2+p^2+2\lambda)^{\frac{3}{2}-\delta}} \frac{1}{(p^2+2\lambda)^{\delta}} \int dq' \, \frac{\left| \widehat{\rho}_\varepsilon\,(p,q') - \widehat{\xi}(q') \right|^2}{(q'^2+p^2+2\lambda)^{\frac{3}{2}-\delta}} \\ &\leq \frac{C_\delta}{\lambda} \sup_p \int dq' \, \frac{\left| \widehat{\rho}_\varepsilon\,(p,q') - \widehat{\xi}(q') \right|^2}{(q'^2+p^2+2\lambda)^{\frac{3}{2}-\delta}}, \end{aligned}$$

for any $0 < \delta \leq 3/2$. In the first inequality we used the bound (8.16), Cauchy-Schwarz inequality in the integral in q', and Remark 8.7. In the second inequality we dropped q from the denominator (here we used the constraint $0 < \delta \leq 3/2$), took the supremum on p and used again Remark 8.7. Bound (8.25) is obtained by changing p in q and q' in p at the r.h.s.. Next we consider Bound (8.23).

$$\begin{aligned} &\sup_q \int dp \, \frac{1}{(q^2+p^2+2\lambda)^{\frac{3}{2}-\delta}} \, |F(q,p)|^2 \\ &= \sup_q \int dp \, \frac{1}{(q^2+p^2+2\lambda)^{\frac{3}{2}-\delta}} \left| \int dq' \, \frac{t_\varepsilon(\lambda+\frac{3}{4}p^2; q+\frac{p}{2}, q'+\frac{p}{2}) - \tau(\lambda+\frac{3}{4}p^2)}{q'^2+p^2+q'p+\lambda} \widehat{f}(q',p) \right|^2 \\ &\leq 4\|f\| \sup_{q,p} \frac{1}{(q^2+p^2+2\lambda)^{\frac{3}{2}-\delta}} \int dq' \, \frac{|t_\varepsilon(\lambda+\frac{3}{4}p^2; q+\frac{p}{2}, q'+\frac{p}{2}) - \tau(\lambda+\frac{3}{4}p^2)|^2}{(q'^2+p^2+2\lambda)^2}. \end{aligned}$$

Here we used first Cauchy-Schwarz inequality in the integral in q', then took the supremum in p, and used Remark 8.7.

Before proceeding further we note that for the term in the Bound (8.24) we have

$$\begin{aligned} &\sup_q \int dp \, \frac{1}{(q^2+p^2+2\lambda)^{\frac{3}{2}-\delta}} \, |X_\pm(q,p)|^2 \\ &= \sup_q \int dp \, \frac{1}{(q^2+p^2+2\lambda)^{\frac{3}{2}-\delta}} \left| \int dq' \, \frac{t_\varepsilon(\lambda+\frac{3}{4}p^2; q+\frac{p}{2}, \pm q' \pm \frac{p}{2}) - \tau(\lambda+\frac{3}{4}p^2)}{q'^2+p^2+q'p+\lambda} \widehat{\xi}\,(q') \right|^2 \\ &\leq \frac{C}{\lambda^{\frac{3}{2}}} \|f\|^2 \sup_q \int dp \, \frac{1}{(q^2+p^2+2\lambda)^{\frac{3}{2}-\delta}} \int dq' \, \frac{|t_\varepsilon(\lambda+\frac{3}{4}p^2; q+\frac{p}{2}, q'+\frac{p}{2}) - \tau(\lambda+\frac{3}{4}p^2)|^2}{(q'^2+p^2+2\lambda)^2}. \end{aligned}$$

Here we used Cauchy-Schwarz inequality and the fact that

$$\|\xi\| \leq \frac{C}{\lambda^{\frac{3}{2}}}\|f\|^2,$$

for λ large enough, which follows easily from Equation (8.11) and Remarks 8.2 and 8.7.
We are left to prove that

$$\sup_{q,p} \frac{1}{(q^2+p^2+2\lambda)^{\frac{3}{2}-\delta}} \int dq' \, \frac{|t_\varepsilon(\lambda+\frac{3}{4}p^2; q+\frac{p}{2}, q'+\frac{p}{2}) - \tau(\lambda+\frac{3}{4}p^2)|^2}{(q'^2+p^2+2\lambda)^2} \leq \varepsilon^a \frac{C_a}{\lambda^{\frac{3}{2}}} \tag{8.26}$$

and

$$\sup_{q} \int dp \, \frac{1}{(q^2+p^2+2\lambda)^{\frac{3}{2}-\delta}} \int dq' \, \frac{|t_\varepsilon(\lambda+\frac{3}{4}p^2; q+\frac{p}{2}, q'+\frac{p}{2}) - \tau(\lambda+\frac{3}{4}p^2)|^2}{(q'^2+p^2+2\lambda)^2} \leq \varepsilon^a \frac{C_a}{\lambda}. \tag{8.27}$$

We start by noticing that

$$\begin{aligned}\int dq' \, \frac{|t_\varepsilon(\lambda+\frac{3}{4}p^2; q+\frac{p}{2}, q'+\frac{p}{2}) - \tau(\lambda+\frac{3}{4}p^2)|^2}{(q'^2+p^2+2\lambda)^2} &\leq C\varepsilon^a \int dq' \, \frac{1+|p|^a+|q|^a+|q'|^a}{(q'^2+p^2+2\lambda)^2} \\ &\leq C_a\varepsilon^a \left(\frac{1+|p|^a+|q|^a}{(p^2+2\lambda)^{\frac{3}{2}}} + \frac{1}{(p^2+2\lambda)^{\frac{3-a}{2}}} \right)\end{aligned}$$

where we used Remark 8.7, Equation (8.17) and the trivial identity
$\int dq' \, \frac{|q'|^a}{(q'^2+\eta)^2} = \eta^{-\frac{3-a}{2}} \int ds \, \frac{s^a}{(s^2+1)^2} = C_a \eta^{-\frac{3-a}{2}}$, which holds true for all $a \in (0,3)$.
Setting $\delta = (3-a)/2$, one has

$$\begin{aligned}&\sup_q \frac{1}{(q^2+p^2+2\lambda)^{\frac{3}{2}-\delta}} \left(\frac{1+|p|^a+|q|^a}{(p^2+2\lambda)^{\frac{3}{2}}} + \frac{1}{(p^2+2\lambda)^{\frac{3-a}{2}}} \right) \\ &= \sup_q \frac{1}{(q^2+p^2+2\lambda)^{\frac{a}{2}}} \left(\frac{1+|p|^a+|q|^a}{(p^2+2\lambda)^{\frac{3}{2}}} + \frac{1}{(p^2+2\lambda)^{\frac{3-a}{2}}} \right) \leq \frac{C}{(p^2+2\lambda)^{\frac{3}{2}}},\end{aligned}$$

in the latter inequality we assumed $2\lambda \geq 1$, which in turn implies $\frac{1}{(p^2+2\lambda)^{\frac{3+a}{2}}} \leq \frac{1}{(p^2+2\lambda)^{\frac{3}{2}}}$.
Then Bounds (8.23) and (8.24) follow from (8.26) and (8.27).
By Equations (8.22), (8.23), (8.24), (8.25) one has that, for λ large enough,

$$\begin{aligned}&\sup_q \int dp \, \frac{1}{(q^2+p^2+2\lambda)^{\frac{3}{2}-\delta}} \left| \widehat{\rho}_\varepsilon(q,p) - \widehat{\xi}(p) \right|^2 \\ &\leq \varepsilon^a \frac{C_a}{\lambda^{\frac{3}{2}}} \|f\|^2 + \frac{C_\delta}{\lambda} \sup_q \int dp \, \frac{1}{(q^2+p^2+2\lambda)^{\frac{3}{2}-\delta}} \left| \widehat{\rho}_\varepsilon(q,p) - \widehat{\xi}(p) \right|^2.\end{aligned}$$

Hence,

$$\begin{aligned}\sup_q \int dp \, \frac{1}{(q^2+p^2+2\lambda)^{\frac{3}{2}-\delta}} \left| \widehat{\rho}_\varepsilon(q,p) - \widehat{\xi}(p) \right|^2 &\leq \varepsilon^a \frac{C_a}{\lambda^{\frac{3}{2}}} \frac{1}{1-C_\delta/\lambda} \|f\|^2 \\ &\leq \varepsilon^a \frac{C_a}{\lambda^{\frac{3}{2}}} \|f\|^2.\end{aligned}$$

□

Proof of Theorem 8.3. We notice that for any $\delta > 0$

$$\begin{aligned}\int dp\, dq\, \frac{|\widehat{\rho}_\varepsilon(q,p) - \widehat{\xi}(p)|^2}{p^2 + q^2 + qp + \lambda} \leq & 4 \int dq\, \frac{1}{(q^2 + 2\lambda)^{\frac{1}{2}+\delta}} \\ & \times \int dp\, \frac{1}{(q^2 + p^2 + 2\lambda)^{\frac{3}{2}-\delta}} \left|\widehat{\rho}_\varepsilon(q,p) - \widehat{\xi}(p)\right|^2 \\ \leq & \frac{C_\delta}{\lambda^\delta} \sup_q \int dp\, \frac{1}{(q^2 + p^2 + 2\lambda)^{\frac{3}{2}-\delta}} \left|\widehat{\rho}_\varepsilon(q,p) - \widehat{\xi}(p)\right|^2 .\end{aligned}$$

Setting $\delta = (3 - a)/2$ the thesis follows immediately from Lemma 8.3. □

9 THREE DIMENSIONS

9.1 THE ENERGY FORM

In this chapter we present the results obtained in [10] on the study of three-particle system with zero-range interaction in dimension 3. The Hamiltonian of the system is $\mathscr{H}_{\nu}^{3\text{body},3}$ defined in Equation (1.4). Removing the center of mass motion, along the same line of Appendix A, we obtain the formal Hamiltonian in $L^2(\mathbb{R}^6)$

$$\mathscr{H}_{\nu}^{3d} = H_0 + \sum_{\alpha} \nu_\alpha \delta(x_\alpha).$$

As in the one dimensional case we want to use a quadratic form approach to give a rigorous definition of $\mathscr{H}_{\nu}^{3d}$. Unfortunately the quadratic form associated to $\mathscr{H}_{\nu}^{3d}$ is no longer a small perturbation of the free form and something different has to be done. The strategy is then to take advantage of the well known two-body case selecting the so-called Skornyakov-Ter-Martirosyan (STM) extensions.

In this coordinates the generalized boundary conditions stated in Equation (7.3) reduced to

$$\psi(x_\alpha, y_\alpha) = \frac{\xi^{(\ell)}(y_\alpha)}{|x_\alpha|} + r_\alpha(y_a) + o(1) \qquad \text{with } r_\alpha = \nu_\alpha \xi^{(\ell)} \tag{9.1}$$

as $|x_\alpha| \to 0$. Where $\xi^{(\ell)}$ is a suitable function defined on the hyperplane $\pi_\alpha = \{x_\alpha = 0\}$. Furthermore we ask the analogous of (7.4):

$$\mathscr{H}_{\nu}^{3d}\psi = H_0\psi \qquad \text{if } |x_\alpha| \neq 0 \quad \forall \alpha. \tag{9.2}$$

Let us introduce the "potential" produced by the "charges" $\xi^{(\ell)}$ as we have done in the one dimensional case in the previous chapter. Let $\boldsymbol{\xi} = (\xi^{(1)}, \xi^{(2)}, \xi^{(3)})$, we set

$$G\boldsymbol{\xi} = \sum_{\alpha} G_\alpha \xi^{(\ell)}$$

where in Fourier transform

$$\widehat{G_\alpha \xi^{(\ell)}}(k_\alpha, p_\alpha) = \frac{\widehat{\xi}^{(\ell)}(p_\alpha)}{\frac{k_\alpha^2}{2\mu_\alpha} + \frac{p_\alpha^2}{2n_\alpha}}.$$

Following the line of Proposition 6.1 in [42] one shows that $G\boldsymbol{\xi}$ solves in the distributional sense the equation

$$H_0 G\boldsymbol{\xi} = (2\pi)^{3/2} \sum_{\alpha} \delta(x_\alpha)\xi^{(\ell)}(y_a). \tag{9.3}$$

Moreover $G\boldsymbol{\xi}$ has the following behavior when $|x_\alpha| \to 0$

$$G\boldsymbol{\xi}(x_\alpha, y_\alpha) = \frac{\sqrt{2\pi}\mu_\alpha \xi_\alpha(y_\alpha)}{|x_\alpha|} - (\Gamma\boldsymbol{\xi})_\alpha(y_a) + o(1) \tag{9.4}$$

where in Fourier transform

$$(\widehat{\Gamma\xi})_\alpha(p_\alpha) = \mu_\alpha\sqrt{2\pi}\,\widehat{\xi}^{(\ell)}(p_a)\sqrt{\frac{\mu_\alpha}{n_\alpha}p_a^2} - \sum_{\alpha'\neq\alpha}\frac{1}{(2\pi)^{3/2}}\int dp_{\alpha'}\frac{\widehat{\xi}^{(\ell')}(p_{\alpha'})}{H_0(p_\alpha,p_{\alpha'})} \tag{9.5}$$

with $H_0(p_\alpha, p_{\alpha'})$ defined in (A.6). Next, proceeding as in the one-dimensional case we decompose the generic element ψ in the domain of $\mathscr{H}^{3\text{d}}$ as

$$\psi = \phi + G\xi \tag{9.6}$$

where ϕ is a smooth function. Then Equations (9.4) and (9.1) yield

$$\phi|_{\pi_\alpha}(y_\alpha) = \sqrt{2\pi}\mu_\alpha\nu_\alpha\xi^{(\ell)}(y_\alpha) + (\Gamma\xi)_\alpha(y_a). \tag{9.7}$$

The next step is to use the decomposition (9.6) to obtain the explicit expression of the quadratic form $\mathcal{B}_\nu$ associated to the operator $\mathscr{H}_\nu^{3\text{d}}$. As in Chapter 8 we will denote with $\mathcal{B}_{\text{free}}$ the quadratic form associated with the free Hamiltonian. We set $\Omega_\varepsilon = \{|x_\alpha| > \varepsilon, \forall\alpha\}$. Then taking into account Equation (9.3) and the request (9.2) we have

$$\begin{aligned}
\mathcal{B}_\nu(\psi) =&(\psi, \mathscr{H}_\nu^{3\text{d}}\psi) = \lim_{\varepsilon\to 0}\int_{\Omega_\varepsilon}\overline{\psi}H_0\psi\\
=&\mathcal{B}_{\text{free}}(\phi) + \lim_{\varepsilon\to 0}\int_{\Omega_\varepsilon}\overline{G\xi}H_0\phi\\
=&\mathcal{B}_{\text{free}}(\phi) + \sum_\alpha\Bigg[(2\pi)^2\mu_\alpha\int dp_\alpha|\widehat{\xi}^{(\ell)}|^2\left(\nu_\alpha + \sqrt{\frac{\mu_\alpha}{n_\alpha}p_\alpha^2}\right)\\
&-\sum_{\alpha'\neq\alpha}\int dp_\alpha dp_{\alpha'}\frac{\widehat{\xi}^{(\ell)}(p_\alpha)\widehat{\xi}^{(\ell')}(p_{\alpha'})}{H_0(p_\alpha,p_{\alpha'})}\Bigg].
\end{aligned}$$

Where we have used also (9.5) and (9.7). Hence we can finally give a rigorous definition of $\mathcal{B}_\nu$:

$$D(\mathcal{B}_\nu) = \{\psi\in L^2(\mathbb{R}^6) : \psi = \phi + G\xi,\ |\nabla\phi|\in L^2(\mathbb{R}^6),\ \xi^{(\ell)}\in H^{1/2}(\mathbb{R}^3)\} \tag{9.8}$$

$$\begin{aligned}
\mathcal{B}_\nu(\psi) = \mathcal{B}_{\text{free}}(\phi) + \sum_\alpha\Bigg[&(2\pi)^2\mu_\alpha\int dp_\alpha|\widehat{\xi}^{(\ell)}|^2\left(\nu_\alpha + \sqrt{\frac{\mu_\alpha}{n_\alpha}p_\alpha^2}\right)\\
&-\sum_{\alpha'\neq\alpha}\int dp_\alpha dp_{\alpha'}\frac{\widehat{\xi}^{(\ell)}(p_\alpha)\widehat{\xi}^{(\ell')}(p_{\alpha'})}{H_0(p_\alpha,p_{\alpha'})}\Bigg].
\end{aligned} \tag{9.9}$$

Remark 9.1. We note that in (9.8) the choice of the charges $\xi^{(\ell)}\in H^{1/2}(\mathbb{R}^3)$ guarantees that all terms in the square brackets of (9.9) are finite. However, it is not a priori clear for which class of charges the *sum* of the last two terms in the square brackets is finite. Therefore our choice has some degree of arbitrariness and in fact, in some relevant cases, a larger class of charges must be considered ([25]).

