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AN ELLIPTIC SYSTEM RELATED TO THE STATIONARY THERMISTOR PROBLEM*

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Abstract. In this paper we prove existence of solutions for an elliptic system related to the stationary thermistor problem with bounded sources, and thermal and electrical condictivities growing as powers of the temperature.

Key words. thermistor, elliptic systems, regularizing effect, nonlinear Dirichlet problems

AMS subject classifications. 35J15, 35J47, 35J60

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1. Introduction and statement of the results.

A *thermistor* is a device in which the electrical resistance is dependent on the temperature. If Ω is a bounded, open subset of $\mathbb{R}^N$, $N \geq 2$, representing the body of the device, u is the temperature of the body, and ψ is its electrical potential, following [2] we have that if $A(u)$ is the thermal conductivity, and $B(u)$ is the electrical conductivity, then u and ψ are solutions of the elliptic-parabolic system

$$-\operatorname{div}(B(u)\,\nabla\psi) = 0\,, \qquad d\,c\,u_t - \operatorname{div}(A(u)\,\nabla u) = B(u)\,|\nabla\psi|^2\,.$$

If one considers the stationary problem, then $u_t = 0$, so that u and ψ are solutions of the system of elliptic equations

$$\begin{cases} -\operatorname{div}(A(u)\,\nabla u) = B(u)\,|\nabla\psi|^2 & \text{in } \Omega, \\ -\operatorname{div}(B(u)\,\nabla\psi) = 0 & \text{in } \Omega. \end{cases} \tag{1.1}$$

Adding suitable boundary conditions, for example, assigning both u and ψ on some subsets of $\partial\Omega$, it is possible to prove existence and uniqueness of solutions, under some assumptions on $A(u)$ and $B(u)$; see, for example, the papers [9, 10] and the references therein.

In this paper, we will study an elliptic system related to the stationary thermistor problem (1.1), under a more general setting with respect to previous works, since we will allow the functions $A(x,t)$ and $B(x,t)$, which will be matrix valued, to depend not only on the unknown u but also on x in Ω, therefore allowing for anisotropic media as well as different properties of the materials. Also, we will consider a positive and bounded source term g for the equation involving the unknown ψ and will suppose that both u and ψ are zero on the boundary of Ω (that is, we will consider homogeneous Dirichlet problems). Thus, we will study existence of solutions for the system

$$\begin{cases} -\operatorname{div}(A(x,u)\,\nabla u) + u = B(x,u)\,\nabla\psi\cdot\nabla\psi & \text{in } \Omega, \\ -\operatorname{div}(B(x,u)\,\nabla\psi) + \psi = g & \text{in } \Omega, \\ u = 0 = \psi & \text{on } \partial\Omega. \end{cases} \tag{1.2}$$

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Note, with respect to (1.1), the presence of the lower order terms "$+u$" and "$+\psi$" in the two equations, which could be used to approximate (using a semigroup theory approach) the corresponding parabolic equations related to the nonstationary thermistor problem. Furthermore, since we will consider possibly degenerate elliptic operators (as u tends to infinity), these terms will allow us to recover some coerciveness on the two equations, yielding a priori estimates; see section 5 below.

We now define the various terms of system (1.2). Let p and q be two real numbers such that

$$p > q - 1\,, \tag{1.3}$$

and define

$$\rho(t) = (1+|t|)^p\,, \qquad \sigma(t) = (1+|t|)^q\,.$$

We will suppose that $A : \Omega \times \mathbb{R} \to \mathbb{R}^{N\times N}$ and $B : \Omega \times \mathbb{R} \to \mathbb{R}^{N\times N}$ are two Carathéodory matrix-valued functions (that is, measurable for x in Ω and continuous for t in $\mathbb{R}$) such that

$$\begin{aligned} &A(x,t)\,\xi\cdot\xi \geq \alpha\,\rho(t)\,|\xi|^2\,, \qquad |A(x,t)| \leq \beta\,\rho(t)\,,\\ &B(x,t)\,\xi\cdot\xi \geq \alpha\,\sigma(t)\,|\xi|^2\,, \qquad |B(x,t)| \leq \beta\,\sigma(t) \end{aligned} \tag{1.4}$$

for almost every x in Ω, for every t in $\mathbb{R}$, and for every ξ in $\mathbb{R}^N$, where $0 < \alpha \leq \beta$ are two real numbers.

Note that we do not assume sign conditions on p or q: they may be positive or negative (in this latter case the differential operators degenerate as u becomes unbounded). We will also define

$$H(t) = \int_0^t \frac{\rho(s)}{\sigma(s)}\,ds = \int_0^t (1+|s|)^{p-q}\,ds = \frac{(1+|t|)^{p-q+1}-1}{p-q+1}\,\mathrm{sgn}(t)\,. \tag{1.5}$$

Note that from assumption (1.3) it follows that $H(t)$ behaves as a positive power of t as t tends to infinity. On the function g we will assume that it is positive, and that it belongs to $L^\infty(\Omega)$.

Our main result is the following.

THEOREM 1.1. *Let $A(x,t)$ and $B(x,t)$ be such that* (1.4) *holds, with p and q such that* (1.3) *holds. Let $g \geq 0$ be a function in $L^\infty(\Omega)$. Then there exist solutions u and ψ of system* (1.2)*, such that*

- *$u \geq 0$ belongs to $W_0^{1,2}(\Omega)$, and, if $H(t)$ is as in* (1.5)*, then there exists $\gamma = \gamma(g) > 0$ such that*
$$\mathrm{e}^{\gamma\,H(u)} \text{ belongs to } L^1(\Omega);$$
- *$\psi \geq 0$ belongs to $W_0^{1,2}(\Omega) \cap L^\infty(\Omega)$;*
- *the vector fields $A(x,u)\,\nabla u$ and $B(x,u)\,\nabla\psi$ belong to $(L^2(\Omega))^N$;*
- *the function $B(x,u)\,\nabla\psi\cdot\nabla\psi$ belongs to $L^1(\Omega)$.*

Furthermore, u and ψ are such that

$$\int_\Omega A(x,u)\,\nabla u\cdot\nabla v + \int_\Omega u\,v = \int_\Omega B(x,u)\,\nabla\psi\cdot\nabla\psi\,v$$

for every v in $W_0^{1,2}(\Omega) \cap L^\infty(\Omega)$ and

$$\int_\Omega B(x,u)\,\nabla\psi\cdot\nabla\varphi + \int_\Omega \psi\,\varphi = \int_\Omega g\,\varphi$$

for every φ *in* $W_0^{1,2}(\Omega)$.

Remark 1.2. We explicitly remark that the solution u of the first equation of (1.2) belongs to $W_0^{1,2}(\Omega)$ even if the right-hand side $B(x,u)\,\nabla\psi\cdot\nabla\psi$ only belongs to $L^1(\Omega)$; thus, the coupling of the two equations in the system yields a *regularizing effect* on the solutions.

Remark 1.3. Assumption (1.4) has been done in order to fix the ideas; for the existence proof to work, it is enough that $H(t)$, defined in (1.5), behaves as a positive power of t as t tends to infinity, and that for every $\gamma > 0$ the function $\mathrm{e}^{\gamma\,H(t)}$ "dominates" both ρ and σ as t tends to infinity (see Lemma 2.2, below). Note that if both $A(x,t)$ and $B(x,t)$ are bounded functions, or if $A(x,t) = B(x,t)$, then assumption (1.3) holds since $p = q > q - 1$, and $H(t) \approx t$ as t tends to infinity.

Remark 1.4. We point out that the approach and the results of the present work are related to those of [14] and [4]. In the paper [14] existence results for system (1.2) are proved in dimension $N \le 4$, under the assumption $p > q - 1$, and $q \ge 0$, with both $A(x,t)$ and $B(x,t)$ independent on x, working in the context of weighted Sobolev spaces. In the paper [4], existence results are proved in the case $p \ge q \ge 0$, but under weaker assumptions on the datum g.

Remark 1.5. There are several results in the literature concerning the thermistor problem, which was introduced in 1899 in the paper [17] (see also [13]). Starting with the paper by Cimatti and Prodi [12], the stationary and the evolution thermistor problem was studied by Cimatti in [9, 10, 11]. The study was then continued by Chipot and Cimatti [8], Antontsev and Chipot [2], and Howison, Rodrigues, and Shillor [16], among others; see also the references in these papers. More recently, the problem has been studied in the evolutionary case in [15], and, in the case of p-Laplacian or $p(x)$-Laplacian differential operators, in [18] and [7].

The plan of the paper is as follows: In the next section we will use Schauder's fixed point theorem to prove an existence result for a system which "approximates" system (1.2), as well as a technical result concerning the function $H(t)$ defined above. In section 3 we will prove Theorem 1.1, using the results of section 2, while in section 4 we will suppose that both $A(x,t)$ and $B(x,t)$ do not depend on x, proving the existence of a bounded solution u for the first equation under the weaker assumption $p \ge q-1$. Finally, section 5 contains some remarks on the case $p < q - 1$, which seems not easy to deal with and—at least in one case—cannot be solved without the lower order term $+u$ in the first equation of system (1.2).

