A WEAK MINIMA APPROACH TO THE STUDY OF THE EXISTENCE OF SADDLE POINTS OF INTEGRAL FUNCTIONALS

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Abstract. We study of the existence of saddle points of the functional J defined in (1.1) both in the regular case, i.e., if E belongs to $(L^N(\Omega))^N$, and in the singular one, i.e., if E belongs to $(L^2(\Omega))^N$.

Carlo: weak minima – strong friend

1. Introduction

The objective of a suitable definition of minima of integral functionals, if the data are not regular, is the aim of the papers [6], by Carlo Sbordone and Tadeusz Iwaniec, where the definition of weak minima is introduced (see also [5]), and [1], where the definition of T-minima is introduced.

In this paper we follow [6], in order to prove the existence of a saddle point of the functional J below, if the vectorial field E is very singular.

Let Ω be a bounded, open subset of $\mathbb{R}^N$, with $N > 2$. Let us define, for (v, ψ) in $(W_0^{1,2}(\Omega), W_0^{1,2}(\Omega))$,

$$J(v,\psi) = \frac{1}{2}\int_\Omega A(x)\,\nabla v\,\nabla v - \frac{1}{2}\int_\Omega M(x)\,\nabla\psi\,\nabla\psi + \int_\Omega v\,E(x)\nabla\psi - \int_\Omega f(x)\,v\,. \tag{1.1}$$

where $A(x)$, $M(x)$ are symmetric measurable matrices such that

$$\begin{cases} A(x)\,\xi\,\xi \ge \alpha|\xi|^2\,, & |A(x)| \le \beta\,,\\ M(x)\,\xi\,\xi \ge \alpha|\xi|^2\,, & |M(x)| \le \beta\,, \end{cases} \tag{1.2}$$

for almost every x in Ω, for every ξ in $\mathbb{R}^N$, with $0 < \alpha \le \beta$, and

$$f \in L^m(\Omega)\,,\ m \ge 2_* = \frac{2N}{N+2}, \tag{1.3}$$

$$E \in (L^N(\Omega))^N. \tag{1.4}$$

It is easy to see, thanks to the assumptions on $A(x)$, $M(x)$, $E(x)$ and $f(x)$, that for every v and ψ in $W_0^{1,2}(\Omega)$ both $I_\psi(\cdot) = J(\cdot,\psi)$ and $I_v(\cdot) = -J(v,\cdot)$ are coercive and weakly lower semicontinuous on $W_0^{1,2}(\Omega)$; hence, by standard result of the Calculus of Variations, for every ψ in $W_0^{1,2}(\Omega)$ there exists a (unique) minimum u_ψ of $I_\psi(\cdot)$, and for every v in $W_0^{1,2}(\Omega)$ there exists a (unique) minimum φ_v of $I_v(\cdot)$. Both u_ψ and φ_v are weak solutions of the corresponding Euler-Lagrange equations, that is

$$\begin{cases} \displaystyle\int_\Omega A(x)\,\nabla u_\psi\,\nabla w + \int_\Omega E(x)\,\nabla\psi\, w = \int_\Omega f(x)\, w\,, & \forall\, w \in W_0^{1,2}(\Omega)\,, \\ \displaystyle\int_\Omega M(x)\,\nabla\varphi_v\,\nabla\eta = \int_\Omega v\, E(x)\,\nabla\eta\,, & \forall\,\eta \in W_0^{1,2}(\Omega)\,. \end{cases}$$

Therefore, and thanks to uniqueness of minima, one can build a map $v \mapsto u_{\varphi_v}$; if such a map has a fixed point, that is if there exists u in $W_0^{1,2}(\Omega)$ such that $u = u_{\varphi_u}$, then the couple (u,φ), with $\varphi = \varphi_u$, is a saddle point of the functional J, in the sense that

$$J(u,\psi) \le J(u,\varphi) \le J(v,\varphi)\,, \qquad \forall\, v\,,\psi \in W_0^{1,2}(\Omega)\,.$$

Furthermore, standard techniques of the Calculus of Variations imply that (u,φ) is a weak solution of the system

