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Resolvent and logarithmic residues of a singular operator pencil in Hilbert spaces

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ABSTRACT

The present paper considers the operator pencil $A(\lambda) = A_0 + A_1\lambda$, where $A_0, A_1 \neq 0$ are bounded linear mappings between complex Hilbert spaces and A_0 is neither one-to-one nor onto. Assuming that 0 is an isolated singularity of $A(\lambda)$ and that the image of A_0 is closed, certain operators are defined recursively starting from A_0 and A_1 and they are shown to provide a characterization of the image and null space of the operators in the principal part of the resolvent and of the logarithmic residues of $A(\lambda)$ at 0. The relations with the classical results based on ascent and descent in [10] are discussed. In the special case of A_0 being Fredholm of index 0, the present results characterize the rank of the operators in the principal part of the resolvent, the dimension of the subspaces that define the ascent and descent, the partial multiplicities, and the algebraic multiplicity of $A(\lambda)$ at 0.

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1. Introduction

Let $\mathscr{B}_{X,Y}$ be the space of bounded linear mappings between two complex Hilbert spaces X and Y (when $X = Y$, the shorthand notation $\mathscr{B}_X = \mathscr{B}_{X,X}$ is employed) and consider the operator pencil[2]

$$A(\lambda) = A_0 + A_1\lambda, \quad A_0, A_1 \in \mathscr{B}_{X,Y}, \quad A_0, A_1 \neq 0, \quad \lambda \in \Omega \subseteq \mathbb{C}. \tag{1.1}$$

Assume that A_0 is neither one-to-one nor onto and that there exists $\rho > 0$ such that $A(\lambda)$ is invertible for all $\lambda \in D_\rho \setminus \{0\}$, where $D_\rho = \{\lambda \in \mathbb{C} : |\lambda| < \rho\}$. That is, 0 is an isolated singularity of $A(\lambda)$.[3]

Under the assumption that the image of A_0 is closed and that the resolvent $A(\lambda)^{-1}$ has a pole of some order $m \in \mathbb{N}$ at 0, the present paper provides a construction of the resolvent and of the logarithmic residues and a characterization of the image and null space of those operators. The quantities are all expressed in terms of the image and null space of certain operators defined recursively by a procedure called 'local subspace decomposition', see Definition 2.2 below, which only requires knowledge of A_0, A_1.

This recursive procedure was introduced in [12,13] for the finite dimensional case $X = Y = \mathbb{C}^p$ and it was called 'local rank factorization'. Franchi and Paruolo [13] also showed that the (extended) local rank factorization coincides with the 'complete reduction process' in [6]; the latter builds on the results of [21] and on the reduction technique in [20], [19] and delivers the order of the pole and the coefficients of the resolvent. The book by [5] contains a thorough treatment of their approach and its many extensions, see e.g. [22] and [1–4] for the infinite dimensional Hilbert and Banach space cases. The local rank factorization was further employed in [14] and [11] for eigenvalues of finite type in the (possibly infinite dimensional) Hilbert space case $X = Y$.

In the setup considered in the present paper, the notion of rank may not be well defined and thus the name 'local rank factorization' is misleading. Because the procedure depends only on the sequence of subspaces determined by the image and null space of certain operators (and not on their dimensions), the name 'local subspace decomposition' is used here instead.

From the theory of essential singularities in Banach spaces in [2], it is well known that the resolvent of (1.1) admits the Laurent series representation

$$A(\lambda)^{-1} = \sum_{n\in\mathbb{Z}} R_n\lambda^n, \quad \lambda \in U_{\sigma,\rho} = \{\lambda \in \mathbb{C} : \sigma < |\lambda| < \rho\}, \quad 0 \leq \sigma < \rho \leq \infty, \tag{1.2}$$

if and only if the operators $R_n \in \mathscr{B}_{Y,X}$ are suitably bounded, i.e.

[2] In the following, 0 and I respectively denote the zero and the identity in the different ambient spaces (the relevant space should be clear from the context).

[3] By the change of variable $(\lambda - \lambda_0) \mapsto \lambda$, this setup handles isolated singularities at points λ_0 other than zero. In fact, if $\lambda_0 \neq 0$ is such that $A(\lambda_0) = A_0 + \lambda_0 A_1$ is not invertible then $A(\lambda) = A_0 + \lambda_0 A_1 + A_1\lambda - \lambda_0 A_1 = \widetilde{A}_0 + A_1(\lambda - \lambda_0)$, where $\widetilde{A}_0 = A_0 + \lambda_0 A_1$ is singular. Hence there is no loss of generality in assuming that $\lambda_0 = 0$ is the point at which $A(\lambda)$ is singular.

$$\lim_{n\to\infty} ||R_{-n}||^{1/n} \leq \sigma, \qquad \lim_{n\to\infty} ||R_n||^{1/n} \leq 1/\rho, \tag{1.3}$$

and satisfy the 'fundamental equations'

$$A_0 R_n + A_1 R_{n-1} = 1_{n=0} I, \qquad R_n A_0 + R_{n-1} A_1 = 1_{n=0} I, \qquad n \in \mathbb{Z}, \tag{1.4}$$

where $1_{n=h}$ is the indicator function (equal to 1 if $n = h$ and to 0 otherwise); these follow from the identities $A(\lambda)A(\lambda)^{-1} = I \in \mathscr{B}_Y$ and $A(\lambda)^{-1}A(\lambda) = I \in \mathscr{B}_X$.

Albrecht et al. [2] show that (1.3) and (1.4) hold if and only if (i) $P = R_{-1}A_1$ and $P^c = I - P = R_0 A_0$ are complementary projections on X and $Q = A_1 R_{-1}$ and $Q^c = I - Q = A_0 R_0$ are complementary projections on Y, (ii) $A_n = QA_nP + Q^cA_nP^c$, $n = 0, 1$, (iii) $R_n = PR_nQ$, $n = -1, -2, \dots$, and $R_n = P^cR_nQ^c$, $n = 0, 1, \dots$,

$$R_n = \begin{cases} (-1)^{-n-1}(R_{-1}A_0)^{-n-1}R_{-1}, & n = -1, -2, \dots \\ (-1)^n (R_0 A_1)^n R_0, & n = 0, 1, \dots \end{cases}, \tag{iv}$$

and (v) $\lim_{n\to\infty} ||(R_{-1}A_0)^n||^{1/n} \leq \sigma$, $\lim_{n\to\infty} ||(R_0A_1)^n||^{1/n} \leq 1/\rho$. The formulas in (iv) extend those derived for the matrix case in [25,26], see also [28]. Note that $A(\lambda)$ (and thus $A(\lambda)^{-1}$) is completely reduced by the complementary projections in (i) and further observe that, when $R_{-1}A_0$ is nilpotent, the theory covers poles as a particular case.

However, as observed in Remark 1 in [4], R_n, $n \in \mathbb{Z}$, is unknown; hence these formulas cannot be used per se to calculate the complementary projections and the coefficients of the resolvent because they are all expressed in terms of R_0, R_{-1}. Albrecht et al. [4] define infinite-length singular and regular Jordan chains that determine the complementary projections P, P^c, Q, Q^c, and, when X, Y are separable, they find R_0, R_{-1} by solving recursively the projected versions of the fundamental equations. They then employ (iv) to calculate the resolvent.

The present paper considers the special case in which X and Y are Hilbert spaces and the resolvent has a pole of some order $m \in \mathbb{N}$ at 0, i.e. $R_{-m} \neq 0$ and $R_{-m-n} = 0$ for $n = 1, 2, \dots$. As shown in [2], in this case the annulus of convergence of (1.2) becomes the punctured disc $U_{0,\rho} = \{\lambda \in \mathbb{C} : 0 < |\lambda| < \rho\} = D_\rho \setminus \{0\}$; here we set $Z_n = R_{-m+n}$ for $n = 0, 1, \dots$ and write (1.2) as

$$A(\lambda)^{-1} = \sum_{n=0}^{\infty} Z_n \lambda^{n-m}, \qquad \lambda \in D_\rho \setminus \{0\}, \qquad Z_0 \neq 0. \tag{1.5}$$

From the classical theory of poles in Banach spaces in Bart and Lay [10], see also [16], [24], [29], [23], [27], it is well known that the resolvent has a pole of order m at 0 if and only if the ascent and the descent (defined therein in terms of certain subspaces $\mathscr{N}_n, \mathscr{R}_n, \mathscr{N}'_n, \mathscr{R}'_n$, $n = 0, 1, \dots$) equal m. In this case, the direct sum decompositions

$$X = \mathscr{N}_m \oplus \mathscr{R}_m, \qquad Y = \mathscr{N}'_m \oplus \mathscr{R}'_m, \tag{1.6}$$

where $\mathscr{N}_m$, $\mathscr{R}_m$, $\mathscr{N}'_m$, and $\mathscr{R}'_m$ are closed subspaces, completely reduce $A(\lambda)$ (and thus $A(\lambda)^{-1}$); that is, relative to them, $A(\lambda)$ admits the operator matrix representation

$$A(\lambda) = \begin{pmatrix} A_{0\mathscr{N}} + A_{1\mathscr{N}}\lambda & 0 \\ 0 & A_{0\mathscr{R}} + A_{1\mathscr{R}}\lambda \end{pmatrix}, \qquad A_{n\mathscr{Z}} = A_n|\mathscr{Z}_m : \mathscr{Z}_m \mapsto \mathscr{Z}'_m,$$

where $A_{n\mathscr{N}}$ denotes the restriction of A_n to $\mathscr{N}_m$ and maps $\mathscr{N}_m$ to $\mathscr{N}'_m$ and $A_{n\mathscr{R}}$ is the restriction of A_n to $\mathscr{R}_m$ mapping $\mathscr{R}_m$ to $\mathscr{R}'_m$; thus

$$A(\lambda)^{-1} = \begin{pmatrix} (A_{0\mathscr{N}} + A_{1\mathscr{N}}\lambda)^{-1} & 0 \\ 0 & (A_{0\mathscr{R}} + A_{1\mathscr{R}}\lambda)^{-1} \end{pmatrix}.$$

Because $A_{1\mathscr{N}}$ and $A_{0\mathscr{R}}$ are invertible and $N = (A_{1\mathscr{N}})^{-1}A_{0\mathscr{N}}$ is nilpotent with index m, i.e. $N^m = 0$ and $N^{m-1} \neq 0$, the principal part is

$$(A_{0\mathscr{N}} + A_{1\mathscr{N}}\lambda)^{-1} = \sum_{n=0}^{m-1} Z_{n\mathscr{N}'}\lambda^{n-m}, \qquad Z_{n\mathscr{N}'} = (-1)^{m-1-n}N^{m-1-n}A_{1\mathscr{N}}^{-1},$$

where $Z_{n\mathscr{N}'}$ is the restriction of Z_n to $\mathscr{N}'_m$ mapping $\mathscr{N}'_m$ to $\mathscr{N}_m$, and the regular part is

$$(A_{0\mathscr{R}} + A_{1\mathscr{R}}\lambda)^{-1} = \sum_{n=m}^{\infty} Z_{n\mathscr{R}'}\lambda^{n-m},$$

$$Z_{n\mathscr{R}'} = \begin{cases} A_{0\mathscr{R}}^{-1} & n = m \\ -A_{0\mathscr{R}}^{-1}A_{1\mathscr{R}}Z_{n-1\mathscr{R}'} & n = m+1, m+2, \dots \end{cases},$$

where $Z_{n\mathscr{R}'}$ is the restriction of Z_n to $\mathscr{R}'_m$ mapping $\mathscr{R}'_m$ to $\mathscr{R}_m$. This shows that, relative to the direct sum decompositions in (1.6), the resolvent admits the operator matrix representation

$$A(\lambda)^{-1} = \sum_{n=m}^{\infty} \begin{pmatrix} Z_{n\mathscr{N}'} & 0 \\ 0 & 0 \end{pmatrix} \lambda^{n-m} + \sum_{n=m}^{\infty} \begin{pmatrix} 0 & 0 \\ 0 & Z_{n\mathscr{R}'} \end{pmatrix} \lambda^{n-m}, \tag{1.7}$$

from which one can construct Z_n in (1.5) by a change of bases. Finally recall that, letting $P_{\mathscr{N},\mathscr{R}}$, $P_{\mathscr{R},\mathscr{N}} = I - P_{\mathscr{N},\mathscr{R}}$ and $P_{\mathscr{N}',\mathscr{R}'}$, $P_{\mathscr{R}',\mathscr{N}'} = I - P_{\mathscr{N}',\mathscr{R}'}$ be the projections associated to the direct sum decompositions in (1.6), one has

$$P_{\mathscr{N},\mathscr{R}} = \frac{1}{2\pi i}\int_{\partial D_\rho} A(\lambda)^{-1}A_1 d\lambda = Z_{m-1}A_1,$$

$$P_{\mathscr{N}',\mathscr{R}'} = \frac{1}{2\pi i}\int_{\partial D_\rho} A_1 A(\lambda)^{-1} d\lambda = A_1 Z_{m-1},$$

where ∂D_ρ is the boundary of D_ρ. That is, $P_{\mathscr{N},\mathscr{R}}$ coincides with the right logarithmic residue of $A(\lambda)$ at 0 and $P_{\mathscr{N}',\mathscr{R}'}$ with the left logarithmic residue of $A(\lambda)$ at 0, see [7–9] and references therein for general results on logarithmic residues and idempotents.

The results of the present paper are the following: first, necessary and sufficient conditions for a pole of order m are given; these conditions are expressed in terms of certain orthogonal direct sum decompositions

$$X = \bigoplus_{h=0}^{m} \beta_h, \qquad \beta_0, \beta_m \neq 0, \qquad Y = \bigoplus_{h=0}^{m} \alpha_h, \qquad \alpha_0, \alpha_m \neq 0,$$

where β_h and α_h are closed subspaces defined recursively by the 'local subspace decomposition', see Definition 2.2 below. This recursion employs the first $m+1$ fundamental equations, i.e.

$$A_0 Z_n + A_1 Z_{n-1} = 1_{n=m} I, \qquad Z_n A_0 + Z_{n-1} A_1 = 1_{n=m} I, \qquad n = 0, \ldots, m, \tag{1.8}$$

and defines certain operators S_h whose images and orthogonal complements of the null spaces, $\alpha_h = \operatorname{Im} S_h$ and $\beta_h = (\operatorname{Ker} S_h)^\perp$, deliver the orthogonal direct sum decompositions that characterize the order of the pole.

The same recursion defines operators $M_\beta \in \mathscr{B}_X$ and $M_\alpha \in \mathscr{B}_Y$ that allow to construct the factors in

$$A(\lambda) = E(\lambda) D(\lambda) F(\lambda),$$

where $E(\lambda), F(\lambda)$ are analytic and invertible on D_ρ and $D(\lambda)$ describes the singularity of $A(\lambda)$ at 0, see Gohberg and Sigal [17]. This in turn delivers chains that allow to construct the subspaces in (1.6) as

$$\mathscr{N}_m = \bigoplus_{h=1}^{m} \bigoplus_{k=0}^{h-1} M_\beta^k \beta_h, \qquad \mathscr{R}_m = \left(A_1^* \bigoplus_{h=1}^{m} \bigoplus_{k=0}^{h-1} M_\alpha^k \alpha_h \right)^\perp,$$

$$\mathscr{N}_m' = A_1 \bigoplus_{h=1}^{m} \bigoplus_{k=0}^{h-1} M_\beta^k \beta_h, \qquad \mathscr{R}_m' = \left(\bigoplus_{h=1}^{m} \bigoplus_{k=0}^{h-1} M_\alpha^k \alpha_h \right)^\perp.$$

This allows to compute the operators $Z_{n\mathscr{N}'}$, $Z_{n\mathscr{R}'}$ and the projections $P_{\mathscr{N},\mathscr{R}}$, $P_{\mathscr{R},\mathscr{N}}$, $P_{\mathscr{N}',\mathscr{R}'}$, $P_{\mathscr{R}',\mathscr{N}'}$, i.e. it provides a construction of the resolvent and of the logarithmic residues in terms of the quantities defined by the local subspace decomposition.

Further it is shown that the same quantities characterize image and null space of each operator in the principal part of the resolvent, i.e.

$$\operatorname{Im} Z_n = \bigoplus_{h=m-n}^{m} \bigoplus_{k=0}^{h-m+n} M_\beta^k \beta_h, \qquad \operatorname{Ker} Z_n = \left(\bigoplus_{h=m-n}^{m} \bigoplus_{k=0}^{h-m+n} M_\alpha^k \alpha_h \right)^\perp,$$

where $n = 0, \ldots, m-1$.

The last part of the present paper considers the special case of A_0 being Fredholm of index 0; it characterizes the rank of the operators in the principal part of the resolvent and links the dimensions of the subspaces in (1.6) to the partial multiplicities and to the algebraic multiplicity of $A(\lambda)$ at 0.

The rest of the paper is organized as follows: the remaining of this section reports notational conventions and preliminaries, Section 2 presents definitions and main results and Section 3 illustrates some of the results for the particular cases of poles of order 1 and 2. Section 4 considers a pole of order $m \in \mathbb{N}$ and corresponding factorizations of $A(\lambda)$, Section 5 describes chains and the associated subspaces that characterize the principal part of the resolvent and the logarithmic residues and Section 6 considers the particular case of A_0 being Fredholm of index 0. Appendix A contains proofs.