2. Some preliminary results. In this section, we will prove existence of solutions for a system which will be used to approximate (1.2). More precisely, we will consider the following system:

$$
\begin{cases}
-\operatorname{div}(R(x,u)\,\nabla u) + u = S(x,u)\,T(x,\nabla\psi)\cdot T(x,\nabla\psi) & \text{in } \Omega,\\
-\operatorname{div}(S(x,u)\,\nabla\psi) + \psi = g & \text{in } \Omega,\\
u = 0 = \psi & \text{on } \partial\Omega.
\end{cases}
\tag{2.1}
$$

Here we will assume the following: $R : \Omega \times \mathbb{R} \to \mathbb{R}^{N\times N}$ and $S : \Omega \times \mathbb{R} \to \mathbb{R}^{N\times N}$ will be two Carathéodory matrix-valued functions (that is, measurable for x in Ω and continuous for t in $\mathbb{R}$) such that

$$
\begin{aligned}
R(x,t)\,\xi\cdot\xi &\ge \mathcal{A}_R\,|\xi|^2\,, \qquad |R(x,t)| \le \mathcal{B}_R\,,\\
S(x,t)\,\xi\cdot\xi &\ge \mathcal{A}_R\,|\xi|^2\,, \qquad |S(x,t)| \le \mathcal{B}_S
\end{aligned}
\tag{2.2}
$$

for some $0 < \mathcal{A}_R \le \mathcal{B}_R$ and $0 < \mathcal{A}_S \le \mathcal{B}_S$, for almost every x in Ω, for every t in $\mathbb{R}$, and for every ξ in $\mathbb{R}^N$; $T : \Omega \times \mathbb{R}^N \to \mathbb{R}^N$ will be a Carathéodory vector-valued function (that is, measurable for x in Ω and continuous for ξ in $\mathbb{R}^N$) such that

$$|T(x,\xi)| \le \mathcal{T} \tag{2.3}$$

for some $\mathcal{T} > 0$, for almost every x in Ω, and for every ξ in $\mathbb{R}^N$. We will furthermore assume that $g \ge 0$ is an $L^\infty(\Omega)$ function. Our result is the following.

THEOREM 2.1. *Let $g \ge 0$ be a function in $L^\infty(\Omega)$, and let R, S, and T satisfy* (2.2) *and* (2.3). *Then there exist weak solutions u and ψ to system* (2.1), *with $u \ge 0$ in $W_0^{1,2}(\Omega) \cap L^\infty(\Omega)$, and $\psi \ge 0$ in $W_0^{1,2}(\Omega) \cap L^\infty(\Omega)$.*

Proof. We are going to prove existence of solutions using Schauder's theorem. Toward this goal, let v be a function in $L^2(\Omega)$, and let φ be the unique weak solution in $W_0^{1,2}(\Omega)$ of

$$-\operatorname{div}(S(x,v)\,\nabla\varphi) + \varphi = g\,.$$

Note that $\varphi \ge 0$ since $g \ge 0$. Following the ideas of [6], we choose $(\varphi - k)^+$ as test function, with $k = \|g\|_{L^\infty(\Omega)}$, to have (after a few straightforward passages)

$$0 \le \int_\Omega (\varphi - k)\,(\varphi - k)^+ \le 0\,.$$

From this inequality we obtain that $(\varphi - k)^+ = 0$, so that, recalling the definition of k,

$$\|\varphi\|_{L^\infty(\Omega)} \le \|g\|_{L^\infty(\Omega)}\,. \tag{2.4}$$

Given v and φ as above, let w be the unique weak solution in $W_0^{1,2}(\Omega)$ of the equation

$$-\operatorname{div}(R(x,v)\,\nabla w) + w = S(x,v)\,T(x,\nabla\varphi)\cdot T(x,\nabla\varphi)\,.$$

Note that $w \ge 0$ since $S(x,v)\,T(x,\nabla\varphi)\cdot T(x,\nabla\varphi) \ge 0$ by (2.2); furthermore, choosing as test function $(w-k)^+$, with $k = \mathcal{B}_S\,\mathcal{T}^2$, and reasoning as above, we have that

$$\|w\|_{L^\infty(\Omega)} \le \mathcal{B}_S\,\mathcal{T}^2 = M\,. \tag{2.5}$$

Thanks to the above estimate, one also has that

$$\|w\|_{L^2(\Omega)} \le M\,\sqrt{\operatorname{meas}(\Omega)} = \mathcal{M}\,.$$

Thus, if $\mathcal{S}$ is the operator from $L^2(\Omega)$ to $L^2(\Omega)$ defined by $\mathcal{S}(v) = u$, one has that the ball $B_{\mathcal{M}}$ of $L^2(\Omega)$ is invariant for $\mathcal{S}$. We are going to prove that $\mathcal{S}$ satisfies the assumptions of Schauder's theorem: that it is continuous from $L^2(\Omega)$ into itself, and that $\mathcal{S}(L^2(\Omega))$ is precompact in $L^2(\Omega)$.

Let $\{v_n\}$ be a sequence of functions in $L^2(\Omega)$ which strongly converge to v in $L^2(\Omega)$, and let $\{\varphi_n\}$ be the sequence of solutions of

$$-\operatorname{div}(S(x,v_n)\,\nabla\varphi_n) + \varphi_n = g\,.$$

Choosing φ_n as test function, and using (2.2), one has that the sequence $\{\varphi_n\}$ is bounded in $W_0^{1,2}(\Omega)$; thus, up to subsequences, it converges to some function φ, weakly in $W_0^{1,2}(\Omega)$, and strongly in $L^2(\Omega)$. Using the continuity and the boundedness of $S(x,\cdot)$, one has that φ is the weak solution of the equation

$$-\operatorname{div}(S(x,v)\,\nabla\varphi)+\varphi=g\,,$$

so that, by uniqueness, the whole sequence $\{\varphi_n\}$ converges to φ. Choosing $\varphi_n-\varphi$ as test function, one easily obtains that the sequence $\{\varphi_n\}$ strongly converges to φ in $W_0^{1,2}(\Omega)$.

Now let $\{w_n\}$ be the sequence of solutions of

$$-\operatorname{div}(R(x,v_n))\,\nabla w_n)+w_n=S(x,v_n)\,T(x,\nabla\varphi_n)\cdot T(x,\nabla\varphi_n)\,,$$

and choose w_n as test function. Using (2.2) one obtains that the sequence $\{w_n\}$ is bounded in $W_0^{1,2}(\Omega)$. Thus, up to subsequences, it converges to some function w weakly in $W_0^{1,2}(\Omega)$ and strongly in $L^2(\Omega)$; using the continuity and the boundedness of $R(x,t)$, $S(x,t)$, $T(x,\xi)$ and the strong convergence of $\{\nabla\varphi_n\}$ in $(L^2(\Omega))^N$, it is easy to see that w is the weak solution of

$$-\operatorname{div}(R(x,v)\,\nabla w)+w=S(x,v)\,T(x,\nabla\varphi)\cdot T(x,\nabla\varphi)\,.$$

Thanks to uniqueness, the whole sequence $\{w_n=\mathcal{S}(v_n)\}$ strongly converges to $w=\mathcal{S}(v)$ in $L^2(\Omega)$, and this implies that $\mathcal{S}$ is continuous. As far as compactness is concerned, if $\{v_n\}$ is bounded in $L^2(\Omega)$, the same calculations performed above imply that the sequence $\{w_n=\mathcal{S}(v_n)\}$ is bounded in $W_0^{1,2}(\Omega)$, so that it strongly converges in $L^2(\Omega)$ up to subsequences by Rellich's theorem.

Since $\mathcal{S}$ satisfies the assumptions of Schauder's theorem, there exists u in $L^2(\Omega)$ such that $u=\mathcal{S}(u)$ (so that u actually belong to $W_0^{1,2}(\Omega)$). Thus, if ψ is the solution of

$$-\operatorname{div}(S(x,u)\,\nabla\psi)+\psi=g\,,$$

then u is the solution of

$$-\operatorname{div}(R(x,u)\,\nabla u)+u=S(x,u)\,T(x,\nabla\psi)\cdot T(x,\nabla\psi)\,.$$

Hence, we have proved the existence of weak solutions $u\geq 0$ and $\psi\geq 0$ to the equations of (2.1), with both u and ψ belonging to $W_0^{1,2}(\Omega)\cap L^\infty(\Omega)$. ∎

We conclude this section with a technical result concerning the function $H(t)$ defined in (1.5).

Lemma 2.2. *Suppose that p and q are such that* (1.3) *holds, and let H be as in* (1.5)*. Then for every $\gamma>0$ and for every r in $\mathbb{R}$ there exists a constant $C=C_{\gamma,r,p,q}>0$ such that*

$$(1+t)^r\leq C\,\mathrm{e}^{\gamma\,H(t)}\qquad\forall t\geq 0\,. \tag{2.6}$$

Furthermore, for every $\gamma>0$ and every r in $\mathbb{R}$ there exists $C=C(\gamma,r,p,q)>0$ such that

$$(1+t)^r\,\mathrm{e}^{\gamma\,H(t)}\geq C\qquad\forall t\geq 0\,. \tag{2.7}$$

Proof. If $r\leq 0$, it is enough to choose $C=1$ to have that (2.6) holds true, so that it remains to prove that (2.6) holds if $r>0$; since $H(t)\approx t^{p-q+1}$ as t tends to infinity, one clearly has, for every $\gamma>0$,

$$\lim_{t\to+\infty}(1+t)^r\,\mathrm{e}^{-\gamma\,H(t)}=0\,.$$

This means that the function $y(t) = (1+t)^r \, \mathrm{e}^{-\gamma\, H(t)}$, which is positive, has a maximum $C = C_{\gamma,r,p,q}$ on $[0,+\infty)$; that is, one has $y(t) \le C$ for every $t \ge 0$, and this inequality is exactly (2.6). As far as (2.7) is concerned, it is clearly true if $r \ge 0$ (it is enough to choose $C = 1$); if $r < 0$, the existence of a constant $C > 0$ as in the statement follows from the fact that, since $H(t) \approx t^{p-q+1}$ as t tends to infinity, one has

$$\lim_{t\to+\infty} (1+t)^r \, \mathrm{e}^{\gamma\, H(t)} = +\infty\,,$$

so that $y(t) = (1+t)^r \, \mathrm{e}^{\gamma\, H(t)}$ has a (strictly positive) minimum $C = C_{\gamma,r,p,q}$ on $[0,+\infty)$. ∎

3. Proof of the main result. As stated in the introduction, we are going to prove Theorem 1.1. During the proof, we will make use of the following result, whose proof can be found in [5].

