

DERIVATION OF A LINE-TENSION MODEL FOR DISLOCATIONS FROM A NONLINEAR THREE-DIMENSIONAL ENERGY: THE CASE OF QUADRATIC GROWTH*

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Abstract. In this paper we derive a line tension model for dislocations in 3D starting from a geometrically nonlinear elastic energy with quadratic growth. In the asymptotic analysis, as the amplitude of the Burgers vectors (proportional to the lattice spacing) tends to zero, we show that the elastic energy linearizes and the line tension energy density, up to an overall constant rotation, is identified by the linearized cell problem formula given in [S. Conti, A. Garroni, and M. Ortiz, *Arch. Ration. Mech. Anal.*, 218 (2015), pp. 699–755].

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1. Introduction. Dislocations are line defects in crystals originated by plastic slips. Their presence, motion, and interaction are considered the key ingredients in understanding plastic behavior of metals, as well as other important effects (e.g., the interface energy at grain boundaries). We refer the reader to [6, 26] for a general introduction to the subject.

In the last decades the mathematical community has shown an increasing interest in the analysis of models for dislocations involving several different approaches and frameworks. Here we focus on a three-dimensional semidiscrete variational model where dislocations can be seen as topological singularities of a continuum strain field. More precisely, while a deformed elastic body can be described by a deformation whose gradient represents locally the distortion of an undeformed reference configuration, in the presence of defects the relevant continuum variable is a field $\beta \in L^1(\Omega; \mathbb{R}^{3 \times 3})$ (the strain field), which may be represented by a gradient only locally. Therefore the defects may be identified with the set in which the curl β is concentrated. Precisely, a distribution of dislocations in an elastic body $\Omega \subset \mathbb{R}^3$ is represented by a matrix-valued measure μ of the form

$$(1.1) \quad \mu = b \otimes t\mathcal{H}^1 \llcorner \gamma,$$

where $b \in \mathcal{B}$ is the (normalized) Burgers vector (a vector-valued multiplicity which describes the kinematics of the line defect), $\mathcal{B} \subset \mathbb{R}^3$ is a discrete lattice generated by the set of admissible Burgers vectors (which depends on the underlying crystalline

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structure of the body), γ is a closed curve in Ω , and t is its unit tangent vector. Therefore, the strain in the presence of a distribution of dislocations μ satisfies

$$(1.2) \quad \operatorname{curl} \beta = \varepsilon \mu \quad \text{in } \Omega,$$

in the sense of distributions. The small parameter ε represents here a length scale which is comparable with the lattice parameter and reveals the discrete nature of the model. The discreteness is indeed highlighted by the presence of the line γ around which crystal defects occur at a microscopic scale, inside the so-called core of the dislocation, while the continuum variable β is actually an approximation of the discrete deformation far from the core. The coexistence of the continuum variable and the discrete one, which makes this model, so to speak, semidiscrete, is a common feature of models showing topological defects (this is also the case of Ginzburg–Landau models for superconductors or Landau–De Gennes models for liquid crystals; see, e.g., [2, 7, 8, 30, 42, 43]).

The validity of the continuum approximation, though, is limited to regimes in which the density of dislocations is not too high and in a region sufficiently far from the dislocations. It is known indeed that an incompatible strain β satisfying (1.2) diverges close to the dislocation line as

$$|\beta(x)| \simeq \frac{1}{\operatorname{dist}(x, \operatorname{supp} \mu)},$$

and in particular is not square integrable. In order to work with the continuum strain field it is then common to perform a regularization of the elastic energy, either by removing the core of the dislocation from the energy by considering

$$(1.3) \quad \int_{\Omega_\varepsilon(\mu)} W(\beta) dx,$$

with $\Omega_\varepsilon(\mu) := \{x \in \Omega : \operatorname{dist}(x, \operatorname{supp} \mu) > \varepsilon\}$, or by regularizing directly the strain field enforcing the following alternative constraint:

$$(1.4) \quad \operatorname{curl} \beta = \varepsilon \mu * \varphi_\varepsilon,$$

with φ_ε a mollifier at scale ε . We will use the latter, which we refer to as the regularization by mollification, while the former is the so-called core cut-off regularization.

In the framework of linear elasticity it is well known that the energy stored by a straight dislocation, with Burgers vector b and direction t in a hollow cylinder T_ε with inner and outer radii, respectively, $0 < \varepsilon < R$, and height h , is given by

$$\int_{T_\varepsilon} \frac{1}{2} \mathbb{C} \eta : \eta \, dx \simeq \Psi_0(b, t) h \log \frac{R}{\varepsilon},$$

where \mathbb{C} denotes the elastic tensor and $\operatorname{curl} \eta = b \otimes t \mathcal{H}^1 \llcorner \mathbb{R}t$ in T_ε . In this linear context the elastic strain and the corresponding dislocation density are normalized by ε , in view of the 2-homogeneity of the energy.

The function Ψ_0 is the so-called self-energy per unit length of a straight infinity dislocation, and, for any Burgers vector $b \in \mathbb{R}^3$ and any direction $t \in S^2$, it is obtained classically by solving the elasticity problem in the whole of \mathbb{R}^3 or it can be characterized, as in (3.6), by a suitable variational formula (see [16, Lemma 5.1]).

In [16] Conti, Garroni, and Ortiz have shown that for general dislocation distributions μ satisfying an appropriate diluteness condition (see Definition 2.3) the rescaled energy

$$\mathcal{F}_\varepsilon(\mu) = \min \left\{ \frac{1}{|\log \varepsilon|} \int_{\Omega_\varepsilon(\mu)} \frac{1}{2} \mathbb{C} \eta : \eta \, dx \quad \text{for } \operatorname{curl} \eta = \mu \text{ in } \Omega \right\}$$

(where we recall $\Omega_\varepsilon(\mu) := \{x \in \Omega : \operatorname{dist}(x, \operatorname{supp} \mu) > \varepsilon\}$) Γ -converges with respect to the weak* convergence to the line-tension energy

$$(1.5) \quad \int_\gamma \tilde{\Psi}_0(b, t) \, d\mathcal{H}^1,$$

where $\tilde{\Psi}_0$ is the \mathcal{H}^1 -elliptic envelope of Ψ_0 and is obtained by a relaxation procedure in [12]. In particular the relaxation process may produce microstructures at mesoscopic scales. Precise definitions of $\tilde{\Psi}_0(b, t)$ and Ψ_0 are recalled in (2.12) and (3.6), respectively. The above-mentioned result is one of the first rigorous asymptotic analyses of the elastic energy induced by dislocations in a quite general three-dimensional framework (see also [27]). Previous results were indeed confined to special geometries where the models could be treated in a two-dimensional framework and studied by means of Γ -convergence in different relevant energetic regimes. In these reduced models dislocations can be seen either as points in the cross section of a cylindrical domain (see [11, 17, 19, 23, 35]) with a strong similarity to the case of filaments of currents in superconductors [42, 43, 30] or as lines confined to a single slip plane and their energy described by nonlocal phase field models (generalizing the Peierls–Nabarro model [31, 32, 24, 10, 12, 13, 14, 20, 15]).

In the context of two-dimensional models with point singularities, several authors have also studied fully discrete models for screw dislocations [41, 3], deriving the same asymptotics obtained by means of the semidiscrete models (which then turns out to be a robust approximation of the discrete framework) and considering further asymptotic expansions able to capture the interaction and to drive the evolution of systems of dislocations [4, 28, 29]; see also [36] and [37].

The classical semidiscrete models of dislocations, including the ones mentioned above, are based on the assumption of the elastic far field being small and therefore of the corresponding elastic energy being linear, which then confines the analysis of systems with dilute dislocations in single grain bodies. A step forward in the direction of having more flexible models that in principle may be suitable for the description of grain boundaries effects or of large deformations in thin materials is to consider nonlinear energies which may incorporate invariance under rigid rotations. This was first done for noncoherent interfaces in rods by [38, 39, 18], where the nonlinearity of the energy, with p growth and $p < 2$, is also used as an alternative regularization of the core region (see also [45, 44]).

In a two-dimensional setting the use of a geometric nonlinear energy has shown already quite interesting features (see [46, 40]). The prototype energy is the following:

$$\int_\omega \operatorname{dist}^2(\beta, SO(2)) \, dx,$$

with ω representing the two-dimensional cross section of the cylindrical body and the field $\beta \in L^1(\omega; \mathbb{R}^{2 \times 2})$ an incompatible field, describing the local distortion of the body in the context of plane elasticity, with curl concentrated on a sum of Dirac masses representing the distribution of dislocations ($\operatorname{curl} \beta = \varepsilon \sum_{i=1}^N b_i \delta_{x_i}$). The analysis

performed in [46, 40] required an assumption of separation of scales (then removed in [25]) that prevents dislocations from being too close in the scale of ε . A crucial ingredient is a rigidity estimate for incompatible fields proved in [40] which is the nonlinear counterpart of Korn’s inequality for incompatible fields proved in [23]. In dilute regimes (in particular in the logarithmic scale) the rigidity estimate shows that the strain is close to a given rotation, and therefore in the asymptotics as the lattice spacing tends to zero the rescaled energy linearizes around such a rotation Γ -converging, as ε goes to zero, to

$$(1.6) \quad \int_{\omega} \frac{1}{2} \mathbb{C} \xi : \xi dx + \sum_i \psi(Q^T b_i) \quad \text{for} \quad \text{curl} \xi = 0, \quad Q \in SO(2),$$

where $\mathbb{C} = \frac{\partial^2 W}{\partial F^2}(I)$ and ψ is the same self-energy which is found in the asymptotics of the linear, above-mentioned, semidiscrete two-dimensional models. We stress here that the result in the limit is still single grain and, up to a rigid motion, coincides with the one obtained in the linear case. Nevertheless the analysis can be pushed to different regimes, and in principle these models can allow for multigrain structures (see [33, 34] for the derivation of the Shockley–Read formula for small-angle grain boundaries).

In the present paper we will combine two important features: the geometric nonlinearity and a full three-dimensional geometry. We will assume, as in [16], that dislocations are separated in the sense of Definition 2.3 and derive via Γ -convergence the line-tension energy for a general three-dimensional distribution of dislocations. We consider an energy density W with quadratic growth and invariant by rigid rotations, then behaving as the $\text{dist}^2(\cdot, SO(3))$, and to any distribution of dislocations μ of the form (1.1) in Ω we associate the rescaled energy

$$(1.7) \quad \mathcal{E}_{\varepsilon}(\beta) := \frac{1}{\varepsilon^2 |\log \varepsilon|} \int_{\Omega} W(\beta) dx,$$

where the incompatible deformation field $\beta: \Omega \rightarrow \mathbb{R}^{3 \times 3}$ satisfies $\text{curl} \beta = \varepsilon \mu * \varphi_{\varepsilon}$. The Γ -convergence analysis shows that the limit functional takes the form

$$(1.8) \quad \int_{\Omega} \frac{1}{2} \mathbb{C} \xi : \xi dx + \int_{\gamma} \tilde{\Psi}_0(Q^T b, t) d\mathcal{H}^1,$$

where $(\xi, Q) \in L^2(\Omega; \mathbb{R}^{3 \times 3}) \times SO(3)$, $\text{curl} \xi = 0$ in Ω , $\tilde{\Psi}_0$ is the self-energy density of the linear case (see (1.5)), and γ is the support of the limit dislocation density $\mu = b \otimes t \mathcal{H}^1 \llcorner \gamma$. The precise statements of the results are given in section 2.

We stress that in the general three-dimensional model the geometry of line dislocations makes the problem of removing the separation scales from the definition of admissible configuration, as it was done in the two-dimensional case [25], substantially more complex (as well as for the linear case in [16]). In particular the crucial use of the rigidity estimate here (and of Korn’s inequality in the linear context) does not permit us to adopt directly the strategy used in the context of Ginzburg–Landau [2]. Therefore, how to obtain a general compactness result and a sharp lower bound for the energy in (1.7) is still an open problem.

Finally, we also point out that the combination of the three-dimensional framework with the geometric nonlinearity is far from being a straightforward adaptation of the techniques used in [16] and [46]. In particular, the proof of the lower bound where linearization, concentration, and relaxation must be dealt with all at once requires us

to show precise quantitative estimates for the optimal energy of straight dislocations. A key step is then the use, in section 3, of an auxiliary cell problem formula which introduces an extra parameter and allows us to separate the linearization (guaranteed by the rigidity estimate and the subsequent compactness) from the relaxation process (see subsection 3.3).

The paper is organized as follows. In section 2 we formulate the model and state the main result. Section 3 is devoted to the study of the cell problem formula in the comparison with the linear cell problem formula. Section 4 is devoted to the L^2 estimate of admissible strain fields which are used to show the compactness in section 5 (where the Γ -limit result is also proved). For the convenience of the reader we collect in Appendix A all the notation we used throughout the paper.

