a PhD Thesis by
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Lamé and ZK:

Spectral Analysis and Unique Continuation

Supervised by
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To my mom

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Preface

The bulk of this thesis focuses on two fields that are being deeply investigated both by the mathematical and physical community, namely spectral theory and unique continuation. Both these theories are extremely rich and nowadays represent inclusive terms covering a wide variety of branches of physics and mathematics. More precisely, the first one, in its general meaning, includes theories which extend the eigenvalues analysis for square matrices to a much broader class of mathematical characters, for instance, due to their relevance in quantum interpretation, to unbounded operators in Hilbert space. The second one is concerned with the search for classes of functions for which the vanishing in a region ensures the vanishing in a larger one, roughly speaking it is the issue to find the correct analogue of harmonic functions for which the Liouville theorem guarantees the stated rigidity.

The structure of this document is roughly the following. The main body of work of this thesis is contained in the first two parts, in which the aforementioned themes are analyzed, specifically spectral properties for the non self-adjoint perturbed Lamé operator of elasticity and unique continuation for Zakharov-Kuznetzov dispersive equation are objects of our investigation. Each part contains an introductory chapter which endeavors to give an overview of the problem in exam and to clarify why it is worthy of attention. Moreover a time-based analysis, involving also the recent developments of these matters, is provided in the same chapters. At times the discussion is chosen to be informal in order to convey the basic underlying ideas. The concise statements together with their proofs, employing the necessary rigor lacking in the introductions, are given in the following chapters.

The third part is slightly different, it is not concerned with achieved results but it involves a future possible project that we would like to deepen. More precisely the prospect presented takes place in the field of inverse problems in elasticity. The possibility to re-adapt some useful tools earned to address the problems described in the first two parts has played a relevant role to motivate solidly this future investigation.

Part I

Spectral Theory for Complex Perturbed Lamé Operators

We will turn to the first part of this thesis. It is devoted to the analysis of spectral properties connected with the Lamé operator of elasticity once it is perturbed by a potential which, possibly, is assumed to be complex-valued, this will lead our way off the well beaten path of self-adjoint operators.

In particular we focus on *two* distinct but intimately related *problems* that will be treated separately:

> PROBLEM 1. *Verify whether the stability of the spectrum, or part of it, occurs under suitable small perturbations.*

and

> PROBLEM 2. *Produce bounds on the distribution of eigenvalues in the complex plane, roughly speaking, obtain the correct analogue to Lieb-Thirring inequalities in a complex setting.*

All the results presented are mainly motivated by the deep connection between the Laplace and Lamé operator that we will clarify later on by means of Helmholtz decomposition.

The results in this part couldn't have been achieved without useful and encouraging conversations with Luca Fanelli.

I.1.| Introduction

This part is concerned with spectral analysis of operators of the form

$$-\Delta^* + V(x)$$

acting on the Hilbert space $[L^2(\mathbb{R}^d)]^d$ that is the Hilbert space of vector fields with components in $L^2(\mathbb{R}^d)$.

$-\Delta^*$ denotes the Lamé operator of elasticity which rules the behavior of solid bodies, or better their reversible deformation, once they are subjected to excitations of various physical natures.

$-\Delta^*$ is a linear symmetric differential operator of second order that acts on smooth L^2 vector fields u on $\mathbb{R}^d$, for example $[C_c^\infty(\mathbb{R}^d)]^d$, in this way:

$$-\Delta^* u := -\mu(\Delta u_1, \Delta u_2, \ldots, \Delta u_d) - (\lambda + \mu)\nabla \operatorname{div}(u_1, u_2, \ldots, u_d), \tag{I.1.1}$$

where λ and μ are the so called Lamé's coefficients.

$V(x)$ is a notation for the multiplication operator by the *complex*-valued potential $V(x)$, this means that the context we are working in is a *non self-adjoint* setting.

Notations: In this part the following notations are used:

- Depending on our potential perturbation V be a scalar $V : \mathbb{R}^d \to \mathbb{C}$ or matrix-valued $V : \mathbb{R}^d \to \mathcal{M}_{d\times d}(\mathbb{C})$ function, the notation $|V(x)|$ would represent the standard absolute value for a complex number or the matrix norm $|V(x)| := \Big(\sum_{i=1}^d \sum_{j=1}^d |V_{ij}(x)|^2\Big)^{\frac{1}{2}}$.
- Since the operator we are dealing with acts on vector-valued functions, we will use the following for the L^p norm of a vector field $u : \mathbb{R}^d \to \mathbb{R}^d$: $\|u\|_p := \|u\|_{[L^p(\mathbb{R}^d)]^d} = \big(\sum_{j=1}^d \|u_j\|_{L^p(\mathbb{R}^d)}^p\big)^{\frac{1}{p}}$, for all $1 < p < \infty$.
- Since treating the first problem only the L^2 norm is needed, we will skip the index 2 in $\|\|_2$ writing just $\|\|$. The tradition notation will appear working on the second problem.
- In this part weighted estimates will appear, we will use the notation $L^p(w\,dx)$ for the L^p- space on $\mathbb{R}^d$ with measure $w(x)\,dx$, under suitable assumptions about w.

The results involved in this part can be thought to belong to the very relevant domain usually put under the “umbrella name” of theory of perturbations.

In general all the disciplines called theory of perturbations are based on the idea of *studying a system deviating possibly slightly from a simple ideal system for which the complete solution of the problem under consideration is known.*

According to this general notion, the ideology under the classical theory of perturbation for linear operators is as follows.

Let H_0 be a self-adjoint operator on a Hilbert space $\mathcal{H}$, then suppose to perturb it, that is to consider the new operator

$$H := H_0 + V,$$

with V still a self-adjoint operator.

The main task of the theory is to deduce information about properties of H from the knowledge of those of H_0.

In general there is no reason to expect that H preserves properties of the unperturbed operator H_0, but it is more reasonable to believe that one can manage changes of these characters if the perturbations are "small" if compared to H_0.

Clearly, a first non trivial issue is how to define properly and reasonably a "small" perturbation.

For our aims, the characters attached to the operator we are interested in are mainly spectral properties.

We would like to start out with a brief and not comprehensive description on the motivations that pushed mathematical and physical community to get involved into spectral analysis. In order to do that we need first to recall some very classical facts arising from quantum mechanics.

According to the postulate of quantum mechanics the state of a physical system is described by a state vector in a certain Hilbert space which changes depending on the physical system we are trying to describe. Moreover we recall that if the initial state, namely state at time $t = 0$, of a general system is represented by a reasonable vector ψ_0, then at any time $t > 0$ the system is represented by a vector

$$\psi(t) = e^{-iHt}\psi_0,$$

where H represents the self-adjoint, time-independent energy operator in the Hilbert space which is chosen to describe our physical system. Moreover, the state $\psi(t)$ so defined solves the Cauchy problem associated to the Schrödinger equation

$$i\partial_t\psi = H\psi.$$

The self-adjointness of the energy operator H ensure the well-posedness of the initial value problem attached with the equation above.

Just to have in mind a particular but relevant situation suppose we want to describe a single particle state of mass $m = \frac{1}{2}$, it is very well known that this is governed by a complex-valued function of position and time $\psi(x,t)$, $x \in \mathbb{R}^d, t \in \mathbb{R}$, the so-called *wave function*, which is a vector of the Hilbert space $L^2(\mathbb{R}^d)$. Moreover, suppose that the particle moves under the force generated by a potential function V then the energy operator, appointed to describe its evolution, is the differential operator

$$H := H_0 + V,$$

where, in this particular situation, H_0 is represented by the Laplacian $-\Delta$.

It is worthy to underline that according to the interpretation of quantum mechanics, the position of a particle cannot be determined as a definite point $x \in \mathbb{R}^d$, in fact with the aid of the wave function ψ we can just obtain information about its probable location. To be more precise the quantity $|\psi(x,t)|^2\,dx$ is treated as a probability measure and specifically, it provides the probability of finding a particle in any space region $\Omega \subset \mathbb{R}^d$ at time t through the computation $\int_\Omega |\psi(x,t)|^2\,dx$. Therefore, the following normalization is required: $\int_{\mathbb{R}^d} |\psi(x,t)|^2\,dx = 1$.

Of a particular importance are the states described by eigenvectors of the energy operator H, namely those vector functions ψ_0 such that there exists $E \in \mathbb{R}$ for which the following

$$H\psi_0 = E\psi_0$$

holds true then, clearly, it follows that the evolved state $\psi(t) = e^{-iEt}\psi_0$. Since ψ differs from ψ_0 just by a phase factor, it describes the same state of the particle indeed, by virtue of the postulate of quantum physics, the only relevant quantity attached to the description of a particle's evolution is represented by the density of probability $|\psi(x,t)|^2$ which does not distinguish between ψ and ψ multiplied by a phase factor. In other words, if the state of the particle is represented by an eigenvector of H, then it is time-independent. In particular, the probability $\int_\Omega |\psi(x,y)|^2\,dx$ to find the particle in some region $\Omega \subseteq \mathbb{R}^d$ is then constant in time. Therefore, a particle in such a state is said to be localized.

It is thus important to know whether there exist real numbers E for which $H\psi_0 = E\psi_0$ holds and, if so, how many of such numbers there are, how large they are, where they are located, etc.

This yields the aforementioned interest in spectral analysis and in particular, since several quantum mechanical systems are described by Hamiltonian of that form, in spectral analysis of operators of type $-\Delta + V$. Clearly depending on the concrete physical problem at hand, the Laplacian may need to be replaced by a more general differential operator H_0.

Spectral analysis for *self-adjoints operators* has been intensively studied for several decades. Unfortunately the generalization of the achieved results in this topic to the *non self-adjoint* framework seems to be not that straightforward. Indeed, the lack of spectral theorem and of a variational characterization of eigenvalues, among other tools, makes the theory of non self-adjoint operators more challenging and therefore very much less unified that the self-adjoint one. On the other hand, recently, there has been a growing interest in this subject, mainly motivated by the surfacing of increasing number of problems in physics requiring the analysis of non self-adjoint operators. We refer to [21] and references therein for an overview of different sources of such problems and in general for a description of methods that have been used to analyze the spectrum of non self-adjoint operators.

Even though the interest in this field is quite recent, the study of spectral properties of non self-adjoint operators already has a bibliography especially in the context of Schrödinger operators and it was precisely the presence of these results for Schrödinger operators that mainly motivates our work. A deeper understanding of the action of Lamé operator on smooth vector fields can explain this sentence. Using the Helmholtz decomposition, which is a standard way to decompose smooth vector fields into a sum of a divergence free vector field and a gradient, we can see that, for any $u = u_P + u_S$, the operator $-\Delta^*$ acts on u in this way:

$$-\Delta^* u = -\mu\Delta u_S - (\lambda + 2\mu)\Delta u_P,$$

where the component u_S is the divergence free vector field and the component u_P is the gradient. In other words, Lamé operator, up to multiplicative constants, is nothing but the sum of two Laplacian acting on distinct components of the Helmholtz decomposition. This means that there is a deep link between Lamé and Laplace operator and therefore, at least at first sight, this fact augurs well for the possibility that results which can be seen as the proper counterpart of ones already gotten for Schrödinger could be still meaningful if concern Lamé operator.

With respect to the existing result in the non self-adjoint landscape, since our aim is concerned mainly with two topics, the stability of the spectrum and bounds of eigenvalues, we limit ourselves to quoting just results regarding these two themes and that moreover were the main source of inspiration for ours. Let us start with state of the art in the first theme.

PROBLEM 1

Fanelli, Krejčiřík and Vega, in a very recent work [38] improved the state of the art in the picture of the stability of spectral properties for *non self-adjoint Schrödinger operators*, namely $H := -\Delta + V$ in $\mathbb{R}^d$, with $V : \mathbb{R}^d \to \mathbb{C}^d$ complex-valued potential perturbation.

It is common knowledge that the spectrum of the Laplace operator is purely continuous and coincides with the non negative semi-axis, in particular, by virtue of the disjoint partition of the spectrum (see Appendix C), this means there are no eigenvalues. For $d \geqslant 3$

in [38] it was proved that, assuming for the complex-valued potential a suitable smallness condition, the point spectrum of Schrödinger operator remains empty.

Our goal has been to obtain a result that represents the analogue for Lamé of the one Fanelli, Krečiřík and Vega proved in the aforementioned work [38]. In other words, since also for free Lamé operator the spectrum results to be purely continuos and coinciding with the positive real line, we want to prove that, under suitable conditions about the potential, this property is preserved, at least in part, in the perturbed setting, specifically it is proved that no eigenvalues can occur. The formal statement is the following:

Theorem I.1. *Let $d \geqslant 3$. Assume that $\lambda, \mu \in \mathbb{R}$ satisfy*

$$\mu > 0,\ \lambda > -\frac{2}{d}\mu \tag{I.1.2}$$

and that $V\colon \mathbb{R}^d \to \mathcal{M}_{d\times d}(\mathbb{C})$ is such that

$$\forall\, u \in [H^1(\mathbb{R}^d)]^d, \qquad \int_{\mathbb{R}^d} |x|^2|V|^2|u|^2 \leqslant \Lambda^2 \int_{\mathbb{R}^d} |\nabla u|^2, \tag{I.1.3}$$

where Λ satisfies

$$\frac{\Lambda}{\min\{\mu, \lambda+2\mu\}} \frac{4(2d-3)}{d-2}(C+1) + \frac{\Lambda^{\frac{3}{2}}}{\min\{\mu, \lambda+2\mu\}^{\frac{3}{2}}} \frac{4\sqrt{2}}{\sqrt{d-2}}(C+1)^{\frac{3}{2}} < 1, \tag{I.1.4}$$

and where $C > 0$ is a suitable constant. Then $\sigma_p(-\Delta^* + V) = \varnothing$.

As a further application of the multipliers technique we have developed to prove Theorem I.1, we are also able to perform uniform resolvent estimates for our operator $-\Delta^* + V$, which generalize the ones obtained, for the Helmholtz equation, by Barceló, Vega and Zubeldia in [6].

Precisely, in this regard we consider the resolvent equation

$$\Delta^* u - Vu + ku = f, \tag{I.1.5}$$

where $k = k_1 + ik_2$ is any complex constant, with $k_1 := \Re k$ and $k_2 := \Im k$, and $f\colon \mathbb{R}^d \to \mathbb{C}^d$ is a measurable function and we will prove, for solution of (I.1.5), the following result:

Theorem I.2. *Let $d \geqslant 3$, $\||\cdot|f\| < \infty$ and assume that V satisfies (I.1.3). Then, there exist $c > 0$ independent of k and f such that for any solution $u \in [H^1(\mathbb{R}^d)]^d$ of the equation (I.1.5) one has*

$$\||x|^{-1}u\| \leqslant c\||x|f\|. \tag{I.1.6}$$

Remark I.1. We remark that the estimate (I.1.6) for the perturbed Lamé operator was already proved in [4]. On the other hand our integral-smallness assumption on the potential is weaker than the one required in that work. Indeed, to be more precise, assuming that $\||x|^2 V\|_{L^\infty} < \infty$, the authors provided the uniform resolvent estimate (I.1.6) for the equation

$$\Delta^* u - \delta V u + k u = f,$$

using a purely perturbative argument, that is, roughly speaking, taking δ as small as needs in order to treat the term $-\delta V u$ as a mere correction.

Actually, in order to prove Theorem I.2 we establish the following stronger result, which shows that a priori estimates for solutions of (I.1.5) hold.

Theorem I.3. *Let $d \geqslant 3$, $\||\cdot|f\| < \infty$ and assume that V satisfies (I.1.3). Then, there exist $c > 0$ independent of k and f such that for any solution $u \in [H^1(\mathbb{R}^d)]^d$ of the equation (I.1.5) one has*

- *for $|k_2| \leqslant k_1$*

$$\|\nabla u_S^-\| \leqslant c\||x|f\|, \qquad \text{and} \qquad \|\nabla u_P^-\| \leqslant c\||x|f\|, \tag{I.1.7}$$

where the vector fields u_S^- and u_P^- will be defined in (I.4.10) and (I.4.12) respectively.

- *for $|k_2| > k_1$*

$$\|\nabla u\| \leqslant c\||x|f\|. \tag{I.1.8}$$

From this theorem, as a straightforward corollary, we easily obtain Theorem I.2.

PROBLEM 2

With respect to the second topic, namely finding quantitative estimates regarding the location in the complex plane of eigenvalues, among others, as it primarily motivated our result, it is worthy to mention the recent work by Frank [40]. In this paper he was concerned with the situation of a non self-adjoint Schrödinger operator $H = -\Delta + V$ in $\mathbb{R}^d$, with $V : \mathbb{R}^d \to \mathbb{C}^d$ which is assumed to decay at infinity (at least in some averaged sense). As in the self-adjoint case, this entails, by mean of a proper generalization of Weyl's theorem in the non self-adjoint situation, that the essential spectrum remains stable, therefore coincides with $[0, \infty)$. In a compact form, we say that the following chain of identities holds: $\sigma_{\text{ess}}(-\Delta + V) = \sigma_{\text{ess}}(-\Delta) = [0, \infty)$. We underline that even if the "behavior" of the essential spectrum does not change replacing real-valued potentials with complex ones; the discrete spectrum represents more subtle issue, indeed unlike the self-adjoint situation, in which we have just 0 as a possible accumulation point, in the non self-adjoint context, since the spectrum is no more necessarily real, we might have *positive* accumulation points and this fact makes the analysis of the discrete spectrum less manageable then the self-adjoint situation.

In this direction in [40] it was improved the knowledge about the location of those eigenvalues, particularly it was shown that the absolute value of non positive eigenvalues for these operators can be bounded in terms of L^p- norms of the potential, in other words a weaker form of classical Lieb-Thirring inequalities is provided also in a non self-adjoint context. This and several other previous results in the same spirit share, as starting point in their proof, the use of the very well known Birman-Schwinger principle which, roughly speaking, permits to re-phrase conveniently the eigenvalue problem for the Schrödinger operator in order to exploit compactness properties that were missing in the original formulation. Coming up the classical formulation of this principle.

Proposition I.1 (Birman-Schwinger principle)**.** *Let $z \notin \sigma(H_0)$. Then*

$$z \in \sigma_p(H_0 + V) \quad \Longleftrightarrow \quad -1 \in \sigma_p(K_z),$$

where $K_z := V_{\frac{1}{2}}(H_0 - z)^{-1}|V|^{\frac{1}{2}}$, *with* $V_{\frac{1}{2}} := |V|^{\frac{1}{2}} \operatorname{sgn}(V)$.

Remark I.2. Clearly, in the context of Schrödinger operators, H_0 is replaced by $-\Delta$ and the assumption about z is $z \in \mathbb{C}\backslash[0, \infty)$.

Remark I.3. Let us observe that, since $(-\Delta - z)^{-1}$ is an integral operator for $z \in \mathbb{C}\backslash[0, \infty)$ with explicit integral kernel for all dimensions $d \geqslant 1$, this formulation has the additional advantage to enable us to treat integral equations instead of the (less easy to handle) partial differential equations.

Our original contribution in the setting of quantitative estimate for the spectrum of $-\Delta^* + V$, was to obtain similar bounds to the ones shown in [40].

Since we want to use the same powerful approach, that is the Birman-Schwinger principle, we need first to get an explicit expression for the resolvent operator associated with $-\Delta^*$. In order to do that Helmholtz's decomposition again plays a relevant role, indeed, making use of this tool, precisely writing $g = g_S + g_P$ where g_S is the divergence free vector field and g_P is the gradient, it turns out, as we will see in more details below, that $(-\Delta^* - z)^{-1}$ has a favorable form

$$(-\Delta^* - z)^{-1} g = \frac{1}{\mu}\big(- \Delta - \tfrac{z}{\mu}\big)^{-1} g_S + \frac{1}{\lambda + 2\mu}\big(- \Delta - \tfrac{z}{\lambda+2\mu}\big)^{-1} g_P.$$

From the previous identity, the action of the resolvent of the Lamé operator on a sufficiently smooth vector field g can be seen to be nothing but a sum of two resolvent operators associated with the Laplacian for each component of g. This fact was responsible in laying solid motivations to the possible success of our project. In fact the following were obtained:

Theorem I.4. *Let $d \geqslant 2$ and $0 < \gamma \leqslant \frac{1}{2}$. Then any eigenvalue $z \in \mathbb{C}\backslash[0, \infty)$ of the perturbed*

Lamé operator $-\Delta^* + V$ *satisfies*

$$|z|^\gamma \leqslant D_{\gamma,d,\lambda,\mu} \int_{\mathbb{R}^d} |V(x)|^{\gamma+\frac{d}{2}}\, dx, \tag{I.1.9}$$

with a constant $D_{\gamma,d,\lambda,\mu}$ *independent of* V.

Theorem I.5. *Let* $d \geqslant 2$, $0 < \gamma \leqslant \frac{1}{2}$ *and* $(d-1)(2\gamma+d)/2(d-2\gamma) < p \leqslant \gamma + \frac{d}{2}$. *Then any eigenvalue* $z \in \mathbb{C}\backslash[0,\infty)$ *of the perturbed Lamé operator* $-\Delta^* + V$ *satisfies*

$$|z|^\gamma \leqslant D_{\gamma,d,p,\lambda,\mu} \sup_{x,r} r^d \left(r^{-d} \int_{B_r(x)} |V(y)|^p dy \right)^{\frac{2\gamma+d}{2p}}. \tag{I.1.10}$$

Theorem I.6. *Let* $d \geqslant 2$ *and* $\alpha > \frac{1}{2}$. *Then any eigenvalue* $z \in \mathbb{C}\backslash[0,\infty)$ *of the perturbed Lamé operator* $-\Delta^* + V$ *satisfies*

$$|z|^{\frac{1}{2}} \leqslant C_{d,\alpha,\lambda,\mu} \sup_{x\in\mathbb{R}^d} (1+|x|^2)^\alpha |V(x)|.$$

Using similar arguments as in the previous results and making use of interpolation theory, the following theorem was also proved.

Theorem I.7. *Let* $d \geqslant 2$, $\gamma > \frac{1}{2}$ *and* $\alpha > \gamma - \frac{1}{2}$. *Then any eigenvalue* $z \in \mathbb{C}\backslash[0,\infty)$ *of the perturbed Lamé operator* $-\Delta^* + V$ *satisfies*

$$|z|^\gamma \leqslant C_{d,\gamma,\alpha,\lambda,\mu} \int_{\mathbb{R}^d} |V(x)|^{2\gamma+\frac{(d-1)}{2}} (1+|x|^2)^\alpha\, dx.$$

The previous four theorems match properly ones proved by Frank in [40].

We want to emphasize that even if the generalization to our context of the proofs by Fanelli, Krečiřík and Vega and Frank seems to be quite natural, mainly looking at the explicit expression of the Lamé operator and its resolvent after the Helmholtz decomposition, this is not entirely obvious. Indeed, as we will see a little further on in more details, the exploitation, at the very beginning, of the Helmholtz decomposition as a fundamental tool to address the problems, gives rise to new highly non-trivial difficulties.

I.2. The perturbation theory in the self-adjoint case

Even if, since we are working in a non self-adjoint setting, the majority of results in classical theory of perturbation which give precise information about whether or not (and if negative also how the changes occur) the preservation of the spectrum, or part of it, occurs cannot be used, we want to dedicate few rows to the description of some useful results that hold in the self-adjoint context. The reason for discussing this more investigated case is twofold. In first place, the theory of perturbation for self-adjoint operator, though classical, is not trivial. Secondly, it essentially embodies certain features of perturbation theory that may arise also in the general case. Moreover, since in this section we are not attempting to be exhaustive, the vast majority of details will be given for the Schrödinger operator (see [58] for a more comprehensive analysis on this topic).

Among several cornerstone results which take part to the classical theory of perturbations for self-adjoint linear operators, we must mention the Weyl's Theorem. In its general form it ensures that the essential spectrum of a self-adjoint operator turns out to be stable under relatively compact perturbations. In the particular situation in which the operator into account is represented by the self-adjoint Schrödinger operator $H = -\Delta + V$, this result has a more handle form, indeed it ensures that if the potential V decays sufficiently fast at infinity then the essential spectrum is preserved and in particular the following holds: $\sigma_{\text{ess}}(-\Delta + V) = \sigma_{\text{ess}}(-\Delta) = [0, \infty)$. This means that sufficiently decaying potentials do not change the essential spectrum, but may create discrete eigenvalues below it. Hence, in this framework, since the essential spectrum is easily determined, we are led to focus on the more particular issue to understand how potential perturbations influence and change the discrete spectrum.

Our problem can be re-phrase in this way:

QUESTION. *Which kind of assumptions about the potential perturbation ensures that*

$$\inf \sigma(-\Delta + V) = \inf \sigma(-\Delta) = 0?$$

In the event of affirmative answer then we would say that the discrete spectrum is also stable, therefore no negative eigenvalues can occur.

Let us start considering potential with definite sign. Clearly if $V \geqslant 0$ then, as $-\Delta$ is a non-negative operator (in sense of quadratic form, namely $\langle \psi, -\Delta\psi \rangle \geqslant 0$ for all $\psi \in \mathscr{D}(-\Delta)$), $H = -\Delta + V \geqslant 0$. Therefore, as a consequence of the mini/max principle which, we recall, gives a variational characterization of the eigenvalues below the bottom of the essential spectrum of an operator H in terms of the minimization problem for the functional $\langle \psi, H\psi \rangle$, $\inf \sigma(-\Delta + V) \geqslant 0$, this means that no negative eigenvalues arise.

Let us continue to consider the simpler case of definite sign potentials, namely $V \leqslant 0$. Since $-\Delta + V \leqslant -\Delta$, exploiting again mini/max principle we can say that $\inf \sigma(-\Delta + V) \leqslant \inf \sigma(-\Delta)$, but in general there is no reason for the inequality being strict.

It turns out that the fact the inequality is strict or not strongly depends on the dimension of the Euclidean space $\mathbb{R}^d$.

We consider first the dimension $d \geqslant 3$. As we will see, the discriminating factor is whether or not a Hardy-type inequality exists. A general Hardy-type inequality for functions in $H^1(\mathbb{R}^d)$ displays this form

$$\int_{\mathbb{R}^d} |\nabla\psi|^2 \, dx \geqslant \int_{\mathbb{R}^d} \rho|\psi|^2 \, dx, \quad \rho \geqslant 0, \tag{I.2.1}$$

where ρ is called Hardy weight.

We recall that in $d \geqslant 3$ the following inequality holds true:

$$\int_{\mathbb{R}^d} |\nabla\psi|^2 \, dx \geqslant \frac{(d-2)^2}{4} \int_{\mathbb{R}^d} \frac{|\psi|^2}{|x|^2} \, dx, \quad \forall\psi \in H^1(\mathbb{R}^d),$$

clearly, this is a particular case of the previous one in which we pick $\rho = \frac{(d-2)^2}{4}\frac{1}{|x|^2}$ From the previous it follows that

$$-\Delta - \frac{(d-2)^2}{4}\frac{1}{|x|^2} \geqslant 0,$$

again in sense of quadratic forms. This entails that if our potential decays at least as fast as $|x|^2$ at infinity and it is sufficiently small then the spectrum of the so perturbed Laplace operator remains empty. We attempt to be more precise: considering the operator

$$H = -\Delta + \varepsilon V,$$

with α to be defined, we have

$$-\Delta + \alpha V = \underbrace{-\Delta - \frac{(d-2)^2}{4}\frac{1}{|x|^2}}_{\substack{\geqslant 0 \\ \text{by Hardy-ineq.}}} + \underbrace{\frac{(d-2)^2}{4}\frac{1}{|x|^2} - \varepsilon|V|}_{\substack{\geqslant 0 \\ \text{if } V \text{ decays at least as } |x|^{-2} \\ \text{and} \\ \alpha \text{ sufficiently small}}}.$$

Hardy inequality shows that the kinetic term $-\Delta$ dominate at infinity if we consider a potential V that behaves at infinity as $V(x) = \varepsilon|x|^{-\beta}$ with $\beta > 2$, or if $\beta = 2$ and $\varepsilon < \frac{d-2}{2}$. In $d \geqslant 3$, this fact put $-\Delta$, in the class of the so-called *subcritical operators*, where we recall that

H is *subcritical* if for all V there exists $\varepsilon_0 > 0$ such that for all $\varepsilon \in [0, \varepsilon_0]$,

$$\inf \sigma(H + \varepsilon V) = \inf \sigma(H).$$

In low dimensions, namely $d = 1, 2$ the situation is completely different, indeed it can be proved that an arbitrarily small perturbation V always generates negative eigenvalue. This fact is mainly due to the lack of existence of a Hardy-type inequality. It is not difficult to see that if one assumes (I.2.1) to hold, then a proper choice of a sequence of test functions shows that ρ is forced to be identically equal to zero. Now we want to give a rough and quick idea on how to prove the so-called *criticality* of the operator $-\Delta$ in low dimensions, where we recall that

H is *critical* if for all V and for all $\varepsilon > 0$,

$$\inf \sigma(H + \varepsilon V) < \inf \sigma(H).$$

Substantially, using again mini/max principle, we want to prove that $-\Delta + \varepsilon V$ is negative for arbitrarily small ε. It is sufficient to build a test function ψ, such that $\|\nabla \psi\|^2 + \varepsilon \langle \psi, V\psi \rangle < 0$. The rigorous way to proceed would be to find a trial function ψ which resemble the constant function $\psi \equiv 1$, indeed with this choice $\nabla 1 = 0$ and $\varepsilon \int_{\mathbb{R}^d} V(x)\, dx < 0$.

Summing up, we have showed, if not in a rigorous way at least giving the main ideas, that the the following equivalence is valid:

$$-\Delta \,\text{is subcritical} \quad \Longleftrightarrow \quad \exists\, \text{Hardy inequality for } -\Delta.$$

I.3.| Preliminaries: Helmholtz decomposition and its consequences

This preliminary section is devoted to a deeper analysis of what has represented a fundamental tool for our scopes: the *Helmholtz decomposition.* As it is well known this is a standard way to decompose a vector field into a sum of a gradient and a divergence free vector field. To be more precise, we have that every smooth vector field u sufficiently rapidly decaying at infinity, can be uniquely decomposed as

$$u = u_S + u_P,$$

where $\operatorname{div} u_S = 0$ and $u_P = \nabla \varphi$, for some smooth scalar function φ.

A very useful property of the two components of the Helmholtz decomposition is summarized in the following lemma.

Lemma I.1. *Let u be a smooth vector field sufficiently rapidly decaying at infinity. Let u_S and u_P be the two components of the Helmholtz decomposition. Then*

- *u_S and u_P are L^2-orthogonal.*
- *u_S and u_P are H^1-orthogonal.*

Proof. The proof of both the sentences makes use of an integration by part argument. The first one immediately follows from the assumption about u_S and u_P.

Let us now consider the second sentence. In order to simplify the notation we call $F := u_S$ and $G := u_P$, thus F is the divergence free vector field and G is the gradient.

$$\langle \nabla F, \nabla G \rangle = \int_{\mathbb{R}^d} \nabla \overline{F} \cdot \nabla G = \sum_{j=1}^{d} \int_{\mathbb{R}^d} \nabla \overline{F}_j \cdot \nabla G_j = -\sum_{j=1}^{d} \int_{\mathbb{R}^d} \overline{F}_j \Delta G_j = -\sum_{j=1}^{d} \int_{\mathbb{R}^d} \overline{F}_j \Delta(\partial_j \varphi)$$
$$= \sum_{j=1}^{d} \int_{\mathbb{R}^d} \partial_j \overline{F}_j \Delta\varphi = \int_{\mathbb{R}^d} \operatorname{div} \overline{F}\, \Delta\varphi = 0.$$

□

Using the Helmholtz decomposition, a straightforward computation shows that for any $u = u_S + u_P$, the operator $-\Delta^*$ acts on u in this way

$$-\Delta^* u = -\mu\Delta u_S - (\lambda + 2\mu)\Delta u_P. \tag{I.3.1}$$

This makes evident the similarity between Lamé and Laplace operator, which primarily motivates our work.

As already said one of our main character in this part will be the resolvent operator associated to $-\Delta^*$, namely $(-\Delta^* - z)^{-1}$, for any $z \notin \sigma(-\Delta^*) = [0, \infty)$.

The following lemma let to a better understanding of this operator.

Lemma I.2. *Let $z \in \mathbb{C}\backslash[0, \infty)$ and $g \in [L^2(\mathbb{R}^d)]^d$, assume that λ and μ satisfy $\mu > 0, \lambda + 2\mu > 0$. Then the following identity holds:*

$$(-\Delta^* - z)^{-1} g = \frac{1}{\mu}\big(-\Delta - \tfrac{z}{\mu}\big)^{-1} g_S + \frac{1}{\lambda + 2\mu}\big(-\Delta - \tfrac{z}{\lambda+2\mu}\big)^{-1} g_P, \tag{I.3.2}$$

where

$$(-\Delta - z)^{-1} g = \big((-\Delta - z)^{-1} g_1, (-\Delta - z)^{-1} g_2, \dots, (-\Delta - z)^{-1} g_d\big).$$

Proof. Given $g \in [L^2(\mathbb{R}^d)]^d$, we want to obtain an explicit expression of the vector field f, where

$$f = (-\Delta^* - z)^{-1} g. \tag{I.3.3}$$

Let us observe that, since $z \notin \sigma(-\Delta^*) = [0, \infty)$, the previous is equivalent to

$$(-\Delta^* - z) f = g.$$

Now, writing $f = f_S + f_P$ and $g = g_S + g_P$ and using (I.3.1), we obtain that the previous can be re-written as

$$-\mu\Delta f_S - (\lambda + 2\mu)\Delta f_P - z f_S - z f_P = g_S + g_P.$$

The H^1- orthogonality of the two components of the decomposition enables us to split this intertwining equation for both the two components into a system of two decoupled equations, i.e

$$\begin{cases} -\mu\Delta f_S - z f_S = g_S, \\ -(\lambda+2\mu)\Delta f_P - z f_P = g_P. \end{cases}$$

The prior system can be written in an equivalent form as

$$\begin{cases} \mu\big(-\Delta - \frac{z}{\mu}\big) f_S = g_S, \\ (\lambda+2\mu)\big(-\Delta - \frac{z}{\lambda+2\mu}\big) f_P = g_P. \end{cases}$$

From the previous identities, since by our hypotheses about Lamé's parameters $\frac{z}{\mu}, \frac{z}{\lambda+2\mu} \notin \sigma(-\Delta)$, we get

$$f_S = \frac{1}{\mu}\big(-\Delta - \tfrac{z}{\mu}\big)^{-1} g_S, \qquad f_P = \frac{1}{\lambda+2\mu}\big(-\Delta - \tfrac{z}{\lambda+2\mu}\big)^{-1} g_P.$$

Making use of this explicit expressions in $f = f_S + f_P$ and of (I.3.3) we obtain (I.3.2). This concludes the proof of the lemma. □

I.4.| Problem 1

The discussion in this chapter is mainly taken from [19] and more precisely it is concerned with the proof of Theorem I.1 and Theorem I.1.6.

I.4.1. Absence of eigenvalues: proof of Theorem I.1

We devote this section to the proof of Theorem I.1 we stated in the introduction.

We recall that our strategy wants to be built in analogy to that one in the recent work [38] of Fanelli, Krejčiříc and Vega, who established the analogous result for the Laplace operator.

First of all, to this end, starting from the eigenvalue equation associated with the perturbed Laplacian, they provided three integral identities which had a crucial role in the proof of their main result; in order to do that they re-adapted to a non self-adjoint setting the standard technique of Morawetz multipliers. This tool was introduced in [71] for the Klein-Gordon equation and then it was developed in several other contexts. For example, with respect to the Helmholtz equation's framework, let us mention the seminal works of Perthame and Vega [77], [78] which are concerned with the purely electric case and then [39, 37, 92, 3, 6, 93], which extend the technique in an electromagnetic setting. We should also quote [15] for an adaptation of multipliers method on exterior domains.

Now we are in position to begin the proof of our result.

The eigenvalue problem for the perturbed Lamé operator is

$$\Delta^* u + ku = Vu, \tag{I.4.1}$$

where k is any complex constant (throughout the paper we will denote by $k_1 = \Re k$ and by $k_2 = \Im k$).

Just to simplify the notations, we start assuming that u is a solution of this more general problem

$$\Delta^* u + ku = f, \tag{I.4.2}$$

where $f \colon \mathbb{R}^d \to \mathbb{C}^d$ is a measurable function.

Clearly, we can identify the problem (I.4.2) with (I.4.1) by setting $f = Vu$.

As we said above, the Helmholtz decomposition has been a fundamental tool for our purposes, according to this, writing $u = u_S + u_P$ and $f = f_S + f_P$, the resolvent equation (I.4.2) associated to the Lamé operator can be re-written as

$$\mu\Delta u_S + (\lambda + 2\mu)\Delta u_P + ku_S + ku_P = f_S + f_P, \tag{I.4.3}$$

where, again, the S component is the divergence free vector field and the P component is the gradient.

Let us observe that the equation written in this form is very far to be easy to handle, indeed the two components has the same frequency of oscillation k but different speed of propagation μ and $\lambda + 2\mu$ respectively, and therefore the first attempt one would like to try is splitting the previous equation into a system of two decoupled equations involving separately the two components u_S and u_P. This attempt is going to work indeed, as a consequence of Lemma I.1 which guarantees the L^2-orthogonality of the S and P components of the Helmholtz decomposition and of their gradients, we are allowed to reduce our "intertwining" equation into a system of two decoupled equations, precisely one has the following result:

Lemma I.3. *Let $u = u_S + u_P$ be a solution to equation* (I.4.3)*, then the two components of the Helmholtz decomposition, u_S and u_P respectively, satisfies this two unrelated problems*

$$\begin{cases} \mu\Delta u_S + ku_S = f_S \\ (\lambda + 2\mu)\Delta u_P + ku_P = f_P. \end{cases} \tag{I.4.4}$$

Proof. As we have already said we basically are going to use the L^2 and H^1-orthogonality of u_S and u_P.

Since u is a solution to (I.4.3), clearly we have

$$\|\mu\Delta u_S + (\lambda + 2\mu)\Delta u_P + ku_S + ku_P - f_S - f_P\|^2 = 0,$$

or more explicitly

$$\int_{\mathbb{R}^d} \overline{(\mu\Delta u_S + (\lambda+2\mu)\Delta u_P + k u_S + k u_P - f_S - f_P)} \\ \cdot (\mu\Delta u_S + (\lambda+2\mu)\Delta u_P + k u_S + k u_P - f_S - f_P) = 0.$$

A straightforward computation allows us to write the previous as

$$\|\mu\Delta u_S + k u_S - f_S\|^2 + \|(\lambda+2\mu)\Delta u_P + k u_P - f_P\|^2 \\ + 2\Re \int \overline{(\mu\Delta u_S + k u_S - f_S)} \cdot ((\lambda+2\mu)\Delta u_P + k u_P - f_P) = 0.$$

In order to obtain the thesis is just needed to show that the third term is zero. Let us consider

$$I := \int \overline{(\mu\Delta u_S + k u_S - f_S)} \cdot ((\lambda+2\mu)\Delta u_P + k u_P - f_P)\,,$$

we can write this explicitly and we have

$$I = \mu(\lambda+2\mu)\int_{\mathbb{R}^d} \Delta\overline{u}_S \cdot \Delta u_P + \mu\, k \int_{\mathbb{R}^d} \Delta\overline{u}_S \cdot u_P - \mu \int_{\mathbb{R}^d} \Delta\overline{u}_S f_P + (\lambda+2\mu)\,\overline{k}\int_{\mathbb{R}^d} \overline{u}_S \cdot \Delta u_P \\ + |k|^2 \int_{\mathbb{R}^d} \overline{u}_S \cdot u_P - \overline{k}\int_{\mathbb{R}^d} \overline{u}_S \cdot f_P - (\lambda+2\mu)\int_{\mathbb{R}^d} \overline{f}_S \cdot \Delta u_P - k\int_{\mathbb{R}^d} \overline{f}_S \cdot u_P + \int_{\mathbb{R}^d} \overline{f}_S \cdot f_P. \tag{I.4.5}$$

The L^2-orthogonality of the S component and P component gives immediately that the first two and the last two integrals in the second row of (I.4.5) vanish. Thus one gets

$$I = \mu(\lambda+2\mu)\underbrace{\int_{\mathbb{R}^d} \Delta\overline{u}_S \cdot \Delta u_P}_{I_1} + \mu\, k \underbrace{\int_{\mathbb{R}^d} \Delta\overline{u}_S \cdot u_P}_{I_2} - \mu \underbrace{\int_{\mathbb{R}^d} \Delta\overline{u}_S f_P}_{I_3} \\ + (\lambda+2\mu)\,\overline{k}\underbrace{\int_{\mathbb{R}^d} \overline{u}_S \cdot \Delta u_P}_{I_4} - (\lambda+2\mu)\underbrace{\int_{\mathbb{R}^d} \overline{f}_S \cdot \Delta u_P}_{I_5}.$$

We are going to consider the five integrals separately. In order to simplify the details, again we use the notation adopted in Lemma I.1, that is $F := u_S$ and $G := u_P$, thus F is the divergence free vector field and G is the gradient.

$$I_1 = \int_{\mathbb{R}^d} \Delta\overline{F} \cdot \Delta G = \sum_{j=1}^{d} \int_{\mathbb{R}^d} \Delta\overline{F}_j\, \Delta G_j = \sum_{j=1}^{d} \int_{\mathbb{R}^d} \Delta\overline{F}_j\, \Delta\partial_j\varphi = -\int_{\mathbb{R}^d} \Delta\, \mathrm{div}\,\overline{F}\, \Delta\varphi = 0.$$

Now we see I_2, we omit the details for I_3, I_4 and I_5, indeed they could be handle in the same manner.

$$I_2 = \int_{\mathbb{R}^d} \Delta\overline{F} \cdot G = \sum_{j=1}^{d} \int_{\mathbb{R}^d} \Delta\overline{F}_j\, G_j = -\sum_{j=1}^{d} \int_{\mathbb{R}^d} \nabla\overline{F}_j \cdot \nabla G_j = (\nabla F, \nabla G) = 0.$$

Putting the previous altogether we obtain that $I = 0$ and consequently we have

$$\|\mu\Delta u_S + ku_S - f_S\|^2 + \|(\lambda+2\mu)\Delta u_P + ku_P - f_P\|^2 = 0$$

which obviously implies (I.4.4). □

As a starting point for the proof of Theorem I.1, we consider the weak formulation of (I.4.4)

$$\forall\, v \in [H^1(\mathbb{R}^d)]^d, \qquad \begin{cases} -\mu(\nabla v, \nabla u_S) + k(v, u_S) = (v, f_S) \\ -(\lambda+2\mu)(\nabla v, \nabla u_P) + k(v, u_P) = (v, f_P). \end{cases} \tag{I.4.6}$$

Following [6] we divide the proof of our result into two cases depending on the relation between real and imaginary part of the eigenvalue k: $|k_2| \leqslant k_1$ and $|k_2| > k_1$.

Let us start by the more technical case $|k_2| \leqslant k_1$.

Case $|k_2| \leqslant k_1$. For the purpose of letting the proof more understandable, we will point out in the following lemma what Fanelli, Krejčiříc and Vega have proved in their paper [38] as the main tool to guarantee the absence of eigenvalues for the perturbed Laplace operator.

Lemma I.4. *Let $u\colon \mathbb{R}^d \to \mathbb{C}$ be a solution to*

$$\Delta u + ku = f,$$

where k is any complex constants, we write $k_1 = \Re k$ and $k_2 = \Im k$ and $f\colon \mathbb{R}^d \to \mathbb{C}$ is a measurable function. If one sets

$$u^-(x) := e^{-i\,\mathrm{sgn}(k_2)k_1^{\frac{1}{2}}|x|}\, u(x)$$

the following estimate holds

$$\|\nabla u^-\|^2 + \frac{d-3}{d-1}\frac{|k_2|}{k_1^{\frac{1}{2}}}\int_{\mathbb{R}^d}|x||\nabla u^-|^2 \leqslant \frac{2(2d-3)}{d-2}\||x|f\|\|\nabla u^-\| + \frac{\sqrt{2}}{\sqrt{d-2}}\||x|f\|^{\frac{3}{2}}\|\nabla u^-\|^{\frac{1}{2}}. \tag{I.4.7}$$

Proof. As we have already mentioned in the introduction, this result is based on an appropriate use of the multipliers technique and we refer to [38] for the details of the proof. □

Remark I.4. Let us just underline that the main tool of the proof is an integration by parts argument, therefore in order to make the calculations rigorous they need to assume u and f sufficiently smooth and then the result will be obtained by a standard density argument.

At this point, the next step is, in some sense, obliged. Indeed the most natural way to proceed is to use directly the estimate which appears in Lemma I.4.7 for our two decoupled equations (I.4.4).

In order to do that we have to make the two equations independent of Lamé's coefficients, for this purpose we need to re-define appropriately k and f (differently in each equations). Precisely calling

$$k_S := \frac{k}{\mu}, \qquad g_S := \frac{f_S}{\mu} \tag{I.4.8}$$

and on the counterpart

$$k_P := \frac{k}{\lambda + 2\mu}, \qquad g_P := \frac{f_P}{\lambda + 2\mu}, \tag{I.4.9}$$

we have that u_S and u_P satisfies

$$\begin{cases} \Delta u_S + k_S\, u_S = g_S \\ \Delta u_P + k_P\, u_P = g_P, \end{cases}$$

that written in components clearly are

$$\begin{cases} \Delta (u_S)_j + k_S\, (u_S)_j = (g_S)_j \\ \Delta (u_P)_j + k_P\, (u_P)_j = (g_P)_j, \end{cases}$$

for all $j = 1, \dots, d$.

First let us handle the equation for u_S.