LEMMA 3.1. *Let $\{\Gamma_n(x)\}$ be a sequence of uniformly elliptic, bounded matrices, almost everywhere convergent to some uniformly elliptic matrix $\Gamma(x)$, and let $\{\Theta_n\}$ be a sequence of functions which is weakly convergent in $(L^2(\Omega))^N$ to some function Θ. If the sequence $\{\Gamma_n(x)\,\Theta_n\cdot\Theta_n\}$ is bounded in $L^1(\Omega)$, then $\Gamma(x)\,\Theta\cdot\Theta$ belongs to $L^1(\Omega)$ and*

$$\int_\Omega \Gamma(x)\,\Theta\cdot\Theta \le \liminf_{n\to+\infty} \int_\Omega \Gamma_n(x)\,\Theta_n\cdot\Theta_n\,. \tag{3.1}$$

During the proof we will also make use of the following functions of one real variable, defined for $k > 0$ and $t \ge 0$:

$$T_k(t) = \min(t,k)\,, \qquad G_k(t) = t - T_k(t) = (t-k)^+\,.$$

In what follows, C will denote a quantity which may depend on α, β, Q, p, q, Ω, and N, while $C(g)$ will denote a quantity that depends on (some or all of) the above parameters and on the norm of g in $L^\infty(\Omega)$; in this case, the dependence of $C(g)$ on the norm will be *bounded*. Note that C and $C(g)$ will *never* depend on the "approximating parameter" n in $\mathbb{N}$.

Proof. The proof will be divided into several steps.

$\boxed{\text{Step 1: Approximation.}}$

For n in $\mathbb{N}$, there exist weak solutions $u_n \ge 0$ and $\psi_n \ge 0$, with both u_n and ψ_n belonging to $W_0^{1,2}(\Omega)\cap L^\infty(\Omega)$, of the system

$$\begin{cases} -\operatorname{div}(A(x,T_n(u_n))\,\nabla u_n) + u_n = \dfrac{B(x,T_n(u_n))\,\nabla\psi_n\cdot\nabla\psi_n}{\left(1+\frac{1}{n}\,|\nabla\psi_n|\right)^2} & \text{in } \Omega,\\ -\operatorname{div}(B(x,T_n(u_n))\,\nabla\psi_n) + \psi_n = g & \text{in } \Omega,\\ u_n = 0 = \psi_n & \text{on } \partial\Omega. \end{cases} \tag{3.2}$$

The existence of such solutions follows from Theorem 2.1, choosing

$$R(x,t) = A(x,T_n(t))\,, \qquad S(x,t) = B(x,T_n(t))\,, \qquad T(x,\xi) = \frac{\xi}{1+\frac{1}{n}\,|\xi|}\,,$$

which satisfy (2.2) with

$$\mathcal{A}_R = \alpha\,\min(1,(1+n)^p)\,, \qquad \mathcal{B}_R = \beta\,\max(1,(1+n)^p)\,,$$

$$\mathcal{A}_S = \alpha\, \min(1, (1+n)^q)\,, \qquad \mathcal{B}_S = \beta\, \max(1, (1+n)^q)\,,$$

and

$$\mathcal{T} = n\,.$$

$\boxed{\text{Step 2: The sequence } \{\psi_n\} \text{ is bounded in } L^\infty(\Omega).}$

This is a straightforward consequence of (2.4), which implies that

$$\|\psi_n\|_{L^\infty(\Omega)} \leq \|g\|_{L^\infty(\Omega)}\,. \tag{3.3}$$

$\boxed{\text{Step 3. A priori estimates on the sequence } \{T_n(u_n)\}.}$

Let $H(t)$ be as in (1.5), let $\gamma > 0$, and choose $v = \mathrm{e}^{\gamma\, H(T_n(u_n))} - 1$ as test function in the first equation of system (3.2). We obtain, using (1.4),

$$\begin{aligned}\gamma \int_\Omega A(x, T_n(u_n))\, \nabla u_n \cdot \nabla T_n(u_n)\, \mathrm{e}^{\gamma\, H(T_n(u_n))}\, H'(T_n(u_n)) + \int_\Omega u_n\, (\mathrm{e}^{\gamma\, H(T_n(u_n))} - 1)& \\ = \int_\Omega \frac{B(x, T_n(u_n))\, \nabla\psi_n \cdot \nabla\psi_n}{\big(1 + \frac{1}{n}\, |\nabla\psi_n|\big)^2}\, (\mathrm{e}^{\gamma\, H(T_n(u_n))} - 1)& \\ \leq \beta \int_\Omega \sigma(T_n(u_n))\, |\nabla\psi_n|^2\, \mathrm{e}^{\gamma\, H(T_n(u_n))}\,.&\end{aligned}$$

Recalling (1.4) and the definition of $H(t)$, from the above inequality we deduce that

$$\begin{aligned}\alpha\,\gamma \int_\Omega \frac{\rho^2(T_n(u_n))}{\sigma(T_n(u_n))}\, |\nabla T_n(u_n)|^2\, \mathrm{e}^{\gamma\, H(T_n(u_n))} + \int_\Omega u_n\, (\mathrm{e}^{\gamma\, H(T_n(u_n))} - 1)& \\ \leq \beta \int_\Omega \sigma(T_n(u_n))\, |\nabla\psi_n|^2\, \mathrm{e}^{\gamma\, H(T_n(u_n))}\,.&\end{aligned} \tag{3.4}$$

Now choose $\varphi = \psi_n\, \mathrm{e}^{\gamma\, H(T_n(u_n))}$ as test function in the second equation of system (3.2). We obtain

$$\begin{aligned}\int_\Omega B(x, T_n(u_n))\, \nabla\psi_n \cdot \nabla\psi_n\, \mathrm{e}^{\gamma\, H(T_n(u_n))} + \int_\Omega \psi_n^2\, \mathrm{e}^{\gamma\, H(T_n(u_n))}& \\ = \int_\Omega g\, \psi_n\, \mathrm{e}^{\gamma\, H(T_n(u_n))}& \\ - \gamma \int_\Omega B(x, T_n(u_n))\, \nabla\psi_n \cdot \nabla T_n(u_n)\, \mathrm{e}^{\gamma\, H(T_n(u_n))}\, H'(T_n(u_n))\, \psi_n\,.&\end{aligned} \tag{3.5}$$

Recalling (1.4), we have

$$\alpha \int_\Omega \sigma(T_n(u_n))\, |\nabla\psi_n|^2\, \mathrm{e}^{\gamma\, H(T_n(u_n))} \leq \int_\Omega B(x, T_n(u_n))\, \nabla\psi_n \cdot \nabla\psi_n\, \mathrm{e}^{\gamma\, H(T_n(u_n))}\,. \tag{3.6}$$

On the other hand, again by (1.4) and by Young's inequality, we have

$$\begin{aligned}
&\left|\gamma \int_\Omega B(x,T_n(u_n))\,\nabla\psi_n\cdot\nabla T_n(u_n)\,\mathrm{e}^{\gamma\,H(T_n(u_n))}\,H'(T_n(u_n))\,\psi_n\right| \\
&\qquad\le \gamma\,\beta \int_\Omega \sigma(T_n(u_n))\,|\nabla\psi_n|\,|\nabla T_n(u_n)|\,\mathrm{e}^{\gamma\,H(T_n(u_n))}\,H'(T_n(u_n))\,\psi_n \\
&\qquad\le \frac{\alpha}{2}\int_\Omega \sigma(T_n(u_n))\,|\nabla\psi_n|^2\,\mathrm{e}^{\gamma\,H(T_n(u_n))} \\
&\qquad\quad + C\,\gamma^2 \int_\Omega \sigma(T_n(u_n))\,|\nabla T_n(u_n)|^2\,\mathrm{e}^{\gamma\,H(T_n(u_n))}\,[H'(T_n(u_n))]^2\,\psi_n^2\,.
\end{aligned}$$

Recalling the definition of H and estimate (3.3), we thus have

$$\begin{aligned}
&\gamma \int_\Omega B(x,T_n(u_n))\,\nabla\psi_n\cdot\nabla T_n(u_n)\,\mathrm{e}^{\gamma\,H(T_n(u_n))}\,H'(T_n(u_n))\,\psi_n \\
&\qquad\le \frac{\alpha}{2}\int_\Omega \sigma(T_n(u_n))\,|\nabla\psi_n|^2\,\mathrm{e}^{\gamma\,H(T_n(u_n))} \\
&\qquad\quad + C\,\gamma^2\,\|g\|^2_{L^\infty(\Omega)} \int_\Omega \frac{\rho^2(T_n(u_n))}{\sigma(T_n(u_n))}\,|\nabla T_n(u_n)|^2\,\mathrm{e}^{\gamma\,H(T_n(u_n))}\,.
\end{aligned}\tag{3.7}$$

Using (3.6) and (3.7) in (3.5) we obtain, dropping a positive term,

$$\begin{aligned}
\alpha \int_\Omega \sigma(T_n(u_n))\,|\nabla\psi_n|^2\,\mathrm{e}^{\gamma\,H(T_n(u_n))} &\le \int_\Omega g\,\psi_n\,\mathrm{e}^{\gamma\,H(T_n(u_n))} \\
&\quad + \frac{\alpha}{2}\int_\Omega \sigma(T_n(u_n))\,|\nabla\psi_n|^2\,\mathrm{e}^{\gamma\,H(T_n(u_n))} \\
&\quad + C\,\gamma^2\,\|g\|^2_{L^\infty(\Omega)} \int_\Omega \frac{\rho^2(T_n(u_n))}{\sigma(T_n(u_n))}\,|\nabla T_n(u_n)|^2\,\mathrm{e}^{\gamma\,H(T_n(u_n))}\,.
\end{aligned}$$