(1.5)
$$\begin{cases} u \in W_0^{1,2}(\Omega): \displaystyle\int_\Omega A(x)\,\nabla u\,\nabla v + \int_\Omega E(x)\,\nabla\varphi\, v = \int_\Omega f(x)\, v\,, & \forall\, v \in W_0^{1,2}(\Omega)\,, \\ \varphi \in W_0^{1,2}(\Omega): \displaystyle\int_\Omega M(x)\,\nabla\varphi\,\nabla\psi = \int_\Omega u\, E(x)\,\nabla\psi\,, & \forall\,\psi \in W_0^{1,2}(\Omega)\,. \end{cases}$$

However, it is not possible to prove that the map $v \mapsto u_{\varphi_v}$ has a fixed point; this is due to the fact that the map does not have an invariant ball, unless the norm of E in $(L^N(\Omega))^N$ is small. Thus, our approach is to pass from the Ptolemaic theory to the Copernican one: *first* we prove the existence of a solution (u,φ) of the system (1.5) and *then* we prove that (u,φ) is a saddle point of the functional J. This will be done in Section 2 of this paper, where uniqueness of solutions for (1.5) will be also proved.

The second part of the paper is devoted to the study the existence a weak solution (u,φ) in $(W_0^{1,2}(\Omega), W_0^{1,2}(\Omega))$ of the system (1.5) even if we only assume

$$E \in (L^2(\Omega))^N\,. \tag{1.6}$$

In this singular case, the summability of the functions is poor, so that the standard formulation of minimum (maximum) is meaningless: as we said before, a way to give a meaning is to use the concept of "weak minima" introduced in [6]: see Section 3.

One final remark: even though the vector field E is not regular, we are nevertheless able to prove existence of solutions (or of weak saddle points) in the "energy space" $W_0^{1,2}(\Omega)$. This fact, which is *false* if one considers the equations separately, is a consequence of the fact that we are dealing with a system of equations. For further results in which the "system structure" of the problem improves the regularity of the solutions with respect to the single equation, see [3], where the link between equations is given by a term of the form

$$\int_\Omega \psi\,|v|^r\,, \qquad r > 1\,,$$

and [2], where the link is of the form

$$\int_\Omega \psi\,|\nabla v|^2\,.$$

2. E belongs to $(L^N(\Omega))^N$

As stated in the Introduction, even though the functional J has good geometric properties, it is not possible to use fixed points theorems since it is not possible to find an invariant ball. Therefore, we need to approximate either J, or the corresponding Euler-Lagrange equations; this latter is the path we have chosen.

We recall the definition of the truncation $T_k : \mathbb{R} \to \mathbb{R}$: if $k \geq 0$ we define

$$T_k(s) = \max(-k, \min(s,k))\,.$$

Let n in $\mathbb{N}$, and let v in $L^2(\Omega)$; then there exists a unique function ψ in $W_0^{1,2}(\Omega)$, weak solution of the Dirichlet problem

$$\psi \in W_0^{1,2}(\Omega):\ -\mathrm{div}(M(x)\,\nabla\psi) = -\mathrm{div}(T_n(v)\,E(x))\,,$$

that is

$$\int_\Omega M(x)\,\nabla\psi\,\nabla\eta = \int_\Omega T_n(v)\,E(x)\,\nabla\eta\,, \qquad \forall\,\eta \in W_0^{1,2}(\Omega)\,.$$

Furthermore, choosing $\eta = \psi$, and using (1.2), we have

$$\alpha\int_\Omega |\nabla\psi|^2 \leq \int_\Omega M(x)\,\nabla\psi\,\nabla\psi = \int_\Omega T_n(v)\,E(x)\,\nabla\psi \leq n\,\|E\|_{L^2(\Omega)}\,\|\nabla\psi\|_{L^2(\Omega)}\,,$$

so that

$$\|\nabla\psi\|_{L^2(\Omega)} \leq \frac{\|E\|_{L^2(\Omega)}}{\alpha}\,n = C_1\,n\,. \tag{2.1}$$

Once ψ is given, there exists a unique weak solution u in $W_0^{1,2}(\Omega)$ of

$$u \in W_0^{1,2}(\Omega):\ -\mathrm{div}(A(x)\,\nabla u) + E(x)\,\nabla\psi = f(x)\,,$$

that is

$$\int_\Omega A(x)\,\nabla u\,\nabla w + \int_\Omega E(x)\,\nabla\psi\,w = \int_\Omega f(x)\,w\,, \qquad \forall\,w \in W_0^{1,2}(\Omega)\,.$$