1.1. Notation and preliminaries

$\mathscr{B}_{X,Y}$ denotes the space of bounded linear mappings between two complex Hilbert spaces X and Y (when $X = Y$, the shorthand notation $\mathscr{B}_X = \mathscr{B}_{X,X}$ is employed). Image and null space of $T \in \mathscr{B}_{X,Y}$ are respectively denoted by $\operatorname{Im} T = \{y \in Y : y = Tx, x \in X\}$ and $\operatorname{Ker} T = \{x \in X : Tx = 0\}$. Note that $\operatorname{Ker} T$ is closed while $\operatorname{Im} T$ may not be so. $T^* \in \mathscr{B}_{Y,X}$ denotes the adjoint of T; recall that $\operatorname{Ker} T = (\operatorname{Im} T^*)^\perp$, $\operatorname{Ker} T^* = (\operatorname{Im} T)^\perp$ and if $\operatorname{Im} T$ is closed, $\operatorname{Im} T = (\operatorname{Ker} T^*)^\perp$, $\operatorname{Im} T^* = (\operatorname{Ker} T)^\perp$. If $\operatorname{Im} T$ is closed, its generalized inverse $T^+ \in \mathscr{B}_{Y,X}$ exists and it is unique; recall that $T^+TT^+ = T^+$, $TT^+T = T$, $TT^+ = P_{\operatorname{Im} T}$, $T^+T = P_{(\operatorname{Ker} T)^\perp}$, $\operatorname{Ker} T^+ = (\operatorname{Im} T)^\perp$, and $\operatorname{Im} T^+ = (\operatorname{Ker} T)^\perp$. A Hilbert space Z is said to be the direct sum of ζ and φ, written $Z = \zeta \oplus \varphi$, if (i) $Z = \zeta + \varphi = \{u + v : u \in \zeta, v \in \varphi\}$ and (ii) $\zeta \cap \varphi = 0$. If ζ and φ are closed subspaces and $Z = \zeta \oplus \varphi$, one has the associated projection identity $I = P_{\zeta,\varphi} + P_{\varphi,\zeta}$, where $P_{\zeta,\varphi} \in \mathscr{B}_Z$ denotes the projection of Z onto ζ along φ, i.e. $P_{\zeta,\varphi} = P^2_{\zeta,\varphi}$, $\operatorname{Im} P_{\zeta,\varphi} = \zeta$ and $\operatorname{Ker} P_{\zeta,\varphi} = \varphi$; similarly for $P_{\varphi,\zeta}$. When $\varphi = \zeta^\perp$, the direct sum is said to be orthogonal and the shorthand notation $P_\zeta = P_{\zeta,\zeta^\perp}$ is employed for the orthogonal projection onto ζ; similarly for $P_{\zeta^\perp}$.

2. Definitions and main results

This section introduces the local subspace decomposition and presents the main results. The analysis is conducted under the following assumption.

Assumption 2.1 *(*$\operatorname{Im} A_0$ *is closed and* A_0 *is singular).* Assume that (i) $\operatorname{Im} A_0$ is closed and (ii) $A_0 \neq 0$ is neither one-to-one nor onto.

Note that, under Assumption 2.1, $\operatorname{Im} A_0, (\operatorname{Im} A_0)^\perp, \operatorname{Ker} A_0, (\operatorname{Ker} A_0)^\perp$ are all nonzero closed subspaces. Hence one has the orthogonal direct sum decompositions

$$\begin{aligned} Y &= \alpha_0 \oplus \alpha_0^\perp, & \alpha_0, \alpha_0^\perp &\neq 0, & \alpha_0 &= \operatorname{Im} A_0, \\ X &= \beta_0 \oplus \beta_0^\perp, & \beta_0, \beta_0^\perp &\neq 0, & \beta_0 &= (\operatorname{Ker} A_0)^\perp, \end{aligned}$$

and the associated orthogonal projections identities

$$\begin{aligned} P_{\alpha_0} + P_{\alpha_0^\perp} &= I, & P_{\alpha_0}, P_{\alpha_0^\perp} &\in \mathscr{B}_Y, & P_{\alpha_0}, P_{\alpha_0^\perp} &\neq 0, \\ P_{\beta_0} + P_{\beta_0^\perp} &= I, & P_{\beta_0}, P_{\beta_0^\perp} &\in \mathscr{B}_X, & P_{\beta_0}, P_{\beta_0^\perp} &\neq 0, \end{aligned}$$

where

$$\begin{aligned} \operatorname{Im} P_{\alpha_0} &= \operatorname{Ker} P_{\alpha_0^\perp} = \alpha_0, & \operatorname{Ker} P_{\alpha_0} &= \operatorname{Im} P_{\alpha_0^\perp} = \alpha_0^\perp, \\ \operatorname{Im} P_{\beta_0} &= \operatorname{Ker} P_{\beta_0^\perp} = \beta_0, & \operatorname{Ker} P_{\beta_0} &= \operatorname{Im} P_{\beta_0^\perp} = \beta_0^\perp. \end{aligned}$$

Moreover, the generalized inverse A_0^+ of A_0 is such that

$$A_0 A_0^+ = P_{\alpha_0}, \qquad A_0^+ A_0 = P_{\beta_0}.$$

Definition 2.2 *(Local subspace decomposition).* Let $m \in \mathbb{N}$,

$$S_0 = A_0, \qquad \alpha_0 = \operatorname{Im} S_0, \qquad \beta_0 = (\operatorname{Ker} S_0)^\perp,$$

and for $h = 1, \ldots, m$ define

$$S_h = P_{a_h^\perp} Q_h P_{b_h^\perp}, \qquad \alpha_h = \operatorname{Im} S_h, \qquad \beta_h = (\operatorname{Ker} S_h)^\perp,$$

where

$$a_h = \bigoplus_{j=0}^{h-1} \alpha_j, \quad b_h = \bigoplus_{j=0}^{h-1} \beta_j, \quad Q_h = \begin{cases} A_1, & h = 1 \\ -Q_{h-1} \sum_{j=0}^{h-2} S_j^+ Q_{j+1}, & h = 2, \ldots, m \end{cases}.$$

Further let

$$M_\beta = -\sum_{h=0}^{m-1} S_h^+ Q_{h+1}, \qquad M_\alpha = \left(-\sum_{h=0}^{m-1} Q_{h+1} S_h^+ \right)^*.$$

Note that

$$\alpha_h \subseteq a_h^\perp = (\alpha_0 \oplus \cdots \oplus \alpha_{h-1})^\perp, \quad \beta_h \subseteq b_h^\perp = (\beta_0 \oplus \cdots \oplus \beta_{h-1})^\perp, \quad h = 1, \ldots, m,$$

i.e. α_h is orthogonal to α_j and β_h is orthogonal to β_j, $h \neq j$. It turns out that for $h \neq 0, m$, it is possible that $\alpha_h = 0$ and $\beta_h = 0$. In what follows, every statement concerning α_h or β_h implicitly assumes that they are nonzero; the modifications required otherwise are straightforward.

Observe that $\beta_h = (\operatorname{Ker} S_h)^\perp$, $\beta_h^\perp = \operatorname{Ker} S_h$ and $\alpha_h^\perp = (\operatorname{Im} S_h)^\perp$ are closed subspaces and also $\alpha_h = \operatorname{Im} S_h$ need to be so. Indeed, let ξ_h be the orthogonal complement of α_h in $a_h^\perp$, i.e. $a_h^\perp = \alpha_h \oplus \xi_h$; because the closed subspace ξ_h complements the image α_h of the bounded operator S_h in the closed subspace $a_h^\perp$, it follows from Theorem IV.1.12 in [18] that $\alpha_h = \operatorname{Im} S_h$ must be closed.

Hence $a_h, a_h^\perp, b_h, b_h^\perp$, $h = 1, \dots, m$, are all nonzero closed subspaces and one has the orthogonal direct sum decompositions

$$Y = a_h \oplus a_h^\perp, \qquad a_h, a_h^\perp \neq 0, \qquad a_h = \bigoplus_{j=0}^{h-1} \alpha_j,$$

$$X = b_h \oplus b_h^\perp, \qquad b_h, b_h^\perp \neq 0, \qquad b_h = \bigoplus_{j=0}^{h-1} \beta_j,$$

and the associated orthogonal projections identities

$$P_{a_h} + P_{a_h^\perp} = I, \qquad P_{a_h}, P_{a_h^\perp} \in \mathscr{B}_Y, \qquad P_{a_h}, P_{a_h^\perp} \neq 0,$$
$$P_{b_h} + P_{b_h^\perp} = I, \qquad P_{b_h}, P_{b_h^\perp} \in \mathscr{B}_X, \qquad P_{b_h}, P_{b_h^\perp} \neq 0.$$

Moreover,

$$\begin{aligned} S_h S_h^+ &= P_{\alpha_h}, & S_h^+ S_h &= P_{\beta_h}, \\ \operatorname{Ker} S_h^+ &= \alpha_h^\perp, & \operatorname{Im} S_h^+ &= \beta_h, \\ S_h^+ P_{a_h^\perp} &= S_h^+, & P_{b_h^\perp} S_h^+ &= S_h^+. \end{aligned} \tag{2.1}$$

The first result links the order of the pole of the resolvent to the α_h and β_h subspaces; further it shows that Z_0 is the generalized inverse of S_m, i.e. $Z_0 = S_m^+$.

Theorem 2.3 *(Pole of order m). The following statements are equivalent:*
(i) $A(\lambda)^{-1}$ has a pole of order $m \geq 1$ at 0,
(ii) $X = \bigoplus_{h=0}^m \beta_h$, where β_h is closed and $\beta_0, \beta_m \neq 0$,
(iii) $Y = \bigoplus_{h=0}^m \alpha_h$, where α_h is closed and $\alpha_0, \alpha_m \neq 0$,
(iv) $Z_0 = S_m^+$.

The second result provides two alternative constructions of the factorization $A(\lambda) = E(\lambda)D(\lambda)F(\lambda)$, where $E(\lambda), F(\lambda)$ are analytic and invertible on D_ρ and $D(\lambda)$ describes the singularity of $A(\lambda)$ at 0.

Theorem 2.4 *(Factorizations). Consider Definition 2.2 and let*

$$E_\beta(\lambda) = \sum_{n=0}^{\infty} E_{\beta,n} \lambda^h, \qquad D_\beta(\lambda) = \sum_{h=0}^{m} P_{\beta_h} \lambda^h, \qquad F_\beta(\lambda) = I - M_\beta \lambda,$$

$$E_\alpha(\lambda) = I - M_\alpha^* \lambda, \qquad D_\alpha(\lambda) = \sum_{h=0}^{m} P_{\alpha_h} \lambda^h, \qquad F_\alpha(\lambda) = \sum_{n=0}^{\infty} F_{\alpha,n} \lambda^h,$$

where

$$E_{\beta,0} = F_{\alpha,0} = \sum_{h=0}^{m} S_h, \qquad \begin{array}{l} E_{\beta,n} = P_{\alpha_0^\perp} A_1 M_\beta^{n-1} B_{\beta,m}, \\ F_{\alpha,n} = C_{\alpha,m} (M_\alpha^*)^{n-1} A_1 P_{\beta_0^\perp} \end{array}, \qquad n = 1, 2, \ldots ,$$

and

$$B_{\beta,n} = \sum_{k=0}^{n} M_\beta^k P_{\beta_{m-n+k}}, \qquad C_{\alpha,n} = \sum_{k=0}^{n} P_{\alpha_{m-n+k}} (M_\alpha^*)^k, \qquad n = 0, \ldots, m.$$

Then

$$A(\lambda) = E_\zeta(\lambda) D_\zeta(\lambda) F_\zeta(\lambda), \qquad \zeta = \alpha, \beta, \tag{2.2}$$

where $E_\zeta(\lambda), F_\zeta(\lambda)$ are analytic and invertible on D_ρ.

This in turn allows to compute the resolvent and the logarithmic residues, see Corollary 4.3 below.

The third result links these results with the classical ones based on ascent and descent in Bart and Lay [10], showing how the subspaces in (1.6) can be constructed using the quantities defined by the local subspace decomposition.

Theorem 2.5 *($\mathscr{N}_m, \mathscr{R}_m, \mathscr{N}'_m, \mathscr{R}'_m$). The subspaces in (1.6) can be constructed as*

$$\mathscr{N}_m = \bigoplus_{h=1}^{m} \bigoplus_{k=0}^{h-1} M_\beta^k \beta_h, \qquad \mathscr{R}_m = \left(A_1^* \bigoplus_{h=1}^{m} \bigoplus_{k=0}^{h-1} M_\alpha^k \alpha_h \right)^\perp,$$

$$\mathscr{N}'_m = A_1 \bigoplus_{h=1}^{m} \bigoplus_{k=0}^{h-1} M_\beta^k \beta_h, \qquad \mathscr{R}'_m = \left(\bigoplus_{h=1}^{m} \bigoplus_{k=0}^{h-1} M_\alpha^k \alpha_h \right)^\perp.$$

One can thus use these formulas to compute the restricted operators $Z_{n\mathscr{N}'}$, $Z_{n\mathscr{R}'}$ and the projections which deliver the resolvent and the logarithmic residues.

The next result characterizes image and null space of each operator in the principal part of the resolvent.

Theorem 2.6 *(Image and null space of Z_n, $n = 0, \ldots, m-1$). For $n = 0, \ldots, m-1$, image and null space of Z_n are given by*

$$\operatorname{Im} Z_n = \bigoplus_{h=m-n}^{m} \bigoplus_{k=0}^{h-m+n} M_\beta^k \beta_h, \qquad \operatorname{Ker} Z_n = \left(\bigoplus_{h=m-n}^{m} \bigoplus_{k=0}^{h-m+n} M_\alpha^k \alpha_h \right)^\perp.$$

The last part of the present paper considers the special case of A_0 being Fredholm of index 0; it characterizes the rank of the operators in the principal part of the resolvent and links the dimensions of the subspaces in (1.6) to the partial multiplicities and to the algebraic multiplicity of $A(\lambda)$ at 0.

Theorem 2.7 *(Special case: A_0 is Fredholm of index 0). If A_0 is Fredholm of index 0, let $r_h = \dim \alpha_h = \dim \beta_h$, $h = 1, \dots, m$. Then*

(i) $0 \le r_h < \infty$, $h = 1, \dots, m-1$, *and* $0 < r_m < \infty$,

(ii) $\operatorname{rank} Z_n = \sum_{h=0}^{n} (h+1) r_{m-n+h} < \infty$, $n = 0, \dots, m-1$,

(iii) $\dim \mathscr{N}_m = \dim \mathscr{N}'_m = \sum_{h=1}^{m} h r_h < \infty$,

(iv) the algebraic multiplicity of $A(\lambda)$ at 0 is equal to $\sum_{h=1}^{m} h r_h < \infty$,

(v) the number of distinct partial multiplicities of $A(\lambda)$ at 0 is equal to $\sum_{h=1}^{m} 1_{r_h > 0}$,

(vi) there are exactly r_h partial multiplicities that are equal to $h = 1, \dots, m$.

3. Poles of order 1 and 2

This section illustrates the main ideas behind the local subspace decomposition considering poles of order 1 and 2. The results in the present section are thus particular cases of those in Section 2 but direct proofs are given here in order to illustrate the working of the general case described in Section 4.

In the following, equations in system (1.8) are labelled according to the value of n; for instance $A_0 Z_0 = 0$ and $Z_0 A_0 = 0$ are equations 0 and the identities appear in equations m, which is the order of the pole. The equations derived from $A(\lambda)A(\lambda)^{-1} = I \in \mathscr{B}_Y$ are called Y versions (and correspond to the first equation in (1.8)) and those that derive from $A(\lambda)^{-1}A(\lambda) = I \in \mathscr{B}_X$ are called X versions (and correspond to the second equation in (1.8)); for instance $A_0 Z_m + A_1 Z_{m-1} = I$ is the Y version of equation m.

Some implications of equations 0 and 1 are next derived; remark that these properties hold for any value of m. From equations 0, $A_0 Z_0 = 0$ and $Z_0 A_0 = 0$, one respectively has

$$\operatorname{Im} Z_0 \subseteq \operatorname{Ker} A_0, \qquad \operatorname{Ker} Z_0 \supseteq \operatorname{Im} A_0,$$

i.e., letting $\beta_0 = (\operatorname{Ker} A_0)^\perp$ and $\alpha_0 = \operatorname{Im} A_0$, one has

$$\operatorname{Im} Z_0 \subseteq \beta_0^\perp, \qquad \operatorname{Ker} Z_0 \supseteq \alpha_0.$$

The l.h.s. of the Y version of equation 1 reads $A_0 Z_1 + A_1 Z_0$; use $P_{\beta_0^\perp} + P_{\beta_0} = I$ to write $A_1 Z_0 = A_1(P_{\beta_0^\perp} + P_{\beta_0}) Z_0 = A_1 P_{\beta_0^\perp} Z_0 + A_1 P_{\beta_0} Z_0 = A_1 P_{\beta_0^\perp} Z_0$, because $P_{\beta_0} Z_0 = 0$ follows from $\operatorname{Im} Z_0 \subseteq \beta_0^\perp$. Applying $P_{\alpha_0^\perp}$ on both sides of $A_1 Z_0 = A_1 P_{\beta_0^\perp} Z_0$ one has

$$P_{\alpha_0^\perp} A_1 Z_0 = S_1 Z_0,$$

having defined $S_1 = P_{\alpha_0^\perp} A_1 P_{\beta_0^\perp}$; note that, by definition,

$$\operatorname{Im} S_1 \subseteq \operatorname{Im} P_{\alpha_0^\perp} = \alpha_0^\perp, \qquad \operatorname{Ker} S_1 \supseteq \operatorname{Ker} P_{\beta_0^\perp} = \beta_0,$$

i.e., letting $\alpha_1 = \operatorname{Im} S_1$ and $\beta_1 = (\operatorname{Ker} S_1)^\perp$, one has

$$\alpha_1 \subseteq \alpha_0^\perp, \qquad \beta_1 \subseteq \beta_0^\perp.$$

This shows that from the l.h.s. of the Y version of equation 1, $A_0 Z_1 + A_1 Z_0$, one has

$$P_{\alpha_0^\perp}(A_0 Z_1 + A_1 Z_0) = S_1 Z_0, \tag{3.1}$$

because $P_{\alpha_0^\perp} A_0 = 0$ follows from $\operatorname{Im} A_0 = \alpha_0$.