Simplifying equal terms, we have thus proved that

$$\begin{aligned}
\int_\Omega \sigma(T_n(u_n))\,|\nabla\psi_n|^2\,\mathrm{e}^{\gamma\,H(T_n(u_n))} &\le C \int_\Omega g\,\psi_n\,\mathrm{e}^{\gamma\,H(T_n(u_n))} \\
&\quad + C(g)\,\gamma^2 \int_\Omega \frac{\rho^2(T_n(u_n))}{\sigma(T_n(u_n))}\,|\nabla T_n(u_n)|^2\,\mathrm{e}^{\gamma\,H(T_n(u_n))}\,.
\end{aligned}\tag{3.8}$$

Using this inequality with (3.4), we have that

$$\begin{aligned}
&\alpha\,\gamma \int_\Omega \frac{\rho^2(T_n(u_n))}{\sigma(T_n(u_n))}\,|\nabla T_n(u_n)|^2\,\mathrm{e}^{\gamma\,H(T_n(u_n))} + \int_\Omega u_n\,(\mathrm{e}^{\gamma\,H(T_n(u_n))}-1) \\
&\qquad\le C \int_\Omega g\,\psi_n\,\mathrm{e}^{\gamma\,H(T_n(u_n))} + C(g)\,\gamma^2 \int_\Omega \frac{\rho^2(T_n(u_n))}{\sigma(T_n(u_n))}\,|\nabla T_n(u_n)|^2\,\mathrm{e}^{\gamma\,H(T_n(u_n))}\,.
\end{aligned}$$

We now choose γ small enough to have

$$C(g)\,\gamma = \frac{\alpha}{2}\,,$$

which implies that $\gamma \approx \|g\|^{-2}_{L^\infty(\Omega)}$; from the previous inequality it thus follows that

$$\begin{aligned}
&\frac{\alpha\,\gamma}{2} \int_\Omega \frac{\rho^2(T_n(u_n))}{\sigma(T_n(u_n))}\,|\nabla T_n(u_n)|^2\,\mathrm{e}^{\gamma\,H(T_n(u_n))} + \int_\Omega u_n\,(\mathrm{e}^{\gamma\,H(T_n(u_n))}-1) \\
&\qquad\le \int_\Omega g\,\psi_n\,\mathrm{e}^{\gamma\,H(T_n(u_n))}\,.
\end{aligned}\tag{3.9}$$

Since the first term of (3.9) is positive, we have

$$\int_\Omega u_n\,(\mathrm{e}^{\gamma\,H(T_n(u_n))}-1)\le C\int_\Omega g\,\psi_n\,\mathrm{e}^{\gamma\,H(T_n(u_n))}\le C(g)\int_\Omega \mathrm{e}^{\gamma\,H(T_n(u_n))}\,,$$

where in the last passage we have used (3.3). This inequality can be rewritten (adding and subtracting the term $C(g)$ meas(Ω)) as

$$\int_\Omega (u_n-C(g))\,(\mathrm{e}^{\gamma\,H(T_n(u_n))}-1)\le C(g)\ \mathrm{meas}(\Omega)=C(g)\,.$$

Splitting the integral on the set where $u_n\ge 2C(g)$ and $u_n<2C(g)$, we have

$$\begin{aligned}C(g)\int_{\{u_n\ge 2C(g)\}}(\mathrm{e}^{\gamma\,H(T_n(u_n))}-1)&\le\int_{\{u_n\ge 2C(g)\}}(u_n-C(g))\,(\mathrm{e}^{\gamma\,H(T_n(u_n))}-1)\\&\le\int_{\{u_n<2C(g)\}}|u_n-C(g)|\,|\mathrm{e}^{\gamma\,H(T_n(u_n))}-1|+C(g)\\&\le C(g)\,\mathrm{e}^{\gamma\,H(C(g))}\ \mathrm{meas}(\Omega)+C(g)=C(g)\,,\end{aligned}$$

which implies, recalling that γ depends on g, that

$$\int_\Omega \mathrm{e}^{\gamma\,H(T_n(u_n))}\le C(g)\,. \tag{3.10}$$

This inequality, together with (3.9), yields that

$$\int_\Omega \frac{\rho^2(T_n(u_n))}{\sigma(T_n(u_n))}\,|\nabla T_n(u_n)|^2\,\mathrm{e}^{\gamma\,H(T_n(u_n))}\le C(g)\,. \tag{3.11}$$

Now we apply (2.7) of Lemma 2.2 with $r=2p-q$, and with γ as above to obtain that there exists $C>0$ such that

$$\frac{\rho^2(t)}{\sigma(t)}\,\mathrm{e}^{\gamma\,H(t)}=(1+t)^{2p-q}\,\mathrm{e}^{\gamma\,H(t)}\ge C\qquad\forall t\ge 0\,.$$

Thus, from (3.11) (and the fact that $u_n\ge 0$) it follows that

$$C\int_\Omega|\nabla T_n(u_n)|^2\le\int_\Omega g\,\psi_n\,\mathrm{e}^{\gamma\,H(T_n(u_n))}\le\|g\|^2_{L^\infty(\Omega)}\int_\Omega \mathrm{e}^{\gamma\,H(T_n(u_n))}\le C(g)\,,$$

so that we have proved that

$$\text{the sequence } \{T_n(u_n)\} \text{ is bounded in } W^{1,2}_0(\Omega). \tag{3.12}$$

Furthermore, applying (2.6) of Lemma 2.2 with $r=-q$ and γ as above, we have that

$$(1+t)^{-q}\,\mathrm{e}^{\gamma\,H(t)}\ge C\qquad\forall t\ge 0\,,$$

so that (3.11) and the fact that $u_n\ge 0$ imply that

$$C\int_\Omega\rho^2(T_n(u_n))\,|\nabla T_n(u_n)|^2\le C(g)\,.$$

From this inequality and the assumptions on $A(x,t)$ it follows that

$$\text{the sequence } \{A(x,T_n(u_n))\,\nabla T_n(u_n)\} \text{ is bounded in } (L^2(\Omega))^N. \tag{3.13}$$

Step 4. A priori estimates on the sequence $\{\psi_n\}$.

Starting from (3.8) and using (3.10) and (3.11), as well as the boundedness of the sequence $\{g\,\psi_n\}$ in $L^\infty(\Omega)$, one has that

$$\int_\Omega \sigma(T_n(u_n))\,|\nabla\psi_n|^2\,\mathrm{e}^{\gamma\,H(T_n(u_n))} \le C(g)\,. \tag{3.14}$$

Using (2.7) of Lemma 2.2 with $r = q$ and γ as above, one has that there exists $C > 0$ such that

$$\sigma(t)\,\mathrm{e}^{\gamma\,H(t)} = (1+t)^q\,\mathrm{e}^{\gamma\,H(t)} \ge C \qquad \forall t \ge 0\,.$$

Using this inequality in (3.14) (together with the fact that $u_n \ge 0$), one has that

$$\int_\Omega |\nabla\psi_n|^2 \le C(g)\,,$$

which implies that

$$\text{the sequence } \{\psi_n\} \text{ is bounded in } W_0^{1,2}(\Omega). \tag{3.15}$$

Furthermore, using again (2.7) of Lemma 2.2 with $r = -q$ and γ as above, one has that there exists $C > 0$ such that

$$\frac{\mathrm{e}^{\gamma\,H(t)}}{\sigma(t)} = (1+t)^{-q}\,\mathrm{e}^{\gamma\,H(t)} \ge C \qquad \forall t \ge 0\,,$$

which implies that

$$\mathrm{e}^{\gamma\,H(t)} \ge C\,\sigma(t) \qquad \forall t \ge 0\,.$$

Using this inequality in (3.14) (and using the fact that $u_n \ge 0$), we thus have that

$$\int_\Omega \sigma^2(T_n(u_n))\,|\nabla\psi_n|^2 \le C(g)\,,$$

which implies, recalling the assumptions on $B(x,t)$, that

$$\text{the sequence } \{B(x,T_n(u_n))\,\nabla\psi_n\} \text{ is bounded in } (L^2(\Omega))^N. \tag{3.16}$$

Step 5. Convergences of u_n and ψ_n.

Thanks to the results of Step 3, we have that the sequence $\{T_n(u_n)\}$ is bounded in $W_0^{1,2}(\Omega)$; thus it converges, up to subsequences, to a function u in $W_0^{1,2}(\Omega)$, weakly in $W_0^{1,2}(\Omega)$, strongly in $L^2(\Omega)$, and almost everywhere.