9.2 THREE BOSONS FOR NON-ZERO ANGULAR MOMENTUM

For a system of three identical bosons of unitary masses, considered in the center of mass reference frame, the Hilbert space of states in momentum space in the coordinates (p_α, p_β) is

$$\mathcal{H}^{\text{sym}} = \{\psi\in L^2(\mathbb{R}^6) : \psi(p_{23}, p_{31}) = \psi(p_{31}, p_{23}) = \psi(p_{23}, -p_{23} - p_{31})\}.$$

Moreover the symmetry condition implies that $\nu_\alpha = \nu$ and $\xi^{(\ell)} = \xi$ for any α, ℓ respectively. Then we have the following expression for the potential

$$(\widehat{G\xi})(p_{23}, p_{31}) = \widehat{G\xi}(p_{23}, p_{31}) = \frac{\hat{\xi}(p_{23}) + \hat{\xi}(p_{31}) + \hat{\xi}(-p_{23} - p_{31})}{p_{23}^2 + p_{31}^2 + p_{23} \cdot p_{31}}$$

With an abuse of notation we define the quadratic form associated to the STM operator in the bosonic case as

$$D(\mathcal{B}_\nu) = \left\{ \psi \in \mathcal{H}^{\mathrm{sym}} \,:\, \psi = \phi + G\xi,\ |\nabla\phi| \in \mathcal{H}^{\mathrm{sym}},\ \xi \in H^{1/2}(\mathbb{R}^3) \right\} \tag{9.10}$$

$$\mathcal{B}_\nu(\psi) = \mathcal{B}_{\mathrm{free}}(\phi) + 6\pi^2\, \Phi_\nu(\xi) \tag{9.11}$$

where the form Φ_ν acting on the charge $\xi \in D(\Phi_\nu) = H^{1/2}(\mathbb{R}^3)$ is given by

$$\Phi_\nu(\xi) = \Phi^{\mathrm{diag}}(\hat{\xi}) + \Phi^{\mathrm{off}}(\hat{\xi}) + \nu \|\xi\|^2$$

and the diagonal part and the off-diagonal part are defined respectively by

$$\Phi^{\mathrm{diag}}(f) = \frac{\sqrt{3}}{2} \int dp\, |p|\, |f(p)|^2$$

$$\Phi^{\mathrm{off}}(f) = -\frac{1}{\pi^2} \int dp_1\, dp_2\, \frac{\overline{f(p_1)} f(p_2)}{p_1^2 + p_2^2 + p_1 \cdot p_2}$$

It is easy to see that if one can find an f_0 such that $\Phi^{\mathrm{diag}}(f_0) + \Phi^{\mathrm{off}}(f_0) < 0$ then, by a scaling argument, one shows that the form (9.11) is unbounded from below. As a matter of fact, such f_0 can be explicitly constructed and it is rotationally invariant (for the proof one can follows the line of [42], Section 4). This fact is not surprising since it is known that the STM operator for three bosons is not self-adjoint and all its self-adjoint extensions are unbounded from below, showing the occurrence of the Thomas effect ([36]). Following [48], we define $\mathcal{H}_0 = \{\psi \in \mathcal{H}^{\mathrm{sym}} \,|\, \hat{\psi} = \hat{\psi}(|p_{23}|, |p_{31}|)\}$, which is an invariant subspace for the STM operator, and we consider its orthogonal complement $\mathcal{H}_0^\perp$. In the next theorem we characterize our quadratic form in $\mathcal{H}_0^\perp$.

Theorem 9.1. *The quadratic form (9.11) restricted to the subspace $\mathcal{H}_0^\perp$ is bounded from below and closed for any $\nu \in \mathbb{R}$.*

We start with some preliminaries, following the line of [24]. Given $f \in L^2(\mathbb{R}^3)$, we consider its expansion in spherical harmonics. Using this expansion one can obtain the following decompositions for Φ^{off} and Φ^{diag} (see [24], Lemma 3.1)

$$\begin{aligned} \Phi^{\mathrm{diag}}(f) &= \sum_{l=0}^{+\infty} \sum_{\nu=-l}^{l} F^{\mathrm{diag}}(f_{l\nu}) \\ \Phi^{\mathrm{off}}(f) &= \sum_{l=0}^{+\infty} \sum_{\nu=-l}^{l} F_l^{\mathrm{off}}(f_{l\nu}) \end{aligned} \tag{9.12}$$

where $f_{l\nu}$ is defined in (6.3) and F^{diag}, F_l^{off} acting on $g \in L^2((0, +\infty), (1+\rho^3)d\rho)$as

$$F^{\mathrm{diag}}(g) = \frac{\sqrt{3}}{2} \int_0^{+\infty} d\rho\, \rho^3\, |g(\rho)|^2$$

$$F_l^{\mathrm{off}}(g) = -\frac{2}{\pi} \int_0^{+\infty} d\rho \int_0^{+\infty} d\rho'\, \rho^2\, \overline{g(\rho)} \rho'^2\, g(\rho') \int_{-1}^{1} dy\, \frac{P_l(y)}{\rho^2 + \rho'^2 + \rho\rho' y}$$

where P_l denotes the Legendre polynomial of order l. Indeed

$$\Phi^{\text{diag}}(f) = \frac{\sqrt{3}}{2}\int_0^{+\infty} d\rho\,\rho^3 \int_{\mathbb{S}^2} d\Omega(\zeta)\left|\sum_{l=0}^{+\infty}\sum_{\nu=-l}^{l} f_{l\nu}(\rho)Y_l^\nu(\zeta)\right|^2$$
$$= \sum_{l=0}^{+\infty}\sum_{\nu=-l}^{l}\frac{\sqrt{3}}{2}\int_0^{+\infty} d\rho\,\rho^3\,|f_{l\nu}(\rho)|^2.$$

While for the off diagonal term we first use the addition formula (6.2) to rewrite

$$\frac{1}{p_1^2+p_2^2+p_1\cdot p_2} = 2\pi\sum_{l=0}^{+\infty}\sum_{\nu=-l}^{l}\overline{Y_l^\nu(\zeta)}Y_l^\nu(\omega)\int_{-1}^{1} dy\,\frac{P_l(y)}{|p_1|^2+|p_2|^2+|p_1||p_2|y}$$

where $p_1 = (|p_1|,\zeta)$, $p_2 = (|p_2|,\omega)$, $\omega,\zeta\in\mathbb{S}^2$, from which it follows

$$\Phi^{\text{off}}(f) = -\frac{2}{\pi}\int_0^{+\infty} d\rho\,\rho^2\int_0^{+\infty} d\rho'\rho'^2\int_{\mathbb{S}^2} d\Omega(\zeta)\int_{\mathbb{S}^2} d\Omega(\omega)\overline{f(\rho,\zeta)}f(\rho',\omega)$$
$$\times\sum_{l=0}^{+\infty}\sum_{\nu=-l}^{l}\overline{Y_l^\nu(\zeta)}Y_l^\nu(\omega)\int_{-1}^{1} dy\,\frac{P_l(y)}{\rho^2+\rho'^2+\rho\rho' y}$$
$$= -\frac{2}{\pi}\sum_{l=0}^{+\infty}\sum_{\nu=-l}^{l}\int_0^{+\infty} d\rho\,\rho^2\int_0^{+\infty} d\rho\,\rho'^2 f_{l\nu}(\rho)f_{l\nu}(\rho')\int_{-1}^{1} dy\,\frac{P_l(y)}{\rho^2+\rho'^2+\rho\rho' y}.$$

Proceeding as in Lemma 3.2 of [24], one proves that

$$F_l^{\text{off}}(g)\geq 0 \qquad \text{for l odd} \tag{9.13}$$
$$F_l^{\text{off}}(g)\leq 0 \qquad \text{for l even} \tag{9.14}$$

Moreover F_l^{off} can be diagonalized. Setting

$$g^\sharp(k) = \frac{1}{\sqrt{2\pi}}\int d\rho\, e^{-ik\rho}e^{2\rho}g(e^\rho)$$

we have (for details see [24], Lemma 3.3)

$$\begin{aligned} F^{\text{diag}}(g) &= \frac{\sqrt{3}}{2}\int dk\,|g^\sharp(k)|^2 \\ F_l^{\text{off}}(g) &= \sqrt{6\pi}\int dk\,\widehat{S}^{(l)}(k)|g^\sharp(k)|^2 \end{aligned} \tag{9.15}$$

where $\widehat{S}^{(l)}(k) = \widehat{S}_1^{(l)}(k)$ is defined in Appendix B. Therefore the comparison between F_l^{off} and F^{diag} is reduced to the study of $S_l(k)$. The following estimate, which is the main tool in the proof of Theorem 9.1, is a direct consequence of Lemmas B.1, B.2.

Proposition 9.1. *Let $f\in D(\Phi_\nu)$. Then*

$$-\Omega\,\Phi^{\text{diag}}(f)\leq\Phi^{\text{off}}(f)\leq\Sigma\,\Phi^{\text{diag}}(f)$$

where

$$\Omega = \frac{100}{27\sqrt{3}}\pi - \frac{20}{3} + \frac{2\sqrt{11}}{9\sqrt{3}} - \frac{20}{9\sqrt{3}}t_0 \simeq 0.101\,, \qquad \Sigma = -\frac{8}{3}+\frac{16}{\sqrt{3}\,\pi}\simeq 0.274.$$

Proof. Using (9.12), (9.13), (9.15), (B.6), we have

$$\begin{aligned}
\Phi^{\text{off}}(f) &= \sum_{l=1}^{+\infty} \sum_{\nu=-l}^{l} F_l^{\text{off}}(f_{l\nu}) \geq \sum_{\substack{l=2 \\ l \text{ even}}}^{+\infty} \sum_{\nu=-l}^{l} F_l^{\text{off}}(f_{l\nu}) \\
&= \sqrt{6}\pi \sum_{\substack{l=2 \\ l \text{ even}}}^{+\infty} \sum_{\nu=-l}^{l} \int dk\, \widehat{S}^{(l)}(k)\, |f^\sharp_{l\nu}(k)|^2 \geq -\Omega \sum_{\substack{l=2 \\ l \text{ even}}}^{+\infty} \sum_{\nu=-l}^{l} \frac{\sqrt{3}}{2} \int dk\, |f^\sharp_{l\nu}(k)|^2 \\
&\geq -\Omega\, \Phi^{\text{diag}}(f)
\end{aligned}$$

and analogously recalling Equations (B.4), (B.5) and (3.8)

$$\begin{aligned}
\Phi^{\text{off}}(f) &\leq \sum_{\substack{l=1 \\ l \text{ odd}}}^{+\infty} \sum_{\nu=-l}^{l} F_l^{\text{off}}(f_{l\nu}) \\
&\leq \sqrt{6}\pi \sum_{\substack{l=1 \\ l \text{ odd}}}^{+\infty} \sum_{\nu=-l}^{l} \int dk\, \widehat{S}^{(l)}(k)\, |f^\sharp_{l\nu}(k)|^2 \leq 2\Lambda(1) \sum_{\substack{l=1 \\ l \text{ odd}}}^{+\infty} \sum_{\nu=-l}^{l} F^{\text{diag}}(f_{l\nu}) \\
&= \Sigma\, \Phi^{\text{diag}}(f)
\end{aligned}$$

□

Proof of Theorem 9.1. We first consider the simpler case $\nu > 0$. From Equation (9.11) and Proposition 9.1 we obtain the positivity of $\mathcal{B}_\nu$

$$\begin{aligned}
\mathcal{B}_\nu(\psi) &= \mathcal{B}_{\text{free}}(\phi) + 6\pi^2 \Phi_\nu(\xi) \\
&\geq 6\pi^2 \left[\Phi^{\text{off}}(\widehat{\xi}) + \Phi^{\text{diag}}(\widehat{\xi}) + \nu \int dp |\widehat{\xi}(p)|^2 \right] \\
&\geq 6\pi^2 \left[(1-\Omega)\Phi^{\text{diag}}(\widehat{\xi}) + \nu \int dp |\widehat{\xi}(p)|^2 \right] \geq 0
\end{aligned}$$

Let us prove that $\mathcal{B}_\nu$ is closed. Let $\{\psi_n\} = \{\phi_n + G\xi_n\}$ be a sequence in $D(\mathcal{B}_\nu)$ such that $\psi_n \to \psi \in L^2_{\text{sym}}(\mathbb{R}^6)$ and $\mathcal{B}_\nu(\psi_n - \psi_m) \to 0$.
From $\mathcal{B}_\nu(\psi_n - \psi_m) \to 0$, the positivity of H_0 and the lower bound for Φ_ν it follows

$$\int dp_{23}\, dp_{31}\, (p_{23}^2 + p_{31}^2) \big|(\widehat{\phi}_n - \widehat{\phi}_m)(p_{23}, p_{31})\big|^2 \to 0$$

$$\|\xi_n - \xi_m\|_{H^{1/2}} \to 0$$

Thus there exist $\widetilde{\phi} \in \mathcal{H}^{\text{sym}}$ and $\xi \in H^{1/2}(\mathbb{R}^3)$ such that

$$\int dp \left| \sqrt{p_{23}^2 + p_{31}^2}\, \phi_n(p_{23}, p_{31}) - \widetilde{\phi}(p_{23}, p_{31}) \right|^2 \to 0$$

$$\|\xi_n - \xi\|_{H^{1/2}} \to 0$$

Defining $\widehat{\phi} = \dfrac{\widehat{\widetilde{\phi}}}{\sqrt{k_1^2 + k_2^2}}$, for any $\varepsilon > 0$ we have

$$\int_{\mathbb{R}^6_\varepsilon} dp_{23}\, dp_{31}\, |(\widehat{\phi}_n - \widehat{\phi})(p_{23}, p_{31})|^2 \to 0 \tag{9.16}$$

$$\int_{\mathbb{R}^6_\varepsilon} dp_{23}\, dp_{31} \big|(\widehat{G\xi}_n - \widehat{G\xi})(p_{23}, p_{31})\big|^2 \to 0 \tag{9.17}$$

where $\mathbb{R}^d_\varepsilon = \{x \in \mathbb{R}^d \,|\, |x| \geq \varepsilon\}$. From (9.16) and (9.17) in particular we obtain

$$\psi = \phi + G\xi \in D(\mathcal{B}_\nu)$$

and also $\mathcal{B}_\nu(\psi_n - \psi) \to 0$. This concludes the proof in the case $\nu > 0$.
In order to study the case $\nu \leq 0$ it is convenient to consider the following decomposition for the generic ψ in the domain of $\mathcal{B}_\nu$

$$\psi = \phi^\lambda + G^\lambda \xi$$

where $\lambda > 0$ and

$$G^\lambda \xi(p_{23}, p_{31}) = \frac{\widehat{\xi}(p_{23}) + \widehat{\xi}(p_{31}) + \widehat{\xi}(-p_{23} - p_{31})}{p_{23}^2 + p_{31}^2 + p_{23} \cdot p_{31} + \lambda}$$

Thus $G^\lambda \xi$ belongs to $\mathcal{H}^{\mathrm{sym}}$ and ϕ^λ is in $H^1(\mathbb{R}^6)$. Moreover the quadratic form can be rewritten as

$$\mathcal{B}_\nu(\psi) = \mathcal{B}_{\mathrm{free}}(\phi^\lambda) + \lambda \|\phi^\lambda\|^2 - \lambda \|\psi\|^2 + 6\pi^2\, \Phi^\lambda_\nu(\xi)$$

where

$$\Phi^\lambda_\nu(\xi) = \left[\Phi^{\mathrm{diag}}_\lambda(\widehat{\xi}) + \Phi^{\mathrm{off}}_\lambda(\widehat{\xi}) + \nu \int dp\, \widehat{\xi}(p)|^2 \right]$$

and

$$\Phi^{\mathrm{diag}}_\lambda(f) = \int dp |f(p)|^2 \sqrt{\frac{3}{4}p^2 + \lambda}$$

$$\Phi^{\mathrm{off}}_\lambda(f) = -\frac{1}{\pi^2} \int dp_{23}\, dp_{31}\, \frac{\overline{f(p_{23})} f(p_{31})}{p_{23}^2 + p_{31}^2 + p_{23} \cdot p_{31} + \lambda}$$

Proceeding as in the case $\lambda = 0$ ([24]), one has

$$-\Omega\, \Phi^{\mathrm{diag}}_\lambda \leq \Phi^{\mathrm{off}}_\lambda \leq \Sigma\, \Phi^{\mathrm{diag}}_\lambda$$

Therefore the quadratic form is bounded from below

$$\mathcal{B}_\nu(\psi) \geq -\frac{\nu^2}{\pi^4(1-\Omega)^2} \|\psi\|^2$$

The proof that $\mathcal{B}_\nu$ is closed follows exactly the same line of the proof of Theorem 2.1 in [24] and it is omitted for the sake of brevity. □

We conclude observing that Theorem 9.1 implies the existence of a self-adjoint operator $H^\perp_{\nu,0}$ in $\mathcal{H}^\perp_0$ which, at least formally, coincides with the STM operator restricted to $\mathcal{H}^\perp_0$. Such operator $H^\perp_{\nu,0}$ is positive for $\nu \geq 0$ and bounded from below by $-\dfrac{\nu^2}{\pi^4(1-\Omega)^2}$ for $\nu < 0$.