Similarly, starting from the l.h.s. of the X version of equation 1, $Z_1 A_0 + Z_0 A_1$, and using $P_{\alpha_0} + P_{\alpha_0^\perp} = I$ one has $Z_0 A_1 = Z_0(P_{\alpha_0^\perp} + P_{\alpha_0})A_1 = Z_0 P_{\alpha_0^\perp} A_1 + Z_0 P_{\alpha_0} A_1 = Z_0 P_{\alpha_0^\perp} A_1$, because $Z_0 P_{\alpha_0}$ follows from $\operatorname{Ker} Z_0 \supseteq \alpha_0$. Hence $Z_0 A_1 P_{\beta_0^\perp} = Z_0 P_{\alpha_0^\perp} A_1 P_{\beta_0^\perp}$, i.e. $Z_0 A_1 P_{\beta_0^\perp} = Z_0 S_1$. This shows that from the l.h.s. of the X version of equation 1 one has

$$(Z_1 A_0 + Z_0 A_1) P_{\beta_0^\perp} = Z_0 S_1, \tag{3.2}$$

because $A_0 P_{\beta_0^\perp} = 0$ follows from $\operatorname{Ker} A_0 = \beta_0^\perp$.

3.1. Poles of order 1

Consider the case in which $A(\lambda)^{-1}$ has a pole of order $m = 1$ at 0. In this case, (1.5) reads

$$A(\lambda)^{-1} = \frac{Z_0}{\lambda} + \sum_{n=1}^{\infty} Z_n \lambda^{n-1}, \qquad \lambda \in D_\rho \setminus \{0\}, \qquad Z_0 \neq 0,$$

and (1.8) reads

$$\begin{aligned} A_0 Z_0 &= 0, & Z_0 A_0 &= 0, \\ A_0 Z_1 + A_1 Z_0 &= I, & Z_1 A_0 + Z_0 A_1 &= I. \end{aligned}$$

Because the identity is in equations 1, (3.1) and (3.2) respectively imply

$$S_1 Z_0 = P_{\alpha_0^\perp}, \qquad Z_0 S_1 = P_{\beta_0^\perp}. \tag{3.3}$$

Recall that $\alpha_0 \subseteq \operatorname{Ker} Z_0$ and $\operatorname{Im} S_1 = \alpha_1 \subseteq \alpha_0^\perp$; from $S_1 Z_0 = P_{\alpha_0^\perp}$ one has

$$\operatorname{Ker} Z_0 = \operatorname{Ker} P_{\alpha_0^\perp} = \alpha_0, \qquad \operatorname{Im} S_1 = \operatorname{Im} P_{\alpha_0^\perp} = \alpha_0^\perp,$$

and recalling that $\operatorname{Im} Z_0 \subseteq \beta_0^\perp$ and $\operatorname{Ker} S_1 = \beta_1^\perp \supseteq \beta_0$, from $Z_0 S_1 = P_{\beta_0^\perp}$ one finds

$$\operatorname{Im} Z_0 = \operatorname{Im} P_{\beta_0^\perp} = \beta_0^\perp, \qquad \operatorname{Ker} S_1 = \operatorname{Ker} P_{\beta_0^\perp} = \beta_0.$$

This shows that if $m = 1$ then

$$\alpha_1 = \alpha_0^\perp, \qquad \beta_1 = \beta_0^\perp,$$

i.e.

$$X = \beta_0 \oplus \beta_1, \qquad Y = \alpha_0 \oplus \alpha_1.$$

Conversely, when $m > 1$ the identity is not in equations 1 and (3.1) and (3.2) respectively imply

$$S_1 Z_0 = 0, \qquad Z_0 S_1 = 0.$$

From $A_0 Z_0 = 0$ and $S_1 Z_0 = 0$ one has

$$\operatorname{Im} Z_0 \subseteq (\operatorname{Ker} A_0 \cap \operatorname{Ker} S_1) = \beta_0^\perp \cap \beta_1^\perp = (\beta_0 \oplus \beta_1)^\perp. \tag{3.4}$$

Recall that $\beta_1 \subseteq \beta_0^\perp$; because $\beta_1 = \beta_0^\perp$ implies $\operatorname{Im} Z_0 = 0$, if $m > 1$ then $\beta_1 \subset \beta_0^\perp$. Thus $\beta_1 = \beta_0^\perp$ holds if and only if $m = 1$.

Similarly, from $Z_0 A_0 = 0$ and $Z_0 S_1 = 0$ one has

$$\operatorname{Ker} Z_0 \supseteq (\operatorname{Im} A_0 \oplus \operatorname{Im} S_1) = \alpha_0 \oplus \alpha_1. \tag{3.5}$$

Recall that $\alpha_1 \subseteq \alpha_0^\perp$; because $\alpha_1 = \alpha_0^\perp$ implies $\operatorname{Ker} Z_0 = Y$, if $m > 1$ then $\alpha_1 \subset \alpha_0^\perp$. Thus $\alpha_1 = \alpha_0^\perp$ holds if and only if $m = 1$.

When $m = 1$, (3.3) thus reads $S_1 Z_0 = P_{\alpha_1}$ and $Z_0 S_1 = P_{\beta_1}$. The former implies $Z_0 S_1 Z_0 = Z_0 P_{\alpha_1}$ and because $Z_0 = Z_0 P_{\alpha_1}$ one finds $Z_0 S_1 Z_0 = Z_0$; similarly, from the latter one has $S_1 Z_0 S_1 = S_1 P_{\beta_1}$ and because $S_1 = S_1 P_{\beta_1}$ one finds $S_1 Z_0 S_1 = S_1$. This shows that, when the pole has order one, Z_0 is the generalized inverse of S_1.

This proves the following statement.

Proposition 3.1 *(Pole of order one). Let*

$$S_0 = A_0, \qquad \alpha_0 = \operatorname{Im} S_0, \qquad \beta_0 = (\operatorname{Ker} S_0)^\perp, \tag{3.6}$$

$$S_1 = P_{\alpha_0^\perp} Q_1 P_{\beta_0^\perp}, \qquad \alpha_1 = \operatorname{Im} S_1, \qquad \beta_1 = (\operatorname{Ker} S_1)^\perp, \tag{3.7}$$

where $Q_1 = A_1$. The following statements are equivalent:

(i) $A(\lambda)^{-1}$ has a pole of order $m = 1$ at 0,

(ii) $X = \beta_0 \oplus \beta_1$, where β_h is closed and $\beta_0, \beta_1 \neq 0$,

(iii) $Y = \alpha_0 \oplus \alpha_1$, where α_h is closed and $\alpha_0, \alpha_1 \neq 0$,

(iv) $Z_0 = S_1^+$.

Thus, when the pole has order one, Z_0 is such that

$$\operatorname{Im} Z_0 = \beta_1, \qquad \operatorname{Ker} Z_0 = \alpha_0.$$

Observe that these quantities are all expressed in terms of A_0 and A_1 via the definitions in (3.6) and (3.7); further note that Proposition 3.1 is found by setting $m = 1$ in Theorem 2.3.

Next we illustrate the derivation of the β factorization $A(\lambda) = E_\beta(\lambda)D_\beta(\lambda)F_\beta(\lambda)$ in Theorem 2.4 for $m = 1$ (the α factorization is obtained similarly from the X versions of the fundamental equations). From the above one has

$$P_{\beta_0} Z_0 = 0, \qquad P_{\beta_1} Z_0 = S_1^+ \tag{3.8}$$

and from $A_0Z_1 + A_1Z_0 = I$ it follows that $A_0^+A_0Z_1 + A_0^+A_1Z_0 = A_0^+$; substituting $A_0^+A_0 = P_{\beta_0}$, see (2.1), and rearranging one finds

$$P_{\beta_0} Z_1 = S_0^+ - S_0^+Q_1Z_0, \qquad S_0 = A_0, \qquad Q_1 = A_1. \tag{3.9}$$

Hence

$$\begin{aligned} P_{\beta_0} A(\lambda)^{-1} &= \frac{P_{\beta_0} Z_0}{\lambda} + \sum_{n=1}^{\infty} P_{\beta_0} Z_n \lambda^{n-1} = P_{\beta_0} Z_1 + \sum_{n=2}^{\infty} P_{\beta_0} Z_n \lambda^{n-1} \\ &= S_0^+ - S_0^+ Q_1 Z_0 + \sum_{n=1}^{\infty} P_{\beta_0} Z_{n+1} \lambda^n. \end{aligned}$$

Applying $\lambda S_0^+ Q_1$ to $A(\lambda)^{-1}$ one has

$$(\lambda S_0^+ Q_1)A(\lambda)^{-1} = S_0^+ Q_1 Z_0 + \sum_{n=1}^{\infty} S_0^+ Q_1 Z_n \lambda^n,$$

and summing up the two expressions one finds

$$(P_{\beta_0} + \lambda S_0^+ Q_1)A(\lambda)^{-1} = S_0^+ + \sum_{n=1}^{\infty} (P_{\beta_0} Z_{n+1} + S_0^+ Q_1 Z_n)\lambda^n,$$

i.e., see Proposition 4.2 below,

$$F_0(\lambda)A(\lambda)^{-1} = \widetilde{F}_0(\lambda),$$

where $F_0(\lambda) = P_{\beta_0} + \lambda S_0^+ Q_1$, $\widetilde{F}_0(\lambda) = S_0^+ + \sum_{n=1}^{\infty} (P_{\beta_0} Z_{n+1} + S_0^+ Q_1 Z_n)\lambda^n$ is analytic on D_ρ and $\widetilde{F}_0(0) = S_0^+ \neq 0$. Moreover,

$$P_{\beta_1}A(\lambda)^{-1} = \frac{P_{\beta_1}Z_0}{\lambda} + \sum_{n=1}^{\infty} P_{\beta_1}Z_n\lambda^{n-1} = \frac{S_1^+}{\lambda} + \sum_{n=1}^{\infty} P_{\beta_1}Z_n\lambda^{n-1},$$

i.e., see Proposition 4.2 below,

$$F_1(\lambda)A(\lambda)^{-1} = \lambda^{-1}\widetilde{F}_1(\lambda),$$

where $F_1(\lambda) = P_{\beta_1}$, $\widetilde{F}_1(\lambda) = S_1^+ + \sum_{n=1}^{\infty} P_{\beta_1}Z_n\lambda^n$ is analytic on D_ρ and $\widetilde{F}_1(0) = S_1^+ \neq 0$. Hence one has

$$\widetilde{F}_h(\lambda)A(\lambda) = \lambda^h F_h(\lambda), \qquad \widetilde{F}_0(\lambda) = S_h^+, \qquad h = 0, 1,$$

where $\operatorname{Im} S_h^+ = \beta_h$ and $\operatorname{Ker} S_h^+ = \alpha_h^\perp$, see (2.1). Summing over h one finds

$$\widetilde{F}(\lambda)A(\lambda) = \sum_{h=0}^{1} \lambda^h F_h(\lambda), \qquad \widetilde{F}(\lambda) = \sum_{h=0}^{1} \widetilde{F}_h(\lambda),$$

where $\widetilde{F}(\lambda)$ is analytic on D_ρ, $\widetilde{F}(0) = \sum_{h=0}^{1} S_h^+$ is invertible and

$$\sum_{h=0}^{1} \lambda^h F_h(\lambda) = P_{\beta_0} + \lambda(P_{\beta_1} + S_0^+Q_1).$$

On the other hand, letting

$$D_\beta(\lambda) = P_{\beta_0} + P_{\beta_1}\lambda, \qquad F_\beta(\lambda) = I - M_\beta\lambda, \qquad M_\beta = -S_0^+Q_1,$$

one has $P_{\beta_0}M_\beta = -P_{\beta_0}S_0^+Q_1 = -S_0^+Q_1$ and $P_{\beta_1}M_\beta = -P_{\beta_1}S_0^+Q_1 = 0$ and hence

$$\begin{aligned} D_\beta(\lambda)F_\beta(\lambda) &= P_{\beta_0} + (P_{\beta_1} - P_{\beta_0}M_\beta)\,\lambda - P_{\beta_1}M_\beta\lambda^2 \\ &= P_{\beta_0} + \left(P_{\beta_1} + S_0^+Q_1\right)\lambda = \sum_{h=0}^{1} \lambda^h F_h(\lambda). \end{aligned}$$

This shows that $\widetilde{F}(\lambda)A(\lambda) = D_\beta(\lambda)F_\beta(\lambda)$, i.e., see Theorem 2.4, one has the β factorization

$$A(\lambda) = E_\beta(\lambda)D_\beta(\lambda)F_\beta(\lambda), \qquad E_\beta(\lambda) = \widetilde{F}(\lambda)^{-1} = \sum_{n=0}^{\infty} E_{\beta,n}\lambda^n,$$

where $E_{\beta,0} = \widetilde{F}(0)^{-1} = \sum_{h=0}^{1} S_h$ and the formula for $E_{\beta,n}$, $n = 1, 2, \dots$, is found from $E_\beta(\lambda) = A(\lambda)F_\beta(\lambda)^{-1}D_\beta(\lambda)^{-1}$, see the proof of Theorem 2.4.

3.2. Poles of order 2

Consider the case in which $A(\lambda)^{-1}$ has a pole of order $m = 2$ at 0. In this case, (1.5) reads

$$A(\lambda)^{-1} = \frac{Z_0}{\lambda^2} + \frac{Z_1}{\lambda} + \sum_{n=2}^{\infty} Z_n \lambda^{n-2}, \qquad \lambda \in D_\rho \setminus \{0\}, \qquad Z_0 \neq 0,$$

and (1.8) reads

$$\begin{aligned} A_0 Z_0 &= 0, & Z_0 A_0 &= 0, \\ A_0 Z_1 + A_1 Z_0 &= 0, & Z_1 A_0 + Z_0 A_1 &= 0, \\ A_0 Z_2 + A_1 Z_1 &= I, & Z_2 A_0 + Z_1 A_1 &= I. \end{aligned}$$

Recall that $m > 1$ implies $\alpha_1 \subset \alpha_0^\perp$ and $\beta_1 \subset \beta_0^\perp$. Let $a_2 = \alpha_0 \oplus \alpha_1$ and $b_2 = \beta_0 \oplus \beta_1$; then one has the orthogonal direct sum decompositions

$$Y = a_2 \oplus a_2^\perp, \qquad a_2, a_2^\perp \neq 0, \qquad X = b_2 \oplus b_2^\perp, \qquad b_2, b_2^\perp \neq 0,$$

and the associated orthogonal projection identities

$$I = P_{a_2} + P_{a_2^\perp}, \qquad I = P_{b_2} + P_{b_2^\perp}, \qquad P_\zeta, P_{\zeta^\perp} \neq 0, \qquad \zeta = a_2, b_2.$$

Recall that if $m > 1$ one has $\operatorname{Im} Z_0 \subseteq b_2^\perp$ and $\operatorname{Ker} Z_0 \supseteq a_2$, see (3.4) and (3.5), so that

$$Z_0 = P_{b_2^\perp} Z_0, \qquad Z_0 = Z_0 P_{a_2^\perp}. \tag{3.10}$$

Next observe that from the Y and X versions of equation 1, $A_0 Z_1 + A_1 Z_0 = 0$ and $Z_1 A_0 + Z_0 A_1 = 0$, one respectively has $A_0^+ A_0 Z_1 = -A_0^+ A_1 Z_0$ and $Z_1 A_0 A_0^+ = -Z_0 A_1 A_0^+$; hence, see (2.1),

$$P_{\beta_0} Z_1 = -A_0^+ A_1 Z_0, \qquad Z_1 P_{\alpha_0} = -Z_0 A_1 A_0^+.$$

The l.h.s. of the Y version of equation 2 reads $A_0 Z_2 + A_1 Z_1 = I$; using $P_{\beta_0^\perp} + P_{\beta_0} = I$ one has $A_1 Z_1 = A_1 (P_{\beta_0^\perp} + P_{\beta_0}) Z_1 = A_1 P_{\beta_0^\perp} Z_1 + A_1 P_{\beta_0} Z_1 = A_1 P_{\beta_0^\perp} Z_1 + Q_2 Z_0$, having defined $Q_2 = -A_1 A_0^+ A_1$. From $P_{\alpha_0^\perp} + P_{\alpha_0} = I$ one finds $A_1 Z_1 = (P_{\alpha_0^\perp} + P_{\alpha_0}) A_1 P_{\beta_0^\perp} Z_1 + Q_2 Z_0 = S_1 Z_1 + P_{\alpha_0} A_1 P_{\beta_0^\perp} Z_1 + Q_2 Z_0$, where $S_1 = P_{\alpha_0^\perp} A_1 P_{\beta_0^\perp}$ and $\alpha_1 = \operatorname{Im} S_1$. Applying $P_{a_2^\perp}$ on both sides of $A_1 Z_1 = S_1 Z_1 + P_{\alpha_0} A_1 P_{\beta_0^\perp} Z_1 + Q_2 Z_0$ one has $P_{a_2^\perp} A_1 Z_1 = P_{a_2^\perp} Q_2 Z_0 = S_2 Z_0$, having used $Z_0 = P_{b_2^\perp} Z_0$ and defined $S_2 = P_{a_2^\perp} Q_2 P_{b_2^\perp}$, $\alpha_2 = \operatorname{Im} S_2$ and $\beta_2 = (\operatorname{Ker} S_2)^\perp$; note that $\operatorname{Im} S_2 \subseteq \operatorname{Im} P_{a_2^\perp} = a_2^\perp = (\alpha_0 \oplus \alpha_1)^\perp$ and $\operatorname{Ker} S_2 \supseteq \operatorname{Ker} P_{b_2^\perp} = \operatorname{Im} P_{b_2} = b_2 = \beta_0 \oplus \beta_1$. This shows that from the l.h.s. of the Y version of equation 2, $A_0 Z_2 + A_1 Z_1$, one has

$$P_{a_2^\perp}(A_0 Z_2 + A_1 Z_1) = S_2 Z_0, \tag{3.11}$$

because $P_{a_2^\perp} A_0 = 0$ follows from $\operatorname{Im} A_0 = \alpha_0$.