Now let x in Ω be such that $u(x) = M < +\infty$ and such that $T_n(u_n(x))$ converges to $u(x)$; almost every x in Ω is such that this happens. If n is large enough, one has that $T_n(u_n(x)) \le M+1$ (thanks to the convergence of $T_n(u_n(x))$ to $u(x) < M$); if n is also larger than $M+1$, from $T_n(u_n(x)) \le M+1 < n$ it follows that $T_n(u_n(x)) = u_n(x)$; thus, the convergence of $T_n(u_n(x))$ to $u(x)$ implies that $u_n(x)$ converges to $u(x)$. In other words, we have proved that

$$\text{the sequence } \{u_n\} \text{ converges to } u(x) \text{ almost everywhere in } \Omega.$$

Now we recall (3.10):

$$\int_\Omega \mathrm{e}^{\gamma\,H(T_n(u_n))} \le C(g)\,.$$

Using the almost everywhere convergence of u_n to u, and Fatou's lemma, we deduce from the previous inequality that

$$\int_\Omega \mathrm{e}^{\gamma\, H(u)} \le C(g)\,,$$

so that u has exponential summability, as desired. Recalling (3.9) and using estimate (3.10), one has that

$$\int_\Omega u_n\,(\mathrm{e}^{\gamma\, H(T_n(u_n))} - 1) \le C(g)\,,$$

which implies that

$$(\mathrm{e}^{\gamma\, H(n)} - 1) \int_{\{u_n \ge n\}} u_n \le \int_{\{u_n \ge n\}} u_n\,(\mathrm{e}^{\gamma\, H(T_n(u_n))} - 1) \le C(g)\,,$$

so that

$$0 \le \lim_{n\to+\infty} \int_{\{u_n \ge n\}} u_n \le \lim_{n\to+\infty} \frac{C(g)}{\mathrm{e}^{\gamma\, H(n)} - 1} = 0\,,$$

which implies that

$$\lim_{n\to+\infty} \int_{\{u_n \ge n\}} u_n = 0\,. \tag{3.17}$$

On the other hand, since $\{\mathrm{e}^{\gamma\, H(T_n(u_n))}\}$ is bounded in $L^1(\Omega)$ by (3.10), the sequence $\{T_n(u_n)\}$ is bounded in $L^s(\Omega)$ for every $s < +\infty$, so that it strongly converges to u in every $L^s(\Omega)$. In particular, it converges to u in $L^1(\Omega)$. Thus, since

$$\int_\Omega u_n = \int_{\{u_n < n\}} u_n + \int_{\{u_n \ge n\}} u_n = \int_{\{u_n < n\}} T_n(u_n) + \int_{\{u_n \ge n\}} u_n\,,$$

using (3.17) and the strong convergence of $T_n(u_n)$ to u one has that

$$\lim_{n\to+\infty} \int_\Omega u_n = \lim_{n\to+\infty} \left[\int_{\{u_n < n\}} T_n(u_n) + \int_{\{u_n \ge n\}} u_n\right] = \int_\Omega u\,.$$

This limit, together with the fact that u_n almost everywhere converges to u, and that $u_n \ge 0$, implies that

$$\text{the sequence } \{u_n\} \text{ strongly converges to } u \text{ in } L^1(\Omega). \tag{3.18}$$

Using again that $T_n(u_n)$ converges to u strongly in $L^s(\Omega)$, for every $s < \infty$, and assumption (1.4), we have that the sequences $\{A(x, T_n(u_n))\}$ and $\{B(x, T_n(u_n))\}$ converge, respectively, to $A(x,u)$ and $B(x,u)$ strongly in $(L^s(\Omega))^{N\times N}$ for every $s < +\infty$.

The strong convergence of $A(x, T_n(u_n))$ to $A(x,u)$ in every $(L^s(\Omega))^{N\times N}$ and the weak convergence of $\nabla T_n(u_n)$ to ∇u in $(L^2(\Omega))^N$ imply that $\{A(x, T_n(u_n))\,\nabla T_n(u_n)\}$ converges to $A(x,u)\,\nabla u$ weakly in $(L^r(\Omega))^N$ for every $r < 2$; on the other hand, by (3.13) the same sequence weakly converges to some vector function G in $(L^2(\Omega))^N$. The uniqueness of the weak limit then implies that $G = A(x,u)\,\nabla u$, so that the sequence

$$\{A(x, T_n(u_n))\,\nabla T_n(u_n)\} \text{ weakly converges to } A(x,u)\,\nabla u \text{ in } (L^2(\Omega))^N. \tag{3.19}$$

Since the sequence $\{\psi_n\}$ is bounded in $W_0^{1,2}(\Omega)$, there exists ψ in $W_0^{1,2}(\Omega)$ such that, up to subsequences, ψ_n converges to ψ weakly in $W_0^{1,2}(\Omega)$, strongly in $L^2(\Omega)$, and almost everywhere in Ω.

The strong convergence of $B(x, T_n(u_n))$ to $B(x,u)$ in every $(L^s(\Omega))^{N\times N}$, together with the weak convergence of $\nabla\psi_n$ to $\nabla\psi$ in $(L^2(\Omega))^N$, implies that $\{B(x,T_n(u_n))\,\nabla\psi_n\}$ weakly converges to $B(x,u)\,\nabla\psi$ in $(L^r(\Omega))^N$ for every $r<2$; on the other hand, thanks to the results of Step 4, the sequence $\{B(x,T_n(u_n))\,\nabla\psi_n\}$ is bounded in $(L^2(\Omega))^N$, so that it weakly converges in $(L^2(\Omega))^N$ to some vector function E; by the uniqueness of the limit, we have that $E = B(x,u)\,\nabla\psi$, so that the sequence

$$\{B(x,T_n(u_n))\,\nabla\psi_n\} \text{ weakly converges to } B(x,u)\,\nabla\psi \text{ in } (L^2(\Omega))^N.$$

STEP 6. Passage to the limit in the second equation.

Now let φ be a function in $W_0^{1,2}(\Omega)$, and choose it as test function in the second equation of (3.2) to obtain

$$\int_\Omega B(x,T_n(u_n))\,\nabla\psi_n\cdot\nabla\varphi + \int_\Omega \psi_n\,\varphi = \int_\Omega g\,\varphi\,.$$

Thanks to the weak convergence of $\{B(x,T_n(u_n))\,\nabla\psi_n\}$ in $(L^2(\Omega))^N$, and the strong convergence of $\{\psi_n\}$ in $L^2(\Omega)$, we can pass to the limit in the three terms, to obtain that

$$\int_\Omega B(x,u)\,\nabla\psi\cdot\nabla\varphi + \int_\Omega \psi\,\varphi = \int_\Omega g\,\varphi \qquad \forall\varphi\in W_0^{1,2}(\Omega)\,. \tag{3.20}$$

STEP 7. Passage to the limit in the first equation.

Recalling that ψ belongs to $L^\infty(\Omega)$, and choosing $\varphi=\psi\,\eta$, with η in $W_0^{1,2}(\Omega)\cap L^\infty(\Omega)$ in (3.20), one has that

$$\int_\Omega B(x,u)\,\nabla\psi\cdot\nabla\psi\,\eta + \int_\Omega B(x,u)\,\nabla\psi\cdot\nabla\eta\,\psi + \int_\Omega \psi^2\,\eta = \int_\Omega g\,\psi\,\eta\,. \tag{3.21}$$

On the other hand, choosing $\varphi=\psi_n\,\eta$ as test function in the second equation of system (3.2), one has

$$\int_\Omega B(x,T_n(u_n))\,\nabla\psi_n\cdot\nabla\psi_n\,\eta + \int_\Omega B(x,T_n(u_n))\,\nabla\psi_n\cdot\nabla\eta\,\psi_n + \int_\Omega \psi_n^2\,\eta = \int_\Omega g\,\psi_n\,\eta\,,$$

which can be rewritten as

$$\int_\Omega B(x,T_n(u_n))\,\nabla\psi_n\cdot\nabla\psi_n\,\eta = \int_\Omega g\,\psi_n\,\eta - \int_\Omega B(x,T_n(u_n))\,\nabla\psi_n\cdot\nabla\eta\,\psi_n - \int_\Omega \psi_n^2\,\eta\,.$$

Since all three terms on the right-hand side are convergent (recall that the sequence $\{B(x,T_n(u_n))\,\nabla\psi_n\}$ is weakly convergent in $(L^2(\Omega))^N$, and that $\{\psi_n\}$ is bounded in $L^\infty(\Omega)$) we have

$$\lim_{n\to+\infty}\int_\Omega B(x,T_n(u_n))\,\nabla\psi_n\cdot\nabla\psi_n\,\eta = \int_\Omega g\,\psi\,\eta - \int_\Omega B(x,u)\,\nabla\psi\cdot\nabla\eta\,\psi - \int_\Omega \psi^2\,\eta\,.$$

Recalling (3.21) we thus have proved that

$$\lim_{n\to+\infty}\int_\Omega B(x,T_n(u_n))\,\nabla\psi_n\cdot\nabla\psi_n\,\eta=\int_\Omega B(x,u)\,\nabla\psi\cdot\nabla\psi\,\eta \tag{3.22}$$

for every η in $W_0^{1,2}(\Omega)\cap L^\infty(\Omega)$. Thus, if $\eta\ge 0$ is a function in $W_0^{1,2}(\Omega)\cap L^\infty(\Omega)$, one has

$$\begin{aligned}\limsup_{n\to+\infty}\int_\Omega \frac{B(x,T_n(u_n))\,\nabla\psi_n\cdot\nabla\psi_n}{\big(1+\frac1n|\nabla\psi_n|\big)^2}\,\eta &\le \limsup_{n\to+\infty}\int_\Omega B(x,T_n(u_n))\,\nabla\psi_n\cdot\nabla\psi_n\,\eta\\ &=\int_\Omega B(x,u)\,\nabla\psi\cdot\nabla\psi\,\eta\,.\end{aligned}$$

Recall now that the sequence $\{\psi_n\}$ is bounded in $W_0^{1,2}(\Omega)$ by the results of Step 4; therefore

$$\lim_{n\to+\infty}\frac1n|\nabla\psi_n|=0\,,\qquad\text{strongly in } L^2(\Omega).$$

Thus, one has that

$$\lim_{n\to+\infty}\frac{1}{\big(1+\frac1n|\nabla\psi_n|\big)^2}=1\,,\qquad\text{almost everywhere in }\Omega.$$

This fact allows us to apply Lemma 3.1 with

$$\Gamma_n(x)=\frac{\eta(x)\,B(x,T_n(u_n))}{\big(1+\frac1n|\nabla\psi_n|\big)^2}\,,$$

which almost everywhere converges to

$$\Gamma(x)=\eta(x)\,B(x,u)\,,$$

and

$$\Theta_n=\nabla\psi_n\,,$$

which is bounded in $L^2(\Omega)$ since $\{\psi_n\}$ is bounded in $W_0^{1,2}(\Omega)$, and weakly converges to $\nabla\psi$; we thus have that

$$\liminf_{n\to+\infty}\int_\Omega \frac{B(x,T_n(u_n))\,\nabla\psi_n\cdot\nabla\psi_n}{\big(1+\frac1n|\nabla\psi_n|\big)^2}\,\eta\ge\int_\Omega B(x,u)\,\nabla\psi\cdot\nabla\psi\,\eta\,,$$

so that we have proved that

$$\lim_{n\to+\infty}\int_\Omega \frac{B(x,T_n(u_n))\,\nabla\psi_n\cdot\nabla\psi_n}{\big(1+\frac1n|\nabla\psi_n|\big)^2}\,\eta=\int_\Omega B(x,u)\,\nabla\psi\cdot\nabla\psi\,\eta \tag{3.23}$$

for every $\eta\ge 0$, η in $W_0^{1,2}(\Omega)\cap L^\infty(\Omega)$.

