A NOTE ON AN AVERAGE ADDITIVE PROBLEM WITH PRIME NUMBERS

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ABSTRACT. We continue our investigations on the average number of representations of a large positive integer as a sum of given powers of prime numbers. The average is taken over a "short" interval, whose admissible length depends on whether or not we assume the Riemann Hypothesis.

1. INTRODUCTION

We pursue recent investigations by the present authors and Alessandro Languasco in [4] and [3]. In this short note we study general average additive problems: let $\mathbf{k} = (k_1, k_2, \dots, k_r)$, where $2 \le k_1 \le k_2 \le \cdots \le k_r$ and k_j is an integer for all $j \in \{1, \dots, r\}$. Let

$$R(n;\mathbf{k}) = \sum_{n=m_1^{k_1}+\cdots+m_r^{k_r}} \Lambda(m_1)\cdots\Lambda(m_r), \tag{1}$$

where Λ is the von Mangoldt function, that is, $\Lambda(p^m) = \log(p)$ if p is a prime number and m is a positive integer, and $\Lambda(n) = 0$ for all other integers. We write $\rho = \rho(\mathbf{k}) = k_1^{-1} + \cdots + k_r^{-1}$, for the "density" of the problem, $\gamma_k = \Gamma(1 + 1/k)$ where Γ is the Euler Gamma-function and $G(\mathbf{k}) = \gamma_{k_1} \cdots \gamma_{k_r}$.

Proving the expected individual asymptotic formula for $R(n;\mathbf{k})$ as $n \to \infty$ along "admissible" residue classes (that is, avoiding those residue classes which can not contain values of the form $p_1^{k_1} + \cdots + p_r^{k_r}$ because of the uneven distribution of prime powers in residue classes) is very difficult if either r or ρ is small. Many authors studied the density of exceptions to the weaker conjecture that $R(n;\mathbf{k}) > 0$ for sufficiently large n belonging to an admissible residue class. The most interesting case is the binary Goldbach problem, which corresponds to $\mathbf{k} = (1,1)$; Pintz [16] proved that the number of even integers $n \le N$ such that $R(n,\mathbf{k}) = 0$ is bounded by $N^{2/3+\varepsilon}$ for large N. Another case, among many, that attracted much attention corresponds to $\mathbf{k} = (2,2,k)$, and strong bounds for "exceptional" integers up to some large N have been given, the latest being due to Liu & Zhang [15].

Other approaches to the "weak" conjecture that $R(n;\mathbf{k}) > 0$, provided that n is large enough and there are no arithmetical obstructions, are also possible. In fact, our main goal here is to give an asymptotic formula for the average value of $R(n;\mathbf{k})$ for $n \in [N+1, N+H]$ where $N \to +\infty$ and $H = H(N;\mathbf{k})$ is as small as possible. Here we assume $r \ge 3$, since binary problems have been thoroughly studied recently in a series of papers by Languasco and the last author [9], [11], [12], [13], and by Suzuki [17]. We also recall that averages of $R(n;\mathbf{k})$ where $\mathbf{k} = (1,1)$, that is, in the case of the binary Goldbach problem, have been studied assuming the Riemann Hypothesis with the goal of giving a development into a main term and also "secondary"terms depending on the zeros of the Riemann zeta-function. This problem has been introduced by Fujii [5]; the most recent result is due to Languasco and the last Author, [7].

In fact, whereas the determination of an upper bound for the exceptional set in additive problems like the ones described above is quite classical, the average ones we are concerned with

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here attracted some attention only recently, especially in the case where ρ is quite small. We recall that *weighted* averages of $R(n;\mathbf{k})$ in the case of the binary Goldbach problem and weights of the Cesàro-Riesz type, were intensively studied in recent times: after some preliminary work of Languasco and the last Author [8], and by Goldston & Yang [6], the problem was completely solved in a recent paper by Brüdern, Kaczorowski & Perelli [1]. For a similar problem, see also Cantarini [2].

We now present our main results.

Theorem 1.1. *Let $\mathbf{k} = (k_1,\dots,k_r)$, where $2 \le k_1 \le \cdots \le k_r$, be an r-tuple of integers with $r \ge 3$. For every $\varepsilon > 0$ there exists a constant $C = C(\varepsilon) > 0$, independent of $\mathbf{k}$, such that*

$$\sum_{n=N+1}^{N+H} R(n;\mathbf{k}) = \frac{G(\mathbf{k})}{\Gamma(\rho)} H N^{\rho-1} + O_{\mathbf{k}}\Big(H N^{\rho-1} \exp\Big\{ -C\Big(\frac{\log N}{\log\log N}\Big)^{1/3} \Big\}\Big)$$

as $N \to +\infty$, uniformly for $N^{1-5/(6k_r)+\varepsilon} < H < N^{1-\varepsilon}$.