Setting

$$u_S^-(x) = e^{-i\,\mathrm{sgn}(k_{S,2}) k_{S,1}^{\frac{1}{2}} |x|} u_S(x), \tag{I.4.10}$$

where $k_{S,1} := \Re(k_S)$ and $k_{S,2} = \Im(k_S)$ and exploiting Lemma I.4.7, we have that

$$\begin{aligned} \|\nabla (u_S^-)_j\|^2 + \frac{d-3}{d-1} \frac{|k_{S,2}|}{k_{S,1}^{\frac{1}{2}}} \int_{\mathbb{R}^d} |x| |\nabla (u_S^-)_j|^2 \leqslant & \frac{2(2d-3)}{d-2} \||x| (g_S)_j\| \|\nabla (u_S^-)_j\| \\ & + \frac{\sqrt{2}}{\sqrt{d-2}} \||x| (g_S)_j\|^{\frac{3}{2}} \|\nabla (u_S^-)_j\|^{\frac{1}{2}}. \end{aligned}$$

Summing on $j = 1, \dots, d$ and using Cauchy-Schwartz and Hölder inequalities for descrete measures in the last two terms respectively, we get

$$\|\nabla u_S^-\|^2 + \frac{d-3}{d-1} \frac{|k_{S,2}|}{k_{S,1}^{\frac{1}{2}}} \int_{\mathbb{R}^d} |x| |\nabla u_S^-|^2 \leqslant \frac{2(2d-3)}{d-2} \||x| g_S\| \|\nabla u_S^-\| + \frac{\sqrt{2}}{\sqrt{d-2}} \||x| g_S\|^{\frac{3}{2}} \|\nabla u_S^-\|^{\frac{1}{2}}.$$

Going back to our old notation, i.e. recalling (I.4.8), it is easy to obtain

$$\begin{aligned} \|\nabla u_S^-\|^2 + \frac{1}{\sqrt{\mu}} \frac{d-3}{d-1} \frac{|k_2|}{k_1^{\frac{1}{2}}} \int_{\mathbb{R}^d} |x| |\nabla u_S^-|^2 \leqslant & \frac{1}{\mu} \frac{2(2d-3)}{d-2} \||x| f_S\| \|\nabla u_S^-\| \\ & + \frac{1}{\mu^{\frac{3}{2}}} \frac{\sqrt{2}}{\sqrt{d-2}} \||x| f_S\|^{\frac{3}{2}} \|\nabla u_S^-\|^{\frac{1}{2}}. \end{aligned} \tag{I.4.11}$$

Now we can provide the same estimate for the P component. Clearly, we mostly omit the details in fact these are the same we have already shown for the divergence free vector field u_S. We define u_P^- in the same way as u_S^-, precisely

$$u_P^-(x) := e^{-i\,\mathrm{sgn}(k_{P,2})k_{P,1}^{\frac{1}{2}}|x|}\, u_P(x), \tag{I.4.12}$$

where $k_{P,1} := \Re(k_P)$ and $k_{p,2} = \Im(k_P)$.

Now proceeding in the same way as the previous case we get

$$\begin{aligned} \|\nabla u_P^-\|^2 + \frac{1}{\sqrt{\lambda+2\mu}}\frac{d-3}{d-1}\frac{|k_2|}{k_1^{\frac{1}{2}}}\int_{\mathbb{R}^d}|x||\nabla u_P^-|^2 \leqslant & \frac{1}{\lambda+2\mu}\frac{2(2d-3)}{d-2}\||x|f_P\|\|\nabla u_P^-\| \\ & + \frac{1}{(\lambda+2\mu)^{\frac{3}{2}}}\frac{\sqrt{2}}{\sqrt{d-2}}\||x|f_P\|^{\frac{3}{2}}\|\nabla u_P^-\|^{\frac{1}{2}}. \end{aligned} \tag{I.4.13}$$

In order to complete the argument and to obtain the thesis, the following elliptic regularity lemma will be useful in the immediate sequel.

Lemma I.5. *Let $f \in [C_c^\infty(\mathbb{R}^d)]^d$ be a smooth-compactly supported vector field in $\mathbb{R}^d$, and let $\psi\colon \mathbb{R}^d \to \mathbb{C}$ be a smooth solution to*

$$\Delta\psi = \mathrm{div}\, f. \tag{I.4.14}$$

Then for any $s \in (-d,d)$ the following estimate holds

$$\||x|^s\nabla\psi\| \leqslant c\, Q_2(|x|^s)^2\||x|^s f\|,$$

for some constant $c>0$ only depending on the dimension d and $Q_2(|x|^s)$ the A_2-characteristic of the weight $|x|^s$ whose definition is recalled below.

Proof. The proof of this result basically relies on the very well known theorem about Calderón-Zygmund operator, which ensures that if T is an operator of Calderón-Zygmund type, which, roughly speaking, is a class of integral operator whose kernel has a singularity of the size $|x-y|^{-d}$ asymptotically as $|x-y|$ goes to zero, then for any weight w in the A_p-class, with $1<p<\infty$, T is bounded on the weighted space $L^p(w\,dx)$ (see for example [25], Thm. 7.11). Actually we are interested on a particular Calderón-Zygmund operator, i.e. the well known Riesz transform defined for any $f \in L^2(\mathbb{R}^d)$, via Fourier transform, by

$$\widehat{R_j f}(\xi) = -i\frac{\xi_j}{|\xi|}\hat{f}(\xi), \qquad \forall\, j = 1,2,\dots,d. \tag{I.4.15}$$

In particular we will use the following result of Petermichl which is concerned with the sharp bound for the operator norm of the Riesz transform in $L^2(w)$. He proved [79] that for all $j = 1,2,\dots d$, if $w \in A_2$ then

$$\|R_j f\|_{L^2(w)} \leqslant c\, Q_2(w)\|f\|_{L^2(w)}, \tag{I.4.16}$$

where $Q_2(w)$ is the A_2-characteristic of the weight w defined as

$$Q_2(w) := \sup_Q \Big(\frac{1}{|Q|}\int_Q w\Big)\Big(\frac{1}{|Q|}\int_Q w^{-1}\Big),$$

with Q any cube in $\mathbb{R}^d$ and c a constant only depending upon the dimension.

We are interested in the weighted boundedness of the Riesz transform because our operator $T(f) := \nabla\psi$, with ψ a solution to (I.4.14), can be written in terms of Riesz transform in this way: for all $j = 1, 2, \ldots, d$

$$\partial_j\psi = -(d-2)\sum_{k=1}^{d} R_j R_k F_k. \tag{I.4.17}$$

Indeed, using the fundamental solution of $-\Delta$ in $\mathbb{R}^d$, we can write

$$\psi = (-\Delta)^{-1}\operatorname{div} F = -\frac{1}{\omega_d}|x|^{-(d-2)} * \operatorname{div} F = -\frac{1}{\omega_d}\sum_{k=1}^{d}\partial_k|x|^{-(d-2)} * F_k,$$

where ω_d is the surface area of the d-dimensional unit sphere, precisely, making use of the gamma function, it is $\omega_d := \frac{2\pi^{d/2}}{\Gamma\left(\frac{d}{2}\right)}$. We are interested in the partial derivatives of ψ, that is

$$\partial_j\psi(x) = -\frac{1}{\omega_d}\sum_{k=1}^{d}\partial_j\partial_k|x|^{-(d-2)} * F_k(x).$$

Consequently, using that the Fourier transform of the homogeneous function $|x|^{-\alpha}$ is $\widehat{|x|^{-\alpha}}(\xi) := c_{d,\alpha}|\xi|^{\alpha-d}$, with $c_{d,\alpha} := \frac{2^{\frac{d}{2}-\alpha}\Gamma\left(\frac{d-\alpha}{2}\right)}{\Gamma\left(\frac{\alpha}{2}\right)}$ and that $\widehat{f*g} = (2\pi)^{d/2}\hat{f}\hat{g}$, we get

$$\widehat{\partial_j\psi}(\xi) = \frac{(2\pi)^{d/2}c_{d,d-2}}{\omega_d}\sum_{k=1}^{d}\xi_j\xi_k|\xi|^{-2}\hat{F}_k(\xi).$$

From the definition of the Riesz transform (I.4.15), given in terms of Fourier transform, it is straightforward to see that, for all $j,k = 1, 2\ldots, d$

$$\widehat{R_jR_kf}(\xi) = -\frac{\xi_j\xi_k}{|\xi|^2}\hat{f}(\xi),$$

this clearly means that

$$\widehat{\partial_j\psi}(\xi) = -\frac{(2\pi)^{d/2}c_{d,d-2}}{\omega_d}\sum_{k=1}^{d}\widehat{R_jR_kF_k}(\xi).$$

Now, using the linearity of the Fourier transform, antitrasforming the previous identity and making explicit the constants, we obtain our claim.

Moreover it can be proved that if $-d < s < d$ then $|x|^s$ belongs to A_2-class. Calling $w := |x|^s$, the statement of our lemma is equivalent to find a constant C such that

$$\|\nabla\psi\|_{[L^2(\mathbb{R}^d)(w)]^d} \leqslant C\|F\|_{[L^2(\mathbb{R}^d)(w)]^d}.$$

Putting all the previous facts together we get

$$\begin{aligned}\|\nabla\psi\|_{[L^2(\mathbb{R}^d)(w)]^d} &:= \Big(\sum_{j=1}^{d}\|\partial_j\psi\|^2_{L^2(\mathbb{R}^d)(w)}\Big)^{\frac{1}{2}} \leqslant d(d-2)\Big(\sum_{j,k=1}^{d}\|R_jR_kF_k\|^2_{L^2(\mathbb{R}^d)(w)}\Big)^{\frac{1}{2}}\\ &\leqslant d\sqrt{d}(d-2)c^2Q_2(w)^2\Big(\sum_{k=1}^{d}\|F_k\|^2_{L^2(\mathbb{R}^d)(w)}\Big)^{\frac{1}{2}}\\ &= d\sqrt{d}(d-2)c^2Q_2(w)^2\|F\|_{[L^2(\mathbb{R}^d)(w)]^d}.\end{aligned}$$

This concludes the proof.

Remark I.5. Let's underline that the constant c that appears in the previous equation is the one stated in the Petermichl result.

□

Let us introduce a trivial decomposition of our f :

$$f = f - \nabla\psi + \nabla\psi,$$

where ψ is the unique solution of (I.4.14); as a consequence we have $\operatorname{div}(f-\nabla\psi) = 0$. By the uniqueness of the Helmholtz decomposition, it follows that $f_S = f - \nabla\psi$, $f_P = \nabla\psi$. Substituting these in (I.4.11) and (I.4.13) respectively, one gets the two following estimates

$$\begin{aligned}\|\nabla u_S^-\|^2 + \frac{1}{\sqrt{\mu}}\frac{d-3}{d-1}\frac{|k_2|}{k_1^{\frac{1}{2}}}\int_{\mathbb{R}^d}|x||\nabla u_S^-|^2 \leqslant &\frac{1}{\mu}\frac{2(2d-3)}{d-2}(\||x|f\| + \||x|\nabla\psi\|)\|\nabla u_S^-\|\\ &+ \frac{1}{\mu^{\frac{3}{2}}}\frac{\sqrt{2}}{\sqrt{d-2}}(\||x|f\| + \||x|\nabla\psi\|)^{\frac{3}{2}}\|\nabla u_S^-\|^{\frac{1}{2}};\end{aligned}$$

and

$$\begin{aligned}\|\nabla u_P^-\|^2 + \frac{1}{\sqrt{\lambda+2\mu}}\frac{d-3}{d-1}\frac{|k_2|}{k_1^{\frac{1}{2}}}\int_{\mathbb{R}^d}|x||\nabla u_P^-|^2 \leqslant &\frac{1}{\lambda+2\mu}\frac{2(2d-3)}{d-2}\||x|\nabla\psi\|\|\nabla u_P^-\|\\ &+ \frac{1}{(\lambda+2\mu)^{\frac{3}{2}}}\frac{\sqrt{2}}{\sqrt{d-2}}\||x|\nabla\psi\|^{\frac{3}{2}}\|\nabla u_P^-\|^{\frac{1}{2}}.\end{aligned}$$

Using the elliptic regularity result I.5 we obtain respectively

$$\begin{aligned}\|\nabla u_S^-\|^2 + \frac{1}{\sqrt{\mu}}\frac{d-3}{d-1}\frac{|k_2|}{k_1^{\frac{1}{2}}}\int_{\mathbb{R}^d}|x||\nabla u_S^-|^2 \leqslant &\frac{1}{\mu}\frac{2(2d-3)}{d-2}(C+1)\||x|f\|\|\nabla u_S^-\|\\ &+ \frac{1}{\mu^{\frac{3}{2}}}\frac{\sqrt{2}}{\sqrt{d-2}}(C+1)^{\frac{3}{2}}\||x|f\|^{\frac{3}{2}}\|\nabla u_S^-\|^{\frac{1}{2}};\end{aligned}$$

and

$$\begin{aligned}\|\nabla u_P^-\|^2 + \frac{1}{\sqrt{\lambda+2\mu}}\frac{d-3}{d-1}\frac{|k_2|}{k_1^{\frac{1}{2}}}\int_{\mathbb{R}^d}|x||\nabla u_P^-|^2 \leqslant &\frac{1}{\lambda+2\mu}\frac{2(2d-3)}{d-2}C\||x|f\|\|\nabla u_P^-\|\\ &+ \frac{1}{(\lambda+2\mu)^{\frac{3}{2}}}\frac{\sqrt{2}}{\sqrt{d-2}}C^{\frac{3}{2}}\||x|f\|^{\frac{3}{2}}\|\nabla u_P^-\|^{\frac{1}{2}}.\end{aligned}$$

Recalling that, at the beginning, $f = Vu$ and using (I.1.3) one has

$$\||x|f\| = \||x|Vu\| \leqslant \||x|Vu_S\| + \||x|Vu_P\| \leqslant \Lambda\|\nabla u_S^-\| + \Lambda\|\nabla u_P^-\|.$$

By virtue of the previous inequality and using the convexity of the function $g(x) = |x|^p$ for $p \geqslant 1$ (in the inequality for the S component), we have

$$\begin{aligned}
&\|\nabla u_S^-\|^2 + \frac{1}{\sqrt{\mu}}\frac{d-3}{d-1}\frac{|k_2|}{k_1^{\frac{1}{2}}}\int_{\mathbb{R}^d}|x||\nabla u_S^-|^2 \\
&\qquad \leqslant \frac{\Lambda}{\mu}\frac{2(2d-3)}{d-2}(C+1)\|\nabla u_S^-\|^2 + \frac{\Lambda^{\frac{3}{2}}}{\mu^{\frac{3}{2}}}\frac{2}{\sqrt{d-2}}(C+1)^{\frac{3}{2}}\|\nabla u_S^-\|^2 \\
&\qquad + \frac{\Lambda}{\mu}\frac{2(2d-3)}{d-2}(C+1)\|\nabla u_S^-\|\|\nabla u_P^-\| + \frac{\Lambda^{\frac{3}{2}}}{\mu^{\frac{3}{2}}}\frac{2}{\sqrt{d-2}}(C+1)^{\frac{3}{2}}\|\nabla u_S^-\|^{\frac{1}{2}}\|\nabla u_P^-\|^{\frac{3}{2}};
\end{aligned}$$

and

$$\begin{aligned}
&\|\nabla u_P^-\|^2 + \frac{1}{\sqrt{\lambda+2\mu}}\frac{d-3}{d-1}\frac{|k_2|}{k_1^{\frac{1}{2}}}\int_{\mathbb{R}^d}|x||\nabla u_P^-|^2 \\
&\qquad \leqslant \frac{\Lambda}{\lambda+2\mu}\frac{2(2d-3)}{d-2}C\|\nabla u_P^-\|^2 + \frac{\Lambda^{\frac{3}{2}}}{(\lambda+2\mu)^{\frac{3}{2}}}\frac{2}{\sqrt{d-2}}C^{\frac{3}{2}}\|\nabla u_P^-\|^2 \\
&\qquad + \frac{\Lambda}{\lambda+2\mu}\frac{2(2d-3)}{d-2}C\|\nabla u_S^-\|\|\nabla u_P^-\| + \frac{\Lambda^{\frac{3}{2}}}{(\lambda+2\mu)^{\frac{3}{2}}}\frac{2}{\sqrt{d-2}}C^{\frac{3}{2}}\|\nabla u_S^-\|^{\frac{3}{2}}\|\nabla u_P^-\|^{\frac{1}{2}}.
\end{aligned}$$

Summing these two inequality together and majoring C with $C+1$, we obtain

$$\begin{aligned}
&\|\nabla u_S^-\|^2 + \|\nabla u_P^-\|^2 + \frac{1}{\sqrt{\mu}}\frac{d-3}{d-1}\frac{|k_2|}{k_1^{\frac{1}{2}}}\int_{\mathbb{R}^d}|x||\nabla u_S^-|^2 + \frac{1}{\sqrt{\lambda+2\mu}}\frac{d-3}{d-1}\frac{|k_2|}{k_1^{\frac{1}{2}}}\int_{\mathbb{R}^d}|x||\nabla u_P^-|^2 \\
&\leqslant \frac{\Lambda}{\mu}\frac{2(2d-3)}{d-2}(C+1)\|\nabla u_S^-\|^2 + \frac{\Lambda}{\lambda+2\mu}\frac{2(2d-3)}{d-2}(C+1)\|\nabla u_P^-\|^2 \\
&+\frac{\Lambda^{\frac{3}{2}}}{\mu^{\frac{3}{2}}}\frac{2}{\sqrt{d-2}}(C+1)^{\frac{3}{2}}\|\nabla u_S^-\|^2 + \frac{\Lambda^{\frac{3}{2}}}{(\lambda+2\mu)^{\frac{3}{2}}}\frac{2}{\sqrt{d-2}}(C+1)^{\frac{3}{2}}\|\nabla u_P^-\|^2 \\
&+\frac{\Lambda}{\mu}\frac{2(2d-3)}{d-2}(C+1)\|\nabla u_S^-\|\|\nabla u_P^-\| + \frac{\Lambda}{\lambda+2\mu}\frac{2(2d-3)}{d-2}(C+1)\|\nabla u_S^-\|\|\nabla u_P^-\| \\
&+\frac{\Lambda^{\frac{3}{2}}}{\mu^{\frac{3}{2}}}\frac{2}{\sqrt{d-2}}(C+1)^{\frac{3}{2}}\|\nabla u_S^-\|^{\frac{1}{2}}\|\nabla u_P^-\|^{\frac{3}{2}} + \frac{\Lambda^{\frac{3}{2}}}{(\lambda+2\mu)^{\frac{3}{2}}}\frac{2}{\sqrt{d-2}}(C+1)^{\frac{3}{2}}\|\nabla u_S^-\|^{\frac{3}{2}}\|\nabla u_P^-\|^{\frac{1}{2}}.
\end{aligned}$$

Making use of the Young's inequality, which state that for all non-negative real numbers a and b holds

$$ab \leqslant \frac{a^p}{p} + \frac{b^q}{q},$$

where p, q are determined by $\frac{1}{p} + \frac{1}{q} = 1$, one gets

$$\|\nabla u_S^-\|\|\nabla u_P^-\| \leqslant \frac{1}{2}\|\nabla u_S^-\|^2 + \frac{1}{2}\|\nabla u_P^-\|^2,$$

$$\|\nabla u_S^-\|^{\frac12}\|\nabla u_P^-\|^{\frac32} \leqslant \frac14\|\nabla u_S^-\|^2+\frac34\|\nabla u_P^-\|^2 \qquad \text{and} \qquad \|\nabla u_S^-\|^{\frac32}\|\nabla u_P^-\|^{\frac12} \leqslant \frac34\|\nabla u_S^-\|^2+\frac14\|\nabla u_P^-\|^2.$$

Using the latter in the former and the fact that $\mu, \lambda+2\mu \geqslant \min\{\mu, \lambda+2\mu\}$, we have

$$\left(1-\frac{\Lambda}{\min\{\mu,\lambda+2\mu\}}\frac{4(2d-3)}{d-2}(C+1)-\frac{\Lambda^{\frac32}}{\min\{\mu,\lambda+2\mu\}^{\frac32}}\frac{4}{\sqrt{d-2}}(C+1)^{\frac32}\right)(\|\nabla u_S^-\|^2+\|\nabla u_P^-\|^2)$$
$$+\frac{1}{\sqrt{\mu}}\frac{d-3}{d-1}\frac{|k_2|}{k_1^{\frac12}}\int_{\mathbb{R}^d}|x||\nabla u_S^-|^2+\frac{1}{\sqrt{\lambda+2\mu}}\frac{d-3}{d-1}\frac{|k_2|}{k_1^{\frac12}}\int_{\mathbb{R}^d}|x||\nabla u_P^-|^2 \leqslant 0.$$

Since the two term in the second row are positive, the last inequality becomes

$$\left(1-\frac{\Lambda}{\min\{\mu,\lambda+2\mu\}}\frac{4(2d-3)}{d-2}(C+1)-\frac{\Lambda^{\frac32}}{\min\{\mu,\lambda+2\mu\}^{\frac32}}\frac{4}{\sqrt{d-2}}(C+1)^{\frac32}\right)(\|\nabla u_S^-\|^2+\|\nabla u_P^-\|^2) \leqslant 0.$$

Clearly, by virtue of (I.1.4), the term in parenthesis is strictly positive, then it follows that u_S^-, u_P^- and thus u_S, u_P are identically equal to zero and, as a consequence of the Helmholtz decomposition, u is identically equal to zero as well.

We treat now the simpler case $|k_2| > k_1$.

Case $|k_2| > k_1$ let $u \in [H^1(\mathbb{R}^d)]^d$ be a solution of (I.4.2), i.e. a solution of (I.4.6). Choosing $v := \pm u_S$ in the first of (I.4.6) and $v := \pm u_P$ in the second of (I.4.6), taking real and imaginary parts of the resulting identities and summing these two identities, we obtain respectively for u_S and u_P

$$(k_1 \pm k_2)\int_{\mathbb{R}^d}|u_S|^2 = \mu\int_{\mathbb{R}^d}|\nabla u_S|^2 + \Re\int_{\mathbb{R}^d}\overline{u}_S\cdot f_S \pm \Im\int_{\mathbb{R}^d}\overline{u}_S\cdot f_S,$$

and

$$(k_1 \pm k_2)\int_{\mathbb{R}^d}|u_P|^2 = (\lambda+2\mu)\int_{\mathbb{R}^d}|\nabla u_P|^2 + \Re\int_{\mathbb{R}^d}\overline{u}_P\cdot f_P \pm \Im\int_{\mathbb{R}^d}\overline{u}_P\cdot f_P.$$

Taking the sum of the previous and making use of the H^1- orthogonality of u_S and u_P, one has

$$(k_1 \pm k_2)\int_{\mathbb{R}^d}|u|^2 = \mu\int_{\mathbb{R}^d}|\nabla u|^2 + (\lambda+\mu)\int_{\mathbb{R}^d}|\nabla u_P|^2 + \Re\int_{\mathbb{R}^d}\overline{u}\cdot f \pm \Im\int_{\mathbb{R}^d}\overline{u}\cdot f. \tag{I.4.18}$$

Now we want to estimate the last two terms on the right hand side of (I.4.18), in order to obtain the bound we are going to make use only of the Schwarz's inequality, the classical Hardy's inequality that reads

$$\forall\, \psi \in H^1(\mathbb{R}^d), \qquad \int_{\mathbb{R}^d}\frac{|\psi(x)|^2}{|x|^2}\,dx \leqslant \frac{4}{(d-2)^2}\int_{\mathbb{R}^d}|\nabla\psi|^2, \tag{I.4.19}$$

and the assumption (I.1.3). Indeed, recalling that $f := Vu$, one has

$$\int_{\mathbb{R}^d}|u||f| \leqslant \frac{2}{d-2}\Lambda\|\nabla u\|^2,$$

using the following trivial chains of inequalities

$$\Re \int_{\mathbb{R}^d} \overline{u} \cdot f \geqslant - \left| \int_{\mathbb{R}^d} \overline{u} \cdot f \right| \geqslant - \int_{\mathbb{R}^d} |u||f|, \qquad \text{and} \qquad \pm \Im \int_{\mathbb{R}^d} \overline{u} \cdot f \geqslant - \left| \int_{\mathbb{R}^d} \overline{u} \cdot f \right| \geqslant - \int_{\mathbb{R}^d} |u||f|,$$

we easily obtain

$$(k_1 \pm k_2) \int_{\mathbb{R}^d} |u|^2 \geqslant \left(\mu - \frac{4}{d-2} \Lambda \right) \|\nabla u\|^2 + (\lambda + \mu) \|\nabla u_P\|^2.$$

Let us recall that, to make the quadratic form associated to the Lamé operator positive, we have assumed for the Lamé coefficients the condition (I.1.2); under this hypothesis immediately follows that $\lambda + \mu > 0$ thus we obtain

$$(k_1 \pm k_2) \int_{\mathbb{R}^d} |u|^2 \geqslant \left(\mu - \frac{4}{d-2} \Lambda \right) \|\nabla u\|^2.$$

It's easy to see that any Λ verifying (I.1.4), necessarily satisfies $\frac{4}{d-2}\Lambda < \mu$, therefore one gets

$$(k_1 \pm k_2) \int_{\mathbb{R}^d} |u|^2 \geqslant 0.$$

Thus from the last inequality follows that $k_1 \pm k_2 \geqslant 0$, unless u is identically equal to zero.

It is a straightforward exercise to prove that, under conditions (I.1.3) and (I.1.4), the possible eigenvalues of $-\Delta^* + V$ have to be included in the right complex plane, that is $k_1 > 0$. Noticing that we are assuming $|k_2| > k_1 > 0$, which implies that the inequality $k_1 \pm k_2 \geqslant 0$ cannot hold, we obtain $u = 0$.

This concludes the proof of Theorem I.1. □

I.4.2. Uniform resolvent estimate

The aim of this section is to investigate about uniform resolvent estimate for the solution $u \colon \mathbb{R}^d \to \mathbb{C}^d$ of (I.1.5).

Just to quote a pair of papers on this topic, in a context of Helmholtz equation, we recall Burq, Planchon, Stalker and Tahvildar-Zadeh [11, 12] and later the work of Barceló, Vega and Zubeldia [6] which generalizes the previous to electromagnetic Hamiltonians. Whereas, for this kind of estimate in an elasticity setting, we can cite [5].

As we have already mentioned in the introduction, as a starting point we are going to prove a stronger result than Theorem I.2, which establishes the validity of a priori estimates, then our theorem will follows as a corollary making use of Hardy's inequality only.

In view of the previous comment, we can now start with the proof of Theorem I.3

I.4.2.1. Proof of Theorem I.3

Since the estimates we want to prove are different according to the relation between the real and imaginary part of the frequency, that is when $|k_2| \leqslant k_1$ or the contrary, we treat the two cases separately.

As a starting point, we will easily show that this kind of estimates holds in the free framework, that is in the setting in which $V = 0$. Secondly we prove the estimates in the perturbed case, assuming about V the same integral-smallness condition of Theorem I.1.

Case $|k_2| \leqslant k_1$ We consider the case $V = 0$. In this framework our equation (I.1.5) reduces to the one we considered in Theorem I.1, precisely (I.4.2). Throughout the proof of Theorem I.1, taking into account the Helmholtz decomposition, we proved for this equation the two estimates (I.4.11) and (I.4.13) respectively for the S and P component of the solution u of (I.4.2) that we are going to rewrite in order to clarify our argument. One had

$$\|\nabla u_S^-\|^2 + \frac{1}{\sqrt{\mu}}\frac{d-3}{d-1}\frac{|k_2|}{k_1^{\frac{1}{2}}}\int_{\mathbb{R}^d}|x||\nabla u_S^-|^2 \leqslant \frac{1}{\mu}\frac{2(2d-3)}{d-2}\||x|f_S\|\|\nabla u_S^-\| + \frac{1}{\mu^{\frac{3}{2}}}\frac{\sqrt{2}}{\sqrt{d-2}}\||x|f_S\|^{\frac{3}{2}}\|\nabla u_S^-\|^{\frac{1}{2}}, \quad \text{(I.4.20)}$$

and

$$\|\nabla u_P^-\|^2 + \frac{1}{\sqrt{\lambda+2\mu}}\frac{d-3}{d-1}\frac{|k_2|}{k_1^{\frac{1}{2}}}\int_{\mathbb{R}^d}|x||\nabla u_P^-|^2 \leqslant \frac{1}{(\lambda+2\mu)}\frac{2(2d-3)}{d-2}\||x|f_P\|\|\nabla u_P^-\| + \frac{1}{(\lambda+2\mu)^{\frac{3}{2}}}\frac{\sqrt{2}}{\sqrt{d-2}}\||x|f_P\|^{\frac{3}{2}}\|\nabla u_P^-\|^{\frac{1}{2}}.$$

Let us consider the first inequality only, the details for the second one will be similar.

We want to estimate the right hand side of the inequality, to this end, let ε, $\delta > 0$, making use of the Young's inequality one has

$$\||x|f_S\|\|\nabla u_S^-\| \leqslant \frac{1}{2\varepsilon^2}\||x|f_S\|^2 + \frac{\varepsilon^2}{2}\|\nabla u_S^-\|^2 \quad \text{and} \quad \||x|f_S\|^{\frac{3}{2}}\|\nabla u_S^-\|^{\frac{1}{2}} \leqslant \frac{3}{4\delta^{\frac{4}{3}}}\||x|f_S\|^2 + \frac{\delta^4}{4}\|\nabla u_S^-\|^2.$$

Putting this two in (I.4.20) and observing that the quantity $\frac{1}{\sqrt{\mu}}\frac{|k_2|}{k_1^{\frac{1}{2}}}\frac{d-3}{d-1}\int_{\mathbb{R}^d}|x||\nabla u_S^-|^2$ is positive, we get

$$\|\nabla u_S^-\|^2 \leqslant \frac{1}{\mu}\frac{1}{\varepsilon^2}\frac{2d-3}{d-2}\||x|f_S\|^2 + \varepsilon^2\frac{1}{\mu}\frac{2d-3}{d-2}\|\nabla u_S^-\|^2 + \frac{3}{4\delta^{\frac{4}{3}}}\frac{1}{\mu^{\frac{3}{2}}}\frac{\sqrt{2}}{\sqrt{d-2}}\||x|f_S\|^2 + \frac{\delta^4}{4}\frac{1}{\mu^{\frac{3}{2}}}\frac{\sqrt{2}}{\sqrt{d-2}}\|\nabla u_S^-\|^2.$$

Thus it may be concluded that

$$\left(1-\varepsilon^2\frac{1}{\mu}\frac{2d-3}{d-2}-\frac{\delta^4}{4}\frac{1}{\mu^{\frac{3}{2}}}\frac{\sqrt{2}}{\sqrt{d-2}}\right)\|\nabla u_S^-\|^2 \leqslant \left(\frac{1}{\mu}\frac{1}{\varepsilon^2}\frac{2d-3}{d-2}+\frac{3}{4\delta^{\frac{4}{3}}}\frac{1}{\mu^{\frac{3}{2}}}\frac{\sqrt{2}}{\sqrt{d-2}}\right)\||x|f_S\|^2.$$

The same calculations done for the P component give

$$\left(1-\varepsilon^2\frac{1}{\lambda+2\mu}\frac{2d-3}{d-2}-\frac{\delta^4}{4}\frac{1}{(\lambda+2\mu)^{\frac{3}{2}}}\frac{\sqrt{2}}{\sqrt{d-2}}\right)\|\nabla u_P^-\|^2$$
$$\leqslant \left(\frac{1}{\lambda+2\mu}\frac{1}{\varepsilon^2}\frac{2d-3}{d-2}+\frac{3}{4\delta^{\frac{4}{3}}}\frac{1}{(\lambda+2\mu)^{\frac{3}{2}}}\frac{\sqrt{2}}{\sqrt{d-2}}\right)\||x|f_P\|^2.$$

Now, since $\mu, \lambda+2\mu \geqslant \min\{\mu, \lambda+2\mu\}$ and choosing ε, δ small enough, one can write

$$\|\nabla u_S^-\| \leqslant D_{\varepsilon,\delta}\||x|f_S\|,$$

and

$$\|\nabla u_P^-\| \leqslant D_{\varepsilon,\delta}\||x|f_P\|,$$

where

$$D_{\varepsilon,\delta} = \left(\frac{\frac{1}{\min\{\mu,\lambda+2\mu\}}\frac{1}{\varepsilon^2}\frac{2d-3}{d-2}+\frac{3}{4\delta^{\frac{4}{3}}}\frac{1}{\min\{\mu,\lambda+2\mu\}^{\frac{3}{2}}}\frac{\sqrt{2}}{\sqrt{d-2}}}{1-\varepsilon^2\frac{1}{\min\{\mu,\lambda+2\mu\}}\frac{2d-3}{d-2}-\frac{\delta^4}{4}\frac{1}{\min\{\mu,\lambda+2\mu\}^{\frac{3}{2}}}\frac{\sqrt{2}}{\sqrt{d-2}}}\right)^{\frac{1}{2}}.$$

At the end, using the trivial Helmholtz decomposition of $f = f - \nabla\psi + \nabla\psi$ and the elliptic regularity Lemma I.5, one easily concludes

$$\|\nabla u_S^-\| \leqslant c\||x|f\| \qquad \text{and} \qquad \|\nabla u_P^-\| \leqslant c\||x|f\|,$$

where $c := (C+1)D_{\varepsilon,\delta}$.

Let us remark that $c>0$ does not depend on the frequency k and on f.

Now we can prove our result in the perturbed setting, namely $V \neq 0$.

First of all we define $g := Vu$ and $h := f + g$. Thus, with this notation, u solves the following equation

$$\Delta^* u + ku = h; \tag{I.4.21}$$

again we have these estimates for the two components of the solution:

$$\|\nabla u_S^-\|^2 \leqslant \frac{1}{\mu}\frac{2(2d-3)}{d-2}\||x|h_S\|\|\nabla u_S^-\| + \frac{1}{\mu^{\frac{3}{2}}}\frac{\sqrt{2}}{\sqrt{d-2}}\||x|h_S\|^{\frac{3}{2}}\|\nabla u_S^-\|^{\frac{1}{2}},$$

and

$$\|\nabla u_P^-\|^2 \leqslant \frac{1}{\lambda+2\mu}\frac{2(2d-3)}{d-2}\||x|h_P\|\|\nabla u_P^-\| + \frac{1}{(\lambda+2\mu)^{\frac{3}{2}}}\frac{\sqrt{2}}{\sqrt{d-2}}\||x|h_P\|^{\frac{3}{2}}\|\nabla u_P^-\|^{\frac{1}{2}}.$$

We only consider the first one. Clearly, since $h_S = f_S + g_S$, it can be rewritten as

$$\begin{aligned}\|\nabla u_S^-\|^2 \leqslant & \frac{1}{\mu}\frac{2(2d-3)}{d-2}\||x|f_S\|\|\nabla u_S^-\| + \frac{1}{\mu^{\frac{3}{2}}}\frac{2}{\sqrt{d-2}}\||x|f_S\|^{\frac{3}{2}}\|\nabla u_S^-\|^{\frac{1}{2}} \\ & + \frac{1}{\mu}\frac{2(2d-3)}{d-2}\||x|g_S\|\|\nabla u_S^-\| + \frac{1}{\mu^{\frac{3}{2}}}\frac{2}{\sqrt{d-2}}\||x|g_S\|^{\frac{3}{2}}\|\nabla u_S^-\|^{\frac{1}{2}}.\end{aligned}$$

Since in the free case we have already bounded the terms in which f appears, now let us only consider the terms involving g.

We introduce the trivial decomposition of $g = g - \nabla\phi + \nabla\phi$, where, as usual, ϕ is the unique solution of the elliptic problem $\Delta\phi = \operatorname{div} g$. Following the strategy in Theorem I.1 about the absence of eigenvalues and, in particular, recalling that formerly $g = Vu$ and that V satisfies (I.1.3), one can show

$$\begin{aligned}\|\nabla u_S^-\|^2 \leqslant & \frac{1}{\mu}\frac{2(2d-3)}{d-2}\||x|f_S\|\|\nabla u_S^-\| + \frac{1}{\mu^{\frac{3}{2}}}\frac{2}{\sqrt{d-2}}\||x|f_S\|^{\frac{3}{2}}\|\nabla u_S^-\|^{\frac{1}{2}} \\ & + \frac{\Lambda}{\mu}\frac{2(2d-3)}{d-2}(C+1)\|\nabla u_S^-\|^2 + \frac{\Lambda^{\frac{3}{2}}}{\mu^{\frac{3}{2}}}\frac{2\sqrt{2}}{\sqrt{d-2}}(C+1)^{\frac{3}{2}}\|\nabla u_S^-\|^2 \\ & + \frac{\Lambda}{\mu}\frac{2(2d-3)}{d-2}(C+1)\|\nabla u_S^-\|\|\nabla u_P^-\| + \frac{\Lambda^{\frac{3}{2}}}{\mu^{\frac{3}{2}}}\frac{2\sqrt{2}}{\sqrt{d-2}}(C+1)^{\frac{3}{2}}\|\nabla u_S^-\|^{\frac{1}{2}}\|\nabla u_P^-\|^{\frac{3}{2}}.\end{aligned}$$

For the P component we have the following analogue estimate

$$\begin{aligned}\|\nabla u_P^-\|^2 \leqslant & \frac{1}{\lambda+2\mu}\frac{2(2d-3)}{d-2}\||x|f_P\|\|\nabla u_P^-\| + \frac{1}{(\lambda+2\mu)^{\frac{3}{2}}}\frac{2}{\sqrt{d-2}}\||x|f_P\|^{\frac{3}{2}}\|\nabla u_P^-\|^{\frac{1}{2}} \\ & + \frac{\Lambda}{\lambda+2\mu}\frac{2(2d-3)}{d-2}C\|\nabla u_P^-\|^2 + \frac{\Lambda^{\frac{3}{2}}}{(\lambda+2\mu)^{\frac{3}{2}}}\frac{2\sqrt{2}}{\sqrt{d-2}}C^{\frac{3}{2}}\|\nabla u_P^-\|^2 \\ & + \frac{\Lambda}{\lambda+2\mu}\frac{2(2d-3)}{d-2}C\|\nabla u_S^-\|\|\nabla u_P^-\| + \frac{\Lambda^{\frac{3}{2}}}{(\lambda+2\mu)^{\frac{3}{2}}}\frac{2\sqrt{2}}{\sqrt{d-2}}C^{\frac{3}{2}}\|\nabla u_S^-\|^{\frac{3}{2}}\|\nabla u_P^-\|^{\frac{1}{2}}.\end{aligned}$$

Now estimating the terms involving f as in the free case, summing these inequalities, and using the Young's inequality, we obtain

$$\begin{aligned}&\Bigg(1 - \frac{\Lambda}{\min\{\mu,\,\lambda+2\mu\}}\frac{4(2d-3)}{d-2}(C+1) - \frac{\Lambda^{\frac{3}{2}}}{\min\{\mu,\,\lambda+2\mu\}^{\frac{3}{2}}}\frac{4\sqrt{2}}{\sqrt{d-2}}(C+1)^{\frac{3}{2}} \\ & -\varepsilon^2\frac{1}{\min\{\mu,\,\lambda+2\mu\}}\frac{2d-3}{d-2} - \frac{\delta^4}{2}\frac{1}{\min\{\mu,\,\lambda+2\mu\}^{\frac{3}{2}}}\frac{1}{\sqrt{d-2}}\Bigg)(\|\nabla u_S^-\|^2 + \|\nabla u_P^-\|^2) \\ & \leqslant \left(\frac{1}{\min\{\mu,\lambda+2\mu\}}\frac{1}{\varepsilon^2}\frac{2d-3}{d-2} + \frac{3}{2\delta^{\frac{4}{3}}}\frac{1}{\min\{\mu,\lambda+2\mu\}^{\frac{3}{2}}}\frac{1}{\sqrt{d-2}}\right)(\||x|f_S\|^2 + \||x|f_P\|^2)\end{aligned}$$

Since V satisfies (I.1.3) and assuming ε, δ sufficiently small, the constant in the left hand side of the previous inequality is positive, thus we can write

$$(\|\nabla u_S^-\|^2 + \|\nabla u_P^-\|^2) \leqslant D_{\varepsilon,\delta}^2(\||x|f_S\|^2 + \||x|f_P\|^2),$$

where, obviously, $D^2_{\varepsilon,\delta}$ is the ratio between the two constants which respectively appear on the right and on the left hand side of the last but one inequality.

Using now the trivial Helmholtz decomposition of $f = f - \nabla\psi + \nabla\psi$ and the elliptic regularity Lemma I.5, one easily has

$$\|\nabla u_S^-\|^2 + \|\nabla u_P^-\|^2 \leqslant c^2 \||x|f\|^2,$$

where

$$c^2 := 2(C+1)^2 D^2_{\varepsilon,\delta}$$

does not depend on the frequency k and on f. Moreover, it is clear that the following hold

$$\|\nabla u_S^-\| \leqslant c\||x|f\| \qquad \text{and} \qquad \|\nabla u_P^-\| \leqslant c\||x|f\|.$$

Now we can treat the less technical case.

Case $|k_2| > k_1$.

First we consider the free setting. As the previous case, our equation (I.1.5) becomes the one we have considered in Theorem I.1, precisely (I.4.2). Choosing $v = u_S$ in the first of (I.4.6) and $v = u_P$ in the second and taking the real part of the resulting identity one obtains respectively

$$k_1 \int_{\mathbb{R}^d} |u_S|^2 - \mu \int_{\mathbb{R}^d} |\nabla u_S|^2 = \Re \int_{\mathbb{R}^d} \overline{u}_S \cdot f_S$$

and

$$k_1 \int_{\mathbb{R}^d} |u_P|^2 - (\lambda + 2\mu) \int_{\mathbb{R}^d} |\nabla u_P|^2 = \Re \int_{\mathbb{R}^d} \overline{u}_P \cdot f_P.$$

Taking the sum of the previous and making use of the L^2 and H^1-orthogonality of u_S and u_P, one has

$$k_1 \int_{\mathbb{R}^d} |u|^2 - \mu \int_{\mathbb{R}^d} |\nabla u|^2 - (\lambda+\mu) \int_{\mathbb{R}^d} |\nabla u_P|^2 = \Re \int_{\mathbb{R}^d} \overline{u} \cdot f.$$

Starting again from the weak formulation (I.4.6), choosing $v = \frac{k_2}{|k_2|} u_S$ in the first and $v = \frac{k_2}{|k_2|} u_P$ in the second, taking the imaginary part and then summing the resulting identities, one obtains

$$|k_2| \int_{\mathbb{R}^d} |u|^2 \leqslant \int_{\mathbb{R}^d} |u||f|.$$

Using the latter in the former (here we need the assumption $|k_2| > k_1$) and observing the positivity of the term $(\lambda+\mu)\int_{\mathbb{R}^d} |\nabla u_P|^2$, we have

$$\mu \int_{\mathbb{R}^d} |\nabla u|^2 \leqslant 2 \int_{\mathbb{R}^d} |u||f|.$$

From the Cauchy Schwarz and Hardy's inequalities follows

$$\mu \|\nabla u\|^2 \leqslant \frac{4}{d-2} \||x|f\| \|\nabla u\|.$$

Thus, it may be concluded that

$$\|\nabla u\| < \frac{1}{\mu}\frac{4}{d-2}\||x|f\|.$$

We now proceed to show the a-priori estimates in the perturbed context. Exploiting the same notation we have used in the case $|k_2| \leqslant k_1$, again u solves the equation (I.4.21). As a consequence of the estimates we have just proved for the free case, recalling that $h = f + g$, one easily obtains

$$\|\nabla u\| < \frac{1}{\mu}\frac{4}{d-2}\||x|h\| \leqslant \frac{1}{\mu}\frac{4}{d-2}\||x|f\| + \frac{1}{\mu}\frac{4}{d-2}\||x|g\|.$$

Writing now explicitly g as Vu, by assumption (I.1.3) we have

$$\|\nabla u\| < \frac{1}{\mu}\frac{4}{d-2}\||x|f\| + \frac{\Lambda}{\mu}\frac{4}{d-2}\|\nabla u\|$$

or, more explicitly

$$\left(1 - \frac{\Lambda}{\mu}\frac{4}{d-2}\right)\|\nabla u\| < \frac{1}{\mu}\frac{4}{d-2}\||x|f\|.$$

The condition (I.1.4) about Λ guarantees the positivity of the parenthesis of the left hand side and then the theorem is proved.

Finally, we are in position to prove the uniform resolvent estimate we are looking for.

I.4.2.2. Proof of Theorem I.2

First of all, we consider the case $|k_2| \leqslant k_1$, as a consequence of (I.1.7), making use of the Hardy's inequality, it is not difficult to show that the following chain of inequalities holds

$$\begin{aligned}\||x|^{-1}u\| &\leqslant \||x|^{-1}u_S^-\| + \||x|^{-1}u_P^-\| \leqslant \frac{2}{d-2}(\|\nabla u_S^-\| + \|\nabla u_P^-\|)\\ &\leqslant \frac{4c}{d-2}\||x|f\|.\end{aligned}$$

Assuming $|k_2| > k_1$, using (I.1.8) and again the Hardy's inequality, we have

$$\||x|^{-1}u\| \leqslant \frac{2c}{d-2}\||x|f\|.$$

I.5.| Problem 2

This chapter is devoted to the proof of the Theorem I.4 - Theorem I.7.

Before moving on in the proof of our results, we would like to give a (not comprehensive) overview concerning the issue of spectral bounds.

As one can see, we will outline the main results obtained in this topic over the years but just for the Schrödinger operators. Indeed, if the literature in the context of spectral bounds for Schrödinger is more than abundant, the one for the perturbed Lamé operator is almost absent. Nevertheless by virtue of the likeness of the two operators highlighted by Helmholtz decomposition, also the only state of the art for Schrödinger is worthy to be recalled.

I.5.1. Historical Background

In this section we will treat separately the state of the art for self-adjoint situation and the less developed non self-adjoint one, in that order.

We will focus our attention in considering some fundamental inequalities in this topic that go under the "umbrella" name of Lieb-Thirring inequalities, providing their classical statement and the successive generalizations.