9.3 SYSTEM OF FERMIONS

In a system of identical fermions the wave function, due to the antisymmetry under exchange of coordinates, vanishes at the coincident points of any pair of particles and therefore the zero-range interaction is ineffective. On the other hand, in physical applications it is relevant the case of a mixture of N identical fermions of one species and M identical fermions of another species. Here the dynamics is non trivial since each fermion of one species feels the zero-range interaction with all the fermions of the other species. In particular, numerical simulations seem to suggest ([49]) that the system is stable at least for mass ratio equal to one but, in this generality, no rigorous result is available (see [42] for a formulation of the problem in terms of quadratic forms). A significant aspect of the fermionic problem is that the stability of the system depends on the value of the mass ratio. This has been explicitly shown in the case of $N \geq 2$ identical fermions of mass one plus a different particle of mass m. In the particular case $N = 2$ an optimal result have been proved in [24]. in agreement with previous heuristic results in the physical literature ([15]) and also with other mathematical results ([53], [54]) they prove stability (instability) for $m > m_*$ ($m < m_*$).

Theorem 9.2. *Let m_* be as in Definition 3.2 and $\Lambda(m)$ as in Equation (3.8). Then*

(Stability) *If $m > m_*$ then $\mathcal{B}_\nu$ is closed and bounded from below. In particular $\mathcal{B}_\nu$ is positive for $\nu \geq 0$ and bounded from below by*

$$-\frac{\nu^2}{4\pi^4(1-\Lambda(m))^2}$$

for $\nu < 0$. Therefore the corresponding STM operator H_ν is self-adjoint and bounded from below, with the same lower bound.

(Instability) *If $m < m_*$ then $\mathcal{B}_\nu$ is unbounded from below for any $\nu \in \mathbb{R}$.*

Remark 9.2. We remark the analogy between Theorem 9.2 and Theorem 3.1: in the mass range $m > m_*$ both systems are stable while for $m < m_*$ they are both unstable exhibiting Efimov effect in the first case and suggesting the occurrence of the Thomas effect in the second case.

Furthermore in the case of general N, in the more recent work [55] it is shown the existence of a mass threshold m^* independent of N such that if $m > m^*$ the system is stable. A numerical evaluation yields $m^* \simeq 0,36$ hence in particular the system is stable for $m = 1$ which is particularly interesting in relation to the case of a spin-1/2 fermionic gas.

Let us now focus on the special case $N = 2$ in the unitary limit, i.e., for $\nu = 0$. Such a case exhibits a further interesting behavior that we want to discuss in the rest of this section. As in Part I the Hilbert space is $\mathcal{H}^{\text{asym}}$ defined in (3.1). Furthermore by the symmetry assumptions one immediately gets $\xi^{(1)} = -\xi^{(2)} := -\xi$ and $\xi^{(3)} = 0$. Hence the potential in the Fourier space in coordinates (p_{23}, p_{31}) takes the form

$$(\widehat{G\xi})(p_{23}, p_{31}) = \widehat{G\xi}(p_{23}, p_{31}) = \frac{\hat{\xi}(p_{31}) - \hat{\xi}(p_{23})}{p_{23}^2 + p_{31}^2 + \frac{2}{m+1}\, p_{23} \cdot p_{31}}$$

and the quadratic form associated to the STM operator is

$$D(\mathcal{B}_0) = \left\{ \psi \in \mathcal{H}^{\text{asym}} \,|\, \psi = \phi + G\xi,\ |\nabla\phi| \in \mathcal{H}^{\text{asym}},\ \xi \in H^{1/2}(\mathbb{R}^3) \right\}$$

$$\mathcal{B}_0(\psi) = \mathcal{B}_{\text{free}}(\phi) + \frac{8\pi^2 m}{m+1}\, \Phi_0(\xi)$$

where

$$\Phi_0(\xi) = \Phi^{\text{diag}}(\hat{\xi}) + \Phi^{\text{off}}(\hat{\xi}) \tag{9.18}$$

$$\Phi^{\text{diag}}(f) = \frac{\sqrt{m(m+2)}}{m+1} \int dp\, |p|\, |f(p)|^2 \tag{9.19}$$

$$\Phi^{\text{off}}(f) = \frac{1}{2\,\pi^2} \int dp_{23}\, dp_{31}\, \frac{\overline{f(p_{23})} f(p_{31})}{p_{23}^2 + p_{31}^2 + \frac{2}{m+1} p_{23} \cdot p_{31}} \tag{9.20}$$

From Theorem 9.2 we know that the form is closed and bounded from below for $m > m_*$ and unbounded from below for $m < m_*$. We also notice the main differences with respect to the form in the bosonic case considered in the previous section, i.e., the dependence on m and, more important, the sign $+$ in front of the integral in (9.20). This implies that for the estimate of $\Phi^{\text{off}}(f)$ one has to study the terms for l odd, and in particular the case $l = 1$, in the expansion in spherical harmonics of f.
As a matter of fact, for suitable values of the mass m the above quadratic form can be modified by enlarging the class of admissible charges and the new quadratic form turns out to be closed and bounded from below. Therefore it defines a Hamiltonian, different from H_ν, describing an additional three-body interaction besides the standard two-body zero-range interaction (see [25] for details).
In order to explain the above assertion, we proceed formally. Let us define

$$\widehat{\xi}_\nu^-(k) = \frac{1}{k^{2-s}}\, Y_1^\nu(\theta,\phi)\,, \qquad 0 < s < 1, \qquad n = 0, \pm 1 \tag{9.21}$$

Notice that $\xi_n^- \notin L^2(\mathbb{R}^3)$ but this fact is not relevant since for $\nu = 0$ the condition $\Phi_0(\xi) < \infty$ does not require square-integrability of ξ. The crucial point is that both $\Phi^{\text{diag}}(\widehat{\xi}_n^-)$ and $\Phi^{\text{off}}(\widehat{\xi}_n^-)$ diverge, due to the behavior of $\widehat{\xi}_n^-(k)$ for large k and the two infinities can compensate for an appropriate value of the mass. Indeed, by a direct computation one finds

$$\begin{aligned}\Phi_0(\xi_\nu^-) &= \frac{1}{\pi}\left[\frac{\pi\sqrt{m(m+2)}}{m+1} + \int_{-1}^{1}\!dt\, t\int_0^\infty\! dq\, \frac{q^s}{q^2+1+\frac{2}{m+1}tq}\right]\int_0^\infty\! dk\, \frac{1}{k^{1-2s}}\\ &:= g(m,s)\int_0^\infty\! dk\,\frac{1}{k^{1-2s}} = \infty \qquad \text{unless} \qquad g(m,s) = 0\end{aligned}$$

The problem is then reduced to the study of the equation $g(m,s) = 0$. One can show that for $s \in [0,1]$ there is a unique solution $m(s)$, monotonically increasing, with $m(0) = m_*$ and $m(1) := m_{**} \simeq 0.116$. For $m \in (m_*, m_{**})$ we can therefore define the inverse function $s(m)$, with $0 < s(m) < 1$, which satisfies $g(m, s(m)) = 0$. This means that for each $m \in (m_*, m_{**})$ the charge (9.21) with $s = s(m)$ can be considered to enlarge the class of admissible charges and to construct a more general quadratic form. Starting from the above argument, one can prove the following result.

Theorem 9.3. *For any $m \in (m_*, m_{**})$ and $\beta := \{\beta_n\}$, $n = 0, \pm 1$, the quadratic form in $\mathcal{H}^{\text{asym}}$*

$$\begin{aligned}D(\mathcal{B}_{0,\beta}) = \Big\{\psi \in \mathcal{H}^{\text{asym}} \,|\, \psi = \phi + G\eta,\ |\nabla\phi| \in \mathcal{H}^{\text{asym}},\ \eta \in H^{-1/2}(\mathbb{R}^3),\\ \eta = \xi + \sum_{\nu=-1}^{1} q_\nu \xi_\nu^-,\ \Phi^{\text{diag}}(\hat{\xi}) < \infty,\ q_\nu \in \mathbb{C}\Big\}\end{aligned}$$

$$\mathcal{B}_{0,\beta}(\psi) = \mathcal{B}_{\text{free}}(\phi) + \frac{8\pi^2 m}{m+1}\Phi_0(\xi) + \sum_{\nu=-1}^{1}\beta_\nu |q_\nu|^2$$

is closed and bounded from below. Then it uniquely defines a self-adjoint and bounded from below Hamiltonian $H_{0,\beta}$, $D(H_{0,\beta})$.

We conclude with some comments.

i) At heuristic level, the Hamiltonian $H_{0,\beta}$ has been introduced and studied in the physical literature (see, e.g., [21]). From the mathematical point of view, an analogous result has been found in [53] using an approach based on the theory of self-adjoint extensions. Nevertheless, in [53] the analysis is done for $\nu \neq 0$, which requires charges in L^2. Therefore the parameter s in (9.21) is chosen in the interval $(0, 1/2)$ and for this reason the new Hamiltonian is constructed only for a smaller range of mass, i. e., for $m \in (m_*, m_{**}^{\text{Minlos}})$, with $m_{**}^{\text{Minlos}} = m(s)|_{s=1/2} < m_{**}$.

ii) The quadratic form $\mathcal{B}_{0,\beta}$ constructed in Theorem 9.3 generalizes the previous one $\mathcal{B}_0$ in the sense that $\lim_{\beta \to \infty} \mathcal{B}_{0,\beta} = \mathcal{B}_0$.

iii) A final, and more important, comment concerns the boundary condition satisfied by an element of $D(H_{0,\beta})$. Denoting $R = \sqrt{x_{23}^2 + x_{31}^2}$ and choosing for simplicity $\xi_{\pm 1}^- = 0$, for $R \to 0$ one finds

$$\psi(x_{23}, x_{31}) = \frac{q_0}{R^{2+s(m)}} + \frac{\gamma(m)\beta_0 q_0}{R^{2-s(m)}} + o(R^{s(m)-2})$$

where $\gamma(m)$ is a given positive function of m. In analogy with the case of a point interaction (see (7.1)), such boundary condition describes an interaction supported in $y_{23} = y_{31} = 0$, i. e., when the positions of all the three particles coincide. Therefore the new Hamiltonian $H_{0,\beta}$ describes the two-body (resonant) zero-range interactions plus an effective three-body point interactions.

Part III

QUANTUM LORENTZ GAS

10 INTRODUCTION AND MAIN RESULT

In the following chapters we study the effective behavior at low energy of a non relativistic quantum particle of mass $1/2$ in $\mathbb{R}^3$ interacting with a system of N randomly distributed obstacles in the limit $N \to \infty$. In order to formulate such Lorentz gas model, we introduce the set $Y_N = \{y_1, \dots y_N\}$ of random variables in $\mathbb{R}^3$, independently chosen according to a common distribution with density W.

We assume that the interaction among the particle and the i-th obstacle is described by the Gross-Pitaevskii potential. Hence, the Hamiltonian of the particle is H_N defined in Equation (1.6) where the potential V_i^N is defined in Equation (1.5). The assumptions on V will guarantee that H_N is a self-adjoint operator in $L^2(\mathbb{R}^3)$. Our aim is to characterize the limit behavior of H_N and the fluctuations around the limit. We reasonably expect that, for $N \to \infty$, the particle "sees" an effective potential depending on the density of obstacles. Moreover, one could be tempted to consider essentially correct the formal manipulation

$$\sum_{i=1}^N V_i^N(x) \sim \frac{1}{N}\sum_{i=1}^N N^3 V(N(x-y_i)) \sim b\,\frac{1}{N}\sum_{i=1}^N \delta(x-y_i) \qquad b = \int dx\, V(x)$$

and to obtain bW as effective potential. In reality, this is not the case and we shall see that the effective potential is the density of scattering length of the system of obstacles $4\pi a W$, where a is the scattering length associated to the potential V (see definition below). The situation is completely analogous to the more difficult case of a gas of n particles interacting via two-body potentials scaling as $n^2 V(nx)$ for $n \to \infty$ as investigated in [11, 18, 31–33, 62]. In particular, we refer to [12, Section 5] for a discussion on the emergence of the scattering length in that context.

Let us recall the definition of scattering length. Assuming that zero is neither an eigenvalue nor a resonance for V the associated scattering length a is finite (see Remark 3.1). Moreover one has

$$a = \frac{1}{4\pi}\int dx\, V(x)\phi_0(x).$$

where ϕ_0 is the solution of the zero energy scattering problem

$$\begin{cases} (-\Delta + V)\phi_0 = 0 \\ \lim\limits_{|x|\to+\infty} \phi_0(x) = 1\,. \end{cases} \tag{10.1}$$

As for the physical meaning, we recall that a represents the effective linear dimension of the scatterer at low energy. It is also easy to check by scaling that the scattering length associated to the rescaled potential V_i^N is $a_i^N = a/N$.

Here we give the proof of the convergence in the strong resolvent sense of H_N to the limiting Hamiltonian H, defined in Equation (1.7), where the convergence is in probability with respect to the distribution of the obstacles. In the following we denote by $\|\cdot\|_p$ the norm in $L^p(\mathbb{R}^3)$, $1 \le p \le \infty$. We give below the precise formulation of our main theorem.

Theorem 10.1. *Let $V \in L^1(\mathbb{R}^3, (1+|x|^4)dx) \cap L^3(\mathbb{R}^3, dx)$ such that zero is not an eigenvalue nor a resonance for $-\Delta + V$ and let $a \in \mathbb{R}$ be the corresponding scattering length. Moreover, let $W \in L^1(\mathbb{R}^3) \cap L^p(\mathbb{R}^3)$, for some $p > 3$, $f \in L^2(\mathbb{R}^3)$ and take $\lambda > 0$ large enough. Then for any $\varepsilon > 0$ and $\beta < 1/2$ we have*

$$\lim_{N\to+\infty} P_N\big(\{Y_N : N^\beta \|(H_N+\lambda)^{-1}f - (H+\lambda)^{-1}f\|_2 > \varepsilon\}\big) = 0\,,$$

where P_N is the product probability measure $\{W(x)dx\}^{\otimes N}$ on the set of configurations of points Y_N.

Remark 10.1. Theorem 10.1 implies the convergence in probability as $N \to +\infty$ of the unitary group e^{-itH_N}, associated to the N dependent Hamiltonian (1.6), to the N independent unitary group e^{-itH}, for any time $t > 0$.

As an immediate consequence of Theorem 10.1 and of previous results [38, 39] we can also characterize the fluctuations around the limit operator, as expressed in the following theorem.

Theorem 10.2. *Under the same assumptions of Theorem 10.1, for any $f, g \in L^2(\mathbb{R}^3)$ the random variable*

$$\eta^\lambda_{f,g}(Y_N) := \sqrt{N}\left(g, \big((H_N+\lambda)^{-1} - (H+\lambda)^{-1}\big)f\right)$$

converges in distribution for $N \to \infty$ to a Gaussian random variable $\bar\eta^\lambda_{f,g}$ of zero mean and covariance

$$\begin{aligned}&E\Big((\bar\eta^\lambda_{f,g})^2\Big)\\ &= (4\pi a)^2 \|(H+\lambda)^{-1}g\,(H+\lambda)^{-1}f\|^2_{L^2_W} - 4\pi a\big((H+\lambda)^{-1}g, (H+\lambda)^{-1}f\big)^2_{L^2_W}\end{aligned}$$

where $E(\cdot)$ means expectation with respect to the probability measure P_N and $L^2_W = L^2(\mathbb{R}^3, W(x)dx)$.

Let us briefly comment on the above results. We find that the asymptotic behavior of our Lorentz gas is completely characterized by the density of the obstacles and by their scattering length. In particular, this means that the dependence of the limit Hamiltonian on the interaction potentials V_i^N is only through the associated scattering length a/N, i.e., a single physical parameter describing the effect of the obstacle as a scatterer at low energy. In this sense, in our scaling and for N large, the Lorentz gas exhibits a universal behavior.

As we already mentioned, in the many-body context the same type of universal behavior of the interaction arises in the effective description of the dynamics of n bosons interacting through Gross-Pitaevskii potentials and undergoing Bose-Einstein condensation. More precisely, under the assumption that at time zero the system exhibits Bose-Einstein condensation into the one-particle wave function $\varphi \in L^2(\mathbb{R}^3)$, one expects condensation to be preserved at any time in the limit $n \to \infty$ and the condensed wave function to evolve according to the Gross-Pitaevskii equation $i\partial_t\varphi_t = -\Delta\varphi_t + 4\pi a|\varphi_t|^2\varphi_t$, with initial condition $\varphi_0 = \varphi$. This fact has been well established mathematically for non negative potentials (see [11, 18, 31–33, 62]) and shows that at the level of the evolution of the condensate wave function and in the limit $n \to \infty$ the interaction enters only through its scattering length.

Indeed, our Lorentz gas can be considered as a simplified model obtained from the more general case of a test particle interacting with other N particles when the masses of these particles are infinite. Nevertheless, we believe that our analysis could have some interest and could give some hints for the general case. The reason is that, because of its simpler structure, our Lorentz gas allows a more detailed analysis. In particular, we obtain the convergence result without any assumption on the sign of the interaction potential V and we can characterize the fluctuations in a relatively simple and explicit way.

11 LINE OF THE PROOF

In this chapter we describe the method of the proof and collect some preliminary results and notation useful in the sequel.