2. The model and the results. In what follows we denote by Ω the material reference configuration, which is a simply connected, bounded domain in \mathbb{R}^3 with boundary of class C^2 .

We identify a distribution of dislocations as a matrix-valued measure supported on a one-dimensional subset of Ω . In this framework the topological nature of these defects can be easily translated in the property for these measures to be divergence-free in the sense of distributions. Precisely, the set of *admissible dislocation densities* is the set of all divergence-free bounded measures $\mu \in \mathcal{M}(\Omega; \mathbb{R}^{3 \times 3})$ of the form

$$\mu = b \otimes t\mathcal{H}^1 \llcorner \gamma,$$

with γ a 1-rectifiable subset of Ω , $t: \gamma \rightarrow S^2$ its tangent vector, and $b \in L^1(\gamma; \mathcal{B}; \mathcal{H}^1 \llcorner \gamma)$ the Burgers vector field. Here $\mathcal{B} \subset \mathbb{R}^3$ is a discrete lattice and represents the space of *admissible renormalized Burgers vectors* (e.g., in the case of the cubic crystal $\mathcal{B} = \mathbb{Z}^3$). In particular, without loss of generality we will assume that

$$(2.1) \quad \min\{|b| : b \in \mathcal{B}\} = 1.$$

The divergence-free condition reads as

$$\int_{\gamma} b \cdot (D\phi)t d\mathcal{H}^1 = 0$$

for all $\phi \in C_0^\infty(\Omega; \mathbb{R}^3)$. We will denote this set of admissible dislocation densities as $\mathcal{M}_{\mathcal{B}}(\Omega)$, i.e.,

$$(2.2) \quad \mathcal{M}_{\mathcal{B}}(\Omega) := \left\{ \mu \in \mathcal{M}(\Omega; \mathbb{R}^{3 \times 3}) : \mu = b \otimes t\mathcal{H}^1 \llcorner \gamma, \right. \\ \left. \operatorname{div} \mu = 0, b \in \mathcal{B}, \gamma \text{ 1-rectifiable} \right\}.$$

REMARK 2.1. *It can be seen (see, for instance, [12, Theorem 2.5]) that for each $\mu \in \mathcal{M}_{\mathcal{B}}(\Omega)$ γ must be the union of a countable number of Lipschitz curves with no endpoints in Ω , b must be constant on each connected component of γ away from branching points, and in each branching point the oriented sum of Burgers vector must be zero.*

In order to associate a semidiscrete elastic energy to a given distribution of dislocations, as already mentioned, we need to regularize the problem inside the core, i.e., at scale ε proportional to the lattice spacing. Among different types of regularizations (that in our analysis give rise to the same asymptotics; see [16]) we decide

to introduce a mollification kernel which has the effect of spreading the mass of μ in a neighborhood of order ε of its support and therefore smearing out the singularity of the corresponding strain. Then we define the class of admissible strains associated with any $\mu \in \mathcal{M}_{\mathcal{B}}(\Omega)$ as

$$(2.3) \quad \mathcal{AS}_\varepsilon(\mu) := \{ \beta \in L^2(\Omega; \mathbb{R}^{3 \times 3}) : \text{curl } \beta = \varepsilon \tilde{\mu} * \varphi_\varepsilon \text{ in } \Omega \},$$

where $\tilde{\mu} \in \mathcal{M}_{\mathcal{B}}(\mathbb{R}^3)$ is an extension of μ , that is, $\tilde{\mu} \llcorner \Omega = \mu$, $\varphi_\varepsilon(x) := \varepsilon^{-3} \varphi(x/\varepsilon)$ is a mollifier, and $\text{curl } \beta$ is intended in a distributional sense. To simplify the arguments we will assume that $\varphi \leq C \chi_{B_1(0)}$, so that $\varphi_\varepsilon(x) \leq c \frac{\chi_{B_\varepsilon(0)}}{|B_\varepsilon(0)|}$.

Note that the scaling ε of the dislocation density $\tilde{\mu} * \varphi_\varepsilon$ reflects the fact that dislocations are defects at the atomic scale, and the support of $\tilde{\mu} * \varphi_\varepsilon$ represents the dislocation core.

To any admissible pair $(\mu, \beta) \in \mathcal{M}_{\mathcal{B}}(\Omega) \times \mathcal{AS}_\varepsilon(\mu)$, we associate an energy of the form

$$(2.4) \quad \mathcal{E}_\varepsilon(\mu, \beta) := \int_{\Omega} W(\beta) dx,$$

where $W: \mathbb{R}^{3 \times 3} \rightarrow [0, +\infty]$ satisfies the classical assumptions for the geometrically nonlinear elastic setting, that is,

- (i) W is C^0 and C^2 in a neighborhood of $SO(3)$;
- (ii) $W(I) = 0$ (stress-free reference configuration);
- (iii) $W(RF) = W(F)$ for every $R \in SO(3)$ and $F \in \mathbb{R}^{3 \times 3}$ (frame indifference);
- (iv) there exist constants $C_1, C_2 > 0$ such that for every $F \in \mathbb{R}^{3 \times 3}$

$$(2.5) \quad C_1 \text{dist}^2(F, SO(3)) \leq W(F) \leq C_2 \text{dist}^2(F, SO(3));$$

- (v) there exists a constant $C_3 > 0$ such that for every $\sigma > 0$ and $F, F' \in \mathbb{R}^{3 \times 3}$ there holds

$$(2.6) \quad W(F') \leq (1 + C_3 \sigma) W(F) + C_3 \left(1 + \frac{1}{\sigma} \right) |F' - F|^2.$$

The latter condition is fulfilled by the prototypical energy $W(F) = \text{dist}^2(F, SO(3))$, and it is for instance guaranteed by

- (v') there exists a constant $C_4 > 0$ such that for every $F \in \mathbb{R}^{3 \times 3}$

$$(2.7) \quad \left| \frac{\partial W}{\partial F}(F) \right| \leq C_4 \text{dist}(F, SO(3)),$$

where $\frac{\partial W}{\partial F}(F) \in \mathbb{R}^{3 \times 3}$ is the Jacobian matrix of W in F .

REMARK 2.2. Note that assumption (v) (or (v')) is used only for the lower bound and that the asymptotics of the energy in this regime depends only on its linearization near the identity. Our result holds true for all energies satisfying (i)–(iv) and

$$W(F) \geq \widetilde{W}(F) \quad \forall F \in \mathbb{R}^{3 \times 3},$$

with $\frac{\partial^2 W}{\partial F^2}(I) = \frac{\partial^2 \widetilde{W}}{\partial F^2}(I)$ and \widetilde{W} satisfying (i)–(v).

The main goal of this paper is to study the asymptotic behavior of the energy in a mesoscopic scale, i.e., a scale at which lines are still visible and in the asymptotics

we recover a line tension. Under our assumption for the energy the natural rescaled functional is given by

$$\frac{1}{\varepsilon^2 |\log \varepsilon|} \mathcal{E}_\varepsilon(\mu, \beta)$$

(see also [46]). As in [16], in order to perform the analysis, we need to assume a diluteness condition for the admissible dislocation densities.

DEFINITION 2.3. *Given two positive parameters $\alpha, h > 0$, a dislocation measure $\mu \in \mathcal{M}_{\mathcal{B}}(\Omega)$, with $\Omega \subset \mathbb{R}^3$ open, is said to be (h, α) -dilute if there are finitely many closed segments $\gamma_j \subset \Omega$ and vectors $b_j \in \mathcal{B}$, $t_j \in S^2$ (with t_j tangent to γ_j) such that*

$$\mu = \sum_j b_j \otimes t_j \mathcal{H}^1 \llcorner \gamma_j,$$

where the closed segments γ_j satisfy the following properties:

- (a) Each γ_j has length at least h ;
- (b) if γ_j and γ_k are disjoint, then their distance is at least αh ;
- (c) if the segments γ_j and γ_k are not disjoint, then they share an endpoint, and the angle between them is at least α .

The set of (h, α) -dilute measures is denoted by $\mathcal{M}_{\mathcal{B}}^{h, \alpha}(\Omega)$.

Moreover, we say that a measure $\mu \in \mathcal{M}_{\mathcal{B}}^{h, \alpha}(\Omega)$ is an (h, α) -dilute measure in $\bar{\Omega}$ if also the following condition holds:

- (d) If $\gamma_j \cap \partial\Omega \neq \emptyset$, then this intersection consists of a single point x_j and the angle between γ_j and the tangent plane to $\partial\Omega$ at x_j is at least α . Moreover, if $\gamma_j \cap \partial\Omega = \emptyset$, the distance between γ_j and $\partial\Omega$ is at least $h\alpha$.

We will denote this space by $\mathcal{M}_{\mathcal{B}}^{h, \alpha}(\bar{\Omega})$.

Condition (d) in Definition 2.3 essentially guarantees that any $\mu \in \mathcal{M}_{\mathcal{B}}^{h, \alpha}(\Omega)$ can be extended to $\tilde{\mu} \in \mathcal{M}_{\mathcal{B}}^{h, \alpha}(\mathbb{R}^3)$ still dilute. This is clearly the case when Ω is a half-space and the extension is obtained by reflection (the general case is more delicate but can be obtained with some suitable generalization stated in Lemma 4.2). Nevertheless, we point out that this assumption is not restrictive. A dilute extension if Ω is of class C^1 can be always obtained with an ad hoc construction that we do not give here.

In addition we choose the diluteness parameters h and α much larger than the core radius ε , namely

$$(2.8) \quad \lim_{\varepsilon \rightarrow 0} \frac{\log(1/(\alpha_\varepsilon h_\varepsilon))}{|\log \varepsilon|} = \lim_{\varepsilon \rightarrow 0} \alpha_\varepsilon = \lim_{\varepsilon \rightarrow 0} h_\varepsilon = 0.$$

For technical reasons (see Proposition 4.4) we also require the following stronger diluteness condition:

$$(2.9) \quad \alpha_\varepsilon^4 h_\varepsilon^6 |\log \varepsilon| > 1.$$

This ensures that $\alpha_\varepsilon, h_\varepsilon$ go to zero very slowly; e.g., $\alpha_\varepsilon = |\log \varepsilon|^{-\alpha}$, $h_\varepsilon = |\log \varepsilon|^{-h}$ with $0 < \alpha < \frac{1}{8}$ and $0 < h < \frac{1}{12}$ will work.

With this choice of the diluteness parameters we will show that the rescaled functional

$$(2.10) \quad \mathcal{F}_\varepsilon(\mu, \beta) := \begin{cases} \frac{1}{\varepsilon^2 |\log \varepsilon|} \mathcal{E}_\varepsilon(\mu, \beta) & \text{if } (\mu, \beta) \in \mathcal{M}_{\mathcal{B}}^{h_\varepsilon, \alpha_\varepsilon}(\bar{\Omega}) \times \mathcal{AS}_\varepsilon(\mu), \\ +\infty & \text{otherwise} \end{cases}$$

Γ -converges to the following functional:

$$(2.11) \quad \mathcal{F}_0(\mu, \xi, Q) := \begin{cases} \int_{\Omega} \frac{1}{2} \mathbb{C} \xi : \xi \, dx + \int_{\gamma} \tilde{\Psi}_0(Q^T b, t) d\mathcal{H}^1 & \text{if } (\mu, \xi, Q) \in \mathcal{AS}, \\ +\infty & \text{otherwise,} \end{cases}$$

where

$$\mathcal{AS} := \{(\mu, \xi, Q) \in \mathcal{M}_{\mathcal{B}}(\Omega) \times L^2(\Omega; \mathbb{R}^{3 \times 3}) \times SO(3) : \mu = b \otimes t \mathcal{H}^1 \llcorner \gamma, \operatorname{curl} \xi = 0 \text{ in } \Omega\}.$$

Here $\mathbb{C} := \frac{\partial^2 W}{\partial F^2}(I)$ is the Hessian of W at the identity, and $A : B = \sum_{i,j} A_{ij} B_{ij}$ denotes the Euclidean scalar product of matrices, while $\tilde{\Psi}_0$ is the \mathcal{H}^1 -elliptic envelope of Ψ_0 , defined as

$$(2.12) \quad \tilde{\Psi}_0(b', t) := \inf \left\{ \int_{\gamma} \Psi_0(\theta(x), \tau(x)) d\mathcal{H}^1(x) : \nu = \theta \otimes \tau \mathcal{H}^1 \llcorner \gamma \in \mathcal{M}_{\mathcal{B}}(B_{1/2}(0)), \right. \\ \left. \operatorname{supp}(\nu - b' \otimes t \mathcal{H}^1 \llcorner (\mathbb{R}t \cap B_{1/2}(0))) \subset\subset B_{1/2}(0) \right\}.$$

The function Ψ_0 is defined in (3.6) and (3.8) below and represents the self-energy per unit length of a straight infinite dislocation. Therefore $\tilde{\Psi}_0$ is the relaxed line-tension energy density (see [12]); we remark that one can show examples where the relaxation is actually needed in order to optimize the energy. Note that the dependence of the limit functional on a rotation Q is due to the geometric nonlinear nature of the energy, namely to its frame indifference, and comes out from a Taylor expansion of the energy density W near a constant rotation Q .