Now let $v\ge 0$ be a function in $W_0^{1,2}(\Omega)\cap L^\infty(\Omega)$, and choose it as test function in the first equation of the system (3.2). We have

$$\int_\Omega A(x,T_n(u_n))\,\nabla u_n\cdot\nabla v+\int_\Omega u_n\,v=\int_\Omega \frac{B(x,T_n(u_n))\,\nabla\psi_n\cdot\nabla\psi_n}{\big(1+\frac1n|\nabla\psi_n|\big)^2}\,v\,.$$

Recalling the weak convergence of $\{A(x,T_n(u_n))\,\nabla T_n(u_n)\}$ in $(L^2(\Omega))^N$, and the strong convergence of $\{u_n\}$ to u in $L^1(\Omega)$ (see (3.18) and (3.19)) as well as (3.23) (written with $\eta = v$), we can pass to the limit as n tends to infinity to obtain that

$$\int_\Omega A(x,u)\,\nabla u \cdot \nabla v + \int_\Omega u\,v = \int_\Omega B(x,u)\,\nabla\psi \cdot \nabla\psi\, v$$

for every $v \geq 0$ in $W_0^{1,2}(\Omega) \cap L^\infty(\Omega)$. If v changes sign, splitting $v = v^+ - v^-$ yields that

$$\int_\Omega A(x,u)\,\nabla u \cdot \nabla v + \int_\Omega u\,v = \int_\Omega B(x,u)\,\nabla\psi \cdot \nabla\psi\, v$$

for every v in $W_0^{1,2}(\Omega) \cap L^\infty(\Omega)$, as desired. ◻

Remark 3.2. We remark that even though the right-hand side $B(x,u)\,\nabla\psi \cdot \nabla\psi$ only belongs to $L^1(\Omega)$, one can choose test functions in $W_0^{1,2}(\Omega)$ in the first equation of system (1.2), and not only in $W_0^{1,2}(\Omega) \cap L^\infty(\Omega)$. Indeed, if $v \geq 0$ is a function in $W_0^{1,2}(\Omega)$, and $k > 0$, one can choose $T_k(v)$ as test function in the first equation of system (1.2) to have that

$$\int_\Omega A(x,u)\,\nabla u \cdot \nabla T_k(v) + \int_\Omega u\,T_k(v) = \int_\Omega B(x,u)\,\nabla\psi \cdot \nabla\psi\, T_k(v)\,.$$

Since the term $B(x,u)\,\nabla\psi \cdot \nabla\psi$ is positive, and the functions $A(x,u)\,\nabla u$ and u belong, respectively, to $(L^2(\Omega))^N$ and $L^2(\Omega)$, we can pass to the limit on k in all terms (using the Lebesgue theorem on the left, and the Beppo Levi theorem on the right) to get that

$$\int_\Omega A(x,u)\,\nabla u \cdot \nabla v + \int_\Omega u\,v = \int_\Omega B(x,u)\,\nabla\psi \cdot \nabla\psi\, v$$

for every $v \geq 0$ in $W_0^{1,2}(\Omega)$; if v changes sign, splitting $v = v^+ - v^-$ yields that

$$\int_\Omega A(x,u)\,\nabla u \cdot \nabla v + \int_\Omega u\,v = \int_\Omega B(x,u)\,\nabla\psi \cdot \nabla\psi\, v$$

for every v in $W_0^{1,2}(\Omega)$.

4. Cases A and B independent on x, and $p \geq q - 1$.

As a consequence of Theorem 1.1 we have that the solution u of the first equation of system (1.2) belongs to $L^s(\Omega)$ for every $s < +\infty$, so that one may wonder whether u belongs to $L^\infty(\Omega)$ or not. In our general case, with $A(x,t)$ and $B(x,t)$ depending also on x, we are not able to do so.

However, if we assume that both $A(x,t)$ and $B(x,t)$ do not depend on x, then u belongs to $L^\infty(\Omega)$. In this case, we follow the ideas of [9] (see also [10, 11, 13, 17]), which will also allow us to deal with the case $p = q - 1$.

LEMMA 4.1. *Let $A(t)$ and $B(t)$ be two continuous matrix-valued functions such that*

$$\begin{aligned} &A(t)\,\xi \cdot \xi \geq \alpha\,\rho(t)\,|\xi|^2\,, \qquad |A(t)| \leq \beta\,\rho(t)\,, \\ &B(t)\,\xi \cdot \xi \geq \alpha\,\sigma(t)\,|\xi|^2\,, \qquad |B(t)| \leq \beta\,\sigma(t)\,, \end{aligned} \tag{4.1}$$

with p and q such that $p \geq q-1$. Let $g \geq 0$ be a function in $L^\infty(\Omega)$, and let $\{u_n\}$ and $\{\psi_n\}$ be the sequence of solutions of

$$\begin{cases} -\operatorname{div}(A(T_n(u_n))\,\nabla u_n) + u_n = \dfrac{B(T_n(u_n))\,\nabla\psi_n\cdot\nabla\psi_n}{\left(1+\frac{1}{n}\,|\nabla\psi_n|\right)^2} & \text{in } \Omega, \\ -\operatorname{div}(B(T_n(u_n))\,\nabla\psi_n) + \psi_n = g & \text{in } \Omega, \\ u_n = 0 = \psi_n & \text{on } \partial\Omega, \end{cases} \tag{4.2}$$

whose existence is guaranteed by Theorem 2.1. Then there exists a constant $C(g)$ such that

$$\|u_n\|_{L^\infty(\Omega)} + \|\psi_n\|_{L^\infty(\Omega)} \leq C(g)\,. \tag{4.3}$$

Proof. Define

$$H_n(t) = \int_0^t \frac{A(T_n(s))}{B(T_n(s))}\,ds\,, \qquad H(t) = \int_0^t \frac{A(s)}{B(s)}\,ds,$$

and

$$w_n = H_n(u_n) + \frac{\psi_n^2}{2}\,,$$

so that

$$\nabla w_n = \frac{A(T_n(u_n))}{B(T_n(u_n))}\,\nabla u_n + \psi_n\,\nabla\psi_n\,.$$

Let $k > 0$, and choose $v = G_k(w_n) = (w_n - k)^+$ as test function in the first equation of system (4.2). We obtain

$$\int_\Omega A(T_n(u_n))\,\nabla u_n\cdot\nabla G_k(w_n) + \int_\Omega u_n\,G_k(w_n) = \int_\Omega \frac{B(T_n(u_n))\,\nabla\psi_n\cdot\nabla\psi_n}{\left(1+\frac{1}{n}\,|\nabla\psi_n|\right)^2}\,G_k(w_n)\,.$$

Defining $A_k = \{w_n \geq k\}$, we can rewrite the above identity as

$$\begin{aligned} &\int_{A_k} B(T_n(u_n))\,\frac{A(T_n(u_n))}{B(T_n(u_n))}\,\nabla u_n\cdot\Big(\frac{A(T_n(u_n))}{B(T_n(u_n))}\,\nabla u_n + \psi_n\,\nabla\psi_n\Big) + \int_\Omega u_n\,G_k(w_n) \\ &\qquad = \int_\Omega \frac{B(T_n(u_n))\,\nabla\psi_n\cdot\nabla\psi_n}{\left(1+\frac{1}{n}\,|\nabla\psi_n|\right)^2}\,G_k(w_n) \leq \int_\Omega B(T_n(u_n))\,\nabla\psi_n\cdot\nabla\psi_n\,G_k(w_n)\,. \end{aligned} \tag{4.4}$$

Now choose $G_k(w_n)\,\psi_n$ as test function in the second equation of (4.2) to obtain that

$$\begin{aligned} \int_\Omega B(T_n(u_n))\,\nabla\psi_n\cdot\nabla\psi_n\,G_k(w_n) + \int_\Omega B(T_n(u_n))\,\nabla\psi_n\cdot\nabla G_k(w_n)\,\psi_n \\ + \int_\Omega \psi_n^2\,G_k(w_n) = \int_\Omega g\,\psi_n\,G_k(w_n)\,, \end{aligned}$$

which can be rewritten as

$$\begin{aligned} &\int_\Omega B(T_n(u_n))\,\nabla\psi_n\cdot\nabla\psi_n\,G_k(w_n) = \int_\Omega g\,\psi_n\,G_k(w_n) \\ &- \int_\Omega \psi_n^2\,G_k(w_n) - \int_{A_k} B(T_n(u_n))\,\psi_n\,\nabla\psi_n\cdot\Big(\frac{A(T_n(u_n))}{B(T_n(u_n))}\,\nabla u_n + \psi_n\,\nabla\psi_n\Big)\,. \end{aligned}$$