Let us start with some notation. Given $\underline{\phi}_N = \{\phi_1, \ldots, \phi_N\} \in \oplus_{i=1}^N L^2(\mathbb{R}^3)$, we define

$$\|\underline{\phi}_N\|^2 = \sum_{i=1}^N \|\phi_i\|_2^2 .$$

Moreover, for $\vec{X}_N = \{X_1, \ldots, X_N\} \in \mathbb{R}^N$ we set

$$\|\vec{X}_N\|^2 = \sum_{i=1}^N X_i^2 .$$

It is useful to write the interaction potential as $V(x) = u(x)v(x)$, where

$$u(x) = |V(x)|^{1/2}, \qquad v(x) = \mathrm{sgn}(V(x))|V(x)|^{1/2} .$$

Using the above factorization, we rewrite the scattering length associated to the potential V as

$$a = \frac{1}{4\pi}(u, \mu) , \tag{11.1}$$

where μ solves

$$\mu + v\mathcal{G}^0 u\mu = v \tag{11.2}$$

and $\mathcal{G}^0$ is the operator with integral kernel $\mathcal{G}^0(x) = (4\pi|x|)^{-1}$. Indeed, under the assumption that zero is neither an eigenvalue nor a resonance for $-\Delta + V$, Equation (11.2) has a unique solution in $L^2(\mathbb{R}^3)$. Moreover, one can check that the function $\phi_0 := 1 - \mathcal{G}^0 u\mu$ solves problem (10.1). To see that this is indeed the case notice that

$$(-\Delta + V)(1 - \mathcal{G}^0 u\mu) = -u\mu + V - V\mathcal{G}^0 u\mu = -u\left(\mu + v\mathcal{G}^0 u\mu - v\right) = 0 .$$

and $\lim_{|x|\to\infty} \phi_0(x) = 1$. Finally

$$4\pi a = \int dx\, V\phi_0 = \int dx\, u(v - v\mathcal{G}^0 u\mu) = \int dx\, u\mu ,$$

so that (11.1) is verified.
Analogously, for the rescaled potentials we set $V_i^N(x) = u_i^N(x)v_i^N(x)$, where

$$u_i^N(x) = |V_i^N(x)|^{1/2} = Nu(N(x - y_i)) ,$$
$$v_i^N(x) = |V_i^N(x)|^{1/2}\,\mathrm{sgn}(V_i^N(x)) = Nv(N(x - y_i))$$

and for the scattering length we have

$$a_i^N = \frac{1}{4\pi}(u_i^N, \mu_i^N) = a/N ,$$

where

$$\mu_i^N + u_i^N \mathcal{G}^0 v_i^N \mu_i^N = v_i^N . \tag{11.3}$$

Let us discuss the line of the proof. We first observe that the proof of Theorem 10.1 is non probabilistic. In fact, we prove the convergence for a fixed set of configurations of obstacles $Y_N = \{y_1, \dots y_N\}$ satisfying the following regularity assumptions

(Y_1) Let $\nu^*(p) = \frac{1}{3}\frac{p-3}{p-1} \in (0, 1/3)$. For any $0 < \nu < \nu^*(p)$ there exists C such that

$$\min_{i \neq j} |y_i - y_j| \geq \frac{C}{N^{1-\nu}}\,.$$

(Y_2) For any $N > 0$ and any $0 < \xi \leq 1$ we have

$$\frac{1}{N^2} \sum_{\substack{i,j=1 \\ i \neq j}}^{N} \frac{1}{|y_i - y_j|^{3-\xi}} \leq C_\xi < \infty\,.$$

The convergence in probability then follows once we show that (Y_1) and (Y_2) hold with probability increasing to one in the limit $N \to \infty$. More precisely the following lemma holds.

Lemma 11.1. *Let $Y_N = \{y_1, \dots y_N\}$ a configuration of N identically distributed random variables in $\mathbb{R}^3$, whose distribution has density $W \in L^1(\mathbb{R}^3) \cap L^p(\mathbb{R}^3)$, for some $p > 3$. Then, the set of configurations on which (Y_1) and (Y_2) hold has probability increasing to one as N goes to infinity.*

Proof. The standard proof of the Lemma (see [59, 60]) can be easily adapted to the situation where $W \in L^1(\mathbb{R}^3) \cap L^p(\mathbb{R}^3)$ with $p > 3$. We start analyzing the assumption (Y_1). Let

$$Z_N = \left\{ Y_N | \min_{i \neq j} |y_i - y_j| \geq \frac{C}{N^{1-\nu}} \right\}$$

the set of configurations of N obstacles for which (Y_1) holds. We show that in the limit N going to infinity the probability of the complement of Z_N (denoted by Z_N^c) goes to zero. We have

$$\begin{aligned} P_N(Z_N^c) &= P_N(\{Y_N | \, \exists\, i, j,\, i \neq j : |y_i - y_j| < C N^{-(1-\nu)}\}) \\ &\leq \frac{N(N-1)}{2} \int_{|x| < CN^{-(1-\nu)}} dx\, W(x)\,. \end{aligned} \tag{11.4}$$

To bound the last integral we use Hölder inequality. For $1/p' + 1/p = 1$, we have

$$\begin{aligned} \int_{|x| < CN^{\nu-1}} dx\, W(x) &\leq \left(\int_{|x| < CN^{\nu-1}} dx \right)^{1/p'} \left(\int_{|x| < CN^{\nu-1}} dx |W(x)|^p \right)^{1/p} \\ &\leq C N^{3(\nu-1)/p'} \|W\|_p\,. \end{aligned}$$

Hence the r.h.s. of (11.4) goes to zero as N goes to infinity for any $\nu < \frac{1}{3}\frac{p-3}{p-1}$. Note that the requirement $W \in L^p(\mathbb{R}^3)$ with $p > 3$ assures that $\nu > 0$.

To show that also (Y_2) holds with probability increasing to one as $N \to \infty$ it is sufficient to note that the $M = N(N-1)$ random variables $|y_i - y_j|$, with $i, j = 1, \dots, N$ and $i \neq j$, are interchangeable and we can reorder them as $\{X_1, \dots, X_M\}$ (e.g. using a diagonal progression as in the Cantor pairing function). Standard results (we refer to [47] for further details) ensure that if

$$E(X_1^{-3+\xi}) = \int dx dy \frac{W(x)W(y)}{|x-y|^{3-\xi}} \leq C\,, \tag{11.5}$$

then
$$\lim_{M\to+\infty}\frac{1}{M}\sum_{k=1}^{M}\frac{1}{X_k^{3-\xi}} = E(X_1^{-3+\xi})\,.$$
The bound (11.5) follows from the assumptions on $W(x)$. Let $R>0$ be arbitrary. Then:
$$\int_{|x-y|>R} dxdy\frac{W(x)W(y)}{|x-y|^{3-\xi}} \le R^{-3+\xi}\|W\|_1^2 \le C_\xi\,.$$
On the other hand, by Cauchy-Schwarz inequality
$$\int_{|x-y|\le R} dxdy\frac{W(x)W(y)}{|x-y|^{3-\xi}} \le \int_{|x-y|\le R} dxdy\frac{|W(x)|^2}{|x-y|^{3-\xi}} \le C_\xi\,.$$
□

By Lemma 11.1 we conclude that, in order to prove Theorem 10.1, it is enough to show that for all $f\in L^2(\mathbb{R}^3)$
$$\lim_{N\to\infty}\|(H_N+\lambda)^{-1}f-(H+\lambda)^{-1}f\|_2 = 0 \tag{11.6}$$
uniformly on configurations Y_N satisfying (Y_1) and (Y_2). In fact, since the measure of the configurations where (Y_1) and (Y_2) do not hold goes to zero as N goes to infinity, we have
$$\begin{aligned}&\lim_{N\to+\infty}P_N\big(\{Y_N : N^\beta\|(H_N+\lambda)^{-1}f-(H+\lambda)^{-1}f\|_2>\epsilon\}\big)\\ =&\lim_{N\to+\infty}P_N\big(\{Y_N^* : N^\beta\|(H_N+\lambda)^{-1}f-(H+\lambda)^{-1}f\|_2>\epsilon\}\big)=0\,,\end{aligned}$$
where $\{Y_N^*\}$ is the set of configurations of obstacles where (Y_1) and (Y_2) hold.

The proof of Theorem 10.2 is obtained with slight modifications of the step followed in [39], [38] for a similar problem, and therefore we refer the reader to those papers.
The following remarks summarize two important consequences of the validity of the assumptions (Y_1) and (Y_2).

Remark 11.1. Notice that from (Y_1) and (Y_2) it follows that for any $0<\nu<\nu^*$
$$\frac{1}{N^{3-\nu^2}}\sum_{\substack{i,j=1\\ i\neq j}}^{N}\frac{1}{|y_i-y_j|^4}\le C_\nu\,. \tag{11.7}$$

Indeed
$$\begin{aligned}\frac{1}{N^{3-\nu^2}}\sum_{\substack{i,j=1\\ i\neq j}}^{N}\frac{1}{|y_i-y_j|^4} &= \frac{1}{N^{1-\nu^2}}\frac{1}{N^2}\sum_{\substack{i,j=1\\ i\neq j}}^{N}\frac{1}{|y_i-y_j|^{3-\nu}}\frac{1}{|y_i-y_j|^{1+\nu}}\\ &\underset{\text{by }(Y_1)}{\le}\frac{1}{N^{1-\nu^2}}N^{(1+\nu)(1-\nu)}\left(\frac{1}{N^2}\sum_{\substack{i,j=1\\ i\neq j}}^{N}\frac{1}{|y_i-y_j|^{3-\nu}}\right)\\ &\underset{\text{by }(Y_2)}{\le} c_\nu\,.\end{aligned}$$

Remark 11.2. For $\lambda\ge 0$ we denote by $\mathcal{G}^\lambda$ the free resolvent $(-\Delta+\lambda)^{-1}$ and, with a slight abuse of notation, also the corresponding integral kernel
$$\mathcal{G}^\lambda(x)=\frac{e^{-\sqrt{\lambda}|x|}}{4\pi|x|}.$$

Moreover, we define the $N \times N$ matrix G^λ with entries

$$G^\lambda_{ij} = \begin{cases} \mathcal{G}^\lambda(y_i - y_j) & i \neq j \\ 0 & i = j\,. \end{cases}$$

Then, due to hypothesis (Y_2) on our configuration of obstacles, we get

$$\frac{1}{N}\left\| G^\lambda \right\| \leq c(\lambda) \to 0 \qquad \text{for } \lambda \to +\infty\,,$$

hence, in particular, there exists $\lambda_0 > 0$ such that

$$\frac{1}{N}\|G^\lambda\| < 1 \qquad \forall \lambda > \lambda_0. \tag{11.8}$$

To see that this is indeed the case we fix $0 < \beta < 1$ and use that $e^{-x} \leq x^{-\beta}$. Then, by (Y_2)

$$\begin{aligned} \frac{1}{N^2}\|G^\lambda\|^2 &\leq \frac{1}{N^2}\sum_{\substack{i,j=1 \\ i\neq j}}^N \frac{e^{-2\sqrt{\lambda}|y_i - y_j|}}{16\pi^2|y_i - y_j|^2} \\ &\leq c\,\lambda^{-\beta/2}\frac{1}{N^2}\sum_{\substack{i,j=1 \\ i\neq j}}^N \frac{1}{|y_i - y_j|^{2+\beta}} \leq c_\beta \lambda^{-\beta/2}\,. \end{aligned}$$

Note that with a slight abuse of notation we denote with the same symbol G^λ_{ij} both the elements of the matrix G^λ and the operator on $L^2(\mathbb{R}^3)$ acting as the multiplication by $G^\lambda(y_i - y_j)$.

Given a set of configurations satisfying (Y_1),(Y_2), the strategy for the proof of (11.6) is based on some ideas and techniques developed in the study of boundary value problems for the Laplacian on randomly perforated domains, see [39], [40] and [41]. For a given $f \in L^2(\mathbb{R}^3)$ we consider the solution ψ_N of the equation

$$(H_N + \lambda)\psi_N = f\,.$$

Next we use the Resolvent Identity to rewrite

$$\psi_N = (H_N + \lambda)^{-1} f = \mathcal{G}^\lambda f + \sum_{i=1}^N \mathcal{G}^\lambda v_i^N \rho_i^N\,, \tag{11.9}$$

where the function

$$\rho_i^N = u_i^N (H_N + \lambda)^{-1} f$$

is solution of

$$\rho_i^N + u_i^N \mathcal{G}^\lambda v_i^N \rho_i^N + u_i^N \sum_{\substack{j=1 \\ j\neq i}}^N \mathcal{G}^\lambda v_j^N \rho_j^N = -u_i^N \mathcal{G}^\lambda f. \tag{11.10}$$

The idea is to represent the "potential" on the r.h.s. of (11.9) by its multipole expansion and to show that, for large N, the only relevant contribution comes from the first term of this expansion, that is the monopole term. According to this program we decompose $\psi_N = \widetilde{\psi}_N + (\psi_N - \widetilde{\psi}_N)$, with

$$\widetilde{\psi}_N(x) = (\mathcal{G}^\lambda f)(x) + \sum_{i=1}^N \mathcal{G}^\lambda(x - y_i) Q_i^N \tag{11.11}$$

where

$$Q_i^N = (v_i^N, \rho_i^N)\,.$$

The problem is then split in two parts: find a limit of $\widetilde{\psi}_N$ and show that the difference $(\psi_N - \widetilde{\psi}_N)$ converges to zero for N going to infinity.

In order to find a limit of $\widetilde{\psi}_N$ we recognize that the equation for the "charge" Q_i^N can be written as

$$\frac{N}{4\pi a}\, Q_i^N + \sum_{\substack{j=1\\ j\neq i}}^{N} G_{ij}^{\lambda} Q_j^N = -(\mathcal{G}^{\lambda} f)(y_i) + R_i^N \tag{11.12}$$

with

$$R_i^N = A_i^N + B_i^N + D_i^N \tag{11.13}$$

and

$$\begin{aligned}
A_i^N &= \int dx\, v_i^N(x) \int dz \frac{e^{-\lambda|x-z|}-1}{4\pi|x-z|} u_i^N(z)\mu_i^N(z)\\
B_i^N &= \sum_{\substack{j=1\\ i\neq j}}^{N} \int dx\, u_i^N(x)\mu_i^N(x) \int dz \big(\mathcal{G}^{\lambda}(x-z) - \mathcal{G}^{\lambda}(y_i - y_j)\big) v_j^N(z)\rho_j^N(z)\\
D_i^N &= \int dx\, \mu_i^N(x) u_i^N(x) \int dz\, \big(\mathcal{G}^{\lambda}(x-z) - \mathcal{G}^{\lambda}(y_i - z)\big) f(z)\,.
\end{aligned} \tag{11.14}$$

Since we expect R_i^N to be an error term, equation (11.12) suggests to study the approximate equation obtained from (11.12) removing R_i^N. With this motivation, we define

$$\widehat{\psi}_N(x) = (\mathcal{G}^{\lambda} f)(x) + \sum_{i=1}^{N} \mathcal{G}^{\lambda}(x - y_i) q_i^N\,. \tag{11.15}$$

where the "charges" q_i^N satisfy

$$\frac{N}{4\pi a} q_i^N + \sum_{\substack{j=1\\ j\neq i}}^{N} G_{ij}^{\lambda} q_j^N = -(\mathcal{G}^{\lambda} f)(y_i)\,, \tag{11.16}$$

As before we rewrite $\widetilde{\psi}_N = \widehat{\psi}_N + (\widetilde{\psi}_N - \widehat{\psi}_N)$ and prove that the difference $(\widetilde{\psi}_N - \widehat{\psi}_N)$ converges to zero as N goes to infinity. Finally, we show that the sequence $\widehat{\psi}_N$ converges to ψ defined by

$$\psi = (-\Delta + 4\pi a W + \lambda)^{-1} f\,. \tag{11.17}$$

This last step is strongly based on the analogy with the Hamiltonian with N zero-range interactions considered in [38].

In fact, the resolvent of an Hamiltonian $H_{N\alpha,Y_N}$ with N point interactions located at the points $Y_N = \{y_1, \dots, y_N\}$ with strength $N\alpha$ is given (see, e. g., [2]) by

$$(H_{N\alpha,Y_N} + \lambda)^{-1} = \mathcal{G}^{\lambda} + \sum_{\substack{i,j=1\\ i\neq j}}^{N} (\,\Xi_{N\alpha,Y_N}(\lambda))^{-1}_{ij} (\mathcal{G}^{\lambda}(\cdot - y_i), \cdot) \mathcal{G}^{\lambda}(\cdot - y_j)\,.$$

where the $N \times N$ matrix $\Xi_{N,Y_N}(\lambda)$ is defined by

$$(\,\Xi_{N\alpha,Y_N}(\lambda))_{ij} = \left(N\alpha + \frac{\sqrt{\lambda}}{4\pi}\right) \delta_{ij} - (1 - \delta_{ij}) G_{ij}^{\lambda}.$$

Hence

$$\phi_N = (H_{N\alpha,Y_N} + \lambda)^{-1} f = (\mathcal{G}^\lambda f)(x) + \sum_{i=1}^{N} \mathcal{G}^\lambda(x - y_i)\widetilde{q}_i \tag{11.18}$$

where

$$\left(N\alpha + \frac{\sqrt{\lambda}}{4\pi}\right)\widetilde{q}_i - \sum_{\substack{j=1 \\ j\neq i}}^{N} G_{ij}^\lambda \widetilde{q}_j = (\mathcal{G}^\lambda f)(y_i). \tag{11.19}$$

Comparing (11.18) and (11.19) with (11.15) and (11.16) respectively and recalling in addition that the scattering length of a point interaction with strength α equals $-1/(4\pi\alpha)$, the analogy between ϕ_N and $\widehat{\psi}_N$ is clear.