The first important result concerns a compactness property for the dislocation measures and associated fields with equibounded energies.

THEOREM 2.4 (compactness). *Let $\varepsilon_j \rightarrow 0$ and $(h_{\varepsilon_j}, \alpha_{\varepsilon_j})$ be as in (2.8) and (2.9).*

If $(\mu_j, \beta_j) \in \mathcal{M}_{\mathcal{B}}^{h_{\varepsilon_j}, \alpha_{\varepsilon_j}}(\bar{\Omega}) \times \mathcal{AS}_{\varepsilon_j}(\mu_j)$ is a sequence such that $\mathcal{F}_{\varepsilon_j}(\mu_j, \beta_j) \leq C$, for some $C > 0$, then the following hold:

- (i) *There exists a measure $\mu \in \mathcal{M}_{\mathcal{B}}(\Omega)$ such that, up to a subsequence,*

$$\mu_j \xrightarrow{*} \mu \quad \text{in } \mathcal{M}_{\mathcal{B}}(\Omega).$$

- (ii) *There exist a sequence $\{Q_j\} \subset SO(3)$ and $\xi \in L^2(\Omega; \mathbb{R}^{3 \times 3})$ with $\operatorname{curl} \xi = 0$ in Ω such that, up to a subsequence,*

$$\frac{Q_j^T \beta_j - I}{\varepsilon_j \sqrt{|\log \varepsilon_j|}} \chi_{\Omega_j} \rightharpoonup \xi \quad \text{in } L^2(\Omega; \mathbb{R}^{3 \times 3}) \quad \text{and } Q_j \rightarrow Q \in SO(3),$$

where χ_{Ω_j} denotes the characteristic function of $\Omega_j := \{x \in \Omega : \operatorname{dist}(x, \partial\Omega) > \varepsilon_j\}$.

Since in the definition of admissible strains we do not specify how we extend the measure outside Ω in order to define the regularized dislocation density, we cannot expect a control on the total mass of $\tilde{\mu}_j * \varphi_{\varepsilon_j}$. This is the reason why in the compactness of the strains (property (ii) of the above result) we need to remove a neighborhood of the boundary. Whether the control on the energy of the β 's is enough to deduce compactness without any further assumption on the extension is not clear to us.

In order to recover compactness in the whole of Ω we can fix a specific extension operator $T: \mathcal{M}_{\mathcal{B}}(\Omega) \rightarrow \mathcal{M}_{\mathcal{B}}(\mathbb{R}^3)$, and replace the class of admissible strains $\mathcal{AS}_{\varepsilon}(\mu)$ defined in (2.3) with the class

$$(2.13) \quad \mathcal{AS}_{\varepsilon}^*(\mu) := \{\beta \in L^2(\Omega; \mathbb{R}^{3 \times 3}) : \text{curl } \beta = \varepsilon T\mu * \varphi_{\varepsilon} \text{ in } \Omega\}.$$

There are many possible choices of the extension operator T ; one that will work is defined by Lemma 4.2.

REMARK 2.5. *If in Theorem 2.4 we assume that β_j belongs to $\mathcal{AS}_{\varepsilon}^*(\mu_j)$, property (ii) in Theorem 2.4 can be replaced by the following:*

- (ii') *There exist a sequence $\{Q_j\} \subset SO(3)$ and $\xi \in L^2(\Omega; \mathbb{R}^{3 \times 3})$ with $\text{curl } \xi = 0$ such that, up to a subsequence,*

$$(2.14) \quad \frac{Q_j^T \beta_j - I}{\varepsilon_j \sqrt{|\log \varepsilon_j|}} \rightharpoonup \xi \quad \text{in } L^2(\Omega; \mathbb{R}^{3 \times 3}) \quad \text{and} \quad Q_j \rightarrow Q \in SO(3).$$

DEFINITION 2.6. *We say that $(\mu_j, \beta_j) \in \mathcal{M}_{\mathcal{B}}^{h_{\varepsilon_j}, \alpha_{\varepsilon_j}}(\bar{\Omega}) \times \mathcal{AS}_{\varepsilon_j}(\mu_j)$ converges to $(\mu, \xi, Q) \in \mathcal{M}_{\mathcal{B}}(\Omega) \times L^2(\Omega; \mathbb{R}^{3 \times 3}) \times SO(3)$ if (i) and (ii) of Theorem 2.4 hold.*

THEOREM 2.7 (Γ -convergence). *The energy functional $\mathcal{F}_{\varepsilon}$ Γ -converges to \mathcal{F}_0 in the following sense.*

- (i) *Lower bound: For any sequence $\varepsilon_j \rightarrow 0$ and any $(\mu_j, \beta_j) \in \mathcal{M}_{\mathcal{B}}^{h_{\varepsilon_j}, \alpha_{\varepsilon_j}}(\bar{\Omega}) \times \mathcal{AS}_{\varepsilon_j}(\mu_{\varepsilon_j})$ converging to (μ, ξ, Q) in the sense of Definition 2.6, one has*

$$\mathcal{F}_0(\mu, \xi, Q) \leq \liminf_{j \rightarrow \infty} \mathcal{F}_{\varepsilon_j}(\mu_j, \beta_j).$$

- (ii) *Upper bound: For any $(\mu, \xi, Q) \in \mathcal{M}_{\mathcal{B}}(\Omega) \times L^2(\Omega; \mathbb{R}^{3 \times 3}) \times SO(3)$ with $\text{curl } \xi = 0$ and any sequence $\varepsilon_j \rightarrow 0$ there exists a sequence $(\mu_j, \beta_j) \in \mathcal{M}_{\mathcal{B}}^{h_{\varepsilon_j}, \alpha_{\varepsilon_j}}(\bar{\Omega}) \times \mathcal{AS}_{\varepsilon_j}(\mu_{\varepsilon_j})$ converging to (μ, ξ, Q) in the sense of Definition 2.6 such that*

$$\limsup_{j \rightarrow \infty} \mathcal{F}_{\varepsilon_j}(\mu_j, \beta_j) \leq \mathcal{F}_0(\mu, \xi, Q).$$

3. Asymptotics for straight dislocations. The elastic energy $\mathcal{F}_{\varepsilon}$ of a given polyhedral measure $\mu \in \mathcal{M}_{\mathcal{B}}^{h, \alpha}(\bar{\Omega})$ is asymptotically equivalent to the sum of the energy contributions of each segment. The latter is obtained by studying a cell problem, which provides the energy per unit length of a straight infinite dislocation. In the following we analyze a three-dimensional cell problem in a nonlinear framework combining and developing techniques used by [16] for the linear case and by [46] for the two-dimensional nonlinear case.

3.1. Linear cell problem. In the three-dimensional linear framework, corresponding to a given elasticity tensor $\mathbb{C}: \mathbb{R}^{3 \times 3} \rightarrow \mathbb{R}^{3 \times 3}$ linear, symmetric, and positive definite, the line-tension density was characterized in [16]. For completeness we give here the main results.

For $t \in S^2$ we fix a matrix

$$(3.1) \quad Q_t \in SO(3) \quad \text{such that } Q_t e_3 = t$$

and let

$$(3.2) \quad \Phi_t(r, \theta, z) := Q_t(r \cos \theta, r \sin \theta, z)$$

be the change of variables to cylindrical coordinates with axis t . The local basis in cylindrical coordinates is

$$(3.3) \quad e_r := (\cos \theta, \sin \theta, 0), \quad e_\theta := (-\sin \theta, \cos \theta, 0), \quad e_3 := (0, 0, 1).$$

We denote by B'_R (respectively, B_R) the ball of radius R in \mathbb{R}^2 (respectively, \mathbb{R}^3) centered at the origin.

For any $b \in \mathbb{R}^3$ and $t \in S^2$ we define $\mu_{b,t} := b \otimes t \mathcal{H}^1 \llcorner \mathbb{R}t$ and denote by $\eta_{b,t} \in L^1(\mathbb{R}^3; \mathbb{R}^{3 \times 3})$ the distributional solution to

$$(3.4) \quad \begin{cases} \operatorname{div} \mathbb{C} \xi = 0 & \text{in } \mathbb{R}^3, \\ \operatorname{curl} \xi = \mu_{b,t} & \text{in } \mathbb{R}^3. \end{cases}$$

The function $\eta_{b,t}$ is of the form

$$(3.5) \quad \eta_{b,t}(\Phi_t(r, \theta, z)) = \frac{1}{r} (f(\theta) \otimes Q_t e_\theta + g \otimes Q_t e_r),$$

where $(f, g) \in L^2((0, 2\pi); \mathbb{R}^3) \times \mathbb{R}^3$, with $\int_0^{2\pi} f(s) ds = b$, are solutions to the following minimization problem:

$$(3.6) \quad \Psi_0(b, t) := \min \left\{ \int_0^{2\pi} \frac{1}{2} \mathbb{C} G(\theta) : G(\theta) \, d\theta \right\},$$

the minimum being taken over all functions $G: (0, 2\pi) \rightarrow \mathbb{R}^{3 \times 3}$ of the form $G(\theta) = f(\theta) \otimes Q_t e_3 + g \otimes Q_t e_r$ as in (3.5) (see [16, Lemma 5.1]). In particular,

$$(3.7) \quad |\eta_{b,t}|(x) \leq c \frac{|b|}{\operatorname{dist}(x, \mathbb{R}t)}$$

for a constant $c > 0$ depending only on \mathbb{C} . The line-tension energy density associated to the measure $\mu_{b,t}$ given in (3.6) can be rewritten as

$$(3.8) \quad \Psi_0(b, t) := \int_0^{2\pi} \frac{1}{2} \mathbb{C} \eta_{b,t}(\Phi_t(1, \theta, 0)) : \eta_{b,t}(\Phi_t(1, \theta, 0)) d\theta.$$

REMARK 3.1. *The function Ψ_0 is continuous and satisfies the following:*

- There exist $c_0, c_1 > 0$ such that

$$(3.9) \quad c_0 |b|^2 \leq \Psi_0(b, t) \leq c_1 |b|^2;$$

- for any $t \in S^2$ the map $b \mapsto \Psi_0(b, t)$ is quadratic;
- there exists $c_2 > 0$ such that for any $t, t' \in S^2$

$$(3.10) \quad \Psi_0(b, t) \leq (1 + c_2 |t - t'|) \Psi_0(b, t').$$

This line-tension energy density for straight infinite dislocations is the starting point in characterizing the asymptotics in [16]. Indeed, it can be shown that the line-tension energy

$$(3.11) \quad \int_\gamma \Psi_0(b, t) d\mathcal{H}^1$$

is not lower semicontinuous; therefore the limiting energy in the linear framework requires a relaxation procedure. More precisely, in [12] the authors show that the relaxed energy is given by

$$\int_{\gamma} \tilde{\Psi}_0(b, t) d\mathcal{H}^1,$$

where $\tilde{\Psi}_0$ is the \mathcal{H}^1 -elliptic envelope of Ψ_0 as defined in (2.12). We stress that $\Psi_0(b, t) \geq \tilde{\Psi}_0(b, t)$ and, in particular, it has linear growth with respect to b ; namely, there exist $\tilde{c}_0, \tilde{c}_1 > 0$ such that

$$(3.12) \quad \tilde{c}_0|b| \leq \tilde{\Psi}_0(b, t) \leq \tilde{c}_1|b|.$$

Next we characterize the elastic energy induced by $\mu_{b,t}$ in a finite hollow cylinder. To this aim, fix $b \in \mathbb{R}^3$, $t \in S^2$, $h, r, R \in (0, \infty)$ with $r < R \leq h$. Let Q_t be as in (3.1) and let $T_h^{r,R}$, $Q_t T_h^R$, T_h^R , $Q_t T_h^R$ denote the corresponding cylinders defined as in (A.1) and (A.2), respectively. Then consider the three-dimensional linear cell problem

$$(3.13) \quad \begin{aligned} \Psi(b, t, h, r, R) &:= \frac{1}{h \log \frac{R}{r}} \min \left\{ \int_{Q_t T_h^{r,R}} \frac{1}{2} \mathbb{C} \eta : \eta \, dx, \eta \in L^1_{loc}(\mathbb{R}^3; \mathbb{R}^{3 \times 3}), \operatorname{curl} \eta = \mu_{b,t} \right\} \\ &= \frac{1}{h \log \frac{R}{r}} \min \left\{ \int_{Q_t T_h^{r,R}} \frac{1}{2} \mathbb{C} \eta : \eta \, dx, \eta \in L^2(Q_t T_h^{r,R}; \mathbb{R}^{3 \times 3}), \operatorname{curl} \eta = 0 \text{ in } Q_t T_h^{r,R}, \right. \\ &\quad \left. \int_0^{2\pi} \eta(\Phi_t(\rho, \theta, z)) Q_t e_{\theta\rho} \, d\theta = b \text{ for } (\rho, z) \in (r, R) \times (0, h) \right\}. \end{aligned}$$

REMARK 3.2. *The condition*

$$\int_0^{2\pi} \eta(\Phi_t(\rho, \theta, z)) Q_t e_{\theta\rho} \, d\theta = b \quad \text{for } (\rho, z) \in (r, R) \times (0, h)$$

is intended in the following integral sense:

$$(3.14) \quad \int_0^h \int_r^R \varphi(\rho, z) \int_0^{2\pi} \eta(\Phi_t(\rho, \theta, z)) Q_t e_{\theta\rho} \, d\theta \, d\rho \, dz = \int_0^h \int_r^R \varphi(\rho, z) b \, d\rho \, dz$$

for all $\varphi \in L^2((r, R) \times (0, h))$.