The standard Lieb-Thirring inequalities, named after E.H. Lieb and W.E. Thirring, give an upper bound on the sums of powers of the absolute value of the negative eigenvalues of a Schrödinger operators in terms of integrals of the potential, that, in the original formulation, is assumed to be real-valued. This sets a self-adjoint framework. More precisely, considering $H = -\Delta + V(x)$ on $L^2(\mathbb{R}^d)$, $d \geqslant 1$ and denoting with $e_1 \leqslant e_2 \leqslant \cdots < 0$ the negative eigenvalues of H (if any), the Lieb-Thirring inequalities state that for suitable constants $L_{\gamma,d}$ the following holds:

$$\sum_{j \geqslant 1} |e_j|^\gamma \leqslant L_{\gamma,d} \int_{\mathbb{R}^d} V_-(x)^{\gamma + \frac{d}{2}}\, dx, \tag{I.5.1}$$

with $V_-(x) := \max\{-V(x), 0\}$, for any γ satisfying

$$\begin{aligned} \gamma &\geqslant \frac{1}{2} \quad \text{if } d = 1, \\ \gamma &> 0 \quad \text{if } d = 2, \\ \gamma &\geqslant 0 \quad \text{if } d \geqslant 3. \end{aligned}$$

The proof of the previous inequalities in the cases $\gamma > \frac{1}{2}$, $d = 1$ and $\gamma > 0$, $d \geqslant 2$ were covered by E.H. Lieb and W.E. Thirring in [64] in connection with their proof of stability of matter. The case $\gamma = \frac{1}{2}$, $d = 1$ was established by T. Weidl in [89]. The further endpoint case $\gamma = 0$, $d \geqslant 3$ was independently obtained by Rozenblyum [82, 83], Cwikel [20] and Lieb [63] by different methods and is usually referred to as the Rozenblyum-Cwikel-Lieb inequality.

Some comments on the previous inequality follow.

Remark I.6. The relevance of this kind of spectral bounds, at least at the birth, comes from physics and, in particular, from quantum mechanics. A sizable role among the estimates (I.5.1) is played by the case $\gamma = 1$. With this choice an upper bound for the sum of the absolute value of negative eigenvalues, that is a lower bound on the sum of negative eigenvalues, namely $\sum_{j\geqslant 1} e_j$, is given. This was one of the essential ingredients in Lieb's and Thirring's proof of the stability of matter: indeed since the energy of the ground state of a system of N interacting fermions is $\sum_{j\geqslant 1} e_j$, from the previous estimate turns out that this quantity can be estimated from below by the integral of the negative part of the potential to the power $1+\frac{d}{2}$, guaranteeing the stability.

Remark I.7. Let us notice that only the negative part of V, namely V_-, plays a role in the Lieb-Thirring inequalities. Of course, since $-\Delta$ is a non-negative operator, if V is also non-negative then $-\Delta + V \geqslant 0$ and therefore no negative eigenvalues can occur. If V change its sign, that is if both the positive and negative part of $V = V_+ - V_-$ are non-trivial, clearly both parts influence negative eigenvalues. On the other hand, one can observe that, since $V \geqslant -V_-$, in particular $-\Delta + V \geqslant -\Delta - V_-$ and by virtue of the mini/max principle an upper estimate for the sum of a suitable power of the absolute values of the negative eigenvalues of $-\Delta - V_-$ provides automatically the same upper estimate for the negative eigenvalues of the complete hamiltonian $-\Delta + V$, indeed the effect of V_+ on the negative eigenvalues is only to increase their size.

Remark I.8. It is not difficult to see that if e is an eigenvalue of $-\Delta + V(x)$ with eigenfunction ψ, then $\phi_\lambda(\cdot) := \psi(\lambda\,\cdot)$ is an eigenfunction of $-\Delta + V_\lambda(x)$ where $V_\lambda(\cdot) = \lambda^2 V(\lambda\,\cdot)$ with eigenvalue $\lambda^2 e$. By a simple scaling this gives that $p = \gamma + \frac{d}{2}$ is the only possible exponent for which a inequality of the following type

$$\sum_{j\geqslant 1} |e_j|^\gamma \leqslant L_{\gamma,d} \int_{\mathbb{R}^d} V_-(x)^p \, dx$$

can hold.

Remark I.9. Let us underline that there are "natural" constraints on the soundness of inequalities of type (I.5.1). We emphasize a pathological behavior in dimension $d = 1, 2$. First of all, regardless of the dimension, since we are assuming V to vanish at infinity, $\sigma_{ess}(-\Delta + V(x)) = \sigma_{ess}(-\Delta) = [0, \infty)$. In some sense this means that the non-negative spectrum is easily determined. Therefore the question is deflected to a deeper analysis of the negative part. It is well known (look at chapter I.2) that from the criticality of $-\Delta$ or, in other words, due to the lack of a Hardy-type inequality in low dimension, namely $d = 1, 2$, for any attractive potentials V, that is V non-trivial and $V \leqslant 0$ (beyond the request to vanish at infinity), negative bound states always exists (actually the assumption V to be attractive can be weakened requiring V

to be just "attractive in the mean", that is $\int_{\mathbb{R}^d} V(x)\,dx < 0$). On the other hand if an inequality of the form (I.5.1) with $\gamma = 0$ holds, we would have that the left-hand side turns out to be the counting function of negative eigenvalues and therefore is a positive integer for any such potential. On the contrary, the right-hand side can be made arbitrarily small, indeed is sufficient to assume $\int_{\mathbb{R}^d} V_-(x)^{\frac{d}{2}}\,dx < L_{0,d}^{-1}$ to obtain an evident contradiction.

Remark I.10. Even if common knowledge, we want to show, at least in two very simple cases, the important and deep relation between the possibility to obtain bounds for eigenvalues of the Schrödinger operators and the validity of Sobolev inequality

$$\int_{\mathbb{R}^d} |\nabla u|^2\,dx \geqslant S_d \Big(\int_{\mathbb{R}^d} |u|^q \Big)^{\frac{2}{q}}, \qquad q = \frac{2d}{d-2}, \quad d \geqslant 3, \tag{I.5.2}$$

in the self-adjoint context.

- We consider first the Rozenblyum-Cwikel-Lieb inequality, that is (I.5.1) with $\gamma = 0$. This can be explicitly written as

  $$\mathcal{N}_-(-\Delta + V) \leqslant L_{0,d} \int_{\mathbb{R}^d} V_-(x)^{\frac{d}{2}}\,dx, \tag{I.5.3}$$

  where $\mathcal{N}_-(-\Delta + V)$ is a notation for the number of negative eigenvalues (if any) of the operator $-\Delta + V$.

  Even if this inequality was first discovered by Rozenblum, Cwikel and Lieb, afterwards other proofs of (I.5.3) were given. It is worth mentioning the proof of Li and Yau [69], indeed it relies only upon the Sobolev inequality and the positivity of the heat kernel. Then Levin and Solomyak [62] generalized the strategy in the aforementioned work in order to obtain, under suitable Markov condition, the equivalence between R-C-L and the Sobolev inequality.

  Now we use partial tools from [62] in order to make less theoretical the relation between the two inequalities. More precisely we will see first that Sobolev inequality provides a condition for the absence of negative eigenvalues of $-\Delta + V$ in $d \geqslant 3$. After that we will show that the absence of eigenvalues follows, "sub conditionem", from the R-C-L. This means that, in essence, we would have performed an equivalence between Sobolev inequality and a weaker form of R-C-L.

  Proposition I.2. *Let $d \geqslant 3$, $q = \frac{2d}{d-2}$ and assume $V = -V_-$ such that $\|V_-\|_{\frac{d}{2}} \leqslant S_d$, then*

  $$\int_{\mathbb{R}^d} |\nabla u|^2\,dx \geqslant S_d \Big(\int_{\mathbb{R}^d} |u|^q \Big)^{\frac{2}{q}} \quad \Longleftrightarrow \quad \text{there are no negative eigenvalues for } -\Delta - V_-.$$

  Proof. The proof proceeds with the following steps:

1. We will prove that

$$\int_{\mathbb{R}^d} |\nabla u|^2 \, dx \geqslant S_d \Big(\int_{\mathbb{R}^d} |u|^q \Big)^{\frac{2}{q}} \quad \Longleftrightarrow \quad \int_{\mathbb{R}^d} V_- |u|^2 \, dx \leqslant \int_{\mathbb{R}^d} |\nabla u|^2$$

2. We notice that

$$\int_{\mathbb{R}^d} V_- |u|^2 \, dx \leqslant \int_{\mathbb{R}^d} |\nabla u|^2 \quad \Longleftrightarrow \quad -\Delta - V_- \geqslant 0.$$

3. By virtue of the min/max principle

$$-\Delta - V_- \geqslant 0 \quad \Longleftrightarrow \quad \text{there are no negative eigenvalues for } -\Delta - V_-.$$

The only part which requires more clarifications is (1). We will explicitly prove this equivalence in the Appendix D. ☐

We observe that if $\|V_-\|_{\frac{d}{2}}^{\frac{d}{2}} < L_{0,d}^{-1}$ then R-C-L inequality implies that $\mathcal{N}_-(-\Delta - V_-) = 0$ that is no negative eigenvalues can occur. In this sence we say that the absence of eigenvalues is a weaker form of R-C-L.

- Now we will see that if one is interested in a weaker result about the bound of a *single* eigenvalue of $-\Delta + V$, assuming V to be *real-valued* (so that possible discrete eigenvalues are negative), such a bound will follow again as an easy application of the Sobolev inequality.

Theorem I.8. *Let V be real-valued, if $d = 1$ and $\gamma \geqslant \frac{1}{2}$ or if $d \geqslant 2$ and $\gamma > 0$, then any non-negative eigenvalue λ of the Schrödinger operator $-\Delta + V$ satisfies*

$$|\lambda|^\gamma \leqslant L_{\gamma,d} \int_{\mathbb{R}^d} |V(x)|^{\gamma + \frac{d}{2}} \, dx,$$

with a constant independent of V.

Proof. To avoid technical computations, we will prove the result just for $\gamma = 1$. Without loss of generality we assume $V = -V_-$. Let us define $H := -\Delta - V_-$, by the variational characterization of the eigenvalues one has

$$\inf \sigma(H) = \inf_{\|u\|_2 = 1} \langle u, Hu \rangle.$$

It follows by integration by parts that

$$\langle u, Hu \rangle = \int_{\mathbb{R}^d} |\nabla u|^2 - \int_{\mathbb{R}^d} V_- |u|^2.$$

Defining $\rho := |u|^2$, making use of Hölder and of the inequality $ab \leqslant \frac{a^p}{p} + \frac{b^q}{q}$ which holds for all positive a, b and $\frac{1}{p} + \frac{1}{q} = 1$, we get

$$\int_{\mathbb{R}^d} V_- \, \rho \leqslant \|V_-\|_{1+\frac{d}{2}} \|\rho\|_{\frac{d+2}{d}} \leqslant \frac{1}{\delta^{1+\frac{d}{2}}} \frac{2}{d+2} \|V_-\|_{1+\frac{d}{2}}^{1+\frac{d}{2}} + \delta^{\frac{d+2}{d}} \frac{d}{d+2} \|\rho\|_{\frac{d+2}{d}}^{\frac{d+2}{d}}. \tag{I.5.4}$$

The Sobolev inequality gives

$$\int_{\mathbb{R}^d} |\nabla u|^2 \geqslant S_d \|\rho\|_{\frac{q}{2}}, \qquad q = \frac{2d}{d-2}.$$

Using again Hölder and the assumption $\|\rho\|_1 := \|u\|_2^2 = 1$ we have

$$\|\rho\|_{\frac{d+2}{d}}^{\frac{d+2}{d}} = \int_{\mathbb{R}^d} \rho^{\frac{2}{d}} \rho \leqslant \|\rho\|_1^{\frac{2}{d}} \|\rho\|_{\frac{q}{2}} = \|\rho\|_{\frac{q}{2}}.$$

From this we conclude that

$$\int_{\mathbb{R}^d} |\nabla u|^2 \geqslant S_d \|\rho\|_{\frac{d+2}{2}}^{\frac{d+2}{2}}. \tag{I.5.5}$$

Using (I.5.4) and (I.5.5) together we obtain the following lower bound

$$\langle u, Hu \rangle \geqslant \left(S_d - \delta^{\frac{d+2}{d}} \frac{d}{d+2} \right) \|\rho\|_{\frac{d+2}{d}}^{\frac{d+2}{d}} - \frac{1}{\delta^{1+\frac{d}{2}}} \frac{2}{d+2} \|V_-\|_{1+\frac{d}{2}}^{1+\frac{d}{2}}. \tag{I.5.6}$$

Choosing suitably δ, one has

$$\langle u, Hu \rangle \geqslant -L_{0,d} \int_{\mathbb{R}^d} V_-^{1+\frac{d}{2}}(x) \, dx.$$

Now let λ any negative eigenvalue, we get

$$\lambda \geqslant \inf \sigma(H) \geqslant -L_{0,d} \int_{\mathbb{R}^d} V_-^{1+\frac{d}{2}}(x) \, dx,$$

which is the thesis. □

Let us observe that until now, in the whole dissertation, the potential V was assumed to be real-valued, leading to a self-adjoint context. Now we are interested in consider the complex-valued frame. As already mentioned in the introduction, the generalization of the spectral results from the self-adjoint to the non self-adjoint picture is very far to be easy. This can be already justified by the previous remark. As it shown, spectral bounds in the "self-adjoint paradise" make strongly use of variational characterizations of eigenvalues which do not hold in the non self-adjoint context; moreover another fundamental tool is Sobolev inequality which, as showed in Frank [40], does not suffice to prove similar bound than Theorem I.8 which may cover complex-valued potentials, indeed more subtle estimates are needed.

We conclude this historical section with a short overview of what is known for the location of discrete eigenvalues for non self-adjoint Schrödinger operators. In order to do that we need to come back to the work by Abramov, Aslanyan and Davies [1], in this paper was proved that in dimension $d = 1$ every eigenvalues $\lambda \in \mathbb{C}\backslash[0, \infty)$ of $\frac{d^2}{dx^2} + V$ satisfies

$$|\lambda|^{\frac{1}{2}} \leqslant \frac{1}{2} \int_{\mathbb{R}} |V(x)|\, dx. \tag{I.5.7}$$

The numerous paper after [1] were primarily motivated by a question posed by E. B. Davies about the possibility to extend the previous estimate to dimension $d \geqslant 2$. In [41] Frank, Laptev, Lieb and Seiringer extended the previous result to higher dimensions and to L^p norm of the potential with $p \neq 1$, finding the non self-adjoint counterpart of the Lieb-Thirring inequality for the eigenvalue power sums. However in this work they were able to prove the bound just for eigenvalues lying sufficiently far from the positive real axis. In [61] Laptev and Safronov overcame this constraint, obtaining a result which covers eigenvalues possibly close to the essential spectrum. In the same work, Laptev and Safronov conjectured the natural generalization of (I.5.7), this reads as

$$|\lambda|^{\gamma} \leqslant D_{\gamma,d} \int_{\mathbb{R}^d} |V(x)|^{\gamma+\frac{d}{2}}\, dx \tag{I.5.8}$$

for $d \geqslant 2$ and $0 < \gamma \leqslant \frac{d}{2}$. This remained a conjecture since 2011 when Frank [40] proved that (I.5.8) holds for $d \geqslant 2$ and $0 < \gamma \leqslant \frac{1}{2}$. After, Frank and Simon [42] proved the conjecture for radial potentials for $d \geqslant 2$ and $\frac{1}{2} < \gamma < \frac{d}{2}$ and "disprove" it in the general case.

Now we are in position to prove our results.

I.5.2. Proofs

Proof of Theorem I.4 As already mentioned in the introduction, several works which treat spectral analysis use as a starting point the Birman-Schwinger principle. In our case this state that if $z \in \mathbb{C}\backslash[0, \infty)$ is an eigenvalue of $-\Delta^* + V$ then -1 is an eigenvalue of $K_z := V_{\frac{1}{2}}(-\Delta^* - z)^{-1}|V|^{\frac{1}{2}}$ and vice-versa, where we defined $V_{\frac{1}{2}} = |V|^{\frac{1}{2}} \operatorname{sgn}(V)$. It is clear that if -1 is an eigenvalue of K_z then the norm of K_z is at least 1. Therefore in order to obtain the thesis of our result, it is sufficient to prove that the following holds

$$\|V_{\frac{1}{2}}(-\Delta^* - z)^{-1}|V|^{\frac{1}{2}}\|^{\gamma+\frac{d}{2}} \leqslant D_{\gamma,d}\, |z|^{-\gamma} \int_{\mathbb{R}^d} |V(x)|^{\gamma+\frac{d}{2}}\, dx. \tag{I.5.9}$$

As we have already seen in Chapter I.3, which is devoted to highlight some useful consequence of the Helmholtz decomposition, the resolvent of Lamé operator has a favorable form in terms of resolvents of the Laplace operator, precisely $(-\Delta^* - z)^{-1}$ can be written as in (I.3.2) (see Lemma I.2 for further details).

In view of this remark now we are in position to compute explicitly the operator norm of $K_z := V_{\frac{1}{2}}(-\Delta^* - z)^{-1}|V|^{\frac{1}{2}}$, in order to do that, for any $f, g \in [L^2(\mathbb{R}^d)]^d$, we estimate the quantity $|\langle f, K_z g\rangle|$. We have

$$|\langle f, K_z g\rangle| \leqslant \frac{1}{\mu}\Big|\langle f, V_{\frac12}\big(-\Delta-\tfrac{z}{\mu}\big)^{-1}(|V|^{\frac12}g)_S\rangle\Big| + \frac{1}{\lambda+2\mu}\Big|\langle f, V_{\frac12}\big(-\Delta-\tfrac{z}{\lambda+2\mu}\big)^{-1}(|V|^{\frac12}g)_P\rangle\Big|.$$

To simplify the notations from now on we will write $G = |V|^{\frac12}g$ and $G_S = (|V|^{\frac12}g)_S$ and $G_P = (|V|^{\frac12}g)_P$ the respective components of the Helmholtz decomposition. Therefore the previous inequality can be re-written as

$$|\langle f, K_z g\rangle| \leqslant \frac{1}{\mu}\Big|\langle f, V_{\frac12}\big(-\Delta-\tfrac{z}{\mu}\big)^{-1}G_S\rangle\Big| + \frac{1}{\lambda+2\mu}\Big|\langle f, V_{\frac12}\big(-\Delta-\tfrac{z}{\lambda+2\mu}\rangle\big)^{-1}G_P\big)\Big|. \quad \text{(I.5.10)}$$

As a starting estimate we consider the first term, we recall that we are dealing with vector-valued function, this will involve the necessity to obtain estimates for components. Using Hölder inequality, with p and p' such that $\frac1p + \frac1{p'} = 1$ and its version for discrete measures, we have

$$\begin{aligned}
|\langle f, V_{\frac12}(-\Delta-\tfrac{z}{\mu})^{-1}G_S\rangle| &\leqslant \sum_{j=1}^d \int_{\mathbb{R}^d} |f_j||V|^{\frac12}|(-\Delta-\tfrac{z}{\mu})^{-1}G_S^j| \\
&\leqslant \sum_{j=1}^d \|f_j|V|^{\frac12}\|_p \|(-\Delta-\tfrac{z}{\mu})^{-1}G_S^j\|_{p'} \\
&\leqslant \Big(\sum_{j=1}^d \|f_j|V|^{\frac12}\|_p^p\Big)^{\frac1p}\Big(\sum_{j=1}^d \|(-\Delta-\tfrac{z}{\mu})^{-1}G_S^j\|_{p'}^{p'}\Big)^{\frac1{p'}} \\
&= \|f|V|^{\frac12}\|_p\Big(\sum_{j=1}^d \|(-\Delta-\tfrac{z}{\mu})^{-1}G_S^j\|_{p'}^{p'}\Big)^{\frac1{p'}}.
\end{aligned}$$

Proceeding as in [40] we will strongly use the "uniform Sobolev inequality" by Kenig, Ruiz and Sogge [57], which adfirms that

$$\|(-\Delta - z)^{-1}\|_{p\to p'} \leqslant C_{p,d}\,|z|^{-\frac{d+2}{2}+\frac{d}{p}}, \quad \text{(I.5.11)}$$

for $\frac{2d}{d+2} \leqslant p \leqslant \frac{2(d+1)}{d+3}$ if $d \geqslant 3$ and for $1 < p \leqslant \frac65$ if $d = 2$.

From (I.5.11) it follows

$$\|(-\Delta-\tfrac{z}{\mu})^{-1}G_S^j\|_{p'} \leqslant \frac{C_{p,d}}{\mu^{-\frac{d+2}{2}+\frac{d}{p}}}|z|^{-\frac{d+2}{2}+\frac{d}{p}}\|G_S^j\|_p.$$

This, along with the superadditivity of the convex function $|x|^p$, namely $\big(\sum_{j=1}^d a_j^p\big)^{\frac1p} \leqslant \sum_{j=1}^d a_j$, which holds for all non-negative a_j, and $p \geqslant 1$ and the Hölder inequality for sums, gives

$$\begin{aligned}
\Big(\sum_{j=1}^{d}\|(-\Delta-\tfrac{z}{\mu})^{-1}G_S^j\|_{p'}^{p'}\Big)^{\frac{1}{p'}} &\leqslant \frac{C_{p,d}}{\mu^{-\frac{d+2}{2}+\frac{d}{p}}}|z|^{-\frac{d+2}{2}+\frac{d}{p}}\Big(\sum_{j=1}^{d}\|G_S^j\|_p^{p'}\Big)^{\frac{1}{p'}}\\
&\leqslant \frac{C_{p,d}}{\mu^{-\frac{d+2}{2}+\frac{d}{p}}}|z|^{-\frac{d+2}{2}+\frac{d}{p}}\sum_{j=1}^{d}\|G_S^j\|_p\\
&\leqslant \frac{C_{p,d}}{\mu^{-\frac{d+2}{2}+\frac{d}{p}}}|z|^{-\frac{d+2}{2}+\frac{d}{p}}d^{\frac{1}{p'}}\Big(\sum_{j=1}^{d}\|G_S^j\|_p^{p}\Big)^{\frac{1}{p}}\\
&= \frac{C_{p,d}}{\mu^{-\frac{d+2}{2}+\frac{d}{p}}}|z|^{-\frac{d+2}{2}+\frac{d}{p}}d^{\frac{1}{p'}}\|G_S\|_p.
\end{aligned}$$

At the end we get

$$|\langle f, V_{\frac{1}{2}}(-\Delta-\tfrac{z}{\mu})^{-1}G_S\rangle| \leqslant \frac{C_{p,d}}{\mu^{-\frac{d+2}{2}+\frac{d}{p}}}|z|^{-\frac{d+2}{2}+\frac{d}{p}}d^{\frac{1}{p'}}\|f|V|^{\frac{1}{2}}\|_p\|G_S\|_p,$$

performing the same computation for the second term in (I.5.10) we have

$$|\langle f, V_{\frac{1}{2}}(-\Delta-\tfrac{z}{\lambda+2\mu})^{-1}G_P\rangle| \leqslant \frac{C_{p,d}}{(\lambda+2\mu)^{-\frac{d+2}{2}+\frac{d}{p}}}|z|^{-\frac{d+2}{2}+\frac{d}{p}}d^{\frac{1}{p'}}\|f|V|^{\frac{1}{2}}\|_p\|G_P\|_p.$$

Plugging the previous two together in (I.5.10) one obtains

$$|\langle f, V_{\frac{1}{2}}(-\Delta^*-z)^{-1}G\rangle| \leqslant C_{p,d,\lambda,\mu}|z|^{-\frac{d+2}{2}+\frac{d}{p}}\|f|V|^{\frac{1}{2}}\|_p\big(\|G_S\|_p+\|G_P\|_p\big),$$

where $C_{p,d,\lambda,\mu}:=C_{p,d}\,d^{\frac{1}{p'}}\max\Big\{\mu^{\frac{d}{2}-\frac{d}{p}},(\lambda+2\mu)^{\frac{d}{2}-\frac{d}{p}}\Big\}$. In order to conclude we need to obtain a bound for the sum of the L^p norm of the S and P component of G in terms of the L^p norm of the whole function G. Following a strategy similar to the one used in [19] and set out in the previous chapter I.4, we write this trivial decomposition for G :

$$G = G-\nabla\psi+\nabla\psi$$

and we assume that ψ is a solution of the elliptic problem

$$\Delta\psi = \operatorname{div} G. \tag{I.5.12}$$

Since ψ is a solution of (I.5.12) then it is clear that $\operatorname{div}(G-\nabla\psi)=0$. By uniqueness of the Helmholtz decomposition it follows that $G_S=G-\nabla\psi$ and $G_P=\nabla\psi$. From this explicit form for the components we obtain

$$\|G_S\|_p+\|G_P\|_p \leqslant \|G\|_p+2\|\nabla\psi\|_p.$$

It remains to obtain an estimate for $\|\nabla\psi\|_p$, more precisely we want to obtain from this norm another contribution of $\|G\|_p$.

Let us define the operator T such that $T(G) := \nabla\psi$, where ψ is the unique solution of (I.5.12), in [19] and also recalled above, following a previous insight in [4], was proved that this operator T is noting but a composition of the Riesz transforms, more precisely, for all $j = 1, 2, \ldots, d$

$$\partial_j \psi = c_d \sum_{k=1}^{d} R_j R_k G_k.$$

This addressed our interest toward finding boundedness result for the Riesz transform. Will be a fundamental tool for our aim the following result by Iwaniec and Martin [51] and Bañuelos and Wang [2]. They assert that for all $j = 1, 2, \ldots, d$

$$\|R_j\|_{p\to p} = \cot\left(\frac{\pi}{2p^*}\right) =: c_p, \qquad 1 < p < \infty, \quad p^* = \max\{p, p'\}. \tag{I.5.13}$$

Using (I.5.13) and the Hölder inequality for discrete measures, we obtain

$$\begin{aligned}\|\nabla\psi\|_p := \Big(\sum_{j=1}^{d} \|\partial_j\psi\|_p^p\Big)^{\frac{1}{p}} &\leqslant c_d \left(\sum_{j=1}^{d}\Big(\sum_{k=1}^{d}\|R_jR_kG_k\|_p\Big)^p\right)^{\frac{1}{p}} \leqslant c_d\, c_p^2\, d^{\frac{1}{p}} \sum_{k=1}^{d}\|G_k\|_p \\ &\leqslant c_d\, c_p^2\, d^{\frac{1}{p}} d^{\frac{1}{p'}} \|G\|_p = c_d\, c_p^2\, d\, \|G\|_p.\end{aligned}$$

Summing up, recalling that at the very beginning $G = |V|^{\frac{1}{2}} g$, we have

$$|\langle f, V_{\frac{1}{2}}(-\Delta^* - z)^{-1}|V|^{\frac{1}{2}} g\rangle| \leqslant \widetilde{C}_{d,p,\lambda,\mu}|z|^{-\frac{d+2}{2}+\frac{d}{p}} \|f|V|^{\frac{1}{2}}\|_p \, \||V|^{\frac{1}{2}}g\|_p.$$

Let's see $\|f\,|V|^{\frac{1}{2}}\|_p$. Using again the discrete Hölder inequality we get

$$\begin{aligned}\|f\,|V|^{\frac{1}{2}}\|_p = \Big(\sum_{j=1}^{d}\|f_j|V|^{\frac{1}{2}}\|_p^p\Big)^{\frac{1}{p}} &\leqslant \Big(\sum_{j=1}^{d}\|f_j\|_2^p\|V\|_{\frac{p}{2-p}}^{\frac{p}{2}}\Big)^{\frac{1}{p}} \leqslant \|V\|_{\frac{p}{2-p}}^{\frac{1}{2}}\sum_{j=1}^{d}\|f_j\|_2 \\ &\leqslant d^{\frac{1}{2}}\|V\|_{\frac{p}{2-p}}^{\frac{1}{2}}\|f\|_2.\end{aligned}$$

Performing the same computations for $\||V|^{\frac{1}{2}}g\|_p$ one has

$$|\langle f, V_{\frac{1}{2}}(-\Delta^* - z)^{-1}|V|^{\frac{1}{2}} g\rangle| \leqslant \widetilde{\widetilde{C}}_{d,p,\lambda,\mu}|z|^{-\frac{d+2}{2}+\frac{d}{p}}\|f\|_2\|g\|_2\|V\|_{\frac{p}{2-p}}.$$

Now for $0 < \gamma \leqslant \frac{1}{2}$ (as for $\gamma = 0$ in $d \geqslant 3$) we can choose $p = \frac{2(2\gamma+d)}{2\gamma+d+2}$, indeed this restriction on γ guarantees that p, chosen as above, satisfies the hypotheses requested in the estimate of Kenig, Ruiz and Sogge. Taking the supremum over all f and $g \in [L^2(\mathbb{R}^d)]^d$ with norm less than or equal to one we obtain (I.5.9). This concludes the the proof of Theorem I.4.

Proof of Theorem I.5 Now we are in position to prove Theorem I.5. We underline that this is a stronger result than Theorem I.4. Indeed as already pointed out in [40] with respect to eigenvalues' bounds for Schrödinger operators, it turns out that the control of the size of the potential in terms of its Morrey-Campanato norm is a sufficient condition in order to obtain a similar bound for the eigenvalues than the one obtained in Theorem I.4. Before turning to the core part of the proof, we recall the standard definition of Morrey-Campanato's norm is given:

$$\|V\|_{\mathcal{L}^{\alpha,p}} := \sup_{x,r} r^{\alpha}\Big(r^{-d}\int_{B_r(x)}|V(y)|^p\,dy\Big)^{\frac{1}{p}}.$$

This assumption about the perturbation V allows us to treat potentials with local stronger singularities then the ones covered by the previous result Theorem I.4, in which the potentials were required to belong to a suitable L^p space.

As we will see in a moment we need to replace the uniform Sobolev estimate by Kenig, Ruiz and Sogge, with L^2- weighted estimates. More precisely we will use the following result, a proof of which can be found in [40].

Lemma I.6. *Let* $\frac{4}{3} < \alpha < 2$ *if* $d = 2$, $\frac{2d}{d+1} < \alpha \leqslant 2$ *if* $d \geqslant 3$ *and let* $\frac{d-1}{2(\alpha-1)} < p \leqslant \frac{d}{\alpha}$. *Then for all* $0 < w \in \mathcal{L}^{\alpha,p}(\mathbb{R}^d)$,

$$\|(-\Delta - z)^{-1}\|_{L^2(w^{-1}\,dx)\to L^2(w\,dx)} \leqslant C_{d,\alpha,p}\|w\|_{\mathcal{L}^{\alpha,p}}|z|^{-1+\frac{\alpha}{2}}. \tag{I.5.14}$$

As in the previous result, the explicit expression (I.3.2) of the resolvent for the Lamé operators in terms of the resolvents of the Laplacian will be of great relevance . Let's start with (I.5.10) and again with the first term in there.

Proceeding in analogy with the work by Frank, we pick a strictly positive function $\phi \in \mathcal{L}^{\alpha,p}$ and we define a strictly positive approximation of our potential, that is $V_\varepsilon(x) := \sup_{x\in\mathbb{R}^d}\{|V(x)|, \varepsilon\phi(x)\}$.

Using Cauchy-Schwartz and Hölder inequalities we have

$$\begin{aligned}|\langle f, V_{\frac{1}{2}}(-\Delta - \tfrac{z}{\mu})^{-1}G_S\rangle| &\leqslant \sum_{j=1}^{d}\|f_j|V|^{\frac{1}{2}}V_\varepsilon^{-\frac{1}{2}}\|_2\|(-\Delta - \tfrac{z}{\mu})^{-1}G_S^j\|_{L^2(V_\varepsilon\,dx)}\\ &\leqslant \Big(\sum_{j=1}^{d}\|f_j\sqrt{|V|/V_\varepsilon}\|_2^2\Big)^{\frac{1}{2}}\Big(\sum_{j=1}^{d}\|(-\Delta - \tfrac{z}{\mu})^{-1}G_S^j\|^2_{L^2(V_\varepsilon\,dx)}\Big)^{\frac{1}{2}}.\end{aligned}$$

Making use of (I.5.14) one obtains

$$\|(-\Delta - \tfrac{z}{\mu})^{-1}G_S^j\|_{L^2(V_\varepsilon\,dx)} \leqslant \frac{C_{d,\alpha,p}}{\mu^{-1+\frac{\alpha}{2}}}|z|^{-1+\frac{\alpha}{2}}\|V_\varepsilon\|_{\mathcal{L}^{\alpha,p}}\|V_\varepsilon^{-\frac{1}{2}}G_S^j\|_2.$$

This gives

$$|\langle f, V_{\frac{1}{2}}(-\Delta - \tfrac{z}{\mu})^{-1}G_S\rangle| \leqslant \frac{C_{d,\alpha,p}}{\mu^{-1+\frac{\alpha}{2}}}|z|^{-1+\frac{\alpha}{2}}\|V_\varepsilon\|_{\mathcal{L}^{\alpha,p}}\|f\sqrt{|V|/V_\varepsilon}\|_2\,\|V_\varepsilon^{-\frac{1}{2}}G_S\|_2,$$

performing the same computation for the second term in (I.5.10) we have

$$|\langle f, V_{\frac{1}{2}}(-\Delta - \tfrac{z}{\lambda+2\mu})^{-1}G_P\rangle| \leqslant \frac{C_{d,\alpha,p}}{(\lambda+2\mu)^{-1+\frac{\alpha}{2}}}|z|^{-1+\frac{\alpha}{2}}\|V_\varepsilon\|_{\mathcal{L}^{\alpha,p}}\|f\sqrt{|V|/V_\varepsilon}\|_2\,\|V_\varepsilon^{-\frac{1}{2}}G_P\|_2.$$

Plugging the previous two together in (I.5.10) one obtains

$$|\langle f, V_{\frac{1}{2}}(-\Delta^*-z)^{-1}G\rangle| \leqslant C_{d,\alpha,p,\lambda,\mu}|z|^{-1+\frac{\alpha}{2}}\|V_\varepsilon\|_{\mathcal{L}^{\alpha,p}}\|f\sqrt{|V|/V_\varepsilon}\|_2\left(\|V_\varepsilon^{-\frac{1}{2}}G_S\|_2+\|V_\varepsilon^{-\frac{1}{2}}G_P\|_2\right),$$

where $C_{d,\alpha,p,\lambda,\mu} := C_{d,\alpha,p}\max\left\{\mu^{-\frac{\alpha}{2}},(\lambda+2\mu)^{-\frac{\alpha}{2}}\right\} = \mu^{-\frac{\alpha}{2}}C_{d,\alpha,p}$.

Again we write the trivial decomposition for G :

$$G = G - \nabla\psi + \nabla\psi$$

and we assume that ψ is a solution of the elliptic problem (I.5.12), namely $\Delta\psi = \operatorname{div} G$.

Since ψ is a solution of the mentioned equation, then $\operatorname{div}(G-\nabla\psi) = 0$ and by uniqueness of the Helmholtz decomposition it follows that $G_S = G - \nabla\psi$ and $G_P = \nabla\psi$. From this explicit form for the components we obtain

$$\|V_\varepsilon^{-\frac{1}{2}}G_S\|_2 + \|V_\varepsilon^{-\frac{1}{2}}G_P\|_2 \leqslant \|V_\varepsilon^{-\frac{1}{2}}G\|_2 + 2\|V_\varepsilon^{-\frac{1}{2}}\nabla\psi\|_2.$$

In order to conclude we need to obtain an estimate for the last term of the previous inequality. For this aim the following lemma, which is a generalization of Lemma I.5, will be useful.

Lemma I.7. *Let $G \in [C_c^\infty(\mathbb{R}^d)]^d$ be a smooth-compactly supported vector field in $\mathbb{R}^d$, and let $\psi\colon \mathbb{R}^d \to \mathbb{C}$ be a smooth solution to*

$$\Delta\psi = \operatorname{div} G.$$

Then for any w belonging to the A_p-class, $1 < p < \infty$ the following estimate holds

$$\|\nabla\psi\|_{L^p(w\,dx)} \leqslant c\|G\|_{L^p(w\,dx)},$$

for some constant $c > 0$ independent on G.

The proof of this result basically follows from the weighted L^p-boundednees of Calderón-Zygmund operator when the weights belong to the A_p-class.

Now if V_ε^{-1} is assumed to belong to the A_2-class then

$$\|V_\varepsilon^{-\frac{1}{2}}\nabla\psi\|_2 \leqslant c\|V_\varepsilon^{-\frac{1}{2}}G\|_2.$$

Summing up, recalling that at the very beginning $G = |V|^{\frac{1}{2}} g$, we have

$$|\langle f, V_{\frac{1}{2}}(-\Delta^* - z)^{-1}|V|^{\frac{1}{2}}\rangle| \leqslant \widetilde{C}_{d,\alpha,p,\lambda,\mu}|z|^{-1+\frac{\alpha}{2}}\|V_\varepsilon\|_{\mathcal{L}^{\alpha,p}}\|f\sqrt{|V|/V_\varepsilon}\|_2\,\|g\sqrt{|V|/V_\varepsilon}\|_2$$
$$\leqslant \widetilde{C}_{d,\alpha,p,\lambda,\mu}|z|^{-1+\frac{\alpha}{2}}\|V_\varepsilon\|_{\mathcal{L}^{\alpha,p}}\|f\|_2\,\|g\|_2.$$

Therefore the theorem is proved once ε goes to zero, taking the supremum over all $f, g \in [L^2(\mathbb{R}^d)]^d$ with norm less than or equal to one and by choosing $\alpha = \frac{2d}{2\gamma+d}$.

Proof of Theorem I.6 As Frank himself underlined for his Schrödinger counterpart's result, the previous theorem is not fully satisfactory because, in essence, is required to the potential to decay as $|x|^{-\rho}$ with $\rho > \frac{2d}{d+1}$, this means that slowly decaying potentials, that is potentials which decay just as $|x|^{-\rho}$ with $\rho > 1$, are not included. For this reason, in the same paper, Frank proved a similar result which allows to consider this decay rate, and we do the same with providing Theorem I.6.

Let us start, as in the previous results, with the inequality (I.5.10) and in particular with the first term in it.

From now on we will use the following notation $\langle x\rangle := (1+|x|^2)^{\frac{1}{2}}$.

$$|\langle f, V_{\frac{1}{2}}(-\Delta - \tfrac{z}{\mu})^{-1}G_S\rangle| \leqslant \sum_{j=1}^d |\langle f_j, V_{\frac{1}{2}}\langle x\rangle^{\alpha}\langle x\rangle^{-\alpha}(-\Delta - \tfrac{z}{\mu})^{-1}G_S^j\rangle|$$
$$\leqslant \sum_{j=1}^d \|f_j|V|^{\frac{1}{2}}\langle x\rangle^{\alpha}\|_2\|\langle x\rangle^{-\alpha}(-\Delta - \tfrac{z}{\mu})^{-1}G_S^j\|_2$$
$$\leqslant \Big(\sum_{j=1}^d \|f_j|V|^{\frac{1}{2}}\langle x\rangle^{\alpha}\|_2^2\Big)^{\frac{1}{2}}\Big(\sum_{j=1}^d \|\langle x\rangle^{-\alpha}(-\Delta - \tfrac{z}{\mu})^{-1}G_S^j\|_2^2\Big)^{\frac{1}{2}}.$$

Now we need the following resolvent estimate

$$\|(-\Delta - z)^{-1}\|_{L^2(\langle x\rangle^{2\alpha}dx)\to L^2(\langle x\rangle^{-2\alpha}dx)} \leqslant C_{d,\alpha}|z|^{-\frac{1}{2}}, \qquad \alpha > \frac{1}{2}. \tag{I.5.15}$$

Using (I.5.15) we have

$$\|\langle x\rangle^{-\alpha}(-\Delta - \tfrac{z}{\mu})^{-1}G_S^j\|_2 \leqslant \frac{C_{d,\alpha}}{\mu^{-\frac{1}{2}}}|z|^{-\frac{1}{2}}\|\langle x\rangle^{\alpha}G_S^j\|_2.$$

Summing up we obtained

$$|\langle f, V_{\frac{1}{2}}(-\Delta - \tfrac{z}{\mu})^{-1}G_S\rangle| \leqslant \frac{C_{d,\alpha}}{\mu^{-\frac{1}{2}}}|z|^{-\frac{1}{2}}\|f|V|^{\frac{1}{2}}\langle x\rangle^{\alpha}\|_2\|\langle x\rangle^{\alpha}G_S\|_2.$$

Performing the same computation for the second term in (I.5.10) we have

$$|\langle f, V_{\frac{1}{2}}(-\Delta - \tfrac{z}{\lambda+2\mu})^{-1}G_P\rangle| \leqslant \frac{C_{d,\alpha}}{(\lambda+2\mu)^{-\frac{1}{2}}}|z|^{-\frac{1}{2}}\|f|V|^{\frac{1}{2}}\langle x\rangle^{\alpha}\|_2\|\langle x\rangle^{\alpha}G_P\|_2.$$

Putting the two estimates together in (I.5.10) one has

$$|\langle f, V_{\frac{1}{2}}(-\Delta^* - z)^{-1}G\rangle| \leqslant C_{d,\alpha,\lambda,\mu}|z|^{-\frac{1}{2}}\|f|V|^{\frac{1}{2}}\langle x\rangle^{\alpha}\|_2(\|\langle x\rangle^{\alpha}G_S\|_2 + \|\langle x\rangle^{\alpha}G_P\|_2),$$

where $C_{d,\alpha,\lambda,\mu} := C_{d,\alpha}\max\{\mu^{-\frac{1}{2}}, (\lambda+2\mu)^{-\frac{1}{2}}\} = \mu^{-\frac{1}{2}}C_{d,\alpha}$.

Again we write the trivial decomposition for G :

$$G = G - \nabla\psi + \nabla\psi,$$

where ψ is a solution of the elliptic problem (I.5.12). As in the previous cases it turns out that G_S and G_P, the components of the Helmholtz decomposition, have to be respectively equal to $G - \nabla\psi$ and $\nabla\psi$. This gives

$$\|\langle x\rangle^{\alpha}G_S\|_2 + \|\langle x\rangle^{\alpha}G_P\|_2 \leqslant \|\langle x\rangle^{\alpha}G\|_2 + 2\|\langle x\rangle^{\alpha}\nabla\psi\|_2.$$

This means that again we need an estimate involving the Calderón-Zygmund operator $\nabla\psi$.

Since $\langle x\rangle^s$ belongs to the A_2-class for all $s \in \mathbb{R}$, in particular this is true for $s = 2\alpha$ with $\alpha > \frac{1}{2}$, this guarantees from Lemma I.7 that

$$\|\langle x\rangle^{\alpha}\nabla\psi\|_2 \leqslant c\|\langle x\rangle^{\alpha}G\|_2.$$

Using the previous estimate and recalling that at first $G = |V|^{\frac{1}{2}}g$, we have

$$\begin{aligned}|\langle f, V_{\frac{1}{2}}(-\Delta^* - z)^{-1}|V|^{\frac{1}{2}}g\rangle| &\leqslant \widetilde{C}_{d,\alpha,\lambda,\mu}|z|^{-\frac{1}{2}}\|f|V|^{\frac{1}{2}}\langle x\rangle^{\alpha}\|_2\|\langle x\rangle^{\alpha}|V|^{\frac{1}{2}}g\|_2\\ &\leqslant \widetilde{C}_{d,\alpha,\mu,\lambda}|z|^{-\frac{1}{2}}\sup_{x\in\mathbb{R}^d}(1+|x|^2)^{\alpha}|V(x)|\|f\|_2\|g\|_2.\end{aligned}$$

Taking the supremum over all $f, g \in [L^2(\mathbb{R}^d)]^d$ with norm less then or equal to one we get the desired result.

Proof of Theorem I.7 As a byproduct, in his work Frank performed the following resolvent estimate for the Laplace operator:

$$\|\langle x\rangle^{-\theta\alpha}(-\Delta - z)^{-1}\langle x\rangle^{-\theta\alpha}\|_{L^q\to L^{q'}} \leqslant C_{d,q,\alpha}|z|^{-\frac{(1-\theta)}{d+1}-\frac{\theta}{2}}, \quad \alpha > \frac{1}{2}, \tag{I.5.16}$$

where $\frac{1}{q} = \frac{1-\theta}{p_d} + \frac{\theta}{2}$ and $p_d = \frac{2(d+1)}{d+3}$.

Starting again from (I.5.10), or better from the term involving the S-component, we have

$$\begin{aligned}|\langle f, V_{\frac{1}{2}}(-\Delta - \tfrac{z}{\mu})^{-1}G_S\rangle| &\leqslant \sum_{j=1}^{d}|\langle f_j, V_{\frac{1}{2}}\langle x\rangle^{\theta\alpha}\langle x\rangle^{-\theta\alpha}(-\Delta - \tfrac{z}{\mu})^{-1}G_S^j\rangle|\\ &\leqslant \sum_{j=1}^{d}\|f_j|V|^{\frac{1}{2}}\langle x\rangle^{\theta\alpha}\|_q\|\langle x\rangle^{-\theta\alpha}(-\Delta - \tfrac{z}{\mu})^{-1}G_S^j\|_{q'}\\ &\leqslant \Big(\sum_{j=1}^{d}\|f_j|V|^{\frac{1}{2}}\langle x\rangle^{\theta\alpha}\|_q^q\Big)^{\frac{1}{q}}\Big(\sum_{j=1}^{d}\|\langle x\rangle^{-\theta\alpha}(-\Delta - \tfrac{z}{\mu})^{-1}G_S^j\|_{q'}^{q'}\Big)^{\frac{1}{q'}}.\end{aligned}$$

Making use of (I.5.16), similarly as our first result, we obtain

$$\Big(\sum_{j=1}^{d}\|\langle x\rangle^{-\theta\alpha}(-\Delta-\tfrac{z}{\mu})^{-1}G_S^j\|_{q'}^{q'}\Big)^{\frac{1}{q'}} \leqslant \frac{C_{d,q,\alpha}}{\mu^{-\frac{1-\theta}{d+1}-\frac{\theta}{2}}}|z|^{-\frac{1-\theta}{d+1}-\frac{\theta}{2}}\Big(\sum_{j=1}^{d}\|\langle x\rangle^{\theta\alpha}G_S^j\|_q^{q'}\Big)^{\frac{1}{q'}}$$
$$\leqslant \frac{C_{d,q,\alpha}}{\mu^{-\frac{1-\theta}{d+1}-\frac{\theta}{2}}}|z|^{-\frac{1-\theta}{d+1}-\frac{\theta}{2}}\sum_{j=1}^{d}\|\langle x\rangle^{\theta\alpha}G_S^j\|_q$$
$$\leqslant \frac{C_{d,q,\alpha}}{\mu^{-\frac{1-\theta}{d+1}-\frac{\theta}{2}}}|z|^{-\frac{1-\theta}{d+1}-\frac{\theta}{2}}d^{\frac{1}{q'}}\Big(\sum_{j=1}^{d}\|\langle x\rangle^{\theta\alpha}G_S^j\|_q^{q}\Big)^{\frac{1}{q}}$$
$$= \frac{C_{d,q,\alpha}}{\mu^{-\frac{1-\theta}{d+1}-\frac{\theta}{2}}}|z|^{-\frac{1-\theta}{d+1}-\frac{\theta}{2}}d^{\frac{1}{q'}}\|\langle x\rangle^{\theta\alpha}G_S\|_q.$$

This gives

$$|\langle f, V_{\frac{1}{2}}(-\Delta-\tfrac{z}{\mu})^{-1}G_S\rangle| \leqslant \frac{C_{d,q,\alpha}}{\mu^{-\frac{1-\theta}{d+1}-\frac{\theta}{2}}}d^{\frac{1}{q'}}|z|^{-\frac{1-\theta}{d+1}-\frac{\theta}{2}}\|f|V|^{\frac{1}{2}}\langle x\rangle^{\theta\alpha}\|_q\|\langle x\rangle^{\theta\alpha}G_S\|_q,$$

performing the same computations for the term involving the P component, one gets

$$|\langle f, V_{\frac{1}{2}}(-\Delta-\tfrac{z}{\lambda+2\mu})^{-1}G_P\rangle| \leqslant \frac{C_{d,q,\alpha}}{(\lambda+2\mu)^{-\frac{1-\theta}{d+1}-\frac{\theta}{2}}}d^{\frac{1}{q'}}|z|^{-\frac{1-\theta}{d+1}-\frac{\theta}{2}}\|f|V|^{\frac{1}{2}}\langle x\rangle^{\theta\alpha}\|_q\|\langle x\rangle^{\theta\alpha}G_P\|_q.$$

Putting the two previous estimate together in (I.5.10) one has

$$|\langle f, V_{\frac{1}{2}}(-\Delta^*-z)^{-1}G\rangle| \leqslant C_{d,q,\alpha,\lambda,\mu}|z|^{-\frac{1-\theta}{d+1}-\frac{\theta}{2}}\|f|V|^{\frac{1}{2}}\langle x\rangle^{\theta\alpha}\|_q(\|\langle x\rangle^{\theta\alpha}G_S\|_q+\|\langle x\rangle^{\theta\alpha}G_P\|_q),$$

where $C_{d,q,\alpha,\lambda,\mu}=C_{d,q,\alpha}d^{\frac{1}{q'}}\max\{\mu^{\frac{1-\theta}{d+1}+\frac{\theta}{2}-1},(\lambda+2\mu)^{\frac{1-\theta}{d+1}+\frac{\theta}{2}-1}\}=C_{d,q,\alpha}d^{\frac{1}{q'}}\mu^{\frac{1-\theta}{d+1}+\frac{\theta}{2}-1}$.