Similarly, starting from the l.h.s. of the X version of equation 2, $Z_2 A_0 + Z_1 A_1$, and using projections (the details are omitted) one finds

$$(Z_1 A_0 + Z_0 A_1) P_{b_2^\perp} = Z_0 S_2. \tag{3.12}$$

Because the identity is in equations 2, (3.11) and (3.12) respectively imply

$$S_2 Z_0 = P_{a_2^\perp}, \qquad Z_0 S_2 = P_{b_2^\perp}. \tag{3.13}$$

From $S_2 Z_0 = P_{a_2^\perp}$ one has

$$\operatorname{Ker} Z_0 = \operatorname{Ker} P_{a_2^\perp} = a_2, \qquad \operatorname{Im} S_2 = \operatorname{Im} P_{a_2^\perp} = a_2^\perp,$$

and from $Z_0 S_2 = P_{b_2^\perp}$ one finds

$$\operatorname{Im} Z_0 = \operatorname{Im} P_{b_2^\perp} = b_2^\perp, \qquad \operatorname{Ker} S_2 = \operatorname{Ker} P_{b_2^\perp} = b_2.$$

This shows that if $m = 2$ then

$$\alpha_2 = (\alpha_0 \oplus \alpha_1)^\perp, \qquad \beta_2 = (\beta_0 \oplus \beta_1)^\perp,$$

i.e.

$$X = \beta_0 \oplus \beta_1 \oplus \beta_2, \qquad Y = \alpha_0 \oplus \alpha_1 \oplus \alpha_2.$$

Conversely, when $m > 2$ the identity is not in equations 2 and (3.11) and (3.12) respectively imply

$$S_2 Z_0 = 0, \qquad Z_0 S_2 = 0.$$

From $A_0 Z_0 = 0$, $S_1 Z_0 = 0$ and $S_2 Z_0 = 0$ one has

$$\operatorname{Im} Z_0 \subseteq (\operatorname{Ker} A_0 \cap \operatorname{Ker} S_1 \cap \operatorname{Ker} S_2) = \beta_0^\perp \cap \beta_1^\perp \cap \beta_2^\perp = (\beta_0 \oplus \beta_1 \oplus \beta_2)^\perp$$

and because $\beta_2 = (\beta_0 \oplus \beta_1)^\perp$ implies $\operatorname{Im} Z_0 = 0$, if $m > 2$ then $\beta_2 \subset (\beta_0 \oplus \beta_1)^\perp$. Thus $\beta_2 = (\beta_0 \oplus \beta_1)^\perp$ holds if and only if $m = 2$.

Similarly, from $Z_0 A_0 = 0$, $Z_0 S_1 = 0$ and $Z_0 S_2 = 0$ one has

$$\operatorname{Ker} Z_0 \supseteq (\operatorname{Im} A_0 \oplus \operatorname{Im} S_1 \oplus \operatorname{Im} S_2) = \alpha_0 \oplus \alpha_1 \oplus \alpha_2$$

and because $\alpha_2 = (\alpha_0 \oplus \alpha_1)^\perp$ implies $\operatorname{Ker} Z_0 = Y$ and hence $Z_0 = 0$, if $m > 2$ then $\alpha_2 \subset (\alpha_0 \oplus \alpha_1)^\perp$. Thus $\alpha_2 = (\alpha_0 \oplus \alpha_1)^\perp$ holds if and only if $m = 2$.

When $m = 2$, (3.13) thus reads $S_2Z_0 = P_{\alpha_2}$ and $Z_0S_2 = P_{\beta_2}$. The former implies $Z_0S_2Z_0 = Z_0P_{\alpha_2}$ and because $Z_0 = Z_0P_{\alpha_2}$ one finds $Z_0S_2Z_0 = Z_0$; similarly, from the latter one has $S_2Z_0S_2 = S_2P_{\beta_2}$ and because $S_2 = S_2P_{\beta_2}$ one finds $S_2Z_0S_2 = S_2$. This shows that, when the pole has order two, Z_0 is the generalized inverse of S_2.

This proves the following statement.

Proposition 3.2 *(Pole of order two). Let S_h, α_h, β_h, $h = 0, 1$, be as in* (3.6) *and* (3.7) *and define*

$$S_2 = P_{a_2^\perp}Q_2P_{b_2^\perp}, \qquad \alpha_2 = \operatorname{Im} S_2, \qquad \beta_2 = (\operatorname{Ker} S_2)^\perp, \tag{3.14}$$

where

$$a_2 = \alpha_0 \oplus \alpha_1, \qquad b_2 = \beta_0 \oplus \beta_1, \qquad Q_2 = -Q_1S_0^+Q_1, \qquad Q_1 = A_1.$$

The following statements are equivalent:

(i) $A(\lambda)^{-1}$ has a pole of order $m = 2$ at 0,

(ii) $X = \beta_0 \oplus \beta_1 \oplus \beta_2$, where β_h is closed and $\beta_0, \beta_2 \neq 0$,

(iii) $Y = \alpha_0 \oplus \alpha_1 \oplus \alpha_2$, where α_h is closed and $\alpha_0, \alpha_2 \neq 0$,

(iv) $Z_0 = S_2^+$.

Thus, when the pole has order two, Z_0 is such that

$$\operatorname{Im} Z_0 = \beta_2, \qquad \operatorname{Ker} Z_0 = \alpha_0 \oplus \alpha_1.$$

Observe that these quantities are all expressed in terms of A_0 and A_1 via the definitions in (3.6), (3.7) and (3.14); further note that Proposition 3.2 is found by setting $m = 2$ in Theorem 2.3.

Next we illustrate the derivation of the β factorization $A(\lambda) = E_\beta(\lambda)D_\beta(\lambda)F_\beta(\lambda)$ in Theorem 2.4 for $m = 2$ (the α factorization is obtained similarly from the X versions of the fundamental equations). From the above one has

$$P_{\beta_0}Z_0 = 0, \qquad P_{\beta_1}Z_0 = 0, \qquad P_{\beta_2}Z_0 = S_2^+, \tag{3.15}$$

from $A_0Z_1 + A_1Z_0 = 0$ it follows that $A_0^+A_0Z_1 + A_0^+A_1Z_0 = 0$ and hence, see (2.1),

$$P_{\beta_0}Z_1 = -S_0^+Q_1Z_0, \qquad S_0 = A_0, \qquad Q_1 = A_1, \tag{3.16}$$

and from $A_0Z_2 + A_1Z_1 = I$ one has $A_0^+A_0Z_2 + A_0^+A_1Z_1 = A_0^+$, i.e.

$$P_{\beta_0}Z_2 = S_0^+ - S_0^+Q_1Z_1. \tag{3.17}$$

Moreover, because $S_1^+\alpha_0 = 0$, $S_1^+(A_0Z_2 + A_1Z_1) = S_1^+A_1Z_1$ and from $A_0Z_2 + A_1Z_1 = I$ one has $S_1^+A_1Z_1 = S_1^+$; because, see above, $A_1Z_1 = S_1Z_1 + P_{\alpha_0}A_1P_{\beta_0^\perp}Z_1 + Q_2Z_0$, one has $S_1^+A_1Z_1 = S_1^+S_1Z_1 + S_1^+P_{\alpha_0}A_1P_{\beta_0^\perp}Z_1 + S_1^+Q_2Z_0 = P_{\beta_1}Z_1 + S_1^+Q_2Z_0$, i.e.

$$P_{\beta_1} Z_1 = S_1^+ - S_1^+ Q_2 Z_0. \tag{3.18}$$

Thus

$$\begin{aligned} P_{\beta_0} A(\lambda)^{-1} &= \frac{P_{\beta_0} Z_0}{\lambda^2} + \frac{P_{\beta_0} Z_1}{\lambda} + P_{\beta_0} Z_2 + \sum_{n=3}^{\infty} P_{\beta_0} Z_n \lambda^{n-2} \\ &= -\frac{S_0^+ Q_1 Z_0}{\lambda} + (S_0^+ - S_0^+ Q_1 Z_1) + \sum_{n=1}^{\infty} P_{\beta_0} Z_{n+2} \lambda^n. \end{aligned}$$

Applying $\lambda S_0^+ Q_1$ to $A(\lambda)^{-1}$ one has

$$(\lambda S_0^+ Q_1) A(\lambda)^{-1} = \frac{S_0^+ Q_1 Z_0}{\lambda} + S_0^+ Q_1 Z_1 + \sum_{n=1}^{\infty} S_0^+ Q_1 Z_{n+1} \lambda^n$$

and summing up the two expressions one has

$$(P_{\beta_0} + \lambda S_0^+ A_1) A(\lambda)^{-1} = S_0^+ + \sum_{n=1}^{\infty} (P_{\beta_0} Z_{n+2} + S_0^+ Q_1 Z_{n+1}) \lambda^n,$$

i.e., see Proposition 4.2 below,

$$F_0(\lambda) A(\lambda)^{-1} = \widetilde{F}_0(\lambda),$$

where $F_0(\lambda) = P_{\beta_0} + \lambda S_0^+ A_1$, $\widetilde{F}_0(\lambda) = S_0^+ + \sum_{n=1}^{\infty} (P_{\beta_0} Z_{n+2} + S_0^+ Q_1 Z_{n+1}) \lambda^n$ is analytic on D_ρ and $\widetilde{F}_0(0) = S_0^+ \neq 0$. Moreover,

$$\begin{aligned} P_{\beta_1} A(\lambda)^{-1} &= \frac{P_{\beta_1} Z_0}{\lambda^2} + \frac{P_{\beta_1} Z_1}{\lambda} + \sum_{n=2}^{\infty} P_{\beta_1} Z_n \lambda^{n-2} \\ &= \frac{S_1^+ - S_1^+ Q_2 Z_0}{\lambda} + \sum_{n=0}^{\infty} P_{\beta_1} Z_{n+2} \lambda^n. \end{aligned}$$

Applying $\lambda S_1^+ Q_2$ to $A(\lambda)^{-1}$ one has

$$(\lambda S_1^+ Q_2) A(\lambda)^{-1} = \frac{S_1^+ Q_2 Z_0}{\lambda} + \sum_{n=0}^{\infty} S_1^+ Q_2 Z_{n+1} \lambda^n$$

and summing up the two expressions one has

$$(P_{\beta_1} + \lambda S_1^+ Q_2) A(\lambda)^{-1} = \frac{S_1^+}{\lambda} + \sum_{n=0}^{\infty} (P_{\beta_1} Z_{n+2} + S_1^+ Q_2 Z_{n+1}) \lambda^n,$$

i.e., see Proposition 4.2 below,

$$F_1(\lambda)A(\lambda)^{-1} = \lambda^{-1}\widetilde{F}_1(\lambda),$$

where $F_0(\lambda) = P_{\beta_1} + \lambda S_1^+ Q_2$, $\widetilde{F}_1(\lambda) = S_1^+ + \sum_{n=0}^{\infty}(P_{\beta_1} Z_{n+2} + S_1^+ Q_2 Z_{n+1})\lambda^{n+1}$ is analytic on D_ρ and $\widetilde{F}_1(0) = S_1^+ \neq 0$. Finally,

$$P_{\beta_2}A(\lambda)^{-1} = \frac{P_{\beta_2}Z_0}{\lambda^2} + \sum_{n=1}^{\infty} P_{\beta_2} Z_n \lambda^{n-2} = \frac{S_2^+}{\lambda^2} + \sum_{n=1}^{\infty} P_{\beta_2} Z_n \lambda^{n-2}$$

i.e., see Proposition 4.2 below,

$$F_2(\lambda)A(\lambda)^{-1} = \lambda^{-2}\widetilde{F}_2(\lambda),$$

where $F_2(\lambda) = P_{\beta_2}$, $\widetilde{F}_2(\lambda) = S_2^+ + \sum_{n=1}^{\infty} P_{\beta_2} Z_n \lambda^n$ is analytic on D_ρ and $\widetilde{F}_2(0) = S_2^+ \neq 0$. Hence one has

$$\widetilde{F}_h(\lambda)A(\lambda) = \lambda^h F_h(\lambda), \qquad \widetilde{F}_0(\lambda) = S_h^+, \qquad h = 0, 1, 2,$$

where $\operatorname{Im} S_h^+ = \beta_h$ and $\operatorname{Ker} S_h^+ = \alpha_h^\perp$, see (2.1). Summing over h one finds

$$\widetilde{F}(\lambda)A(\lambda) = \sum_{h=0}^{2} \lambda^h F_h(\lambda), \qquad \widetilde{F}(\lambda) = \sum_{h=0}^{2} \widetilde{F}_h(\lambda),$$

where $\widetilde{F}(\lambda)$ is analytic on D_ρ, $\widetilde{F}(0) = \sum_{h=0}^{2} S_h^+$ is invertible and

$$\sum_{h=0}^{2} \lambda^h F_h(\lambda) = P_{\beta_0} + \lambda(P_{\beta_1} + S_0^+ Q_1) + \lambda^2(P_{\beta_2} + S_1^+ Q_2).$$

On the other hand, letting

$$D_\beta(\lambda) = P_{\beta_0} + P_{\beta_1}\lambda + P_{\beta_2}\lambda^2, \qquad F_\beta(\lambda) = I - M_\beta \lambda, \qquad M_\beta = -\sum_{h=0}^{1} S_h^+ Q_{h+1},$$

one has $P_{\beta_h} M_\beta = -P_{\beta_h} S_h^+ Q_{h+1} = -S_h^+ Q_{h+1}$, $h = 0, 1$, and $P_{\beta_2} M_\beta = 0$ and hence

$$\begin{aligned} D_\beta(\lambda)F_\beta(\lambda) &= P_{\beta_0} + (P_{\beta_1} - P_{\beta_0} M_\beta)\,\lambda + (P_{\beta_2} - P_{\beta_1} M_\beta)\,\lambda^2 - P_{\beta_2} M_\beta \lambda^3 \\ &= P_{\beta_0} + \left(P_{\beta_1} + S_0^+ Q_1\right)\lambda + \left(P_{\beta_1} + S_1^+ Q_2\right)\lambda^2 = \sum_{h=0}^{2} \lambda^h F_h(\lambda). \end{aligned}$$

This shows that $\widetilde{F}(\lambda)A(\lambda) = D_\beta(\lambda)F_\beta(\lambda)$, i.e., see Theorem 2.4, one has the β factorization

$$A(\lambda) = E_\beta(\lambda)D_\beta(\lambda)F_\beta(\lambda), \qquad E_\beta(\lambda) = \widetilde{F}(\lambda)^{-1} = \sum_{n=0}^{\infty} E_{\beta,n}\lambda^n,$$

where $E_{\beta,0} = \widetilde{F}(0)^{-1} = \sum_{h=0}^{2} S_h$ and the formula for $E_{\beta,n}$, $n = 1, 2, \ldots$, is found from $E_\beta(\lambda) = A(\lambda)F_\beta(\lambda)^{-1}D_\beta(\lambda)^{-1}$, see the proof of Theorem 2.4.

As the results in the next section show, the same structure applies to poles of order $m \in \mathbb{N}$. The key result is presented in Lemma 4.1: this provides the subspace decompositions of the fundamental equations that describe the underlying structure.

4. Order of the pole and factorizations

This section reports results in Lemma 4.1, which is of central relevance in the analysis and provides the subspace decompositions of the fundamental equations in (1.8), and in Theorem 2.3, which characterizes the order of the pole in terms of the orthogonal direct sum decompositions $X = \bigoplus_{h=0}^{m} \beta_h$ and $Y = \bigoplus_{h=0}^{m} \alpha_h$. A construction of root functions of $A(\lambda)^{-1}$ at 0 is provided in Proposition 4.2 and this leads to two alternative constructions in Theorem 2.4 of the factorization $A(\lambda) = E(\lambda)D(\lambda)F(\lambda)$, where $E(\lambda), F(\lambda)$ are analytic and invertible on D_ρ and $D(\lambda)$ describes the singularity of $A(\lambda)$ at 0. This in turn allows to provide formulas for the coefficients of the resolvent in Corollary 4.3.

The next statement reports consequences of system (1.8) that are derived using the projection identities that correspond to the subspaces defined recursively by the local subspace decomposition. This lemma motivates Definition 2.2 and plays a central role in the analysis, as all the results in the paper are consequences of (4.1) and (4.2) below.