The following asymptotic analysis is proved in [16, Lemmas 5.6, 5.10, and 5.11].

LEMMA 3.3. *There is a constant $c > 0$ such that for every $M \geq 1$, there is a function $\omega_M: (0, \infty) \rightarrow (0, \infty)$ with*

$$\lim_{r \rightarrow 0} \omega_M(r) = 0,$$

and

$$\left(1 - \frac{c}{M} - \omega_M\left(\frac{r}{R}\right)\right) \Psi_0(b, t) \leq \Psi(b, t, h, r, R) \leq \Psi_0(b, t)$$

for all $b \in \mathbb{R}^3$, $t \in S^2$, $r, R, h > 0$ such that $MR \leq h$. In particular,

$$(3.15) \quad \lim_{h \rightarrow \infty} \lim_{r \rightarrow 0} \Psi(b, t, h, r, R) = \Psi_0(b, t).$$

Furthermore, there exists $c_* > 0$ such that, for all b, t, h, r, R with $2r \leq R \leq h$,

$$(3.16) \quad c_* |b|^2 \leq \Psi(b, t, h, r, R).$$

3.2. Rigidity. In this section we state and prove a rigidity estimate (see [21]) for a hollow cylinder with a constant which does not depend on the radius of the central hole. This result combines the two-dimensional nonlinear version proved in [46] with the three-dimensional Korn inequality in [16, Lemma 5.9].

LEMMA 3.4 (rigidity with a hole). *For every $h > 0$ there exists $c = c(h) > 0$ with the following property: let $R > 0$, $\varepsilon \in (0, \frac{\min\{h,1\}}{2}R]$, $T_{hR}^{\varepsilon,R}$ be as in (A.1) and $u \in W^{1,2}(T_{hR}^{\varepsilon,R}; \mathbb{R}^3)$; then there exists a rotation $Q \in SO(3)$ such that*

$$\|\nabla u - Q\|_{L^2(T_{hR}^{\varepsilon,R})} \leq c \|\text{dist}(\nabla u, SO(3))\|_{L^2(T_{hR}^{\varepsilon,R})}.$$

Proof. In order to obtain the estimate we need to extend the function u in the inner cylinder. This should be done in two steps. By scaling we can take $R = 1$. We set $N := \lfloor h/\varepsilon \rfloor - 1$ and $z_j := j\varepsilon$ for $j = 0, \dots, N - 1$, $z_N := h - 2\varepsilon$, so that $z_N - z_{N-1} \in [0, \varepsilon)$. For every j , we now apply the rigidity estimate [21, Theorem 3.1] on the domain

$$T_j := T_{2\varepsilon}^{\varepsilon,2\varepsilon} + z_j e_3$$

and get

$$(3.17) \quad \|\nabla u - Q_j\|_{L^2(T_j)} \leq c \|\text{dist}(\nabla u, SO(3))\|_{L^2(T_j)}$$

for a rotation $Q_j \in SO(3)$, with a constant that, by scaling and translation, does not depend on ε and j . Moreover, applying Poincaré’s inequality, we find $q_j \in \mathbb{R}^3$ such that setting $a_j(x) := Q_j x + q_j$ there holds

$$\frac{1}{\varepsilon} \|u - a_j\|_{L^2(T_j)} \leq c \|\text{dist}(\nabla u, SO(3))\|_{L^2(T_j)}.$$

Therefore, by applying the triangular inequality we derive for $k = j - 1$ and $k = j + 1$

$$(3.18) \quad \frac{1}{\varepsilon} \|a_k - a_j\|_{L^2(T_j \cap T_k)} \leq c \|\text{dist}(\nabla u, SO(3))\|_{L^2(T_j \cup T_k)}.$$

Now, using, e.g., Lemma 2.6 in [1], for any j we can extend u to a function u_j on the full cylinder

$$(3.19) \quad \widehat{T}_j := T_{2\varepsilon}^{2\varepsilon} + z_j e_3$$

satisfying

$$(3.20) \quad \frac{1}{\varepsilon} \|u_j - a_j\|_{L^2(\widehat{T}_j)} + \|\nabla u_j - Q_j\|_{L^2(\widehat{T}_j)} \leq c \|\text{dist}(\nabla u, SO(3))\|_{L^2(T_j)}.$$

Note that by the definition of the cylinders \widehat{T}_j we have

$$T_h^{2\varepsilon} = \cup_j \widehat{T}_j \quad \text{and} \quad \widehat{T}_j \cap \widehat{T}_{j-1} = T_\varepsilon^{2\varepsilon} + (z_j - \varepsilon)e_3.$$

Again, by scaling and translation, the constant does not depend on ε and j . Finally, we interpolate the different extensions by choosing a partition of unity in $[0, h]$ with $\varphi_j \in C_c^\infty(\mathbb{R}; [0, 1])$ such that

$$\sum_j \varphi_j = 1 \quad \text{on } (0, h), \quad \varphi_j = 0 \quad \text{on } (0, h) \setminus (z_j, z_j + 2\varepsilon), \quad |\nabla \varphi_j| \leq \frac{c}{\varepsilon}.$$

In particular, $\varphi_0(0) = \varphi_N(h) = 1$. We define $\tilde{u} := \sum_j \varphi_j u_j$ in T_h^ε and write

$$(3.21) \quad \nabla \tilde{u} = \sum_j \varphi_j \nabla u_j + \sum_j (u_j - a_j) \otimes \nabla \varphi_j + \sum_j a_j \otimes \nabla \varphi_j.$$

Observe that, since $\sum_j \nabla \varphi_j = 0$, for every k we have

$$(3.22) \quad \sum_j a_j \otimes \nabla \varphi_j = \sum_j (a_j - a_k) \otimes \nabla \varphi_j.$$

We can now write, by the subadditivity of the norm and the definition of \hat{T}_j in (3.19),

$$(3.23) \quad \begin{aligned} \|\text{dist}(\nabla \tilde{u}, SO(3))\|_{L^2(T_h^\varepsilon)} &\leq \sum_j \|\text{dist}(\nabla \tilde{u}, SO(3))\|_{L^2(\hat{T}_j)} \\ &\leq \sum_j \|\nabla \tilde{u} - Q_j\|_{L^2(\hat{T}_j)}. \end{aligned}$$

In turn, by (3.21), (3.22), and recalling that $\varphi_k = 0$ on $(z_j, z_j + 2\varepsilon)$ for $k \notin \{j-1, j, j+1\}$, we have

$$(3.24) \quad \begin{aligned} \|\nabla \tilde{u} - Q_j\|_{L^2(\hat{T}_j)} &\leq \left\| \sum_{k=j-1}^{j+1} \varphi_k (\nabla u_k - Q_j) \right\|_{L^2(\hat{T}_j)} \\ &+ \left\| \sum_{k=j-1}^{j+1} (u_k - a_k) \otimes \nabla \varphi_k \right\|_{L^2(\hat{T}_j)} + \left\| \sum_{k=j-1}^{j+1} (a_j - a_k) \otimes \nabla \varphi_k \right\|_{L^2(\hat{T}_j)} \\ &\leq \sum_{k=j-1}^{j+1} \|\varphi_k (\nabla u_k - Q_k)\|_{L^2(\hat{T}_j)} + \sum_{k=j-1}^{j+1} \|\varphi_k (Q_k - Q_j)\|_{L^2(\hat{T}_j)} \\ &+ \sum_{k=j-1}^{j+1} \|(u_k - a_k) \otimes \nabla \varphi_k\|_{L^2(\hat{T}_j)} + \sum_{k=j-1}^{j+1} \|(a_j - a_k) \otimes \nabla \varphi_k\|_{L^2(\hat{T}_j)}. \end{aligned}$$

Using (3.20) in the first term on the right-hand side of (3.24), we get

$$\begin{aligned} \sum_{k=j-1}^{j+1} \|\varphi_k (\nabla u_k - Q_k)\|_{L^2(\hat{T}_j)} &\leq \sum_{k=j-1}^{j+1} \|(\nabla u_k - Q_k)\|_{L^2(\hat{T}_j \cap \hat{T}_k)} \\ &\leq c \sum_{k=j-1}^{j+1} \|\text{dist}(\nabla u, SO(3))\|_{L^2(T_k)}. \end{aligned}$$

Similarly we estimate the third term

$$\begin{aligned} \sum_{k=j-1}^{j+1} \|(u_k - a_k) \otimes \nabla \varphi_k\|_{L^2(\widehat{T}_j)} &\leq \frac{c}{\varepsilon} \sum_{k=j-1}^{j+1} \|(u_k - a_k)\|_{L^2(\widehat{T}_j \cap \widehat{T}_k)} \\ &\leq c \sum_{k=j-1}^{j+1} \|\text{dist}(\nabla u, SO(3))\|_{L^2(T_k)}. \end{aligned}$$

The fourth term is instead bounded by

$$\begin{aligned} \sum_{k=j-1}^{j+1} \|(a_j - a_k) \otimes \nabla \varphi_k\|_{L^2(\widehat{T}_j)} &\leq \frac{c}{\varepsilon} \|a_j - a_{j+1}\|_{L^2(\widehat{T}_j \cap \widehat{T}_{j+1})} + \frac{c}{\varepsilon} \|a_j - a_{j-1}\|_{L^2(\widehat{T}_j \cap \widehat{T}_{j-1})} \\ &\leq c \|\text{dist}(\nabla u, SO(3))\|_{L^2(T_j \cup T_{j+1})} + c \|\text{dist}(\nabla u, SO(3))\|_{L^2(T_j \cup T_{j-1})}, \end{aligned}$$

where the second inequality follows by (3.18) and by (3.19) we have

$$(3.25) \quad \frac{1}{\varepsilon} \|a_j - a_k\|_{L^2(\widehat{T}_j \cap \widehat{T}_k)} \leq \frac{c}{\varepsilon} \|a_j - a_k\|_{L^2(T_j \cap T_k)}$$

for $k = j - 1, j + 1$. It remains to estimate the second term on the right-hand side of (3.24). To this end we observe that

$$\begin{aligned} \sum_{k=j-1}^{j+1} \|\varphi_k(Q_k - Q_j)\|_{L^2(\widehat{T}_j)} &\leq |\widehat{T}_j|^{1/2} \sum_{k=j-1}^{j+1} |Q_k - Q_j| \\ &= 2\sqrt{2\pi\varepsilon}^{3/2} \sum_{k=j-1}^{j+1} |Q_k - Q_j| \\ &\leq c(\|\nabla u - Q_{j+1}\|_{L^2(T_{j+1} \cap T_j)} + \|\nabla u - Q_j\|_{L^2(T_{j+1} \cap T_j)} \\ &\quad + \|\nabla u - Q_{j-1}\|_{L^2(T_{j-1} \cap T_j)} + \|\nabla u - Q_j\|_{L^2(T_{j-1} \cap T_j)}) \\ &\leq c \sum_{k=j-1}^{j+1} \|\nabla u - Q_k\|_{L^2(T_k)} \\ &\leq c \sum_{k=j-1}^{j+1} \|\text{dist}(\nabla u, SO(3))\|_{L^2(T_k)}, \end{aligned}$$

where the last inequality follows from (3.17).