The rest of this part of the the work is organized as follows. In Chapter 12 we collect some properties of μ_i^N and ρ_i^N that will be used along the proof. In Chapter 13 we give the proof of the Theorem 10.1. In particular in Sections 13.1 and 13.2 we show that the differences $(\psi_N - \widetilde{\psi}_N)$ and $(\widetilde{\psi}_N - \widehat{\psi}_N)$ go to zero as N goes to infinity and finally in Section 13.3 we study the convergence of $\widehat{\psi}_N$.
To not overwhelm the notation, from now on we skip the dependence on N where not strictly necessary.

12 A PRIORI ESTIMATES

We prove some useful a priori estimates for the solutions of Equations (11.3) and (11.10).

Lemma 12.1. *Let $\mu_i = \mu_i^N \in L^2(\mathbb{R}^3)$ and $\rho_i = \rho_i^N \in L^2(\mathbb{R}^3)$ defined in (11.3) and (11.10) respectively. Under the assumptions of Theorem 10.1 there exists a constant $C > 0$ such that*

$$\sup_i \|\mu_i\|_2 \leq CN^{-1/2}$$

Moreover

$$\|\underline{\mu}\| \leq C\,, \qquad \|\underline{\rho}\,\| \leq C\|f\|_2\,.$$

Proof. It is simple to check by scaling that $\mu_i(x) = N\mu(N(x-y_i))$. On the other hand, since μ satisfies Equation (11.2) and zero is not an eigenvalue nor a resonance for V we can invert the operator $(\mathbb{1} + u\,\mathcal{G}^0 v)$ and get

$$\|\mu\|_2^2 \leq C\,. \tag{12.1}$$

Hence

$$\|\mu_i\|_2^2 = \int dx |\mu_i(x)|^2 = \frac{1}{N}\int dx|\mu(x)|^2 = \frac{1}{N}\|\mu\|_2^2\,,$$

which leads to $\sup_i \|\mu_i\|_2 \leq CN^{-1/2}$ and $\|\underline{\mu}\| \leq C$.

We prove now the bound for $\|\underline{\rho}\,\|$. We set

$$\widehat{\rho}_i(x) := N^{-1}\rho_i\,(y_i + x/N)\;.$$

From Equation (11.10) we have

$$\begin{aligned}(\mathbb{1} + u\,\mathcal{G}^0 v)\widehat{\rho}_i(x) + (u(\mathcal{G}^{\lambda/N^2} - \mathcal{G}^0)v\widehat{\rho}_i)(x) + \frac{1}{N}\sum_{\substack{i,j\\ i\neq j}} G^\lambda_{ij}(u\,v\widehat{\rho}_j)(x)&\\ + \frac{1}{N}\sum_{\substack{i,j\\ i\neq j}}\left(u\left(\mathcal{G}^{\lambda,N}_{ij} - G^\lambda_{ij}\right)v\widehat{\rho}_j\right)(x) = -u(x)\left(\mathcal{G}^\lambda f\right)(y_i + x/N)\;,&\end{aligned} \tag{12.2}$$

where $\mathcal{G}^{\lambda,N}_{ij}$ denotes the operator in $L^2(\mathbb{R}^3)$ with integral kernel

$$\mathcal{G}^{\lambda,N}_{ij}(x-z) = \frac{e^{-\sqrt{\lambda}|y_i - y_j - (x-z)/N|}}{4\pi^2|y_i - y_j - (x-z)/N|}\,. \tag{12.3}$$

Our goal is to show that the operator M^λ acting on $\oplus_{i=1}^N L^2(\mathbb{R}^3)$ defined by

$$\begin{aligned}(M^\lambda)_{ij} = \left[(\mathbb{1} + u\,\mathcal{G}^0 v) + u(\mathcal{G}^{\lambda/N^2} - \mathcal{G}^0)v\right]\delta_{ij}&\\ + \frac{1}{N}\left[uG^\lambda_{ij}v + u\left(\mathcal{G}^{\lambda,N}_{ij} - G^\lambda_{ij}\right)v\right](1-\delta_{ij})&\end{aligned}$$

is invertible. Due to the assumptions on V the operator $(\mathbb{1} + u\mathcal{G}^0 v)$ is invertible. Then, in order to prove that M^λ is invertible it suffices to show that there exists λ_0 such that the operators M_1^λ, M_2^λ and M_3^λ defined by

$$\begin{aligned}
\left(M_1^\lambda\right)_{ij} &= u(\mathcal{G}^{\lambda/N^2} - \mathcal{G}^0)v\,\delta_{ij} \\
\left(M_2^\lambda\right)_{ij} &= N^{-1} u\, G_{ij}^\lambda\, v \\
\left(M_3^\lambda\right)_{ij} &= N^{-1} u\big(\mathcal{G}_{ij}^{\lambda,N} - G_{ij}^\lambda\big)v\,(1-\delta_{ij})\,.
\end{aligned}$$

have a norm going to zero as N goes to infinity for any $\lambda > \lambda_0$. Denoting with $\|\cdot\|_{\mathrm{HS}}$ the Hilbert-Schmidt norm in $L^2(\mathbb{R}^3)$ and using the definition of $u(x)$ and $v(x)$, we obtain

$$\|M_1^\lambda\|^2 \le \|u(\mathcal{G}^{\lambda/N^2} - \mathcal{G}^0)v\|_{HS}^2 \le C\frac{\lambda}{N^2}\|V\|_1^2\,, \tag{12.4}$$

which is small for any λ in the limit $N \to \infty$. To bound the norm of the second matrix, we fix $0 < \beta < 1$. Then, using (Y_2)

$$\begin{aligned}
\|M_2^\lambda\|^2 \le \frac{1}{N^2}\sum_{\substack{i,j\\ i\neq j}} \|u\,G_{ij}^\lambda\, v\|_{HS}^2 &= \frac{1}{N^2}\sum_{\substack{i,j\\ i\neq j}} \int dxdz|V(x)|\frac{e^{-2\sqrt{\lambda}|y_i-y_j|}}{16\pi^2|y_i-y_j|^2}|V(z)| \\
&\le c\|V\|_1^2\,\lambda^{-\beta/2}\frac{1}{N^2}\sum_{\substack{i,j\\ i\neq j}}\frac{1}{|y_i-y_j|^{2+\beta}} \le c_\beta \lambda^{-\beta/2}\,.
\end{aligned} \tag{12.5}$$

The r.h.s. of (12.5) can be made small by choosing λ sufficiently large.
To bound the third term we use that for any $\xi < 1$ the following bound holds true:

$$\|u\,(\mathcal{G}_{ij}^{\lambda,N} - G_{ij}^\lambda)\,v\|_{HS}^2 \le C\left[\frac{1}{N^2|y_i-y_j|^4} + \frac{1}{N^2|y_i-y_j|^2} + \frac{1}{N^{1-\xi}|y_i-y_j|^{3-\xi}}\right] \tag{12.6}$$

From (12.6) and using the assumptions on the charge distribution and (11.7) we have

$$\begin{aligned}
\|M_3^\lambda\|^2 &\le \frac{1}{N^2}\sum_{\substack{i,j=1\\ i\neq j}}^N \|u\,(\mathcal{G}_{ij}^{\lambda,N} - G_{ij}^\lambda)\,v\|_{HS}^2 \\
&\le C\sum_{\substack{i,j=1\\ i\neq j}}^N\left[\frac{1}{N^4|y_i-y_j|^4} + \frac{1}{N^4|y_i-y_j|^2} + \frac{1}{N^{3-\xi}|y_i-y_j|^{3-\xi}}\right] \le \frac{C}{N^{1-\xi}}\,.
\end{aligned} \tag{12.7}$$

To prove (12.6) we define the cutoff function $\chi_{ij}^N(x)$ to be equal to one if $|x| \le \frac{N}{2}|y_i - y_j|$ and zero otherwise and write

$$\begin{aligned}
&\|u(\mathcal{G}_{ij}^{\lambda,N} - \mathcal{G}_{ij}^\lambda)v\|_{HS}^2 \\
&\le C\!\int dxdz\,\chi_{ij}^N(x-z)|V(x)||V(z)|\left(\frac{e^{-\sqrt{\lambda}|y_i-y_j-(x-z)/N|}}{|y_i-y_j-(x-z)/N|} - \frac{e^{-\sqrt{\lambda}|y_i-y_j|}}{|y_i-y_j|}\right)^2 \\
&\quad + C\!\int dxdz|V(x)||V(z)|(1-\chi_{ij}^N(x-z))\left(\frac{e^{-2\sqrt{\lambda}|y_i-y_j-(x-z)/N|}}{|y_i-y_j-(x-z)/N|^2} + \frac{e^{-2\sqrt{\lambda}|y_i-y_j|}}{|y_i-y_j|^2}\right)
\end{aligned} \tag{12.8}$$

To bound the term on the second line of (12.8) we exploit the fact that whenever $\chi_{ij}^N(x)$ is different from zero the difference in the round brackets is small. In particular, we have

$$
\begin{aligned}
&\int dxdz\, \chi_{ij}^N(x-z)|V(x)||V(z)| \left(\frac{e^{-\sqrt{\lambda}|y_i-y_j-(x-z)/N|}}{|y_i-y_j-(x-z)/N|} - \frac{e^{-\sqrt{\lambda}|y_i-y_j|}}{|y_i-y_j|} \right)^2 \\
&\leq C \int dx\, dz\, \chi_{ij}^N(x-z)|V(x)||V(z)| \frac{1}{|y_i-y_j|^2} \left(e^{-\sqrt{\lambda}|y_i-y_j-(x-z)/N|} - e^{-\sqrt{\lambda}|y_i-y_j|} \right)^2 \\
&\quad + C \int dx\, dz\, \chi_{ij}^N(x-z)|V(x)||V(z)|\, e^{-\sqrt{\lambda}|y_i-y_j-(x-z)/N|} \\
&\quad \times \left(\frac{1}{|y_i-y_j-(x-z)/N|} - \frac{1}{|y_i-y_j|} \right)^2 .
\end{aligned}
\tag{12.9}
$$

To bound the first term on the r.h.s. of (12.9) we use the bound

$$
\left| e^{-|y-w/N|} - e^{-|y|} \right| \leq C \frac{|w|}{N}
$$

obtaining

$$
C \int dx\, dz\, \chi_{ij}^N(x-z)|V(x)||V(z)| \frac{|x-z|^2}{N^2|y_i-y_j|^2} \leq \frac{C}{N^2} \frac{1}{|y_i-y_j|^2} . \tag{12.10}
$$

To bound the second term on the r.h.s. of (12.9) we note that $|y_i - y_j - (x-z)/N| \geq |y_i - y_j| - |x-z|/N$, and moreover on the support of $\chi_{ij}^N(x-z)$ we also have $|y_i - y_j - (x-z)/N \geq |y_i - y_j|/2$. Hence:

$$
\chi_{ij}^N(x-z) \Big(\frac{1}{|y_i-y_j-(x-z)/N|} - \frac{1}{|y_i-y_j|} \Big) \leq \frac{2|x-z|}{N|y_i-y_j|^2}
$$

We obtain

$$
\begin{aligned}
&\int dx\, dz\, \chi_{ij}^N(x-z)|V(x)||V(z)| e^{-\sqrt{\lambda}|y_i-y_j-(x-z)/N|} \left(\frac{1}{|y_i-y_j-(x-z)/N|} - \frac{1}{|y_i-y_j|} \right)^2 \\
&\qquad \leq C \int dx\, dz\, \chi_{ij}^N(x-z)|V(x)||V(z)| \frac{|x-z|^2}{N^2|y_i-y_j|^4} \leq \frac{C}{N^2|y_i-y_j|^4} .
\end{aligned}
\tag{12.11}
$$

We are left with the bound of the term on the third line of (12.8), for which we exploit the fast decaying behavior of the potential. We start from the term which does not contain the singularity. We fix α such that $1-\alpha \in (0,1)$ and we multiply and divide by $|x-z|^\alpha$; then we use that $|x-z| \geq CN|y_i - y_j|$ on the support of the integral and that $|x-z| \geq 1$ for N large enough, due to (Y_1):

$$
\begin{aligned}
&\int dxdz\, |V(x)||V(z)|(1-\chi_{ij}^N(x-z)) \frac{e^{-2\sqrt{\lambda}|y_i-y_j|}}{|y_i-y_j|^2} \\
&\leq \frac{C}{|y_i-y_j|^2} \int dxdz |V(x)||V(z)|(1-\chi_{ij}^N(x-z)) \frac{|x-z|^\alpha}{N^\alpha|y_i-y_j|^\alpha} \\
&\leq \frac{C}{N^\alpha|y_i-y_j|^{2+\alpha}} \int dxdz |V(x)||V(z)||x-z| \\
&\leq \frac{C}{N^\alpha|y_i-y_j|^{2+\alpha}} .
\end{aligned}
\tag{12.12}
$$

To bound the remaining term on the third line of (12.8) we use a similar idea; we obtain

$$\begin{aligned}
&\int dxdz|V(x)||V(z)|(1-\chi_{ij}^N(x-z))\frac{e^{-2\sqrt{\lambda}|y_i-y_j-(x-z)/N|}}{|y_i-y_j-(x-z)/N|^2}\\
&\leq \frac{C}{(N|y_i-y_j|)^{2+\alpha}}\int dxdz|V(x)||V(z)|(1-\chi_{ij}^N(x-z))\frac{|x-z|^{2+\alpha}}{|y_i-y_j-(x-z)/N|^2}\\
&\leq \frac{C}{N^\alpha|y_i-y_j|^{2+\alpha}}\int dxdz|V(x)||V(z)|(1-\chi_{ij}^N(x-z))\frac{|x|^3+|z|^3}{|N(y_i-y_j)-(x-z)|^2}\\
&\leq \frac{C}{N^\alpha|y_i-y_j|^{2+\alpha}}\int dx'dz'\frac{|V(x'+Ny_i)||V(z'+Ny_j)|}{|x'-z'|^2}(|x'+Ny_i|^3+|z'+Ny_j|^3)
\end{aligned}\tag{12.13}$$

where in the last line we used the change of variables $x'=x-Ny_i$ and $z'=z-Ny_j$ and we removed the cutoff function.

To bound the r.h.s. of (12.13) we use the Hardy-Littlewood-Sobolev inequality: let $f\in L^p(\mathbb{R}^n)$ and $h\in L^q(\mathbb{R}^n)$ with $p,q>1$ and let $0<\lambda<n$ with $1/p+\lambda/n+1/q=2$, then there exists a constant C independent of f and h such that

$$\left|\int_{\mathbb{R}^n} f(x)|x-y|^{-\lambda}h(y)dxdy\right|\leq C\|f\|_p\|h\|_q.$$

Hence

$$\int dx\,dz\,\frac{|V(x+Ny_i)||V(z+Ny_j)||x+Ny_i|^3}{|x-z|^2}\leq C\||\cdot|^3V\|_{3/2}\|V\|_{3/2},$$

and exactly in the same way we bound also the other term in the r.h.s. of Equation (12.13). Then, due to our assumptions on the potential, we obtain

$$\begin{aligned}
&\int dxdz|V(x)||V(z)|(1-\chi_{ij}^N(x-z))\frac{e^{-2\sqrt{\lambda}|y_i-y_j-(x-z)/N|}}{|y_i-y_j-(x-z)/N|^2}\\
&\qquad\leq\frac{C}{N^\alpha|y_i-y_j|^{2+\alpha}}.
\end{aligned}\tag{12.14}$$

Putting together (12.10), (12.11), (12.12) and (12.14) we prove (12.6).

The bounds (12.4), (12.5) and (12.7) together with the assumptions on V prove that there exists a $\lambda_0>0$ such that for any $\lambda>\lambda_0$ and N large enough the operator M^λ is invertible. From Equation (12.2) we obtain

$$\begin{aligned}
\|\widehat{\rho}_i\|_2^2 &\leq C\sum_{i=1}^N\int dx|u(x)|^2|(\mathcal{G}^\lambda f)(y_i+x/N)|^2\\
&\leq CN\Big(\sup_x|(\mathcal{G}^\lambda f)(x)|^2\Big)\\
&\leq CN\|f\|_2^2.
\end{aligned}$$

It follows that

$$\|\underline{\rho}\|^2=\sum_{i=1}^N\int dx|N\widehat{\rho}_i(N(x-y_i))|^2=\frac{1}{N}\sum_{i=1}^N\|\widehat{\rho}_i\|_2^2\leq C\|f\|_2^2.$$

□

13 PROOF OF THEOREM 10.1

13.1 MONOPOLE EXPANSION

In this section we analyze the difference between the solution ψ_N defined by (11.9) and the approximate solution $\widetilde{\psi}_N$ obtained considering the first term of a multipole expansion for the potential defined in (11.11). This is the content of the following proposition.