It is well known that the Riemann Hypothesis (RH for short) implies that prime numbers are fairly regularly distributed. In this problem, it has the effect of allowing far wider ranges for H, that is, much smaller values of H are admissible than in Theorem 1.1. The final error term is also smaller, as it is to be expected.

We use throughout the paper the convenient notation $f = \infty(g)$ as equivalent to $g = o(f)$.

Theorem 1.2. *Assume the Riemann Hypothesis. Let $\mathbf{k} = (k_1,\dots,k_r)$, where $2 \le k_1 \le \cdots \le k_r$, be an r-tuple of integers with $r \ge 3$. For every $\varepsilon > 0$ there exists a constant $C = C(\varepsilon) > 0$, independent of $\mathbf{k}$, such that*

$$\sum_{n=N+1}^{N+H} R(n;\mathbf{k}) = \frac{G(\mathbf{k})}{\Gamma(\rho)} H N^{\rho-1} + O_{\mathbf{k}}\Big(H^2 N^{\rho-2} + H^{1/2} N^{\rho-1/2-1/(2k_r)} L^3 \Big)$$

as $N \to +\infty$, uniformly for $H = \infty\big(N^{1-1/k_r}(\log N)^6\big)$ with $H < N^{1-\varepsilon}$.

Theorem 1.1 contains as special cases all results in [4] and [3], whereas Theorem 1.2 is occasionally slightly weaker because our basic combinatorial identity here, equation (2), is less efficient than the identities we used in the papers mentioned above. The results we obtain are essentially what one expects in view of earlier work in this field, though here we have some combinatorial complications arising from the decomposition of our exponential sums as "main term" plus "remainder term," see §2 for details, and in handling the ensuing terms, which were overcome in previous papers such as [4] by means of "ad hoc" identities. Here we need to work somewhat more in order to get the corresponding results in general, because of the number of "error terms" that we have to deal with.

2. Definitions and preparation for the proofs

We rewrite $R(N;\mathbf{k})$ as the integral over the unit interval of the product of suitable exponential sums. We then proceed to "replace" each exponential sum by its approximation, which is given by the leading term of the Prime Number Theorem. This gives rise to the main term and also to a number of additional terms that we have to bound in various ways. Let $S_j = x_j + y_j$ for $j \in \{1, \dots, r\}$: then we have

$$\prod_{j=1}^{r} S_j = \prod_{j=1}^{r} (x_j + y_j) = \prod_{j=1}^{r} x_j + \mathfrak{A} + \mathfrak{B}, \tag{2}$$

where

$$\mathfrak{A} = \sum_{i=1}^{r} y_i \Bigl(\prod_{j\neq i} S_j\Bigr), \tag{3}$$

$$\mathfrak{B} = \sum_{\substack{I\subseteq\{1,\dots,r\}\\ |I|\geq 2}} c_r(I) \Bigl(\prod_{i\in\{1,\dots,r\}\setminus I} x_i\Bigr)\Bigl(\prod_{i\in I} y_i\Bigr), \tag{4}$$

for suitable coefficients $c_r(I)$. In fact, according to the definitions (5) and (10) below, we will choose $\widetilde{S}_{k_j}(\alpha) = S_j = x_j + y_j$ where $x_j = x_j(\alpha) = \gamma_j z^{-1/k_j}$ and $y_j = y_j(\alpha) = \widetilde{\mathcal{E}}_{k_j}(\alpha)$, so that we can exploit the fact that $S_j, x_j, y_j \ll N^{1/k_j}$ and that y_j is small in L^2-norm by Lemma 3.1 below.