Using this identity in (4.4) we obtain, after grouping similar terms,

$$\int_{A_k} B(T_n(u_n))\Big(\frac{A(T_n(u_n))}{B(T_n(u_n))}\,\nabla u_n + \psi_n\,\nabla\psi_n\Big)\cdot\Big(\frac{A(T_n(u_n))}{B(T_n(u_n))}\,\nabla u_n + \psi_n\,\nabla\psi_n\Big) + \int_\Omega (u_n+\psi_n^2)\,G_k(w_n) \le \int_\Omega g\,\psi_n\,G_k(w_n)\,.$$

Recalling once again the expression for ∇w_n, the previous inequality implies that

$$\int_{A_k} B(T_n(u_n))\,\nabla w_n\cdot\nabla w_n + \int_\Omega (u_n+\psi_n^2)\,G_k(w_n) \le \int_\Omega g\,\psi_n\,G_k(w_n)\,,$$

so that, dropping a positive term, we have

$$\int_\Omega (u_n+\psi_n^2)\,G_k(w_n) \le \int_{A_k} g\,\psi_n\,G_k(w_n)\,,$$

which implies, recalling that g belongs to $L^\infty(\Omega)$, that

$$\int_\Omega (u_n+\psi_n^2 - \|g\|_{L^\infty(\Omega)}\,\psi_n)\,G_k(w_n) \le 0\,. \tag{4.5}$$

We now recall that from (2.4) it follows that

$$\|\psi_n\|_{L^\infty(\Omega)} \le \|g\|_{L^\infty(\Omega)}\,, \tag{4.6}$$

which gives the desired estimate on the sequence $\{\psi_n\}$ in $L^\infty(\Omega)$; from (2.4) one has that

$$-\frac{\|g\|^2_{L^\infty(\Omega)}}{4} \le \psi_n^2 - \|g\|_{L^\infty(\Omega)}\,\psi_n \le 0\,;$$

define $M=\frac14\,\|g\|^2_{L^\infty(\Omega)}$, so that (4.5) implies, recalling the definition of $G_k(w_n)$, that

$$\int_\Omega (u_n-M)\,(w_n-k)^+ \le 0\,. \tag{4.7}$$

We are now going to prove that if $n\ge M$, there exists $k>0$ large enough such that

$$(w_n-k)^+ = 0 \qquad \text{on } \{0\le u_n\le M\}. \tag{4.8}$$

Indeed, recalling the definition of w_n, if $0\le u_n\le M$ and $n\ge M$, and using the definition of M as well as (4.6) and the fact that $H_n(t)$ is increasing, one has

$$w_n = H_n(u_n) + \frac{\psi_n^2}{2} \le H_n(M) + 2M = \int_0^M \frac{A(s)}{B(s)}\,ds + 2M = H(M) + 2M\,.$$

Thus, choosing $k > H(M)+2M = C(g)$ we have (4.8). We now write (4.7) as

$$\begin{aligned}\int_\Omega (u_n-M)\,(w_n-k)^+ &= \int_{\{u_n>M\}} (u_n-M)\,(w_n-k)^+ + \int_{\{0\le u_n\le M\}} (u_n-M)\,(w_n-k)^+ \\ &= \int_{\{u_n>M\}} (u_n-M)\,(w_n-k)^+\,,\end{aligned}$$

where we have used (4.8) in the last passage. Thus, we have that

$$0 \le \int_{\{u_n > M\}} (u_n - M)\,(w_n - k)^+ \le 0\,,$$

which implies that $(u_n - M)\,(w_n - k)^+ = 0$ almost everywhere on the set $\{u_n > M\}$. We now have two possibilities: either $n \ge M$ is such that the set $\{u_n > M\}$ has zero measure, or it is such that the function $(w_n - k)^+ = 0$ almost everywhere on the set $\{u_n > M\}$. In the first case, we have $0 \le u_n \le M$ in Ω almost everywhere in Ω, so that

$$\|u_n\|_{L^\infty(\Omega)} \le M \qquad \forall n \ge M \text{ such that } \mathrm{meas}(\{u_n > M\}) = 0. \tag{4.9}$$

In the second case, from (4.8) it follows that $(w_n - k)^+ = 0$ almost everywhere in Ω for every $k > C(g) = H(M) + 2M$. Therefore,

$$0 \le w_n = H_n(u_n) + \frac{\psi_n^2}{2} \le C(g) \qquad \forall n \ge M\,,$$

which implies that

$$0 \le H_n(u_n) \le C(g) \qquad \forall n \ge M \text{ such that } (w_n - k)^+ = 0. \tag{4.10}$$

Recalling (4.1), we have that

$$H_n(t) = \int_0^t \frac{A(T_n(s))}{B(T_n(s))}\,ds \ge \int_0^{T_n(t)} \frac{A(T_n(s))}{B(T_n(s))}\,ds = \int_0^{T_n(t)} \frac{A(s)}{B(s)}\,ds \ge \frac{\alpha}{\beta} \int_0^{T_n(t)} \frac{\rho(s)}{\sigma(s)}\,ds\,,$$

so that

$$H_n(t) \ge \begin{cases} \dfrac{\alpha}{\beta}\,\dfrac{(1+T_n(t))^{p-q+1} - 1}{p-q+1} & \text{if } p > q-1, \\ \dfrac{\alpha}{\beta}\,\ln(1+T_n(t)) & \text{if } p = q-1. \end{cases}$$

Thus, from (4.10) we have that either

$$\frac{\alpha}{\beta}\,\frac{(1+T_n(u_n))^{p-q+1} - 1}{p-q+1} \le C(g)$$

if $p > q-1$ or

$$\frac{\alpha}{\beta}\,\ln(1+T_n(u_n)) \le C(g)$$

if $p = q-1$. In both cases, there exists a constant $C(g)$, independent of n, such that

$$0 \le T_n(u_n) \le C(g)\,.$$

Choosing $n \ge C(g)$, the above inequality implies that

$$0 \le u_n \le C(g)\,,$$

so that

$$\|u_n\|_{L^\infty(\Omega)} \le C(g) \qquad \forall n \ge M \text{ such that } (w_n - k)^+ = 0. \tag{4.11}$$

Putting together (4.9) and (4.11), we have that for every $n \ge M$ the norm of u_n in $L^\infty(\Omega)$ is bounded by a constant independent of n, so that the sequence $\{u_n\}$ is bounded in $L^\infty(\Omega)$, as desired. ∎

Remark 4.2. Note that we used the assumption $p \geq q-1$ only at the end of the proof, when dealing with the consequences of the fact that $(w_n - k)^+ = 0$ on the set $\{u_n > M\}$.

As a consequence of Lemma 4.1 and of the proof of Theorem 1.1, we have that if $p > q-1$, the solution u of system (1.2) belongs to $L^\infty(\Omega)$ if both $A(x,t)$ and $B(x,t)$ do not depend on x and satisfy (4.1). If $p = q-1$, Lemma 4.1 allows us to prove an existence result for system (1.2).

THEOREM 4.3. *Let $A(t)$ and $B(t)$ be such that* (4.1) *holds, with $p = q-1$. Let $g \geq 0$ be a function in $L^\infty(\Omega)$. Then there exist solutions u and ψ of system* (1.2), *such that*

- *$u \geq 0$ belongs to $W_0^{1,2}(\Omega) \cap L^\infty(\Omega)$;*
- *$\psi \geq 0$ belongs to $W_0^{1,2}(\Omega) \cap L^\infty(\Omega)$.*

Furthermore, u and ψ are such that

$$\int_\Omega A(u)\,\nabla u \cdot \nabla v + \int_\Omega u\,v = \int_\Omega B(u)\,\nabla\psi \cdot \nabla\psi\, v$$

for every v in $W_0^{1,2}(\Omega) \cap L^\infty(\Omega)$ and

$$\int_\Omega B(u)\,\nabla\psi \cdot \nabla\varphi + \int_\Omega \psi\,\varphi = \int_\Omega g\,\varphi$$

for every φ in $W_0^{1,2}(\Omega)$.

Proof. Let $\{u_n\}$ and $\{\psi_n\}$ be the sequences of solutions of system (4.2); by Lemma 4.1, both sequences are bounded in $L^\infty(\Omega)$. Choosing u_n as test function in the first equation and ψ_n as test function in the second equation, we have that

$$\int_\Omega A(T_n(u_n))\,\nabla u_n \cdot \nabla u_n + \int_\Omega u_n^2 = \int_\Omega \frac{B(T_n(u_n))\,\nabla\psi_n \cdot \nabla\psi_n}{\left(1 + \frac{1}{n}\,|\nabla\psi_n|\right)^2}\, u_n$$

and

$$\int_\Omega B(T_n(u_n))\,\nabla\psi_n \cdot \nabla\psi_n + \int_\Omega \psi_n^2 = \int_\Omega g\,\psi_n\,.$$

Since by (4.1) and by the boundedness of $\{u_n\}$ in $L^\infty(\Omega)$ there exist constants $\mathcal{A} > 0$ and $\mathcal{B}$ such that

$$\mathcal{A}\,|\nabla u_n|^2 \leq A(T_n(u_n))\,\nabla u_n \cdot \nabla u_n \leq \mathcal{B}\,|\nabla u_n|^2$$

and

$$\mathcal{A}\,|\nabla\psi_n|^2 \leq B(T_n(u_n))\,\nabla\psi_n \cdot \nabla\psi_n \leq \mathcal{B}\,|\nabla\psi_n|^2\,,$$

from the above identities we have that

$$\mathcal{A}\int_\Omega |\nabla\psi_n|^2 \leq \int_\Omega g\,\psi_n \leq C(g)\,,$$

and then that

$$\mathcal{A}\int_\Omega |\nabla u_n|^2 \leq \mathcal{B}\,C(g)\int_\Omega |\nabla\psi_n|^2 = C(g)\,.$$