Lemma 4.1 *(Subspace decompositions of (1.8)). Consider Definition 2.2 and let $P_{a_0^\perp} = P_{b_0^\perp} = I$ and $Z_{-1} = 0$; then the Y version of equation $n + h \leq m$ in system (1.8) implies*

$$S_h Z_n + P_{a_h^\perp} Q_{h+1} Z_{n-1} = 1_{n+h=m} P_{a_h^\perp}, \qquad h = 0, \ldots, m - n. \tag{4.1}$$

Similarly, the X version of equation $n + h \leq m$ in system (1.8) implies

$$Z_n S_h + Z_{n-1} Q_{h+1} P_{b_h^\perp} = 1_{n+h=m} P_{b_h^\perp}, \qquad h = 0, \ldots, m - n. \tag{4.2}$$

First note that for $h = 0$, (4.1) and (4.2) coincide with (1.8); in fact, because $P_{a_0^\perp} = I$, $P_{b_0^\perp} = I$, $S_0 = A_0$, $Q_1 = A_1$, and $Z_{-1} = 0$, one has

$$A_0 Z_n + A_1 Z_{n-1} = 1_{n=m} I, \qquad Z_n A_0 + Z_{n-1} A_1 = 1_{n=m} I, \qquad n = 0, \ldots, m.$$

More informative relations arise from (4.1) and (4.2) by setting, for example, $n = 0$ and $h = m = 1, 2$; for instance, (4.1) respectively delivers (3.3) and (3.13), which led to prove Propositions 3.1, 3.2. The same structure is found for poles of any order, i.e. setting $n = 0$ and $h = m$ in (4.1) and (4.2), one has

$$S_m Z_0 = P_{a_m^\perp}, \qquad Z_0 S_m = P_{b_m^\perp}.$$

This leads to the statement in Theorem 2.3, see the proof in Appendix A; note that from (iv) in Theorem 2.3 one has

$$\operatorname{Im} Z_0 = \beta_m, \qquad \operatorname{Ker} Z_0 = \alpha_m^\perp.$$

Further observe that (4.1) and (4.2) show how to linearly combine Z_n and Z_{n-1} in order to get the zero operator; that is, they provide a construction of the root functions of $A(\lambda)^{-1}$ at 0. In fact, applying S_h^+ to (4.1) and (4.2) to S_h^+ and using (2.1) one has

$$P_{\beta_h} Z_n + S_h^+ Q_{h+1} Z_{n-1} = 1_{n+h=m} S_h^+, \qquad h = 0, \ldots, m-n, \tag{4.3}$$

$$Z_n P_{\alpha_h} + Z_{n-1} Q_{h+1} S_h^+ = 1_{n+h=m} S_h^+, \qquad h = 0, \ldots, m-n. \tag{4.4}$$

Note that by setting $m = 1$ and $n = 0, 1$ in (4.3), one has

$$P_{\beta_0} Z_0 = 0, \qquad P_{\beta_1} Z_0 = S_1^+, \qquad P_{\beta_0} Z_1 = S_0^+ - S_0^+ Q_1 Z_0,$$

i.e. (3.8) and (3.9), which led to prove the β factorization when $m = 1$. Similarly, setting $m = 2$ and $n = 0, 1, 2$ in (4.3), one has

$$\begin{gathered} P_{\beta_0} Z_0 = 0, \qquad P_{\beta_1} Z_0 = 0, \qquad P_{\beta_2} Z_0 = S_2^+, \\ P_{\beta_0} Z_1 = -S_0^+ Q_1 Z_0, \qquad P_{\beta_1} Z_1 = S_1^+ - S_1^+ Q_2 Z_0, \\ P_{\beta_0} Z_2 = S_0^+ - S_0^+ Q_1 Z_1, \end{gathered}$$

i.e. (3.15), (3.16), (3.17), and (3.18), which led to prove the β factorization when $m = 2$.

The same structure is found for poles of any order and leads to the following statement.

Proposition 4.2 *(Root functions). Consider Definition 2.2 and for $h = 0, \ldots, m$, let*

$$F_h(\lambda) = P_{\beta_h} + \lambda 1_{h \neq m} S_h^+ Q_{h+1}, \qquad E_h(\lambda) = P_{\alpha_h} + \lambda 1_{h \neq m} Q_{h+1} S_h^+. \tag{4.5}$$

Then for $h = 0, \ldots, m$, one has

$$F_h(\lambda) A(\lambda)^{-1} = \lambda^{-h} \widetilde{F}_h(\lambda), \qquad A(\lambda)^{-1} E_h(\lambda) = \lambda^{-h} \widetilde{E}_h(\lambda), \tag{4.6}$$

where $\widetilde{F}_h(\lambda)$ and $\widetilde{E}_h(\lambda)$ are analytic on D_ρ and $\widetilde{F}_h(0) = \widetilde{E}_h(0) = S_h^+ \neq 0$.

This shows that $F_h(\lambda) \in \mathscr{B}_X$ and $E_h(\lambda) \in \mathscr{B}_Y$ are root functions of order $-h$ of $A(\lambda)^{-1}$ at 0. In fact, $F_h(\lambda)$ and $E_h(\lambda)$ factor out λ^{m-h} from $Z(\lambda) = \sum_{n=0}^\infty Z_n \lambda^n$ in $A(\lambda)^{-1} = \lambda^{-m} Z(\lambda)$ hence decreasing the order of the pole in $F_h(\lambda) A(\lambda)^{-1}$ and $A(\lambda)^{-1} E_h(\lambda)$ to $h = 0, \ldots, m-1$.

This leads to the statement in Theorem 2.4, see the proof in Appendix A. Theorem 2.4 provides two alternative constructions of the factorization $A(\lambda) = E(\lambda) D(\lambda) F(\lambda)$, where $E(\lambda), F(\lambda)$ are analytic and invertible on D_ρ and $D(\lambda)$ describes the singularity of $A(\lambda)$ at 0. The existence of the factorization $A(\lambda) = E(\lambda) D(\lambda) F(\lambda)$ for an operator function with values in the algebra of all bounded linear operators acting in a Banach space which are holomorphic in some neighbourhood of some point, except possibly at this point itself, is discussed in Gohberg and Sigal [17] in the special case of A_0 being Fredholm of index 0, i.e. $\dim \operatorname{Ker} A_0$ and $\operatorname{codim} \operatorname{Im} A_0 = \dim(Y/\operatorname{Im} A_0)$ are finite and equal. The same kind of factorization holds in the present setup and Theorem 2.4 provides two alternative constructions of its factors.

Note that from $E_{\beta,0} = F_{\alpha,0} = \sum_{h=0}^{m} S_h$ one has $E_{\beta,0}^{-1} = F_{\alpha,0}^{-1} = \sum_{h=0}^{m} S_h^+$; further observe that because $S_h S_h^+ = P_{\alpha_h}$ and $S_h^+ S_h = P_{\beta_h}$ one has $S_h P_{\beta_h} S_h^+ = P_{\alpha_h}$ and $S_h^+ P_{\alpha_h} S_h = P_{\beta_h}$; hence

$$E_{\beta,0}^{-1} D_\alpha(\lambda) E_{\beta,0} = \left(\sum_{h=0}^{m} S_h^+\right) \sum_{h=0}^{m} P_{\alpha_h} \lambda^h \left(\sum_{h=0}^{m} S_h\right) = \sum_{h=0}^{m} (S_h^+ P_{\alpha_h} S_h) \lambda^h = D_\beta(\lambda)$$

and thus

$$A(\lambda) = E_\beta(\lambda) D_\beta(\lambda) F_\beta(\lambda) = \widetilde{E}_\beta(\lambda) D_\alpha(\lambda) \widetilde{F}_\beta(\lambda),$$

where $\widetilde{E}_\beta(0) = E_\beta(0) E_{\beta,0}^{-1} = I$ and $\widetilde{F}_\beta(0) = E_{\beta,0} F_\beta(0) = \sum_{h=0}^{m} S_h$. This illustrates that the factors in (2.2) are not unique.

Finally note that from (2.2) one has

$$A(\lambda)^{-1} = F_\zeta(\lambda)^{-1} D_\zeta(\lambda)^{-1} E_\zeta(\lambda)^{-1} = \sum_{n=0}^{\infty} Z_n \lambda^{n-m}, \qquad \zeta = \alpha, \beta;$$

hence the following statement.

Corollary 4.3 *(A formula for Z_n, $n = 0, 1, \ldots$). For $n = 0, 1, \ldots$, Z_n can be computed as*

$$Z_n = \sum_{k=0}^{n} B_{\beta,k} C_{\beta,n-k} = \sum_{k=0}^{n} B_{\alpha,n-k} C_{\alpha,k},$$

where

$$B_{\beta,n} = \begin{cases} \sum_{k=0}^{n} M_\beta^k P_{\beta_{m-n+k}} \\ M_\beta^{n-m} B_{\beta,m} \end{cases},$$

$$C_{\alpha,n} = \begin{cases} \sum_{k=0}^{n} P_{\alpha_{m-n+k}} (M_\alpha^*)^k \\ C_{\alpha,m} (M_\alpha^*)^{n-m} \end{cases}, \qquad \begin{matrix} n = 0, \ldots, m \\ n = m+1, m+2, \ldots \end{matrix},$$

$$\begin{matrix} C_{\beta,n} = -\sum_{k=1}^{n} E_{\beta,0}^{-1} E_{\beta,k} C_{\beta,n-k} \\ B_{\alpha,n} = -\sum_{k=1}^{n} F_{\alpha,0}^{-1} F_{\alpha,k} B_{\alpha,n-k} \end{matrix}, \qquad n = 1, 2, \ldots,$$

and

$$C_{\beta,0} = B_{\alpha,0} = E_{\beta,0}^{-1} = F_{\alpha,0}^{-1} = \sum_{h=0}^{m} S_h^+, \qquad \begin{matrix} E_{\beta,k} = P_{\alpha_0^\perp} A_1 M_\beta^{k-1} B_{\beta,m} \\ F_{\alpha,k} = C_{\alpha,m} (M_\alpha^*)^{k-1} A_1 P_{\beta_0^\perp} \end{matrix}.$$

5. Chains and subspaces

This section presents the following results: Proposition 5.1 shows that M_β and M_α deliver chains for $A(\lambda)$ and $A(\lambda)^*$ at 0 and Theorem 2.5 links these chains to the subspaces in the [10] direct sum decompositions. Theorem 2.6 characterizes the image and the null space of the operators in the principal part of the resolvent via the same quantities and Corollary 5.2 provides recursions for $\operatorname{Im} Z_n$ and $\operatorname{Ker} Z_n$, $n = 0, \dots, m-1$.

From (2.2) one has $A(\lambda)F_\beta(\lambda)^{-1} = E_\beta(\lambda)D_\beta(\lambda)$ and $E_\alpha(\lambda)^{-1}A(\lambda) = D_\alpha(\lambda)F_\alpha(\lambda)$; direct manipulations of these equations allow to show that the operators M_β and M_α introduced in Definition 2.2 deliver chains for $A(\lambda)$ and $A(\lambda)^*$ at 0.

Proposition 5.1 *(Chains). Consider Definition 2.2 and for $h = 1, \dots, m$, let $x_{h,0} \in \beta_h$ and $y_{h,0} \in \alpha_h$ be nonzero vectors and further define*

$$x_{h,n} = M_\beta x_{h,n-1}, \qquad y_{h,n} = M_\alpha y_{h,n-1}, \qquad n = 1, 2, \dots .$$

Then

$$\begin{aligned} &A_0 x_{h,0} = 0, \qquad A_0 x_{h,n} + A_1 x_{h,n-1} = 0, \qquad n = 1, \dots, h-1, \\ &A_0^* y_{h,0} = 0, \qquad A_0^* y_{h,n} + A_1^* y_{h,n-1} = 0, \qquad n = 1, \dots, h-1, \end{aligned}$$

and

$$A_0 x_{h,h} + A_1 x_{h,h-1} \neq 0, \qquad A_0^* y_{h,h} + A_1^* y_{h,h-1} \neq 0.$$

Moreover, the vectors in the X and Y-chains

$$x_{h,0}, \dots, x_{h,h-1}, \qquad y_{h,0}, \dots, y_{h,h-1}, \qquad h = 1, \dots, m,$$

are linearly independent.

This shows that $x_{h,0}, x_{h,1}, \dots, x_{h,h-1}$ is a chain in X of maximal length $h = 1, \dots, m$ for $A(\lambda) = A_0 + A_1\lambda$ at 0 and similarly, $y_{h,0}, y_{h,1}, \dots, y_{h,h-1}$ is a chain in Y of maximal length $h = 1, \dots, m$ for $A(\lambda)^* = A_0^* + A_1^*\lambda$ at 0. Remark that, because $\operatorname{Ker} A_0 = \beta_0^\perp$, no X-chain is associated to the subspace β_0 and similarly, because $\operatorname{Ker} A_0^* = (\operatorname{Im} A_0)^\perp = \alpha_0^\perp$, no Y-chain is associated to the subspace α_0; moreover, the orthogonal direct sum decompositions $\operatorname{Ker} A_0 = \bigoplus_{h=1}^m \beta_h$ and $\operatorname{Ker} A_0^* = (\operatorname{Im} A_0)^\perp = \bigoplus_{h=1}^m \alpha_h$ isolate the subspaces that deliver chains that have the same maximal length.

Consider the X-chains (the interpretation of the Y-chain is similar) and the orthogonal direct sum decomposition $\operatorname{Ker} A_0 = \bigoplus_{h=1}^m \beta_h$. Fix a nonzero vector $x_{h,0} \in \beta_h$ and repeatedly apply the M_β operator to define $x_{h,n} = M_\beta^n x_{h,0}$, $n = 1, 2, \dots$. Proposition 5.1 shows that such a chain ends after $h-1$ repetitions, i.e. the chain associated to $x_{h,0} \in \beta_h$ has maximal length h. Next take another nonzero vector $\widetilde{x}_{h,0} \in \beta_h$ which is linearly independent of $x_{h,0}$ and construct in the same way the chain of maximal length h associated to $\widetilde{x}_{h,0} \in \beta_h$. Proposition 5.1 shows that the vectors in these two chains are linearly independent. This process is repeated as long as there are vectors in β_h from which one can start a chain. Because of linear independence, the subspace defined by the X-chains of maximal length h associated to the subspace β_h is thus $\bigoplus_{n=0}^{h-1} M_\beta^n \beta_h$. Collecting together the subspaces associated to the different β_h, one has that the subspace associated to $\operatorname{Ker} A_0 = \bigoplus_{h=1}^{m} \beta_h$ is

$$\bigoplus_{h=1}^{m} \bigoplus_{n=0}^{h-1} M_\beta^n \beta_h.$$

Similarly, the subspace defined by the Y-chains of maximal length h associated to the subspace α_h is $\bigoplus_{n=0}^{h-1} M_\alpha^n \alpha_h$ and the one associated to $\operatorname{Ker} A_0^* = (\operatorname{Im} A_0)^\perp = \bigoplus_{h=1}^{m} \alpha_h$ is

$$\bigoplus_{h=1}^{m} \bigoplus_{n=0}^{h-1} M_\alpha^n \alpha_h.$$

This leads to the statement in Theorem 2.5, see the proof in Appendix A. Theorem 2.5 links the present results with the ones in Bart and Lay [10], showing how the subspaces in (1.6) can be constructed using the quantities defined by the local subspace decomposition. This allows to construct the operators $Z_{n\mathscr{N}'}$, $Z_{n\mathscr{R}'}$ in (1.7) and the projections $P_{\mathscr{N},\mathscr{R}}$, $P_{\mathscr{R},\mathscr{N}}$, $P_{\mathscr{N}',\mathscr{R}'}$, $P_{\mathscr{R}',\mathscr{N}'}$ associated to the direct sum decompositions in (1.6). It thus provides a way to compute the resolvent and the logarithmic residues in terms of the quantities defined by the local subspace decomposition.

The linear independence of the vectors in the X and Y-chains leads to the statement in Theorem 2.6, see the proof in Appendix A. Note that for $n = 0$, Theorem 2.6 gives

$$\operatorname{Im} Z_0 = \beta_m, \qquad \operatorname{Ker} Z_0 = \alpha_m^\perp,$$

which are already known from Theorem 2.3, and for $n = m-1$,

$$\operatorname{Im} Z_{m-1} = \bigoplus_{h=1}^{m} \bigoplus_{k=0}^{h-1} M_\beta^k \beta_h = \mathscr{N}_m, \qquad \operatorname{Ker} Z_{m-1} = \left(\bigoplus_{h=1}^{m} \bigoplus_{k=0}^{h-1} M_\alpha^k \alpha_h \right)^\perp = \mathscr{R}'_m,$$

which are already known from Theorem 2.5 since $P_{\mathscr{N},\mathscr{R}} = Z_{m-1} A_1$, $P_{\mathscr{N}',\mathscr{R}'} = A_1 Z_{m-1}$ and $\operatorname{Im} P_{\mathscr{N},\mathscr{R}} = \operatorname{Im} Z_{m-1}$, $\operatorname{Ker} P_{\mathscr{N}',\mathscr{R}'} = \operatorname{Ker} Z_{m-1}$.

Further note that for $n = 1, \dots, m-2$, one has

$$\begin{aligned} \operatorname{Im} Z_1 &= \beta_{m-1} \oplus \beta_m \oplus M_\beta \beta_m, & \operatorname{Ker} Z_1 &= (\alpha_{m-1} \oplus \alpha_m \oplus M_\alpha \alpha_m)^\perp, \\ &\vdots & &\vdots \end{aligned}$$

$$\operatorname{Im} Z_{m-2} = \bigoplus_{h=2}^{m} \bigoplus_{k=0}^{h-(m-2)+n} M_\beta^k \beta_h, \qquad \operatorname{Ker} Z_n = \left(\bigoplus_{h=2}^{m} \bigoplus_{k=0}^{h-(m-2)+n} M_\alpha^k \alpha_h \right)^\perp .$$

This shows in which way $\operatorname{Im} Z_n$ and $\operatorname{Ker} Z_n$, $n = 0, \dots, m-1$, respectively form an increasing and a decreasing sequence of nested subspaces, see the next proposition.

Corollary 5.2 *(Recursions for* $\operatorname{Im} Z_n$ *and* $\operatorname{Ker} Z_n$*,* $n = 0, \dots, m-1$*). Let*

$$\xi_{\beta,n} = \bigoplus_{k=0}^{n} M_\beta^k \beta_{m-n+k}, \qquad \xi_{\alpha,n} = \bigoplus_{k=0}^{n} M_\alpha^k \alpha_{m-n+k}, \qquad n = 1, \dots, m-1.$$

Then $\operatorname{Im} Z_0 = \beta_m$*,* $\operatorname{Ker} Z_0 = \alpha_m^\perp$ *and*

$$\operatorname{Im} Z_n = \operatorname{Im} Z_{n-1} \oplus \xi_{\beta,n}, \quad \operatorname{Ker} Z_n = \ker Z_{n-1} \cap \xi_{\alpha,n}^\perp, \quad n = 1, \dots, m-1.$$

Remark that, because $0 \neq M_\beta^n \beta_m \subseteq \xi_{\beta,n}$ and $0 \neq M_\alpha^n \alpha_m \subseteq \xi_{\beta,n}$, one has the strict inclusions

$$\operatorname{Im} Z_n \supset \operatorname{Im} Z_{n-1}, \qquad \operatorname{Ker} Z_n \subset \operatorname{Ker} Z_{n-1}, \qquad n = 1, \dots, m-1.$$

6. Special case: A_0 is Fredholm of index 0

This sections considers the special case of A_0 being Fredholm of index 0, i.e. $\dim \operatorname{Ker} A_0$ and $\operatorname{codim} \operatorname{Im} A_0 = \dim(Y/\operatorname{Im} A_0)$ are finite and equal.