Again we write the trivial decomposition for G :

$$G = G-\nabla\psi+\nabla\psi,$$

where ψ is a solution of the elliptic problem (I.5.12). Arguing as above, the next inequality easily follows:

$$\|\langle x\rangle^{\theta\alpha}G_S\|_q+\|\langle x\rangle^{\theta\alpha}G_P\|_q \leqslant \|\langle x\rangle^{\theta\alpha}G\|_q+2\|\langle x\rangle^{\theta\alpha}\nabla\psi\|_q.$$

Since $\langle x\rangle^s$ belongs to the A_q-class, using again Lemma I.7 we get

$$\|\langle x\rangle^{\theta\alpha}\nabla\psi\|_q \leqslant c\|\langle x\rangle^{\theta\alpha}G\|_q.$$

Using this estimate and the fact that at the very beginning $G:=|V|^{\frac{1}{2}}g$, we obtain

$$|\langle f, V_{\frac{1}{2}}(-\Delta^*-z)^{-1}|V|^{\frac{1}{2}}g\rangle| \leqslant \widetilde{C}_{d,p,\alpha,\lambda,\mu}|z|^{-\frac{1-\theta}{d+1}-\frac{\theta}{2}}\|f|V|^{\frac{1}{2}}\langle x\rangle^{\theta\alpha}\|_q\|\langle x\rangle^{\theta\alpha}|V|^{\frac{1}{2}}g\|_q.$$

Let's see $\|f|V|^{\frac{1}{2}}\langle x\rangle^{\theta\alpha}\|_q$.

$$\|f|V|^{\frac{1}{2}}\langle x\rangle^{\theta\alpha}\|_q = \Big(\sum_{j=1}^{d}\|f_j|V|^{\frac{1}{2}}\langle x\rangle^{\theta\alpha}\|_q^q\Big)^{\frac{1}{q}} \leqslant \Big(\sum_{j=1}^{d}\|f_j\|_2^q\||V|\langle x\rangle^{2\theta\alpha}\|_{\frac{q}{2-q}}^{\frac{q}{2}}\Big)^{\frac{1}{q}} \leqslant \||V|\langle x\rangle^{2\theta\alpha}\|_{\frac{q}{2-q}}^{\frac{1}{2}}\sum_{j=1}^{d}\|f_j\|_2.$$

Performing the same computations for $\|\langle x\rangle^{\theta\alpha}|V|^{\frac{1}{2}}g\|_q$ one gets

$$|\langle f, V_{\frac{1}{2}}(-\Delta^* - z)^{-1}|V|^{\frac{1}{2}}g\rangle| \leqslant \tilde{\tilde{C}}_{d,p,\alpha,\lambda,\mu}|z|^{-\frac{1-\theta}{d+1}-\frac{\theta}{2}}\||V|\langle x\rangle^{2\theta\alpha}\|_{\frac{q}{2-q}}\|f\|_2\|g\|_2.$$

Taking the supremum over all f and $g \in [L^2(\mathbb{R}^d)]^d$ with norm less than or equal to one we obtain

$$\|V_{\frac{1}{2}}(-\Delta^* - z)^{-1}|V|^{\frac{1}{2}}\|_{L^2\to L^2}^{\frac{q}{2-q}} \leqslant \tilde{\tilde{C}}_{d,p,\alpha,\lambda,\mu}|z|^{\left[-\frac{1-\theta}{d+1}-\frac{\theta}{2}\right]\frac{q}{2-q}}\||V|\langle x\rangle^{2\alpha\theta}\|_{\frac{q}{2-q}}^{\frac{q}{2-q}}.$$

Calling

$$\gamma := \left[\frac{1-\theta}{d+1} + \frac{\theta}{2}\right]\frac{q}{2-q},$$

this clearly gives $\frac{q}{2-q} = 2\gamma\,\frac{d+1}{2-\theta+d\theta}$, since we also have $\frac{1}{q} = \frac{1-\theta}{p_d} + \frac{\theta}{2}$ where $p_d = \frac{2(d+1)}{d+3}$, this provides a constraint in the choice of θ, precisely $\theta = 1 - \frac{d+1}{4\gamma+d-1}$. Using this explicit expression for θ and the fact that, by virtue of Birman-Schwinger principle, -1 is an eigenvalue of our operator $V_{\frac{1}{2}}(-\Delta^* - z)^{-1}|V|^{\frac{1}{2}}$, one has

$$|z|^{\gamma} \leqslant \tilde{\tilde{C}}_{d,p,\alpha,\lambda,\mu}\int_{\mathbb{R}^d}|V|^{2\gamma+\frac{d-1}{2}}(1+|x|^2)^{\alpha(2\gamma-1)}.$$

Renaming $\alpha(2\gamma-1) = \alpha$ we obtain the aforementioned result.

A.| Self-adjointness

First of all we want to give a rigorous meaning to the free Lamé operator as a self-adjoint operator, i.e. we want to build the self-adjoint extension of the operator $-\Delta^*$; in order to do that we proceed using a quadratic form approach. After that we will treat the perturbed setting, distinguishing the case of real-valued perturbation from complex-valued one.

Let us introduce the quadratic form associated with the operator $-\Delta^*$,

$$Q_0[u] = \int_{\mathbb{R}^d} q_0[u]\, dx,$$

where

$$q_0[u] = \lambda \left| \sum_{j=1}^d \frac{\partial u_j}{\partial x_j} \right|^2 + \frac{\mu}{2} \sum_{j,k=1}^d \left| \frac{\partial u_j}{\partial x_k} + \frac{\partial u_k}{\partial x_j} \right|^2, \qquad u \in [C_c^\infty(\mathbb{R}^d)]^d.$$

Remark I.11. Since a complex setting will be needed once the perturbation V will come into play, from now on we assume u to be complex-valued; although this assumption is not yet necessary in the free framework, namely $V = 0$.

A straightforward computation, made explicit in Appendix B shows that under the physical assumption (I.1.2) the quadratic form $q_0[u]$, and thus $-\Delta^*$, is positive.

Remark I.12. The quadratic form clearly remains positive under the stronger condition $\lambda, \mu > 0$.

We recall that, since our form Q_0 is associated with a densely defined positive and symmetric operator, this form is closable.

Let $\overline{Q}_0$ be the closure of our form. Even though completely standard, for reason of completeness, we will show the closedness of our form $\overline{Q}_0$ with form domain the Sobolev space of H^1- vector fields. In order to do that we need to prove that $[H^1(\mathbb{R}^d)]^d$ equipped with the norm

$$\|u\|_{\overline{Q}_0} := \big(\overline{Q}_0[u] + \|u\|^2_{[L^2(\mathbb{R}^d)]^d}\big)^{\frac{1}{2}}$$

is complete. For this purpose we just have to prove that $\|u\|_{\overline{Q}_0}$ is equivalent to $\|u\|_{[H^1(\mathbb{R}^d)]^d} := \big(\|u\|^2_{[L^2(\mathbb{R}^d)]^d} + \|\nabla u\|^2_{[L^2(\mathbb{R}^d)]^d}\big)^{\frac{1}{2}}$. We need the following trivial chain of inequalities, for every $d \times d$ matrix ξ

$$\tfrac{1}{d} |\mathrm{Tr}(\xi)|^2 \leqslant \big|\tfrac{1}{2}(\xi + \xi^T)\big|^2 \leqslant |\xi|^2. \tag{A.1}$$

Calling ξ the Jacobian matrix, $\xi_i^j := \frac{\partial u_i}{\partial x_j}$ we can rewrite $\overline{Q}_0$ in terms of ξ in this way:

$$\overline{Q}_0[u] := 2\mu \int_{\mathbb{R}^d} \big|\tfrac{1}{2}(\xi + \xi^T)\big|^2 + \lambda \int_{\mathbb{R}^d} |\mathrm{Tr}(\xi)|^2.$$

Assuming λ and μ to satisfy (I.1.2) and using (A.1), we get

$$\overline{Q}_0[u] \leqslant 2\mu \int_{\mathbb{R}^d} \big|\tfrac{1}{2}(\xi + \xi^T)\big|^2 + (2\mu + \lambda d) \int_{\mathbb{R}^d} \tfrac{1}{d} |\mathrm{Tr}(\xi)|^2 \leqslant (4\mu + \lambda d) \int_{\mathbb{R}^d} |\xi|^2 = (4\mu + \lambda d) \|\nabla u\|^2_{[L^2(\mathbb{R}^d)]^d}.$$

Summing up we proved that

$$\|u\|_{\overline{Q}_0}^2 \leqslant (4\mu+\lambda d)\|\nabla u\|^2_{[L^2(\mathbb{R}^d)]^d} + \|u\|^2_{[L^2(\mathbb{R}^d)]^d} \leqslant \max\{4\mu+\lambda d, 1\}\|u\|^2_{[H^1(\mathbb{R}^d)]^d}. \tag{A.2}$$

As a starting point for proving the opposite inequality, we show that there exists $c>0$ such that

$$\overline{Q}_0[u] \geqslant c\int_{\mathbb{R}^d}\left|\tfrac{1}{2}(\xi+\xi^T)\right|^2 = c\sum_{j,k=1}^d \int_{\mathbb{R}^d} \left|\tfrac{1}{2}\left(\tfrac{\partial u_j}{\partial x_k}+\tfrac{\partial u_k}{\partial x_j}\right)\right|^2. \tag{A.3}$$

Since the Lamé parameters have to satisfy (I.1.2), there exists $c>0$ such that $2\mu \geqslant c$ and $2\mu+\lambda d \geqslant c$, using this fact we obtain

$$\overline{Q}_0[u] - c\int_{\mathbb{R}^d}\left|\tfrac{1}{2}(\xi+\xi^T)\right|^2 \geqslant (2\mu-c)\int_{\mathbb{R}^d}\left[\left|\tfrac{1}{2}(\xi+\xi^T)\right|^2 - \tfrac{1}{d}\left|\mathrm{Tr}(\xi)\right|^2\right] \geqslant 0,$$

and this clearly gives the claim. In order to conclude we make use of the *Korn's inequality* that reads

$$\|u\|^2_{[H^1(\mathbb{R}^d)]^d} \leqslant \tilde{c}\Big(\sum_{j=1}^d\int_{\mathbb{R}^d}|u_j|^2 + \sum_{j,k=1}^d\int_{\mathbb{R}^d}\left|\tfrac{1}{2}\left(\tfrac{\partial u_j}{\partial x_k}+\tfrac{\partial u_k}{\partial x_j}\right)\right|^2\Big), \tag{A.4}$$

for some constant $\tilde{c} \geqslant 0$.

Exploiting (A.3) and (A.4) we easily obtain

$$\|u\|^2_{\overline{Q}_0} \geqslant c\sum_{j,k=1}^d\int_{\mathbb{R}^d}\left|\tfrac{1}{2}\left(\tfrac{\partial u_j}{\partial x_k}+\tfrac{\partial u_k}{\partial x_j}\right)\right|^2 + \sum_{j=1}^d\int_{\mathbb{R}^d}|u_j|^2 \geqslant \tfrac{\min\{c,1\}}{\tilde{c}}\|u\|^2_{[H^1(\mathbb{R}^d)]^d}. \tag{A.5}$$

Observe that from (A.2) and (A.5) we have the anticipated equivalence, that is $\overline{Q}_0$ is closed.

Therefore, as $\overline{Q}_0$ is a densely defined lower semi-bounded (actually positive) closed form on an Hilbert space, then there is a canonical way to build from it a distinguished self-adjoint extension, called Friedrichs extension, of the symmetric operator $-\Delta^*$, that is the self-adjoint operator we are looking for and that, with abuse of notation, we again write as $-\Delta^*$.

In order to handle the perturbed operator, we want to use the operator written after the use of the Helmholtz decomposizion:

$$-\Delta^* u = -\mu\Delta u_S - (\lambda+2\mu)u_P.$$

The quadratic form associated with the operator $-\Delta^*$, explicitly written in the previous form, is

$$Q_0[u] = \int_{\mathbb{R}^d} q_0[u]\,dx,$$

with

$$q_0[u] = \mu|\nabla u_S|^2 + (\lambda+2\mu)|\nabla u_P|^2 \qquad \text{and} \qquad \mathscr{D}(Q_0) = [H^1(\mathbb{R}^d)]^d,$$

where $|\nabla v|^2$, when $v=(v_1,v_2,\dots,v_d)$ is a vector field, denotes $\sum_{j=1}^d|\nabla v_j|^2$.

We observe that here, with an abuse of notation, we have used the same symbol Q_0 for the quadratic form associated with $-\Delta^*$, both when its action is written explicitly using the Helmholtz decomposition and when the operator is defined in its classical way.

Now we are in position to consider the perturbed operator

$$-\Delta^* + V,$$

where $V\colon \mathbb{R}^d \to \mathcal{M}_{d\times d}(\mathbb{C})$ is the perturbation term.

Clearly, in the Helmholtz decomposition, this operator acts on a smooth vector fields u in this way

$$(-\Delta^* + V)u = -\mu\Delta u_S - (\lambda + 2\mu)\Delta u_P + Vu.$$

The corresponding perturbed quadratic form associated with this operator is

$$Q_{\text{pert}}[u] = Q_0[u] + Q_V[u] = Q_0[u] + \int_{\mathbb{R}^d} q_V[u]\, dx,$$

where

$$q_V[u] = \overline{Vu} \cdot u \qquad \text{and} \qquad \mathscr{D}(Q_V) = \left\{ u \in [L^2(\mathbb{R}^d)]^d : \int_{\mathbb{R}^d} |V||u|^2 < \infty \right\}.$$

Let us suppose now to assume the following smallness condition about V :

$$\exists\, a < \min\{\mu, \lambda + 2\mu\}, \qquad \forall\, u \in [H^1(\mathbb{R}^d)]^d, \qquad \int_{\mathbb{R}^d} |V||u|^2 \leqslant a \int_{\mathbb{R}^d} |\nabla u|^2. \tag{A.6}$$

It's not difficult to see that, as a consequence of the constrictions on a, Q_V is relatively bounded with respect to Q_0 with bound less than one.

Let us suppose, for a moment, that our potential V is real-valued. As a consequence, the sesquilinear form, associated with the quadratic form Q_V, is symmetric. By virtue of these remarks, we are able to build from Q_{pert} an associated self-adjoint operator on $[L^2(\mathbb{R}^d)]^d$ exploiting the well known forms counterpart of the Kato-Rellich perturbation result for operators, namely the KLMN theorem (see for example [80], Thm X.17, or [85], Thm 10.21).

If one is dealing with complex-valued potentials, as our setting, instead of real-valued ones, the scenario turns out to be quite different. In fact, assuming now that V is a complex-valued potential, the sesquilinear form Q_V is no more symmetric and, as a consequence, we clearly cannot expect to be able to build from Q_{pert} a self-adjoint extension of $-\Delta^* + V$. Nevertheless, even though we are dealing with non symmetric forms, we can obtain useful information about the operator $-\Delta^* + V$ by exploiting the theory about sectorial forms (resp. operators). Precisely we can use the representation theorem (see [52], Thm. VI.2.1) to build an m-sectorial operator from a densely defined, sectorial and closed form.

B.| Ellipticity properties

The aim of this section is to point out how the notion of ellipticity has been generalized for systems, in particular for systems of second order differential operators. Let's consider the system of second order constant coefficients operator L acting on $u\colon \mathbb{R}^n \to \mathbb{C}^m$ and defined as

$$Lu := -\sum_{\alpha=1}^{n}\sum_{\beta=1}^{n} A^{\alpha\beta}\partial_\alpha\partial_\beta u; \tag{B.1}$$

more precisely the j-th component of Lu is

$$[Lu]_{1\leqslant j\leqslant m} := -\sum_{\alpha=1}^{n}\sum_{\beta=1}^{n}\sum_{k=1}^{m} A_{jk}^{\alpha\beta}\partial_\alpha\partial_\beta u_k,$$

where $A = \big(A_{jk}^{\alpha\beta}\big)_{\substack{1\leqslant\alpha,\beta\leqslant n\\ 1\leqslant j,k\leqslant m}}$ is the coefficient tensor associated with L. Via Fourier transformation we associate to L its principal symbol $p(\xi)$ that is an $m\times m$ matrix of this form

$$p(\xi) := \sum_{\alpha=1}^{n}\sum_{\beta=1}^{n} A^{\alpha\beta}\xi_\alpha\xi_\beta.$$

By virtue of the previous definition, the classical notion of ellipticity for operators is naturally generalized to systems as the condition of *invertibility* of the symbol $p(\xi)$.

Now we introduce a stronger notion of ellipticity. An operator as in (B.1) is strongly elliptic if satisfies the so called *Legendre-Hadamard* condition, that is, if there exists $c > 0$ such that

$$\Re\Big(\sum_{\alpha,\beta=1}^{n}\sum_{j,k=1}^{m} A_{jk}^{\alpha\beta}\xi_\alpha\xi_\beta\eta_j\overline{\eta}_k\Big) \geqslant c|\xi|_{\mathbb{R}^n}^2|\eta|_{\mathbb{C}^m}^2 \qquad \forall\,\xi\in\mathbb{R}^n,\ \eta\in\mathbb{C}^m. \tag{B.2}$$

To conclude our survey about elliptic properties for systems of constant coefficients second order operators, we state the notion of very strong ellipticity: an operator as in (B.1) is very strongly elliptic if satisfies the *Legendre* condition, that is, if there exists $c > 0$ such that

$$\Re\Big(\sum_{\alpha,\beta=1}^{n}\sum_{j,k=1}^{m} A_{jk}^{\alpha\beta}\tau_\beta^k\overline{\tau}_\alpha^j\Big) \geqslant c|\tau|_{\mathbb{C}^{n\times m}}^2 \qquad \forall\,\tau\in\mathbb{C}^{n\times m}.$$

Remark I.13. However for many applications the Legendre condition is too strong. This comes from the fact that the tensor A usually, for example in elasticity theory and in compressible fluids, has symmetries like

$$A_{jk}^{\alpha\beta} = A_{\alpha\beta}^{jk} = A_{j\beta}^{\alpha k} = A_{\alpha k}^{j\beta}. \tag{B.3}$$

These symmetries are called hyperelastic and mean that A only acts on the symmetric part of a matrix and yields again a symmetric matrix. For this situation the appropriate condition reads

$$\Re\Big(\sum_{\alpha,\beta=1}^{n}\sum_{j,k=1}^{m} A_{jk}^{\alpha\beta}\sigma_\beta^k\overline{\sigma}_\alpha^j\Big) \geqslant c|\sigma|_{\mathbb{C}^{m\times m}}^2 \qquad \forall\,\sigma\in\mathrm{Sym}(\mathbb{C}^{m\times m}), \tag{B.4}$$

for some $c > 0$, where $\mathrm{Sym}(\mathbb{C}^{m\times m})$ is the space of symmetric matrices.

Now we are in position to consider our Lamé system $-\Delta^*$. We want to write $-\Delta^*$ in the form of a general system of d second order constant coefficients operator, precisely we want to rewrite $-\Delta^*$ as in (B.1). Let's underline that two different coefficient tensors A may have the same associated operator L, indeed, it's easy to see that both the tensors

$$A_{jk}^{\alpha\beta} := (\lambda + \mu)\delta_{j\alpha}\delta_{k\beta} + \mu\delta_{\alpha\beta}\delta_{jk}$$

and

$$A_{jk}^{\alpha\beta} := \mu(\delta_{\alpha\beta}\delta_{jk} + \delta_{\alpha k}\delta_{\beta j}) + \lambda\delta_{\alpha j}\delta_{\beta k}$$

give the Lamé operator of elasticity (here we used the δ-Kronecker formalism).

The rest of the section will be used to highlight under which conditions about Lamé parameters λ and μ, our operator is elliptic (in sense of the three definitions stated above).

First of all we want to point out that the characteristic matrix (or principal symbol) of $-\Delta^*$ is invertible. This guarantees the ellipticity of the operator in the most classical sense. A straightforward computation shows that the principal symbol of $-\Delta^*$, is a matrix of this form:

$$p(\xi) := \mu|\xi|^2 I + (\lambda + \mu)\xi \otimes \xi, \qquad \forall\, \xi \in \mathbb{R}^d;$$

where $\xi \otimes \xi$ is the dyadic product of ξ and ξ that is defined as $(v \otimes w)_{jk} = v_j w_k$ for all $v = (v_1, v_2, \ldots, v_d)$ and $w = (w_1, w_2, \ldots, w_d)$, and I denotes the $d \times d$ identity matrix. For our convenience we rewrite $p(\xi)$ in this way:

$$p(\xi) = \mu|\xi|^2 \left(I + \frac{\lambda + \mu}{\mu} l(\xi) \right),$$

now the matrix $l(\xi) := \frac{\xi}{|\xi|} \otimes \frac{\xi}{|\xi|}$ is idempotent, i.e $l^2(\xi) = l(\xi)$, and therefore $p(\xi)$ is invertible and it is quite easy to find its inverse, that is

$$p^{-1}(\xi) := \frac{1}{\mu|\xi|^2} \left(I - \frac{\lambda + \mu}{\lambda + 2\mu} l(\xi) \right).$$

Remark I.14. Let's underline that everything makes sense if μ, $\lambda + 2\mu \neq 0$, however our condition (I.1.2) guarantees that the previous assumption is fulfilled.

Now let's see that assuming $\mu > 0$ and $\lambda + 2\mu > 0$, the operator $-\Delta^*$ satisfies the Legendre-Hadamard condition(B.2). Using one of the two possible definitions of the tensor associated with the Lamé operator, we easily have

$$\sum_{\alpha,\beta=1}^{d} \sum_{j,k=1}^{d} A_{jk}^{\alpha\beta} \xi_\alpha \xi_\beta \eta_j \overline{\eta}_k = \sum_{\alpha,\beta=1}^{d} (\lambda + \mu)\, \xi_\alpha \eta_\alpha \xi_\beta \overline{\eta}_\beta + \sum_{\alpha,j=1}^{d} \mu\, \xi_\alpha^2 |\eta_j|^2 = (\lambda + \mu)|\xi \cdot \eta|^2 + \mu|\xi|^2|\eta|^2.$$

At this point, we just need to prove that there exists a constant $c > 0$ such that

$$(\lambda + \mu)|\xi \cdot \eta|^2 + \mu|\xi|^2|\eta|^2 \geqslant c\,|\xi|^2|\eta|^2.$$

It's not difficult to see ([70] Lemma 3.1) that the previous is equivalent to

$$\min\{\mu, \lambda + 2\mu\} > 0,$$

which proves the claim.

Remark I.15. We observe that the Legendre-Hadamard condition can be reformulated in terms of the characteristic matrix saying that our operator is strongly elliptic if the symbol $p(\xi)$ is a positive defined matrix, that is, if there exists $c > 0$ such that

$$(p(\xi)\eta, \eta) = \mu|\xi|^2|\eta|^2 + (\lambda + \mu)|\xi \cdot \eta|^2 \geqslant c|\xi|^2|\eta|^2.$$

Now we analyze the very strong ellipticity. Considering the second form of the tensor A, one can observe that the hyperelasticity condition (B.3) holds. This means that we just need to prove that $-\Delta^*$ satisfies (B.4).

Considering the second form of the tensor A, assuming $\mu > 0$ and $2\mu + d\lambda > 0$, by the same argument exploited in the previous section for proving the closedness of the quadratic form $\overline{Q}_0$, it is easy to see the validity of the condition (B.4), indeed for all $\sigma \in \mathrm{Sym}(\mathbb{C}^{d\times d})$, we have

$$\sum_{\alpha,\beta=1}^{d} \sum_{j,k=1}^{d} A_{jk}^{\alpha\beta} \sigma_\beta^k \overline{\sigma}_\alpha^j = 2\mu \sum_{\alpha,j=1}^{d} |\sigma_\alpha^j|^2 + \lambda |\sum_{\alpha=1}^{d} \sigma_\alpha^\alpha|^2 = 2\mu|\sigma|^2 + \lambda|\mathrm{Tr}(\sigma)|^2 \geqslant c|\sigma|^2.$$

Remark I.16. Note that if one assume A to satisfy (B.4), that is if $-\Delta^*$ is very strongly elliptic, immediately follows the positivity of the quadratic form Q_0 associated with the operator. Let's consider our quadratic form Q_0, we want to write Q_0 in terms of the simmetric matrix $\sigma = (\sigma_j^k)_{1\leqslant j,k\leqslant d} := \frac{1}{2}\big(\frac{\partial u_j}{\partial x_k} + \frac{\partial u_k}{\partial x_j}\big)$. A straightforward computation gives

$$Q_0[u] = \lambda \int_{\mathbb{R}^d} |\sum_{j=1}^{d} \frac{1}{2}\Big(\frac{\partial u_j}{\partial x_j} + \frac{\partial u_j}{\partial x_j}\Big)|^2 + 2\mu \sum_{j,k=1}^{d} \int_{\mathbb{R}^d} |\frac{1}{2}\Big(\frac{\partial u_j}{\partial x_k} + \frac{\partial u_k}{\partial x_j}\Big)|^2 = 2\mu \int_{\mathbb{R}^d} |\sigma|^2 + \lambda \int_{\mathbb{R}^d} |\mathrm{Tr}(\sigma)|^2,$$

since $\sigma \in \mathrm{Sym}(\mathbb{C}^{d\times d})$ and $-\Delta^*$ satisfies (B.4), we conclude that $Q_0[u] \geqslant c\int_{\mathbb{R}^d} |\sigma|^2$ which, in particular, gives the positivity. Summing up, if $-\Delta^*$ is very strongly elliptic, that is if $\mu > 0$ and $2\mu + d\lambda > 0$, our quadratic form is positive.

Remark I.17. Let's underline that if one assume for the Lamé operator just to be strongly elliptic, the quadratic form associated with the operator is still positive under weaker conditions about the Lamé parameters. Indeed by Plancherel theorem we can see that

$$\begin{aligned} Q_0[u] &:= \lambda \|\sum_{j=1}^{d} \frac{\partial u_j}{\partial x_j}\|_{L^2}^2 + \frac{\mu}{2} \sum_{j,k=1}^{d} \|\frac{\partial u_j}{\partial x_k} + \frac{\partial u_k}{\partial x_j}\|_{L^2}^2 = \lambda \|\sum_{j=1}^{d} \xi_j \hat{u}_j\|_{L^2}^2 + \frac{\mu}{2} \sum_{j,k=1}^{d} \|\xi_k \hat{u}_j + \xi_j \hat{u}_k\|_{L^2}^2 \\ &= \lambda \int_{\mathbb{R}^d} |\sum_{j=1}^{d} \xi_j \hat{u}_j|^2 + \mu \sum_{j,k=1}^{d} \int_{\mathbb{R}^d} \xi_k^2 |\hat{u}_j|^2 + \mu \int_{\mathbb{R}^d} |\xi \cdot \hat{u}|^2 = (\lambda + \mu) \int_{\mathbb{R}^d} |\xi \cdot \hat{u}|^2 + \mu \int_{\mathbb{R}^d} |\xi|^2 |\hat{u}|^2. \end{aligned}$$

Now, if $\mu > 0$ and $\lambda + 2\mu > 0$, the Lamé operator satisfies (B.2), therefore there exists $c > 0$ such that

$$(\lambda + \mu) \int_{\mathbb{R}^d} |\xi \cdot \hat{u}|^2 + \mu \int_{\mathbb{R}^d} |\xi|^2 |\hat{u}|^2 \geqslant c \int_{\mathbb{R}^d} |\xi|^2 |\hat{u}|^2,$$

and clearly this provides the positivity of the quadratic form.

C.| Spectrum of closed linear operator

We want to recall, although totally classical, the definition of the spectrum of a *closed* linear operator H on a Hilbert space $\mathcal{H}$ and of its usual partitions.

First of all we give the notion of resolvent of H : a complex number z belongs to the *resolvent set* $\rho(H)$ of H if the operator $H - zI$ has a bounded everywhere on $\mathcal{H}$ defined inverse $(H - zI)^{-1}$, called the resolvent of H.

Let us observe that since H is assumed to be close, then the requirement that the inverse $(H - zI)^{-1}$ is bounded can be omitted (this readily follows from closed graph Theorem). Therefore we can give the following definition for the resolvent of a closed linear operator H :

$$\rho(H) := \{z \in \mathbb{C} | \quad H - zI \colon \mathscr{D}(H) \to \mathcal{H} \quad \text{is bijective}\}.$$

Here, as usual $\mathcal{H}$ is a notation for the domain of the closed operator H. Its complement

$$\sigma(H) := \mathbb{C} \backslash \rho(H)$$

is called the *spectrum* of H and denoted by $\sigma(H)$. In other words the spectrum of a closed operator H in a complex Hilbert space $\mathcal{H}$ is determined by the set of points $z \in \mathbb{C}$ for which $H - zI \colon \mathscr{D}(H) \to \mathcal{H}$ is not bijective.

It is customary to have the following partition of the spectrum by means of three disjoint subsets of $\sigma(H)$ which saturate the spectrum itself: they are respectively *point spectrum* that is the set of all eigenvalues of H

$$\sigma_p(H) := \{z \in \mathbb{C} | \quad H - zI \quad \text{is not injective}\},$$

the *continuous* spectrum

$$\sigma_c(H) := \{z \in \sigma(H) \backslash \sigma_p(H) | \quad \overline{R(H - zI)} = \mathcal{H}\},$$

and the *residual* spectrum

$$\sigma_r(H) := \{z \in \sigma(H) \backslash \sigma_p(H) | \quad \overline{R(H - zI)} \neq \mathcal{H}\}.$$

The following relation holds true

$$\sigma(H) = \sigma_p(H) \cup \sigma_c(H) \cup \sigma_r(H).$$

We notice that for self-adjoint operators H, $\sigma(H) \subset \mathbb{R}$ and $\sigma_r(H) = \varnothing$.

We add a last remark: for a self-adjoint operators a different disjoint partition can be given:

$$\sigma(H) = \sigma_{\text{disc}}(H) \cup \sigma_{\text{ess}}(H).$$

$\sigma_{\text{disc}}(H)$ is the so called *discrete spectrum* and it is the set of isolated eigenvalues of H which have finite multiplicity. Its complement $\sigma_{\text{ess}}(H)$, called *essential spectrum* consists of either accumulation points of $\sigma(H)$ or isolated eigenvalues of H which have infinite multiplicity.

D. | An equivalent formulation for Sobolev inequality

In this appendix we will prove the following equivalence:

Lemma I.8. *Let $d \geqslant 3$ and let $\widetilde{V}$ be such that $\|\widetilde{V}\|_{\frac{d}{2}} \leqslant S_d$, then the Sobolev inequality*

$$\int_{\mathbb{R}^d} |\nabla u|^2 \, dx \geqslant S_d \Big(|u|^q \Big)^{\frac{2}{q}}, \qquad q = \frac{2d}{d-2} \tag{D.1}$$

is equivalent to

$$\int_{\mathbb{R}^d} \widetilde{V} - |u|^2 \, dx \leqslant \int_{\mathbb{R}^d} |\nabla u|^2 \, dx. \tag{D.2}$$

Proof. (D.1) $\Rightarrow$ (D.2) By Hölder inequality and our hypotheses we easily obtain

$$\int_{\mathbb{R}^d} \widetilde{V}|u|^2 \, dx \leqslant \|\widetilde{V}\|_{\frac{d}{2}} \Big(\int_{\mathbb{R}^d} |u|^q \, dx \Big)^{\frac{2}{q}} dx \leqslant S_d \Big(\int_{\mathbb{R}^d} |u|^q \, dx \Big)^{\frac{2}{q}} \leqslant \int_{\mathbb{R}^d} |\nabla u|^2 \, dx.$$

(D.2)$\Rightarrow$ (D.1) For any $V \in L^{\frac{d}{2}}(\mathbb{R}^d)$, by virtue of our hypothesis we have

$$\left| \int_{\mathbb{R}^d} V|u|^2 \, dx \right| = \frac{\|V\|_{\frac{d}{2}}}{S_d} \int_{\mathbb{R}^d} \widetilde{V}|u|^2 \, dx \leqslant \frac{1}{S_d} \|V\|_{\frac{d}{2}} \int_{\mathbb{R}^d} |\nabla u|^2 \, dx,$$

where we defined $\widetilde{V} := \frac{S_d}{\|V\|_{\frac{d}{2}}} V$ in order to use (D.2).

The last inequality shows that for any $u \in H^1(\mathbb{R}^d)$ the integral $\int_{R^d} V|u|^2 \, dx$ is a linear functional on $L^{\frac{d}{2}}(\mathbb{R}^d)$, with norm less then or equal to $\frac{1}{S_d} \int_{\mathbb{R}^d} |\nabla u|^2 \, dx$. Then, using the Riesz representation theorem, we conclude that $|u|^2 \in L^{\frac{q}{2}}(\mathbb{R}^d)$, with q as in the statement and that

$$\Big(\int_{\mathbb{R}^d} |u|^q \, dx \Big)^{\frac{2}{q}} = \|u^2\|_{\frac{q}{2}} = \|T_u\|_{\left(L^{\frac{d}{2}}\right)^*} \leqslant \frac{1}{S_d} \int_{\mathbb{R}^d} |\nabla u|^2 \, dx,$$

where $T_u(V) := \int_{\mathbb{R}^d} V|u|^2 \, dx$.

□

Part II

Unique Continuation for ZK Equation

We we will turn to the second and deeper part of this thesis, namely the study of unique continuation's properties connected with the Zakharov-Kuznetsov (ZK) equation arises from plasma physics. The discussion in this part is the fruit of a collaboration with Felipe Linares.

II.1.| Introduction

In the present work, we shall provide a result in matter of unique continuation for the so called Zakharov-Kuznetsov equation

$$\partial_t u + \partial_x^3 u + \partial_x \partial_y^2 u + u \partial_x u = 0, \qquad (x, y) \in \mathbb{R}^2, \quad t \in [0, 1]. \tag{II.1.1}$$

Actually our designs concern the analysis of unique continuation properties attached to a symmetric version of the previous, more precisely

$$\partial_t u + (\partial_x^3 + \partial_y^3) u + 4^{-\frac{1}{3}} u (\partial_x + \partial_y) u = 0, \qquad (x, y) \in \mathbb{R}^2, \quad t \in [0, 1]. \tag{II.1.2}$$

Here will be treated the most recent notion of unique continuation, in other words we crave to give an answer to the following question:

> QUESTION. *Let u_1 and u_2 be two solutions of (II.1.2) which kind of assumptions for the behavior of their difference $u_1 - u_2$ at two distinct times we need in order to ensure the uniqueness $u_1 \equiv u_2$?*

Before stating our main theorem which contains the answer to the above question, we want to make a quick description of the equation we are dealing with. Equation (II.1.1) is one of the variants of the $(2+1)$-dimensional generalization of the Korteweg-de Vries (KdV) equation that reads

$$\partial_t u + \partial_x^3 u + u\partial_x u = 0, \qquad x \in \mathbb{R}, \quad t \in [0,1]. \tag{II.1.3}$$

The equation (II.1.1) was introduced in the context of plasma physic by Zakharov and Kuznetsov, indeed in [90] they formally deduced that the propagation of nonlinear ion-acoustic waves in magnetized plasma is governed by this mathematical model. Further, this equation became known as the Zakharov-Kuznetsov equation.

The problem of local and global well-posedness for the Cauchy problem associated to (II.1.1) has extensively been studied in the literature. In [35] Faminskii showed local and global well-posedness for initial data in $H^m(\mathbb{R}^2)$, $m \geqslant 1$, integer. His method of proof was inspired by the argument developed by Kenig, Ponce and Vega [53] to prove local well-posedness for the initial value problem associated to the KdV equation. Indeed he proved the local smoothing effect, a maximal function estimate as well as a Strichartz type inequality for the linear equation to obtain local well-posedness by the contraction mapping principle. Then, as usual, the global result follows as a consequence of the presence of L^2 and H^1 conserved quantities for solutions of (II.1.1). In [66] Linares and Pastor established the local well-posedness for initial data in $H^s(\mathbb{R}^2)$, $s > \frac{3}{4}$. Moreover, even though it can be shown, performing a scaling argument, that the critical space for this equation is $L^2(\mathbb{R}^2)$, they also proved that well-posedness is not possible in such space. Last but not least it is worthy to be mentioned the work by Grünrock and Herr [43] in which an improvement of the latest threshold was given. Precisely they proved the local well-posedness for in $H^s(\mathbb{R}^2)$ with $s > \frac{1}{2}$. Without attempting to be complete we refer to [7, 66, 68, 67] and references therein for other result of this type and several additional remarks concerning with properties of this equation.

Now we want to give mention to the motivations which primarily are under our intent. In order to do that we shall comment on a previous related result. For the case of the original ZK equation (II.1.1) in a recent work [14] Bustamante, Isaza and Mejía proved the following result:

Theorem II.1. *Suppose that for some small* $\varepsilon > 0$

$$u_1, u_2 \in C\big([0,1]; H^4(\mathbb{R}^2) \cap L^2((1+x^2+y^2)^{\frac{4}{3}+\varepsilon}\,dxdy)\big) \cap C^1\big([0,1]; L^2(\mathbb{R}^2)\big),$$

are solutions of (II.1.1)*. Then there exists a universal constant* $a_0 > 0$*, such that if for some* $a > a_0$

$$u_1(0) - u_2(0), u_1(1) - u_2(1) \in L^2(e^{a(x^2+y^2)^{\frac{3}{4}}}\,dxdy),$$

then $u_1 \equiv u_2$.

This result was obtained following the scheme in [29], that is by applying two types of estimates: a lower estimate which follows after performing a suitable Carleman estimate and an upper bound for the L^2- norm of the solution and its derivatives.

This result, as the authors themselves pointed out, can't be optimal, indeed the symmetric character in x and y of the decay assumption does not reflect the non symmetric form of the equation (II.1.1). To justify this sentence we want to come back to discuss the structure of the ZK equation, or better the structure of the associated linear problem

$$\partial_t u + \partial_x^3 u + \partial_x \partial_y^2 u = 0. \tag{II.1.4}$$

Looking at the structure of the equation (II.1.4) that, roughly speaking, resembles a KdV equation in the x variable and a parabolic equation in y, the result that should naively be expected as optimal is that one in which the decay assumed is of the form $e^{-ax^{\frac{3}{2}}-by^2}$, i.e. a no more symmetric decay in the two variables. Indeed the exponent $\frac{3}{2}$ in the x variable would reflect the decay of the fundamental solution to the linearized KdV (the Airy function) and in the y variable we would suppose to have a gaussian decay which comes out from the parabolic heritage.

This uncertainty about the right decay of the fundamental solution of the linearized ZK comes out from the fact that, contrary to KdV equation, ZK was very much less investigated. Recently Faminskii and Antonova in [36] cleared up any confusion: indeed in this quoted work they proved that actually the fundamental solution to the operator $\partial_t + \partial_x^3 + \partial_x \partial_y^2$ still displays an exponential decay but just in the x variable. Let us consider the initial value problem

$$\begin{cases} \partial_t u + \partial_x^3 u + \partial_x \partial_y^2 u = 0, \\ u(x,y,0) = u_0(x,y), \end{cases}$$

it is easy to see that the solution of this linear problem is given by

$$u(x,y,t) = \frac{\theta(t)}{t^{\frac{2}{3}}} S\left(\frac{x}{t^{\frac{1}{3}}}, \frac{y}{t^{\frac{1}{3}}}\right) * u_0(x,y),$$

where

$$S(x,y) := \frac{1}{2\pi}\mathcal{F}^{-1}\big[(\xi,\eta) \mapsto e^{i(\xi^3+\xi\eta^2)}\big] = \frac{1}{4\pi^2}\int_{\mathbb{R}^2} e^{i\xi x + i\eta y} e^{i(\xi^3+\xi\eta^2)}\, d\xi d\eta, \tag{II.1.5}$$

θ is the Heaviside function and $\mathcal{F}^{-1}$ represents the inverse Fourier transform. The rigorous result in [36] (Lemma 7) in which the correct decay of the fundamental solution turns out is the following:

Lemma II.1. *Let $S(x,y)$ be as in (II.1.5), for any $x \in \mathbb{R}$ and integer $k \geqslant 0$ the derivative $\partial_x^k S(x,y)$ belongs to the Schwartz space $\mathcal{S}(\mathbb{R})$ with respect to y and there exists a constant $c_0 > 0$ such that for any $x_0 \in \mathbb{R}$, integer $m \geqslant 0$ and multi-index ν*

$$(1+|y|)^m |D^\nu S(x,y)| \leqslant c(m,|\nu|,x_0) e^{-c_0(x-x_0)^{\frac{3}{2}}} \quad \forall\, x \geqslant x_0,\ \forall\, y \in \mathbb{R}. \tag{II.1.6}$$

Motivated by this deeper knowledge of the ZK equation, our first attempt was to draw inspiration from the strategy in [14] and, as a starting point, try to perform a Carleman estimate of this form

$$\|e^{\alpha\theta(x,y,t)}g\|_X \leqslant c(\alpha)\|e^{\alpha\theta(x,y,t)}(\partial_t + \partial_x^3 + \partial_x\partial_y^2)g\|_X.$$

where $e^{\alpha\theta}$, clearly, must remind the *right* exponential decay of the fundamental solution associated with the operator $\partial_t + \partial_x^3 + \partial_x\partial_y^2$ that first came to light in the aforementioned work [36].

In [14] the estimate above was proved by taking as θ the following function

$$\theta(x,y,t) = \left(\frac{x}{R} + \phi(t)\right)^2 + \frac{y}{R}^2,$$

and requiring for α to equal (up to a multiplicative constant) the quantity $R^{\frac{3}{2}}$. But clearly, by virtue of the previous remarks, with this choice no optimal result can occur.

Therefore our attempt was to choose as θ the function

$$\theta(x,y,t) = \theta(x,t) = \left(\frac{x}{R} + \phi(t)\right)^2$$

in order to let appear the decay just in the x coordinate.

Unfortunately it turns out that the absence of the y component in our choice of θ, or better of a nonlinear dependance of θ by y, does not let the argument work, in other word we are not able to obtain the desired Carleman estimate. This means that new ideas to tackle this problem have still to be found and this will be matter of future investigations.

All these facts brought us to a slightly alternative analysis which find its source of inspiration, at least at the beginning, in already quoted work of Grünrock and Herr [43]. Even if the problem addressed therein relates to a different topic then one we aim to solve, namely the local well-posedness for the Cauchy problem associated to the ZK equation, there it was shown that making use of a very simple tool that is a linear change of variable, essentially a rotation, equation (II.1.1) can be written in a symmetric form, precisely (II.1.2).

It is worthy to be underlined that for an equation of this form it is reasonable to believe that the correct and optimal decay to possibly guarantee the unique continuation principle should be exactly the one that appears in the paper [14], that is $[(x^2+y^2)^{\frac{1}{2}}]^{\frac{3}{2}}$. Indeed in (II.1.2) we can recognize the structure of a two dimensional KdV equation and the decay $r^{\frac{3}{2}}$, with $r := (x^2 + y^2)^{\frac{1}{2}}$, resembles the asymptotic behavior of the Airy function, the fundamental solution of the linearized KdV.

Encouraged by the previous fact we proved the following result which sharpness follows analogously to [29] where the construction of local solutions with the estimated decay for 1-dimensional KdV is provided.

Theorem II.2. *Suppose that for some $\varepsilon > 0$,*

$$u_1, u_2 \in C\big([0,1]; H^4(\mathbb{R}^2) \cap L^2((1+x^2+y^2)^{\frac{4}{3}+\varepsilon}\, dxdy)\big) \cap C^1([0,1]; L^2(\mathbb{R}^2)), \tag{II.1.7}$$

are solutions of the equation (II.1.2).