For real α we write $\mathrm{e}(\alpha) = \mathrm{e}^{2\pi i\alpha}$. We take N as a large positive integer, and write $L = \log N$ for brevity. In this and in the following section k denotes any positive real number. Let $z = 1/N - 2\pi i\alpha$ and

$$\widetilde{S}_k(\alpha) = \sum_{n\geq 1} \Lambda(n)\mathrm{e}^{-n^k/N}\mathrm{e}(n^k\alpha) = \sum_{n\geq 1} \Lambda(n)\mathrm{e}^{-n^k z}. \tag{5}$$

Thus, recalling definition (1) and using (5), for all $n \geq 1$ we have

$$R(n;\mathbf{k}) = \sum_{n_1^{k_1}+\cdots+n_r^{k_r}=n} \Lambda(n_1)\cdots\Lambda(n_r) = \mathrm{e}^{n/N}\int_{-1/2}^{1/2} \widetilde{S}_{k_1}(\alpha)\cdots\widetilde{S}_{k_r}(\alpha)\,\mathrm{e}(-n\alpha)\,\mathrm{d}\alpha. \tag{6}$$

It is clear from the above identity that we are only interested in the range $\alpha \in [-1/2, 1/2]$. We record here the basic inequality

$$|z|^{-1} \ll \min\{N, |\alpha|^{-1}\}. \tag{7}$$

We also need the following exponential sum over the "short interval" $[1,H]$

$$U(\alpha,H) = \sum_{m=1}^{H} \mathrm{e}(m\alpha),$$

where $H \leq N$ is a large integer. We recall the simple inequality

$$|U(\alpha,H)| \leq \min\{H, |\alpha|^{-1}\}. \tag{8}$$

With these definitions in mind and recalling (6), our starting point is the identity

$$\sum_{n=N+1}^{N+H} \mathrm{e}^{-n/N} R(n;\mathbf{k}) = \int_{-1/2}^{1/2} \widetilde{S}_{k_1}(\alpha)\cdots\widetilde{S}_{k_r}(\alpha) U(-\alpha,H)\,\mathrm{e}(-N\alpha)\,\mathrm{d}\alpha. \tag{9}$$

The basic strategy is to replace each factor $\widetilde{S}_k(\alpha)$ by its expected main term, which is $\gamma_k/z^{1/k}$, and estimating the ensuing error term by means of a combination of techniques and bounds for exponential sums, with the aid of (2). One key ingredient is the L^2-bound in Lemma 3.1, which we may use only in a restricted range, and we need a different argument on the remaining part of the integration interval. This leads to some complications in details. The conditional case, when the Riemann Hypothesis is assumed, has a somewhat simpler proof, as we see in §5.

3. Lemmas

For brevity, we set

$$\widetilde{\mathcal{E}}_k(\alpha) := \widetilde{S}_k(\alpha) - \frac{\gamma_k}{z^{1/k}} \qquad \text{and} \qquad A(N;c) := \exp\Bigl\{c\Bigl(\frac{\log N}{\log\log N}\Bigr)^{1/3}\Bigr\}, \tag{10}$$

where c is a real constant.

Lemma 3.1 (Lemma 3 of [10]). *Let ε be an arbitrarily small positive constant, $k \ge 1$ be an integer, N be a sufficiently large integer and $L = \log N$. Then there exists a positive constant $c_1 = c_1(\varepsilon)$, which does not depend on k, such that*

$$\int_{-\xi}^{\xi} \left|\widetilde{\mathcal{E}}_k(\alpha)\right|^2 \mathrm{d}\alpha \ll_k N^{2/k-1} A(N;-c_1)$$

uniformly for $0 \le \xi < N^{-1+5/(6k)-\varepsilon}$. Assuming the Riemann Hypothesis we have

$$\int_{-\xi}^{\xi} \left|\widetilde{\mathcal{E}}_k(\alpha)\right|^2 \mathrm{d}\alpha \ll_k N^{1/k} \xi L^2$$

uniformly for $0 \le \xi \le 1/2$.

We remark that the proof of Lemma 3 in [10] contains oversights which are corrected in [14]. The next result is a variant of Lemma 4 of [10], which is fully proved in [4].

Lemma 3.2. *Let N be a positive integer, $z = z(\alpha) = 1/N - 2\pi i\alpha$, and $\mu > 0$. Then, uniformly for $n \ge 1$ and $X > 0$ we have*

$$\int_{-X}^{X} z^{-\mu} \mathrm{e}(-n\alpha)\,\mathrm{d}\alpha = \mathrm{e}^{-n/N} \frac{n^{\mu-1}}{\Gamma(\mu)} + O_\mu\left(\frac{1}{nX^\mu}\right).$$

Lemma 3.3 (Lemma 3.3 of [4]). *We have $\widetilde{S}_k(\alpha) \ll_k N^{1/k}$.*

We record an immediate consequence of (7), (10) and Lemma 3.3:

$$\widetilde{\mathcal{E}}_k(\alpha) \ll_k N^{1/k}. \tag{11}$$

Our next tool is the extension to $\widetilde{S}_k$ of Lemma 7 of Tolev [18]. The proof can be found in [3].