It is well known, see e.g. Chapter XI in [15], that if A_0 is Fredholm of index 0 then Z_n, $n = 0, \dots, m-1$, has finite rank and Z_m is Fredholm of index 0. Moreover, in the Gohberg and Sigal [17] factorization $A(\lambda) = E(\lambda)D(\lambda)F(\lambda)$ one can choose

$$D(\lambda) = D_0 + D_1 \lambda^{\kappa_1} + \dots + D_s \lambda^{\kappa_s},$$

where $0 < \kappa_1 \leq \kappa_2 \leq \dots \leq \kappa_s < \infty$ are positive integers, D_0 is Fredholm of index 0, $D_1, \dots, D_s$ have rank one, and $D_0, D_1, \dots, D_s$ are mutually disjoint projections that decompose the identity.

$D(\lambda)$ is the local Smith form of $A(\lambda)$ at 0 and $E(\lambda)$, $F(\lambda)$ are extended canonical systems of root functions. The integers $0 < \kappa_1 \leq \kappa_2 \leq \dots \leq \kappa_s < \infty$ are the partial multiplicities of $A(\lambda)$ at 0; remark that they are uniquely defined and not necessarily distinct. Their sum $\sum_{n=1}^{s} \kappa_n < \infty$ is the algebraic multiplicity of $A(\lambda)$ at 0 and it is denoted by $\kappa(A)$.

The operator version of the logarithmic residue theorem, see [17], states that

$$\kappa(A) = \operatorname{trace} \left(\frac{1}{2\pi i} \int_{\partial D_\rho} A(\lambda)^{-1} A_1 d\lambda \right) = \operatorname{trace} \left(\frac{1}{2\pi i} \int_{\partial D_\rho} A_1 A(\lambda)^{-1} d\lambda \right).$$

Because $P_{\mathscr{N},\mathscr{R}} = \frac{1}{2\pi i}\int_{\partial D_\rho} A(\lambda)^{-1}A_1 d\lambda$ and $P_{\mathscr{N}',\mathscr{R}'} = \frac{1}{2\pi i}\int_{\partial D_\rho} A_1 A(\lambda)^{-1} d\lambda$, this implies

$$\kappa(A) = \operatorname{rank} P_{\mathscr{N},\mathscr{R}} = \operatorname{rank} P_{\mathscr{N}',\mathscr{R}'} = \dim \mathscr{N}_m = \dim \mathscr{N}'_m. \tag{6.1}$$

Remark that if A_0 is Fredholm of index 0, $\operatorname{Im} A_0$ is closed and $\alpha_0^\perp = (\operatorname{Im} A_0)^\perp$, $\beta_0^\perp = \operatorname{Ker} A_0$ are finite dimensional; hence Assumption 2.1 is satisfied.

A direct consequence of Theorems 2.5 and 2.6 is the following.

Corollary 6.1 *(Rank of Z_n, $n = 0, \dots, m-1$, and dimension of $\mathscr{N}_m, \mathscr{N}'_m$). Assume that A_0 is Fredholm of index 0 and let $r_h = \dim \alpha_h = \dim \beta_h$, $h = 1, \dots, m$. Then*

$$0 \le r_h < \infty, \qquad h = 1, \dots, m-1, \qquad 0 < r_m < \infty,$$

$$\operatorname{rank} Z_n = \sum_{h=0}^{n} (h+1) r_{m-n+h} < \infty, \qquad n = 0, \dots, m-1,$$

and

$$\dim \mathscr{N}_m = \dim \mathscr{N}'_m = \sum_{h=1}^{m} h r_h < \infty.$$

This expresses the rank of the operators in the principal part of the resolvent and the dimension of the subspaces in (1.6) in terms of the quantities defined by the local subspace decomposition.

Combining (6.1) and Corollary 6.1, one can express $\kappa(A)$ as follows.

Corollary 6.2 *(Algebraic multiplicity of $A(\lambda)$ at 0). If A_0 is Fredholm of index 0, then the algebraic multiplicity of $A(\lambda)$ at 0 is equal to $\sum_{h=1}^{m} h r_h < \infty$.*

Finally it is shown that the local subspace decomposition fully describes the partial multiplicities; this is a direct consequence of Theorem 2.4, which states that in the Gohberg and Sigal [17] factorization one can choose $D(\lambda) = \sum_{h=0}^{m} P_{\beta_h} \lambda^h$ or $D(\lambda) = \sum_{h=0}^{m} P_{\alpha_h} \lambda^h$.

Corollary 6.3 *(Partial multiplicities of $A(\lambda)$ at 0). If A_0 is Fredholm of index 0, then the number of distinct partial multiplicities of $A(\lambda)$ at 0 is equal to $\sum_{h=1}^{m} 1_{r_h > 0}$ and there are exactly $r_h < \infty$ partial multiplicities that are equal to $h = 1, \dots, m$.*

Remark that $r_m > 0$ but r_h can be 0 for some $h = 1, \dots, m-1$; in this case, no partial multiplicities is equal to h. The results in Corollaries 6.1, 6.2 and 6.3 are collected together in Theorem 2.7.

Declaration of competing interest

There is no competing interest.

Appendix A. Proofs

Proof of Lemma 4.1. The proof of (4.1) is by induction and consists in showing that the Y version of equation $n = 0, \dots, m$ in system (1.8) implies

$$S_h Z_{n-h} + P_{a_h^\perp} Q_{h+1} Z_{n-h-1} = 1_{n=m} P_{a_h^\perp}, \qquad h = 0, 1, \dots, n, \qquad n = 0, \dots, m; \quad \text{(A.1)}$$

replacing n with $n+h$ one finds (4.1). In order to show that (A.1) holds for $h = 0$, observe that the Y version of equation n in system (1.8) reads $A_0 Z_n + A_1 Z_{n-1} = 1_{n=m} I$. By definition, $P_{a_0^\perp} = I$, $S_0 = A_0$ and $Q_1 = A_1$ and this shows that (A.1) holds for $h = 0$. Next assume that (A.1) holds for $h = 0, \dots, \ell - 1$ and show that it also holds for $h = \ell \leq n$. First note that $S_h^+ S_h = P_{\beta_h}$ and $S_h^+ P_{a_h^\perp} = S_h^+$; hence applying S_h^+ on both sides of (A.1), by the induction assumption one has that the Y version of equation $n = 0, \dots, m$ in system (1.8) implies

$$P_{\beta_h} Z_{n-h} + S_h^+ Q_{h+1} Z_{n-h-1} = 1_{n=m} S_h^+, \qquad h = 0, 1, \dots, \ell - 1, \qquad n = 0, \dots, m.$$

Observe that, because $n - \ell + h \leq n - 1 < m$ for $h = 0, 1, \dots, \ell - 1$, in equation $n - \ell + h$ one has $1_{n-\ell+h=m} = 0$ and hence

$$P_{\beta_h} Z_{n-\ell} = -S_h^+ Q_{h+1} Z_{n-\ell-1}, \qquad h = 0, 1, \dots, \ell - 1. \tag{A.2}$$

Next write (A.1) for $h = \ell - 1$,

$$S_{\ell-1} Z_{n-\ell+1} + P_{a_{\ell-1}^\perp} Q_\ell Z_{n-\ell} = 1_{n=m} P_{a_{\ell-1}^\perp},$$

where $\operatorname{Im} S_{\ell-1} = \alpha_{\ell-1}$, see Definition 2.2; applying $P_{a_\ell^\perp}$, where $a_\ell = \bigoplus_{h=0}^{\ell-1} \alpha_h$, on both sides of this equation and using $P_{a_\ell^\perp} S_{\ell-1} = 0$ and $P_{a_\ell^\perp} P_{a_{\ell-1}^\perp} = P_{a_\ell^\perp}$ one has

$$P_{a_\ell^\perp} Q_\ell Z_{n-\ell} = 1_{n=m} P_{a_\ell^\perp}. \tag{A.3}$$

Next consider $b_\ell = \bigoplus_{h=0}^{\ell-1} \beta_h$ and use projections, inserting $I = P_{b_\ell^\perp} + P_{b_\ell}$ between Q_ℓ and $Z_{n-\ell}$; one finds

$$P_{a_\ell^\perp} Q_\ell Z_{n-\ell} = \left(P_{a_\ell^\perp} Q_\ell P_{b_\ell^\perp} \right) Z_{n-\ell} + P_{a_\ell^\perp} Q_\ell P_{b_\ell} Z_{n-\ell}.$$

Because $P_{b_\ell} = \sum_{j=0}^{\ell-1} P_{\beta_j}$, one has $P_{a_\ell^\perp} Q_\ell P_{b_\ell} Z_{n-\ell} = P_{a_\ell^\perp} Q_\ell \sum_{j=0}^{\ell-1} P_{\beta_j} Z_{n-\ell}$ and by the induction assumption, see (A.2), one finds

$$P_{a_\ell^\perp} Q_\ell P_{b_\ell} Z_{n-\ell} = -P_{a_\ell^\perp}\left(Q_\ell \sum_{j=0}^{\ell-1} S_j^+ Q_{j+1}\right) Z_{n-\ell-1}.$$

By Definition 2.2, one has $P_{a_\ell^\perp} Q_\ell P_{b_\ell^\perp} = S_\ell$ and $Q_{\ell+1} = -Q_\ell \sum_{j=0}^{\ell-1} S_j^+ Q_{j+1}$ and hence one rewrites (A.3) as

$$S_\ell Z_{n-\ell} + P_{a_\ell^\perp} Q_{\ell+1} Z_{n-\ell-1} = 1_{n=m} P_{a_\ell^\perp}.$$

This shows that (A.1) holds for $h = \ell$ and completes the proof of (4.1). The proof of (4.2) applies similar arguments to the X version of system (1.8) and it is omitted. ∎

The proof of Theorem 2.3 makes use of the following lemma.

Lemma A.1. *Consider Definition 2.2. Then* $\operatorname{Im} Z_0 \subseteq b_m^\perp$ *and* $a_m \subseteq \operatorname{Ker} Z_0$*.*

Proof. For $n = 0$, (4.1) and (4.2) read

$$S_h Z_0 = 1_{h=m} P_{a_h^\perp}, \qquad Z_0 S_h = 1_{h=m} P_{b_h^\perp} \qquad h = 0, \dots, m, \tag{A.4}$$

and imply

$$S_h Z_0 = Z_0 S_h = 0, \qquad h = 0, \dots, m-1.$$

From $S_h Z_0 = 0$, $h = 0, \dots, m-1$, one has $\operatorname{Im} Z_0 \subseteq \left(\bigcap_{h=0}^{m-1} \operatorname{Ker} S_h\right)$. By Definition 2.2, $\operatorname{Ker} S_h = \beta_h^\perp$ and hence $\operatorname{Im} Z_0 \subseteq \bigcap_{h=0}^{m-1} \beta_h^\perp = \left(\bigoplus_{h=0}^{m-1} \beta_h\right)^\perp = b_m^\perp$. This proves the first statement. From $Z_0 S_h = 0$, $h = 0, \dots, m-1$, one has $\bigoplus_{h=0}^{m-1} \operatorname{Im} S_h \subseteq \operatorname{Ker} Z_0$. By Definition 2.2, $\operatorname{Im} S_h = \alpha_h$ and hence $\bigoplus_{h=0}^{m-1} \alpha_h = a_m \subseteq \operatorname{Ker} Z_0$. This completes the proof. ∎

Proof of Theorem 2.3. First note that by definition, (i) holds if and only if the identity is in equation m of system (1.8).
$(i) \Leftrightarrow (ii)$: It is shown that $(i) \Rightarrow (ii) \Rightarrow (\operatorname{Im} Z_0 = \beta_m) \Rightarrow (i)$. Under (i), one has $h = m$ in (A.4), i.e. $Z_0 S_m = P_{b_m^\perp}$, $b_m^\perp \neq 0$; by Definition 2.2, $b_m \subseteq \operatorname{Ker} S_m$ and because $b_m \subset \operatorname{Ker} S_m$ contradicts $Z_0 S_m = P_{b_m^\perp}$, one has $b_m = \operatorname{Ker} S_m$. By Definition 2.2, $\operatorname{Ker} S_m = \beta_m^\perp$ and $b_m = \beta_0 \oplus \cdots \oplus \beta_{m-1}$, and hence (ii). Moreover, by Lemma A.1, $\operatorname{Im} Z_0 \subseteq b_m^\perp$ and because $\operatorname{Im} Z_0 \subset b_m^\perp$ contradicts $Z_0 S_m = P_{b_m^\perp}$, one has $\operatorname{Im} Z_0 = b_m^\perp$. Using $b_m^\perp = \beta_m$, see (ii), one finds $\operatorname{Im} Z_0 = \beta_m$. Next let $\operatorname{Im} Z_0 = \beta_m$ and proceed by contradiction, assuming that the identity is not in equation m, so that the first equation in (A.4) reads $S_m Z_0 = 0$, which implies $\operatorname{Im} Z_0 \subseteq \operatorname{Ker} S_m$, where $\operatorname{Im} Z_0 = \beta_m$ and $\operatorname{Ker} S_m = \beta_m^\perp$. Hence $\beta_m \subseteq \beta_m^\perp$, so that $\beta_m = 0$. This contradicts $Z_0 \neq 0$, i.e. that the pole has order m, and proves that (i) holds.

$(i) \Leftrightarrow (iii)$: It is shown that $(i) \Rightarrow (iii) \Rightarrow (\operatorname{Ker} Z_0 = \alpha_m^\perp) \Rightarrow (i)$. Under (i), one has $h = m$ in (A.4), i.e. $S_m Z_0 = P_{a_m^\perp}$, $a_m^\perp \neq 0$; by Definition 2.2, $\operatorname{Im} S_m \subseteq a_m^\perp$ and because $\operatorname{Im} S_m \subset a_m^\perp$ contradicts $S_m Z_0 = P_{a_m^\perp}$, one has $\operatorname{Im} S_m = a_m^\perp$. By Definition 2.2, $\operatorname{Im} S_m = \alpha_m$ and $a_m^\perp = (\alpha_0 \oplus \cdots \oplus \alpha_{m-1})^\perp$, and hence (iii). Moreover, by Lemma A.1, $a_m \subseteq \operatorname{Ker} Z_0$ and because $a_m \subset \operatorname{Ker} Z_0$ contradicts $S_m Z_0 = P_{a_m^\perp}$, one has $a_m = \operatorname{Ker} Z_0$. Using $a_m = \alpha_m^\perp$, see (iii), one finds $\operatorname{Ker} Z_0 = \alpha_m^\perp$. Next let $\operatorname{Ker} Z_0 = \alpha_m^\perp$ and proceed by contradiction, assuming that the identity is not in equation m, so that the second equation in (A.4) reads $Z_0 S_m = 0$, which implies $\operatorname{Im} S_m \subseteq \operatorname{Ker} Z_0$, where $\operatorname{Im} S_m = \alpha_m$ and $\operatorname{Ker} Z_0 = \alpha_m^\perp$. Hence $\alpha_m \subseteq \alpha_m^\perp$, so that $\alpha_m = 0$ and thus $\alpha_m^\perp = Y$. This contradicts $Z_0 \neq 0$, i.e. that the pole has order m, and proves that (i) holds.

Finally note that $S_m Z_0 = P_{\alpha_m}$ implies $Z_0 S_m Z_0 = Z_0 P_{\alpha_m}$ and because $Z_0 = Z_0 P_{\alpha_m}$ one finds $Z_0 S_m Z_0 = Z_0$; similarly, from $Z_0 S_m = P_{\beta_m}$ one has $S_m Z_0 S_m = S_m P_{\beta_m}$ and because $S_m = S_m P_{\beta_m}$ one finds $S_m Z_0 S_m = S_m$. Hence Z_0 is the generalized inverse of S_m. ■

Proof of Proposition 4.2. It is first shown that $F_h(\lambda) A(\lambda)^{-1} = \lambda^{-h} \widetilde{F}_h(\lambda)$, $h = 0, \ldots, m$. Applying P_{β_h} on both sides of (1.5) one obtains

$$P_{\beta_h} A(\lambda)^{-1} = \sum_{n=0}^{\infty} P_{\beta_h} Z_n \lambda^{-m+n}. \tag{A.5}$$

First consider $h = m$; setting $n = 0$ and $h = m$ in (4.3) one finds $P_{\beta_m} Z_0 = S_m^+$. This shows that (A.5) implies $F_m(\lambda) A(\lambda)^{-1} = \lambda^{-m} \widetilde{F}_m(\lambda)$, where $F_m(\lambda) = P_{\beta_m}$ and $\widetilde{F}_m(0) = S_m^+$.

Next consider $h = m - 1$; setting $n = 0$ and $h = m - 1$ in (4.3) one finds $P_{\beta_{m-1}} Z_0 = 0$ and setting $n = 1$ and $h = m - 1$ in (4.3) one has $P_{\beta_{m-1}} Z_1 = S_{m-1}^+ - S_{m-1}^+ Q_m Z_0$. This shows that (A.5) implies

$$P_{\beta_{m-1}} A(\lambda)^{-1} = (S_{m-1}^+ - S_{m-1}^+ Q_m Z_0) \lambda^{-m+1} + \sum_{n=2}^{\infty} P_{\beta_{m-1}} Z_n \lambda^{-m+n}.$$

Applying $\lambda S_{m-1}^+ Q_m$ on both sides of $A(\lambda)^{-1} = \sum_{n=0}^{\infty} Z_n \lambda^{-m+n}$ one obtains

$$(\lambda S_{m-1}^+ Q_m) A(\lambda)^{-1} = S_{m-1}^+ Q_m Z_0 \lambda^{-m+1} + \sum_{n=1}^{\infty} S_{m-1}^+ Q_m Z_n \lambda^{-m+n+1}$$

and summing these two expressions one has

$$\left(P_{\beta_{m-1}} + \lambda S_{m-1}^+ Q_m\right) A(\lambda)^{-1} = S_{m-1}^+ \lambda^{-m+1} + \sum_{n=2}^{\infty} \left(P_{\beta_{m-1}} Z_n + S_{m-1}^+ Q_m Z_{n-1}\right) \lambda^{-m+n}.$$

This shows that (A.5) implies $F_{m-1}(\lambda) A(\lambda)^{-1} = \lambda^{-m+1} \widetilde{F}_{m-1}(\lambda)$, where $F_{m-1}(\lambda) = P_{\beta_{m-1}} + \lambda S_{m-1}^+ Q_m$ and $\widetilde{F}_{m-1}(0) = S_{m-1}^+$.