Going back to (3.23), we have that

$$\begin{aligned} \|\text{dist}(\nabla \tilde{u}, SO(3))\|_{L^2(T_h^\varepsilon)} &\leq c \sum_j \|\text{dist}(\nabla u, SO(3))\|_{L^2(T_j)} \\ &\leq c \|\text{dist}(\nabla u, SO(3))\|_{L^2(T_h^{\varepsilon,1})}. \end{aligned}$$

Therefore, we infer

$$\begin{aligned} &\|\text{dist}(\nabla \tilde{u}, SO(3))\|_{L^2(T_h^1)} \\ &\leq \|\text{dist}(\nabla u, SO(3))\|_{L^2(T_h^{\varepsilon,1})} + \|\text{dist}(\nabla \tilde{u}, SO(3))\|_{L^2(T_h^\varepsilon)} \\ &\leq c \|\text{dist}(\nabla u, SO(3))\|_{L^2(T_h^{\varepsilon,1})} \end{aligned}$$

for a constant c independent of ε . We can conclude the proof by applying the rigidity estimate on the fixed domain T_h^1 (see [22, Theorem 6] and [5]). \square

REMARK 3.5. *The estimate in Lemma 3.4 still holds if we replace $T_{hR}^{\varepsilon,R}$ with a set $S_\varepsilon(\vartheta) \times (0, hR)$ where $S_\varepsilon(\vartheta) := \{(r, \theta) : \varepsilon < r < R, 0 < \theta < \vartheta\}$ for $0 < \vartheta < 2\pi$ with a constant depending on ϑ .*

Lemma 3.4 cannot be directly applied to strains β whose curl is concentrated on a line, as they are not globally gradients in the hollow cylinder. They become gradients if we cut the domain to let it be simply connected. Therefore, as in [46] we need a variant of Lemma 3.4 for a domain with a hole and a cut.

For $r > 0$ we fix the segment $L_r := \{(x, 0) : r < x < R\} \subset \mathbb{R}^2$ and define the hollow cylinder with a cut

$$(3.26) \quad \tilde{T}_h^{r,R} := [(B'_R \setminus B'_r) \setminus L_r] \times (0, h).$$

COROLLARY 3.6 (rigidity with a “hole” and a “cut”). *For every $h, R > 0$, $\varepsilon \in (0, \frac{\min\{h,1\}}{2}R]$ let $\tilde{T}_{hR}^{\varepsilon,R}$ be as in (3.26). Then there exists a constant $c = c(h) > 0$ such that for every $u \in W^{1,2}(\tilde{T}_{hR}^{\varepsilon,R}; \mathbb{R}^3)$ there exists a rotation $Q \in SO(3)$ that satisfies*

$$(3.27) \quad \|\nabla u - Q\|_{L^2(\tilde{T}_{hR}^{\varepsilon,R})} \leq c \|\text{dist}(\nabla u, SO(3))\|_{L^2(\tilde{T}_{hR}^{\varepsilon,R})}.$$

Proof. The proof is analogous to the two-dimensional case (see [46, Proposition 3.3]) and consists in partitioning the domain into four cylindrical subdomains of the form $S_\varepsilon(\pi/2) \times (0, hR)$ (as in Remark 3.5) and consequently applying the rigidity estimate to each of them and finding four constant rotations. The thesis follows by showing that for these rotations the estimate also holds in the whole domain. \square

COROLLARY 3.7. *For every $h > 0$ there exists $c = c(h) > 0$ with the following property: let $R > 0$, $\varepsilon \in (0, \frac{\min\{h,1\}}{2}R]$, and $T_{hR}^{\varepsilon,R}$ be as in (A.1), and let $b \in \mathbb{R}^3$ be fixed; then for every $\beta \in L^2(T_{hR}^{\varepsilon,R}; \mathbb{R}^{3 \times 3})$ with $\text{curl } \beta = 0$ in $T_{hR}^{\varepsilon,R}$ and $\int_0^{2\pi} \beta(\rho, \theta, z) e_{\theta\rho} d\theta = \varepsilon b$ for every $\rho \in (\varepsilon, R)$, $z \in (0, h)$, there exists a rotation $Q \in SO(3)$ such that*

$$\|\beta - Q\|_{L^2(T_{hR}^{\varepsilon,R})} \leq c \|\text{dist}(\beta, SO(3))\|_{L^2(T_{hR}^{\varepsilon,R})}.$$

Proof. Let $\tilde{T}_{hR}^{\varepsilon,R}$ be the cut cylinder as defined in (3.26), and let β be as in the statement. Then $\text{curl } \beta = 0$ in the simply connected domain $\tilde{T}_{hR}^{\varepsilon,R}$; therefore there exists $u \in W^{1,2}(\tilde{T}_{hR}^{\varepsilon,R}; \mathbb{R}^3)$ such that $\beta = \nabla u$ in $\tilde{T}_{hR}^{\varepsilon,R}$. We conclude by applying Corollary 3.6 to the function u . \square

Analogously to the two-dimensional case derived in [46], Corollary 3.7 points out that any elastic strain β whose curl is concentrated on the vertical axis of a cylinder with multiplicity εb and whose energy tends to zero is close to a constant rotation $Q \in SO(3)$ in $L^2(T_{hR}^{\varepsilon,R}; \mathbb{R}^{3 \times 3})$. This suggests that in the limit as $\varepsilon \rightarrow 0$ the energy density $W(\beta)$ linearizes near such a rotation, or, equivalently, by frame indifference, $W(Q^T \beta)$ linearizes near the identity I . This will imply that the nonlinear energy of a straight dislocation $\mu_{b,t}$ on a hollow cylinder is asymptotically comparable (as the inner radius goes to zero) to the self-energy (3.8) where the normalized Burgers vector is rotated by Q^T .

In order to deal with this additional variable that appears in the asymptotics it is convenient to define an auxiliary cell problem formula which describes the nonlinear elastic energy of a straight dislocation in a cylinder with the constraint that the elastic strain is close to a fixed rotation.

3.3. Nonlinear auxiliary cell problem formula. For every $Q \in SO(3)$, $b \in \mathbb{R}^3$, $t \in S^2$, $R > r > 0$, $h > 0$, and $\lambda > 0$ we consider the variational problem

$$(3.28) \quad \Psi_\lambda^{nl}(Q, b, t, h, r, R) := \frac{1}{hr^2 \log \frac{R}{r}} \inf \left\{ \int_{Q_t T_h^{r,R}} W(\beta) + \lambda |\beta - Q|^2 dx : \right. \\ \left. \beta \in L^2(Q_t T_h^{r,R}; \mathbb{R}^{3 \times 3}), \operatorname{curl} \beta = 0 \text{ in } Q_t T_h^{r,R}, \right. \\ \left. \int_0^{2\pi} \beta(\Phi_t(\rho, \theta, z)) Q_t e_\theta \rho \, d\theta = rb \text{ for all } \rho \in (r, R), z \in (0, h) \right\},$$

where $Q_t T_h^{r,R}$ is as defined in (A.1). The dependence on $Q \in SO(3)$ introduced in this auxiliary cell problem formula is needed since, a priori, due to the frame indifference of the energy, the latter can linearize close to any rotation giving rise to different optimal energies. This dependence, which helps to control the convergence, will not appear in the Γ -limit and will be removed by taking $\lambda \rightarrow 0$.

REMARK 3.8. *By scaling, it holds that*

$$(3.29) \quad \Psi_\lambda^{nl}(Q, b, t, kh, kr, kR) = \Psi_\lambda^{nl}(Q, b, t, h, r, R) \quad \text{for all } k > 0.$$

Indeed, for every β , admissible competitor for $\Psi_\lambda^{nl}(Q, b, t, h, r, R)$, the function $\tilde{\beta}(x) := \beta(x/k)$ is admissible for $\Psi_\lambda^{nl}(Q, b, t, kh, kr, kR)$, since $\int_0^{2\pi} \tilde{\beta}(\Phi_t(\rho, \theta, z)) Q_t e_\theta \, d\theta = krb$. Furthermore, by the frame indifference of W , we have

$$(3.30) \quad \Psi_\lambda^{nl}(Q, b, t, h, r, R) = \Psi_\lambda^{nl}(I, Q^T b, t, h, r, R) \quad \forall Q \in SO(3).$$

The main result of this section regards the asymptotic analysis of the cell problem; more precisely, we show that $\Psi_\lambda^{nl}(Q, b, t, h, r, R)$ converges to $\Psi_0(Q^T b, t)$ as $r \rightarrow 0$, $h \rightarrow \infty$, $\lambda \rightarrow 0$.

LEMMA 3.9 (lower bound). *There exists a constant $C > 0$ such that, for every $Q \in SO(3)$, $b \in \mathbb{R}^3$, $t \in S^2$, $h \geq R > 0$, and $\lambda > 0$, it holds that*

$$\liminf_{r \rightarrow 0} \Psi_\lambda^{nl}(Q, b, t, h, r, R) \geq \Psi_0(Q^T b, t) - C|b|^2 \frac{R}{h}.$$

Proof. Fix Q , b , and t . To simplify the notation we assume $t = e_3$ (the general case being similar).

For every $0 < r < R$, let β_r be admissible for the definition of Ψ_λ^{nl} (see (3.28)) and be such that

$$(3.31) \quad \frac{1}{hr^2 \log \frac{R}{r}} \int_{T_h^{r,R}} W(\beta_r) + \lambda |\beta_r - Q|^2 dx \leq \Psi_\lambda^{nl}(Q, b, t, h, r, R) + |b|^2 \frac{R}{h}.$$

Next fix $\delta \in (0, 1/2)$ and divide $T_h^{r,R}$ into dyadic cylinders

$$(3.32) \quad C_i^k := T_{h\delta^{k-1}}^{R\delta^k, R\delta^{k-1}} + (i-1)e_3 = (B'_{R\delta^{k-1}} \setminus B'_{R\delta^k}) \times ((i-1)h\delta^{k-1}, ih\delta^{k-1}),$$

with $k = 1, \dots, \tilde{k}_r$, where $\tilde{k}_r := \lfloor k_r \rfloor + 1$,

$$(3.33) \quad k_r := s \frac{\log \frac{R}{r}}{|\log \delta|} \quad \text{for some fixed } s \in (0, 1),$$

and $i = 1, \dots, i_k := \lfloor 1/\delta^{k-1} \rfloor$. Since δ , R , and s are fixed, the smallest inner radius of the dyadic annuli, namely $R\delta^{\tilde{k}_r}$, is much bigger than r as $r \rightarrow 0$; indeed

$$(3.34) \quad R\delta^{\tilde{k}_r} \geq R\delta^{k_r+1} = \delta R^{1-s} r^s \gg r.$$

In particular,

$$(3.35) \quad \begin{aligned} & \frac{1}{hr^2 \log \frac{R}{r}} \int_{T_h^{r,R}} W(\beta_r) + \lambda |\beta_r - Q|^2 dx \\ & \geq \frac{1}{\log \frac{R}{r}} \sum_{k=1}^{\tilde{k}_r} \sum_{i=1}^{i_k} \frac{1}{h} \int_{C_i^k} \frac{W(\beta_r) + \lambda |\beta_r - Q|^2}{r^2} dx. \end{aligned}$$

We stress that the cylinders C_i^k defined in (3.32) (and the corresponding indices i and k) depend on r . We will denote by $I(r)$ the set of such pairs of indices, i.e.,

$$(3.36) \quad I(r) = \{(i, k) : i = 1, \dots, i_k, k = 1, \dots, \tilde{k}_r\}.$$

For every $0 < r < R$ and $k = 1, \dots, \tilde{k}_r$ we denote by $i(k, r)$ an index i such that

$$(3.37) \quad \sum_{i=1}^{i_k} \int_{C_i^k} W(\beta_r) + \lambda |\beta_r - Q|^2 dx \geq i_k \int_{C_{i(k,r)}^k} W(\beta_r) + \lambda |\beta_r - Q|^2 dx.$$

Let $M := h/R$, and let $\omega_M : (0, \infty) \rightarrow (0, \infty)$ be the function given in the statement of Lemma 3.3.

We will show that there exist a sequence of positive numbers σ_r , infinitesimal for $r \rightarrow 0$, and a positive constant $C > 0$, such that

$$(3.38) \quad \begin{aligned} & \frac{1}{h\delta^{k-1}} \int_{C_{i(k,r)}^k} \frac{W(\beta_r) + \lambda |\beta_r - Q|^2}{r^2} dx \\ & \geq |\log \delta| \Psi_0(Q^T b, t) \left(1 - \frac{C}{M} - \omega_M(\delta)\right) - \sigma_r \end{aligned}$$

for every $r > 0$ and for every $k = 1, \dots, \tilde{k}_r$. Equivalently we will show that

$$(3.39) \quad \begin{aligned} & \lim_{r \rightarrow 0} \left(|\log \delta| \Psi_0(Q^T b, t) \left(1 - \frac{C}{M} - \omega_M(\delta)\right) \right. \\ & \left. - \frac{1}{h\delta^{k-1}} \int_{C_{i(k,r)}^k} \frac{W(\beta_r) + \lambda |\beta_r - Q|^2}{r^2} dx \right)_+ = 0, \end{aligned}$$

uniformly in k , where $(a)_+$ denotes the positive part of a . We establish (3.39) by arguing by contradiction. The argument is based on the following claim, which is a variation of the one used in [46] for the proof of the lower bound.