Proposition 13.1. *Let ψ_N and $\widetilde{\psi}_N$ be defined in (11.9) and (11.11) respectively. Then, under the same assumptions as in Theorem 10.1,*

$$\lim_{N\to+\infty} N^\beta \|\psi_N - \widetilde{\psi}_N\|_2 = 0 \qquad \forall \beta < 1 .$$

Proof. Using (11.9) and (11.11) we write

$$\psi_N(x) - \widetilde{\psi}_N(x) = \sum_{i=1}^N \int dz\, v_i(z)\rho_i(z)\big(\mathcal{G}^\lambda(x-z) - \mathcal{G}^\lambda(x-y_i)\big) := \sum_{i=1}^N K_i(x) .$$

We have

$$\|\psi_N - \widetilde{\psi}_N\|^2 \le \sum_{i=1}^N \int dx\, K_i^2(x) + \sum_{\substack{i,j=1\\ i\neq j}}^N \int dx\, K_i(x)K_j(x) . \tag{13.1}$$

We first bound the diagonal term on the r.h.s. of (13.1).

$$\begin{aligned}\sum_{i=1}^N \int dx K_i^2(x) = \sum_{i=1}^N \int dz dz' v_i(z) v_i(z')\rho_i(z)\rho_i(z') \\ \times \int dx \big(\mathcal{G}^\lambda(x-z) - \mathcal{G}^\lambda(x-y_i)\big)\big(\mathcal{G}^\lambda(x-z') - \mathcal{G}^\lambda(x-y_i)\big)\end{aligned}$$

Using elliptic coordinates we can explicitly calculate the integral over x of the products of Green's functions on the last line. For instance let us consider the product $\mathcal{G}^\lambda(x-z)\mathcal{G}^\lambda(x-z')$. We set $r_1 = |x-z|$, $r_2 = |x-z'|$ and $R = |z-z'|$ and consider the new variables $\{\mu,\nu,\varphi\}$ with $\mu = (r_1+r_2)/R \in [1,+\infty)$, $\nu = (r_1-r_2)/R \in [-1,1]$ and $\varphi \in [0,2\pi)$ the rotation angle with respect to the axis zz'.

Then

$$\begin{aligned}\int dx\, \mathcal{G}^\lambda(x-z)\mathcal{G}^\lambda(x-z') &= \int dx\, \frac{e^{-\sqrt{\lambda}(|x-z|+|x-z'|)}}{16\pi^2|x-z||x-z'|} \\ &= \int_1^{+\infty} d\mu \int_{-1}^1 d\nu \int_0^{2\pi} d\varphi \frac{R^3}{8}(\mu^2-\nu^2) \\ &\quad \times \frac{e^{-\sqrt{\lambda}(\mu+\nu+\mu-\nu)R/2}}{16\pi^2\, \frac{R}{2}\,(\mu+\nu)\,\frac{R}{2}\,(\mu-\nu)} \\ &= \frac{1}{8\pi\sqrt{\lambda}} e^{-\sqrt{\lambda}|z-z'|} .\end{aligned}$$

Proceeding analogously for the other terms we obtain

$$\sum_{i=1}^{N}\int dx\, K_i^2(x)$$
$$\leq C\sum_{i=1}^{N}\int dz\, dz'\, v_i(z)v_i(z')\rho_i(x)\rho_i(z')\Big(e^{-\lambda|z-z'|}-e^{-\lambda|z-y_i|}-e^{-\lambda|z'-y_i|}+1\Big)\,.$$

We use the definition of v_i and rescale the integration variables as follows $N(z-y_i)\to z$ and $N(z'-y_i)\to z'$. Hence

$$\begin{aligned}\sum_{i=1}^{N}\int dxK_i^2(x) &\leq CN^{-3}\sum_{i=1}^{N}\int dzdz'|V(z)|^{1/2}|V(z')|^{1/2}\widehat{\rho}_i(z)\widehat{\rho}_i(z')(|z|+|z'|)\\ &\leq CN^{-3}\|(1+|\cdot|^2)V\|_1\sum_{i=1}^{N}\|\widehat{\rho}_i\|_2^2\leq CN^{-2}\end{aligned}\tag{13.2}$$

where we recall that $\widehat{\rho}_i(z)=N^{-1}\rho_i(y_i+z/N)$ and $\|\underline{\rho}\|^2\leq CN$ from Lemma 12.1.

To bound the non diagonal term we use Cauchy-Schwarz inequality and the elementary estimate $ab\leq 1/2(a^2+b^2)$

$$\begin{aligned}\sum_{\substack{i,j=1\\ i\neq j}}^{N}\int dxK_i(x)K_j(x) &\leq \frac{\varepsilon}{2}\sum_{\substack{i,j=1\\ i\neq j}}^{N}\|\rho_i\|_2^2\,\|\rho_j\|_2^2+\frac{1}{2\varepsilon}\sum_{\substack{i,j=1\\ i\neq j}}^{N}\int dz\, dz'|v_i(z)|^2|v_j(z')|^2\\ &\quad\times\Big|\int dx\big(\mathcal{G}^\lambda(x-z)-\mathcal{G}^\lambda(x-y_i)\big)\big(\mathcal{G}^\lambda(x-z')-\mathcal{G}^\lambda(x-y_j)\big)\Big|^2\end{aligned}\tag{13.3}$$

with $\varepsilon>0$ to be fixed. To bound the second term in the r.h.s. of Equation (13.3) we first integrate over x using elliptic coordinates, then we use the definition of v_i and rescale the integration variables z and z'. We obtain

$$\frac{1}{2\varepsilon N^2}\sum_{\substack{i,j=1\\ i\neq j}}^{N}\int dzdz'|V(z)||V(z')||\zeta_{ij}^N(z,z')|^2$$

with

$$\zeta_{ij}^N(z,z')=e^{-\sqrt{\lambda}|y_i-y_j+(z-z')/N|}-e^{-\sqrt{\lambda}|y_i-y_j+z/N|}-e^{-\sqrt{\lambda}|y_i-y_j-z'/N|}+e^{-\sqrt{\lambda}|y_i-y_j|}\,.\tag{13.4}$$

With a Taylor expansion at first order, it is easy to check that the function $f(x)=e^{-\sqrt{\lambda}|x+a|}$ satisfies

$$\Big|e^{-\sqrt{\lambda}|x+a|}-e^{-\sqrt{\lambda}|a|}\Big(1-\sqrt{\lambda}\,\frac{x\cdot a}{|a|}\Big)\Big|\leq C|x|^2\,.$$

with C independent on a. Hence

$$|\zeta_{ij}^N(z,z')|\leq\frac{C}{N^2}(|z|+|z'|)^2\,,\tag{13.5}$$

and

$$\frac{1}{2\varepsilon N^2}\sum_{\substack{i,j=1\\ i\neq j}}^{N}\int dzdz'|V(z)||V(z')|\,|\zeta_{ij}^N(z,z')|^2$$
$$\leq\frac{C}{\varepsilon N^4}\int dzdz'|V(z)||V(z')|(|z|+|z'|)^4\leq C\varepsilon^{-1}N^{-4}\,.\tag{13.6}$$

Using Equation (13.3) and (13.6), the bound $\|\underline{\rho}\| \leq C$ and the assumptions on the potential, and choosing $\varepsilon = N^{-2}$ we obtain

$$\sum_{\substack{i,j=1 \\ i\neq j}}^{N} \int dx K_i(x) K_j(x) \ \leq CN^{-2}\,. \tag{13.7}$$

Equation (13.2) and (13.7), together with (13.1) conclude the proof of the proposition. □

13.2 POINT CHARGE APPROXIMATION

In this section we analyze the difference between $\widetilde{\psi}_N$ and $\widehat{\psi}_N$ and show that it is small for large N. More precisely our aim to prove the next proposition.

Proposition 13.2. *Let $\widehat{\psi}_N$ and $\widetilde{\psi}_N$ be defined in (11.15) and (11.11) respectively. Under the assumptions of Theorem 10.1 and for λ large enough*

$$\lim_{N\to\infty} N^{-\beta}\|\widehat{\psi}_N - \widetilde{\psi}_N\|_2 = 0 \qquad \forall\beta < 3/2$$

The proposition follows from the next two lemmas.

Lemma 13.1. *Let $\widehat{\psi}_N$ and $\widetilde{\psi}_N$ be defined in (11.15) and (11.11) respectively. Then,*

$$\|\widehat{\psi}_N - \widetilde{\psi}_N\|_2 \leq c\sqrt{N}\,\|\vec{q}-\vec{Q}\|$$

Lemma 13.2. *Let $Q_i = (v_i, \rho_i)$ and q_i be defined in (11.16). Then there exists $\delta > 0$ such that*

$$\|\vec{Q}-\vec{q}\,\| \leq CN^{-2-\delta}\,.$$

Proof of Lemma 13.1. We notice that

$$\begin{aligned}\|\widetilde{\psi}_N - \widehat{\psi}_N\|_2^2 &\leq \int dx\left|\sum_{i=1}^N \mathcal{G}^\lambda(x-y_i)(q_i-Q_i)\right|^2\\ &= \sum_{i=1}^N (q_i-Q_i)^2\int dx \frac{e^{-2\sqrt{\lambda}|x-y_i|}}{16\pi^2|x-y_i|^2}\\ &\quad+\sum_{\substack{i,j=1\\ i\neq j}}^N (q_i-Q_i)(q_j-Q_j)\int dx\,\frac{e^{-\sqrt{\lambda}(|x-y_i|+|x-y_j|)}}{16\pi^2|x-y_i||x-y_j|}.\end{aligned}\tag{13.8}$$

The term on the second line of Equation (13.8) is clearly bounded by $C\|\vec{Q}-\vec{q}\|^2$.
To evaluate the integral in the last line of (13.8) we use an explicit integration as in the proof of Proposition 13.1 and Cauchy-Schwarz inequality. We get:

$$\begin{aligned}\sum_{\substack{i,j=1\\ i\neq j}}^N (q_i-Q_i)(q_j-Q_j)\int dx\,\frac{e^{-\sqrt{\lambda}(|x-y_i|+|x-y_j|)}}{16\pi^2|x-y_i||x-y_j|} &\leq c_\lambda\sum_{\substack{i,j=1\\ i\neq j}}^N (q_i-Q_i)(q_j-Q_j)e^{-\sqrt{\lambda}|y_i-y_j|}\\ &\leq c_\lambda\sum_{\substack{i,j=1\\ i\neq j}}^N (q_i-Q_i)^2 \leq c_\lambda N\|\vec{q}-\vec{Q}\|^2\,.\end{aligned}$$

□

Proof of Lemma 13.2. Equations (11.12) and (11.16) for the charges Q_i and q_i give

$$\frac{N}{4\pi a}(Q_i-q_i)+\sum_{\substack{j=1\\ j\neq i}}^N G_{ij}^\lambda(Q_j-q_j) = R_i\,,\tag{13.9}$$

where R_i is defined by Equations (11.13), (11.14). We denote

$$\Gamma_{ij}^\lambda := \left(\delta_{ij}+\frac{4\pi a}{N}(1-\delta_{ij})G_{ij}^\lambda\right)\,,\tag{13.10}$$

so that (13.9) becomes

$$\sum_{\substack{j=1\\j\neq i}}^{N} \Gamma^{\lambda}_{ij}(Q_j - q_j) = \frac{4\pi a}{N} R_i \,.$$

On the other hand the bound (11.8) yields immediately the invertibility of Γ^{λ}_{ij} for $\lambda > \lambda_0$. Therefore

$$\|\vec{Q} - \vec{q}\,\| \le \frac{C}{N}\|\vec{R}\|$$

Lemma 13.2 is proved showing that there exists $\delta > 0$ such that

$$\|\vec{R}\,\| \le N^{-1-\delta} \,. \tag{13.11}$$

We start from $\|\vec{A}\,\|$. By Cauchy-Schwarz inequality we get

$$|A_i| \le \|\rho_i\|_2 \|\mu_i\|_2 \|v_i(\mathcal{G}^{\lambda} - \mathcal{G}^{0})u_i\|_{HS} \,.$$

Using the definitions of u_i and v_i

$$\begin{aligned}
\|v_i(\mathcal{G}^{\lambda} - \mathcal{G}^{0})u_i\|^2_{HS} &= \int dxdz N^4 |V(N(x-y_i))||V(N(z-y_i))| \frac{(e^{-\sqrt{\lambda}|x-z|}-1)^2}{16\pi^2|x-z|^2} \\
&= C\int dxdz \frac{|V(x)||V(z)|}{|x-z|^2}\left(e^{-\sqrt{\lambda}|x-z|/N}-1\right)^2 \\
&\le C\lambda N^{-2}\|V\|_1^2 \,.
\end{aligned}$$

With the bounds in Lemma 12.1 we have

$$\|\vec{A}\,\| \le \|\underline{\rho}\,\| \Big(\sup_i \|v_i(\mathcal{G}^{\lambda} - \mathcal{G}^{0})u_i\|^2_{HS}\Big)^{1/2} \Big(\sup_i \|\mu_i\|_2^2\Big)^{1/2} \le CN^{-3/2}\|f\|_2 \,. \tag{13.12}$$

Next, we analyze $\|\vec{B}\,\|$. We define

$$B_{ij} = \int dxdz\, u_i(x)\mu_i(x)\big(\mathcal{G}^{\lambda}(x-z) - G^{\lambda}_{ij}\big)v_j(z)\rho_j(z) \,.$$

Then

$$\|\vec{B}\,\|^2 = \sum_{i=1}^{N}\Bigg(\sum_{\substack{j=1\\j\neq i}}^{N} B_{ij}\Bigg)^2 .$$

Using twice Cauchy-Schwarz inequality, first in the x and z variables and then in the sum over j, we get

$$\begin{aligned}
\|\vec{B}\,\|^2 &\le \sum_{i=1}^{N}\Bigg[\sum_{\substack{j=1\\j\neq i}}^{N} \|\mu_i\|_2\|\rho_j\|_2 \Big(\int dxdz |u_i(x)|^2 \big(\mathcal{G}^{\lambda}(x-z) - G^{\lambda}_{ij}\big)^2 |v_j(z)|^2\Big)^{1/2}\Bigg]^2 \\
&\le \sum_{i=1}^{N}\|\mu_i\|_2^2 \sum_{\substack{k=1\\k\neq i}}^{N}\|\rho_k\|_2^2 \sum_{\substack{j=1\\j\neq i}}^{N} \|v_j(\mathcal{G}^{\lambda} - G^{\lambda}_{ij})u_i\|^2_{HS} \\
&\le \big(\sup_i \|\mu_i\|_2^2\big)\|\underline{\rho}\,\|^2 \sum_{i=1}^{N}\sum_{\substack{j=1\\j\neq i}}^{N} \|v_j(\mathcal{G}^{\lambda} - G^{\lambda}_{ij})u_i\|^2_{HS} \,.
\end{aligned}$$

Rescaling variables and recalling the definition (12.3) for $\mathcal{G}_{ij}^{\lambda,N}$ we have

$$\|v_j(\mathcal{G}^\lambda - G_{ij}^\lambda)u_i\|_{HS}^2 = N^{-2}\|v(\mathcal{G}_{ij}^{\lambda,N} - G_{ij}^\lambda)u\|_{HS}. \tag{13.13}$$

Using the bounds (11.7), (12.9) and (13.13), together with Lemma 12.1 we obtain

$$\|\vec{B}\,\| \leq C\Bigg(\frac{1}{N^5}\sum_{\substack{i,j=1\\ i\neq j}}^{N}\frac{1}{|y_i-y_j|^4}\Bigg)^{1/2} \leq c_\nu N^{-1-\frac{1}{2}\nu^2}\,. \tag{13.14}$$

To conclude we consider $\|\vec{D}\|$. We have

$$\begin{aligned}
\|\vec{D}\,\|^2 &= \sum_{i=1}^{N}\Big|\int dxdz\mu_i(x)u_i(x)\left(\mathcal{G}^\lambda(x-z) - \mathcal{G}^\lambda(y_i-z)\right)f(z)\Big|^2 \\
&= \sum_{i=1}^{N}\int dzdz' f(z)f(z')\xi_i(z)\xi_i(z')\,,
\end{aligned}$$

with

$$\xi_i(z) = \int dx\mu_i(x)u_i(x)\big(\mathcal{G}^\lambda(x-z) - \mathcal{G}^\lambda(y_i-z)\big)\,.$$

Using Cauchy-Schwarz inequality

$$\|\vec{D}\,\|^2 \leq \|f\|_2^2\left[\sum_{i=1}^{N}\Big(\int dz\xi_i^2(z)\Big)^2 + \sum_{\substack{i,j=1\\ i\neq j}}^{N}\Big(\int dz\xi_i(z)\xi_j(z)\Big)^2\right]^{1/2}. \tag{13.15}$$

We proceed as in the proof of Proposition 13.1. As for the diagonal term, using the scaling property $\mu(x) = N^{-1}\mu_i(y_i + x/N)$ we obtain

$$\begin{aligned}
&\sum_{i=1}^{N}\Big(\int dz\,\xi_i^2(z)\Big)^2 \\
&\leq \sum_{i=1}^{N}\Bigg[\frac{C}{N^2}\int dxdx'\mu(x)\mu(x')|V(x)|^{1/2}|V(x')|^{1/2} \\
&\times\Big(e^{-\sqrt{\lambda}\frac{|x-x'|}{N}} - e^{-\sqrt{\lambda}\frac{|x|}{N}} - e^{-\sqrt{\lambda}\frac{|x'|}{N}} + 1\Big)\Bigg]^2 \\
&\leq C\sum_{i=1}^{N}\left[N^{-3}\int dxdx'\mu(x)\mu(x')|V(x)|^{1/2}|V(x')|^{1/2}(|x|+|x'|)\right]^2 \\
&\leq CN^{-6}\|(1+|\cdot|^2)V\|_1\sum_{i=1}^{N}\|\mu\|_2^4 \leq CN^{-5}\,.
\end{aligned} \tag{13.16}$$

Here we used $\|\mu\|_2^2 \leq C$, see (12.1).