Then there exists a universal constant a_0, such that if for some $a > a_0$

$$u_1(0) - u_2(0), u_1(1) - u_2(1) \in L^2\big(e^{a(x^2+y^2)^{\frac{3}{4}}} dxdy\big), \tag{II.1.8}$$

then $u_1 \equiv u_2$.

As it is customary in these contexts, we will see that our nonlinear result, Theorem II.2, will be a consequence of the following *linear* result:

Theorem II.3. *Suppose that for some $\varepsilon > 0$,*

$$v \in C\big([0,1]; H^3(\mathbb{R}^2) \cap L^2((1+x^2+y^2)^{\frac{4}{3}+\varepsilon}\, dxdy)\big) \cap C^1([0,1]; L^2(\mathbb{R}^2)),$$

is a solution of

$$\partial_t v + (\partial_x^3 + \partial_y^3)v + a_1(x,y,t)(\partial_x + \partial_y)v + a_0(x,y,t)v = 0, \tag{II.1.9}$$

where $a_0 \in L^\infty \cap L^2_x L^\infty_{y,t}$ and $a_1 \in L^\infty \cap L^2_x L^\infty_{yt} \cap L^1_x L^\infty_{yt}$.

Then there exists a universal constant $a_0 > 0$ such that if for some $a > a_0$

$$v(0), v(1) \in L^2(e^{a(x^2+y^2)^{\frac{3}{4}}}\, dxdy), \tag{II.1.10}$$

then $v \equiv 0$.

Remark II.1. The linear equation (II.1.9) comes out from the fact that we are interested in a result involving the difference of two solutions u_1 and u_2 of (II.1.2). It is easy to see that defining $v := u_1 - u_2$ this satisfies

$$\partial_t v + (\partial_x^3 + \partial_y^3)v + 4^{-\frac{1}{3}} u_1(\partial_x + \partial_y)v + 4^{-\frac{1}{3}}(\partial_x + \partial_y)u_2 v = 0,$$

that clearly is a particular case of (II.1.9) choosing $a_0 = 4^{-\frac{1}{3}}(\partial_x + \partial_y)u_2$ and $a_1 = 4^{-\frac{1}{3}} u_1$.

Before moving on in outlining the main steps in the proof of our result, as an matter of keen interest for the mathematical community, we want to devoted the coming section to give an overview, from its birth to the more recent developments, of the main issues and results in unique continuation.

II.1.1. Unique continuation

Now we are in position to discuss in greater depth the notion of unique continuation.

It is well known that a real analytic function has the property that if it vanishes sufficiently fast at a point then it is vanishing all over its definition domain. This property is called unique continuation property of real analytic functions.

The following question comes naturally: which other classes of functions enjoy this property?

A first further example of such a class of functions is represented by the harmonic functions, indeed it is common knowledge that harmonic functions are real analytic and therefore they still display the unique continuation property.

Actually, it was shown, see for instance the classical Holmgren's theorem, that this property is shared by solutions of more general elliptic partial differential equations, more precisely if $P(x, D)$ is an elliptic differential operator with *real analytic* coefficients and $P(x, D)u = 0$ in a bounded open connected set, then u is real analytic and again we can conclude that the unique continuation property holds.

Therefore, now, when one is dealing with a unique continuation result it is customary to refer to any statement of the following type:

> *Let $\Omega \subset \mathbb{R}^d$ be a bounded connected open set. Given a linear partial differential operator P, if a solution u to $Pu = 0$ in the region Ω satisfies that u vanishes to infinite order at $x_0 \in \Omega$, in the sense that*
>
> $$\lim_{r\to 0} \frac{1}{r^N} \int_{B(x_0,r)} |u|^2 \, dx = 0, \quad \textit{for all} \quad N \geqslant 0,$$
>
> *then $u = 0$ in Ω.*

The previous statement is known as *Strong Unique Continuation Principle.*

Enlarging the point x_0 to an open set B we can get the same conclusion if we assume the vanishing of the solution in that region B. This provides a weaker version of the unique continuation principle stated above:

> *Let $\Omega \subset \mathbb{R}^d$ be a bounded connected open set. Given a linear partial differential operator P, if a solution u to $Pu = 0$ in the region Ω satisfies that $u = 0$ in some ball B contained in Ω, then $u = 0$ in Ω.*
> Namely *the solution u is uniquely determined in the larger set Ω by its behavior in the smaller region B.*

This version is known as *Weak Unique Continuation Principle.*

A naive explanation of why the name "unique continuation" is used can be easily given. In this regard one can just observe that since the operator P is linear, the unique continuation principles, as stated above, ensure that any two solutions u_1 and u_2 must coincide in the whole domain Ω once they coincide in a smaller region.

In other words this means that the unique continuation principle guarantees the problem to have a unique solution.

From the cornerstone work by Hadamard [44] in which the notion of well-posed Cauchy problems was coined, it took three decades to realize that it would be desirable, for example for the applications to nonlinear problems, to establish the unique continuation property for operator for which is not required a strong analyticity structure as, instead, was done so far.

The first results in establishing the strong unique continuation property for elliptic operators whose coefficients are not necessarily real analytic, are to be found in the pioneering work [17] by Torsten Carleman dating back to 1939. Here he proved the strong unique continuation property for

$$P(x, D) = \Delta + V(x), \qquad \text{with} \qquad V \in L^\infty_{\text{loc}}(\mathbb{R}^2).$$

To avoid analyticity conditions, Carleman introduced the type of estimates that bear his name and that have permeated essentially all the subsequent works in the subject. Roughly speaking these are weighted estimates in which weights are chosen to be extremely concentrated in certain parts of the underlying domain and it is precisely for this reason that they represent a successful tool in proving uniqueness' results, indeed concentrations can be created close to points at which informations of the function under consideration are given.

Carleman's method was improved and extended beyond the elliptic operators to address the unique continuation principle for several other equations, even for evolution equations such as parabolic and dispersive equations.

Even though in this work we are mainly interested in dispersive equations, we are going to mention briefly how the unique continuation results can be phrased for parabolic equations, greater attention to the dispersive equation will be devoted later on.

II.1.1.1. Unique continuation for parabolic equations

As already pointed out by Escauriaza in [26], for second order linear parabolic operators with *time-independent* coefficients, such as

$$\partial_t u - \Delta u + V(x)u = 0, \tag{II.1.11}$$

the strong unique continuation property is reduced to the previously established elliptic counterparts as shown in [65]. This reduction in essence relies on a representation formula for solutions of parabolic equations in terms of eigenfunctions of the corresponding elliptic operator. This clearly means that this technique cannot be applied to more general equations with *time-dependent* coefficients. Therefore in order to accomplish our aim that, roughly speaking, is to find sufficient conditions for a solution of a parabolic equation to vanish, we need to examine the structure of the equation thoroughly. As a first sight we consider the prototypical example of parabolic PDE, the heat equation

$$\partial_t u - \Delta u = f, \tag{II.1.12}$$

in $\mathbb{R}^d \times (0,1]$,

As it is well-known the following uniqueness for the heat equation holds:

If $|u(x,t)| \leqslant Ce^{\lambda|x|^2}$ *for all* $t \in (0,1]$ *and* $u(x,0) \equiv 0$, *then* $u \equiv 0$ *in* $\mathbb{R}^d \times [0,1]$.

This kind of *forward in time* uniqueness for the heat equation is quite classical and easily follows by an application of the maximum principle for unbounded domains (see [34]). A rather more subtle question, due to the lack of time-reversal symmetry for the heat equation which describes irreversible processes or, in other words, phenomena with a preferential direction of time, concerns uniqueness *backward in time*. Although unexpected, a backward uniqueness result still holds for the heat equation. We will present the statement for the more general parabolic equation with *time-dependent* coefficients

$$\partial_t u - \Delta u + W(x,t) \cdot \nabla u + V(x,t)u = 0, \tag{II.1.13}$$

in $\mathbb{R}^d \times (0,1]$, with $|W| \leqslant N, |V| \leqslant M$.

Let u *be a solution of* (II.1.13) *such that* $|u(x,t)| \leqslant C_0$ *and* $u(x,1) \equiv 0$, *then* $u \equiv 0$ *in* $\mathbb{R}^d \times [0,1]$

(see [60]). This result has been extended by Escauriaza, Kenig, Ponce and Vega in [27] in which they proved that the backward uniqueness still holds if, instead of assuming at $t = 1$ that $u(x,1) \equiv 0$, one assumes that $|u(x,1)| \leqslant Ce^{-C|x|^{2+\varepsilon}}$, for some $\varepsilon > 0$. The proof of this result, as in the elliptic setting, makes use of the parabolic version of Carleman estimates.

Now we are in position to treat in greater details dispersive equations, making a deeper analysis of how to rephrase the unique continuation principle in this setting. Moreover the historical developments and the achieved results in matter of unique continuation for this kind of equations will be resumed.

II.1.1.2. Unique continuation for dispersive equations

As already mentioned, particular attention has been paid to the unique continuation results for nonlinear dispersive equations, especially for Schrödinger and KdV equations. This two equations, together with most dispersive models, enjoy a time-reversal symmetry. Roughly speaking, this means that every solution to these equations comes with a counterpart which evolves backward in time if compared to the original solution, this entails that the forward behavior of solutions is typically very similar to the backward one. The time-reversibility represents the first obstacle into understanding what is the analog of the parabolic unique continuation results for dispersive equations, indeed, clearly, backward in time does not make any sense in this context. Let us start considering linear Schrödinger equation of the form

$$i\partial_t u + \Delta u + V(x,t)u = 0, \tag{II.1.14}$$

in $\mathbb{R}^d \times [0,1]$. Then, choosing $V(x,t) = F(u(x,t), \overline{u}(x,t))$ we are allowed to consider nonlinear equations of the type

$$i\partial_t u + \Delta u + F(u, \overline{u}) = 0. \tag{II.1.15}$$

In order to understand how to formulate unique continuation results in this setting and which kind of assumptions about the solution has to be made in order to get those results, particular relevance can be attributed to the Heisenberg uncertainty principle and its connection with the so-called Fourier uncertainty.

The Heisenberg uncertainty principle represents one of the fundamental implications of quantum theory. Vaguely speaking it states that *certain pairs of physical quantities cannot be measured simultaneously with arbitrary accuracy.* Before moving on in giving the precise mathematical statement of Heisenberg uncertainty principle, we would like to recall some very basic facts from quantum mechanics.

Let us turn our attention to observables. Generally speaking *an observable is a quantity that can be experimentally measured in a given physical framework.* The interpretive rules in quantum mechanics dictate that a physical observable a (position, momentum, energy etc.) has a quantum mechanics counterpart that is a self-adjoint operator A on the state space $L^2(\mathbb{R}^d)$.

For our aims, as we will see in a moment, it is interesting to see how quantum mechanics predicts mean value, or expectation, $\langle a \rangle_\psi$ and variance $\mathrm{Var}(a)_\psi$ of a physical observable a prepared in the state ψ in terms of analogue quantities involving the associated self-adjoint operator A. We recall that in a probability setting the mean value of a certain non deterministic quantity a is the best guess of the value of the quantity, strictly connected to expectation is the notion of variance, which quantifies the uncertainty on the quantity for which a guess of its "real" value is given.

Quantum theory gives precise rules in order to predict those quantities:

- $\langle a \rangle_\psi = \langle A \rangle_\psi := \langle \psi, A\psi \rangle$,
- $\mathrm{Var}(a)_\psi = \langle (A - \langle A \rangle_\psi \mathbb{1})^2 \rangle_\psi := \langle \psi, (A - \langle A \rangle_\psi \mathbb{1})^2 \psi \rangle = \|(A - \langle A \rangle_\psi \mathbb{1})\psi\|^2_{L^2(\mathbb{R}^d)}$,

where, also in this part, the brackets $\langle \cdot, \cdot \rangle$ is the notation used to indicate the inner product of $L^2(\mathbb{R}^d)$.

Now we are in position to state the Heisenberg uncertainty principle of quantum mechanics.

Let A, B be two densely defined self-adjoint operators on $L^2(\mathbb{R}^d)$. Suppose that there exists a linear, dense subspace $S \subset L^2(\mathbb{R}^d)$ invariant for both A and B, therefore

$$\mathrm{Var}(A)_\psi \,\mathrm{Var}(B)_\psi \geqslant \frac{1}{4} |\langle \psi, [A, B]\psi \rangle|^2,$$

or, what is equivalent

$$\sigma_\psi(A)\, \sigma_\psi(B) \geqslant \frac{1}{2} \,|\langle \psi, [A, B]\psi \rangle|, \tag{II.1.16}$$

where σ_ψ represents the standard deviation, namely the square root of the variance.

Recalling that the standard deviation quantifies the precision of an observable's measurement (the smaller it is, the more precise is the measurement), the previous statement says that the possibility to measure simultaneously two observables with arbitrary precision is strictly connected with the commutation relations between the two corresponding self-adjoint operators. More precisely, if the operators which theoretically represent the physical observables we are focusing on have non-vanishing commutator, this yields a non trivial lower bound for the product of the precisions of the observables' measurements, so the more certain we are about the measurement of one of the observable, the less certain we can be about the other one.

For our purpose we are mainly interested in two observables: momentum and position or better in the j^{th} component of them $j = 1, 2, \ldots, d$. Quantum theory represents the two observables respectively by the operator $A = P_j := -i\frac{\partial}{\partial_{x_j}}$ and by $B = X_j$ the multiplication operator by the j^{th}−coordinate x_j. More precisely the action of the two operators is as follows:

$$P_j \colon \psi(x) \mapsto -i\frac{\partial}{\partial_{x_j}}\psi(x),$$
$$X_j \colon \psi(x) \mapsto x_j \psi(x).$$

A straightforward computation gives the that the commutator of these two operators is

$$[A, B] := [P_j, X_l] = -i\delta_{j,l}\mathbb{1},$$

where $\delta_{j,l}$ is the kronecker symbol, defined as it is usual.

By virtue of Heisenberg uncertainty principle as stated in (II.1.16) we can conclude

$$\sigma_\psi(P_j)\,\sigma_\psi(X_l) \geqslant \frac{1}{2}\delta_{j,l},$$

roughly speaking, this means that if we want to measure position and momentum in the same direction, namely $j = l$, we have a non-trivial lower bound for the precision of their simultaneous measurements. In particular the following lower bound

$$\sigma_\psi(P_j)\,\sigma_\psi(X_j) \geqslant \frac{1}{2} \tag{II.1.17}$$

holds.

In order to see explicitly the deep attachment between the uncertainty principle of quantum mechanics and the not yet stated Fourier uncertainty, we want to compute explicitly $\sigma_\psi(P_j)$ and $\sigma_\psi(X_j)$ making use of the interpretative rules which appear above.

In order to do that we formerly compute $\mathrm{Var}(P_j)_\psi$ and $\mathrm{Var}(X_j)_\psi$. Using Plancherel theorem for the operator P_j and simply the definition of variance for X_j we have

$$\mathrm{Var}(P_j) := \|P_j^2\psi\|^2_{L^2(\mathbb{R}^d)} = \|\widehat{(P_j)^2\psi}\|^2_{L^2(\mathbb{R}^d)} = \int_{\mathbb{R}^d}(\xi_j - \bar{\xi}_j)^2\,|\widehat{\psi}(\xi)|^2\,d\xi =: \mathrm{Var}(\widehat{\psi}),$$
$$\mathrm{Var}(X_j) := \|X_j^2\psi\|^2_{L^2(\mathbb{R}^d)} = \int_{\mathbb{R}^d}(x_j - \bar{x}_j)^2\,|\psi(x)|^2\,dx =: \mathrm{Var}(\psi),$$

where here we used the following notation $\bar{\xi}_j = \langle P_j\rangle_\psi$ and $\bar{x}_j = \langle X_j\rangle_\psi$.

From the previous identities we can infer that the variance of the observable momentum for a particle in the state ψ is the same as the variance of the observable position for a particle in the state $\widehat{\psi}$.

Notice that if ψ is highly concentrated near the mean value $\bar{x}_j$, by virtue of our interpretation of the wave function ψ, this means that there is a high probability that the location of the $j^{th}-$ coordinate x_j of the particle is near $\bar{x}_j$ and so we expect the uncertainty to be small. This is precisely what happens, indeed looking at the quantity $\int_{\mathbb{R}^d}(x_j - \bar{x}_j)^2\,|\psi(x)|^2\,dx$ which represents the variance, in other words the uncertainty, this quantity is small because most of the contribution to the integral arises from values of x_j near to $\bar{x}_j$. Clearly, the same reasoning can be done for the momentum.

Now plugging these two quantities in (II.1.17) or better in the square of the quoted identity we get

$$\left(\int_{\mathbb{R}^d}(x_j - \bar{x}_j)^2|\psi(x)|^2\,dx\right)\left(\int_{\mathbb{R}^d}(\xi_j - \bar{\xi}_j)^2|\widehat{\psi}(\xi)|^2\,d\xi\right) \geqslant \frac{1}{4}. \tag{II.1.18}$$

The previous represents the rigorous statement of the Fourier uncertainty. Indeed, by virtue of the lower bound (II.1.18), if the first term in the product on the left-hand side is small and this can occur making ψ more concentrated near $\bar{x}_j$, therefore the second term in the product cannot be small as well and this forces $\widehat{\psi}$ to be sufficiently spread. We can re-phrase the previous result in this way: *a function and its Fourier transform cannot both be essentially localized.*

Now we are interested into see what inequality (II.1.18) says if our particle is described by a function ψ which resembles a gaussian. First of all for notational simplicity we assume $\bar{\xi}_j := \langle P_j \rangle_\psi = 0$ and that $\bar{x}_j := \langle X_j \rangle_\psi = 0$.

Let us consider an L^2- normalized gaussian function, that is $\psi_\sigma(x) = c_\sigma e^{-\frac{x^2}{4\sigma^2}}$, with c_σ such that $\|\psi\|_{L^2(\mathbb{R}^d)} = 1$. A straightforward computation gives $\mathrm{Var}(\psi_\sigma)$ as defined above is related to the parameter σ, more precisely equals σ^2.

Now, without attempting to be rigorous, assuming our particle described by $\psi(x) = \mathcal{O}(e^{-\frac{|x|^2}{\beta^2}})$ and that its Fourier transform $\widehat{\psi}(\xi) = \mathcal{O}(e^{-\frac{4|\xi|^2}{\alpha^2}})$, therefore we expect the following values for their variances:

$$\mathrm{Var}(\psi) = \frac{\beta^2}{4}, \qquad \mathrm{Var}(\widehat{\psi}) = \frac{\alpha^2}{16},$$

that, in particular, gives

$$\sigma(\psi) = \frac{\beta}{2}, \qquad \sigma(\widehat{\psi}) = \frac{\alpha}{4}.$$

Plugging these two explicit values of the standard deviations in (II.1.17) we get a constraint for the parameters α and β, namely

$$\alpha\beta \geqslant 4.$$

Also in this setting we can re-phrase the constraint above saying that if we want to peak our "gaussian" ψ at its mean value (the origin in this case), that is letting α become more and more small, then this coerces β to be remarkably large, i.e the gaussian $\widehat{\psi}$ has to become increasingly flattened.

Now in view of the above, leaving aside the quantum mechanical interpretation, the following result should appear reasonable:

If $f(x) = \mathcal{O}(e^{-\frac{|x|^2}{\beta^2}})$ *and its Fourier transform* $\widehat{f}(\xi) = \mathcal{O}(e^{-\frac{4|\xi|^2}{\alpha^2}})$, *then*

- *If* $\alpha\beta < 4 \Rightarrow f \equiv 0$.
- *If* $\alpha\beta = 4 \Rightarrow f$ *is a constant multiple of* $e^{-\frac{x^2}{\beta^2}}$.

The previous is known as the Hardy uncertainty principle.

It can be seen that from the Hardy uncertainty principle one can easily obtain a PDE's counterpart of this. In order to do that we need to recall that the solution to the free Schrödinger equation $i\partial_t u + \Delta u = 0$ with initial datum $f \in L^2$ has the following form:

$$u(x,t) := e^{it\Delta} f(x) = (4\pi i t)^{-\frac{d}{2}} \int_{\mathbb{R}^d} e^{\frac{i|x-y|^2}{4t}} f(y)\, dy = (2it)^{-\frac{d}{2}} e^{i\frac{|x|^2}{4t}} \mathcal{F}(y \mapsto e^{i\frac{|y|^2}{4t}} f(y))\big(\frac{x}{2t}\big),$$

where $\mathcal{F}$ represents the Fourier transform as well as the hat notation $\widehat{\cdot}$. This representation formula tells us that, in essence, up to multiplication by a phase factor, the solution $u(x,t)$ of the free Schrödinger equation is a multiple of a *rescaled* Fourier transform of the initial datum f. In view of this remark, the PDE's counterpart of the Hardy uncertainty principle can be stated as follows:

If $u(x,0) = \mathcal{O}(e^{-\frac{|x|^2}{\beta^2}})$ *and* $u(x,T) := e^{iT\Delta}u(x,0) = \mathcal{O}(e^{-\frac{|x|^2}{\alpha^2}})$, *then*

- *If* $\alpha\beta < 4T \Rightarrow f \equiv 0$.
- *If* $\alpha\beta = 4T \Rightarrow f$ *is a constant multiple of* $e^{-(\frac{1}{\beta^2}+\frac{i}{4T})x^2}$.

Clearly, without loss of generality, we can restrict ourselves to the case $T = 1$.

Thus, for time-reversible equations, *the analog of backward uniqueness will be uniqueness from behavior of the solution at two distinct times.* To be more precise, we are interested in such results with data eventually 0 or even with data which decay very fast.

Just to go off a slight but remarkable tangent, as already mentioned by Escauriaza, Kenig, Ponce and Vega in [31], the Hardy uncertainty principle can be applied to prove unique continuation for the free heat equation by only assuming the solution to satisfy at time $t = 1$ a sufficiently strong decay, together with the square-integrability of the initial datum $u(x,0)$. Precisely the following sharp result holds: if $u(x,1) := e^{\Delta}u(x,0) = \mathcal{O}(e^{-\frac{|x|^2}{\delta^2}})$, with $\delta \leqslant 2$, then $f \equiv 0$. Indeed let us consider $f(x) := u(x,1)$, from our hypothesis we have $f(x) = \mathcal{O}(e^{-\frac{|x|^2}{\delta^2}})$, moreover it is easy to see that $\widehat{f}(\xi) = \widehat{u}(\xi,1) := e^{-|\xi|^2}\widehat{u}(\xi,0) = e^{-\frac{4|\xi|^2}{2^2}}\widehat{u}(\xi,0)$. That is $\widehat{f}(\xi) = \mathcal{O}(e^{-\frac{4\xi^2}{2^2}})$. The application of Hardy uncertainty principle to f gives that if $2\delta \leqslant 4$, that is if $\delta \leqslant 2$, then f, and so $u(x,1), \equiv 0$. Then backward uniqueness arguments guarantee that $u(x,t) \equiv 0$ for all $t \in [0,1]$.

Going back to Schrödinger equation, there is a large literature concerning the uniqueness question for data eventually zero. For the one dimensional cubic Schrödinger equation, $i\partial_t u + \partial_x^2 u \mp |u|^2 u = 0$, in $\mathbb{R} \times [0,1]$, in [91] Zhang showed that if $u = 0$ in $(-\infty, a) \times \{t_0, t_1\}$ (or in $(a,\infty) \times \{t_0, t_1\}$) for some $a \in \mathbb{R}$, then $u \equiv 0$.

Then, the result of Zhang was extended, under more general assumption on the potential V in (II.1.14) and on the domain where the vanishing of $u(0)$ and $u(1)$ is assumed, first by Kenig, Ponce and Vega in [54] and then by Ionescu and Kenig in [46, 47].

As anticipated unique continuation of the kind described above has also been established for other dispersive equations. In [84] the unique continuation principle was proved for a general class of dispersive equations, including the KdV equation. More precisely it was proved that if a sufficiently smooth solution u of

$$\partial_t u + \partial_x^3 u + \sum_{j=0}^{2} r(x,t)\partial_x^j u = 0, \qquad x \in (a,b), \quad t \in (t_1,t_2)$$

with $r_j \in L_t^\infty L_{\mathrm{loc}}^2$, vanishes in an open set Ω of the space-time space, then u vanishes in all horizontal components of Ω.

In [8] Bourgain proved that if a solution u of the KdV equation is supported in a compact set on a non trivial time interval, then u must be identically zero.

In [55], Kenig, Ponce and Vega considered a solution of the generalized KdV equation

$$\partial_t u + \partial_x^3 u + u^k \partial_x u = 0, \qquad x \in \mathbb{R}, \quad t \in [0,1]$$

which vanishes only in two half-line $[B,+\infty) \times \{t_0\}$ and $[B,+\infty) \times \{t_1\}$, they proved that u vanishes identically. A result in the same spirit but for the difference of two solutions of the equation above was proved by the same authors in [56]. They proved that, the solutions of the generalized KdV equation are uniquely determined by their values on a semi-line at two different instant of times.

The historical path of unique continuation results for the bi-dimensional Z-K equation followed the same landmarks of the KdV equation. Indeed in [79], Panthee proved that if a solution u of this equation is supported in a set $[-M,M] \times [-M,M]$ for a non trivial time interval, then u must be identically zero. Following the method introduced for KdV equation by Kenig,Ponce and Vega in [55], Bustamante, Isaza and Mejía in [13] showed, improving the result in [79], that one can conclude the same even just assuming u to be compactly supported only at two distinct times.

So far we have only mentioned uniqueness results assuming data eventually identically zero at two distinct times. Actually, taking into account our intents, we want to employ the remaining part of this historical background presenting the main advances in the field of unique continuation principles for dispersive equations where, instead of requiring the solution to be zero on large sub-domains of $\mathbb{R}^d$, we just assume a sufficiently rapid decay for two distinct time. Roughly speaking, we will show the extensions of Hardy uncertainty principle for no more free dispersive equations.

Let us start with Schrödinger equation. In [28] Escauriaza, Kenig, Ponce and Vega proved that if at two distinct times the solution u of the Schrödinger equation (II.1.14) and its first spatial derivatives decay faster than any quadratic exponential, that is decay as $e^{-a|x|^\alpha}$ with $\alpha > 2$ and $a > 0$ sufficiently large and providing suitable assumptions on the potential V, then u has to be identically equal to zero. This linear result was then applied by the same authors to nonlinear equation of the form (II.1.15). They showed that if the difference of two sufficiently smooth solution u_1 and u_2 of (II.1.15) and the difference of their gradients have a stronger decay than the gaussian one for two different instants of time, then the two solutions are forced to agree in $\mathbb{R}^d$ in the whole time interval. Later, a deeper analysis of the solutions of (II.1.14) displaying gaussian decay and in particular of the convexity properties of the L^2-exponentially weighted norm of those solutions, allowed Escauriaza, Kenig, Ponce and Vega in [31] to obtain an improvement upon the results in [28], their main contribution is as follows: let u be a solution of the linear Schrödinger equation (II.1.14), if $\|e^{\frac{|\cdot|^2}{\beta^2}}u(0)\|_{L^2(\mathbb{R}^d)}$ and $\|e^{\frac{|\cdot|^2}{\alpha^2}}u(1)\|_{L^2(\mathbb{R}^d)}$ are both finite and assuming appropriate and not too much restrictive hypothesis on the potential V, therefore if $\alpha\beta < 2$ the $u \equiv 0$. Moreover, as already done in their previous work, this result was applied to the nonlinear equation (II.1.15), this shows that two regular solutions u_1 and u_2 must agree in $\mathbb{R}^d \times [0, 1]$ when one requests that $\|e^{\frac{|\cdot|^2}{\beta^2}}(u_1(0) - u_2(0))\|_{L^2(\mathbb{R}^d)}$ and $\|e^{\frac{|\cdot|^2}{\alpha^2}}(u_1(1) - u_2(1))\|_{L^2(\mathbb{R}^d)}$ are both finite. It is worth noting that this result cannot be the optimal one, indeed the decay requested is stronger than the one that appears in Hardy uncertainty principle, in other words this result can be considered as a *weaker* variant of the Hardy uncertainty principle. Only later the same authors in [32] showed that for many general bounded potentials the optimal version of Hardy uncertainty principle holds, that is just requiring for the decay that $\alpha\beta < 4$.

For the KdV equation, in [29], Escauriaza, Kenig, Ponce and Vega, making use of suitable Carleman inequalities which replace the "energy" estimates that were used for the unique continuation for Schrödinger and, as in that context, introducing lower estimates that comes from the work of Isakov [48], proved that if the difference of two solutions of the KdV equation decays as $e^{-ax^{\frac{3}{2}}}$ for $x > 0$ and $a > 0$ sufficiently large for two distinct times, then the solutions must agree.

With respect to the ZK equation, as already said in the introduction, results in this direction were already obtained. Bustamante, Isaza and Mejía in [14] proved that if the difference of two sufficiently smooth solutions u_1 and u_2 of (II.1.1) decays as $e^{-a(x^2+y^2)^{\frac{3}{4}}}$ for a large enough $a > 0$, at two distinct instant of times, the the solutions must agree.

This concludes our overview over the unique continuation results.

II.2.| The proof of Theorem II.2 and Theorem II.3

This chapter is devoted to focus on a very detailed and comprehensive discussion about the main tools we used to achieve our result Theorem II.3 and as a consequence Theorem II.2 and then the endgame will consist to the proof of both the theorems.

As already said we want to proceed embracing the now well rooted strategy underlying the proof of this kind of results that, for instance, is exploited in [29] to treat unique continuation for KdV.

In essence those results follows by making a comparison between two estimates: a lower bound for the L^2- norm of a solution in an annular domain, which follows after performing a suitable Carleman estimate and an upper bound for the L^2- norm of the solution and its derivatives up to order two.

As a starting point, for sake of completeness, we want to make explicit the computations carried out in [43] in order to pass from ZK equation to its symmetric counterpart.

Getting down into details, for the space variables the following linear transformation was used:

$$\begin{cases} x' = \mu x + \lambda y \\ y' = \mu x - \lambda y. \end{cases}$$

Let $v(x', y') = u(x, y)$, then it is easy to see that

$$\partial_x u(x,y) = \mu(\partial_{x'} + \partial_{y'})v(x', y')$$
$$\partial_y u(x,y) = \lambda(\partial_{x'} - \partial_{y'})v(x', y').$$

This implies

$$\begin{aligned}(\partial_x^3 + \partial_x\partial_y^2)u(x,y) &= \mu^3(\partial_{x'} + \partial_{y'})^3 v(x', y') + \mu\lambda^2(\partial_{x'} + \partial_{y'})(\partial_{x'} - \partial_{y'})^2 v(x', y') \\ &= (\mu^3 + \mu\lambda^2)(\partial_{x'}^3 + \partial_{y'}^3)v(x', y') + (3\mu^3 - \mu\lambda^2)(\partial_{x'}^2\partial_{y'} + \partial_{x'}\partial_{y'}^2)v(x', y').\end{aligned}$$

In order to symmetrize the equation, we want to make the last term of the previous equal to zero, this leads us in choosing $\lambda = \sqrt{3}\mu$, then fixing also $\mu = 4^{-\frac{1}{3}}$ we get

$$(\partial_x^3 + \partial_x\partial_y^2)u(x,y) = (\partial_{x'}^3 + \partial_{y'}^3)v(x', y')$$

which implies that, without changing the well-posedness theory, we can reduce equation (II.1.1) to the symmetric-type equation (II.1.1).

Now we are in position to perform our first crucial estimate, namely the lower bound in the annulus domain.

II.2.1. Lower bound in the annulus domain

This section is concerned with a lower bound estimate for the L^2-norm of the difference between two solutions u_1 and u_2 of (II.1.2), its first order and second order space derivative in the annular region $\{(x, y) \colon R - 1 \leqslant \sqrt{x^2 + y^2} \leqslant R\} \times [0, 1]$ with an exponential of the form $e^{-cR^{\frac{3}{2}}}$.

The result we proved has the following precise statement:

Theorem II.4. *Let $v \in C([0,1]; H^3(\mathbb{R}^2))$ be a solution of (II.1.9) with $a_0, a_1 \in L^\infty(\mathbb{R}^3)$. Assume that*

$$\int_{\mathbb{R}^2}\int_0^1 \left(|v|^2 + |\nabla v|^2 + |\Delta v|^2\right) dx\, dy\, dt \leqslant A^2.$$

Let $\delta > 0$, $r \in (0, \frac{1}{2})$ and $Q := \{(x,y,t)\colon \sqrt{x^2+y^2} \leqslant 1,\ t \in [r, 1-r]\}$ and suppose that $\|v\|_{L^2(Q)} \geqslant \delta$. Then there exist constants $\widetilde{R}_0, c_0, c_1$ depending on $A, \|a_0\|_\infty$ and $\|a_1\|_\infty$, such that for $R \geqslant \widetilde{R}_0$

$$A_R(v) := \Big(\int_0^1 \int_{R-1 \leqslant \sqrt{x^2+y^2} \leqslant R} \left(|v|^2 + |\nabla v|^2 + |\Delta v|^2\right) dx\, dy\, dt\Big)^{\frac{1}{2}} \geqslant c_0 e^{-c_1 R^{\frac{3}{2}}}.$$

The previous idea of establishing lower bounds for the asimptotic behavior of a suitable norm of the solution in an annulus domain stems from a work by Bourgain and Kenig [9] on a class of stationary Schrödinger operators $-\Delta + V(x)$ in which the property of spectral localization, that is the phenomenon for which the point spectrum of the analyzed operator presents exponentially decaying eigenfunctions, is studied.

In that work they needed precise quantitative information on the rate of local vanishing for eigenfunctions, more precisely, local bounds on the eigenfunctions both from above and from below were required. Unlike the upper bound, which just needs classical tools to be achieved, the lower bound is a more subtle issue. The statement (Lemma 3.10 in [9]) is as follows

Lemma II.2. *Let u be a bounded solution of $\Delta u + Vu = 0$ in $\mathbb{R}$ with suitable additional assumptions about V. Let $x_0 \in \mathbb{R}^d, |x_0| = R > 1$. Then*

$$\max_{R-1<x<R} |u(x)| > c_0 e^{-c_1 (\log R) R^{\frac{4}{3}}}$$

This was derived from the following Carleman type estimate.

Lemma II.3. *There are constants C_1, C_2, C_3, depending only on d and an increasing function $w = w(r)$ for $0 < r < 10$ such that*

$$\frac{1}{C_1} < \frac{w(r)}{r} < C_1$$

and for all $f \in C_0^\infty(B_{10}\backslash\{0\})$, $\alpha > C_2$, we have

$$\alpha^3 \int_{\mathbb{R}^d} w^{-1-2\alpha} f^2 \leqslant C_3 \int_{\mathbb{R}^d} w^{2-2\alpha} (\Delta f)^2.$$

In order to obtain our lower bound Theorem II.4 we will perform a Carleman estimate as the one stated in Lemma II.3.

Before proving in our case such a Carleman estimate, we start out with a brief discussion about this cornerstone tool.

A typical Carleman estimate for a, usually linear, differential operator P looks like

$$\|u\|_{X(wdx)} \leqslant C\|Pu\|_{X(wdx)},$$

where $X(wdx)$ is a weighted space with weight w which has to be chosen differently for each operator P taken in consideration; typically, as in our case, the weight is in an exponential form, that is $w := e^{\alpha\,\theta(x)}$, for a suitable function θ and a parameter α. In this situation a Carleman estimate assumes the following form

$$\alpha\,\|e^{\alpha\,\theta(x)}u\|_{X(dx)} \leqslant C\|e^{\alpha\,\theta(x)}Pu\|_{X(dx)},$$

with a constant C independent of α.

Remark II.2. A very relevant point in the Carleman inequality is the presence of the multiplicative parameter α on the left-hand side. Indeed by taking α very large we can make the term on the left-hand side as large as we need in order to absorb potential error terms.

This fact can be seen at work explicitly in the proof of Lemma II.5 below, indeed, by virtue of (II.1.9), we are interested in proving a Carleman estimate in which the linear operator taken into account is $P = \partial_t + \partial_x^3 + \partial_y^3 + a_1(\partial_x + \partial_y) + a_0$. Actually we will prove a Carleman inequality for the "principal" operator $P = \partial_t + \partial_x^3 + \partial_y^3$ and then, by Remark II.2, with no effort, we will include in the estimate lower order derivatives.

II.2.1.1. Carleman estimates

Precisely we are going to prove first the following lemma.

Lemma II.4. *Assume that* $\varphi\colon [0,1] \to \mathbb{R}$ *is a smooth function. Then, there exist two constants* $c > 0$ *and* $M_1 = M_1(\|\varphi'\|_\infty, \|\varphi''\|_\infty) > 0$ *such that the inequality*

$$\frac{\alpha^{\frac{5}{2}}}{R^3}\left\|e^{\alpha\theta(x,y,t)}\theta(x,y,t)\,g\right\|_{L^2(\mathbb{R}^2\times[0,1])} + \frac{\alpha^{\frac{3}{2}}}{R^2}\|e^{\alpha\theta(x,y,t)}|\nabla g|\|_{L^2(\mathbb{R}^2\times[0,1])} \leqslant c\|e^{\alpha\theta(x,y,t)}(\partial_t + \partial_x^3 + \partial_y^3)g\|_{L^2(\mathbb{R}^2\times[0,1])} \quad \text{(II.2.1)}$$

holds, for $R \geqslant 1$, α *such that* $\alpha^2 \geqslant M_1 R^3$, $g \in C_0^\infty(\mathbb{R}^2 \times [0,1])$ *supported in*

$$\left\{(x,y,t) \in \mathbb{R}^2 \times [0,1]\colon \left|\frac{\varkappa}{R} + \varphi(t)\xi\right| \geqslant 1\right\}$$

and $\theta(x,y,t) = \left|\frac{\varkappa}{R} + \varphi(t)\xi\right|^2 = \left(\frac{x}{R} + \varphi(t)\right)^2 + \left(\frac{y}{R} + \varphi(t)\right)^2$, *with* $\varkappa = (x,y)$ *and* $\xi = (1,1)$.

Proof. From now on with an abuse of notation we will write $L^2(\mathbb{R}^2 \times [0,1])$ as L^2.

Cause to the difficulty to prove an exponentially weighted estimate, as usual in this context, we reduce ourselves into proving an estimate for the conjugated operator

$$e^{\alpha\theta(x,y,t)}(\partial_t + \partial_x^3 + \partial_y^3)e^{-\alpha\theta(x,y,t)}.$$

The main point in the proof is, roughly speaking, a "positive commutator argument" that will give a lower bound for the conjugated operator $e^{\alpha\theta}(\partial_t + \partial_x^3 + \partial_y^3)e^{-\alpha\theta}$ once it is decomposed as a sum of its symmetric and skew-symmetric part.

In order to do that we define $f = e^{\alpha\theta(x,y,t)}g$, now we want to see how (II.2.1) can be re-written in terms of this auxiliary function f.

Let us first consider the term

$$\begin{aligned} e^{2\alpha\theta}|\nabla g|^2 &= e^{2\alpha\theta}\big[(e^{-\alpha\theta}\partial_x f - \alpha\partial_x\theta e^{-\alpha\theta}f)^2 + (e^{-\alpha\theta}\partial_y f - \alpha\partial_y\theta e^{-\alpha\theta}f)^2\big] \\ &= (\partial_x f - \alpha\partial_x\theta f)^2 + (\partial_y f - \alpha\partial_y\theta f)^2. \end{aligned}$$

By virtue of the previous identity, instead of (II.2.1), it is sufficient to prove

$$c\|e^{\alpha\theta}(\partial_t + \partial_x^3 + \partial_y^3)e^{-\alpha\theta}f\|_{L^2} \geqslant \frac{\alpha^{\frac{5}{2}}}{R^3}\|\theta f\|_{L^2} + \frac{\alpha^{\frac{3}{2}}}{R^2}\|\partial_x f - \alpha\partial_x\theta f\|_{L^2} + \frac{\alpha^{\frac{3}{2}}}{R^2}\|\partial_y f - \alpha\partial_y\theta f\|_{L^2}. \quad \text{(II.2.2)}$$

A straightforward computation gives

$$\begin{aligned} e^{\alpha\theta}(\partial_t + \partial_x^3 + \partial_y^3)e^{-\alpha\theta}f = &- \alpha\partial_t\theta f + \partial_t f - \alpha\partial_x^3\theta f + 3\alpha^2(\partial_x\theta)(\partial_x^2\theta)f - \alpha^3(\partial_x\theta)^3 f \\ &- 3\alpha\partial_x^2\theta\partial_x f + 3\alpha^2(\partial_x\theta)^2\partial_x f - 3\alpha\partial_x\theta\partial_x^2 f + \partial_x^3 f \\ &- \alpha\partial_y^3\theta f + 3\alpha^2(\partial_y\theta)(\partial_y^2\theta)f - \alpha^3(\partial_y\theta)^3 f - 3\alpha\partial_y^2\theta\partial_y f \\ &+ 3\alpha^2(\partial_y\theta)^2\partial_y f - 3\alpha\partial_y\theta\partial_y^2 f + \partial_y^3 f. \end{aligned}$$

As already mentioned it is customary to write the operator in the following way

$$e^{\alpha\theta}(\partial_t + \partial_x^3 + \partial_y^3)e^{-\alpha\theta}f = A_\alpha f + S_\alpha f,$$

where A_α and S_α are respectively skew-symmetric and symmetric operators and depend on α.

The two operators we are looking for are the following:

$$\begin{aligned} &A_\alpha := \partial_t + \partial_x^3 + \partial_y^3 + 3\alpha^2(\partial_x\theta)^2\partial_x + 3\alpha^2(\partial_y\theta)^2\partial_y + 3\alpha^2(\partial_x\theta)(\partial_x^2\theta) + 3\alpha^2(\partial_y\theta)(\partial_y^2\theta) \\ &S_\alpha := -3\alpha\partial_x(\partial_x\theta\partial_x\cdot) - 3\alpha\partial_y(\partial_y\theta\partial_y\cdot) + \big(-\alpha^3(\partial_x\theta)^3 - \alpha\partial_x^3\theta\big) + \big(-\alpha^3(\partial_y\theta)^3 - \alpha\partial_y^3\theta\big) - \alpha\partial_t\theta. \end{aligned}$$

Indeed, since the first three operators of A_α are derivatives of odd order, they are skew-symmetric, moreover, a straightforward computation shows that $3\alpha^2(\partial_x\theta)^2\partial_x + 3\alpha^2(\partial_x\theta)(\partial_x^2\theta)$ are also skew-symmetric and the same holds for the corresponding operators in the y variable. With respect to S_α, it is very easy to prove the symmetry of the operators $-3\alpha\partial_x(\partial_x\theta\partial_x\cdot) -$

$3\alpha\partial_y(\partial_y\theta\partial_y\cdot)$, moreover as the rest part is constituted by operators of multiplication by real-valued functions it is symmetric.

Thus,

$$A_\alpha^* = -A_\alpha, \qquad S_\alpha^* = S_\alpha,$$

therefore one get

$$\begin{aligned}
\|e^{\alpha\theta(x,y,t)}(\partial_t + \partial_x^3 + \partial_y^3)e^{-\alpha\theta(x,y,t)}f\|_{L^2}^2 &= \|(A_\alpha + S_\alpha)f\|_{L^2}^2 \\
&= \langle (A_\alpha + S_\alpha)f, (A_\alpha + S_\alpha)f\rangle \\
&= \|A_\alpha f\|_{L^2}^2 + \|S_\alpha f\|_{L^2}^2 + \langle A_\alpha f, S_\alpha f\rangle + \langle S_\alpha f, A_\alpha f\rangle \\
&\geqslant \langle [S_\alpha, A_\alpha]f, f\rangle.
\end{aligned}$$

Remark II.3. From now on, to save space, we abbreviate $\int := \iiint_{\mathbb{R}^2\times[0,1]}$ and omit the arguments of integrated functions.

A computation shows

$$\begin{aligned}
\langle [S_\alpha, A_\alpha]f, f\rangle = \int \Big(&\alpha\partial_t^2\theta + 2\alpha\partial_x^3\partial_t\theta + 6\alpha^3(\partial_x\theta)^2\partial_x\partial_t\theta + 2\alpha\partial_y^3\partial_t\theta + 6\alpha^3(\partial_y\theta)^2\partial_y\partial_t\theta \\
&- 18\alpha^3\partial_x\theta\partial_x^2\theta\partial_x^3\theta - 3\alpha^3(\partial_x^2\theta)^3 - 3\alpha^3(\partial_x\theta)^2\partial_x^4\theta + \alpha\partial_x^6\theta + 9\alpha^5(\partial_x\theta)^4\partial_x^2\theta \\
&- 18\alpha^3\partial_y\theta\partial_y^2\theta\partial_y^3\theta - 3\alpha^3(\partial_y^2\theta)^3 - 3\alpha^3(\partial_y\theta)^2\partial_y^4\theta + \alpha\partial_y^6\theta + 9\alpha^5(\partial_y\theta)^4\partial_y^2\theta \\
&- 18\alpha^3\partial_x^2\theta\partial_y^2\theta\partial_x\partial_y\theta - 9\alpha^3\partial_x^2\theta\partial_y\theta\partial_x\partial_y^2\theta \\
&\qquad - 9\alpha^3\partial_x\theta\partial_y^2\theta\partial_x^2\partial_y\theta \\
&- 18\alpha^3\partial_x\theta\partial_y\theta\partial_x^2\partial_y^2\theta - 9\alpha^3\partial_x\theta\partial_y^2\theta\partial_x^2\partial_y\theta \\
&\qquad - 9\alpha^3\partial_x^2\theta\partial_y\theta\partial_x\partial_y^2\theta \\
&+ 12\alpha^3(\partial_x\partial_y\theta)^3 + 6\alpha^3(\partial_x\theta)^2\partial_x\partial_y^3\theta + 2\alpha\partial_x^3\partial_y^3\theta + 18\alpha^5(\partial_x\theta)^2(\partial_y\theta)^2\partial_x\partial_y\theta \\
&\qquad + 6\alpha^3(\partial_y\theta)^2\partial_x^3\partial_y\theta\Big)f^2 \\
+ \int \Big(&-6\alpha\partial_x\partial_t\theta - 6\alpha\partial_x^4\theta + 18\alpha^3(\partial_x\theta)^2\partial_x^2\theta - 6\alpha\partial_x\partial_y^3\theta - 18\alpha^3(\partial_y\theta)^2\partial_x\partial_y\theta\Big)(\partial_x f)^2 \\
+ \int \Big(&-6\alpha\partial_y\partial_t\theta - 6\alpha\partial_y^4\theta + 18\alpha^3(\partial_y\theta)^2\partial_y^2\theta - 6\alpha\partial_x^3\partial_y\theta - 18\alpha^3(\partial_x\theta)^2\partial_x\partial_y\theta\Big)(\partial_y f)^2 \\
+ \int &64\alpha^3\partial_x\theta\partial_y\theta\partial_x\partial_y\theta\,\partial_x f\,\partial_y f \\
+ \int &9\alpha\partial_x^2\theta(\partial_x^2 f)^2 + \int 9\alpha\partial_y^2\theta(\partial_y^2 f)^2 \\
+ \int &18\alpha\partial_x\partial_y\theta(\partial_x\partial_y f)^2.
\end{aligned}$$

Now we choose

$$\theta(x,y,t) = \left|\frac{\varkappa}{R} + \varphi(t)\xi\right|^2 = \left(\frac{x}{R} + \varphi(t)\right)^2 + \left(\frac{y}{R} + \varphi(t)\right)^2,$$

where clearly $\varkappa = (x,y)$ and $\xi = (1,1)$.