Claim. Given a sequence $r_j \rightarrow 0$, assume that there exists a constant $\tilde{C} > 0$ such that

$$(3.40) \quad \frac{1}{h\delta^{k_j-1}} \int_{C_{i_j}^{k_j}} \frac{W(\beta_j) + \lambda |\beta_j - Q|^2}{r_j^2} dx < \tilde{C} < +\infty$$

for some sequence $(i_j, k_j) \in I(r_j)$ as defined in (3.36), $\beta_j \in L^2(C_{i_j}^{k_j}; \mathbb{R}^{3 \times 3})$ with $\text{curl } \beta_j = 0$ in $C_{i_j}^{k_j}$, and

$$(3.41) \quad \int_0^{2\pi} \beta_j(\rho, \theta, z) e_{\theta\rho} d\theta = r_j b$$

for $(\rho, z) \in (R\delta^{k_j}, R\delta^{k_j-1}) \times ((i-1)h\delta^{k_j-1}, ih\delta^{k_j-1})$. Then it holds that

$$(3.42) \quad \liminf_{j \rightarrow +\infty} \frac{1}{h\delta^{k_j-1}} \int_{C_{i_j}^{k_j}} \frac{W(\beta_j) + \lambda|\beta_j - Q|^2}{r_j^2} dx \geq |\log \delta| \Psi_0(Q^T b, t) \left(1 - \frac{C}{M} - \omega_M(\delta)\right).$$

Before proving the claim, we show that it implies (3.39) and how this concludes the proof of the lemma. If, by contradiction, (3.39) does not hold, then there exists an $\varepsilon_0 > 0$ such that for all $\sigma > 0$ there exist $r_\sigma \leq \sigma$ and k_σ such that

$$(3.43) \quad \left(|\log \delta| \Psi_0(Q^T b, t) \left(1 - \frac{C}{M} - \omega_M(\delta)\right) - \frac{1}{h\delta^{k_\sigma-1}} \int_{C_{i(k_\sigma, r_\sigma)}^{k_\sigma}} \frac{W(\beta_{r_\sigma}) + \lambda|\beta_{r_\sigma} - Q|^2}{r_\sigma^2} dx \right)_+ > \varepsilon_0,$$

and therefore we can assume that there exist a sequence $r_j \rightarrow 0$ and k_j such that for the corresponding $i_j := i(k_j, r_j)$, (3.40) holds with

$$\tilde{C} = |\log \delta| \Psi_0(Q^T b, t) \left(1 - \frac{C}{M} - \omega_M(\delta)\right) - \varepsilon_0,$$

and $\beta_j := \beta_{r_j}$. Namely,

$$(3.44) \quad \frac{1}{h\delta^{k-1}} \int_{C_{i_j}^{k_j}} \frac{W(\beta_j) + \lambda|\beta_j - Q|^2}{r_j^2} dx \leq |\log \delta| \Psi_0(Q^T b, t) \left(1 - \frac{C}{M} - \omega_M(\delta)\right) - \varepsilon_0.$$

Combining this with (3.42), we get the contradiction, and we then prove (3.38).

Therefore, by (3.35), (3.37), and (3.38) we have

$$\begin{aligned} \liminf_{r \rightarrow 0} \frac{1}{hr^2 \log \frac{R}{r}} \int_{T_h^{r,R}} W(\beta_r) + \lambda|\beta_r - Q|^2 dx &\geq \liminf_{r \rightarrow 0} \frac{1}{\log \frac{R}{r}} \sum_{k=1}^{\tilde{k}_r} \sum_{i=1}^{i_k} \frac{1}{h} \int_{C_i^k} \frac{W(\beta_r) + \lambda|\beta_r - Q|^2}{r^2} dx \\ &\geq \liminf_{r \rightarrow 0} \frac{(1-\delta)}{\log \frac{R}{r}} \sum_{k=1}^{\tilde{k}_r} \frac{1}{h\delta^{k-1}} \int_{C_{i(k,r)}^k} \frac{W(\beta_r) + \lambda|\beta_r - Q|^2}{r^2} dx \\ &\geq \liminf_{r \rightarrow 0} s(1-\delta) \Psi_0(Q^T b, t) \left(1 - \frac{C}{M} - \omega_M(\delta)\right) - \sigma_r \\ &= s(1-\delta) \Psi_0(Q^T b, t) \left(1 - \frac{C}{M} - \omega_M(\delta)\right), \end{aligned}$$

where the quantity $(1-\delta)$ comes from the fact that $i_k \geq (1-\delta^{k-1})/\delta^{k-1} \geq (1-\delta)/\delta^{k-1}$. Since $\delta \in (0, 1/2)$ and $s \in (0, 1)$ are arbitrary, we can pass to the limit as $\delta \rightarrow 0$ and $s \rightarrow 1$ to get

$$\liminf_{r \rightarrow 0} \Psi_\lambda^{nl}(Q, b, t, h, r, R) \geq \Psi_0(Q^T b, t) \left(1 - C \frac{R}{h}\right) \geq \Psi_0(Q^T b, t) - C|b|^2 \frac{R}{h},$$

where the last inequality follows by (3.9) and the statement is proved.

It remains to prove the claim. To simplify the notation we drop the dependence of i and k on j . We proceed in steps. First we scale the problem to a fixed cylinder (Step 1); then we estimate the nonlinear term of the energy (Step 2) and conclude.

Step 1: Scaling and compactness. Here we reduce the analysis to the fixed cylinder $T_{h/R}^{\delta,1}$ defined as in (A.1), and we consider the scaled function

$$\tilde{\eta}_j(x) := R\delta^{k-1}\eta_j\left(R\delta^{k-1}\left(x + (i-1)\frac{h}{R}e_3\right)\right) \quad \text{for } x \in T_{h/R}^{\delta,1},$$

where

$$(3.45) \quad \eta_j := \frac{Q^T\beta_j - I}{r_j}.$$

In this step we show that up to a subsequence

$$(3.46) \quad \tilde{\eta}_j \rightharpoonup \tilde{\eta} \quad \text{in } L^2(T_{h/R}^{\delta,1}; \mathbb{R}^{3 \times 3}),$$

and that $\tilde{\eta}$ is admissible for the cell problem defining $\Psi(Q^Tb, t, h/R, \delta, 1)$ in (3.13).

Indeed, from (3.40) we have

$$(3.47) \quad \frac{\lambda}{h\delta^{k-1}} \int_{C_i^k} \frac{|\beta_j - Q|^2}{r_j^2} dx < \tilde{C},$$

and then by a change of variable we have

$$\int_{T_{h/R}^{\delta,1}} |\tilde{\eta}_j|^2 dx = \frac{1}{R\delta^{k-1}} \int_{C_i^k} |\eta_j|^2 dx \leq \frac{h}{R\lambda} \tilde{C},$$

which implies the weak convergence in (3.46) up to a subsequence. To verify the admissibility of the limiting function we recall that by (3.41) it holds that

$$\int_0^h \int_r^R \varphi(\rho, z) \int_0^{2\pi} \eta_j(\rho, \theta, z) e_{\theta\rho} d\theta d\rho dz = \int_0^h \int_r^R \varphi(\rho, z) Q^T b d\rho dz$$

for all $\varphi \in L^2((R\delta^k, R\delta^{k-1}) \times ((i-1)h\delta^{k-1}, ih\delta^{k-1}))$, and therefore

$$(3.48) \quad \int_0^{h/R} \int_\delta^1 \varphi(\rho, z) \int_0^{2\pi} \tilde{\eta}_j(\rho, \theta, z) e_{\theta\rho} d\theta d\rho dz = \int_0^h \int_r^R \varphi(\rho, z) Q^T b d\rho dz$$

for all $\varphi \in L^2((\delta, 1) \times (0, h/R))$. The conclusion follows by taking the limit as $j \rightarrow \infty$, obtaining

$$(3.49) \quad \int_0^{h/R} \int_\delta^1 \varphi(\rho, z) \int_0^{2\pi} \tilde{\eta}(\rho, \theta, z) e_{\theta\rho} d\theta d\rho dz = \int_0^h \int_r^R \varphi(\rho, z) Q^T b d\rho dz$$

for all $\varphi \in L^2((\delta, 1) \times (0, h/R))$, and recalling (3.14).

Step 2: Estimate of the nonlinear term. We will show, following the ideas in [46], that the scaled problem linearizes in the limit, giving rise to the expected estimate. From (3.33) we have

$$\lambda_j^k := \frac{r_j}{R\delta^{k-1}} \leq \frac{r_j}{R\delta^{k_r}} \leq \delta^{-1} \left(\frac{r_j}{R}\right)^{1-s},$$

and then the sequence λ_j^k is infinitesimal for $j \rightarrow \infty$. We also consider the sequence χ_j of characteristic functions

$$(3.50) \quad \chi_j := \begin{cases} 1 & \text{if } |\tilde{\eta}_j| \leq r_j^{(s-1)/2}, \\ 0 & \text{otherwise in } T_{h/R}^\delta. \end{cases}$$

By the boundedness of $\tilde{\eta}_j$ in $L^2(T_{h/R}^{\delta,1}; \mathbb{R}^{3 \times 3})$ it follows that $\chi_j \rightarrow 1$ in measure, so that, by (3.46), $\chi_j \tilde{\eta}_j \rightharpoonup \tilde{\eta}$ in $L^2(T_{h/R}^{\delta,1}; \mathbb{R}^{3 \times 3})$. By a Taylor expansion, using assumptions (ii) and (iv) on W (see section 2), we get

$$W(I + F) = \frac{1}{2} \mathbb{C}F : F + \sigma(F),$$

where $\mathbb{C} := \frac{\partial^2 W}{\partial F^2}(I)$ and $\sigma(F)/|F|^2 \rightarrow 0$ as $|F| \rightarrow 0$. Setting $\omega(t) := \sup_{|F| \leq t} |\sigma(F)|$, we have

$$(3.51) \quad W(I + rF) \geq \frac{1}{2} r^2 \mathbb{C}F : F - \omega(r|F|),$$

with $\omega(t)/t^2 \rightarrow 0$ as $t \rightarrow 0$. Thus, by the frame indifference of W and using (3.51) and (3.45) we obtain

$$(3.52) \quad \begin{aligned} \frac{1}{h\delta^{k-1}} \int_{C_i^k} \frac{W(\beta_j)}{r_j^2} dx &= \frac{1}{h\delta^{k-1}} \int_{C_i^k} \frac{W(I + r_j \eta_j)}{r_j^2} dx \geq \frac{R}{h} \int_{T_{h/R}^{\delta,1}} \chi_j \frac{W(I + \lambda_j^k \tilde{\eta}_j)}{(\lambda_j^k)^2} dx \\ &\geq \frac{R}{h} \int_{T_{h/R}^{\delta,1}} \left(\frac{1}{2} \mathbb{C}(\chi_j \tilde{\eta}_j) : (\chi_j \tilde{\eta}_j) - \chi_j \frac{\omega(\lambda_j^k |\tilde{\eta}_j|)}{(\lambda_j^k)^2} \right) dx. \end{aligned}$$

Again the first term on the right-hand side of (3.52) is lower semicontinuous with respect to the weak $L^2(T_{h/R}^{\delta,1}; \mathbb{R}^{3 \times 3})$ -convergence, that is,

$$(3.53) \quad \liminf_{j \rightarrow \infty} \frac{R}{h} \int_{T_{h/R}^{\delta,1}} \frac{1}{2} \mathbb{C}(\chi_j \tilde{\eta}_j) : (\chi_j \tilde{\eta}_j) dx \geq \frac{R}{h} \int_{T_{h/R}^{\delta,1}} \frac{1}{2} \mathbb{C}\tilde{\eta} : \tilde{\eta} dx,$$

while the last term in (3.52) converges to zero as $j \rightarrow \infty$. Indeed, we can rewrite its integrand as

$$\chi_j \frac{\omega(\lambda_j^k |\tilde{\eta}_j|)}{(\lambda_j^k)^2} = |\tilde{\eta}_j|^2 \cdot \chi_j \frac{\omega(\lambda_j^k |\tilde{\eta}_j|)}{(\lambda_j^k |\tilde{\eta}_j|)^2},$$

which is the product of a bounded sequence in $L^1(T_{h/R}^{\delta,1})$ and a sequence converging to zero in $L^\infty(T_{h/R}^\delta)$, since $\lambda_j^k |\tilde{\eta}_j| \leq \delta^{-1} R^{s-1} r_j^{(1-s)/2}$ for every k , when $\chi_j \neq 0$. This combined with (3.53) yields

$$\liminf_{j \rightarrow \infty} \frac{1}{h\delta^{k-1}} \int_{C_i^k} \frac{W(\beta_j)}{r_j^2} dx \geq \frac{R}{h} \int_{T_{h/R}^{\delta,1}} \frac{1}{2} \mathbb{C}\tilde{\eta} : \tilde{\eta} dx.$$

Finally, since $\tilde{\eta}$ is admissible for $\Psi(Q^T b, t, h/R, \delta, 1)$, recalling the estimate in Lemma 3.3 with $M := \frac{h}{R}$, we get

$$(3.54) \quad \begin{aligned} \liminf_{j \rightarrow \infty} \frac{1}{h\delta^{k-1}} \int_{C_i^k} \frac{W(\beta_j)}{r_j^2} dx &\geq |\log \delta| \Psi(Q^T b, t, h/R, \delta, 1) \\ &\geq |\log \delta| \Psi_0(Q^T b, t) \left(1 - \frac{C}{M} - \omega_M(\delta)\right). \end{aligned}$$

This concludes the proof of the claim since

$$\begin{aligned} \liminf_{j \rightarrow \infty} \frac{1}{h\delta^{k-1}} \int_{C_i^k} \frac{W(\beta_j) + \lambda|\beta_j - Q|^2}{r_j^2} dx \\ \geq \liminf_{j \rightarrow \infty} \frac{1}{h\delta^{k-1}} \int_{C_i^k} \frac{W(\beta_j)}{r_j^2} dx. \end{aligned} \quad \square$$

LEMMA 3.10 (upper bound). *There exists a constant $C > 0$ such that for every $Q \in SO(3)$, $b \in \mathbb{R}^3$, $t \in S^2$, $h \geq R > 0$, it holds that*

$$\limsup_{r \rightarrow 0} \Psi_\lambda^{nl}(Q, b, t, h, r, R) \leq (1 + \lambda C) \Psi_0(Q^T b, t).$$

Proof. Analogously to the previous lemma, we perform the proof for $t = e_3$ (the general case is similar).