To estimate the non diagonal term in (13.15) we use the bound (13.5) for the function ζ_{ij}^N defined in (13.4). By the scaling properties of $u_i(x)$ we have:

$$
\begin{aligned}
&\sum_{\substack{i,j=1\\ i\neq j}}^{N} \Big(\int dz\, \xi_i(z)\xi_j(z) \Big)^2 \\
&= \sum_{\substack{i,j=1\\ i\neq j}}^{N} \Big[N^{-2} \int dxdx' |V(x)|^{1/2} |V(x')|^{1/2} \mu(x)\mu(x') \\
&\qquad \times \int dz \Big(\mathcal{G}^\lambda(y_i - z + x/N) - \mathcal{G}^\lambda(y_i - z) \Big) \Big(\mathcal{G}^\lambda(y_j - z + x'/N) - \mathcal{G}^\lambda(y_j - z) \Big) \Big]^2 \\
&= \sum_{\substack{i,j=1\\ i\neq j}}^{N} \Big[N^{-2} \int dxdx' |V(x)|^{1/2} |V(x')|^{1/2} \mu(x)\mu(x')\, \zeta_{ij}^N(x,x') \Big]^2 \\
&\leq C \sum_{\substack{i,j=1\\ i\neq j}}^{N} \Big[N^{-4} \int dxdx' |V(x)|^{1/2} |V(x')|^{1/2} \mu(x)\mu(x') (|x| + |x'|)^2 \Big]^2 \\
&\leq C\, N^{-8} \sum_{\substack{i,j=1\\ i\neq j}}^{N} \Big(\int dx |V(x)| (1 + |x|^2) \int dx' |\mu(x')|^2 \Big)^2 \leq CN^{-6}\,.
\end{aligned}
\tag{13.17}
$$

Putting together (13.15), (13.16) and (13.17) we obtain

$$
\|\vec{D}\,\|^2 \leq CN^{-5}\,. \tag{13.18}
$$

The bound (13.11) for $\|\vec{R}\,\|$ follows from (13.12), (13.14) and (13.18). □

13.3 LIMIT OF $\widehat{\psi}_N$

In this section we conclude the proof of the main result stated in Theorem 10.1. By Propositions 13.1 and 13.2 it remains to show the convergence of $\widehat{\psi}_N$ to ψ. Although the proof is a slight modification of the step followed in [38] (see also [17]) we report the details here for the sake of completeness.

Proposition 13.3. *Let $\widehat{\psi}_N$ and ψ be defined as in (11.15) and (11.17) respectively. Then under the assumptions of Theorem 10.1 and for $\lambda > \lambda_0$*

$$\lim_{N\to+\infty} N^\beta \|\widehat{\psi}_N - \psi\| = 0 \qquad \forall \beta < 1/2\,.$$

To prove the proposition we first introduce q defined by

$$-\frac{1}{4\pi a} q = \psi = (-\Delta + 4\pi a W + \lambda)^{-1} f. \tag{13.19}$$

Then by the second resolvent identity we get

$$\frac{1}{4\pi a} q(x) + \int dz\, \mathcal{G}^\lambda(x-z) W(z) q(z) = -(\mathcal{G}^\lambda f)(x). \tag{13.20}$$

In the following lemma we compare $q(y_i)$ with q_i.

Lemma 13.3. *Let q_i and q be defined as in (11.16) and (13.20) respectively. Then under the same assumptions as in Theorem 10.1 and for $\lambda > \lambda_0$*

$$\lim_{N\to+\infty} N^\beta \left\{ \frac{1}{N} \sum_{i=1}^N [Nq_i - q(y_i)]^2 \right\}^{1/2} = 0 \qquad \forall \beta < 1/2.$$

Proof. From (11.16) and (13.20) we get

$$\sum_{i=1}^N \left(\frac{1}{4\pi a}\delta_{ij} + \frac{1}{N}(1-\delta_{ij}) G^\lambda_{ij} \right) \left(\sqrt{N} q_j - \frac{1}{\sqrt{N}} q(y_j) \right) = L_i \tag{13.21}$$

where

$$L_i = \frac{1}{N^{3/2}} \sum_{\substack{j=1 \\ j\neq i}}^N G^\lambda_{ij} q(y_j) - \frac{1}{\sqrt{N}} (\mathcal{G}^\lambda W q)(y_i). \tag{13.22}$$

Recalling the definition of Γ^λ_{ij} given in Equation (13.10) we rewrite Equation (13.21) as

$$\sum_{j=1}^N \Gamma^\lambda_{ij} \left(\sqrt{N} q_j - \frac{1}{\sqrt{N}} q(y_j) \right) = (4\pi a) L_i.$$

Using invertibility of Γ^λ for $\lambda > \lambda_0$ (see (11.8)) and multiplying by N^β with $\beta < 1/2$ we get

$$N^\beta \left\{ \sum_{i=1}^N \left| \sqrt{N} q_i - \frac{1}{\sqrt{N}} q(y_i) \right|^2 \right\}^{1/2} \leq C N^\beta \|\vec{L}\,\|\,.$$

It remains to prove that $N^\beta \|\vec{L}\|$ goes to zero. In particular noticing that $E(\|\vec{L}\|) = 0$ and applying Chebyshev inequality it is enough to show $N^{2\beta} E(\|\vec{L}\|^2) \to 0$. We use that

$$\begin{aligned} &E\big(\, \mathcal{G}^\lambda(x-y_i) q(y_i) \,\big) = (\mathcal{G}^\lambda W q)(x) \\ &E\big(\, (\mathcal{G}^\lambda W q)^2(y_i) \,\big) = \|\mathcal{G}^\lambda W q\|^2_{L^2_W} \\ &E\big(\, (\mathcal{G}^\lambda(y_i - y_j) q(y_j))^2 \,\big) = (1, (\mathcal{G}^\lambda)^2 * (W q^2))_{L^2_W}\,, \end{aligned} \tag{13.23}$$

where we used the notation $(f * g)(x) = \int dy f(x-y)g(y)$. From (13.22) and (13.23) we obtain

$$
\begin{aligned}
N^{2\beta} E\left(\|\vec{L}\|^2\right) &= N^{2\beta-1} E\Bigg(\sum_{i=1}^{N} \Bigg(\frac{1}{N^2} \sum_{\substack{j,k=1\\ j\neq i, k\neq i,j}}^{N} G_{ij}^{\lambda} G_{ik}^{\lambda} q(y_j) q(y_k) + \frac{1}{N^2} \sum_{\substack{j=1\\ j\neq i}}^{N} \left(G_{ij}^{\lambda} q(y_j)\right)^2 \\
&\quad - \frac{2}{N} \sum_{\substack{j=1\\ j\neq i}}^{N} G_{ij}^{\lambda}\, q(y_j) (\mathcal{G}^{\lambda} W q)(y_i) \Bigg) + (\mathcal{G}^{\lambda} W q)^2 (y_i) \Bigg) \Bigg) \\
&= N^{2\beta-1} \Bigg(\frac{(N-1)}{N} E\big((\mathcal{G}^{\lambda}(y_1 - y_2) q(y_2))^2 \big) \\
&\quad + \left(\frac{(N-1)(N-2)}{N} - 2(N-1) + N \right) E\big((\mathcal{G}^{\lambda} W q)^2 (y_1) \big) \Bigg) \\
&= \frac{N-1}{N^{2-2\beta}} \left(1, (\mathcal{G}^{\lambda})^2 * (W q^2)\right)_{L^2_W} - \frac{N-2}{N^{2-2\beta}} \|\mathcal{G}^{\lambda} W q\|^2_{L^2_W},
\end{aligned}
$$

which goes to zero for $N \to \infty$ for any $\beta < 1/2$. This concludes the proof of the lemma. □

Proof of Proposition 13.3. Let us consider $g \in L^2(\mathbb{R}^3)$. Then by (11.16), (11.15), (13.19) and (13.20) we get

$$
\begin{aligned}
|(g, \widehat{\psi}_N - \psi)| &= |(g, \sum_{i=1}^{N} q_i\, \mathcal{G}^{\lambda}(\cdot - y_i)) - (g, \mathcal{G}^{\lambda} W q)| \\
&\leq |\sum_{i=1}^{N} (q_i - \frac{q(y_i)}{N}) \mathcal{G}^{\lambda} g(y_i)| + |\frac{1}{N} \sum_{i=1}^{N} q(y_i) \mathcal{G}^{\lambda} g(y_i) - (g, \mathcal{G}^{\lambda} W q)|.
\end{aligned}
\tag{13.24}
$$

Then using Cauchy-Schwarz inequality and multiplying both sides by $\frac{N^{\beta}}{\|g\|}$ we obtain

$$
\begin{aligned}
\frac{|N^{\beta}(g, \widehat{\psi}_N - \psi)|}{\|g\|} \leq & \frac{\sup_x |\mathcal{G}^{\lambda} g(x)|}{\|g\|} N^{\beta} \left\{ \frac{1}{N} \sum_{i=1}^{N} (N q_i - q(y_i))^2 \right\}^{1/2} \\
& + N^{\beta} \frac{|\eta(Y_N) - E(\eta(Y_N))|}{\|g\|}
\end{aligned}
\tag{13.25}
$$

where

$$
\eta(Y^N) = \frac{1}{N} \sum_{i=1}^{N} (\mathcal{G}^{\lambda} g)(y_i) q(y_i).
$$

The first term in (13.25) goes to zero by Lemma 13.3. Furthermore

$$
\begin{aligned}
E\left(\frac{|\eta(Y_N) - E(\eta(Y_N))|^2}{\|g\|^2} \right) &= \frac{E\left(\eta(Y_N)^2\right)}{\|g\|^2} - \frac{E\left(\eta(Y_N)\right)^2}{\|g\|^2} \\
&= \left(\frac{\int dx\, W(x) q^2(x) (\mathcal{G}^{\lambda} g)^2(x)}{\|g\|^2 N} + \frac{N-1}{N} \frac{E(\eta(Y_N))^2}{\|g\|^2} \right) \\
&\quad - \frac{E(\eta(Y_N))^2}{\|g\|^2} \\
&= \frac{\int dx\, W(x) q^2(x) (\mathcal{G}^{\lambda} g)^2(x) - \left(\int dy\, (\mathcal{G}^{\lambda} g)(y) q(y) W(y) \right)^2}{\|g\|^2\, N} \\
&\leq \frac{C}{N} \left(\|q\|_{L^2_W} + (1, q)^2_{L^2_W} \right).
\end{aligned}
$$

Then by Chebyshev inequality also the second term in (13.25) goes to zero uniformly in $\|g\|$. Taking the supremum over $g \in L^2(\mathbb{R}^3)$ we get the thesis. $\square$

We are now ready to prove our main result

Proof of Theorem 10.1. It follows immediately from Propositions 13.1, 13.2 and 13.3. $\square$

Part IV

APPENDIX

A THREE-PARTICLE SYSTEMS

A.1 THREE-BODY QUANTUM PROBLEM IN THE CENTER OF MASS FRAME

Let us consider a generic three-particle quantum system in dimension $d = 1, 2, 3$ interacting via pairwise interactions. We denote with $x_i \in \mathbb{R}^d$ the coordinate of the $i-$th particle and with m_i its mass. The Hilbert space is then $L^2(\mathbb{R}^{3d})$ and the Hamiltonian describing the dynamic of the system is

$$H^{3\text{body}} = -\sum_{i=1}^{3} \frac{1}{2m_i} \Delta_{x_i} + \sum_{\alpha} v_\alpha(x_i - x_j)$$

where Δ_{x_i} is the Laplacian with respect to x_i and $\alpha = ij$ runs over the pair $\{12, 23, 31\}$. We will often use an "odd-man-out" notation common in the study of three-body systems denoting equivalently the pair α with the index $\ell = 1, 2, 3$ such that $\ell \notin \alpha$. Throughout this work the Hamiltonian $H^{3\text{body}}$ will be either a well-defined self-adjoint Hamiltonian with regular potential or a formal Hamiltonian with singular interaction of $\delta-$type (see Part II).

As usual in the study of three-particle systems it is convenient to extract the center of mass motion. To do this we introduce the coordinates (X, x_α, y_α), where X is the coordinate of the center of mass and (x_α, y_α) is any pair of Jacobi coordinates (see Figure A.1): x_α is the relative coordinate between the pair of particles α while y_α is the relative coordinate between the third particle and center of mass of the pair, i. e.,

$$\begin{aligned} x_{23} &= x_2 - x_3, \quad y_{23} = \frac{m_2 x_2 + m_3 x_3}{m_2 + m_3} - x_1 \\ x_{31} &= x_3 - x_1, \quad y_{31} = \frac{m_1 x_1 + m_3 x_3}{m_1 + m_3} - x_2 \\ x_{12} &= x_1 - x_2, \quad y_{12} = \frac{m_1 x_1 + m_2 x_2}{m_1 + m_2} - x_3. \end{aligned}$$

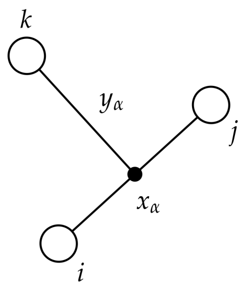

Figure A.1: Jacobi coordinates associated to the pair $\alpha = ij$

These coordinate systems define a decomposition $L^2(\mathbb{R}^{3d}) = L^2(\mathbb{R}^d) \otimes L^2(\mathbb{R}^{2d})$ and a tensor decomposition of $H^{3\text{body}}$ which is independent of the choice of the pair in the Jacobi coordinates. Denoting with $\mathbb{1}$ the identity operator in $L^2(\mathbb{R}^n)$

$$H^{3\text{body}} = H_{\text{CM}} \otimes \mathbb{1} + \mathbb{1} \otimes H$$

where

$$H_{\text{CM}} = -\frac{1}{2(m_1 + m_2 + m_3)} \Delta_X$$

and

$$H = H_0 + \sum_{\alpha'} v_{\alpha'}(x_{\alpha'}) = -\frac{1}{2\mu_\alpha}\Delta_{x_\alpha} - \frac{1}{2n_\alpha}\Delta_{y_\alpha} + \sum_{\alpha'} v_{\alpha'}(x_{\alpha'}). \tag{A.1}$$

In the above formula x_β, for $\beta \neq \alpha$, is expressed in terms of (x_α, y_α) and the new masses are defined by

$$\mu_\alpha = \frac{m_i m_j}{m_i + m_j}, \qquad n_\alpha = \frac{m_\ell(m_i + m_j)}{m_1 + m_2 + m_3}, \quad \alpha = ij,\ \ell \neq i, j. \tag{A.2}$$

H_{CM} is just the free Hamiltonian and we can reduce our analysis to $\widetilde{H}$ acting on $L^2(\mathbb{R}^{2d})$.

Moreover it is often useful to work in momentum space. Let k_α, p_α be the conjugate variables of x_α, y_α respectively. Denoting with k_i the conjugate variable of x_i, they are explicitly defined by

$$\begin{aligned}
k_{23} &= \frac{m_3 k_2 - m_2 k_3}{m_2 + m_3} & p_{23} &= \frac{m_1(k_2 + k_3) - (m_2 + m_3)k_1}{m_1 + m_2 + m_3} \\
k_{31} &= \frac{m_1 k_3 - m_3 k_1}{m_1 + m_3} & p_{31} &= \frac{m_2(k_1 + k_3) - (m_1 + m_3)k_2}{m_1 + m_2 + m_3} \\
k_{12} &= \frac{m_2 k_1 - m_1 k_2}{m_1 + m_2} & p_{12} &= \frac{m_3(k_1 + k_2) - (m_1 + m_2)k_3}{m_1 + m_2 + m_3}.
\end{aligned} \tag{A.3}$$

With a slight abuse of the notation we denote with the same symbols the Hamiltonians H, H_0 written in momentum space. For any choice of pair α, we have

$$H = H_0 + \sum_\beta V_\beta$$

where

$$H_0 = H_0(k_\alpha, p_\alpha) = \frac{k_\alpha^2}{2\mu_\alpha} + \frac{p_\alpha^2}{2n_\alpha}$$

is the free Hamiltonian and

$$(V_\beta \psi)(k_\beta, p_\beta) = \frac{1}{(2\pi)^{d/2}} \int dk\, \widehat{v}_\beta(k - k_\beta)\psi(k, p_\beta) \tag{A.4}$$

describes each interaction term. In (A.4) we denote by $\widehat{f}$ the Fourier transform of f.