Lemma 3.4. *Let $k > 1$ and $\tau > 0$. Then*

$$\int_{-\tau}^{\tau} |\widetilde{S}_k(\alpha)|^2 \mathrm{d}\alpha \ll \left(\tau N^{1/k} + N^{2/k-1}\right) L^3.$$

Lemma 3.5 (Lemma 3.6 of [4]). *For $N \to +\infty$, $H \in [1,N]$ and a real number λ we have*

$$\sum_{n=N+1}^{N+H} \mathrm{e}^{-n/N} n^\lambda = \frac{1}{\mathrm{e}} H N^\lambda + O_\lambda\left(H^2 N^{\lambda-1}\right).$$

4. Proof of Theorem 1.1

We need to introduce another parameter $B = B(N)$, defined as

$$B = N^{2\varepsilon}. \tag{12}$$

We can not take $B = 1$, because of the estimate in §4.4. We let $\mathcal{C} = \mathcal{C}(B,H) = [-1/2, -B/H] \cup [B/H, 1/2]$, and write $\widetilde{S}_{k_j}(\alpha) = x_j + y_j$ where $x_j = x_j(\alpha) = \gamma_j z^{-1/k_j}$ and $y_j = y_j(\alpha) = \widetilde{\mathcal{E}}_{k_j}(\alpha)$ in (2), so that

$$\widetilde{S}_{k_1}(\alpha) \cdots \widetilde{S}_{k_r}(\alpha) = \prod_{j=1}^{r} (x_j + y_j) = \prod_{j=1}^{r} x_j(\alpha) + \mathfrak{A}(\alpha) + \mathfrak{B}(\alpha), \tag{13}$$

where $\mathfrak{A}(\alpha)$ and $\mathfrak{B}(\alpha)$ are defined by (3) and (4) respectively. We multiply (13) by $U(-\alpha,H) \times \mathrm{e}(-N\alpha)$ and integrate over the interval $[-B/H, B/H]$. Recalling (9) we have

$$\sum_{n=N+1}^{N+H} \mathrm{e}^{-n/N} R(n;\mathbf{k}) = G(\mathbf{k}) \int_{-B/H}^{B/H} \frac{U(-\alpha,H)}{z^\rho} \mathrm{e}(-N\alpha)\,\mathrm{d}\alpha$$

$$
\begin{aligned}
&+\int_{-B/H}^{B/H} \mathfrak{A}(\alpha) U(-\alpha,H) \mathrm{e}(-N\alpha)\,\mathrm{d}\alpha \\
&+\int_{-B/H}^{B/H} \mathfrak{B}(\alpha) U(-\alpha,H) \mathrm{e}(-N\alpha)\,\mathrm{d}\alpha \\
&+\int_{C} \widetilde{S}_{k_1}(\alpha)\cdots\widetilde{S}_{k_r}(\alpha) U(-\alpha,H) \mathrm{e}(-N\alpha)\,\mathrm{d}\alpha \\
=\ & G(\mathbf{k}) I_1 + I_2 + I_3 + I_4,
\end{aligned}
$$

say. The first summand gives rise to the main term via Lemma 3.2, the next two are majorised in §4.2–4.3 by means of Lemma 3.3 and the L^2-estimate provided by Lemma 3.1. Finally, I_4 is easy to bound using Lemma 3.4.

4.1. **Evaluation of I_1.** It is a straightforward application of Lemma 3.2: we have

$$\int_{-B/H}^{B/H} \frac{U(-\alpha,H)}{z^{\rho}} \mathrm{e}(-N\alpha)\,\mathrm{d}\alpha = \frac{1}{\Gamma(\rho)} \sum_{n=N+1}^{N+H} \mathrm{e}^{-n/N} n^{\rho-1} + O_{\mathbf{k}}\Big(\frac{H}{N}\Big(\frac{H}{B}\Big)^{\rho}\Big). \tag{14}$$