It is not difficult to see that the following hold:

$$\begin{aligned}
&\partial_x\theta = \frac{2}{R}\Big(\frac{x}{R} + \varphi(t)\Big), \qquad (\partial_x\theta)^2 = \frac{4}{R^2}\Big(\frac{x}{R} + \varphi(t)\Big)^2, \qquad (\partial_x\theta)^4 = \frac{16}{R^4}\Big(\frac{x}{R} + \varphi(t)\Big)^4.\\
&\partial_y\theta = \frac{2}{R}\Big(\frac{y}{R} + \varphi(t)\Big), \qquad (\partial_y\theta)^2 = \frac{4}{R^2}\Big(\frac{y}{R} + \varphi(t)\Big)^2, \qquad (\partial_y\theta)^4 = \frac{16}{R^4}\Big(\frac{y}{R} + \varphi(t)\Big)^4.\\
&\partial_x^2\theta = \frac{2}{R^2}, \qquad (\partial_x^2\theta)^3 = \frac{8}{R^6} \qquad \text{and} \qquad \partial_x^3\theta = 0, \qquad \partial_x^4\theta = 0, \qquad \partial_x^6\theta = 0.\\
&\partial_y^2\theta = \frac{2}{R^2}, \qquad (\partial_y^2\theta)^3 = \frac{8}{R^6} \qquad \text{and} \qquad \partial_y^3\theta = 0, \qquad \partial_y^4\theta = 0, \qquad \partial_y^6\theta = 0.\\
&\partial_x\partial_y\theta = 0.\\
&\partial_t\theta = 2\varphi'(t)\Big[\Big(\frac{x}{R} + \varphi(t)\Big) + \Big(\frac{y}{R} + \varphi(t)\Big)\Big], \qquad \partial_x\partial_t\theta = \frac{2}{R}\varphi'(t), \qquad \partial_y\partial_t\theta = \frac{2}{R}\varphi'(t).\\
&\partial_t^2\theta = 4(\varphi'(t))^2 + 2\varphi''(t)\Big[\Big(\frac{x}{R} + \varphi(t)\Big) + \Big(\frac{y}{R} + \varphi(t)\Big)\Big].
\end{aligned} \tag{II.2.3}$$

Using (II.2.3), adding and subtracting the terms $\frac{\alpha^3}{R^4}\|\partial_x f - \alpha\partial_x\theta f\|_{L^2}^2 + \frac{\alpha^3}{R^4}\|\partial_y f - \alpha\partial_y\theta f\|_{L^2}^2$, we get

$$\begin{aligned}
\langle[S_\alpha, A_\alpha]f, f\rangle =& \frac{18\alpha}{R^2}\|\partial_x^2 f\|_{L^2}^2 + \frac{18\alpha}{R^2}\|\partial_y^2 f\|_{L^2}^2\\
&- \frac{12\alpha}{R}\int \varphi'(t)(\partial_x f)^2 - \frac{12\alpha}{R}\int \varphi'(t)(\partial_y f)^2 && I_1 + I_1^*\\
&+ \frac{144\alpha^3}{R^4}\int \Big(\frac{x}{R} + \varphi(t)\Big)^2(\partial_x f)^2 + \frac{144\alpha^3}{R^4}\int \Big(\frac{y}{R} + \varphi(t)\Big)^2(\partial_y f)^2\\
&- \frac{24\alpha^3}{R^6}\int f^2 + 2\alpha\int(\varphi'(t)f)^2 + 2\alpha\int \Big(\frac{x}{R} + \varphi(t)\Big)\varphi''(t)f^2\\
&- \frac{24\alpha^3}{R^6}\int f^2 + 2\alpha\int(\varphi'(t)f)^2 + 2\alpha\int \Big(\frac{y}{R} + \varphi(t)\Big)\varphi''(t)f^2\\
&+ \frac{48\alpha^3}{R^3}\int \Big(\frac{x}{R} + \varphi(t)\Big)^2\varphi'(t)f^2 + \frac{288\alpha^5}{R^6}\int \Big(\frac{x}{R} + \varphi(t)\Big)^4 f^2\\
&+ \frac{48\alpha^3}{R^3}\int \Big(\frac{y}{R} + \varphi(t)\Big)^2\varphi'(t)f^2 + \frac{288\alpha^5}{R^6}\int \Big(\frac{y}{R} + \varphi(t)\Big)^4 f^2\\
&+ \frac{\alpha^3}{R^4}\|\partial_x f - \alpha\partial_x\theta f\|_{L^2}^2 + \frac{\alpha^3}{R^4}\|\partial_y f - \alpha\partial_y\theta f\|_{L^2}^2\\
&- \frac{\alpha^3}{R^4}\|\partial_x f - \alpha\partial_x\theta f\|_{L^2}^2 - \frac{\alpha^3}{R^4}\|\partial_y f - \alpha\partial_y\theta f\|_{L^2}^2. && I_2 + I_2^*
\end{aligned}$$

Let us consider $(I_1 + I_1^*)$, for

$$\alpha^2 \geqslant \|\varphi'\|_\infty R^3,$$

it follows that

$$\begin{aligned} I_1 + I_1^* &\geqslant -\frac{12\alpha}{R} \int \|\varphi'\|_\infty (\partial_x f)^2 - \frac{12\alpha}{R} \int \|\varphi'\|_\infty (\partial_y f)^2 \\ &\geqslant -\frac{12\alpha^3}{R^4} \int (\partial_x f)^2 - \frac{12\alpha^3}{R^4} \int (\partial_y f)^2. \end{aligned} \tag{II.2.4}$$

We compute ($I_2 + I_2^*$) using the explicit expression for $\partial_x \theta$ and $\partial_y \theta$:

$$\begin{aligned} I_2 + I_2^* = &- \frac{\alpha^3}{R^4} \int (\partial_x f)^2 - \frac{4\alpha^5}{R^6} \int \Big(\frac{x}{R} + \varphi(t)\Big)^2 f^2 + \frac{4\alpha^4}{R^5} \int \Big(\frac{x}{R} + \varphi(t)\Big) f \partial_x f \\ &- \frac{\alpha^3}{R^4} \int (\partial_y f)^2 - \frac{4\alpha^5}{R^6} \int \Big(\frac{y}{R} + \varphi(t)\Big)^2 f^2 + \frac{4\alpha^4}{R^5} \int \Big(\frac{y}{R} + \varphi(t)\Big) f \partial_y f. \end{aligned}$$

Now let us just consider the last terms in the first and the second rows of the previous identity, using the classical Young inequality

$$a\,b \leqslant \frac{a^p}{p} + \frac{b^q}{q}, \qquad a, b > 0, \qquad \frac{1}{p} + \frac{1}{q} = 1, \tag{II.2.5}$$

we obtain

$$\begin{aligned} &\frac{4\alpha^4}{R^5} \int \Big(\frac{x}{R} + \varphi(t)\Big) f \partial_x f + \frac{4\alpha^4}{R^5} \int \Big(\frac{y}{R} + \varphi(t)\Big) f \partial_y f \\ &\qquad \geqslant -\frac{4\alpha^4}{R^5} \int \Big|\frac{x}{R} + \varphi(t)\Big| |f| |\partial_x f| - \frac{4\alpha^4}{R^5} \int \Big|\frac{y}{R} + \varphi(t)\Big| |f| |\partial_y f| \\ &\qquad \geqslant -\frac{2\alpha^5}{R^6} \int \Big(\frac{x}{R} + \varphi(t)\Big)^2 f^2 - \frac{2\alpha^3}{R^4} \int (\partial_x f)^2 \\ &\qquad\quad - \frac{2\alpha^5}{R^6} \int \Big(\frac{y}{R} + \varphi(t)\Big)^2 f^2 - \frac{2\alpha^3}{R^4} \int (\partial_y f)^2. \end{aligned}$$

Since $\left|\frac{\varkappa}{R} + \varphi(t)\xi\right| \geqslant 1$, then one obtains

$$\begin{aligned} I_2 + I_2^* &\geqslant -\frac{6\alpha^5}{R^6} \int \Big|\frac{\varkappa}{R} + \varphi(t)\xi\Big|^2 f^2 - \frac{3\alpha^3}{R^4} \int (\partial_x f)^2 - \frac{3\alpha^3}{R^4} \int (\partial_y f)^2 \\ &\geqslant -\frac{6\alpha^5}{R^6} \int \Big|\frac{\varkappa}{R} + \varphi(t)\xi\Big|^4 f^2 - \frac{3\alpha^3}{R^4} \int (\partial_x f)^2 - \frac{3\alpha^3}{R^4} \int (\partial_y f)^2. \end{aligned}$$

Gathering altogether we get

$$\begin{aligned}\langle [S_\alpha, A_\alpha] f, f\rangle =& \frac{18\alpha}{R^2}\|\partial_x^2 f\|_{L^2}^2 + \frac{18\alpha}{R^2}\|\partial_y^2 f\|_{L^2}^2 && I_1 + I_1^* \\ &- \frac{15\alpha^3}{R^4}\int (\partial_x f)^2 - \frac{15\alpha^3}{R^4}\int (\partial_y f)^2 && I_2 + I_2^* \\ &+ \frac{144\alpha^3}{R^4}\int \Big(\frac{x}{R}+\varphi(t)\Big)^2 (\partial_x f)^2 + \frac{144\alpha^3}{R^4}\int \Big(\frac{y}{R}+\varphi(t)\Big)^2 (\partial_y f)^2 \\ &- \frac{24\alpha^3}{R^6}\int f^2 + 2\alpha\int (\varphi'(t) f)^2 + 2\alpha \int \Big(\frac{x}{R}+\varphi(t)\Big)\varphi''(t) f^2 \\ &- \frac{24\alpha^3}{R^6}\int f^2 + 2\alpha\int (\varphi'(t) f)^2 + 2\alpha \int \Big(\frac{y}{R}+\varphi(t)\Big)\varphi''(t) f^2 \\ &+ \frac{48\alpha^3}{R^3}\int \Big(\frac{x}{R}+\varphi(t)\Big)^2 \varphi'(t) f^2 + \frac{48\alpha^3}{R^3}\int \Big(\frac{y}{R}+\varphi(t)\Big)^2 \varphi'(t) f^2 \\ &+ \frac{288\alpha^5}{R^6}\int \Big(\frac{x}{R}+\varphi(t)\Big)^4 f^2 + \frac{288\alpha^5}{R^6}\int \Big(\frac{y}{R}+\varphi(t)\Big)^4 f^2 && I_3 + I_3^* \\ &+ \frac{\alpha^3}{R^4}\|\partial_x f - \alpha\partial_x\theta f\|_{L^2}^2 + \frac{\alpha^3}{R^4}\|\partial_y f - \alpha\partial_y\theta f\|_{L^2}^2 \\ &- \frac{6\alpha^5}{R^6}\int \Big|\frac{\varkappa}{R}+\varphi(t)\xi\Big|^4 f^2. \end{aligned}$$

We consider ($I_2 + I_2^*$), using again that $\left|\frac{\varkappa}{R}+\varphi(t)\xi\right| \geqslant 1$, we obtain

$$\begin{aligned} I_2 + I_2^* \geqslant& - \frac{15\alpha^3}{R^4}\int \Big(\frac{x}{R}+\varphi(t)\Big)^2 (\partial_x f)^2 \underbrace{- \frac{15\alpha^3}{R^4}\int \Big(\frac{y}{R}+\varphi(t)\Big)^2 (\partial_x f)^2}_{=:I} \\ & \underbrace{- \frac{15\alpha^3}{R^4}\int \Big(\frac{x}{R}+\varphi(t)\Big)^2 (\partial_y f)^2}_{=:II} - \frac{15\alpha^3}{R^4}\int \Big(\frac{y}{R}+\varphi(t)\Big)^2 (\partial_y f)^2. \end{aligned}$$

First let us observe that, making use of integrazion by parts, I can be re-written as

$$I = + \frac{15\alpha^3}{R^4}\int \Big(\frac{y}{R}+\varphi(t)\Big)^2 f\partial_x^2 f. \tag{II.2.6}$$

Now we want to rebuild some "positivity" from $I_1 + I + I_3^*$, using (II.2.6), observing that $18 = \frac{47}{4} + \frac{25}{4}$ and that $288 = 9 + 279$ we have

$$\begin{aligned} I_1 + I + I_3^* =& \int \Big(\frac{5}{2}\frac{\alpha^{\frac12}}{R}\partial_x^2 f\Big)^2 + 2\int \frac{15}{2}\frac{\alpha^3}{R^4}\Big(\frac{y}{R}+\varphi(t)\Big)^2 f\partial_x^2 f + \int \Big[3\frac{\alpha^{\frac52}}{R^3}\Big(\frac{y}{R}+\varphi(t)\Big)^2 f\Big]^2 \\ &+ \frac{47}{4}\frac{\alpha}{R^2}\int (\partial_x^2 f)^2 + 279\frac{\alpha^5}{R^6}\int \Big(\frac{y}{R}+\varphi(t)\Big)^4 f^2 \\ =& \int \Big[\frac{5}{2}\frac{\alpha^{\frac12}}{R}\partial_x^2 f + 3\frac{\alpha^{\frac52}}{R^3}\Big(\frac{y}{R}+\varphi(t)\Big)^2 f\Big]^2 + \frac{47}{4}\frac{\alpha}{R^2}\int (\partial_x^2 f)^2 \\ &+ 279\frac{\alpha^5}{R^6}\int \Big(\frac{y}{R}+\varphi(t)\Big)^4 f^2. \end{aligned}$$

Proceeding in the same way for $I_1^* + II + I_3$ we get

$$I_1^* + II + I_3 \geqslant \int \Big[\frac{5}{2}\frac{\alpha^{\frac{1}{2}}}{R}\partial_y^2 f + 3\frac{\alpha^{\frac{5}{2}}}{R^3}\Big(\frac{x}{R}+\varphi(t)\Big)^2 f\Big]^2 + \frac{47}{4}\frac{\alpha}{R^2}\int(\partial_y^2 f)^2 + 279\frac{\alpha^5}{R^6}\int\Big(\frac{x}{R}+\varphi(t)\Big)^4 f^2.$$

Summing up, neglecting the two squares of binomial, that clearly are non negative, one has

$$\begin{aligned}
\langle [S_\alpha, A_\alpha] f, f\rangle = & \frac{47}{4}\frac{\alpha}{R^2}\|\partial_x^2 f\|_{L^2}^2 + \frac{47}{4}\frac{\alpha}{R^2}\|\partial_y^2 f\|_{L^2}^2 \\
& \frac{129\alpha^3}{R^4}\int\Big(\frac{x}{R}+\varphi(t)\Big)^2(\partial_x f)^2 + \frac{129\alpha^3}{R^4}\int\Big(\frac{y}{R}+\varphi(t)\Big)^2(\partial_y f)^2 \\
& - \frac{24\alpha^3}{R^6}\int f^2 + 2\alpha\int(\varphi'(t)f)^2 + 2\alpha\int\Big(\frac{x}{R}+\varphi(t)\Big)\varphi''(t)f^2 \qquad I_1+I_2+I_3 \\
& - \frac{24\alpha^3}{R^6}\int f^2 + 2\alpha\int(\varphi'(t)f)^2 + 2\alpha\int\Big(\frac{y}{R}+\varphi(t)\Big)\varphi''(t)f^2 \qquad I_1^*+I_2^*+I_3^* \\
& + \frac{48\alpha^3}{R^3}\int\Big(\frac{x}{R}+\varphi(t)\Big)^2\varphi'(t)f^2 + \frac{279\alpha^5}{R^6}\int\Big(\frac{x}{R}+\varphi(t)\Big)^4 f^2 \qquad I_4+I_5 \\
& + \frac{48\alpha^3}{R^3}\int\Big(\frac{y}{R}+\varphi(t)\Big)^2\varphi'(t)f^2 + \frac{279\alpha^5}{R^6}\int\Big(\frac{y}{R}+\varphi(t)\Big)^4 f^2 \qquad I_4^*+I_5^* \\
& + \frac{\alpha^3}{R^4}\|\partial_x f - \alpha\partial_x\theta f\|_{L^2}^2 + \frac{\alpha^3}{R^4}\|\partial_y f - \alpha\partial_y\theta f\|_{L^2}^2 \\
& - \frac{6\alpha^5}{R^6}\int\Big|\frac{\varkappa}{R}+\varphi(t)\xi\Big|^4 f^2.
\end{aligned}$$

Now we compute $I_2 + I_2^* + I_4 + I_4^* + I_5 + I_5^*$, using that $(a^2+b^2) \geqslant \frac{1}{2}(a+b)^2$ for all $a, b > 0$ and that $\frac{279}{2} = 144 - \frac{9}{2}$ we have

$$\begin{aligned}
& I_2 + I_2^* + I_4 + I_4^* + I_5 + I_5^* \\
& \quad \geqslant 4\alpha\int(\varphi'(t)f)^2 + \frac{48\alpha^3}{R^3}\int\Big|\frac{\varkappa}{R}+\varphi(t)\xi\Big|^2\varphi'(t)f^2 + \frac{279}{2}\frac{\alpha^5}{R^6}\int\Big|\frac{\varkappa}{R}+\varphi(t)\xi\Big|^4 f^2 \\
& \quad = \int\Big(2\alpha^{\frac{1}{2}}\varphi'(t) + 12\frac{\alpha^{\frac{5}{2}}}{R^3}\Big|\frac{\varkappa}{R}+\varphi(t)\xi\Big|^2\Big)^2 f^2 - \frac{9}{2}\frac{\alpha^5}{R^6}\int\Big|\frac{\varkappa}{R}+\varphi(t)\xi\Big|^4 f^2.
\end{aligned}$$

Since we are assuming $\alpha^2 \geqslant \|\varphi'\|_\infty R^3$ and since $\left|\frac{\varkappa}{R}+\varphi(t)\xi\right| \geqslant 1$, therefore

$$2\alpha^{\frac{1}{2}}\varphi'(t) \geqslant -\frac{2\alpha^{\frac{5}{2}}}{R^3} \geqslant -\frac{2\alpha^{\frac{5}{2}}}{R^3}\Big|\frac{\varkappa}{R}+\varphi(t)\xi\Big|^2.$$

This gives

$$I_2 + I_2^* + I_4 + I_4^* + I_5 + I_5^* \geqslant \frac{100\alpha^5}{R^6}\int\Big|\frac{\varkappa}{R}+\varphi(t)\xi\Big|^4 f^2 - \frac{9}{2}\frac{\alpha^5}{R^6}\int\Big|\frac{\varkappa}{R}+\varphi(t)\xi\Big|^4 f^2.$$

With regards to $I_3 + I_3^*$, assuming

$$\alpha^2 \geqslant \|\varphi''\|_\infty^{\frac{1}{2}} R^3,$$

and recalling that $|\frac{\varkappa}{R} + \varphi(t)\xi| \geqslant 1$, we have

$$I_3 + I_3^* \geqslant -2\alpha \int \Big[\Big|\frac{x}{R} + \varphi(t)\Big| + \Big|\frac{y}{R} + \varphi(t)\Big|\Big] \|\varphi''\|_\infty f^2 \geqslant -\frac{2\sqrt{2}\alpha^5}{R^6} \int \Big|\frac{\varkappa}{R} + \varphi(t)\xi\Big| f^2$$
$$\geqslant -\frac{2\sqrt{2}\alpha^5}{R^6} \int \Big|\frac{\varkappa}{R} + \varphi(t)\xi\Big|^4 f^2.$$

Moreover

$$I_1 + I_1^* = -\frac{48\alpha^3}{R^6} \int f^2 \geqslant -\frac{48\alpha^3}{R^6} \int \Big|\frac{\varkappa}{R} + \varphi(t)\xi\Big|^4 f^2.$$

Putting everything together and neglecting positive terms, we obtain the following estimate for the quantity $\langle [S_\alpha, A_\alpha] f, f\rangle$:

$$\begin{aligned}\langle [S_\alpha, A_\alpha] f, f\rangle \geqslant\ & \frac{47}{4}\frac{\alpha}{R^2} \int (\partial_x^2 f)^2 + \frac{47}{4} \int (\partial_y^2 f)^2 \\ & + \frac{129\alpha^3}{R^4} \int \Big(\frac{x}{R} + \varphi(t)\Big)^2 (\partial_x f)^2 + \frac{129\alpha^3}{R^4} \int \Big(\frac{y}{R} + \varphi(t)\Big)^2 (\partial_y f)^2 \\ & + \big(100 - \frac{9}{2} - 2\sqrt{2} - 48 - 6\big) \int \Big|\frac{\varkappa}{R} + \varphi(t)\xi\Big|^4 f^2 \\ & + \frac{\alpha^3}{R^4} \|\partial_x f - \alpha\partial_x\theta f\|_{L^2}^2 + \frac{\alpha^3}{R^4} \|\partial_y f - \alpha\partial_y\theta f\|_{L^2}^2 \\ \geqslant\ & \big(\frac{83}{2} - 2\sqrt{2}\big)\frac{\alpha^5}{R^6} \int \Big|\frac{\varkappa}{R} + \varphi(t)\xi\Big|^4 f^2 \\ & + \frac{\alpha^3}{R^4} \|\partial_x f - \alpha\partial_x\theta f\|_{L^2}^2 + \frac{\alpha^3}{R^4} \|\partial_y f - \alpha\partial_y\theta f\|_{L^2}^2.\end{aligned}$$

Gathering the above information we conclude that

$$\begin{aligned}&\|e^{\alpha\theta(x,y,t)}(\partial_t + \partial_x^3 + \partial_y^3) e^{-\alpha\theta(x,y,t)} f\|_{L^2}^2 \\ &\qquad \geqslant \frac{\alpha^5}{R^6} \int \Big|\frac{\varkappa}{R} + \varphi(t)\xi\Big|^4 f^2 + \frac{\alpha^3}{R^4} \|\partial_x f - \alpha\partial_x\theta f\|_{L^2}^2 + \frac{\alpha^3}{R^4} \|\partial_y f - \alpha\partial_y\theta f\|_{L^2}^2\end{aligned}$$

holds. Then a straightforward computation shows that this easily gives (II.2.1) in terms of g with $c = \sqrt{3}$. □

Remark II.4. We would like to spend few words on the costants in the Carleman estimate (II.2.1). The constants $\frac{\alpha^{\frac{5}{2}}}{R^3}$ and $\frac{\alpha^{\frac{3}{2}}}{R^2}$, which appear respectively as coefficients of the L^2- norm involving the 0 and first order derivatives, are of crucial importance and come out precisely from the structure of the equation we are working with, indeed also doing not too much effort as the one spent in the proof of the previous lemma, we can bet that the higher power of α that can shows up is really $\alpha^{\frac{5}{2}}$. We will attempt to clarify this fact considering just the one dimensional case.

The estimate requires the computation of the quantity $\|e^{-\alpha\theta}(\partial_t + \partial_x^3) e^{\alpha\theta} f\|_{L^2}^2$ (actually the exponentials are in the reverse form but with the aim of merely try to fix some ideas, it is easier to have this form instead of the other one, otherwise we would have had to take care of minus signs.

The only possibility for α to appear is after derivatives with respect to the x variable of $e^{\alpha\theta}$, for this reason we will consider just $e^{-\alpha\theta}\partial_x^3(e^{\alpha\theta}f)$. We recall that from the proof above we have seen that, writing $e^{-\alpha\theta}\partial_x^3 e^{\alpha\theta}f = (\mathcal{S}+\mathcal{A})f$, the relevant contributions for the estimate to be achieved come from the commutator $[\mathcal{S},\mathcal{A}]f := \mathcal{S}\mathcal{A}f - \mathcal{A}\mathcal{S}f$. Therefore, roughly speaking, we need to find the way to obtain the highest power of α applying, one after the other, a symmetric and then a skew-symmetric operator or the reverse. Clearly the best we can do by the action of a first operator comes from picking $e^{-\alpha\theta}\partial_x^3(e^{\alpha\theta})f$ and to choose always (three times) to derive $e^{\alpha\theta}$ getting the following starting contribution $e^{-\alpha\theta}\alpha^3 e^{\alpha\theta}(\partial_x\theta)^3 f = \alpha^3(\partial_x\theta)^3 f$.

Summing up, the contribution for the largest power of α from the application of a first operator arises from the action of the operator $\alpha^3(\partial_x\theta)^3$ that, as a multiplication operator, is clearly symmetric. Now, since the only possibility is then to apply a skew-symmetric operator, we need to understand which part of $\partial_x^3(e^{\alpha\theta}f)$ once it is explicitly computed, or better of its skew-symmetric part, would give the highest power of α once applied to $\alpha^3(\partial_x\theta)^3 f$ and would involve first derivative with respect to x. Since we need a skew-symmetric operator, we cannot proceed as before to obtain again a term like α^3, our hope is to find a way to obtain at least α^2. This means that we want something like $e^{-\alpha\theta}\partial_x^2(e^{\alpha\theta})\partial_x\cdot$ and choose to make derivatives just of the term $e^{\alpha\theta}$, this leads to the operator $\alpha^2(\partial_x\theta)^2\partial_x\,\cdot$.

So at the end we have $\alpha^2(\partial_x\theta)^2\partial_x[\alpha^3(\partial_x\theta)^3 f]$ which, up to constants, gives $\alpha^5(\partial_x\theta)^4\partial_x^2\theta$ that, after plugging the explicit expression for θ, yields the predicted constant $\frac{\alpha^5}{R^6}$.

Clearly provided suitable changes, proceeding in a similar way we can understand which kind of parts of the operator we are dealing with we have to involve in order to obtain $\frac{\alpha^3}{R^4}$ as a coefficient in front of the first derivative-dependent term. We will skip details in this case.

As already mentioned, our work is concerned with the proof of a uniqueness result, therefore we are interested in the difference of two solutions of (II.1.2), precisely, considering $v := u_1 - u_2$, where u_1 and u_2 are solutions of (II.1.2), it is not difficult to see that v satisfies the following equation:

$$\partial_t v + (\partial_x^3 + \partial_y^3)v + 4^{-\frac{1}{3}}u_1(\partial_x + \partial_y)v + 4^{-\frac{1}{3}}(\partial_x + \partial_y)u_2 v = 0. \tag{II.2.7}$$

This can be seen as a particular case of the following equation

$$\partial_t v + (\partial_x^3 + \partial_y^3)v + a_1(x,y,t)(\partial_x + \partial_y)v + a_0(x,y,t)v = 0.$$

This means that the linear operator we are interested in is

$$P = \partial_t + (\partial_x^3 + \partial_y^3) + a_1(x,y,t)(\partial_x + \partial_y) + a_0(x,y,t), \tag{II.2.8}$$

where $a_0, a_1 \in L^\infty(\mathbb{R}^3)$.

Next, we shall extend the result of II.4 to operators of the form (II.2.8). Precisely we prove

Lemma II.5. *Assume that* $\varphi\colon [0,1] \to \mathbb{R}$ *is a smooth function. Then, there exists* $c > 0$, $R_0 = R_0(\|\varphi'\|_\infty, \|\varphi''\|_\infty, \|a_0\|_\infty, \|a_1\|_\infty) > 1$ *and* $M_1 = M_1(\|\varphi'\|_\infty, \|\varphi''\|_\infty) > 0$ *such that the inequality*

$$\frac{\alpha^{\frac{5}{2}}}{R^3}\|e^{\alpha\theta(x,y,t)}\theta(x,y,t)\,g\|_{L^2(\mathbb{R}^2\times[0,1])} + \frac{\alpha^{\frac{3}{2}}}{R^2}\|e^{\alpha\theta(x,y,t)}|\nabla g|\|_{L^2(\mathbb{R}^2\times[0,1])} \leqslant c\|e^{\alpha\theta(x,y,t)}(\partial_t + \partial_x^3 + \partial_y^3 + a_1(x,y,t)(\partial_x+\partial_y) + a_0(x,y,t))g\|_{L^2(\mathbb{R}^2\times[0,1])} \quad \text{(II.2.9)}$$

holds for $R \geqslant R_0$, α *such that* $\alpha^2 \geqslant M_1 R^3$, $g \in C_0^\infty(\mathbb{R}^2\times[0,1])$ *supported in*

$$\left\{(x,y,t)\in\mathbb{R}^2\times[0,1] \colon \left|\frac{\varkappa}{R} + \varphi(t)\xi\right| \geqslant 1\right\}$$

and $\theta(x,y,t) = \left|\frac{\varkappa}{R}+\varphi(t)\xi\right|^2 = \left(\frac{x}{R}+\varphi(t)\right)^2 + \left(\frac{y}{R}+\varphi(t)\right)^2$, *with* $\varkappa = (x,y)$ *and* $\xi = (1,1)$.

Proof. First of all let us say that we are going to hide the dependence of our functions on x, y and t.

From the estimate (II.2.1) of Lemma II.4, adding and subtracting what is missing, it follows that

$$\begin{aligned}
\frac{\alpha^{\frac{5}{2}}}{R^3}\|e^{\alpha\theta}\theta g\|_{L^2} &+ \frac{\alpha^{\frac{3}{2}}}{R^2}\|e^{\alpha\theta}|\nabla g|\|_{L^2} \\
&\leqslant c\|e^{\alpha\theta}(\partial_t+\partial_x^3+\partial_y^3)g\|_{L^2} \\
&\leqslant c\|e^{\alpha\theta}(\partial_t+\partial_x^3+\partial_y^3+a_1(\partial_x+\partial_y)+a_0)g\|_{L^2} + c\|e^{\alpha\theta}(a_1(\partial_x+\partial_y)+a_0)g\|_{L^2} \\
&\leqslant c\|e^{\alpha\theta}(\partial_t+\partial_x^3+\partial_y^3+a_1(\partial_x+\partial_y)+a_0)g\|_{L^2} + \sqrt{2}c\|e^{\alpha\theta}|\nabla g|\|_{L^2}\|a_1\|_{L^\infty} \\
&\quad + c\|e^{\alpha\theta}\theta g\|_{L^2}\|a_0\|_{L^\infty},
\end{aligned}$$

where the last inequality follows from the assumption $\left|\frac{\varkappa}{R}+\varphi(t)\xi\right| \geqslant 1$.

In order to hide the last two terms on the right-hand side by the ones on the left-hand side it is necessary first to be handling finite quantities, but this is ensured by our strong assumptions about g. Moreover, as anticipated in our previous treatise about Carleman estimates, the constants in front of terms in the left-hand side should be sufficiently large to allow absorptions of eventual correction terms. Let us observe that to let ratios $\frac{\alpha^{\frac{3}{2}}}{R^2}$ and $\frac{\alpha^{\frac{3}{2}}}{R^2}$ grow as at least *positive* power of R, we need to require $\frac{\alpha^3}{R^4} \geqslant M_1 R^\varepsilon$ for some $\varepsilon > 0$, that is $\alpha^3 \geqslant M_1 R^{4+\varepsilon}$. We observe that taking $\varepsilon = \frac{1}{2}$ we find $\alpha^2 \geqslant M_1 R^3$, that is our hypothesis, therefore the last two terms can be absorbed on the left-hand side, yielding the desired result.

Remark II.5. Our hypothesis about α, namely $\alpha^2 \geqslant M_1 R^3$, turns out to be fundamental mainly to obtain that the term $\frac{\alpha^{\frac{3}{2}}}{R^2}$ grows as a positive fractional power of R. Indeed we observe that without the presence of $\|e^{\alpha\theta}|\nabla g|\|_{L^2}$, we would have had to choose α just in such a way to obtain that $\frac{\alpha^{\frac{5}{2}}}{R^3}$ grows as a positive fractional power of R. In this case, for instance, it would be sufficient to assume $\alpha^4 \geqslant M_1 R^5$ that clearly is not enough to conclude the same for $\frac{\alpha^{\frac{3}{2}}}{R^2}$. This means that we could have put weaker hypothesis about the decay of the solution in two distinct times, that is $u_1(0) - u_2(0)$; $u_1(1) - u_2(1) \in L^2(e^{a(x^2+y^2)^{\frac{1}{2}\frac{5}{4}}}\, dxdy)$.

Unfortunately, the presence on the left-hand side of the term $\|e^{\alpha\theta}|\nabla g|\|_{L^2}$ is crucial, indeed we want to obtain from a Carleman estimate for $P = \partial_t + \partial_x^3 + \partial_y^3$, a similar estimate for an operator in which the first derivatives appear and since, in order to do that, we apply an "adding and subtracting argument", we have the need of a term for the gradient and this influence the assumption about the decay of the solution. It is here that the form of the operator plays a role in the decay necessary in the hypotheses. □

Before moving on in the proof of the lower bound estimate, one of the two fundamental groundworks for proving our unique continuation result, we want to say a few words more about the strict link between the decay assumption necessary to let the continuation argument work and the form of the operator taken in exam. In order to do that we give mention to the following three works [29], [22] and [49] (in chronological order) all on unique continuation for KdV type equations.

Since our work, in essence, represents the two-dimensional counterpart of the one by Escauriaza, Kenig, Ponce and Vega [29], for their result the same observations made above for commenting on our case hold. Therefore we shall move on the work of Liana Dawson [22]. There the following result was proved:

Theorem II.5. *Let u_1, u_2 two sufficiently smooth solutions of*

$$\partial_t u + \partial_x^5 u + 10u\partial_x^3 u + 20\partial_x u\partial_x^2 u + 30u^2\partial_x u = 0, \qquad x \in \mathbb{R}, \quad t \in [0,1].$$

If there exists an $\varepsilon > 0$ such that

$$u_1(0) - u_2(0), u_1(1) - u_2(1) \in H^2(e^{ax_+^{\frac{4}{3}+\varepsilon}}\, dx)$$

for $a > 0$ sufficiently large, then $u_1 \equiv u_2$.

Actually, as it is customary, the previous comes out as a consequence of the analogue linear result for the equation

$$\partial_t v + \partial_x^5 v + a_4(x,t)\partial_x^4 v + a_3(x,t)\partial_x^3 v + a_2(x,t)\partial_x^2 v + a_1(x,t)\partial_x v + a_0(x,t)v = 0,$$

or better, since it is always possible to eliminate the fourth order term by considering $w(x,t) := u(x,y)e^{\frac{1}{5}\int_0^x a_4(s,t)\,ds}$, the attention was turned to the equation

$$\partial_t v + \partial_x^5 v + a_3(x,t)\partial_x^3 v + a_2(x,t)\partial_x^2 v + a_1(x,t)\partial_x v + a_0(x,t)v = 0. \tag{II.2.10}$$

Going through the already paved way presented in [29], a Carleman estimate for the leading order terms of the operator, namely $\partial_t + \partial_x^5$, was shown:

$$\frac{\alpha^{\frac{1}{2}}}{R}\left\|e^{\alpha\left(\frac{x}{R}+\varphi(t)\right)^2}\partial_x^4 g\right\|_{L^2} + \frac{\alpha^{\frac{3}{2}}}{R^2}\left\|e^{\alpha\left(\frac{x}{R}+\varphi(t)\right)^2}\left(\frac{x}{R}+\varphi(t)\right)\partial_x^3 g\right\|_{L^2}$$
$$+ \frac{\alpha^{\frac{5}{2}}}{R^3}\left\|e^{\alpha\left(\frac{x}{R}+\varphi(t)\right)^2}\left(\frac{x}{R}+\varphi(t)\right)^2\partial_x^2 g\right\|_{L^2} + \frac{\alpha^{\frac{7}{2}}}{R^4}\left\|e^{\alpha\left(\frac{x}{R}+\varphi(t)\right)^2}\left(\frac{x}{R}+\varphi(t)\right)^3\partial_x g\right\|_{L^2}$$
$$+ \frac{\alpha^{\frac{9}{2}}}{R^5}\left\|e^{\alpha\left(\frac{x}{R}+\varphi(t)\right)^2}\left(\frac{x}{R}+\varphi(t)\right)^4 g\right\|_{L^2} \leqslant c\|e^{\alpha\left(\frac{x}{R}+\varphi(t)\right)^2}(\partial_t+\partial_x^5)g\|_{L^2}.$$

As in our case, in order to obtain a Carleman estimate for the operator involving the lower order derivatives, that is $\partial_t + \partial_x^5 + a_3\partial_x^3 + a_2\partial_x^2 + a_1\partial_x + a_0$, an "adding and subtracting argument" is performed. To let this argument hold, since the first lower order than the fifth one that appears is the third derivative, we need to put conditions on α in such a way the ratios $\frac{\alpha^{\frac{3}{2}}}{R^2}, \frac{\alpha^{\frac{5}{2}}}{R^3}, \frac{\alpha^{\frac{7}{2}}}{R^4}, \frac{\alpha^{\frac{9}{2}}}{R^5}$ grow as fractional powers of R because, therefore, for R sufficiently large the "error" terms on the right-hand side, which come out from the addiction of derivatives up to order three, can be absorbed on the left-hand side. This entails the restriction $\alpha^3 \geqslant M_1 R^{4+\varepsilon}$ about α which strongly influences the exponential decay rate assumed about data in the unique continuation result Theorem II.5.

Let us observe that also in this fifth order setting, if one considered a differential equation in which *third* and fourth derivatives do not appear

$$\partial_t v + \partial_x^5 v + a_2(x,t)\partial_x^2 v + a_1(x,t)\partial_x v + a_0(x,t)v = 0 \tag{II.2.11}$$

we need to find conditions about α just to guarantee that $\frac{\alpha^{\frac{5}{2}}}{R^3}, \frac{\alpha^{\frac{7}{2}}}{R^4}, \frac{\alpha^{\frac{9}{2}}}{R^5}$ grow as a fractional positive power of R, then would be enough to assume a weaker condition about α, namely $\alpha^4 \geqslant M_1 R^5$. This means that in this case a stronger unique continuation result could be achieved requiring a weaker decay rate for the solutions at two distinct times.

In [49] was proved that this fact holds for a quite general class of high order equations of KdV type, which includes the KdV hierarchy. Precisely that work is concerned with unique continuation results for the equation

$$\partial_t v + (-1)^{k+1}\partial_x^n v + P(v, \partial_x v, \ldots, \partial_x^p v) = 0, \qquad x \in \mathbb{R}, \quad t \in [0,1], \tag{II.2.12}$$

where $n = 2k+1$, $k = 1, 2, \ldots$ and P is a polynomial in $v, \partial_x v, \ldots, \partial_x^p v$, with $p \leqslant n-1$. Of particular interest in that work were the cases $p = n-2$ and $p \leqslant k$ with $n \geqslant 5$. For these situations it was proved that if the difference of two sufficiently smooth solutions of the equation (II.2.12) with $p = n-2$ decays as $e^{-x_+^{\frac{4}{3}+\varepsilon}}$ at two distinct times, then $u_1 \equiv u_2$. Moreover when $p \leqslant k$ a similar result was got assuming the weaker decay $e^{-ax_+^{\frac{n}{n-1}}}$ for $a > 0$ sufficiently large.

Let us underline that the results in [22] are a particular case of the ones in [49], indeed equation (II.2.11) is nothing but (II.2.12) once it is assumed $n = 5$ ($k = 2$), and $p = 3$ ($p = n-2$), instead equation (II.2.10) is a particular case of (II.2.12) with $n = 5$ and $p \leqslant 2$.

II.2.1.2. Proof of lower bound

Now we are in position to prove the lower bound.

Before doing that in a rigorous way, we would like to deserve the first part of this section underlining, for dimension one (just for notational simplicity), that is pretending to be working with the classical KdV equation as in [29], the main steps which lead to our lower bound.

- In this framework the equation we are delaing with is

$$\partial_t v + \partial_x^3 v + a_1(x,t)\partial_x v + a_0(x,t)v = 0.$$

- Let us start from the Carleman estimate that in this simplified setting reads

$$\frac{\alpha^{\frac{5}{2}}}{R^3}\left\| e^{\alpha\left(\frac{x}{R}+\varphi(t)\right)^2} g \right\|_{L^2} \leqslant c\| e^{\alpha\left(\frac{x}{R}+\varphi(t)\right)^2}(\partial_t + \partial_x^3 + a_1\partial_x + a_0)g\|_{L^2}.$$

  Remark II.6. Let us observe that once the estimate is performed we can neglect terms on the left-hand side depending on our interests, for this reason in the previous the term involving the first derivative is missing.

  We know that Carleman estimate holds for a sufficiently smooth function g with suitable additional hypotheses but with no requirement about g to be solution of any equation. Our next aim is to apply Carleman estimate to our solution v of the KdV equation, actually to a function which resembles v but for which all the hypotheses required to apply Carleman estimates are fulfilled.

- We define as our candidate g the following function

$$g(x,t) := \theta_R(x)\mu\left(\frac{x}{R} + \varphi(t)\right) v(x,t),$$

  where

$$\theta_R(x) = \begin{cases} 1 & x < R-1 \\ 0 & x > R \end{cases}, \qquad \mu(x) = \begin{cases} 0 & x < 1 \\ 1 & x > 2 \end{cases}, \qquad \varphi(t) = \begin{cases} 0 & t \in \left[0, \frac{r}{2}\right] \cup \left[1-\frac{r}{2}, 1\right] \\ 3 & t \in [r, 1-r] \end{cases}.$$

  Let us observe that if for all $t \in [0,1]$ $\varphi(t) \equiv 0$ then $\theta_R(x)\mu\left(\frac{x}{R}\right) \equiv 0$, indeed the two functions would have had disjoint supports.

With these assumptions about functions into play, g satisfies the hypothesis for the Carleman to be applied. It is easy to see that g satisfies

$$
\begin{aligned}
&(\partial_t + \partial_x^3 + \partial_y^3 + a_1\partial_x + a_0)g \\
&\qquad = \chi_{B_R \backslash B_{R-1}\times[0,1]}(\partial_x^2 v + \partial_x v + v) + \chi_{\{(x,t)\colon 1\leqslant \left|\frac{x}{R}+\varphi(t)\right|\leqslant 2,\, t\in[0,1]\}}(\partial_x^2 v + \partial_x v + v).
\end{aligned}
$$

Note that the characteristic functions are due to the fact that we have collected in the first term on the right-hand side terms involving derivatives of θ_R, which are supported in $B_R\backslash B_{R-1}\times[0,1]$, where $\left|\frac{x}{R}+\varphi(t)\right| \leqslant 4$ instead in the second one, terms involving derivatives of $\mu\left(\frac{x}{R}+\varphi(t)\right)$, which are supported in $\{(x,t)\colon 1\leqslant \left|\frac{x}{R}+\varphi(t)\right|\leqslant 2,\ t\in[0,1]\}$. We observe also that the set $B_R\backslash B_{R-1}\times[0,1]$ will be the annular domain we want to work in. The Carleman estimate now gives

$$
\begin{aligned}
\frac{\alpha^{\frac{5}{2}}}{R^3}\|e^{\alpha\left(\frac{x}{R}+\varphi(t)\right)^2} g\|_{L^2} &\leqslant c_1\|e^{\alpha\left(\frac{x}{R}+\varphi(t)\right)^2}\chi_{B_R\backslash B_{R-1}\times[0,1]}(\partial_x^2 v + \partial_x v + v)\|_{L^2} \\
&\quad + c_2\|e^{\alpha\left(\frac{x}{R}+\varphi(t)\right)^2}\chi_{\{(x,t)\colon 1\leqslant\left|\frac{x}{R}+\varphi(t)\right|\leqslant 2\}}(\partial_x^2 v + \partial_x v + v)\|_{L^2}.
\end{aligned}
$$

- On the left-hand side we would have our solution u instead of g, in this regards it is sufficiently to observe that for $(x,t)\in(0,R-1)\times[r,1-r]$, $g(x,t)=u(x,t)$ and $\left|\frac{x}{R}+\varphi(t)\right|\geqslant 2$. So one obtains
$$\frac{\alpha^{\frac{5}{2}}}{R^3}e^{4\alpha} \leqslant \tilde{c}_1 e^{16\alpha}A_R(v) + \tilde{c}_2 e^{4\alpha},$$
where in this particular case $A_R(v) := \left(\int_0^1\int_{B_R\backslash B_{R-1}} |v|^2 + |\partial_x v|^2 + |\partial_x^2 v|^2\right)^{\frac{1}{2}} dx.$
- At the end, taking $\alpha = M_1 R^{\frac{3}{2}}$, assuming R sufficiently large, the lower bound readily follows.

Now we are in position to give the precise proof of the lower bound in our more general case, clearly we will follow the steps given above.

For $R>2$ let $\theta_R\in C^\infty(\mathbb{R}^2)$ with $\theta_R(x,y)=1$ if $\sqrt{x^2+y^2}<R-1$ and $\theta_R(x,y)=0$ if $\sqrt{x^2+y^2}>R$.

Let $\mu\in C^\infty(\mathbb{R}^2)$ with $\mu(x,y)=0$ if $\sqrt{x^2+y^2}<1$ and $\mu(x,y)=1$ if $\sqrt{x^2+y^2}>2$ and $\varphi\colon\mathbb{R}\to[0,2\sqrt{2}]$, $\varphi\in C_0^\infty(\mathbb{R})$ with

$$
\varphi(t) = \begin{cases} 0 & t\in\left[0,\frac{r}{2}\right]\cup\left[1-\frac{r}{2},1\right], \\ 2\sqrt{2} & t\in[r,1-r], \end{cases}
$$

increasing in $[\frac{r}{2},r]$ and decreasing in $[1-r,1-\frac{r}{2}]$.