We define the sequence

$$(3.55) \quad \beta_r := Q(I + r\eta_{Q^T b, t}) \quad \text{in } T_h^{r, R},$$

where $\eta_{Q^T b, t}$ is the solution to (3.5) with $\mu_{Q^T b, t} = Q^T b \otimes t\mathcal{H}^1 \llcorner \mathbb{R}t$. Then β_r is admissible for $\Psi_\lambda^{nl}(Q, b, t, h, r, R)$. We observe that from (3.7) we immediately get

$$(3.56) \quad \int_{T_h^{r, R}} |\beta_r - Q|^2 dx = \int_{T_h^{r, R}} |r\eta_{Q^T b, t}|^2 dx \leq C|b|^2 hr^2 \log \frac{R}{r}.$$

Hence by (3.9) we have

$$(3.57) \quad \lambda \int_{T_h^{r, R}} |\beta_r - Q|^2 dx \leq \lambda C \Psi_0(Q^T b, t) hr^2 \log \frac{R}{r}.$$

Next, fix $s \in (0, 1)$ and set

$$(3.58) \quad C_r^1 := T_h^{r^{1-s}, R} \quad \text{and} \quad C_r^2 := T_h^{r, r^{1-s}}.$$

In view of the frame indifference of W , we have

$$\begin{aligned} \frac{1}{hr^2 \log \frac{R}{r}} \int_{T_h^{r, R}} W(\beta_r) dx &= \frac{1}{hr^2 \log \frac{R}{r}} \int_{T_h^{r, R}} W(I + r\eta_{Q^T b, t}) dx \\ &= \frac{1}{hr^2 \log \frac{R}{r}} \int_{C_r^1} W(I + r\eta_{Q^T b, t}) dx + \frac{1}{hr^2 \log \frac{R}{r}} \int_{C_r^2} W(I + r\eta_{Q^T b, t}) dx =: I_r^1 + I_r^2. \end{aligned}$$

We now estimate I_r^1 and I_r^2 . Regarding I_r^1 , by a Taylor expansion of W near the identity we get

$$I_r^1 = \frac{1}{h \log \frac{R}{r}} \int_{C_r^1} \frac{1}{2} \mathbb{C} \eta_{Q^T b, t} : \eta_{Q^T b, t} dx + \frac{1}{h} \int_{C_r^1} \frac{\sigma(r\eta_{Q^T b, t})}{r^2 \log \frac{R}{r}} dx,$$

where $\sigma(F)/|F|^2 \rightarrow 0$ as $|F| \rightarrow 0$. By (3.5) and (3.8) we deduce

$$\int_{C_r^1} \frac{1}{2} \mathbb{C} \eta_{Q^T b, t} : \eta_{Q^T b, t} \, dx = \int_0^h \int_{r^{1-s}}^R \frac{1}{\rho^2} \Psi_0(Q^T b, t) \rho \, d\rho dz = h \log \frac{R}{r^{1-s}} \Psi_0(Q^T b, t),$$

and then

$$(3.59) \quad \limsup_{r \rightarrow 0} \frac{1}{h \log \frac{R}{r}} \int_{C_r^1} \frac{1}{2} \mathbb{C} \eta_{Q^T b, t} : \eta_{Q^T b, t} \, dx \leq \Psi_0(Q^T b, t).$$

Moreover, setting $\omega(t) := \sup_{|F| \leq t} |\sigma(F)|$, we have

$$(3.60) \quad \lim_{r \rightarrow 0} \left| \frac{1}{h} \int_{C_r^1} \frac{\sigma(r \eta_{Q^T b, t})}{r^2 \log \frac{R}{r}} \, dx \right| \leq \lim_{r \rightarrow 0} \frac{1}{h} \int_{T_h^R} \chi_{C_r^1} \frac{\omega(r |\eta_{Q^T b, t}|)}{|r \eta_{Q^T b, t}|^2} \cdot \frac{|r \eta_{Q^T b, t}|^2}{r^2 \log \frac{R}{r}} \, dx = 0.$$

In fact, the above integrand is the product of a sequence converging to zero in $L^\infty(T_h^R)$ and a bounded sequence in $L^1(T_h^R)$ by (3.56). Thus, combining (3.59) and (3.60), we infer

$$(3.61) \quad \limsup_{r \rightarrow 0} I_r^1 \leq \Psi_0(Q^T b, t).$$

Finally, the growth assumption on W and the definition of $\eta_{Q^T b, t}$ give

$$I_r^2 \leq \frac{C}{hr^2 \log \frac{R}{r}} \int_{C_r^2} |r \eta_{Q^T b, t}|^2 \, dx \leq C \frac{\log r^{-s}}{\log \frac{R}{r}};$$

then, as $s < 1$, we get

$$(3.62) \quad \limsup_{r \rightarrow 0} I_r^2 \leq Cs.$$

In view of (3.57), (3.61), and (3.62), we conclude that

$$\limsup_{r \rightarrow 0} \Psi_\lambda^{nl}(Q, b, t, h, r, R) \leq (1 + \lambda C) \Psi_0(Q^T b, t) + Cs;$$

hence the thesis follows by the arbitrariness of $s \in (0, 1)$. □

COROLLARY 3.11. *For any $Q \in SO(3)$, $b \in \mathbb{R}^3$, $t \in S^2$, $R > 0$, one has*

$$(3.63) \quad \lim_{\lambda \rightarrow 0} \lim_{h \rightarrow \infty} \lim_{r \rightarrow 0} \Psi_\lambda^{nl}(Q, b, t, h, r, R) = \Psi_0(Q^T b, t).$$

Proof. The thesis is an immediate consequence of Lemmas 3.9 and 3.10. □

Property (3.63) cannot be directly applied in the analysis of the Γ -limit in section 5; in fact, we need a uniform bound from below (which does not depend on the parameters) for the cell formula in order to deal with the relaxation process. The uniform bound is proved in the following lemma.

LEMMA 3.12. *For every $\lambda > 0$, $M \geq 1$, $L > 0$, there exist $C > 0$ and $\omega_{M,L} : (0, \infty) \rightarrow (0, \infty)$ with*

$$\lim_{r \rightarrow 0} \omega_{M,L}(r) = 0,$$

such that

$$(3.64) \quad \Psi_\lambda^{nl}(Q, b, t, h, r, R) \geq \Psi_0(Q^T b, t) - \frac{CL^2}{M} - \omega_{M,L} \left(\frac{r}{R} \right)$$

for all $Q \in SO(3)$, $b \in \bar{B}_L(0) \subset \mathbb{R}^3$, $t \in S^2$, $r, R, h > 0$ with $MR \leq h$. Furthermore, there exists $c_* > 0$ (which does not depend on the parameters), such that, for all λ , Q, b, t, h, r, R with $2r \leq R \leq h$,

$$(3.65) \quad \Psi_\lambda^{nl}(Q, b, t, h, r, R) \geq \left(1 - \frac{R}{h} \right) c_* |b|^2.$$

REMARK 3.13. As a consequence of the above estimate we also deduce a uniform bound from below of the elastic energy which will be needed for the compactness argument. Namely, if $\beta \in L^2(Q_t T_h^{r,R}; \mathbb{R}^{3 \times 3})$ is a test function for $\Psi_\lambda^{nl}(Q, b, t, h, r, R)$, then from (3.65) we get

$$(3.66) \quad \frac{1}{hr^2 \log \frac{R}{r}} \int_{Q_t T_h^{r,R}} W(\beta) dx \geq \lim_{\lambda \rightarrow 0} \Psi_\lambda^{nl}(Q, b, t, h, r, R) \geq \left(1 - \frac{R}{h} \right) c_* |b|^2.$$

For the proof of Lemma 3.12 we follow the approach proposed in [16, Lemma 5.7], which requires the following preliminary result.

LEMMA 3.14. Let $\lambda > 0$, let $H := \{(b, t, h) \in \mathbb{R}^3 \times S^2 \times (0, \infty)\}$, and let $K \subset H$ be compact. The family of functions $\Psi_\lambda^{nl}(I, \cdot, \cdot, \cdot, r, 1)$, $r \in (0, 1)$, is equicontinuous on K .

Proof. Let $\lambda > 0$ and $K \subset H$ compact be fixed. It suffices to prove equicontinuity of Ψ_λ^{nl} with respect to r separately in each variable b, t , and h .

Step 1. Continuity in b . We show that there exists a constant $c > 0$ (not depending on r) such that for all $b, b' \in \mathbb{R}^3$, it holds that

$$\Psi_\lambda^{nl}(I, b', t, h, r, 1) - \Psi_\lambda^{nl}(I, b, t, h, r, 1) \leq c|b' - b|(1 + c|b' - b|)$$

for all $t \in S^2, h > 0, r \in (0, 1)$.

Let $b, b' \in \mathbb{R}^3$, consider β as a test function for $\Psi_\lambda^{nl}(I, b, t, h, r, 1)$, and define

$$\beta' := \beta + r\eta_{b'-b,t},$$

with $\eta_{b'-b,t}$ the solution of (3.4) for $\mu_{b'-b,t}$. Then for every $\sigma > 0$ we have

$$\begin{aligned} \int_{Q_t T_h^{r,1}} |\beta' - I|^2 dx &\leq (1 + \sigma) \int_{Q_t T_h^{r,1}} |\beta - I|^2 dx + \left(1 + \frac{1}{\sigma} \right) \int_{Q_t T_h^{r,1}} |\beta' - \beta|^2 dx \\ &= (1 + \sigma) \int_{Q_t T_h^{r,1}} |\beta - I|^2 dx + \left(1 + \frac{1}{\sigma} \right) \int_{Q_t T_h^{r,1}} |r\eta_{b'-b,t}|^2 dx. \end{aligned}$$

Moreover, by (2.6) we get

$$\begin{aligned} \int_{Q_t T_h^{r,1}} W(\beta') dx - \int_{Q_t T_h^{r,1}} W(\beta) dx \\ \leq C_3 \sigma \int_{Q_t T_h^{r,1}} W(\beta) dx + C_3 \left(1 + \frac{1}{\sigma} \right) \int_{Q_t T_h^{r,1}} |\eta_{b'-b,t}|^2 dx. \end{aligned}$$

Then setting $\sigma = |b' - b|$, after rescaling and using the arbitrariness of the test function β , we finally get

$$\Psi_\lambda^{nl}(I, b', t, h, r, 1) - \Psi_\lambda^{nl}(I, b, t, h, r, 1) \leq c|b' - b|(\Psi_\lambda^{nl}(I, b, t, h, r, 1) + 1) + c|b' - b|^2.$$

We conclude the proof of Step 1 observing that by the growth condition on W

$$\begin{aligned} \Psi_\lambda^{nl}(I, b, t, h, r, 1) &\leq \frac{1}{hr^2 \log \frac{1}{r}} \int_{Q_t T_h^{r,1}} W(I + r\eta_{b,t}) + \lambda|r\eta_{b,t}|^2 dx \\ (3.67) \quad &\leq \frac{c + \lambda}{hr^2 \log \frac{1}{r}} \int_{Q_t T_h^{r,1}} |r\eta_{b,t}|^2 dx \leq c(1 + \lambda)|b|^2. \end{aligned}$$

Step 2. Continuity in t. We now show that there is a constant $c > 0$ such that for all $t, t' \in S^2$,

$$\Psi_\lambda^{nl}(I, b, t', h, r, 1) - \Psi_\lambda^{nl}(I, b, t, h, r, 1) \leq c(|t' - t| + |t' - t|^2)$$

for all $h > 0, r \in (0, 1)$.