We evaluate the sum on the right-hand side of (14) by means of Lemma 3.5 with $\lambda = \rho - 1$. Summing up, we have

$$\int_{-B/H}^{B/H} \frac{U(-\alpha,H)}{z^{\rho}} \mathrm{e}(-N\alpha)\,\mathrm{d}\alpha = \frac{1}{\mathrm{e}\Gamma(\rho)} H N^{\rho-1} + O_{\mathbf{k}}\Big(H^2 N^{\rho-2} + \frac{H}{N}\Big(\frac{H}{B}\Big)^{\rho}\Big). \tag{15}$$

We now choose the range for H: since we will need Lemma 3.1, we see that we can take

$$H > N^{1-5/(6k_r)+3\varepsilon}. \tag{16}$$

4.2. **Bound for I_2.** We recall the bound (8), and Lemmas 3.3 and 3.4. Using Lemma 3.1 and the Cauchy-Schwarz inequality where appropriate, we see that the contribution from $\widetilde{S}_{k_1}(\alpha)\times \cdots \widetilde{S}_{k_{r-1}}(\alpha) y_r$, say, is

$$
\begin{aligned}
&\ll_{\mathbf{k}} H \max_{\alpha\in[-1/2,1/2]} |\widetilde{S}_{k_1}(\alpha)\cdots\widetilde{S}_{k_{r-2}(\alpha)}| \Big(\int_{-B/H}^{B/H} |\widetilde{S}_{k_{r-1}}(\alpha)|^2\,\mathrm{d}\alpha \int_{-B/H}^{B/H} |\widetilde{\mathfrak{E}}_{k_r}(\alpha)|^2\,\mathrm{d}\alpha\Big)^{1/2} \\
&\ll_{\mathbf{k}} H N^{1/k_1+\cdots+1/k_{r-2}} L^{3/2} \Big(\frac{B}{H} N^{1/k_{r-1}} + N^{2/k_{r-1}-1}\Big)^{1/2} \big(N^{2/k_r-1} A(N;-c_1)\big)^{1/2} \\
&\ll_{\mathbf{k}} H N^{\rho-1} A\Big(N; -\frac{1}{3}c_1\Big),
\end{aligned} \tag{17}
$$

where $c_1 = c_1(\varepsilon) > 0$ is the constant provided by Lemma 3.1, which we can use on the interval $[-B/H, B/H]$ since B and H satisfy (12) and (16) respectively. The other summands in I_2 are treated in the same way.

4.3. **Bound for I_3.** We remark that, by definition (4), each summand in $\mathfrak{B}(\alpha)$ is the product of r factors chosen among the x_js and the y_js, with at least two of the latter type. Using (7), (8) and Lemma 3.1, by the Cauchy-Schwarz inequality, we see that the contribution from the term $y_1 y_2 x_3 \ldots x_r$, say, is

$$
\begin{aligned}
&= \gamma_{k_3}\cdots\gamma_{k_r} \int_{-B/H}^{B/H} \frac{\widetilde{\mathfrak{E}}_{k_1}(\alpha)\widetilde{\mathfrak{E}}_{k_2}(\alpha)}{z^{1/k_3+\cdots+1/k_r}} U(-\alpha,H)\mathrm{e}(-N\alpha)\,\mathrm{d}\alpha \\
&\ll_{\mathbf{k}} H N^{1/k_3+\cdots+1/k_r} \Big(\int_{-B/H}^{B/H} |\widetilde{\mathfrak{E}}_{k_1}(\alpha)|^2\,\mathrm{d}\alpha \int_{-B/H}^{B/H} |\widetilde{\mathfrak{E}}_{k_2}(\alpha)|^2\,\mathrm{d}\alpha\Big)^{1/2} \\
&\ll_{\mathbf{k}} H N^{\rho-1} A(N; -c_1).
\end{aligned} \tag{18}
$$

Furthermore, we recall the bound $\widetilde{\mathcal{E}}_k(\alpha) \ll_k N^{1/k}$ in (11). Hence we may treat the other summands in I_3 in the same way, since $x_j, y_j \ll_{k_j} N^{1/k_j}$ for $j \in \{1,\dots,r\}$.