Finally consider $h = 0, \ldots, m-2$. Write $A(\lambda)^{-1} = \sum_{n=0}^{\infty} Z_n \lambda^{-m+n}$ as

$$A(\lambda)^{-1} = Z_0 \lambda^{-m} + \sum_{n=1}^{m-h-1} Z_n \lambda^{-m+n} + \lambda^{-h} R_0(\lambda), \qquad R_0(0) = Z_h,$$

and apply P_{β_h} on both sides to find

$$P_{\beta_h} A(\lambda)^{-1} = P_{\beta_h} Z_0 \lambda^{-m} + \sum_{n=1}^{m-h-1} P_{\beta_h} Z_n \lambda^{-m+n} + \lambda^{-h} P_{\beta_h} R_0(\lambda).$$

Setting $n = 0$ in (4.3) one has $P_{\beta_h} Z_0 = 0$ and hence

$$P_{\beta_h} A(\lambda)^{-1} = \sum_{n=1}^{m-h-1} P_{\beta_h} Z_n \lambda^{-m+n} + \lambda^{-h} P_{\beta_h} R_0(\lambda). \tag{A.6}$$

Because $1_{n+h=m} = 0$ for $n = 1, \ldots, m-h-1$, from (4.3) one has $P_{\beta_h} Z_n = -S_h^+ Q_{h+1} Z_{n-1}$ and hence

$$\sum_{n=1}^{m-h-1} P_{\beta_h} Z_n \lambda^{-m+n} = -S_h^+ Q_{h+1} \left(\sum_{n=1}^{m-h-1} Z_{n-1} \lambda^{-m+n} \right).$$

Next write

$$\lambda A(\lambda)^{-1} = \left(\sum_{n=1}^{m-h-1} Z_{n-1} \lambda^{-m+n} \right) + \lambda^{-h} R_1(\lambda), \qquad R_1(0) = Z_{h-1},$$

so that

$$\sum_{n=1}^{m-h-1} P_{\beta_h} Z_n \lambda^{-m+n} = -\lambda S_h^+ Q_{h+1} A(\lambda)^{-1} + \lambda^{-h} S_h^+ Q_{h+1} R_1(\lambda).$$

Substituting this expression in (A.6) and rearranging one finds $F_h(\lambda) A(\lambda)^{-1} = \lambda^{-h} \widetilde{F}_h(\lambda)$, where

$$F_h(\lambda) = P_{\beta_h} + \lambda S_h^+ Q_{h+1}, \qquad \widetilde{F}_h(\lambda) = P_{\beta_h} R_0(\lambda) + S_h^+ Q_{h+1} R_1(\lambda).$$

Note that, because $R_0(0) = Z_h$ and $R_1(0) = Z_{h-1}$, one has

$$\widetilde{F}_h(0) = P_{\beta_h} Z_h + S_h^+ Q_{h+1} Z_{h-1} = S_h^+,$$

where the last equality follows setting $n = m - h$ in (4.3). This shows that

$$F_h(\lambda) A(\lambda)^{-1} = \lambda^{-h} \widetilde{F}_h(\lambda), \qquad h = 0, \ldots, m,$$

where $F_h(\lambda) = P_{\beta_h} + 1_{h \neq m} S_h^+ Q_{h+1}\lambda$ and $\widetilde{F}_h(0) = S_h^+$, i.e. $F_h(\lambda) \in \mathscr{B}_X$ is a root function of order $-h$ of $A(\lambda)^{-1}$ at 0. This completes the proof of the second equality in (4.6). The proof of the first equality in (4.6) applies similar arguments to $A(\lambda)^{-1}P_{\alpha_h} = \sum_{n=0}^{\infty} Z_n P_{\alpha_h} \lambda^{-m+n}$ using (4.4) and it is omitted. ■

Proof of Theorem 2.4. It is first shown that $A(\lambda) = E_\beta(\lambda) D_\beta(\lambda) F_\beta(\lambda)$. From (4.6) one has

$$\widetilde{F}_h(\lambda)A(\lambda) = \lambda^h F_h(\lambda), \qquad \widetilde{F}_h(0) = S_h^+, \qquad h = 0, \dots, m,$$

and summing over h one finds

$$\widetilde{F}(\lambda)A(\lambda) = \sum_{h=0}^{m} \lambda^h F_h(\lambda), \qquad \widetilde{F}(\lambda) = \sum_{h=0}^{m} \widetilde{F}_h(\lambda). \tag{A.7}$$

First it is shown that $\widetilde{F}(0) = \sum_{h=0}^{m} \widetilde{F}_h(0) = \sum_{h=0}^{m} S_h^+$ is invertible: because $\operatorname{Im} S_h^+ = \beta_h$ and $X = \bigoplus_{h=0}^{m} \beta_h$, see (*ii*) in Theorem 2.3, $\widetilde{F}(0)$ is onto. Because $\operatorname{Ker} S_h^+ = \alpha_h^\perp$ and $Y = \bigoplus_{h=0}^{m} \alpha_h$, see (*iii*) in Theorem 2.3, one has

$$0 = \left(\bigoplus_{h=0}^{m} \alpha_h \right)^{\perp} = \bigcap_{h=0}^{m} \alpha_h^\perp = \bigcap_{h=0}^{m} \operatorname{Ker} S_h^+ = \operatorname{Ker} \widetilde{F}(0),$$

i.e. $\widetilde{F}(0)$ is one-to-one. This shows that $\widetilde{F}(0)$ is invertible and thus $\widetilde{F}(\lambda)$ is analytic and invertible on D_ρ. Note that

$$\left(\sum_{h=0}^{m} S_h \right) \left(\sum_{h=0}^{m} S_h^+ \right) = \sum_{h=0}^{m} S_h S_h^+ = \sum_{h=0}^{m} P_{\alpha_h} = I,$$
$$\left(\sum_{h=0}^{m} S_h^+ \right) \left(\sum_{h=0}^{m} S_h \right) = \sum_{h=0}^{m} S_h^+ S_h = \sum_{h=0}^{m} P_{\beta_h} = I,$$

i.e.

$$\left(\sum_{h=0}^{m} S_h \right)^{-1} = \sum_{h=0}^{m} S_h^+. \tag{A.8}$$

Next it is shown that $\widetilde{F}(\lambda)A(\lambda) = D_\beta(\lambda)F_\beta(\lambda)$. Consider the r.h.s. of (A.7); using the definition of $F_h(\lambda)$ in (4.5), this can be written as

$$\sum_{h=0}^{m} \lambda^h F_h(\lambda) = P_{\beta_0} + \sum_{h=1}^{m} \left(P_{\beta_h} + S_{h-1}^+ Q_h \right) \lambda^h.$$

On the other hand, the definitions of $D_\beta(\lambda)$ and $F_\beta(\lambda)$ imply

$$D_\beta(\lambda)F_\beta(\lambda) = P_{\beta_0} + \sum_{h=1}^{m} \left(P_{\beta_h} - P_{\beta_{h-1}} M_\beta\right) \lambda^h - P_{\beta_m} M_\beta \lambda^{m+1},$$

where $P_{\beta_{h-1}} M_\beta = -P_{\beta_{h-1}} \sum_{j=0}^{m-1} S_j^+ Q_{j+1} = -P_{\beta_{h-1}} S_{h-1}^+ Q_h = -S_{h-1}^+ Q_h$ for $h = 1,\dots,m$ and $P_{\beta_m} M_\beta = -P_{\beta_m} \sum_{j=0}^{m-1} S_j^+ Q_{j+1} = 0$ because $\operatorname{Im} S_h^+ = \beta_h$. Hence

$$D_\beta(\lambda)F_\beta(\lambda) = P_{\beta_0} + \sum_{h=1}^{m} \left(P_{\beta_h} + S_{h-1}^+ Q_h\right) \lambda^h = \sum_{h=0}^{m} \lambda^h F_h(\lambda).$$

This shows that $\widetilde{F}(\lambda)A(\lambda) = D_\beta(\lambda)F_\beta(\lambda)$, i.e. $A(\lambda) = E_\beta(\lambda)D_\beta(\lambda)F_\beta(\lambda)$, where

$$E_\beta(\lambda) = \widetilde{F}(\lambda)^{-1}, \quad D_\beta(\lambda) = \sum_{h=0}^{m} P_{\beta_h} \lambda^h, \quad F_\beta(\lambda) = I - M_\beta \lambda.$$

Note that $E_\beta(\lambda)$ and $F_\beta(\lambda)$ are analytic and invertible on D_ρ because $F_\beta(0) = I$ and $E_\beta(0) = \widetilde{F}(0)^{-1} = \left(\sum_{h=0}^{m} S_h^+\right)^{-1} = \sum_{h=0}^{m} S_h$. This completes the proof of (2.2) for $\zeta = \beta$. The proof of $A(\lambda) = E_\alpha(\lambda)D_\alpha(\lambda)F_\alpha(\lambda)$ applies similar arguments to $A(\lambda)\widetilde{E}_h(\lambda) = \lambda^h E_h(\lambda)$, $\widetilde{E}_h(0) = S_h^+$, $h = 0,\dots,m$, and it is omitted.

Next it is shown that the operators $E_{\beta,n}$ in $E_\beta(\lambda) = \sum_{n=0}^{\infty} E_{\beta,n}\lambda^n$ satisfy the formulas given in the statement. From $E_\beta(0) = \sum_{h=0}^{m} S_h$ one has $E_{\beta,0} = \sum_{h=0}^{m} S_h$ and because $E_\beta(\lambda) = A(\lambda)F_\beta(\lambda)^{-1}D_\beta(\lambda)^{-1} = \sum_{n=0}^{\infty} E_{\beta,n}\lambda^n$, the formula for $E_{\beta,n}$, $n = 1,2,\dots,$ is found as follows. Let $B_\beta(\lambda) = F_\beta(\lambda)^{-1}D_\beta(\lambda)^{-1}$; from $F_\beta(\lambda)^{-1} = \sum_{n=0}^{\infty} M_\beta^n \lambda^n$ and $D_\beta(\lambda)^{-1} = \sum_{h=0}^{m} P_{\beta_h} \lambda^{-h}$ one has

$$B_\beta(\lambda) = F_\beta(\lambda)^{-1}D_\beta(\lambda)^{-1} = \left(\sum_{n=0}^{\infty} M_\beta^n \lambda^n\right)\left(\sum_{h=0}^{m} P_{\beta_h} \lambda^{-h}\right) = \sum_{n=0}^{\infty} B_{\beta,n} \lambda^{n-m},$$

where

$$B_{\beta,n} = \begin{cases} \sum_{k=0}^{n} M_\beta^k P_{\beta_{m-n+k}}, & n = 0,\dots,m \\ M_\beta^{n-m} B_{\beta,m}, & n = m+1, m+2,\dots \end{cases}. \tag{A.9}$$

Hence

$$E_\beta(\lambda) = A(\lambda)F_\beta(\lambda)^{-1}D_\beta(\lambda)^{-1} = (A_0 + A_1\lambda) \sum_{n=0}^{\infty} B_{\beta,n}\lambda^{n-m}$$

implies

$$\begin{aligned} A_0 B_{\beta,n} + A_1 B_{\beta,n-1} &= 0, && n = 0,\dots,m, \\ E_{\beta,n} &= A_0 B_{\beta,m+n} + A_1 B_{\beta,m+n-1}, && n = 0,1,\dots, \end{aligned}$$

having defined $B_{\beta,-1} = 0$, i.e.

$$E_{\beta,n} = (A_0 M_\beta + A_1) M_\beta^{n-1} B_{\beta,m}, \qquad n = 1, 2, \dots .$$

Because $M_\beta = -\sum_{h=0}^{m-1} S_h^+ Q_{h+1}$, see Definition 2.2, $\operatorname{Ker} A_0 = \beta_1 \oplus \cdots \oplus \beta_m$ and $\operatorname{Im} S_h^+ = \beta_h$, one has $A_0 M_\beta = -A_0 S_0^+ Q_1$. Recall that $S_0 = A_0$ and $Q_1 = A_1$, see Definition 2.2, so that $A_0 S_0^+ Q_1 = P_{\alpha_0} A_1$; hence $A_0 M_\beta + A_1 = (I - P_{\alpha_0}) A_1 = P_{\alpha_0^\perp} A_1$. Thus

$$E_{\beta,n} = P_{\alpha_0^\perp} A_1 M_\beta^{n-1} B_{\beta,m}, \qquad n = 1, 2, \dots . \tag{A.10}$$

This completes the proof for $E_{\beta,n}$, $n = 0, 1, \dots$. The derivation of the formulas for $F_{\alpha,n}$ in the statement applies similar arguments to $F_\alpha(\lambda) = D_\alpha(\lambda)^{-1} E_\alpha(\lambda)^{-1} A(\lambda) = \sum_{n=0}^{\infty} F_{\alpha,n} \lambda^n$ and it is omitted. It introduces the notation

$$C_\alpha(\lambda) = D_\alpha(\lambda)^{-1} E_\alpha(\lambda)^{-1} = \sum_{n=0}^{\infty} C_{\alpha,n} \lambda^{n-m}, \tag{A.11}$$

where

$$C_{\alpha,n} = \begin{cases} \sum_{k=0}^{n} P_{\alpha_{m-n+k}} (M_\alpha^*)^k, & n = 0, \dots, m \\ C_{\alpha,m} (M_\alpha^*)^{n-m}, & n = m+1, m+2, \dots \end{cases} . \quad \blacksquare \tag{A.12}$$

Proof of Corollary 4.3. From (2.2) one has

$$A(\lambda)^{-1} = B_\beta(\lambda) C_\beta(\lambda) = B_\alpha(\lambda) C_\alpha(\lambda) = \sum_{n=0}^{\infty} Z_n \lambda^{n-m},$$

where

$$B_\beta(\lambda) = F_\beta(\lambda)^{-1} D_\beta(\lambda)^{-1} = \sum_{n=0}^{\infty} B_{\beta,n} \lambda^{n-m},$$

$$C_\beta(\lambda) = E_\beta(\lambda)^{-1} = \sum_{n=0}^{\infty} C_{\beta,n} \lambda^n,$$

$$B_\alpha(\lambda) = F_\alpha(\lambda)^{-1} = \sum_{n=0}^{\infty} B_{\alpha,n} \lambda^n,$$

$$C_\alpha(\lambda) = D_\alpha(\lambda)^{-1} E_\alpha(\lambda)^{-1} = \sum_{n=0}^{\infty} C_{\alpha,n} \lambda^{n-m}.$$

The expressions of $B_{\beta,n}$ and $C_{\alpha,n}$ are given in (A.9) and in (A.12) and those of $E_{\beta,k}$ and $F_{\alpha,k}$ are taken from Theorem 2.4. Because $E_\beta(0) = F_\alpha(0) = \sum_{h=0}^m S_h$ is invertible, those of $C_{\beta,n}$ and $B_{\alpha,n}$ are found from $E_\beta(\lambda)C_\beta(\lambda) = F_\alpha(\lambda)B_\alpha(\lambda) = I$, i.e. $\sum_{k=0}^n E_{\beta,k}C_{\beta,n-k} = \sum_{k=0}^n F_{\alpha,k}B_{\alpha,n-k} = 1_{n=0}I$. ■

Proof of Proposition 5.1. We first prove the statement for the X-chains. From (2.2) one has $A(\lambda)F_\beta(\lambda)^{-1} = E_\beta(\lambda)D_\beta(\lambda)$, and substituting $F_\beta(\lambda)^{-1} = \sum_{n=0}^\infty M_\beta^n\lambda^n$ and $D_\beta(\lambda) = \sum_{h=0}^m P_{\beta_h}\lambda^h$ one finds $A(\lambda)\left(\sum_{n=0}^\infty M_\beta^n\lambda^n\right) = E_\beta(\lambda)\left(\sum_{h=0}^m P_{\beta_h}\lambda^h\right)$, which implies

$$A(\lambda)\left(\sum_{n=0}^{\infty} M_\beta^n P_{\beta_h}\lambda^n\right) = \lambda^h E_\beta(\lambda)P_{\beta_h}, \tag{A.13}$$

where $E_\beta(0)P_{\beta_h} = S_h$. Substituting $A(\lambda) = A_0 + \lambda A_1$ and rearranging, one rewrites the l.h.s. of (A.13) as

$$(A_0+\lambda A_1)\left(\sum_{n=0}^{\infty} M_\beta^n P_{\beta_h}\lambda^n\right) = A_0P_{\beta_h} + \sum_{n=1}^{\infty}(A_0M_\beta^n + A_1M_\beta^{n-1})P_{\beta_h}\lambda^n;$$

thus (A.13) implies $A_0P_{\beta_h} = 0$ and

$$(A_0M_\beta^n + A_1M_\beta^{n-1})P_{\beta_h} = \begin{cases} 0, & n = 1,\dots,h-1 \\ S_h, & n = h \end{cases}, \qquad h = 1,\dots,m.$$

That is,

$$A_0x_{h,0} = 0, \qquad A_0x_{h,n} + A_1x_{h,n-1} = \begin{cases} 0, & n = 1,\dots,h-1 \\ S_hx_{h,0} \neq 0, & n = h \end{cases}, \tag{A.14}$$

where $x_{h,0}$ is a nonzero vector in β_h and $x_{h,n} = M_\beta^n x_{h,0}$. This completes the proof of the first part of the statement.