As usual we define the auxiliary function

$$g(x,y,t) = \theta_R(x,y)\,\mu\Big(\frac{x}{R}+\varphi(t),\frac{y}{R}+\varphi(t)\Big)\,v(x,y,t), \qquad (x,y)\in\mathbb{R}^2,\ t\in[0,1].$$

It is easy to see that g satisfies

$$\begin{aligned}
&(\partial_t+\partial_x^3+\partial_y^3+a_1(\partial_x+\partial_y)+a_0)g\\
&= \mu\Big(\frac{x}{R}+\varphi(t),\frac{y}{R}+\varphi(t)\Big)\Big[3\partial_x\theta_R\partial_x^2v+3\partial_y\theta_R\partial_y^2v+3\partial_x^2\theta_R\partial_xv+3\partial_y^2\theta_R\partial_yv+\partial_x^3\theta_Rv+\partial_y^3\theta_Rv\\
&\qquad\qquad +a_1\partial_x\theta_Rv+a_1\partial_y\theta_Rv\Big]\\
&\quad +3R^{-1}\theta_R\partial_x\mu\,\partial_x^2v+3R^{-1}\theta_R\partial_y\mu\,\partial_y^2v\\
&\quad +3R^{-1}\big[(R^{-1}\theta_R\partial_x^2\mu+2\partial_x\theta_R\partial_x\mu)\partial_xv+(R^{-1}\theta_R\partial_y^2\mu+2\partial_y\theta_R\partial_y\mu)\partial_yv\big]\\
&\quad +\Big[\theta_R\partial_x\mu\Big(\varphi'+\frac{a_1}{R}\Big)+\theta_R\partial_y\mu\Big(\varphi'+\frac{a_1}{R}\Big)+R^{-3}\partial_x^3\mu+R^{-3}\partial_y^3\mu+3R^{-1}\partial_x^2\theta_R\partial_x\mu\\
&\qquad +3R^{-1}\partial_y^2\theta_R\partial_y\mu+3R^{-2}\partial_x\theta_R\partial_x^2\mu+3R^{-2}\partial_y\theta_R\partial_y^2\mu\Big]v.
\end{aligned}$$

Remark II.7. Let us observe that since in the first term in the right-hand side of the previous equation the derivatives of θ_R appear, therefore this term is supported in $\{(x,y)\colon R-1\leqslant\sqrt{x^2+y^2}\leqslant R\}\times[0,1]$ where $\big|\frac{\varkappa}{R}+\varphi(t)\xi\big|\leqslant 5$. Moreover one can observe that all the remaining terms, sorted with respect to their dependance on the derivatives of our solution v, contain the derivatives of μ, this means that they are supported in $\{(x,y)\colon 1\leqslant\big|\frac{\varkappa}{R}+\varphi(t)\xi\big|\leqslant 2\}\times[0,1]$.

The next step is to apply Lemma II.5 to our function g. In order to do that we have to check if the hypotheses of the lemma are fulfilled.

First of all we want to prove that g is compactly supported. Let us observe that

- if $\sqrt{x^2+y^2}>R$, then we fall outside the support of θ_R, this means that $g(x,y,t)=0$.
- if $\sqrt{x^2+y^2}<R$ and $t\in[0,\frac{r}{2}]\cup[1-\frac{r}{2},1]$, then $g(x,y,t)=0$, indeed where $t\in[0,\frac{r}{2}]\cup[1-\frac{r}{2},1]$ then $\varphi(t)=0$, this gives $\big|\frac{\varkappa}{R}+\varphi(t)\xi\big|<1$, in that case we are out of the support of $\mu\big(\frac{x}{R}+\varphi(t),\frac{y}{R}+\varphi(t)\big)$, therefore $g(x,y,t)=0$.

From the previous facts we conclude that g is compactly supported.

Now we need to prove that g is supported in $\{(x,y,t)\in\mathbb{R}^2\times[0,1]\colon\big|\frac{\varkappa}{R}+\varphi(t)\xi\big|\geqslant 1\}$. This is true simply from the definition of g, indeed if $\big|\frac{\varkappa}{R}+\varphi(t)\xi\big|<1$ then $\mu\big(\frac{x}{R}+\varphi(t),\frac{y}{R}+\varphi(t)\big)=0$ and so $g(x,y,t)=0$. Summing up g can be assumed to satisfy the hypothesis of Lemma II.5. This means that there exist $c>0, R_0$ and M_1 such that

$$c\frac{\alpha^{\frac{5}{2}}}{R^3}\|e^{\alpha\theta}g\|_{L^2(\mathbb{R}^2\times[0,1])}\leqslant\|e^{\alpha\theta}(\partial_t+\partial_x^3+\partial_y^3+a_1(\partial_x+\partial_y)+a_0)g\|_{L^2(\mathbb{R}^2\times[0,1])}. \tag{II.2.13}$$

Making use of Remark II.7 it is easy to see that

$$\|e^{\alpha\theta}(\partial_t+\partial_x^3+\partial_y^3+a_1(\partial_x+\partial_y)+a_0)g\|_{L^2(\mathbb{R}^2\times[0,1])}\leqslant c_1\,e^{25\alpha}A_R(v)+c_2\,e^{4\alpha}A. \tag{II.2.14}$$

We observe that in $Q = \{(x,y,t)\colon \sqrt{x^2+y^2} \leqslant 1,\ t \in [r, 1-r]\}$, the product $\theta_R(x,y)\mu\big(\frac{x}{R} + \varphi(t), \frac{y}{R} + \varphi(t)\big) = 1$, indeed it is not difficult to see that if we are in Q, since we are assuming $R > 2$, then $\left|\frac{x}{R} + \varphi(t)\xi\right| > \sqrt{12} > 2$ and clearly $\sqrt{x^2+y^2} \leqslant R-1$. This means that $g(x,y,t) = v(x,y,t)$ in Q.

Using this fact we obtain the following chain of inequalities:

$$c\frac{\alpha^{\frac{5}{2}}}{R^3}\|e^{\alpha\theta}g\|_{L^2(\mathbb{R}^2\times[0,1])} \geqslant c\frac{\alpha^{\frac{5}{2}}}{R^3}\|e^{\alpha\theta}g\|_{L^2(Q)} = c\frac{\alpha^{\frac{5}{2}}}{R^3}\|e^{\alpha\theta}v\|_{L^2(Q)} \geqslant c\frac{\alpha^{\frac{5}{2}}}{R^3}e^{4\alpha}\|v\|_{L^2(Q)}. \tag{II.2.15}$$

Using (II.2.13), (II.2.14), (II.2.15) and the assumption $\|v\|_{L^2(Q)} > \delta$ we obtain

$$c\frac{\alpha^{\frac{5}{2}}}{R^3}e^{4\alpha}\delta \leqslant c_1\, e^{25\alpha}A_R(v) + c_2\, e^{4\alpha}A,$$

therefore

$$c\frac{\alpha^{\frac{5}{2}}}{R^3}\delta \leqslant c_1\, e^{21\alpha}A_R(v) + c_2 A.$$

Taking $\alpha = M_1^{\frac{1}{2}}R^{\frac{3}{2}}$ with M_1 as in Lemma II.5 we obtain

$$cM_1^{\frac{5}{4}}R^{\frac{3}{4}}\delta \leqslant c_1 e^{21M_1^{\frac{1}{2}}R^{\frac{3}{2}}}A_R(v) + c_2 A.$$

Now if we take R large enough, the second term on the right-hand side of the previous inequality can be absorbed by the term on the left-hand side, so we can conclude that there exists $\widetilde{R}_0 > 0$ such that for $R \geqslant \widetilde{R}_0$ the following holds

$$A_R(v) \geqslant \frac{c}{2}e^{-21M_1^{\frac{1}{2}}R^{\frac{3}{2}}}.$$

This yields the desired result. □

II.2.2. Upper estimates

Now we will turn on the proof of the upper bound. Precisely the result that we will prove is the following.

Theorem II.6. *Assume that the coefficients of* (II.1.9) *satisfy* $a_0 \in L^\infty \cap L^2_x L^\infty_{yt}$ *and* $a_1 \in L^\infty \cap L^2_x L^\infty_{yt} \cap L^1_x L^\infty_{yt}$.

Let $v \in C([0,1]; H^4(\mathbb{R}^2))$ *be a solution of* (II.1.9) *satisfying that*

$$v(0), v(1) \in L^2(e^{a(x^2+y^2)^{\frac{3}{4}}}dxdy)$$

for some $a > 0$, *then there exists* c *and* $R_0 > 0$ *sufficiently large such that for* $R \geqslant R_0$

$$\|v\|_{L^2(\{R-1<\sqrt{x^2+y^2}<R\}\times[0,1])} + \sum_{0<k+l\leqslant 2}\|\partial_x^k\partial_y^l v\|_{L^2(\{R-1<\sqrt{x^2+y^2}<R\}\times[0,1])} \leqslant ce^{-a\left(\frac{R}{36}\right)^{\frac{3}{2}}}.$$

II.2.2.1. Groundwork results

As in [14] we will prove first the following lemma

Lemma II.6. *Let* $w \in C([0,1]; H^4(\mathbb{R}^2)) \cap C^1([0,1]; L^2(\mathbb{R}^2))$ *such that for all* $t \in [0,1]$ $\operatorname{supp} w(t) \subseteq K$, *where* K *is a compact subset of* $\mathbb{R}^2$. *Then*

1. *For* $\lambda > 0$ *and* $\beta > 0$,

$$\begin{aligned}\|e^{\lambda|x|}e^{\beta|y|}w\|_{L_t^\infty L^2_{xy}(\mathbb{R}^2\times[0,1])} \leqslant &\|e^{\lambda|x|}e^{\beta|y|}w(0)\|_{L^2(\mathbb{R}^2)} + \|e^{\lambda|x|}e^{\beta|y|}w(1)\|_{L^2(\mathbb{R}^2)} \\ &+ \|e^{\lambda|x|}e^{\beta|y|}(\partial_t + \partial_x^3 + \partial_y^3)w\|_{L_t^1 L^2_{xy}(\mathbb{R}^2\times[0,1])}.\end{aligned} \tag{II.2.16}$$

2. *There exists* $c > 0$, *independent of the set* K, *such that for* $\beta \geqslant 1$ *and* $\lambda \geqslant 2\beta$

$$\begin{aligned}&\|e^{\lambda|x|}e^{\beta|y|}Lw\|_{L_x^\infty L^2_{yt}(\mathbb{R}^2\times[0,1])} \\ &\quad\leqslant c\,(\lambda^2+\beta^2)\big(\|J^3(e^{\lambda|x|}e^{\beta|y|}w(0))\|_{L^2(\mathbb{R}^2)} + \|J^3(e^{\lambda|x|}e^{\beta|y|}w(1))\|_{L^2(\mathbb{R}^2)}\big) \\ &\quad\quad+ \|e^{\lambda|x|}e^{\beta|y|}(\partial_t + \partial_x^3 + \partial_y^3)w\|_{L_x^1 L^2_{yt}(\mathbb{R}^2\times[0,1])},\end{aligned} \tag{II.2.17}$$

where L *denotes any operator in the set* $\{\partial_x, \partial_y, \partial_x^2, \partial_y^2\}$ *and* J *is such that* $\widehat{Jg}(\xi,\eta) := (1+\xi^2+\eta^2)^{\frac{1}{2}}\widehat{g}(\xi,\eta)$.

Remark II.8. It's a very fundamental fact that in (II.2.17) there is no dependance upon λ and β except for the terms involving the data, indeed, as for Carleman estimates, we will need to hide some correction terms in the right hand side; since λ and β will grow as R which is supposed to go to infinity, we cannot expect to succeed if in front of the norm involving the operator it will be λ or β.

First of all we will need the subsequent notations

$$H_{\lambda,\beta}\cdot := e^{\lambda x}e^{\beta y}(\partial_t + \partial_x^3 + \partial_y^3)e^{-\lambda x}e^{-\beta y}\cdot = \big[\partial_t + (\partial_x - \lambda)^3 + (\partial_y - \beta)^3\big]\cdot\,. \tag{II.2.18}$$

It is easy to see from the previous definition that $H_{\lambda,\beta}$ is defined through the space-time Fourier transform by the multiplier

$$i\tau + (i\xi - \lambda)^3 + (i\eta - \beta)^3.$$

We can define the inverse operator T_0 of $H_{\lambda,\beta}$ by the symbol

$$m_0(\xi,\eta,\tau) := \frac{1}{i\tau + (i\xi-\lambda)^3 + (i\eta-\beta)^3}, \tag{II.2.19}$$

this means that

$$\widehat{T_0\,h} := m_0(\xi,\eta,\tau)\widehat{h},$$

where, in order to simplify the notation, we use $\widehat{\cdot}$ to denote the Fourier transform $\mathcal{F}$ in $S'(\mathbb{R}^3)$. The proof of Lemma II.6 is based on two previous lemmas, these lemmas express respectively the boundedness of the operator T_0 and $(\partial_x - \lambda)^k(\partial_y - \beta)^l T_0$ where k, l are non negative integers with $0 \leqslant k + l \leqslant 2$ (actually we need just the decoupled options, that is $(k, l) = (0,0), (1,0), (0,1), (2,0)$ and $(0,2)$).

Lemma II.7. *Let $h \in L^1(\mathbb{R}^3)$ with $\|h\|_{L^1_t L^2_{xy}}(\mathbb{R}^3) < \infty$. Then for all $(\lambda, \beta) \neq (0,0)$, $m_0\widehat{h} \in S'(\mathbb{R}^3)$ and $[m_0\widehat{h}]^{\vee}$ defines a bounded function from $\mathbb{R}_t$ with values in L^2_{xy}. Besides,*

$$\|[m_0\widehat{h}]^{\vee}(t)\|_{L^2_{xy}(\mathbb{R}^2)} \leqslant \|h\|_{L^1_t L^2_{xy}(\mathbb{R}^3)} \quad \forall t \in \mathbb{R}, \tag{II.2.20}$$

where $^{\vee}$ denotes the inverse Fourier transform in $S'(\mathbb{R}^3)$.

Remark II.9. Clearly the previous inequality gives the boundedness of the operator T_0 indeed, by its definition, from (II.2.20) follows that

$$\|[T_0 h](t)\|_{L^2_{xy}(\mathbb{R}^2)} \leqslant \|h\|_{L^1_t L^2_{xy}(\mathbb{R}^3)} \quad \forall t \in \mathbb{R}.$$

Proof. First of all we want to write the symbol $m_0(\xi, \eta, \tau)$ in a more useful way, precisely it is not difficult to see that the following holds:

$$m_0(\xi,\eta,\tau) = \frac{-i}{\tau + a(\xi,\eta) + ib(\xi,\eta)},$$

where

$$a(\xi,\eta) = -\xi^3 + 3\xi\lambda^2 - \eta^3 + 3\eta\beta^2 \qquad \text{and} \qquad b(\xi,\eta) = \lambda^3 - 3\xi^2\lambda + \beta^3 - 3\eta^2\beta.$$

Before going any further we want to quote the subsequent fact about Fourier transform.

Remark II.10. Our definition for the 1-dimensional Fourier transform is

$$\widehat{f}(\tau) = \frac{1}{\sqrt{2\pi}} \int_{\mathbb{R}} e^{-i\tau t} f(t)\, dt. \tag{II.2.21}$$

Making a straightforward computation it is not difficult to see that, defining

$$g(\tau) = \frac{-i}{\tau + ib}, \qquad b \neq 0,$$

the inverse Fourier transform of g has this form

$$\check{g}(t) = \begin{cases} \sqrt{2\pi}\, \chi_{(0,+\infty)}(t) e^{tb} & b < 0, \\ -\sqrt{2\pi}\, \chi_{(-\infty,0)}(t) e^{tb} & b > 0, \end{cases} \tag{II.2.22}$$

where, as usual, for a set A, χ_A denotes the characteristic function of A.

Considering the translation by the real number a of g, that is defining $G(\tau) = g(\tau + a)$, from (II.2.22) and the property that the translation in the moment space is a multiplication by a phase factor in the position space and vice-versa, in other words

$$\check{g}(\cdot + a)(t) = e^{-ita}\check{g}(t),$$

one has

$$\check{G}(t) = \begin{cases} \sqrt{2\pi}\,\chi_{(0,+\infty)}(t)e^{tb}e^{-ita} & b < 0, \\ -\sqrt{2\pi}\,\chi_{(-\infty,0)}(t)e^{tb}e^{-ita} & b > 0. \end{cases}$$

With the previous remark in mind we can say that for a fixed pair (ξ, η) with $b(\xi, \eta) \neq 0$ and $t \in \mathbb{R}$ we have

$$[m_0(\xi, \eta, \cdot_\tau)]^{\vee\tau}(t) = \begin{cases} \sqrt{2\pi}\,\chi_{(0,+\infty)}(t)e^{tb(\xi,\eta)}e^{-ita(\xi,\eta)} & b(\xi, \eta) < 0, \\ -\sqrt{2\pi}\,\chi_{(-\infty,0)}(t)e^{tb(\xi,\eta)}e^{-ita(\xi,\eta)} & b(\xi, \eta) > 0. \end{cases}$$

Clearly the magnitude of the right-hand side is bounded by $\sqrt{2\pi}$.

Now we need to compute the quantity $\big[m_0(\xi, \eta, \cdot_\tau)\widehat{h}(\xi, \eta, \cdot_\tau)\big]^{\vee\tau}(t)$.

In order to do that we recall that under our definition of the Fourier transform (II.2.21) and its inverse, the following property holds:

$$\widecheck{fg}(t) = \frac{\check{f}(t) * \check{g}(t)}{\sqrt{2\pi}},$$

moreover using that $\widehat{h} = h^{\wedge xy \wedge \tau}$, one easily obtains

$$\begin{aligned}
&\big[m_0(\xi, \eta, \cdot_\tau)\widehat{h}(\xi, \eta, \cdot_\tau)\big]^{\vee\tau}(t) \\
&\quad= \frac{[m_0(\xi, \eta, \cdot_\tau)]^{\vee\tau}(t) * h(\cdot_x, \cdot_y, t)^{\wedge xy}(\xi, \eta)}{\sqrt{2\pi}} \\
&\quad= \frac{1}{\sqrt{2\pi}}\int_{\mathbb{R}_s} [m_0(\xi, \eta, \cdot_\tau)]^{\vee\tau}(t - s)h(\cdot_x, \cdot_y, s)^{\wedge xy}(\xi, \eta)\,ds \\
&\quad= \begin{cases} \int_{\mathbb{R}_s} \chi_{(0,+\infty)}(t - s)e^{(t-s)b(\xi,\eta)}e^{-i(t-s)a(\xi,\eta)}h(\cdot_x, \cdot_y, s)^{\wedge xy}(\xi, \eta)\,ds & b(\xi, \eta) < 0, \\ -\int_{\mathbb{R}_s} \chi_{(-\infty,0)}(t - s)e^{(t-s)b(\xi,\eta)}e^{-i(t-s)a(\xi,\eta)}h(\cdot_x, \cdot_y, s)^{\wedge xy}(\xi, \eta)\,ds & b(\xi, \eta) > 0. \end{cases}
\end{aligned}$$

Let us observe that for $(\lambda, \beta) \neq (0, 0)$ since the set $\{(\xi, \eta) \colon b(\xi, \eta) = 0\}$ represents an ellipse, it has measure zero in $\mathbb{R}^2$, this gives, by applying Plancherel's formula and Minkowski's integral inequality, that for all $t \in \mathbb{R}$

$$\begin{aligned}
\|[m_0\widehat{h}]^{\vee}(\cdot_x, \cdot_y, t)\|_{L^2_{xy}(\mathbb{R}^2)} = \|[m_0\widehat{h}]^{\vee\tau}(\cdot_\xi, \cdot_\eta, t)\|_{L^2_{\xi\eta}(\mathbb{R}^2)} &\leqslant \int_{\mathbb{R}_s} \|h^{\wedge xy}(\cdot_\xi, \cdot_\eta, s)\|_{L^2_{\xi\eta}(\mathbb{R}^2)} \\
&= \|h(\cdot_x, \cdot_y, \cdot_t)\|_{L^1_t L^2_{xy}(\mathbb{R}^3)} < \infty.
\end{aligned}$$

□

As previously anticipated, we are going to prove the boundedness of the operator $(\partial_x - \lambda)^k(\partial_y - \beta)^l T_0$. precisely, we will prove the following lemma

Lemma II.8. *Let* $h \in L^1(\mathbb{R}^3)$ *with* $\|h\|_{L^1_x L^2_{yt}}(\mathbb{R}^3) < \infty$. *For* $\beta \geqslant 1$, $\lambda \geqslant 2\beta$, $k, l \in \{0, 1, 2\}$, *and* $0 \leqslant k + l \leqslant 2$, *let*

$$m_{k,l}(\xi, \eta, \tau) := (i\xi - \lambda)^k (i\eta - \beta)^l m_0(\xi, \eta, \tau),$$

with m_0 *as in* (II.2.19), *the symbol associated with the operator* $(\partial_x - \lambda)^k(\partial_y - \beta)^l\, T_0$. *Then* $m_{k,l}\widehat{h} \in S'(\mathbb{R}^3)$ *and*

$$\|[m_{k,l}\widehat{h}]^{\vee}(t)\|_{L^\infty_x L^2_{ty}(\mathbb{R}^3)} \leqslant \|h\|_{L^1_x L^2_{yt}(\mathbb{R}^3)}.$$

Remark II.11. As in Lemma II.7, from the previous inequality we can conclude the boundedness of the operator $(\partial_x - \lambda)^k(\partial_y - \beta)^l T_0$, indeed it is a trivial consequence that

$$\|[(\partial_x - \lambda)^k(\partial_y - \beta)^l T_0]h\|_{L^\infty_x L^2_{ty}(\mathbb{R}^3)} \leqslant C\|h\|_{L^1_x L^2_{yt}(\mathbb{R}^3)}.$$

Proof. We will only consider the case $k = 2$ and $l = 0$. Since the proofs of other cases are similar, for brevity, we will omit them. First of all let us note that

$$m_{2,0}(\xi, \eta, \tau) = \frac{-i(\xi + i\lambda)^2}{[(\xi + i\lambda)^3 + (\eta + i\beta)^3 - \tau]}.$$

Defining $v := \xi + i\lambda$ and $w := \eta + i\beta$ we can re-write the preceding as

$$m_{2,0}(\xi, \eta, \tau) = \frac{-iv^2}{v^3 + w^3 - \tau}.$$

The polynomial $P(v) := v^3 + w^3 - \tau$ has got, as a multiple root, just $v = 0$, but since under our hypothesis v is always different from zero, we can assume $P(v)$ not to have multiple roots. This allows us to use the following decomposition in partial fractions

$$m_{2,0} = \sum_{j=1}^{3} \frac{-iv_j^2}{3(v - v_j)v_j^2} = \sum_{j=1}^{3} \frac{-i}{3(\xi - \Re(v_j) + i[\lambda - \Im(v_j)])} = \frac{1}{3}\sum_{j=1}^{3} \frac{-i}{\xi + a_j(\eta, \tau) + ib_j(\eta, \tau)}$$

where v_j, $j = 1, 2, 3$ are the different roots of $P, a_j(\eta, \tau) = -\Re(v_j)$ and $b_j(\eta, \tau) = \lambda - \Im(v_j)$. Moving on as in Lemma II.7, that is using the Remark II.10, for a fixed pair (η, τ) such that $b(\eta, \tau) \neq 0$, making use of the linearity of the inverse Fourier transform we have

$$[m_{2,0}(\cdot_\xi, \eta, \tau)]^{\vee_\xi}(x) = \begin{cases} \frac{1}{3}\sum_{j=1}^{3} \sqrt{2\pi}\, \chi_{(0,+\infty)}(x) e^{xb_j(\eta,\tau)} e^{-ixa_j(\eta,\tau)} & b_j(\eta, \tau) < 0, \\ -\frac{1}{3}\sum_{j=1}^{3} \sqrt{2\pi}\, \chi_{(-\infty,0)}(x) e^{xb_j(\eta,\tau)} e^{-ixa_j(\eta,\tau)} & b_j(\eta, \tau) > 0. \end{cases}$$

Clearly the magnitude of the right-hand side is bounded by $\sqrt{2\pi}$.

Let us observe that the set $\{(\eta,\tau)\colon \Im(v_j)-\lambda=0\}$ has two-dimensional measure zero. Therefore using similar computations to those performed in Lemma II.7 we get that for all $x\in\mathbb{R}$

$$\|[m_{2,0}\widehat{h}]^{\vee}(x,\cdot_y,\cdot_t)\|_{L^2_{yt}(\mathbb{R}^2)} = \|[m_{2,0}\widehat{h}]^{\vee\xi}(x,\cdot_\eta,\cdot_\tau)\|_{L^2_{\eta\tau}(\mathbb{R}^2)} \leqslant \int_{\mathbb{R}_z} \|h^{\wedge yt}(z,\cdot_\eta,\cdot_\tau)\|_{L^2_{\eta\tau}(\mathbb{R}^2)}$$
$$= \|h(\cdot_x,\cdot_y,\cdot_t)\|_{L^1_x L^2_{yt}(\mathbb{R}^3)} < \infty.$$

□

Now we are in position to prove Lemma II.6. Even if the proof of this lemma is similar to the one for the corresponding result in [14], we will provide it for sake of completeness.

Proof of Lemma II.6. The proof of (II.2.16) follows from Lemma II.7 and the proof of the estimate (II.2.17) follows from Lemma II.8. We only prove the estimate (II.2.17) for $L=\partial_x^2$.

For $\varepsilon\in(0,\frac{1}{4})$ let η_ε be a function in $C_0^\infty(\mathbb{R})$ of the time variable t such that $\eta_\varepsilon(t)=1$ if $t\in[2\varepsilon,1-2\varepsilon]$, $\operatorname{supp}\eta_\varepsilon\subset[\varepsilon,1-\varepsilon]$, η_ε increasing in $[\varepsilon,2\varepsilon]$ and decreasing in $[1-2\varepsilon,1-\varepsilon]$. Let us define for all $t\in\mathbb{R}$

$$w_\varepsilon(t) := \eta_\varepsilon(t)w(t),$$

where with an abuse of notation w represents the extension of w which is identically zero outside $[0,1]$. We define

$$h_\varepsilon := e^{\lambda x}e^{\beta y}(\partial_t+\partial_x^3+\partial_y^3)w_\varepsilon,$$

then, more explicitly

$$h_\varepsilon = \eta_\varepsilon' e^{\lambda x}e^{\beta y}w + h_0, \tag{II.2.23}$$

where

$$h_0 := \eta_\varepsilon e^{\lambda x}e^{\beta y}(\partial_t+\partial_x^3+\partial_y^3)w.$$

It is not difficult to see that h_ε can be re-written as

$$h_\varepsilon = [e^{\lambda x}e^{\beta y}(\partial_t+\partial_x^3+\partial_y^3)e^{-\lambda x}e^{-\beta y}]e^{\lambda x}e^{\beta y}w_\varepsilon = H_{\lambda,\beta}(e^{\lambda x}e^{\beta y}w_\varepsilon).$$

This means that

$$e^{\lambda x}e^{\beta y}w_\varepsilon = T_0 h_\varepsilon = [m_0\widehat{h_\varepsilon}]^{\vee}.$$

Now we consider $e^{\lambda x}e^{\beta y}\,\partial_x^2 w_\varepsilon$. It is easy to see that

$$e^{\lambda x}e^{\beta y}\partial_x^2 w_\varepsilon = (e^{\lambda x}e^{\beta y}\,\partial_x^2\,e^{-\lambda x}e^{-\beta y})e^{\lambda x}e^{\beta y}w_\varepsilon = (\partial_x-\lambda)^2 e^{\lambda x}e^{\beta y}w_\varepsilon = (\partial_x-\lambda)^2 T_0 h_\varepsilon = [m_{2,0}\widehat{h_\varepsilon}]^{\vee}.$$

From the previous identity and (II.2.23), one gets

$$\begin{aligned}\|e^{\lambda x}e^{\beta y}\partial_x^2 w_\varepsilon\|_{L^\infty_x L^2_{yt}} &= \|[m_{2,0}\widehat{h_\varepsilon}]^\vee\|_{L^\infty_x L^2_{yt}}\\ &\leqslant \|\chi_{[0,1]}(\cdot_t)[m_{2,0}(\eta'_\varepsilon e^{\lambda x}e^{\beta y}w)^\wedge]^\vee\|_{L^\infty_x L^2_{yt}} + \|[m_{2,0}\widehat{h_0}]^\vee\|_{L^\infty_x L^2_{yt}}.\end{aligned} \tag{II.2.24}$$

First of all let us consider the second term on the right-hand side, using the hypotheses of Lemma II.6 we can apply Lemma II.8 to h_0, this gives

$$\|[m_{2,0}\widehat{h_0}]^\vee\|_{L^\infty_x L^2_{yt}} \leqslant \|h_0\|_{L^1_x L^2_{yt}}. \tag{II.2.25}$$

Now we need to provide an estimate for the first term on the right-hand side of (II.2.24). Using our definition of $m_{k,l}(\xi,\eta,\tau)$ we get

$$\begin{aligned}\|\chi_{[0,1]}(\cdot_t)[m_{2,0}(\eta'_\varepsilon e^{\lambda x}e^{\beta y}w)^\wedge]^\vee\|_{L^\infty_x L^2_{yt}} &= \|\chi_{[0,1]}(\cdot_t)[-(\xi+i\lambda)^2 m_0(\eta'_\varepsilon e^{\lambda x}e^{\beta y}w)^\wedge]^\vee\|_{L^\infty_x L^2_{yt}}\\ &= \|\chi_{[0,1]}(\cdot_t)[m_0\widehat{g}]^\vee\|_{L^\infty_x L^2_{yt}},\end{aligned}$$

where $\widehat{g} = -(\xi+i\lambda)^2(\eta'_\varepsilon e^{\lambda x}e^{\beta y}w)^\wedge$.

For a fixed pair $(y,t)\in\mathbb{R}^2$ one has

$$\begin{aligned}\|\chi_{[0,1]}(t)[m_0\widehat{g}]^\vee(\cdot_x,y,t)\|_{H^1_x} &= \|(1+(\cdot_\xi)^2)^{\frac12}\chi_{[0,1]}(t)[m_0\widehat{g}]^{\vee^{\eta\tau}}(\cdot_\xi,y,t)\|_{L^2_\xi}\\ &= \|(1+(\cdot_\xi)^2)^{\frac12}(\cdot_\xi+i\lambda)^2\chi_{[0,1]}(t)[m_0(\eta'_\varepsilon e^{\lambda x}e^{\beta y}w)^\wedge]^{\vee^{\eta\tau}}(\cdot_\xi,y,t)\|_{L^2_\xi}.\end{aligned}$$

Since

$$(1+\xi^2)^{\frac12}|\xi+i\lambda|^2 \leqslant (1+\xi^2)^{\frac12}(1+\xi^2+\lambda^2) \leqslant (1+\xi^2)^{\frac32}(1+\lambda^2)$$

we obtain

$$\|\chi_{[0,1]}(t)[m_0\widehat{g}]^\vee(\cdot_x,y,t)\|_{H^1_x} \leqslant (1+\lambda^2)\|J^3_x\,\chi_{[0,1]}(t)[m_0(\eta'_\varepsilon e^{\lambda x}e^{\beta y}w)^\wedge]^\vee(\cdot_x,y,t)\|_{L^2_x}.$$

Remark II.12. We emphasize that here J^3_x denotes the operator defined through the Fourier transform just in the x variable by

$$\widehat{J^3_x g}(\xi) := (1+\xi^2)^{\frac32}\widehat{g}(\xi).$$

Now, using that $H^1_x(\mathbb{R}) \hookrightarrow L^\infty_x(\mathbb{R})$ we have

$$\begin{aligned}|\chi_{[0,1]}(t)[m_0\widehat{g}]^\vee(x,y,t)| &\leqslant c\|\chi_{[0,1]}(t)[m_0\widehat{g}]^\vee(\cdot_x,y,t)\|_{H^1_x}\\ &\leqslant c(1+\lambda^2)\|J^3_x\chi_{[0,1]}(t)[m_0(\eta'_\varepsilon e^{\lambda x}e^{\beta y}w)^\wedge]^\vee(\cdot_x,y,t)\|_{L^2_x}.\end{aligned}$$

Therefore, for $x \in \mathbb{R}$, by virtue of Lemma II.7 one obtains

$$\begin{aligned}
\|\chi_{[0,1]}(\cdot_t)[m_0\widehat{g}]^{\vee}(x,\cdot_y,\cdot_t)\|_{L^2_{yt}} &\leqslant c(1+\lambda^2)\|J^3_x\chi_{[0,1]}(\cdot_t)[m_0(\eta'_\varepsilon e^{\lambda x}e^{\beta y}w)^{\wedge}]^{\vee}\|_{L^2} \\
&\leqslant c(1+\lambda^2)\|J^3_x[m_0(\eta'_\varepsilon e^{\lambda x}e^{\beta y}w)^{\wedge}]^{\vee}\|_{L^\infty_t L^2_{xy}} \\
&\leqslant c(1+\lambda^2)\|(1+(\cdot_\xi)^2+(\cdot_\eta)^2)^{\frac{3}{2}}[m_0(\eta'_\varepsilon e^{\lambda x}e^{\beta y}w)^{\wedge}]^{\vee\tau}\|_{L^\infty_t L^2_{\xi\eta}} \\
&= c(1+\lambda^2)\|[m_0(\eta'_\varepsilon J^3(e^{\lambda x}e^{\beta y}w))^{\wedge}]^{\vee}\|_{L^\infty_t L^2_{xy}} \\
&\leqslant c(1+\lambda^2)\|\eta'_\varepsilon J^3(e^{\lambda x}e^{\beta y}w)\|_{L^1_t L^2_{xy}}.
\end{aligned} \tag{II.2.26}$$

Now plugging (II.2.25) and (II.2.26) in (II.2.24) and using the explicit definition of h_0, it follows that

$$\|e^{\lambda x}e^{\beta y}\partial_x^2 w_\varepsilon\|_{L^\infty_x L^2_{yt}} \leqslant c(1+\lambda^2)\|\eta'_\varepsilon J^3(e^{\lambda x}e^{\beta y}w)\|_{L^1_t L^2_{xy}} + \|\eta_\varepsilon e^{\lambda x}e^{\beta y}(\partial_t+\partial_x^3+\partial_y^3)w\|_{L^1_x L^2_{yt}}. \tag{II.2.27}$$

First of all we want to prove that the left-hand side of (II.2.27) goes to $\|e^{\lambda x}e^{\beta y}\partial_x^2 w\|_{L^\infty_x L^2_{yt}}$ as ε tends to 0^+. Since by our hypotheses we are assuming $w(t)$ to be compactly supported, without loss of generality we may suppose $\operatorname{supp} w(t) \subset [-M,M]\times[-M,M]$ for all $t\in[0,1]$. Making use that $\partial_x^2 w(t) \in H^2(\mathbb{R}^2) \hookrightarrow L^\infty(\mathbb{R}^2)$, we get

$$\begin{aligned}
&\|e^{\lambda x}e^{\beta y}\partial_x^2 w_\varepsilon - e^{\lambda x}e^{\beta y}\partial_x^2 w\|_{L^\infty_x L^2_{yt}} \\
&\qquad = \operatorname*{ess\,sup}_{x\in[-M,M]}\Big[\int_0^1\int_{-M}^{M} e^{2\lambda x}e^{2\beta y}(\eta_\varepsilon(t)-1)^2(\partial_x^2 w)^2(x,y,t)\,dy\,dt\Big]^{\frac{1}{2}} \\
&\qquad \leqslant ce^{\lambda M}e^{\beta M}\|\partial_x^2 w\|_{C([0,1];H^2(\mathbb{R}^2))}(2M)^{\frac{1}{2}}\Big[\int_0^{2\varepsilon}dt+\int_{1-2\varepsilon}^1 dt\Big]^{\frac{1}{2}} \xrightarrow{\varepsilon\to 0^+} 0.
\end{aligned}$$

With respect to the first term of the right-hand side of (II.2.27) we can show that

$$\begin{aligned}
\|\eta'_\varepsilon J^3(e^{\lambda x}e^{\beta y}w)\|_{L^1_t L^2_{xy}} &= \int_0^1 |\eta'_\varepsilon(t)|\|J^3(e^{\lambda x}e^{\beta y}w(t))\|_{L^2_{xy}}\,dt \\
&= \int_\varepsilon^{2\varepsilon}\eta'_\varepsilon(t)\|J^3(e^{\lambda x}e^{\beta y}w(t))\|_{L^2_{xy}}\,dt - \int_{1-2\varepsilon}^{1-\varepsilon}\eta'_\varepsilon(t)\|J^3(e^{\lambda x}e^{\beta y}w(t))\|_{L^2_{xy}}\,dt \\
&= \int_\varepsilon^{2\varepsilon}\eta'_\varepsilon(t)\big(\|J^3(e^{\lambda x}e^{\beta y}w(t))\|_{L^2_{xy}} - \|J^3(e^{\lambda x}e^{\beta y}w(0))\|_{L^2_{xy}}\big)\,dt \\
&\quad + \|J^3(e^{\lambda x}e^{\beta y}w(0))\|_{L^2_{xy}} \\
&\quad - \int_{1-2\varepsilon}^{1-\varepsilon}\eta'_\varepsilon(t)\big(\|J^3(e^{\lambda x}e^{\beta y}w(t))\|_{L^2_{xy}} - \|J^3(e^{\lambda x}e^{\beta y}w(1))\|_{L^2_{xy}}\big)\,dt \\
&\quad + \|J^3(e^{\lambda x}e^{\beta y}w(1))\|_{L^2_{xy}},
\end{aligned}$$

since $e^{\lambda x}e^{\beta y}w \in C([0,1];H^3(\mathbb{R}^2))$, it is easy to see that

$$\|\eta'_\varepsilon J^3(e^{\lambda x}e^{\beta y}w)\|_{L^1_t L^2_{xy}} \xrightarrow{\varepsilon\to 0^+} \|J^3(e^{\lambda x}e^{\beta y}w(0))\|_{L^2_{xy}} + \|J^3(e^{\lambda x}e^{\beta y}w(1))\|_{L^2_{xy}}.$$

Now only the estimate of the second term of the right-hand side of (II.2.27) is missing. Taking into account that $\operatorname{supp} w \subset [-M, M] \times [-M, M] \times [0, 1]$ and using the dominated convergence theorem we can conclude that

$$\|(\eta_\varepsilon - 1)e^{\lambda x}e^{\beta y}(\partial_t + \partial_x^3 + \partial_y^3)w\|_{L_x^1 L_{yt}^2} \leqslant (2M)^{\frac{1}{2}} e^{\lambda M} e^{\beta M} \|(\eta_\varepsilon - 1)(\partial_t + \partial_x^3 + \partial_y^3)w\|_{L^2} \xrightarrow{\varepsilon \to 0^+} 0.$$

Putting all these estimates together and using $\beta \geqslant 1$ we obtain

$$\begin{aligned}\|e^{\lambda x}e^{\beta y}\partial_x^2 w\|_{L_x^\infty L_{yt}^2} \leqslant & c(\lambda^2 + \beta^2)\big(\|J^3(e^{\lambda x}e^{\beta y}w(0))\|_{L^2} + \|J^3(e^{\lambda x}e^{\beta y}w(1))\|_{L^2}\big) \\ & + \|e^{\lambda x}e^{\beta y}(\partial_t + \partial_x^3 + \partial_y^3)w\|_{L_x^1 L_{yt}^2}.\end{aligned} \tag{II.2.28}$$

In order to conclude the proof we need the following remark.

An equivalent way to write the estimate (II.2.17) is the following

$$\begin{aligned}\|e^{j\lambda x}e^{k\beta y}\partial_x^2 w\|_{L_x^\infty L_{yt}^2} \leqslant & c(\lambda^2 + \beta^2)\big(\|J^3(e^{j\lambda x}e^{k\beta y}w(0))\|_{L^2} + \|J^3(e^{j\lambda x}e^{k\beta y}w(1))\|_{L^2}\big) \\ & + \|e^{j\lambda x}e^{k\beta y}(\partial_t + \partial_x^3 + \partial_y^3)w\|_{L_x^1 L_{yt}^2},\end{aligned}$$

for $j \in \{-1, 1\}$ and $k \in \{-1, 1\}$.

We have already proved the former estimate for $j = k = 1$. Our aim is to show that the other cases follow in a similar way and so omit them.

The first step we have to perform is to modify the definition of the multipliers m_0 and $m_{k,l}$ considering, instead of $(i\xi - \lambda)$ and $(i\eta - \beta)$, the other three possible pairs: $(i\xi + \lambda)$ and $(i\eta + \beta)$ if we want to estimate $\|e^{-\lambda x}e^{-\beta y}Lw\|_{L_x^\infty L_{yt}^2}$, $(i\xi + \lambda)$ and $(i\eta - \beta)$ if we want to estimate $\|e^{-\lambda x}e^{\beta y}Lw\|_{L_x^\infty L_{yt}^2}$, $(i\xi - \lambda)$ and $(i\eta + \beta)$ for the estimate of $\|e^{\lambda x}e^{-\beta y}Lw\|_{L_x^\infty L_{yt}^2}$.

Since in order to prove (II.2.28) we strongly used the estimates in Lemma II.7 and II.8, we would like them to hold also for the modified versions of m_0 and $m_{k,l}$ written above. But one can easily see that this is true just retracing the proof of the two lemmas with the new definitions. This concludes the proof of our lemma.

As for the Carleman's estimates, our next step is to extend the estimates (II.2.16) and (II.2.17) in Lemma II.6 to operators of the form (II.2.8).

More precisely we are going to prove the following result.

Lemma II.9. *Let $w \in C([0,1]; H^4(\mathbb{R}^2)) \cap C^1([0,1]; L^2(\mathbb{R}^2))$ such that for all t $\operatorname{supp} w(t) \subseteq K$, where K is a compact subset of $\mathbb{R}^2$.*

Assume that $a_0 \in L^\infty \cap L_x^2 L_{yt}^\infty$ and $a_1 \in L^\infty \cap L_x^2 L_{yt}^\infty \cap L_x^1 L_{yt}^\infty$, with small norms in these spaces.

Then there exists $c>0$, independent of the set K, such that for $\beta \geqslant 1$ and $\lambda \geqslant 2\beta$

$$\|e^{\lambda|x|}e^{\beta|y|}w\|_{L^2(\mathbb{R}^2\times[0,1])} + \sum_{0<k+l\leqslant 2}\|e^{\lambda|x|}e^{\beta|y|}\partial_x^k\partial_y^l w\|_{L^\infty_x L^2_{yt}(\mathbb{R}^2\times[0,1])}$$
$$\leqslant c(\lambda^2+\beta^2)\big(\|J^3(e^{\lambda|x|}e^{\beta|y|}w(0))\|_{L^2(\mathbb{R}^2)} + \|J^3(e^{\lambda|x|}e^{\beta|y|}w(1))\|_{L^2(\mathbb{R}^2)}\big)$$
$$+ c\|e^{\lambda|x|}e^{\beta|y|}(\partial_t+\partial_x^3+\partial_y^3+a_1(\partial_x+\partial_y)+a_0)w\|_{L^1_tL^2_{xy}\cap L^1_xL^2_{yt}(\mathbb{R}^2\times[0,1])} \quad \text{(II.2.29)}$$

holds.

Proof. From Lemma II.6 and using the fact that $\|\cdot\|_{L^2(\mathbb{R}^2\times[0,1])} \leqslant \|\cdot\|_{L^\infty_t L^2_{xy}(\mathbb{R}^2\times[0,1])}$, it follows that

$$\|e^{\lambda|x|}e^{\beta|y|}w\|_{L^2} \leqslant \|e^{\lambda|x|}e^{\beta|y|}w(0)\|_{L^2} + \|e^{\lambda|x|}e^{\beta|y|}w(1)\|_{L^2}$$
$$+ \|e^{\lambda|x|}e^{\beta|y|}(\partial_t+\partial_x^3+\partial_y^3+a_1(\partial_x+\partial_y)+a_0)w\|_{L^1_tL^2_{xy}}$$
$$+ \|e^{\lambda|x|}e^{\beta|y|}(a_1(\partial_x+\partial_y)+a_0)w\|_{L^1_tL^2_{xy}}, \quad \text{(II.2.30)}$$

and

$$\|e^{\lambda|x|}e^{\beta|y|}Lw\|_{L^\infty_x L^2_{yt}}$$
$$\leqslant c\,(\lambda^2+\beta^2)\big(\|J^3(e^{\lambda|x|}e^{\beta|y|}w(0))\|_{L^2} + \|J^3(e^{\lambda|x|}e^{\beta|y|}w(1))\|_{L^2}\big)$$
$$+ \|e^{\lambda|x|}e^{\beta|y|}(\partial_t+\partial_x^3+\partial_y^3+a_1(\partial_x+\partial_y)+a_0)w\|_{L^1_xL^2_{yt}}$$
$$+ \|e^{\lambda|x|}e^{\beta|y|}(a_1(\partial_x+\partial_y)+a_0)w\|_{L^1_xL^2_{yt}}. \quad \text{(II.2.31)}$$

We are interested in considering the last terms in the former estimates.