4.4. **Bound for I_4.** Using a partial integration from Lemma 3.4 and the Cauchy-Schwarz inequality, we have

$$\begin{aligned} I_4 &= \int_C \widetilde{S}_{k_1}(\alpha)\cdots\widetilde{S}_{k_r}(\alpha)U(-\alpha,H)\mathrm{e}(-N\alpha)\,\mathrm{d}\alpha \\ &\ll_{\mathbf{k}} \max_{\alpha\in[-1/2,1/2]} |\widetilde{S}_{k_1}(\alpha)\cdots\widetilde{S}_{k_{r-2}}(\alpha)|\Big(\int_C |\widetilde{S}_{k_{r-1}}(\alpha)|^2\frac{\mathrm{d}\alpha}{|\alpha|}\int_C |\widetilde{S}_{k_r}(\alpha)|^2\frac{\mathrm{d}\alpha}{|\alpha|}\Big)^{1/2} \\ &\ll_{\mathbf{k}} N^{1/k_1+\cdots+1/k_{r-2}}\Big(\frac{H^2}{B^2}N^{2/k_{r-1}+2/k_r-2}L^6\Big)^{1/2} \ll_{\mathbf{k}} \frac{H}{B}N^{\rho-1}L^3, \end{aligned} \tag{19}$$

because of (16). This is $\ll_{\mathbf{k}} HN^{\rho-1}A(N;-c_1/3)$, by our choice in (12).

4.5. **Completion of the proof.** For simplicity, from now on we assume that $H \le N^{1-\varepsilon}$. Summing up from (15), (17), (18) and (19), we proved that

$$\sum_{n=N+1}^{N+H} \mathrm{e}^{-n/N}R(n;\mathbf{k}) = \frac{G(\mathbf{k})}{\mathrm{e}\Gamma(\rho)}HN^{\rho-1} + O_{\mathbf{k}}\left(HN^{\rho-1}A\Big(N;-\frac{1}{3}c_1\Big)\right), \tag{20}$$

provided that (12) and (16) hold, since the other error terms are smaller in our range for H. In order to achieve the proof, we have to remove the exponential factor on the left-hand side, exploiting the fact that, since H is "small," it does not vary too much over the summation range. Since $\mathrm{e}^{-n/N} \in [\mathrm{e}^{-2},\mathrm{e}^{-1}]$ for all $n \in [N+1,N+H]$, we can easily deduce from (20) that

$$\mathrm{e}^{-2}\sum_{n=N+1}^{N+H} R(n;\mathbf{k}) \le \sum_{n=N+1}^{N+H} \mathrm{e}^{-n/N}R(n;\mathbf{k}) \ll_{\mathbf{k}} HN^{\rho-1}.$$

We can use this weak upper bound to majorise the error term arising from the development $\mathrm{e}^{-x} = 1+O(x)$ that we need in the left-hand side of (20). In fact, we have

$$\begin{aligned} \sum_{n=N+1}^{N+H} \mathrm{e}^{-n/N}R(n;\mathbf{k}) &= \sum_{n=N+1}^{N+H} \big(\mathrm{e}^{-1} + O\big((n-N)N^{-1}\big)\big)R(n;\mathbf{k}) \\ &= \mathrm{e}^{-1}\sum_{n=N+1}^{N+H} R(n;\mathbf{k}) + O_{\mathbf{k}}\big(H^2N^{\rho-2}\big). \end{aligned}$$

Finally, substituting back into (20), we obtain the required asymptotic formula for H as in the statement of Theorem 1.1.

5. Proof of Theorem 1.2

Here we assume the Riemann Hypothesis: as we mentioned above, we obtain stronger results (wider ranges for H, better error term) and the proof is simpler because Lemma 3.1 applies to the whole unit interval. In fact, we use identity (13) over $[-1/2,1/2]$. Recalling (9) we have

$$\begin{aligned} \sum_{n=N+1}^{N+H} \mathrm{e}^{-n/N}R(n;\mathbf{k}) = G(\mathbf{k})&\int_{-1/2}^{1/2} \frac{U(-\alpha,H)}{z^\rho}\mathrm{e}(-N\alpha)\,\mathrm{d}\alpha \\ &+ \int_{-1/2}^{1/2} \mathfrak{A}(\alpha)U(-\alpha,H)\mathrm{e}(-N\alpha)\,\mathrm{d}\alpha \\ &+ \int_{-1/2}^{1/2} \mathfrak{B}(\alpha)U(-\alpha,H)\mathrm{e}(-N\alpha)\,\mathrm{d}\alpha \end{aligned}$$

$$= G(\mathbf{k})I_1 + I_2 + I_3,$$

say. For the main term we use Lemma 3.2 over $[-1/2,1/2]$ and then Lemma 3.5 with $\lambda = \rho - 1$, obtaining

$$\int_{-1/2}^{1/2} \frac{U(-\alpha,H)}{z^{\rho}} e(-N\alpha)\, d\alpha = \frac{1}{e\Gamma(\rho)} HN^{\rho-1} + O_{\mathbf{k}}\Big(H^2N^{\rho-2} + \frac{H}{N}\Big). \tag{21}$$