Finally we note that one can also choose as independent variables any pair (p_α, p_β) with $\alpha \neq \beta$. The connections between the various coordinate system is given by the following relations

$$\begin{aligned}
k_{23} &= -p_{31} - \frac{m_2}{m_2 + m_3} p_{23} = p_{12} + \frac{m_3}{m_3 + m_2} p_{23} \\
k_{31} &= -p_{12} - \frac{m_3}{m_1 + m_3} p_{31} = p_{23} + \frac{m_1}{m_1 + m_3} p_{31} \\
k_{12} &= -p_{23} - \frac{m_1}{m_1 + m_2} p_{12} = p_{31} + \frac{m_2}{m_1 + m_2} p_{12} \\
& p_{23} + p_{31} + p_{12} = 0.
\end{aligned} \tag{A.5}$$

Using them we find

$$H_0(p_\alpha, p_{\alpha'}) = \frac{p_\alpha^2}{2\mu_{\alpha'}} + \frac{p_\alpha \cdot p_{\alpha'}}{m_k} + \frac{p_{\alpha'}^2}{2\mu_\alpha}, \quad k \neq \ell, \ell'. \tag{A.6}$$

A.2 FADDEEV EQUATIONS FOR THE EIGENVALUES

A useful tool in the study of the point spectrum of the three-particle Schrödinger operator are the so-called Faddeev equation for the eigenvalues, introduced by L. D. Faddeev in [34] (see also [35]), which are the analogous of the Birman-Schwinger equation for the one-body problem. We briefly recall here their derivation following the clear and simple presentation contained in [56]. Let us suppose $z < 0$ to be an eigenvalue for the operator H defined in (A.1) and ψ to be the corresponding eigenfunction. We assume in addition $z \notin \sigma(H_0 + V_\alpha)$, $\forall \alpha$ where $\sigma(A)$ denotes the spectrum of the operator A. Then

$$\big(H_0 + \sum_\alpha V_\alpha\big)\psi = z\,\psi. \tag{A.7}$$

Note that the free resolvent

$$R_0(z) = (H_0 - z)^{-1}$$

is bounded in $L^2(\mathbb{R}^{2d})$, so that (A.7) can be equivalently rewritten as

$$\psi = -R_0(z)\sum_\alpha V_\alpha \psi. \tag{A.8}$$

We decompose ψ in the Faddeev components

$$\psi = \sum_\alpha \eta_\alpha$$

where, for each pair α, from (A.8), we have

$$\eta_\alpha = -R_0(z)V_\alpha\psi. \tag{A.9}$$

In order to find the equations for η_α, we introduce the resolvent of $H_\alpha = H_0 + V_\alpha$

$$R_\alpha(z) = (H_\alpha - z)^{-1}$$

which is bounded in $L^2(\mathbb{R}^{2d})$ by our assumptions. Then we rewrite (A.9) in the form

$$\eta_\alpha = -R_0(z)V_\alpha\sum_\beta \eta_\beta$$

or

$$\eta_\alpha + R_0(z)V_\alpha\eta_\alpha = -R_0(z)V_\alpha\sum_{\beta\neq\alpha}\eta_\beta$$

Applying the operator $R_\alpha(z)(H_0 - z)$ to both sides of the above equation, we obtain the Faddeev equations

$$\eta_\alpha = -R_\alpha(z)V_\alpha\sum_{\beta\neq\alpha}\eta_\beta, \quad \forall\alpha \tag{A.10}$$

Thus, we conclude that if ψ is a solution of (A.7), then $\psi = \sum_\alpha \eta_\alpha$ and η_α are solutions of (A.10). The converse is also true, i.e., if η_α are solutions of (A.10), then $\psi = \sum_\alpha \eta_\alpha$ is a solution of (A.7). As it is well known, a suitable iterated form of the Faddeev equations is characterized by a compact operator and this is the main advantage of the Faddeev equations with respect to Equation (A.8).

A.3 FADDEEV EQUATIONS FOR THE RESOLVENT

In this section we apply a method analogous to the one adopted in the previous section to the study of the resolvent of H. Let $z < 0$ be such that $z \notin \sigma(H)$ and $z \notin \sigma(H_\alpha)$, $\forall\alpha$. By the resolvent identity $R(z) = (H - z)^{-1}$ satisfies

$$R(z) = R_0(z) - R_0(z) \sum_\alpha V_\alpha R(z) = R_0(z) + \sum_\ell R^{(\ell)}(z) \tag{A.11}$$

where $R^{(\ell)}(z) = -R_0(z) V_\alpha R(z)$, $\ell \notin \alpha$. Plugging Equation (A.11) into the definition of $R^{(\ell)}(z)$ we find

$$R^{(\ell)}(z) = -R_0(z) V_\alpha R_0(z) - R_0(z) V_\alpha \sum_{\ell'} R^{(\ell')}(z)$$

or equivalently

$$R^{(\ell)}(z) + R_0(z) V_\alpha R^{(\ell)}(z) = -R_0(z) V_\alpha R_0(z) - R_0(z) V_\alpha \sum_{\ell' \neq \ell} R^{(\ell')}(z).$$

Applying $R_\alpha(z)(H_0 - z)$ to both sides of the equation above we get

$$R^{(\ell)}(z) = -R_\alpha(z) V_\alpha R_0(z) - R_\alpha(z) \sum_{\ell' \neq \ell} R^{(\ell)}(z). \tag{A.12}$$

Let us introduce the operator

$$T_\alpha(z) = V_\alpha - V_\alpha R_a(z) V_\alpha \tag{A.13}$$

and note the identity

$$R_\alpha(z) V_\alpha = R_0(z) T_\alpha(z) \tag{A.14}$$

which is a consequence of the resolvent identity. Using Equation (A.14) in Equation (A.12) we find

$$R^{(\ell)}(z) = -R_0(z) T_\alpha(z) R_0(z) - R_0(z) T_\alpha(z) \sum_{\ell' \neq \ell} R^{(\ell')}(z).$$

Or equivalently, again by resolvent identity,

$$R^{(\ell)}(z) = -R_\alpha(z) - R_0(z) - R_0(z) T_\alpha(z) \sum_{\ell' \neq \ell} R^{(\ell')}(z).$$

Remark A.1. One can prove that

$$T_\alpha(z) \psi(k_\alpha, p_\alpha) = \int dk'_\alpha \, t_\alpha\left(z - \frac{p_\alpha^2}{2n_\alpha}; k_\alpha, k'_\alpha\right) \psi(k'_\alpha, p_\alpha) \tag{A.15}$$

where $t_\alpha(z; k_a, k'_\alpha)$ is the integral kernel in Fourier space of the two-particle operator $t_\alpha(z)$ on $L^2(\mathbb{R}^d)$ defined by

$$t_\alpha(z) = v_\alpha - v_\alpha r_\alpha(z) v_\alpha. \tag{A.16}$$

Here v_α denotes the multiplication operator in $L^2(\mathbb{R}^d)$ and $r_\alpha(z) = (h_0 + v_\alpha - z)^{-1}$ where $h_0 = -\frac{1}{2\mu_\alpha} \Delta_{x_\alpha}$.

Let us check Equation (A.15). By Equation (A.16) and the resolvent identity we get

$$t_\alpha(z) = v_\alpha - v_\alpha r_0(z) v_a$$

where $r_0(z) = (h_0 - z)^{-1}$. From the equation above we obtain the following equation for the kernel $t_\alpha(z; k_\alpha, k'_\alpha)$:

$$t_\alpha(z; k_\alpha, k'_\alpha) = \frac{1}{\sqrt{2\pi}} v_\alpha(k_\alpha - k'_\alpha) - \frac{1}{\sqrt{2\pi}} \int dk''_\alpha \frac{v_\alpha(k_\alpha - k''_\alpha)}{\frac{(k''_\alpha)^2}{2n_\alpha} - z} t_\alpha(z; k''_\alpha, k'_\alpha). \tag{A.17}$$

From Equation (A.17), recalling Equations (A.13), (A.14) and (A.4) we immediately get (A.15).

B PROPERTIES OF $S_m^{(l)}$

We collect here some properties of functions $S_m^{(l)}(x)$ (see also [24]) defined by

$$S_m^{(l)}(x) = -2\pi b(m) \int_{-1}^{1} dy \, \frac{P_l(y)}{\cosh(x) + \frac{y}{m+1}}, \qquad x \in \mathbb{R} \tag{B.1}$$

where

$$b(m) = \frac{1}{4\pi^2} \frac{m+1}{\sqrt{m(m+2)}}$$

is a function of the mass and P_l denotes the l-th Legendre polynomial.
First we evaluate their Fourier transform

$$\begin{aligned}
\sqrt{2\pi}\, \widehat{S}_m^{(l)}(k) &= -2\pi\, b(m) \int_{-\infty}^{+\infty} dx\, e^{-ikx} \int_{-1}^{1} dy \, \frac{P_l(y)}{\cosh x + \frac{y}{m+1}} \\
&= -\frac{1}{2\pi} \frac{m+1}{\sqrt{m(m+2)}} \int_{-1}^{1} dy\, P_l(y) \int_{-\infty}^{+\infty} dx \, \frac{e^{-ikx}}{\cosh x + \frac{y}{m+1}} \\
&= -\frac{1}{\pi} \frac{m+1}{\sqrt{m(m+2)}} \int_{-1}^{1} dy\, P_l(y) \int_{0}^{+\infty} dx \, \frac{1}{\cosh x + \frac{y}{m+1}} \cos(kx) \\
&= -\frac{m+1}{\sqrt{m(m+2)}} \int_{-1}^{1} dy\, P_l(y) \frac{\sinh\left(k \arccos\left(\frac{y}{m+1}\right)\right)}{\sinh(k\pi) \sin\left(\arccos\left(\frac{y}{m+1}\right)\right)}
\end{aligned}$$

where, for the computation of the last integral, we refer the reader to [30, p. 30]. Moreover using the elementary relations $\arccos(\alpha) = \frac{\pi}{2} - \arcsin(\alpha)$ and $\sinh(\alpha \pm \beta) = \sinh(\alpha)\cosh(\beta) \pm \sinh(\beta)\cosh(\alpha)$, we find

$$\begin{aligned}
\widehat{S}_m^{(l)}(k) = -\frac{1}{\sqrt{2\pi}} \frac{m+1}{\sqrt{m(m+2)}} \int_{-1}^{1} dy\, P_l(y) \Bigg[&\frac{\cosh\left(k \arcsin\left(\frac{y}{m+1}\right)\right)}{2\cosh\left(k\frac{\pi}{2}\right) \cos\left(\arcsin\left(\frac{y}{m+1}\right)\right)} \\
&- \frac{\sinh\left(k \arcsin\left(\frac{y}{m+1}\right)\right)}{2\sinh\left(k\frac{\pi}{2}\right) \cos\left(\arcsin\left(\frac{y}{m+1}\right)\right)} \Bigg].
\end{aligned}$$

Using the parity of the Legendre polynomials in the expression above we obtain

$$\widehat{S}_m^{(l)}(k) = \begin{cases} \frac{1}{\sqrt{2\pi}} \frac{m+1}{\sqrt{m(m+2)}} \int_{0}^{1} dy\, P_l(y) \frac{\sinh\left(k \arcsin\left(\frac{y}{m+1}\right)\right)}{\sinh\left(k\frac{\pi}{2}\right) \cos\left(\arcsin\left(\frac{y}{m+1}\right)\right)} & l \text{ odd} \\ \\ -\frac{1}{\sqrt{2\pi}} \frac{m+1}{\sqrt{m(m+2)}} \int_{0}^{1} dy\, P_l(y) \frac{\cosh\left(k \arcsin\left(\frac{y}{m+1}\right)\right)}{\cosh\left(k\frac{\pi}{2}\right) \cos\left(\arcsin\left(\frac{y}{m+1}\right)\right)} & l \text{ even.} \end{cases} \tag{B.2}$$

Note that $\widehat{S}^{(l)}(-k) = \widehat{S}^{(l)}(k)$. Furthermore in [24] the following properties are proved.

Lemma B.1.

$$\begin{aligned} \widehat{S}_m^{(l)}(k) &\geq 0 \qquad l \text{ odd} \\ \widehat{S}_m^{(l)}(k) &\leq 0 \qquad l \text{ even}, \end{aligned} \tag{B.3}$$

$$\begin{aligned} \widehat{S}_m^{(l+2)}(k) &\leq \widehat{S}^{(l)}(k) \qquad l \text{ odd} \\ \widehat{S}_m^{(l+2)}(k) &\geq \widehat{S}^{(l)}(k) \qquad l \text{ even.} \end{aligned} \tag{B.4}$$

Moreover $\widehat{S}^{(1)}(k)$ *is monotone decreasing for* $k \geq 0$ *and*

$$\max_{k\in\mathbb{R}} \widehat{S}_m^{(1)}(k) = \widehat{S}_m^{(1)}(0) = \frac{1}{\sqrt{2\pi}}\Lambda(m) \tag{B.5}$$

where $\Lambda(m)$ *is defined in* (3.8).

Finally in the case $m = 1$, setting $\widehat{S}^{(l)}(k) := \widehat{S}_1^{(l)}(k)$, we have

Lemma B.2. *For* l *even and any* $k \in \mathbb{R}$

$$-\frac{1}{\sqrt{6\pi}}\left(\frac{50}{27}\pi - \frac{10}{3}\sqrt{3} + \frac{\sqrt{11}}{9} - \frac{10}{9}t_0\right) \leq \widehat{S}^{(l)}(k) \leq 0, \qquad l \neq 0 \tag{B.6}$$

where $t_0 = \arcsin(1/\sqrt{12}) \simeq 0.293$ *and*

$$-\frac{4}{\sqrt{6\pi}} \leq \widehat{S}^{(0)}(k) \leq 0. \tag{B.7}$$

Proof. Let us consider the case $l \neq 0$ and even. Since $\widehat{S}^{(l)}(k)$ is decreasing in l by (B.4), we have $\widehat{S}^{(l)}(k) \leq \widehat{S}^{(2)}(k)$, where $\widehat{S}^{(2)}(k)$ is an even function. An explicit integration gives

$$\begin{aligned} \widehat{S}^{(2)}(0) &= -\frac{1}{\sqrt{6\pi}}\int_0^1 dy\,(3y^2-1)\frac{1}{\cos\left(\arcsin\frac{y}{2}\right)} \\ &= -\frac{2}{\sqrt{6\pi}}\int_0^{\pi/6} dx\,\left(12\sin^2 x - 1\right) = -\frac{1}{\sqrt{6\pi}}\left(\frac{5}{3}\pi - 3\sqrt{3}\right) \end{aligned}$$

Let us estimate the difference $\widehat{S}^{(2)}(k) - \widehat{S}^{(2)}(0)$ for any positive k. We have

$$\widehat{S}^{(2)}(k) - \widehat{S}^{(2)}(0) = \frac{1}{\sqrt{6\pi}}\int_0^1 dy\,\frac{3y^2-1}{\cos\left(\arcsin\frac{y}{2}\right)}\left(1 - \frac{\cosh\left(k\arcsin\frac{y}{2}\right)}{\cosh\left(k\frac{\pi}{2}\right)}\right) \tag{B.8}$$

$$= \frac{2}{\sqrt{6\pi}} I(k) \tag{B.9}$$

where

$$I(k) = \int_0^{\pi/6} dt\,(12\sin^2 t - 1)\left(1 - \frac{\cosh(kt)}{\cosh\left(k\frac{\pi}{2}\right)}\right)$$

Since $s(t) = 12\sin^2 t - 1$ is negative if $t < t_0$ and positive otherwise we can write

$$\begin{aligned} I(k) &= -\int_0^{t_0} dt\,|s(t)| + \int_{t_0}^{\pi/6} dt\,s(t) \\ &\quad + \frac{1}{\cosh\left(k\frac{\pi}{2}\right)}\left[\int_0^{t_0} dt\,|s(t)|\cosh(kt) - \int_{t_0}^{\pi/6} dt\,s(t)\cosh(kt)\right] \\ &\geq \frac{1}{\cosh\left(k\frac{\pi}{2}\right)}\left[(b-a)\cosh\left(k\frac{\pi}{2}\right) + a - b\cosh\left(k\frac{\pi}{6}\right)\right] \end{aligned} \tag{B.10}$$

where $a = \int_0^{t_0} dt\,|s(t)|$ and $b = \int_{t_0}^{\pi/6} dt\,s(t)$, with $b - a > 0$. Denoting

$$g(k) = a + \left(\frac{10}{9}b - a\right)\cosh\left(k\frac{\pi}{2}\right) - b\cosh\left(k\frac{\pi}{6}\right)$$

we can rewrite (B.10) as

$$I(k) \geq \frac{g(k)}{\cosh\left(k\frac{\pi}{2}\right)} - \frac{b}{9}.$$

Let us show that $g(k) \geq 0$. We have

$$g'(k) = \frac{\pi}{2}\left(\frac{10}{9}b - a\right)\sinh\left(k\frac{\pi}{2}\right)\left[1 - A\,\frac{3\sinh\left(k\frac{\pi}{6}\right)}{\sinh\left(k\frac{\pi}{2}\right)}\right] \tag{B.11}$$

where

$$A = \frac{b}{10b - 9a}.$$

The term in square bracket in (B.11) is positive, then $g'(k) \geq 0$ which, together with $g(0) = \frac{b}{9}$, yields $g(k) \geq 0$. Thus we find $\widehat{S}^{(2)}(k) - \widehat{S}^{(2)}(0) \geq -\frac{2}{9\sqrt{6\pi}}b$. Inserting the explicit expression for b, we obtain the estimate (B.6).

In the case $l = 0$ the estimate (B.7) is straightforward.

□

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