Furthermore, choosing $w = u$ and using both (1.2) and the Sobolev embedding, we have

$$\begin{aligned}\alpha\int_\Omega |\nabla u|^2 &\leq \int_\Omega A(x)\,\nabla u\,\nabla u = \int_\Omega f(x)\,u - \int_\Omega E(x)\,\nabla\psi\,u \\ &\leq C_2\Big(\|f\|_{L^{2*}(\Omega)}\,\|u\|_{W_0^{1,2}(\Omega)} + \|E\|_{L^N(\Omega)}\,\|\nabla\psi\|_{L^2(\Omega)}\,\|u\|_{W_0^{1,2}(\Omega)}\Big)\,,\end{aligned}$$

which implies that

$$\|u\|_{W_0^{1,2}(\Omega)} \leq \frac{C_3}{\alpha}\left[\|f\|_{L^{2*}(\Omega)} + \|E\|_{L^N(\Omega)}\|\nabla\psi\|_{L^2(\Omega)}\right]. \tag{2.2}$$

Recalling (2.1) we then obtain

$$\|u\|_{W_0^{1,2}(\Omega)} \leq C_4\Big(\|f\|_{L^{2*}(\Omega)} + n\,\|E\|_{L^N(\Omega)}\Big). \tag{2.3}$$

Recalling Poincaré inequality, we thus have that

$$\|u\|_{L^2(\Omega)} \leq C_5\,(1+n) = R.$$

Therefore, the ball in $L^2(\Omega)$ of center the origin and radius R is invariant for the map $v \mapsto u$. Furthermore, from (2.3) and Rellich theorem it follows that the map $v \mapsto u$ is compact between $L^2(\Omega)$ and $L^2(\Omega)$, while the fact that the map $v \to \nabla\psi$ is continuous from $L^2(\Omega)$ to $(L^2(\Omega))^N$, and the map $\nabla\psi \mapsto u$ is continuous as well between $(L^2(\Omega))^N$ and $L^2(\Omega)$ yields that the map $v \mapsto u$ is continuous between $L^2(\Omega)$ and $L^2(\Omega)$. By Schauder fixed point theorem, there exists a couple (u_n, φ_n) of weak solutions of

$$\begin{cases} u_n \in W_0^{1,2}(\Omega): \ -\mathrm{div}(A(x)\nabla u_n) + E(x)\,\nabla\varphi_n = f(x)\,, \\ \varphi_n \in W_0^{1,2}(\Omega): \ -\mathrm{div}(M(x)\nabla\varphi_n) = -\mathrm{div}\big(T_n(u_n)\,E(x)\big)\,. \end{cases} \tag{2.4}$$

In the next theorem, we prove that the sequence (u_n, φ_n) converges to a solution of (1.5).

Theorem 2.1. *Assume* (1.2), (1.3), (1.4). *Then there exists a weak solution* (u, φ), *belonging to* $(W_0^{1,2}(\Omega), W_0^{1,2}(\Omega))$, *of the system* (1.5).

Proof. Let (u_n, φ_n) be the solutions of (2.4). Using $(T_n(u_n), \varphi_n)$ as test functions in the weak formulation of (2.4), we obtain

$$\begin{cases} \displaystyle\int_\Omega A(x)\nabla u_n \nabla T_n(u_n) + \int_\Omega E_n(x)\nabla\varphi_n\, T_n(u_n) = \int_\Omega f(x)\, T_n(u_n), \\ \displaystyle\int_\Omega M(x)\nabla\varphi_n\nabla\varphi_n = \int_\Omega T_n(u_n)\, E_n(x)\nabla\varphi_n\,, \end{cases}$$

which then yields, substituting the second identity in the first one,

$$\int_\Omega A(x)\nabla T_n(u_n)\,\nabla T_n(u_n) + \int_\Omega M(x)\,\nabla\varphi_n\,\nabla\varphi_n = \int_\Omega f(x)\,T_n(u_n)\,.$$

Recalling (1.2) and Sobolev embedding, we thus have

$$\alpha\int_\Omega |\nabla T_n(u_n)|^2 + \alpha\int_\Omega |\nabla\varphi_n|^2 \leq \int_\Omega |f(x)|\,|T_n(u_n)| \leq C_6\|f\|_{L^{2*}(\Omega)}\,\|T_n(u_n)\|_{W_0^{1,2}(\Omega)}\,.$$