For the other terms, we split the integration range at $1/H$. We use Lemma 3.1 and (8) on the interval $[-1/H,1/H]$, and a partial-integration argument from Lemma 3.1 in the remaining range. In view of future constraints (see (30) below) we assume that

$$H \ge N^{1-1/k_r}L. \tag{22}$$

We start bounding the contribution of the term $\widetilde{S}_{k_1}(\alpha)\cdots\widetilde{S}_{k_{r-1}}(\alpha)y_r$ in $\mathfrak{A}(\alpha)$ over $[-1/H,1/H]$. We have that it is

$$\begin{aligned}
&\ll_{\mathbf{k}} H \max_{\alpha\in[-1/2,1/2]} |\widetilde{S}_{k_1}(\alpha)\cdots\widetilde{S}_{k_{r-2}}(\alpha)| \Big(\int_{-1/H}^{1/H} |\widetilde{S}_{k_{r-1}}(\alpha)|^2\, d\alpha \int_{-1/H}^{1/H} |\widetilde{\mathcal{E}}_{k_r}(\alpha)|^2\, d\alpha\Big)^{1/2}\\
&\ll_{\mathbf{k}} HN^{1/k_1+\cdots+1/k_{r-2}}L^{3/2}\Big(\frac{1}{H}N^{1/k_{r-1}} + N^{2/k_{r-1}-1}\Big)^{1/2}\big(N^{1/k_r}H^{-1}L^2\big)^{1/2}\\
&\ll_{\mathbf{k}} H^{1/2}N^{\rho-1/2-1/(2k_r)}L^{5/2},
\end{aligned} \tag{23}$$

by Lemma 3.1, since we assumed (22). The same bound holds for other summands in I_2. As above, we remark that $\mathfrak{B}$ is a finite sum of summands which are products of x_js and y_js, with at least two factors of the latter type. For example, we bound the contribution from the term $x_1\ldots x_{r-2}y_{r-1}y_r$ in $\mathfrak{B}(\alpha)$ on the same interval: it is

$$\begin{aligned}
&= \gamma_{k_1}\cdots\gamma_{k_{r-2}} \int_{-1/H}^{1/H} \frac{\widetilde{\mathcal{E}}_{k_{r-1}}(\alpha)\widetilde{\mathcal{E}}_{k_r}(\alpha)}{z^{1/k_1+\cdots+1/k_{r-2}}} U(-\alpha,H) e(-N\alpha)\, d\alpha\\
&\ll_{\mathbf{k}} HN^{1/k_1+\cdots+1/k_{r-2}}\Big(\int_{-1/H}^{1/H} |\widetilde{\mathcal{E}}_{k_{r-1}}(\alpha)|^2\, d\alpha \int_{-1/H}^{1/H} |\widetilde{\mathcal{E}}_{k_r}(\alpha)|^2\, d\alpha\Big)^{1/2}\\
&\ll_{\mathbf{k}} HN^{1/k_1+\cdots+1/k_{r-2}}\Big(N^{1/k_{r-1}+1/k_r}\frac{1}{H^2}L^4\Big)^{1/2} \ll_{\mathbf{k}} N^{\rho-1/(2k_{r-1})-1/(2k_r)}L^2.
\end{aligned} \tag{24}$$

The other summands in I_3 can be treated in the same way, by (11).

We now deal with the remaining range $[-1/2,1/2]\setminus[-1/H,1/H]$: by symmetry, it is enough to treat the interval $[1/H,1/2]$. Arguing as in (16) of [4] by partial integration from Lemma 3.1, for $k > 1$ we have

$$\int_{1/H}^{1/2} \big|\widetilde{\mathcal{E}}_k(\alpha)\big|^2 \frac{d\alpha}{\alpha} \ll_k N^{1/k}L^3. \tag{25}$$

A partial integration from Lemma 3.4 also yields

$$\int_{1/H}^{1/2} |\widetilde{S}_k(\alpha)|^2 \frac{d\alpha}{\alpha} \ll_k N^{1/k}L^4 + HN^{(2-k)/k}L^3. \tag{26}$$