In order to prove the second part of the statement, we wish to show that the vectors $v_1, v_2, \dots, v_m$, where $v_h = \sum_{n=1}^h c_{h,n}x_{h,n-1}$ for some scalars $c_{h,n}$, are linearly independent. Suppose not, i.e. assume that there exists a linear combination $w = \sum_{h=1}^m b_hv_h$ such that $w = 0$ for some scalars $b_1,\dots,b_m$ not all equal to 0. Letting $\mathcal{H} = \{h = 1,\dots,m : b_h \neq 0\}$ be the set of indices of the nonzero scalars in the linear combination, one has $w = \sum_{h\in\mathcal{H}} b_hv_h$, $b_h \neq 0$.

Consider $P_{\alpha_0^\perp}A_1w = \sum_{h\in\mathcal{H}} b_hw_h$, where $w_h = P_{\alpha_0^\perp}A_1v_h$, and observe that (A.14) implies

$$P_{\alpha_0^\perp}A_1x_{h,n-1} = \begin{cases} 0, & n = 1,\dots,h-1 \\ S_hx_{h,0} \neq 0, & n = h \end{cases}, \tag{A.15}$$

because $P_{\alpha_0^\perp} A_0 = 0$ and $P_{\alpha_0^\perp} S_h = S_h$.

Hence $w_h = P_{\alpha_0^\perp} A_1 v_h = \sum_{n=1}^{h} c_{h,n} P_{\alpha_0^\perp} A_1 x_{h,n-1} = c_{h,h} S_h x_{h,0}$, see (A.15). From $w = 0$ it follows that $0 = P_{\alpha_0^\perp} A_1 w = \sum_{h\in\mathcal{H}} b_h w_h$, $b_h \neq 0$, and because the α-subspaces are orthogonal and $w_h \in \alpha_h$, this contradicts $w = 0$ unless $w_h = 0$. Since $w_h = c_{h,h} S_h x_{h,0} = 0$ if and only if $c_{h,h} = 0$, it must be that $v_h = \sum_{n=1}^{h-1} c_{h,n} x_{h,n-1}$.

Then consider $P_{\alpha_0^\perp} A_1 M_\beta w = \sum_{h\in\mathcal{H}} b_h u_h$, where $u_h = P_{\alpha_0^\perp} A_1 M_\beta v_h$; because $M_\beta v_h = \sum_{n=1}^{h-1} c_{h,n} M_\beta x_{h,n-1} = \sum_{n=1}^{h-1} c_{h,n} x_{h,n}$, one has $u_h = P_{\alpha_0^\perp} A_1 M_\beta v_h = \sum_{n=1}^{h-1} c_{h,n} P_{\alpha_0^\perp} A_1 x_{h,n} = c_{h,h-1} S_h x_{h,0}$, see (A.15). From $w = 0$ it follows that $0 = P_{\alpha_0^\perp} A_1 M_\beta w = \sum_{h\in\mathcal{H}} b_h u_h$, $b_h \neq 0$, and because the α-subspaces are orthogonal and $u_h \in \alpha_h$, this contradicts $w = 0$ unless $u_h = 0$. Since $u_h = c_{h,h-1} S_h x_{h,0} = 0$ if and only if $c_{h,h-1} = 0$, it must be that $v_h = \sum_{n=1}^{h-2} c_{h,n} x_{h,n}$. Iterating on the same reasoning, one finds $c_{h,h-2} = \cdots = c_{h,1} = 0$. This shows that $0 = w = \sum_{h\in\mathcal{H}} b_h v_h$, $b_h \neq 0$, implies $v_h = 0$ hence reaching a contradiction. This proves that the vectors $v_1, v_2, \ldots, v_m$ are linearly independent and completes the proof of the statement for the X-chains.

The proof of the statement on the Y-chains (the explicit steps are omitted) applies similar arguments to $E_\alpha(\lambda)^{-1} A(\lambda) = D_\alpha(\lambda) F_\alpha(\lambda)$, see (2.2), and arrives at $P_{\alpha_h} A_0 = 0$ and

$$P_{\alpha_h}((M_\alpha^*)^n A_0 + (M_\alpha^*)^{n-1} A_1) = \begin{cases} 0, & n = 1, \ldots, h-1 \\ S_h, & n = h \end{cases}, \qquad h = 1, \ldots, m,$$

and taking adjoint one has $A_0^* P_{\alpha_h} = 0$ and

$$(A_0^* M_\alpha^n + A_1^* M_\alpha^{n-1}) P_{\alpha_h} = \begin{cases} 0, & n = 1, \ldots, h-1 \\ S_h^*, & n = h \end{cases}, \qquad h = 1, \ldots, m.$$

Hence

$$A_0^* y_{h,0} = 0, \qquad A_0^* y_{h,n} + A_1^* y_{h,n-1} \begin{cases} 0, & n = 1, \ldots, h-1 \\ S_h^* y_{h,0} \neq 0, & n = h \end{cases},$$

where $y_{h,0}$ is a nonzero vector in α_h and $y_{h,n} = M_\alpha^n y_{h,0}$. ■

Proof of Theorem 2.5. Given $n \geq 1$ and vectors $\xi_0, \ldots, \xi_n$ in X, Bart and Lay [10] call $(\xi_0, \ldots, \xi_n)$ is a chain of length $n+1$ if $A_0 \xi_j + A_1 \xi_{j+1} = 0$ for $j = 0, \ldots, n-1$. Here we let $\varphi_{n-j} = \xi_j$, $j = 0, \ldots, n$, and write the chain in reverse order $(\varphi_0, \ldots, \varphi_n) = (\xi_n, \ldots, \xi_0)$, so that $A_0 \varphi_{j+1} + A_1 \varphi_j = 0$ for $j = 0, \ldots, n-1$. In the present notation, the characterization of $\mathscr{N}_n$ given in Bart and Lay [10] is as follows

$$\mathscr{N}_n = \{x \in X : \exists \text{ chain } (\varphi_0, \ldots, \varphi_n) \text{ such that } \varphi_n = x \text{ and } \varphi_0 = 0\}.$$

That is, the chains that are used to construct $\mathscr{N}_n$, $n = 1, \ldots, m$, are such that

$$A_0\varphi_1 = 0, \qquad A_0\varphi_{j+1} + A_1\varphi_j = 0, \qquad j = 1, \dots, n-1, \qquad n = 1, \dots, m.$$

From Proposition 5.1, (replacing n with k) one has

$$A_0 x_{h,0} = 0, \qquad A_0 x_{h,k} + A_1 x_{h,k-1} = 0, \qquad k = 1, \dots, h-1, \qquad h = 1, \dots, m,$$

where $x_{h,0}$ a nonzero vector in β_h and $x_{h,k} = M_\beta^k x_{h,0}$, and

$$A_0 x_{h,h} + A_1 x_{h,h-1} \neq 0, \qquad h = 1, \dots, m.$$

Hence φ_1 can be chosen equal to $x_{h,0}$ for $h = 1, \dots, m$, i.e. $\mathscr{N}_1 = \bigoplus_{h=1}^m \beta_h$, φ_2 can be chosen equal to $x_{h,1} = M_\beta x_{h,0}$ for $h = 2, \dots, m$ and to $x_{h,0}$ for $h = 1, \dots, m$ (setting $\varphi_1 = 0$), i.e. $\mathscr{N}_2 = (\bigoplus_{h=2}^m M_\beta \beta_h) \oplus \mathscr{N}_1 = \bigoplus_{h=1}^m \bigoplus_{k=0}^{\min(h-1,1)} M_\beta \beta_h$, φ_3 can be chosen equal to $x_{h,2} = M_\beta^2 x_{h,0}$ for $h = 3, \dots, m$, to $x_{h,1} = M_\beta x_{h,0}$ for $h = 2, \dots, m$ (setting $\varphi_2 = 0$) and to $x_{h,0}$ for $h = 1, \dots, m$ (setting $\varphi_1 = \varphi_2 = 0$), i.e. $\mathscr{N}_3 = \left(\bigoplus_{h=3}^m M_\beta^2 \beta_h\right) \oplus \mathscr{N}_2 = \bigoplus_{h=1}^m \bigoplus_{k=0}^{\min(h-1,2)} M_\beta^k \beta_h$, and so on. This shows that

$$\mathscr{N}_n = \bigoplus_{h=1}^{m} \bigoplus_{k=0}^{\min(h-1,n-1)} M_\beta^k \beta_h, \qquad n = 1, \dots, m.$$

Hence

$$\mathscr{N}_m = \bigoplus_{h=1}^{m} \bigoplus_{k=0}^{h-1} M_\beta^k \beta_h, \qquad \mathscr{N}'_m = A_1 \mathscr{N}_m = A_1 \bigoplus_{h=1}^{m} \bigoplus_{k=0}^{h-1} M_\beta^k \beta_h.$$

Applying a similar reasoning to $A(\lambda)^* = A_0^* + A_1^*\lambda$ with decompositions $Y = \widetilde{\mathscr{N}_m} \oplus \widetilde{\mathscr{R}}_m$, $X = \widetilde{\mathscr{N}'_m} \oplus \widetilde{\mathscr{R}}'_m$, where $\widetilde{\mathscr{N}}$, $\widetilde{\mathscr{R}}$ correspond to $\mathscr{N}$, $\mathscr{R}$ in the Bart and Lay [10] construction, one finds

$$\widetilde{\mathscr{N}_n} = \bigoplus_{h=1}^{m} \bigoplus_{k=0}^{\min(h-1,n-1)} M_\alpha^k \alpha_h, \qquad n = 1, \dots, m,$$

and hence

$$\widetilde{\mathscr{N}_m} = \bigoplus_{h=1}^{m} \bigoplus_{k=0}^{h-1} M_\alpha^k \alpha_h, \qquad \widetilde{\mathscr{N}'_m} = A_1^* \widetilde{\mathscr{N}_m} = A_1^* \bigoplus_{h=1}^{m} \bigoplus_{k=0}^{h-1} M_\alpha^k \alpha_h.$$

Because $Y = \widetilde{\mathscr{N}_m} \oplus \widetilde{\mathscr{R}}_m = \mathscr{N}'_m \oplus \mathscr{R}'_m$ and $X = \widetilde{\mathscr{N}'_m} \oplus \widetilde{\mathscr{R}}'_m = \mathscr{N}_m \oplus \mathscr{R}_m$, where $\widetilde{\mathscr{N}_m}$ ($\widetilde{\mathscr{R}}_m$) is isomorphic to $\mathscr{N}'_m$ ($\mathscr{R}'_m$) and $\widetilde{\mathscr{N}'_m}$ ($\widetilde{\mathscr{R}}'_m$) is isomorphic to $\mathscr{N}_m$ ($\mathscr{R}_m$), one can take

$$\mathscr{R}'_m = (\widetilde{\mathscr{N}_m})^\perp = \left(\bigoplus_{h=1}^{m} \bigoplus_{k=0}^{h-1} M_\alpha^k \alpha_h\right)^\perp, \quad \mathscr{R}_m = (\widetilde{\mathscr{N}'_m})^\perp = \left(A_1^* \bigoplus_{h=1}^{m} \bigoplus_{k=0}^{h-1} M_\alpha^k \alpha_h\right)^\perp. \quad \blacksquare$$

Proof of Theorem 2.6. From Corollary 4.3 one has

$$Z_n = \sum_{k=0}^{n} B_{\beta,k} C_{\beta,n-k} = \sum_{k=0}^{n} B_{\alpha,n-k} C_{\alpha,k}, \qquad n = 0, 1, \dots ,$$

where

$$B_{\beta,k} = \begin{cases} \sum_{j=0}^{k} M_\beta^j P_{\beta_{m-k+j}} \\ M_\beta^{k-m} B_{\beta,m} \end{cases},$$

$$C_{\alpha,k} = \begin{cases} \sum_{j=0}^{k} P_{\alpha_{m-k+j}} (M_\alpha^*)^j \\ C_{\alpha,m} (M_\alpha^*)^{k-m} \end{cases}, \qquad \begin{matrix} k = 0, \dots, m \\ k = m+1, m+2, \dots \end{matrix} .$$

Observe that, because of linear independence of the X and Y-chains, one has

$$\operatorname{Im} Z_n = \bigoplus_{k=0}^{n} \operatorname{Im} B_{\beta,k}, \qquad \operatorname{Im} Z_n^* = \bigoplus_{k=0}^{n} \operatorname{Im} C_{\alpha,k}^*, \qquad n = 0, \dots, m-1, \tag{A.16}$$

where

$$\operatorname{Im} B_{\beta,k} = \bigoplus_{j=0}^{k} M_\beta^j \beta_{m-k+j}, \qquad \operatorname{Im} C_{\alpha,k}^* = \bigoplus_{j=0}^{k} M_\alpha^j \alpha_{m-k+j}, \qquad k = 0, \dots, m-1.$$

Thus

$$\operatorname{Im} Z_n = \bigoplus_{h=m-n}^{m} \bigoplus_{k=0}^{h-m+n} M_\beta^k \beta_h, \quad \operatorname{Im} Z_n^* = \bigoplus_{h=m-n}^{m} \bigoplus_{k=0}^{h-m+n} M_\alpha^k \alpha_h, \quad n = 0, \dots, m-1.$$

Because $\operatorname{Ker} Z_n = (\operatorname{Im} Z_n^*)^\perp$, this completes the proof of the statement. ∎

Proof of Corollary 5.2. From $\operatorname{Im} Z_n = \bigoplus_{k=0}^{n} \operatorname{Im} B_{\beta,k}$, $n = 0, \dots, m-1$, see (A.16), one has

$$\operatorname{Im} Z_n = \left(\bigoplus_{k=0}^{n-1} \operatorname{Im} B_{\beta,k} \right) \oplus \operatorname{Im} B_{\beta,n} = \operatorname{Im} Z_{n-1} \oplus \xi_{\beta,n}, \qquad n = 1, \dots, m-1,$$

where $\xi_{\beta,n} = \operatorname{Im} B_{\beta,n} \supseteq M_\beta^n \beta_m \neq 0$ because of linear independence of the X-chains. This proves the statement about $\operatorname{Im} Z_n$. Similarly, from the second equation in (A.16) one has

$$\operatorname{Im} Z_n^* = \operatorname{Im} Z_{n-1}^* \oplus \xi_{\alpha,n}, \qquad n = 1, \dots, m-1,$$

where $\xi_{\alpha,n} = \operatorname{Im} C_{\alpha,n}^* \supseteq M_\alpha^n \alpha_m \neq 0$ because of linear independence of the Y-chains. Hence

$$(\operatorname{Im} Z_n^*)^\perp = (\operatorname{Im} Z_{n-1}^* \oplus \xi_{\alpha,n})^\perp = (\operatorname{Im} Z_{n-1}^*)^\perp \cap \xi_{\alpha,n}^\perp,$$

i.e.

$$\operatorname{Ker} Z_n = \ker Z_{n-1} \cap \xi_{\alpha,n}^\perp, \qquad n = 1, \dots, m-1. \quad \blacksquare$$

Proof of Corollary 6.1. Because of linear independence of the vectors in the X-chains, one has $\dim(M_\beta^k \beta_h) = \dim \beta_h = r_h$ and

$$\dim \operatorname{Im} Z_n = \sum_{h=m-n}^{m} \sum_{k=0}^{h-m+n} \dim(M_\beta^k \beta_h) = \sum_{h=m-n}^{m} (h-m+n+1) r_h,$$

i.e. $\operatorname{rank} Z_n = \sum_{h=0}^{n} (h+1) r_{m-n+h}$, $n = 0, \dots, m-1$. Because $\dim \mathscr{N}_m = \dim \mathscr{N}_m' = \operatorname{rank} Z_{m-1}$, this proves the statement. $\blacksquare$

Proof of Corollary 6.3. Let q be the number of distinct partial multiplicities of $A(\lambda)$ at 0 and for $h = 1, \dots, q$, let m_h be one of the distinct partial multiplicities. That is, write $0 < \kappa_1 \le \kappa_2 \le \dots \le \kappa_s < \infty$ as $0 < m_1 < m_2 < \dots < m_q < \infty$, where $m_1 = \kappa_1$ and $m_q = \kappa_s$ and let s_h be the number of non distinct partial multiplicities κ_j that are equal to m_h. Consider $D(\lambda) = D_0 + D_1 \lambda^{\kappa_1} + \dots + D_s \lambda^{\kappa_s}$ and sum the projections that load the same partial multiplicity into $D(\lambda)$, i.e. let

$$M_0 = D_0, \qquad M_1 = D_1 + \dots + D_{s_1},$$
$$M_2 = D_{s_1+1} + \dots + D_{s_1+s_2}, \qquad \dots \qquad M_q = D_{\sum_{h=1}^{q-1} s_h + 1} + \dots + D_s,$$

so that $D(\lambda) = M_0 + M_1 \lambda^{m_1} + \dots + M_q \lambda^{m_q}$, where $M_0, M_1, \dots, M_q$ are mutually disjoint projections that decompose the identity, M_0 is Fredholm of index 0, and $\operatorname{rank} M_h = s_h < \infty$, $h = 1, \dots, q$. Remark that q, $\{m_1, m_2, \cdots, m_q\}$ and $\{s_1, s_2, \cdots, s_q\}$ are uniquely defined. By Theorem 2.4 one can take $D(\lambda) = \sum_{h=0}^{m} P_{\beta_h} \lambda^h$ (or $D(\lambda) = \sum_{h=0}^{m} P_{\alpha_h} \lambda^h$), where $\dim \alpha_h = \dim \beta_h = r_h$; hence one has that $q = \sum_{h=1}^{m} 1_{\{r_h > 0\}}$, $\{m_1, m_2, \cdots, m_q\} = \{h = 1, \dots, m : r_h > 0\}$ and $\{s_1, s_2, \cdots, s_q\} = \{r_h, h = 1, \dots, m : r_h > 0\}$. Thus the statement. $\blacksquare$

Proof of Theorem 2.7. See the proofs of Corollaries 6.1, 6.2 and 6.3. $\blacksquare$

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