From this inequality it follows that both $\{T_n(u_n)\}$ and $\{\varphi_n\}$ are bounded sequences in $W_0^{1,2}(\Omega)$. Using (2.2), with $\psi = \varphi_n$, we have

$$\|u_n\|_{W_0^{1,2}(\Omega)} \leq \frac{C_7}{\alpha}\left[\|f\|_{L^{2_*}(\Omega)} + \|E\|_{L^N(\Omega)}\|\nabla\varphi_n\|_{L^2(\Omega)}\right] \leq C_8\,,$$

so that also $\{u_n\}$, and not only $\{T_n(u_n)\}$, is a bounded sequence in $W_0^{1,2}(\Omega)$. Using the boundedness of both u_n and φ_n we have that, up to subsequences, there exist u and φ in $W_0^{1,2}(\Omega)$ such that

$$u_n \rightharpoonup u\,, \qquad \varphi_n \rightharpoonup \varphi\,, \qquad\qquad \text{in } W_0^{1,2}(\Omega).$$

These convergences, and Sobolev embedding theorem, imply that

$$E(x)\,\nabla\varphi_n \rightharpoonup E(x)\,\nabla\varphi\,, \quad \text{in } L^{2_*}(\Omega), \qquad T_n(u_n)\,E(x) \rightharpoonup u\,E(x)\,, \quad \text{in } L^2(\Omega).$$

Thus, it is possible to pass to the limit in the weak form of (2.4):

$$\begin{cases} \displaystyle\int_\Omega A(x)\,\nabla u_n\,\nabla v + \int_\Omega E(x)\,\nabla\varphi_n\,v = \int_\Omega f(x)\,v\,, & \forall\, v \in W_0^{1,2}(\Omega)\,, \\ \displaystyle\int_\Omega M(x)\,\nabla\varphi_n\,\nabla\psi = \int_\Omega T_n(u_n)\,E(x)\,\nabla\psi\,, & \forall\, \psi \in W_0^{1,2}(\Omega)\,, \end{cases}$$

to obtain that (u,φ) is a solution of (1.5). □

We now turn to uniqueness of solutions.

Theorem 2.2. *The solution of* (1.5) *given by Theorem 2.1 is unique.*

Proof. If (u_1,φ_1) and (u_2,φ_2) are two solutions of (1.5), then their differences $u = u_1 - u_2$ and $\varphi = \varphi_1 - \varphi_2$ are such that

$$\begin{cases} \displaystyle\int_\Omega A(x)\,\nabla u\,\nabla w + \int_\Omega E(x)\,\nabla\varphi\,w = 0\,, & \forall\, w \in W_0^{1,2}(\Omega)\,, \\ \displaystyle\int_\Omega M(x)\,\nabla\varphi\,\nabla\eta = \int_\Omega u\,E(x)\,\nabla\eta\,, & \forall\, \eta \in W_0^{1,2}(\Omega)\,. \end{cases}$$

Choosing u as test function in the first equation, and φ as test function in the second, we obtain

$$\begin{cases} \displaystyle\int_\Omega A(x)\,\nabla u\,\nabla u + \int_\Omega E(x)\,\nabla\varphi\,u = 0\,, \\ \displaystyle\int_\Omega M(x)\,\nabla\varphi\,\nabla\varphi = \int_\Omega u\,E(x)\,\nabla\varphi\,, \end{cases}$$

which then implies

$$\int_\Omega A(x)\,\nabla u\,\nabla u + \int_\Omega M(x)\,\nabla\varphi\,\nabla\varphi = 0\,.$$

Using (1.2) one then has $u = 0 = \varphi$, so that $u_1 = u_2$ and $\varphi_1 = \varphi_2$. □

As stated in the Introduction, the solution of (1.5) is a saddle point of J.

Theorem 2.3. *The solution of* (1.5) *given by Theorem 2.1 is a saddle point of the functional* J*, that is*

$$J(u,\psi) \leq J(u,\varphi) \leq J(v,\varphi)\,, \qquad \forall\, v\,,\psi \in W_0^{1,2}(\Omega)\,. \tag{2.5}$$