Proceeding as above, we start bounding the contribution of the term $\widetilde{S}_{k_1}(\alpha)\cdots\widetilde{S}_{k_{r-1}}(\alpha)y_r$ in $\mathfrak{A}(\alpha)$ over $[-1/2,1/2]\setminus[-1/H,1/H]$. We have that it is

$$\begin{aligned}
&\ll_{\mathbf{k}} \max_{\alpha\in[-1/2,1/2]} |\widetilde{S}_{k_1}(\alpha)\cdots\widetilde{S}_{k_{r-2}}(\alpha)| \Big(\int_{1/H}^{1/2} |\widetilde{S}_{k_{r-1}}(\alpha)|^2 \frac{d\alpha}{\alpha} \int_{1/H}^{1/2} |\widetilde{\mathcal{E}}_{k_r}(\alpha)|^2 \frac{d\alpha}{\alpha}\Big)^{1/2}\\
&\ll_{\mathbf{k}} N^{1/k_1+\cdots+1/k_{r-2}}\Big(N^{1/k_{r-1}}L^4 + HN^{(2-k_{r-1})/k_{r-1}}L^3\Big)^{1/2}\big(N^{1/k_r}L^3\big)^{1/2}\\
&\ll_{\mathbf{k}} H^{1/2}N^{\rho-1/2-1/(2k_r)}L^3,
\end{aligned} \tag{27}$$

since we assumed (22). The other summands in I_2 can be estimated in the same way. Finally, we bound the contribution from the term $x_1 \dots x_{r-2} y_{r-1} y_r$ in $\mathfrak{B}(\alpha)$ on the same interval: this is enough in view of our remarks above. By (25) we may say that it is

$$\begin{aligned}
&\ll_{\mathbf{k}} \int_{1/H}^{1/2} \frac{|\widetilde{\mathcal{E}}_{k_{r-1}}(\alpha) \widetilde{\mathcal{E}}_{k_r}(\alpha)|}{|z|^{1/k_1+\cdots+1/k_{r-2}}} \frac{\mathrm{d}\alpha}{\alpha} \\
&\ll_{\mathbf{k}} N^{1/k_1+\cdots+1/k_{r-2}} \Bigl(\int_{1/H}^{1/2} |\widetilde{\mathcal{E}}_{k_{r-1}}(\alpha)|^2 \frac{\mathrm{d}\alpha}{\alpha} \int_{1/H}^{1/2} |\widetilde{\mathcal{E}}_{k_r}(\alpha)|^2 \frac{\mathrm{d}\alpha}{\alpha} \Bigr)^{1/2} \\
&\ll_{\mathbf{k}} N^{1/k_1+\cdots+1/k_{r-2}} \bigl(N^{1/k_{r-1}+1/k_r} L^6 \bigr)^{1/2} \ll_{\mathbf{k}} N^{\rho-1/(2k_{r-1})-1/(2k_r)} L^3. \qquad (28)
\end{aligned}$$

The other summands in I_3 can be treated in the same way, by (11) again.

Summing up from (21), (23), (24), (27), (28) and recalling that $2 \le k_1 \le \cdots \le k_r$, we proved that

$$\sum_{n=N+1}^{N+H} \mathrm{e}^{-n/N} R(n;\mathbf{k}) = \frac{G(\mathbf{k})}{\mathrm{e}\Gamma(\rho)} H N^{\rho-1} + O_{\mathbf{k}}\left(\Phi_{\mathbf{k}}(N,H)\right),$$

where, dropping terms that are smaller in view of the constraint in (22), we set

$$\Phi_{\mathbf{k}}(N,H) = H^2 N^{\rho-2} + H^{1/2} N^{\rho-1/2-1/(2k_r)} L^3. \qquad (29)$$

Since we want an asymptotic formula, we need to impose the restriction

$$H = \infty\bigl(N^{1-1/k_r} L^6\bigr), \qquad (30)$$

which supersedes (22).

We remark that when $k_1 = 2$ we can use Lemma 2 of [9] instead of Lemma 3.4 in the partial integration leading to (26), and we can replace the right-hand side by $N^{1/2}L^2 + HL^2$. This means, in particular, that, in this case, we may replace L^3 in the far right of (29) by $L^{5/2}$.

Next, we remove the exponential weight, arguing essentially as in §4.5. This completes the proof of Theorem 1.2.

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