Hence, the sequences $\{u_n\}$ and $\{\psi_n\}$ are bounded in $W_0^{1,2}(\Omega)$, so that there exist u and ψ in $W_0^{1,2}(\Omega) \cap L^\infty(\Omega)$ such that the sequence $\{u_n\}$ converges to u (weakly in

$W_0^{1,2}(\Omega)$ and strongly in $L^s(\Omega)$ for every $s \geq 1$), and the sequence $\{\psi_n\}$ converges to ψ (weakly in $W_0^{1,2}(\Omega)$ and strongly in $L^s(\Omega)$ for every $s \geq 1$). Choosing $\psi_n - \psi$ as test function in the second equation, one easily obtains that the sequence $\{\psi_n\}$ strongly converges to ψ in $W_0^{1,2}(\Omega)$, so that (recalling that the sequence $\{u_n\}$ is bounded in $L^\infty(\Omega)$), the Lebesgue theorem implies that

$$\lim_{n\to+\infty} \frac{B(T_n(u_n))\,\nabla\psi_n \cdot \nabla\psi_n}{\left(1 + \frac{1}{n}\,|\nabla\psi_n|\right)^2} = B(u)\,\nabla\psi \cdot \nabla\psi\,, \qquad \text{strongly in } L^1(\Omega).$$

Using these convergences, one can pass to the limit in the equations satisfied by u_n and ψ_n to prove that u and ψ are weak solutions (in the sense specified in the statement) of the equations of (1.2). ∎

5. Some comments on the case $p < q - 1$. In this section we are going to explain why the case $p < q - 1$ is very different from the case $p \geq q - 1$ in terms of a priori estimates and of existence of solutions for system (1.2). We will confine ourselves to the model cases:

$$A(x,t) = (1 + |t|)^p\, I\,, \qquad B(x,t) = (1 + |t|)^q\, I\,, \qquad p < q - 1\,,$$

where I is the $N \times N$ identity matrix.

As a first remark, we observe that also in this case one can perform the same arguments as in the proof of Theorem 1.1 to obtain that (3.10) and (3.11) hold true; however, if $p < q - 1$, then

$$H(t) = \frac{1 - (1 + |t|)^{p-q+1}}{q - p - 1}\,\mathrm{sgn}(t)$$

is a *bounded* function; thus, the fact that the sequence $\{\mathrm{e}^{\gamma\, H(T_n(u_n))}\}$ is bounded in $L^1(\Omega)$ is trivially true, and from (3.11) one can only obtain that

$$\int_\Omega (1 + T_n(u_n))^{2p-q} |\nabla T_n(u_n)|^2 \leq C(g)\,.$$

This inequality, if p and q are such that $2p - q < 0$, does not yield any estimate on $\{T_n(u_n)\}$ in $W_0^{1,2}(\Omega)$, so that any solution of the first equation (if it exists) does not have finite energy. Furthermore, and worse, from (3.14) one only has that

$$\int_\Omega (1 + T_n(u_n))^q\, |\nabla\psi_n|^2 \leq C(g)\,,$$

which, even though it gives an estimate on $\{\psi_n\}$ in $W_0^{1,2}(\Omega)$ under the assumption $q > 0$, does not allow us to prove that the sequence $\{(1 + T_n(u_n))^q\,\nabla\psi_n\}$ is bounded in $(L^2(\Omega))^N$, a key fact in order to pass to the limit in the approximate equations (see Step 7 of the proof of Theorem 1.1). A possible alternative approach would be to prove that the sequence $\{(1 + T_n(u_n))^q\,\nabla\psi_n \cdot \nabla\psi_n\}$ is strongly convergent in $L^1(\Omega)$ but, once again, the a priori estimates are too weak to prove this using any of the known techniques.

Another possible approach is to prove that the sequence $\{u_n\}$ of solutions is bounded in $L^\infty(\Omega)$, so that existence of solutions for system (1.2) will easily follow as in the proof of Theorem 4.3. Thus, one may think to apply to the case $p < q - 1$ the same ideas as in the proof of Lemma 4.1. And indeed, the proof works, but the final result is that the sequence $\{H_n(u_n)\}$ is bounded in $L^\infty(\Omega)$. In our case, we have that

$$H_n(t) = \begin{cases} \dfrac{1-(1+T_n(t))^{p-q+1}}{q-p-1} & \text{if } 0 \le t \le n, \\ \dfrac{1-(1+n)^{p-q+1}}{q-p-1} + \dfrac{t-n}{(1+n)^{p-q}} & \text{if } t > n, \end{cases}$$

which is an *unbounded* function. However, from the inequality $0 \le H_n(u_n) \le C(g)$ one can only obtain that

$$0 \le u_n \le n + (1+n)^{q-p}\left[C(g) - \frac{1-(1+n)^{p-q+1}}{q-p-1}\right] \approx C(g)\, n^{q-p}\,,$$

and $q-p>1$ by the assumption $p<q-1$. In other words, this method does not yield an a priori estimate on the sequence $\{u_n\}$ in $L^\infty(\Omega)$.

There is a further problem which arises in the case $p<q-1$, and that mathematically justifies the presence of the lower order terms "$+u$" and "$+\psi$" in the two equations of the system. Indeed, under the assumption $p<q-1$, among the possible values there are $q=0$ and $p<-1$; in this case the matrix-valued functions $A(t)=(1+|t|)^p\, I$ and $B(t)=I$, where I is the $N\times N$ identity matrix, satisfy assumption (1.4). Therefore, setting $\gamma=-p>1$, system (1.2) becomes

$$\begin{cases} -\operatorname{div}\Big(\dfrac{\nabla u}{(1+|u|)^\gamma}\Big) + u = |\nabla\psi|^2 & \text{in } \Omega, \\ -\Delta\psi + \psi = g & \text{in } \Omega, \\ u = 0 = \psi & \text{on } \partial\Omega, \end{cases}$$

with the two equations "uncoupled." Since clearly the second equation has a solution in $W_0^{1,2}(\Omega)\cap L^\infty(\Omega)$, the question becomes whether the first one has a solution, taking into account that the datum is an $L^1(\Omega)$ function. In this case, there is a sharp difference between the case where a lower order term "$+u$" is considered or not. We quote here some results contained in [1] (for the "nonexistence" part) and [3] (for the "existence" part).

(A) *If the norm of g in $L^\infty(\Omega)$ is large enough, there is no solution $w \ge 0$ of the equation*

$$-\operatorname{div}\Big(\frac{\nabla w}{(1+w)^\gamma}\Big) = |\nabla\psi|^2\,. \tag{5.1}$$

Indeed, suppose that g and Ω are smooth enough so that ψ belongs to $W_0^{1,\infty}(\Omega)$, and let M be the norm of $|\nabla\psi|$ in $L^\infty(\Omega)$. Let z be the weak solution in $W_0^{1,2}(\Omega)$ of

$$-\Delta z = |\nabla\psi|^2\,, \tag{5.2}$$

and suppose that M is large enough in order to have $\|z\|_{L^\infty(\Omega)} > \frac{1}{\gamma-1}$. Since M depends linearly on g, this can be done by choosing the norm of g large enough. Suppose now that there exists a solution $w\ge 0$ of (5.1), and set

$$z = \frac{1-(1+w)^{1-\gamma}}{\gamma-1}\,.$$

Since

$$\nabla z = \frac{\nabla w}{(1+w)^\gamma}\,,$$

one has that z solves (5.2). However, this is not possible, since by definition $0 \le z \le \frac{1}{\gamma-1}$, while the choice of g is such that $\|z\|_{L^\infty(\Omega)} > \frac{1}{\gamma-1}$. Therefore, w does not exist (actually, one can prove as in [1] that if one approximates (5.1), one obtains a function which is infinite on a subset of positive measure).

(B) *For every g in $L^\infty(\Omega)$ there exists a solution $u \ge 0$ of the equation*

$$-\operatorname{div}\Big(\frac{\nabla u}{(1+u)^\gamma}\Big) + u = |\nabla\psi|^2\,. \tag{5.3}$$

Indeed, reasoning as above and setting

$$v = \frac{1-(1+u)^{1-\gamma}}{\gamma-1}\,,$$

finding a solution of (5.3) is equivalent to finding a solution of equation

$$-\Delta v + \frac{1}{[1-(\gamma-1)\,v]^{\frac{1}{\gamma-1}}} = |\nabla\psi|^2\,, \tag{5.4}$$

which has a lower order term $G(v) = \frac{1}{[1-(\gamma-1)\,v]^{\frac{1}{\gamma-1}}}$ such that

$$\lim_{t\to(\frac{1}{\gamma-1})^-} G(t) = +\infty\,.$$

This is exactly the case studied in [3], where it is proved that these equations have a solution v, such that $0 \le v < \frac{1}{\gamma-1}$ almost everywhere in Ω for every datum in $L^1(\Omega)$. Thus, since $|\nabla\psi|^2$ belongs to $L^1(\Omega)$ for every possible datum g (since ψ belongs to $W_0^{1,2}(\Omega)$), it turns out that (5.3), and so system (1.2) in this "uncoupled" case, has a solution for every datum g.

In other words, in the case $p < q-1$ one can have that the differential operator of the first equation is a very noncoercive one (noncoercive in general, and with respect to the differential operator of the second equation). For this reason, the term "$+u$," which guarantees that at least one has some estimates in $L^1(\Omega)$, is necessary, from a mathematical point of view, in order to prove existence of solutions for system (1.2).

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