Proof. A straightforward consequence of the theory of quadratic functionals in Hilbert spaces is that the first equation of (1.5) implies that

$$\frac{1}{2}\int_\Omega A(x)\nabla u\nabla u - \int_\Omega f\,u + \int_\Omega u\,E(x)\,\nabla\varphi \leq \frac{1}{2}\int_\Omega A(x)\nabla v\nabla v - \int_\Omega f\,v + \int_\Omega v\,E(x)\,\nabla\varphi\,,$$

for every v in $W_0^{1,2}(\Omega)$. Hence, adding a constant term, we have

$$\begin{aligned}\frac{1}{2}\int_\Omega A(x)\nabla u\nabla u - \frac{1}{2}\int_\Omega M(x)\nabla\varphi\nabla\varphi - \int_\Omega f\,u + \int_\Omega u\,E(x)\,\nabla\varphi \\ \leq \frac{1}{2}\int_\Omega A(x)\nabla v\nabla v - \frac{1}{2}\int_\Omega M(x)\nabla\varphi\nabla\varphi - \int_\Omega f\,v + \int_\Omega v\,E(x)\,\nabla\varphi\,.\end{aligned}$$

The last inequality can be rewritten as

$$J(u,\varphi) \leq J(v,\varphi)\,, \qquad \forall\, v \in W_0^{1,2}(\Omega)\,. \tag{2.6}$$

On the other hand, the fact that φ solves the second equation of (1.5) implies that

$$-\frac{1}{2}\int_\Omega M(x)\nabla\psi\nabla\psi + \int_\Omega u\,E(x)\,\nabla\psi \leq -\frac{1}{2}\int_\Omega M(x)\nabla\varphi\nabla\varphi + \int_\Omega u\,E(x)\,\nabla\varphi\,,$$

for every ψ in $W_0^{1,2}(\Omega)$, so that (adding two constant terms)

$$\begin{aligned}\frac{1}{2}\int_\Omega A(x)\nabla u\nabla u - \int_\Omega f\,u - \frac{1}{2}\int_\Omega M(x)\nabla\psi\nabla\psi + \int_\Omega u\,E(x)\,\nabla\psi \\ \leq \frac{1}{2}\int_\Omega A(x)\nabla u\nabla u - \int_\Omega f\,u - \frac{1}{2}\int_\Omega M(x)\nabla\varphi\nabla\varphi + \int_\Omega u\,E(x)\,\nabla\varphi\,.\end{aligned}$$

This second inequality can be rewritten as

$$J(u,\psi) \leq J(u,\varphi)\,, \qquad \forall\, \psi \in W_0^{1,2}(\Omega)\,. \tag{2.7}$$

Putting together (2.6) and (2.7), we get (2.5). □

3. E belongs to $(L^2(\Omega))^N$

In this section, we assume (1.6), so that the functional J is not well defined due to the presence of the term

$$\int_\Omega v\,E\,\nabla\psi\,.$$

Nevertheless, we will be able to prove results similar to those of Theorem 2.1 and Theorem 2.3 (this latter thanks to the definition of weak minimum), but not uniqueness results as that of Theorem 2.2.

Due to the fact that J is not well defined on $(W_0^{1,2}(\Omega), W_0^{1,2}(\Omega))$, we need to approximate it by regularizing the vector field E. Indeed, let $\{E_n\}$ be a sequence of functions in $(L^N(\Omega))^N$ strongly convergent to E in $(L^2(\Omega))^N$ (for example $E_n(x) = \frac{E(x)}{1+\frac{1}{n}|E(x)|}$), and let

$$J_n(v,\psi) = \frac{1}{2}\int_\Omega A(x)\nabla v\nabla v - \frac{1}{2}\int_\Omega M(x)\nabla\psi\nabla\psi + \int_\Omega v\,E_n(x)\nabla\psi - \int_\Omega f(x)\,v\,, \tag{3.1}$$

for every v and ψ in $W_0^{1,2}(\Omega)$.

Applying the results of Section 2, there exists a unique couple (u_n, φ_n) of weak solutions of

$$\begin{cases} u_n \in W_0^{1,2}(\Omega): \ -\mathrm{div}(A(x)\nabla u_n) + E_n(x)\,\nabla\varphi_n = f(x)\,, \\ \varphi_n \in W_0^{1,2}(\Omega): \ -\mathrm{div}(M(x)\nabla\varphi_n) = -\mathrm{div}\big(u_n\,E_n(x)\big)\,, \end{cases} \tag{3.2}$$

with (u_n, φ_n) also being a saddle point of J_n.

We are now going to prove that (u_n, φ_n) converges to a (weaker) solution of (1.5).