We first see $\|e^{\lambda|x|}e^{\beta|y|}(a_1(\partial_x+\partial_y)+a_0)w\|_{L^1_tL^2_{xy}}$ using that $\|\cdot\|_{L^1_tL^2_{xy}(\mathbb{R}^2\times[0,1])} \leqslant \|\cdot\|_{L^2(\mathbb{R}^2\times[0,1])}$, we easily obtain

$$\|e^{\lambda|x|}e^{\beta|y|}(a_1(\partial_x+\partial_y)+a_0)w\|_{L^1_tL^2_{xy}}$$
$$\leqslant \|e^{\lambda|x|}e^{\beta|y|}(a_1(\partial_x+\partial_y)+a_0)w\|_{L^2}$$
$$\leqslant \|a_1\|_{L^2_xL^\infty_{yt}}\|e^{\lambda|x|}e^{\beta|y|}(\partial_x+\partial_y)w\|_{L^\infty_xL^2_{yt}} + \|a_0\|_{L^\infty}\|e^{\lambda|x|}e^{\beta|y|}w\|_{L^2}.$$

Let us consider now $\|e^{\lambda|x|}e^{\beta|y|}(a_1(\partial_x+\partial_y)+a_0)w\|_{L^1_xL^2_{yt}}$, making use of the Hölder's inequality, one gets

$$\|e^{\lambda|x|}e^{\beta|y|}(a_1(\partial_x+\partial_y)+a_0)w\|_{L^1_xL^2_{yt}} \leqslant \|a_1\|_{L^1_xL^\infty_{yt}}\|e^{\lambda|x|}e^{\beta|y|}(\partial_x+\partial_y)w\|_{L^\infty_xL^2_{yt}} + \|a_0\|_{L^2_xL^\infty_{yt}}\|e^{\lambda|x|}e^{\beta|y|}w\|_{L^2}.$$

Plugging the previous estimates into (II.2.30) and (II.2.31) and summing them together we have

$$\begin{aligned}\|e^{\lambda|x|}e^{\beta|y|}w\|_{L^2} &+ \sum_{0<k+l\leqslant 2}\|e^{\lambda|x|}e^{\beta|y|}\partial_x^k\partial_y^l w\|_{L^\infty_x L^2_{yt}}\\ &\leqslant c\,(\lambda^2+\beta^2)\big(\|J^3(e^{\lambda|x|}e^{\beta|y|}w(0))\|_{L^2}+\|J^3(e^{\lambda|x|}e^{\beta|y|}w(1))\|_{L^2}\big)\\ &\quad+\|e^{\lambda|x|}e^{\beta|y|}(\partial_t+\partial_x^3+\partial_y^3+a_1(\partial_x+\partial_y)+a_0)w\|_{L^1_tL^2_{xy}\cap L^1_xL^2_{yt}}\\ &\quad+\|a_0\|_{L^\infty\cap L^2_xL^\infty_{yt}}\|e^{\lambda|x|}e^{\beta|y|}w\|_{L^2}\\ &\quad+\|a_1\|_{L^2_xL^\infty_{yt}\cap L^1_xL^\infty_{yt}}\|e^{\lambda|x|}e^{\beta|y|}(\partial_x+\partial_y)w\|_{L^\infty_xL^2_{yt}}.\end{aligned}$$

Under our hypotheses about a_0 and a_1 we have

$$\begin{aligned}\|e^{\lambda|x|}e^{\beta|y|}w\|_{L^2} &+ \sum_{0<k+l\leqslant 2}\|e^{\lambda|x|}e^{\beta|y|}\partial_x^k\partial_y^l w\|_{L^\infty_x L^2_{yt}}\\ &\leqslant c\,(\lambda^2+\beta^2)\big(\|J^3(e^{\lambda|x|}e^{\beta|y|}w(0))\|_{L^2}+\|J^3(e^{\lambda|x|}e^{\beta|y|}w(1))\|_{L^2}\big)\\ &\quad+\|e^{\lambda|x|}e^{\beta|y|}(\partial_t+\partial_x^3+\partial_y^3+a_1(\partial_x+\partial_y)+a_0)w\|_{L^1_tL^2_{xy}\cap L^1_xL^2_{yt}}\\ &\quad+\frac{1}{2}\Big(\|e^{\lambda|x|}e^{\beta|y|}w\|_{L^2}+\sum_{0<k+l\leqslant 2}\|e^{\lambda|x|}e^{\beta|y|}\partial_x^k\partial_y^l w\|_{L^\infty_x L^2_{yt}}\Big).\end{aligned}\tag{II.2.32}$$

Hence, absorbing the last term on the left-hand side, we have

$$\begin{aligned}\|e^{\lambda|x|}e^{\beta|y|}w\|_{L^2} &+ \sum_{0<k+l\leqslant 2}\|e^{\lambda|x|}e^{\beta|y|}\partial_x^k\partial_y^l w\|_{L^\infty_x L^2_{yt}}\\ &\leqslant c\,(\lambda^2+\beta^2)\big(\|J^3(e^{\lambda|x|}e^{\beta|y|}w(0))\|_{L^2}+\|J^3(e^{\lambda|x|}e^{\beta|y|}w(1))\|_{L^2}\big)\\ &\quad+c\,\|e^{\lambda|x|}e^{\beta|y|}(\partial_t+\partial_x^3+\partial_y^3+a_1(\partial_x+\partial_y)+a_0)w\|_{L^1_tL^2_{xy}\cap L^1_xL^2_{yt}},\end{aligned}$$

which yields the desired result. □

Remark II.13. Although we have assumed w to be compactly supported, it is clear that the argument in Lemma II.9 can be extended to a larger class of functions. Indeed the only we have to ensure is the finiteness of the norm

$$\|e^{\lambda|x|}e^{\beta|y|}w\|_{L^2}+\sum_{0<k+l\leqslant 2}\|e^{\lambda|x|}e^{\beta|y|}\partial_x^k\partial_y^l w\|_{L^\infty_x L^2_{yt}}$$

in order to be able to perform, without contradictions, the same computations as in the estimate (II.2.32). We will see in Section 5 that there exists a class of solutions $w = w(x, y, t)$ of (II.1.9) for which the previous norm is finite. This fact enables us to extend for this kind of solutions the *a priori* estimate (II.2.29).

Now we are in position to prove the upper estimate Theorema II.6 for solutions of (II.1.9).

II.2.2.2. Proof of upper bound

We construct a C^∞ truncation function μ_R with $\mu_R(x,y) = 0$ if $\sqrt{x^2+y^2} \leqslant R$ and $\mu_R(x,y) = 1$ if $\sqrt{x^2+y^2} \geqslant \frac{36R-1}{8}$.

Let us define

$$w(x,y,t) := \mu_R(x,y)v(x,y,t).$$

Now we want to see what kind of equation is satisfied by w. It is easy to see that, since v is a solution of (II.1.9), the following holds

$$\big(\partial_t + \partial_x^3 + \partial_y^3 + a_1(x,y,t)(\partial_x + \partial_y) + a_0(x,y,t)\big)w = e_R(x,y,t),$$

where

$$\begin{aligned} e_R(x,y,t) =& \partial_x^3\mu_R v + 3\partial_x^2\mu_R\partial_x v + 3\partial_x\mu_R\partial_x^2 v + \partial_y^3\mu_R v + 3\partial_y^2\mu_R\partial_y v + 3\partial_y\mu_R\partial_y^2 v \\ &+ a_1(x,y,t)\partial_x\mu_R v + a_1(x,y,t)\partial_y\mu_R v. \end{aligned}$$

Substantially this means that our function w solves an equation like (II.1.9) but with a correction term e_R. As a next step we want to apply Lemma II.9 to our function w. First of all we need a_0, a_1 to have small norms, therefore we introduce $\widetilde{\mu_R}$ such that $\widetilde{\mu_R}\,\mu_R(x,y) = \mu_R(x,y)$, and $\widetilde{a}_j := a_j(x,y,t)\widetilde{\mu}_R$ with $j = 0,1$ have small norms in the corresponding spaces for $R \geqslant R_0$.

Let us consider the operator

$$\widetilde{L} := \partial_t + \partial_x^3 + \partial_y^3 + \widetilde{a}_1(\partial_x + \partial_y) + \widetilde{a}_0, \tag{II.2.33}$$

Now we are in position to apply (II.9) with the operator $\widetilde{L}$. This gives

$$\begin{aligned} &\|e^{\lambda|x|}e^{\beta|y|}w\|_{L^2} + \sum_{0<k+l\leqslant 2}\|e^{\lambda|x|}e^{\beta|y|}\partial_x^k\partial_y^l w\|_{L_x^\infty L_{yt}^2} \\ &\qquad \leqslant c\,(\lambda^2+\beta^2)\big(\|J^3(e^{\lambda|x|}e^{\beta|y|}w(0))\|_{L^2} + \|J^3(e^{\lambda|x|}e^{\beta|y|}w(1))\|_{L^2}\big) \\ &\qquad\qquad + c\,\|e^{\lambda|x|}e^{\beta|y|}e_R\|_{L_t^1L_{xy}^2 \cap L_x^1L_{yt}^2}. \end{aligned} \tag{II.2.34}$$

Remark II.14. With an abuse of notation we have called $\widetilde{e}_R$ as e_R, where $\widetilde{e}_R$ would be

$$\begin{aligned} e_R(x,y,t) =& \partial_x^3\mu_R v + 3\partial_x^2\mu_R\partial_x v + 3\partial_x\mu_R\partial_x^2 v + \partial_y^3\mu_R v + 3\partial_y^2\mu_R\partial_y v + 3\partial_y\mu_R\partial_y^2 v \\ &+ \widetilde{a}_1(x,y,t)\partial_x\mu_R v + \widetilde{a}_1(x,y,t)\partial_y\mu_R v. \end{aligned}$$

For $\lambda \geqslant 2$, let

$$\beta := \frac{\lambda}{2} \geqslant 1.$$

We consider the term $c\,(\lambda^2+\beta^2)\|J^3(e^{\lambda|x|}e^{\beta|y|}w(0))\|_{L^2}$.

Since w is supported in the set $\{(x,y,t)\colon \sqrt{x^2+y^2} \geqslant R,\, t \in [0,1]\}$ and using that the μ_R and its derivatives are bounded by a constant independent of R, it follows

$$\begin{aligned}
c\,(\lambda^2+\beta^2)\|J^3(e^{\lambda|x|}e^{\beta|y|}w(0))\|_{L^2} &\leqslant c\,\lambda^5 \sum_{0\leqslant k+l\leqslant 3} \|e^{\lambda|x|}e^{\beta|y|}\partial_x^k\partial_y^l w(0)\|_{L^2} \\
&\leqslant c\,\lambda^5 \sum_{0\leqslant k+l\leqslant 3} \|e^{\lambda|x|}e^{\beta|y|}\partial_x^k\partial_y^l w(0)\|_{L^2(\sqrt{x^2+y^2}\geqslant R)} \\
&\leqslant c\,\lambda^5 \sum_{0\leqslant k+l\leqslant 3} \|e^{\lambda|x|}e^{\beta|y|}\partial_x^k\partial_y^l v(0)\|_{L^2\left(\sqrt{x^2+y^2}\geqslant R\right)}.
\end{aligned}$$

Now we want to choose λ in such a way to obtain in the right-hand side of the previous estimate the weighted norm of $v(0)$ with the right exponential weight. Let

$$\lambda = \frac{4aR^{\frac{3}{2}}}{36R-1}.$$

We will use the following inequality, the proof of which easily follows applying the classical young inequality:

$$(|x| + b|y|) \leqslant \sqrt{x^2+y^2}\,\sqrt{1+b^2}.$$

This gives

$$\lambda|x| + \beta|y| = \lambda\left(|x| + \frac{|y|}{2}\right) \leqslant \lambda\sqrt{x^2+y^2}\sqrt{1+\frac{1}{4}}.$$

Using the explicit expression of λ we have

$$\lambda|x| + \beta|y| \leqslant \frac{4aR^{\frac{3}{2}}}{36R-1}\sqrt{1+\frac{1}{4}}\,\sqrt{x^2+y^2}.$$

For R sufficiently large depending on a it can be seen that

$$\lambda^5 e^{\lambda|x|+\beta|y|} \leqslant \left(\frac{4aR^{\frac{3}{2}}}{36R-1}\right)^5 e^{\frac{4aR^{\frac{3}{2}}}{36R-1}\sqrt{1+\frac{1}{4}}\,\sqrt{x^2+y^2}} \leqslant c_a e^{\frac{a}{8}(x^2+y^2)^{\frac{3}{4}}}, \qquad \text{for} \quad \sqrt{x^2+y^2} \geqslant R.$$

Using the previous estimate one has

$$c(\lambda^2+\beta^2)\|J^3(e^{\lambda|x|}e^{\beta|y|}w(0))\|_{L^2} \leqslant c_a \sum_{0\leqslant k+l\leqslant 3} \|e^{\frac{a}{8}(x^2+y^2)^{\frac{3}{4}}}\partial_x^k\partial_y^l v(0)\|_{L^2(\sqrt{x^2+y^2}\geqslant R)}.$$

Let us recall that under our hypothesis $v(0) \in L^2(e^{a(x^2+y^2)^{\frac{3}{4}}}dxdy)$, this can be rephrase saying that

$$\|e^{\frac{a}{2}(x^2+y^2)^{\frac{3}{4}}}v(0)\|_{L^2} \tag{II.2.35}$$

is finite.

Using an interpolation argument and the finiteness of (II.2.35), it can be seen that the quantity $\|e^{\frac{a}{8}(x^2+y^2)^{\frac{3}{4}}}\partial_x^k\partial_y^l v(0)\|_{L^2}$ is finite .

Getting down into details, the following interpolation result can be proved.

Lemma II.10. *For $s>0$ and $a>0$, let $f\in H^s(\mathbb{R}^2)\cap L^2\big(e^{a(x^2+y^2)^{\frac34}}\,dxdy\big)$. Then, for $\theta\in[0,1]$,*

$$\|J^{s(1-\theta)}\big(e^{\theta\frac{a}{2}(x^2+y^2)^{\frac34}}f\big)\|_{L^2}\leqslant C\|J^sf\|_{L^2}^{1-\theta}\|e^{\frac{a}{2}(x^2+y^2)^{\frac34}}f\|_{L^2}^{\theta},$$

for $C=C(a,s)$.

Observe that by our hypotheses $v(0)\in L^2(e^{a(x^2+y^2)^{\frac34}}dxdy)$ and $v(t)\in C\big([0,1];H^4(\mathbb{R}^2)\big)$ hence Lemma II.10 with $s=4$ and $\theta=\frac14$ ensures that $\|e^{\frac{a}{8}(x^2+y^2)^{\frac34}}\partial_x^k\partial_y^l v(0)\|_{L^2}$ is finite.

Using this fact we obtain

$$c\,(\lambda^2+\beta^2)\|J^3(e^{\lambda|x|}e^{\beta|y|}w(0))\|_{L^2}\leqslant c_a. \tag{II.2.36}$$

A similar argument shows that

$$c\,(\lambda^2+\beta^2)\|J^3(e^{\lambda|x|}e^{\beta|y|}w(1))\|_{L^2}\leqslant c_a. \tag{II.2.37}$$

It remains to bound the third term in the right-hand side of (II.2.34).

Since e_R is supported in $\Omega_R:=\{(x,y,t)\colon R\leqslant\sqrt{x^2+y^2}\leqslant\frac{36R-1}{8},\ t\in[0,1]\}$, we find that

$$\begin{aligned}\|e^{\lambda|x|}e^{\beta|y|}e_R\|_{L^1_tL^2_{xy}\cap L^1_xL^2_{yt}}&\leqslant e^{(\lambda+\beta)\frac{36R-1}{8}}\|e_R\chi_{\Omega_R}\|_{L^1_tL^2_{xy}\cap L^1_xL^2_{yt}}\\&\leqslant c\,e^{(\lambda+\beta)\frac{36R-1}{8}}\|(|v|+|\partial_xv|+|\partial_yv|+|\partial_x^2v|+|\partial_y^2v|)\chi_{\Omega_R}\|_{L^1_tL^2_{xy}\cap L^1_xL^2_{yt}}\\&\leqslant cR^{\frac12}e^{(\lambda+\beta)\frac{36R-1}{8}},\end{aligned} \tag{II.2.38}$$

where in the last inequality we have used Hölder inequality and the fact that the area of the region Ω_R is of order R.

Summing up, using (II.2.36),(II.2.37) and (II.2.38) we have

$$\|e^{\lambda|x|}e^{\beta|y|}w\|_{L^2}+\sum_{0<k+l\leqslant2}\|e^{\lambda|x|}e^{\beta|y|}\partial_x^k\partial_y^lw\|_{L^\infty_xL^2_{yt}}\leqslant c_a+cR^{\frac12}e^{(\lambda+\beta)\frac{36R-1}{8}}\leqslant c_aR^{\frac12}e^{(\lambda+\beta)\frac{36R-1}{8}}.$$

Defining $D_R:=\{(36R-1\leqslant\sqrt{x^2+y^2}\leqslant36R\}\times[0,1]$, we observe that $D_R\subset\{\sqrt{x^2+y^2}\geqslant R\}\times[0,1]$, the set in which w is supported, observing that in D_R we have $w=v$, one obtains

$$\begin{aligned}\|e^{\lambda|x|}e^{\beta|y|}v\|_{L^2(D_R)}&+\sum_{0<k+l\leqslant2}\|e^{\lambda|x|}e^{\beta|y|}\partial_x^k\partial_y^lv\|_{L^2(D_R)}\\&\leqslant R^{\frac12}\big(\|e^{\lambda|x|}e^{\beta|y|}w\|_{L^2}+\sum_{0<k+l\leqslant2}\|e^{\lambda|x|}e^{\beta|y|}\partial_x^k\partial_y^lw\|_{L^\infty_xL^2_{yt}}\big)\\&\leqslant c_aRe^{(\lambda+\beta)\frac{36R-1}{8}}.\end{aligned}$$

If $\sqrt{x^2+y^2}\geqslant36R-1$, then

$$\lambda|x|+\beta|y|\geqslant\frac{\lambda}{2}\sqrt{x^2+y^2}\geqslant\lambda\frac{36R-1}{2}=2aR^{\frac32}.$$

Moreover since $\lambda \geqslant 2$, one gets

$$Re^{(\lambda+\beta)\frac{36R-1}{8}} \leqslant e^{(\lambda+\beta+1)\frac{36R-1}{8}} \leqslant e^{\lambda(1+\frac{1}{2}+\frac{1}{2})\frac{36R-1}{8}} \leqslant e^{aR^{\frac{3}{2}}}.$$

This means that

$$e^{2aR^{\frac{3}{2}}}\left(\|v\|_{L^2(D_R)} + \sum_{0<k+l\leqslant 2} \|\partial_x^k \partial_y^l v\|_{L^2(D_R)}\right) \leqslant c_a e^{aR^{\frac{3}{2}}}.$$

Making explicit the expression of the set D_R, the previous can be written as

$$\|v\|_{L^2\left(\{36R-1\leqslant\sqrt{x^2+y^2}\leqslant 36R\}\times[0,1]\right)} + \sum_{0<k+l\leqslant 2} \|\partial_x^k \partial_y^l v\|_{L^2\left(\{36R-1\leqslant\sqrt{x^2+y^2}\leqslant 36R\}\times[0,1]\right)} \leqslant c_a e^{-aR^{\frac{3}{2}}},$$

or equivalently

$$\|v\|_{L^2\left(\{R-1\leqslant\sqrt{x^2+y^2}\leqslant R\}\times[0,1]\right)} + \sum_{0<k+l\leqslant 2} \|\partial_x^k \partial_y^l v\|_{L^2\left(\{R-1\leqslant\sqrt{x^2+y^2}\leqslant R\}\times[0,1]\right)} \leqslant c_a e^{-a\left(\frac{R}{36}\right)^{\frac{3}{2}}},$$

which yields the desired upper bound. □

II.2.3. The persistence properties

Even if we would have all the tools to prove our result Theorem II.3 (which gives quite straightforwardly Theorem II.2), taking in mind Remark II.13, we actually need to clarify some more details.

As already mentioned, in order to obtain the fundamental tool, that is Lemma II.9, for proving the upper bound expressed in Theorem II.6, we assumed the solution to satisfy the overabundant hypothesis of being compactly supported.

We underline again that the only hypothesis one has to assume in order to let the argument in Lemma II.9 work is the finiteness of the norm

$$\|e^{\lambda|x|}e^{\beta|y|}u\|_{L^2(\mathbb{R}^2\times[0,1])} + \sum_{0<k+l\leqslant 2} \|e^{\lambda|x|}e^{\beta|y|}\partial_x^k \partial_y^l u\|_{L^\infty_x L^2_{yt}(\mathbb{R}^2\times[0,1])}. \tag{II.2.39}$$

For this purpose we will prove that a solution u of (II.1.2) satisfies a kind of persistence property (in time) (we recall that, in general, a persistence property in the function space X means that the solution $t \mapsto u(t)$ describes a continuous curve on X, that is, $u \in C([0,1];X)$). More precisely we will show that if a solution of the symmetrized ZK equation is such that at two different times $t = 0$ and $t = 1$ has exponential decay, then the solution presents exponential decay for every $t \in [0,1]$.

Getting down into details we will prove the following result

Theorem II.7. *Let $u \in C[0,1]; H^4(\mathbb{R}^2)) \cap C^1([0,1]; L^2(\mathbb{R}^2))$ be a solution of the equation (II.1.2) such that for all $\beta > 0$, $u(0), u(1) \in L^2(e^{2\beta|x|}e^{2\beta|y|}\,dxdy)$. Then u is a bounded function from $[0,1]$ with values in $H^3(e^{2\beta|x|}e^{2\beta|y|}\,dxdy)$ for all $\beta > 0$.*

Proof. First of all we recall the following useful result (see Theorem 1.3 in [13]) concerning the decay preservation property for solutions to the ZK equation.

Theorem II.8. *Let* $u \in C([0,1]; H^4(\mathbb{R}^2)) \cap C^1([0,1]; L^2(\mathbb{R}^2))$ *be a solution of* (II.1.1)*. If for all* $\beta > 0$, $u(0), u(1) \in L^2(e^{2\beta|x|}e^{2\beta|y|}\,dxdy)$*. Then* u *is a bounded function from* $[0,1]$ *with values in* $H^3(e^{2\beta|x|}e^{2\beta|y|}\,dxdy)$ *for all* $\beta > 0$.

Remark II.15. Even if our preservation property have to hold for solutions of the *symmetric* version of the Zakharov-Kuznetsov equation that, roughly speaking, seems to resemble more the behavior of the KdV equation than the ZK' one, the aforementioned result for ZK turns out to be worthy for our purpose if one reminds that in the way to pass from the non-symmetric ZK to the symmetric one what we exploited was just a linear change of variables.

We will make this remark more precise in a moment.

We consider the following change of variables:

$$\begin{cases} x' = \mu x + \lambda y \\ y' = \mu x - \lambda y, \end{cases} \qquad \text{and its inverse} \qquad \begin{cases} x = \dfrac{x'+y'}{2\mu} \\ y = \dfrac{x'-y'}{2\lambda} \end{cases}$$

where $\mu = 4^{-\frac{1}{3}}$ and $\lambda = \sqrt{3}\mu = \sqrt{3}4^{-\frac{1}{3}}$.

We underline that the second one led us to pass from equation (II.1.1) to (II.1.2).

We define

$$\widetilde{u}(t,x,y) := u(t,x',y') = u(t,\mu x + \lambda y, \mu x - \lambda y).$$

A straightforward computation shows that $\widetilde{u}$ solves the Z-K equation

$$\partial_t \widetilde{u} + \partial_x^3 \widetilde{u} + \partial_x \partial_y^2 \widetilde{u} + \widetilde{u}\partial_x \widetilde{u} = 0.$$

Since $u(0,x,y) \in L^2(e^{2\beta|x|}e^{2\beta|y|}\,dxdy)$, then $u(0,\mu x+\lambda y,\mu x-\lambda y) \in L^2(e^{2\beta|\mu x+\lambda y|}e^{2\beta|\mu x-\lambda y|}\,dxdy)$ (clearly the same holds for $t=1$ instead of $t=0$). This guarantee that

$$\widetilde{u}(0,x,y), \widetilde{u}(1,x,y) \in L^2(e^{2\beta|\mu x+\lambda y|}e^{2\beta|\mu x-\lambda y|}\,dxdy).$$

Since we want to apply result II.8 we need to ensure that for all $\beta > 0$, we have $\widetilde{u}(0), \widetilde{u}(1) \in L^2(e^{2\beta|x|}e^{2\beta|y|}\,dxdy)$.

Recalling the parallelogram law in an Euclidean space, that reads

$$|x+y|^2 + |x-y|^2 = 2|x|^2 + 2|y|^2, \qquad \text{for all} \quad x,y \in \mathbb{R}, \tag{II.2.40}$$

and making use of the following trivial inequalities

$$a+b \geqslant (a^2+b^2)^{\frac{1}{2}}, \qquad a+b \leqslant \sqrt{2}(a^2+b^2)^{\frac{1}{2}}, \tag{II.2.41}$$

for all $a, b \geqslant 0$, it is easy to obtain

$$\begin{aligned}|\mu x' + \lambda y'| + |\mu x' - \lambda y'| &\geqslant (|\mu x' + \lambda y'|^2 + |\mu x' - \lambda y'|^2)^{\frac{1}{2}} = (2\mu^2|x'|^2 + 2\lambda^2|y'|^2)^{\frac{1}{2}} \\ &\geqslant \sqrt{2}\min\{\mu, \lambda\}(|x'|^2 + |y'|^2)^{\frac{1}{2}} \\ &\geqslant \min\{\mu, \lambda\}(|x'| + |y'|).\end{aligned}$$

This guarantees that $\widetilde{u}(0), \widetilde{u}(1) \in L^2(e^{2\gamma|x|}e^{2\gamma|y|}\,dxdy)$, where we defined $\gamma = \min\{\mu, \lambda\}\beta$.

Now we can use the decay preservation property for Z-K, Theorem II.8, and obtain that $t \mapsto \widetilde{u}(t)$ is bounded from $[0, 1]$ with values in $L^2(e^{2\gamma|x|}e^{2\gamma|y|}\,dxdy)$. This fact re-phrased in terms of $u(t, x', y')$ gives that u is a bounded function from $[0, 1]$ with values in $L^2(e^{2\gamma\left|\frac{x'+y'}{2\mu}\right|}e^{2\gamma\left|\frac{x'-y'}{2\lambda}\right|}\,dx'dy')$. Now, using again the parallelogram law (II.2.40) and the two trivial inequalities (II.2.41), we get

$$\begin{aligned}\left|\frac{x+y}{2\mu}\right| + \left|\frac{x-y}{2\lambda}\right| &\geqslant \min\left\{\frac{1}{2\mu}, \frac{1}{2\lambda}\right\}(|x+y|^2 + |x-y|^2)^{\frac{1}{2}} = \min\left\{\frac{1}{2\mu}, \frac{1}{2\lambda}\right\}(2|x|^2 + 2|y|^2)^{\frac{1}{2}} \\ &\geqslant \min\left\{\frac{1}{2\mu}, \frac{1}{2\lambda}\right\}(|x| + |y|).\end{aligned}$$

From this follows that $t \mapsto u(t)$ is bounded from $[0, 1]$ to $L^2(e^{2\gamma\min\{\mu,\lambda\}|x'|}e^{2\gamma\min\left\{\frac{1}{2\mu},\frac{1}{2\lambda}\right\}|y'|}dx'dy')$, and using the explicit expressions for λ, μ and γ we obtain that the boundedness holds from $[0, 1]$ to $L^2(e^{2\beta\theta|x'|}e^{2\beta\theta|y'|}dx'dy')$, where $\theta = \frac{2^{\frac{1}{3}}}{3^{\frac{1}{2}}} < 1$. Since this bound holds for each $\beta > 0$, at the end we can conclude that for all $\beta > 0$, $u(t)$ is a bounded function from $[0, 1]$ with values in $L^2(e^{2\beta|x'|}e^{2\beta|y'|}dx'dy')$.

In order to conclude we need another interpolation's type result.

Lemma II.11. *For $s > 0$ and $\beta > 0$, let $f \in H^s(\mathbb{R}^2) \cap L^2(e^{2\beta|x|}e^{2\beta|y|}\,dxdy)$. Then for any $\theta \in (0, 1)$,*

$$\|J^{\theta s}\big(e^{(1-\theta)(\beta|x|+\beta|y|)}f\big)\|_{L^2} \leqslant C\|J^s f\|_{L^2}^{\theta}\|e^{\beta|x|+\beta|y|}f\|_{L^2}^{1-\theta}. \tag{II.2.42}$$

Since the already proved boundedness holds for all $\beta > 0$, and, on the other hand, $u \in C([0, 1]; H^4(\mathbb{R}^2))$, we can apply the interpolation inequality (II.2.42) with $s = 4, \theta = \frac{3}{4}$, to conclude that $t \mapsto u(t)$ is bounded from $[0, 1]$ with values in $H^3(e^{2\beta|x|}e^{2\beta|y|}\,dxdy)$, which completes our proof. □

As we already mentioned the proof of our main Theorem II.2 follows as a consequence of Theorem II.3. Therefore we provide the proof of Theorem II.3 first.

II.2.4. Proof of Theorem II.3

If $v \not\equiv 0$ we can assume after a possible translation, dilation and multiplication by a constant that v satisfies the hypothesis of Theorem II.4. This means that for R sufficiently large

$$A_R(v) \geqslant c_0 e^{-c_1 R^{\frac{3}{2}}}. \tag{II.2.43}$$

Moreover applying Theorem II.6 we can say that

$$\|v\|_{L^2(\{R-1<\sqrt{x^2+y^2}<R\}\times[0,1])} + \sum_{0<k+l\leqslant 2} \|\partial_x^k \partial_y^l v\|_{L^2(\{R-1<\sqrt{x^2+y^2}<R\}\times[0,1])} \leqslant ce^{-a\left(\frac{R}{36}\right)^{\frac{3}{2}}}.$$

It is easy to see that the left-hand side of the previous expression can be bounded from below by the quantity $A_R(v)$, this gives

$$A_R(v) \leqslant ce^{-\frac{a}{6^3}R^{\frac{3}{2}}}. \tag{II.2.44}$$

If one assumes $a > a_0 := 6^3 c_1$, combining (II.2.43) and (II.2.44) and making R tends to infinity we get a contradiction.

Therefore $v \equiv 0$ and Theorem II.3 is proved. □

II.2.5. Proof of Theorem II.2

In order to prove our main result, Theorem II.2, we just need to show that Theorem II.3 applies when we consider as v the difference $u_1 - u_2$ of the solutions.

First of all we have already shown that if u_1 and u_2 are solutions of (II.1.2) then the difference v satisfies

$$\partial_t v + (\partial_x^3 + \partial_y^3)v + a_1(\partial_x + \partial_y)v + a_0 v = 0,$$

where

$$a_0 = 4^{-\frac{1}{3}}(\partial_x + \partial_y)u_2 \qquad \text{and} \qquad a_1 = 4^{-\frac{1}{3}}u_1. \tag{II.2.45}$$

As one can see from the statement of Theorem II.3 no smallness conditions about a_0 and a_1 are assumed to hold. Indeed Theorem II.3, in order to be proved, needs the upper bound presented in Theorem II.6. Retracing the proof of Theorem II.6 one can notice that the reason for which no smallness assumptions are requested, relies on the following fact: we introduced, in no way explicit, the auxiliary function $\widetilde{\mu}_R$ in such a way $\widetilde{\mu}_R a_j$, for $j = 0, 1$ have small norms in the corresponding spaces for R sufficiently large as requested for proving the preliminary and fundamental estimate (II.2.29).

Now we want to make this choice more explicit, precisely defining

$$\widetilde{\mu}_R(x,y) = \chi_{\{(x,y):\sqrt{x^2+y^2}\geqslant R\}}(x,y),$$

we will see that under this definition, $\tilde{\mu}_R a_j$ for $j = 0, 1$ have small norms, where a_j are as in (II.2.45).

For this aim, proceeding as in [14], we use the following interpolation result (see [72]).

Lemma II.12. *For $s > 0$ and $a > 0$, let $f \in H^s(\mathbb{R}^2) \cap L^2((1 + x^2 + y^2)^a \, dxdy)$. Then for any $\theta \in (0, 1)$,*

$$\|J^{\theta s}\big((1 + x^2 + y^2)^{(1-\theta)\frac{a}{2}} f\big)\|_{L^2} \leqslant C\|J^s f\|_{L^2}^{\theta} \|(1 + x^2 + y^2)^{\frac{a}{2}} f\|_{L^2}^{1-\theta}, \tag{II.2.46}$$

for $C = C(a, s)$.

Applying (II.2.46) with $s = 4, a = \frac{4}{3} + \varepsilon$ and $\theta = \frac{1}{4} + \frac{3}{16}\varepsilon$ with ε as in the statement of the theorem, we have that

$$\|J^{1+\frac{3}{4}\varepsilon}\big((1 + x^2 + y^2)^{\frac{1}{2}(1+\varepsilon_1)} f\big)\|_{L^2} \leqslant C\|J^4 f\|_{L^2}^{\theta} \|(1 + x^2 + y^2)^{\frac{1}{2}\left(\frac{4}{3}+\varepsilon\right)} f\|_{L^2}^{1-\theta}, \tag{II.2.47}$$

where $\varepsilon_1 := \frac{\varepsilon}{2} - \frac{3}{16}\varepsilon^2 > 0$.

Applying (II.2.47) with $f = a_1 = 4^{-\frac{1}{3}} u_1(t)$, from our hypothesis about the solution u_1 and from the embedding $H^{1+\frac{3}{4}\varepsilon}(\mathbb{R}^2) \hookrightarrow L^\infty(\mathbb{R}^2) \cap C(\mathbb{R}^2)$ we obtain

$$|u_1(x, y, t)| \leqslant \frac{c}{(1 + x^2 + y^2)^{\frac{1}{2}(1+\varepsilon_1)}}, \tag{II.2.48}$$

for all $(x, y, t) \in \mathbb{R}^2 \times [0, 1]$.

Since $1 + \frac{3}{4}\varepsilon > 1$, the estimate (II.2.47) is also true for J^1 instead of $J^{1+\frac{3}{4}\varepsilon}$ with $f = 4^{-\frac{1}{2}} u_2$, using the product rule for the derivatives we obtain that $\|(1 + x^2 + y^2)^{\frac{1}{2}(1+\varepsilon_1)} 4^{-\frac{1}{3}} \partial_x u_2(t)\|_{L^2(\mathbb{R}^2)}$ and $\|(1 + x^2 + y^2)^{\frac{1}{2}(1+\varepsilon_1)} 4^{-\frac{1}{3}} \partial_y u_2(t)\|_{L^2(\mathbb{R}^2)}$ are bounded function of $t \in [0, 1]$. This let us apply (II.2.46) with $f = 4^{-\frac{1}{3}} \partial_x u_2(t)$ and $f = 4^{-\frac{1}{3}} \partial_y u_2(t)$, $s = 3, a = 1 + \varepsilon_1$ and $\theta = \frac{1}{3} + \varepsilon_2$ with $\varepsilon_2 > 0$ small to obtain

$$\|J^{1+3\varepsilon_2}\big((1 + x^2 + y^2)^{\frac{1}{3}} 4^{-\frac{1}{3}} \partial_x u_2(t)\big)\|_{L^2} \leqslant C\|J^3(4^{-\frac{1}{3}} \partial_x u_2(t))\|_{L^2}^{\theta} \|(1 + x^2 + y^2)^{\frac{1+\varepsilon_1}{2}}\big) 4^{-\frac{1}{3}} \partial_x u_2(t)\|_{L^2}^{1-\theta}$$

and the same for the derivative with respect to y.

Using this estimate and again the Sobolev embeddings one has

$$|4^{-\frac{1}{3}}(\partial_x + \partial_y) u_2(t)| \leqslant \frac{c}{(1 + x^2 + y^2)^{\frac{1}{3}}} \tag{II.2.49}$$

for all $(x, y, t) \in \mathbb{R}^2 \times [0, 1]$.

From (II.2.48) and (II.2.49) it is easy to see that the following four terms

$$\|a_0 \chi_{\{(x,y): \sqrt{x^2+y^2} \geqslant R\}}\|_{L^\infty \cap L_x^2 L_{yt}^\infty}, \qquad \|a_1 \chi_{\{(x,y): \sqrt{x^2+y^2} \geqslant R\}}\|_{L_x^2 L_{yt}^\infty \cap L_x^1 L_{yt}^\infty}$$

tends to zero as R tends to ∞.

This guarantees the validity of the smallness property we need to prove.

Part III

Future Perspectives: Inverse Problem for Lamé Operators

Since the very beginning of my PhD's career, I came across the mathematical research's field related to inverse problems. Roughly and generically speaking,

> *An inverse problem is the process of reconstruction from a set of observations their causal factors that usually cannot directly be observed.*

As far as we know the first formulation of a problem in this topic was posed in the context of electricity by A.P. Calderón in 1980. More precisely, in his pioneering work, the author introduced the problem of whether it was possible to determine the electrical conductivity of a medium by making voltage and current measurements on its boundary.

The mathematical formulation of the Calderón problem was as follows. Let Ω be a bounded Lipschitz domain and let γ be a sufficiently smooth and positive function describing the distribution of the electric conductivity within Ω; it is well known that a voltage potential f at the boundary $\partial\Omega$ induces a voltage potential u in Ω which solves the following Dirichlet problem for the conductivity equation

$$\begin{cases} L_\gamma u = 0 & \Omega, \\ u = f & \partial\Omega; \end{cases}$$

where $L_\gamma u := \nabla \cdot \gamma \nabla u$.

Boundary measurements are defined as the map that takes any Dirichlet boundary value f on the boundary, i.e. the voltage distribution, to the corresponding outflowing current, that is to the term $\Lambda_\gamma(f) := \gamma\frac{\partial u}{\partial\nu}\big|_{\partial\Omega}$ where u is the solution to the Dirichlet problem with boundary data f. In literature this map is known as the Dirichlet to Neumann map.

By view of this we can mathematically re-phrase the Calderón problem as:

Is it possible to recover γ from the knowledge of the Dirichlet to Neumann map Λ_γ?

Clearly an obvious condition for this recovery to be possible is that the map $\gamma \mapsto \Lambda_\gamma$ is injective. Therefore, when one is dealing with an inverse problem, the first purpose is to guarantee the injectivity of Λ_γ.

The general strategy under this proposal follows the starting ideas of Sylvester and Uhlmann [88] and can be summarized into few steps:

- *Reduction to Schrödinger equation.*

  The inverse problem for the conductivity equation is reduced to an inverse problem for Schrödinger equation. Precisely, if u is a solution to the equation $L_\gamma u = 0$, then $v := \gamma^{1/2}u$ satisfies

  $$(-\Delta + q)v = 0, \tag{50}$$

  with $q = \gamma^{-1/2}\Delta\gamma^{1/2}$.

  The corresponding Dirichlet to Neumann map is defined by $\Lambda_q(f) := \frac{\partial v}{\partial\nu}\big|_{\partial\Omega}$, where v is now a solution to $(-\Delta + q)v = 0$ with boundary data f.

  It is easy to see that if γ_1 and γ_2 satisfy $\Lambda_{\gamma_1} = \Lambda_{\gamma_2}$ then, by boundary identification result, we have $\Lambda_{q_1} = \Lambda_{q_2}$ for $q_j = \gamma_j^{-1/2}\Delta\gamma_j^{1/2}$. In this way the uniqueness problem for the conductivity equation is addressed to the same problem for the Schröedinger equation.

- *Intermediate identity.*

  Now if u_j, $j = 1, 2$ are two weak solutions to the equation $(-\Delta + q)u_j = 0$ and if we assume that $\Lambda_{q_1} = \Lambda_{q_2}$, then a simple integration by parts shows that

  $$\int_\Omega (q_1 - q_2)u_1 u_2 \, dx = 0. \tag{51}$$

  It follows that one way to show that the potentials q_1 and q_2 coincide is to produce enough solutions to the corresponding Schrödinger equations such that their product is dense in some sense.

- *Carleman estimates for CGO solutions.*

To this end, Sylvester and Uhlmann provided special solutions to the Schrödinger equation, known as complex geometrical optics (CGO) solutions, that have the form

$$u_j = e^{i\zeta_j \cdot x}(1 + r_j). \tag{52}$$

Here the complex vectors $\zeta_j := \lambda(\beta_j + i\alpha_j)$, with α_j, β_j orthogonal unit vectors in $\mathbb{R}^d$ and λ a *large* parameter, are chosen so that $\zeta_j \cdot \zeta_j = 0$ (thus $e^{i\zeta_j \cdot x}$ is harmonic) and so that $e^{i\zeta_1 \cdot x} e^{i\zeta_2 \cdot x} = e^{ik \cdot x}$ for some fixed frequency $k \in \mathbb{R}^d$.

Now supposing to neglect the remainder's terms r_j, that is supposing to have just $u_j = e^{i\zeta_j \cdot x}$, from the intermediate equality (51) we would obtain

$$\int_\Omega (q_1 - q_2) e^{ik \cdot x} \, dx = 0,$$

which would give the uniqueness via inverse Fourier transform.

Therefore the only part still left is to verify that (52) are solutions and prove that the remainder's terms r_j go to zero as ζ_j goes to infinity and that this occurs in such a way the previous identity is fulfilled.

The main tool used to reach this aim is a suitable Carleman estimate. By virtue of my interests in this topic, I will treat this part more in details.

Let us observe that $u_j := e^{i\zeta_j \cdot x}(1 + r_j)$ is a solution of (50) if and only if

$$e^{-i\zeta_j \cdot x}(-\Delta + q) e^{i\zeta_j \cdot x}(1 + r_j) = 0,$$

that is if and only if r_j solves

$$e^{-i\zeta_j \cdot x}(-\Delta + q) e^{i\zeta_j \cdot x} r_j = -q$$

So the problem to verify the "ansatz" (52) is diverted to find r_j which solves the previous equation.

Find such r_j is nothing but proving that the operator $e^{-i\zeta_j \cdot x}(-\Delta + q) e^{i\zeta_j \cdot x} \cdot$ is surjective or, what is the same with Hanh-Banach Theorem, the adjoint $e^{i\zeta_j \cdot x}(-\Delta + q) e^{-i\zeta_j \cdot x} \cdot$ is injective.

In order to do that, ignoring the imaginary part of the exponent which gives rise to a phase term that is irrelevant within an L^p-norm, we are going to prove the following Carleman type estimate

$$\|r\|_{L^2(\Omega)} \leqslant \frac{C}{|\lambda|} \| e^{-\lambda \alpha \cdot x}(-\Delta + q) e^{\lambda \alpha \cdot x} r \|_{L^2(\Omega)}, \tag{53}$$

indeed this can be seen as a quantitative estimate for the injectivity of the operator $e^{-\lambda\alpha\cdot x}(-\Delta+q)e^{\lambda\alpha\cdot x}\cdot$.

Actually, instead of proving directly the previous estimate, one can prove a reduced version of it:

$$\|r\|_{L^2(\Omega)} \leqslant \frac{C}{|\lambda|}\|e^{-\lambda\alpha\cdot x}(-\Delta)e^{\lambda\alpha\cdot x}r\|_{L^2(\Omega)}, \tag{54}$$

that is the estimate involving just the principal part of the operator. Then (53) is easily obtained. This fact is mainly due to the constant $\frac{C}{|\lambda|}$, indeed by taking λ very large we can make the constant $\frac{C}{|\lambda|}$ as small as we need to absorb eventual extra-terms such as $\|q\|_{L^\infty(\Omega)}\|r\|_{L^2(\Omega)}$.

To achieve estimates like (54), in essence, as we have already seen in Part II, the main goal is to obtain a lower bound for the quantity $\|e^{-\lambda\alpha\cdot x}(-\Delta)e^{\lambda\alpha\cdot x}r\|^2_{L^2(\Omega)}$, this means that we have to be able to focus on positive and negative contributions coming from the explicit action of the operator $P := e^{-\lambda\alpha\cdot x}(-\Delta)e^{\lambda\alpha\cdot x}$. To this end, it is customary to decompose the operator P as a sum of its symmetric $\mathcal{S}$ and skew-symmetric part $\mathcal{A}$. In our case we have:

$$P = \mathcal{S} + \mathcal{A}, \quad \mathcal{S} := -\Delta - |\lambda|^2, \quad \mathcal{A} := -2\lambda\alpha\cdot\nabla.$$

Therefore, one has

$$\|e^{-\lambda\alpha\cdot x}(-\Delta)e^{\lambda\alpha\cdot x}r\|^2_{L^2(\Omega)} = \|\mathcal{S}r\|^2_{L^2(\Omega)} + \|\mathcal{A}r\|^2_{L^2(\Omega)} + \langle[\mathcal{S},\mathcal{A}]r, r\rangle_{L^2(\Omega)}.$$

The terms $\|\mathcal{S}r\|^2_{L^2(\Omega)}$ and $\|\mathcal{A}r\|^2_{L^2(\Omega)}$ are non-negative, this means that the only negative contribution could come from $\langle[\mathcal{S},\mathcal{A}]r, r\rangle_{L^2(\Omega)}$, but, since $\mathcal{S}$ and $\mathcal{A}$ are constant coefficients differential operator, then $[\mathcal{S},\mathcal{A}] \equiv 0$. At the end, by virtue of this remark, it is easy to conclude just by means of Poincaré's inequality.

III.1. Research statement and proposed research approach

One of my future prospect could be to study, in a deeper way, inverse problems related to elasticity setting, indeed, even if a considerable interest in this topic is being developed in recent years (see, for instance, Nakamura-Uhlmann [73, 74, 75], Eskin-Ralston [33]), the literature concerning this field is very much less unified than the one in the electricity framework.

The most natural approach to face this problem could be to try to re-adapt the well-oiled strategy shown above.

First of all we need a slightly different formulation, that is we have to study the equation

$$(-\Delta^* + q)u = 0, \tag{III.1.1}$$

instead of (50), where $-\Delta^*$ represents the Lamé operator of elasticity (I.1.1).

At first sight, a preliminary issue to be considered could be that Lamé operators act on vector-valued functions, so that the resulting differential models are systems. In this case, as it is very well known, Carleman estimates are hard to be proved, mainly due to the difficulties in finding appropriate weights. On the other hand, the Helmholtz decomposition strongly comes into play in overcoming the problem and also in laying solid motivations to the possible success of this project. Indeed, using this tool we have already seen that, for any $u = u_P + u_S$, the operator $-\Delta^*$ acts on u in this way:

$$-\Delta^* u = -\mu\Delta u_S - (\lambda + 2\mu)\Delta u_P,$$

where the component u_S is the divergence free vector field and the component u_P is the gradient. This means that there is a deep link between Lamé and Laplace operator and therefore, beyond technical details that could occur from the no-scalar form of the problem, the resolution should not be too distant from the one that works for Schrödinger. Moreover, once one has this explicit action, by virtue of the H^1- orthogonality of the two components of the decomposition, it can be seen that the equation (III.1.1) can be decoupled into two distinct equations involving separately the two components:

$$\begin{cases} \big(-\Delta + \frac{q}{\mu}\big)u_S = 0 \\ \big(-\Delta + \frac{q}{\lambda+2\mu}\big)u_P = 0. \end{cases}$$

This two equations strongly resemble equation (50) that, we recall, was the starting point of the powerful machinery under the resolution of the Calderón inverse problem and this fact augur well for the possibility to obtain meaningful results within the inverse problems' landscape for elasticity.

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