Theorem 3.1. *Assume* (1.2), (1.3), (1.6). *Then there exists a solution* (u, φ), *belonging to* $(W_0^{1,2}(\Omega), W_0^{1,2}(\Omega))$, *of the system*

$$\begin{cases} \displaystyle\int_\Omega A(x)\,\nabla u\,\nabla w + \int_\Omega E(x)\,\nabla\varphi\,w = \int_\Omega f(x)\,w\,, \qquad \forall\, w \in W_0^{1,2}(\Omega)\cap L^\infty(\Omega) \\ \displaystyle\int_\Omega M(x)\,\nabla\varphi\,\nabla\eta = \int_\Omega u\,E(x)\,\nabla\eta\,, \qquad \forall\, \eta \in C_0^1(\Omega)\,. \end{cases} \tag{3.3}$$

Proof. Choosing u_n and φ_n as test functions in the weak formulation of (3.2) one gets, after substituting the second equation in the first,

$$\int_\Omega A(x)\,\nabla u_n\,\nabla u_n + \int_\Omega M(x)\,\nabla\varphi_n\,\nabla\varphi_n = \int_\Omega f(x)\,u_n\,, \tag{3.4}$$

which then yields (using (1.2)) that both the sequences $\{u_n\}$ and $\{\varphi_n\}$ are bounded in $W_0^{1,2}(\Omega)$. Thus, if u and φ are the weak limits (up to subsequences) of $\{u_n\}$ and $\{\varphi_n\}$ respectively, we have that (u, φ) is a solution of the system (3.3), since one has that

$$E_n\,w \to E\,w \quad \text{in } (L^2(\Omega))^N, \qquad \nabla\varphi_n \rightharpoonup \nabla\varphi \quad \text{in } (L^2(\Omega))^N, \qquad \forall\, w \in W_0^{1,2}(\Omega)\cap L^\infty(\Omega)\,,$$

and (by Rellich theorem)

$$E_n \to E \quad \text{in } (L^2(\Omega))^N, \qquad u_n\,\nabla\eta \to u\,\nabla\eta \quad \text{in } (L^2(\Omega))^N, \qquad \forall\, \eta \in C_0^1(\Omega)\,.$$

□

Note that since the term $u\,E$ belongs to $(L^{\frac{N}{N-1}}(\Omega))^N$, one can choose test functions η belonging to $W_0^{1,N}(\Omega)$ in the weak formulation of the second equation of (3.3).

Remark 3.2. We point out a double regularizing effect on the solution (u, φ) of (3.3):

A) the solution u of the first equation belongs to $W_0^{1,2}(\Omega)$, even if the "right hand side" $f(x) - E(x)\nabla\varphi$ only belongs to $L^1(\Omega)$;
B) the solution φ of the second equation belongs to $W_0^{1,2}(\Omega)$, even if the "right hand side" $-\mathrm{div}(u\,E(x))$ only belongs to $W^{-1,\frac{N}{N-1}}(\Omega)$.

We recall now, and we have used it in the previous section, that in a "regular data" framework minima of variational integrals are solutions of the corresponding Euler-Lagrange equations and *viceversa*; however, if the data are not regular enough for the functional to be defined, the link between functionals and equations is lost. For example, there exist distributional solutions of elliptic equations with (say) $L^1(\Omega)$ data, but the corresponding functional has no minimum: neither on the energy space, nor on a suitable subspace of it.

For this reason, in order to restore the link between functionals and equations, the concept of "weak minimum" has been introduced in [6]: the idea is to rewrite the definition of minimum $I(u) \le I(v)$ in another form. More precisely, to write it as $I(u) - I(u-w) \le 0$, with w smooth enough, and use the assumptions on I to remark that, even though neither $I(u)$ nor $I(u-w)$ are finite, their difference is thanks to cancelations. To be more precise, we have the following defintion.

DEFINITION 3.3. Let I be defined as

$$I(v) = \frac{1}{2}\int_\Omega B(x)\,\nabla v\,\nabla v - \int_\Omega g(x)\,v\,,$$

with B a symmetric matrix satisfying (1.2), and g a function in $L^1(\Omega)$. A function u in $W_0^{1,1}(\Omega)$ is a *weak minimum* of I if

$$\int_\Omega B(x)\,\nabla u\,\nabla w \le \frac{1}{2}\int_\Omega B(x)\,\nabla w\,\nabla w + \int_\Omega g(x)\,w\,, \qquad \forall\, w \in C_0^1(\Omega)\,. \tag{3.5}$$

Note that if one formally writes $I(u) \le I(u-w)$, one has (expanding the right hand side)

$$\begin{aligned}\frac{1}{2}\int_\Omega B(x)\,\nabla u\,\nabla u &- \int_\Omega g(x)\,u \\ &\le \frac{1}{2}\int_\Omega B(x)\,\nabla u\,\nabla u - \int_\Omega B(x)\,\nabla u\,\nabla w + \frac{1}{2}\int_\Omega B(x)\,\nabla w\,\nabla w - \int_\Omega g(x)\,(u-w)\,,\end{aligned}$$

which yields (3.5) after canceling equal terms. Also note that all terms in (3.5) are finite due to the assumptions on u and w.

Theorem 3.4. *If (u,φ) is the solution of* (3.3) *given by Theorem 3.1, then u is a weak minimum of $J(\cdot,\varphi)$, and φ is a weak minimum of $-J(u,\cdot)$.*

Proof. Let $\{E_n\}$, u_n and φ_n be as in the proof of Theorem 3.1; then, by Theorem 2.3, one has

$$J_n(u_n,\varphi) \leq J_n(z,\varphi)\,, \qquad \forall\, z \in W_0^{1,2}(\Omega)\,,$$

where J_n has been defined in (3.1). Simplifying equal terms only depending on φ_n, one has that $J(u_n,\varphi) \leq J(z,\varphi)$ is equivalent to

$$\begin{aligned}\frac{1}{2}\int_\Omega A(x)\nabla u_n\,\nabla u_n + \int_\Omega u_n\,E_n(x)\nabla\varphi_n - \int_\Omega f(x)\,u_n \\ \leq \frac{1}{2}\int_\Omega A(x)\nabla z\,\nabla z + \int_\Omega z\,E_n(x)\nabla\varphi_n - \int_\Omega f(x)\,z\,.\end{aligned}$$

Choosing $z = u_n - w$, with w in $C_0^1(\Omega)$, and simplifying equal terms, one has

$$\int_\Omega A(x)\,\nabla u_n\,\nabla w \leq \frac{1}{2}\int_\Omega A(x)\,\nabla w\,\nabla w - \int_\Omega w\,E_n(x)\,\nabla\varphi_n + \int_\Omega f\,w\,.$$

The boundedness of both u_n and φ_n in $W_0^{1,2}(\Omega)$ (proved in Theorem 3.1), allows to pass to the limit in the above inequality, to prove that

$$\int_\Omega A(x)\,\nabla u\,\nabla w \leq \frac{1}{2}\int_\Omega A(x)\,\nabla w\,\nabla w - \int_\Omega w\,E(x)\,\nabla\varphi + \int_\Omega f\,w\,, \qquad \forall\, w \in C_0^1(\Omega)\,,$$

so that u is a weak minimum of $J(\cdot,\varphi)$. An analogous calculation proves that φ is a weak minimum of $-J(u,\cdot)$. □

Differently from the case E in $(L^N(\Omega))^N$, we are not able to prove that (u,φ) is unique, neither "directly" from the formulation, nor using alternate definitions as, for example, the one of Solution Obtained as Limit of Approximations (see [4]).

Acknowledgements

This work was partially supported by GNAMPA-INdAM.

References

[1] L. Boccardo: T-minima, an approach to minimization problems in L^1. Contributi dedicati alla memoria di Ennio De Giorgi; *Ricerche di Matematica*, **49** (2000), 135–154.

[2] L. Boccardo, L. Orsina: Existence via regularity of solutions for elliptic systems and saddle points of functionals of the calculus of variations; *Adv. Nonlinear Anal.*, **6** (2017), 99–120.

[3] L. Boccardo, L. Orsina: Regularizing effect for a system of Schrödinger-Maxwell equations; *Adv. Calc. Var.*, **11** (2018), 75–87.

[4] A. Dall'Aglio: Approximated solutions of equations with L^1 data. Application to the H-convergence of quasi-linear parabolic equations; *Ann. Mat. Pura Appl.*, **170** (1996), 207–240.

[5] L. Orsina: Weak minima for some functionals and elliptic equations with measure data; *C. R. Acad. Sci. Paris*, **322** (1996), 1151–1156.

[6] T. Iwaniec, C. Sbordone: Weak minima of variational Integrals; *J. Reine Angew. Math.*, **454** (1994), 143–161.

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