We choose $t, t' \in S^2$ and fix as above β to be an admissible strain in the definition of $\Psi_\lambda^{nl}(I, b, t, h, r, 1)$. The function

$$\beta'(x) := \beta(S^T x)S^T - (S^T - I), \quad S := Q_{t'}Q_t^T \in SO(3),$$

is admissible for $\Psi_\lambda^{nl}(Q, b, t', h, r, 1)$ and

$$(3.68) \quad |S^T - I| \leq c|t' - t| \quad \forall t', t \in S^2.$$

If we let $\tilde{\beta}(x) := \beta'(Sx)$, then a change of variable gives

$$\int_{Q_{t'} T_h^{r,1}} W(\beta') + \lambda|\beta' - I|^2 dx = \int_{Q_t T_h^{r,1}} W(\tilde{\beta}) + \lambda|\tilde{\beta} - I|^2 dx.$$

For every $\sigma > 0$, writing $\tilde{\beta} - I = (\tilde{\beta} - \beta) + (\beta - I)$ and using (3.68), we get

$$\int_{Q_t T_h^{r,1}} |\tilde{\beta} - I|^2 \leq \left(1 + \frac{1}{\sigma}\right) c|t' - t|^2 \int_{Q_t T_h^{r,1}} |\beta - I|^2 dx + (1 + \sigma) \int_{Q_t T_h^{r,1}} |\beta - I|^2 dx.$$

Now choosing $\sigma = |t' - t|$ in the above inequality, we find

$$(3.69) \quad \lambda \int_{Q_t T_h^{r,1}} (|\tilde{\beta} - I|^2 - |\beta - I|^2) dx \leq \lambda c(|t' - t| + |t' - t|^2) \int_{Q_t T_h^{r,1}} |\beta - I|^2 dx.$$

Moreover, by (2.6) and (3.68) there holds

$$\begin{aligned} &\int_{Q_t T_h^{r,1}} W(\tilde{\beta}) dx - \int_{Q_t T_h^{r,1}} W(\beta) dx \\ &\leq C_3 \sigma \int_{Q_t T_h^{r,1}} W(\beta) dx + C_3 \left(1 + \frac{1}{\sigma}\right) |t' - t|^2 \int_{Q_t T_h^{r,1}} |\beta - I|^2 dx. \end{aligned}$$

Hence we obtain

$$\int_{Q_t T_h^{r,1}} W(\tilde{\beta}) - W(\beta) \, dx \leq c(|t' - t| + |t' - t|^2) \int_{Q_t T_h^{r,1}} W(\beta) + |\beta - I|^2 \, dx.$$

Combining this with (3.69), we have

$$\Psi_\lambda^{nl}(I, b, t', h, r, 1) - \Psi_\lambda^{nl}(I, b, t, h, r, 1) \leq c(|t' - t| + |t' - t|^2) \Psi_{\lambda+1}^{nl}(I, b, t, h, r, 1),$$

which again gives the conclusion using (3.67).

Step 3. Continuity in h. We assume without loss of generality $h \geq \delta > 0$ (where δ depends on the compact set K). We show that there exists a constant $c > 0$ such that for $h, h' \geq \delta$,

$$|\Psi_\lambda^{nl}(I, b, t, h', r, 1) - \Psi_\lambda^{nl}(I, b, t, h, r, 1)| \leq c|h' - h|$$

for all $t \in S^2$ with $(t, h) \in K$, and for all $r \in (0, 1)$.

Assume that $h' > h$ and notice that any test function for $\Psi_\lambda^{nl}(I, b, t, h', r, 1)$ can be restricted to get a test function for $\Psi_\lambda^{nl}(I, b, t, h, r, 1)$ and in particular

$$(3.70) \quad \Psi_\lambda^{nl}(I, b, t, h, r, 1) \leq \frac{h'}{h} \Psi_\lambda^{nl}(I, b, t, h', r, 1),$$

or equivalently, using (3.67),

$$\begin{aligned} \Psi_\lambda^{nl}(I, b, t, h, r, 1) - \Psi_\lambda^{nl}(I, b, t, h', r, 1) &\leq \frac{h' - h}{h} \Psi_\lambda^{nl}(I, b, t, h', r, 1) \\ &\leq \frac{c(1 + \lambda)|b|^2}{\delta} |h' - h|. \end{aligned}$$

It remains to show the other inequality. As usual we assume $t = e_3$. Let β be admissible for $\Psi_\lambda^{nl}(I, b, t, h, r, 1)$ and let $u \in W^{1,2}(T_h^{r,R}; \mathbb{R}^3)$ be such that $\nabla u = \beta - r\eta_{b,t}$. Then we estimate

$$\begin{aligned} \int_{T_h^{r,R}} |\nabla u - I|^2 \, dx &\leq 2 \int_{T_h^{r,R}} |\beta - I|^2 \, dx + 2 \int_{T_h^{r,R}} |r\eta_{b,t}|^2 \, dx \\ &\leq 2 \int_{T_h^{r,R}} |\beta - I|^2 \, dx + c|b|^2 hr^2 \log \frac{1}{r}. \end{aligned}$$

We consider $w \in W^{1,2}(T_h^{r,R}; \mathbb{R}^3)$ the function such that $\nabla w = \nabla u - I$ and we choose $y_3 \in (0, h)$ such that

$$(3.71) \quad \begin{aligned} \int_{B'_1 \setminus B'_r} |\nabla w(x', y_3)|^2 \, dx' &\leq \frac{1}{h} \int_{T_h^{r,R}} |\nabla w(x', x_3)|^2 \, dx \\ &\leq \frac{2}{h} \int_{T_h^{r,R}} |\beta - I|^2 \, dx + c|b|^2 r^2 \log \frac{1}{r}. \end{aligned}$$

Next we define the extension $\tilde{w} \in W^{1,2}(T_{h'}^{r,R}; \mathbb{R}^3)$ as

$$\tilde{w}(x', x_3) := \begin{cases} w(x', x_3) & \text{if } x_3 \in (0, y_3), \\ w(x', y_3) & \text{if } x_3 \in (y_3, y_3 + h' - h), \\ w(x', x_3 - (h' - h)) & \text{if } x_3 \in (y_3 + h' - h, h'). \end{cases}$$

The function $\tilde{\beta} := r\eta_{b,t} + \nabla\tilde{w} + I$ is admissible for $\Psi_\lambda^{nl}(I, b, t, h', r, 1)$ and satisfies

$$\begin{aligned} \int_{T_{h'}^{r,R}} |\tilde{\beta} - I|^2 dx &= \int_{T_h^{r,R}} |\beta - I|^2 dx + (h' - h) \int_{B'_1 \setminus B'_r} |r\eta_{b,t} + \nabla\tilde{w}(x', y_3)|^2 dx' \\ &\leq \int_{T_h^{r,R}} |\beta - I|^2 dx + (h' - h) \left(\frac{2}{h} \int_{T_h^{r,R}} |\beta - I|^2 dx + c|b|^2 r^2 \log \frac{1}{r} \right), \end{aligned}$$

where we have used (3.71). Furthermore, using the growth assumption on W we find

$$\begin{aligned} \int_{T_{h'}^{r,R}} W(\tilde{\beta}) dx &= \int_{T_h^{r,R}} W(\beta) dx + (h' - h) \int_{B'_1 \setminus B'_r} W(r\eta_{b,t} + \nabla\tilde{w}(x', y_3) + I) dx' \\ &\leq \int_{T_h^{r,R}} W(\beta) dx + c(h' - h) \int_{B'_1 \setminus B'_r} |r\eta_{b,t} + \nabla\tilde{w}(x', y_3)|^2 dx' \\ &\leq \int_{T_h^{r,R}} W(\beta) dx + c(h' - h) \left(\frac{c}{h} \int_{T_h^{r,R}} |\beta - I|^2 dx + c|b|^2 r^2 \log \frac{1}{r} \right). \end{aligned}$$

Then we conclude

$$\begin{aligned} \Psi_\lambda^{nl}(I, b, t, h', r, 1) - \Psi_\lambda^{nl}(I, b, t, h, r, 1) &\leq c \frac{h' - h}{h} (|b|^2 + \Psi_{\lambda+1}^{nl}(I, b, t, h, r, 1)) \\ &\leq \frac{c(1 + \lambda)|b|^2}{\delta} |h' - h|. \quad \square \end{aligned}$$

Proof of Lemma 3.12. By property (3.29) we can reduce to $R = 1$. In addition we can assume $h \in [M, 2M]$. In fact for any $N \in \mathbb{N}$, we can subdivide $Q_t T_h^{r,1}$ into N cylinders T_i , $i = 1, \dots, N$, of length h/N , and for any β test function for $\Psi_\lambda^{nl}(Q, b, t, h, r, 1)$ we have

$$\int_{Q_t T_h^{r,1}} W(\beta) + \lambda |\beta - Q|^2 dx = \sum_{i=1}^N \int_{T_i} W(\beta) + \lambda |\beta - Q|^2 dx \geq N \min_i \int_{T_i} W(\beta) + \lambda |\beta - Q|^2 dx$$

and therefore

$$(3.72) \quad \Psi_\lambda^{nl}(Q, b, t, h, r, 1) \geq \Psi_\lambda^{nl}\left(Q, b, t, \frac{h}{N}, r, 1\right).$$

Consider the compact set $H_{M,L} := \bar{B}_L(0) \times S^2 \times [M, 2M]$ and recall that by the previous lemma $\Psi_\lambda^{nl}(I, \cdot, \cdot, \cdot, r, 1)$ is equicontinuous on $H_{M,L}$. Therefore for any $j \in \mathbb{N}$ there exists $\delta_j > 0$ such that

$$|\Psi_\lambda^{nl}(I, b, t, h, r, 1) - \Psi_\lambda^{nl}(I, b', t', h', r, 1)| \leq 1/j$$

if $|b' - b| + |t' - t| + |h' - h| \leq \delta_j$, $(b, t, h), (b', t', h') \in H_{M,L}$, $r \in (0, 1)$.

Hence from (3.30) it follows that

$$(3.73) \quad |\Psi_\lambda^{nl}(Q, b, t, h, r, 1) - \Psi_\lambda^{nl}(Q, b', t', h', r, 1)| \leq 1/j$$

if $|b' - b| + |t' - t| + |h' - h| \leq \delta_j$, $Q \in SO(3)$, $(b, t, h), (b', t', h') \in H_{M,L}$, $r \in (0, 1)$.

We fix $Q \in SO(3)$ and cover $H_{M,L}$ with finitely many balls of radius δ_j . By Lemma 3.9, at every center (b_l, t_l, h_l) of these balls of radius δ_j , we have

$$\liminf_{r \rightarrow 0} \Psi_\lambda^{nl}(Q, b_l, t_l, h_l, r, 1) \geq \Psi_0(Q^T b_l, t_l) - \frac{C}{M} |b_l|^2.$$

Therefore for every l there exists $r_j(l) \in (0, r_{j-1}(l))$ such that for all $r < r_j(l)$ it holds that

$$(3.74) \quad \Psi_\lambda^{nl}(Q, b_l, t_l, h_l, r, 1) \geq \Psi_0(Q^T b_l, t_l) - \frac{C}{M} |b_l|^2 - \frac{1}{j}.$$

Define $r_j := \min_l r_j(l)$ and for any $(b, t, h) \in H_{M,L}$ let (b_l, t_l, h_l) be the center of the ball that contains (b, t, h) . Then for every $r < r_j$ we get

$$(3.75) \quad \begin{aligned} \Psi_\lambda^{nl}(Q, b, t, h, r, 1) - \Psi_0(Q^T b, t) &= (\Psi_\lambda^{nl}(Q, b, t, h, r, 1) - \Psi_\lambda^{nl}(Q, b_l, t_l, h_l, r, 1)) \\ &\quad + (\Psi_\lambda^{nl}(Q, b_l, t_l, h_l, r, 1) - \Psi_0(Q^T b, t)) \\ &\geq -\frac{3}{j} - \frac{c}{M} L^2, \end{aligned}$$

where we have used (3.74), (3.73), and the continuity of Ψ_0 .

From this we deduce that

$$\Psi_\lambda^{nl}(Q, b, t, h, r, 1) \geq \Psi_0(Q^T b, t) - \frac{c}{M} L^2 - \omega_{M,L}(r)$$

for all $Q \in SO(3)$, $(b, t, h) \in H_{M,K}$, and $r < r_j$, with $\omega_{M,L}(r) := 3/j$ for $r \in (r_{j+1}, r_j]$, and (3.64) is proven.

It remains to prove (3.65). We assume $t = e_3$. Using first (3.70) and then (3.72) for $N = \lfloor h \rfloor$ we get

$$(3.76) \quad \Psi_\lambda^{nl}(Q, b, t, h, r, 1) \geq \frac{\lfloor h \rfloor}{h} \Psi_\lambda^{nl}(Q, b, t, \lfloor h \rfloor, r, 1) \geq \frac{\lfloor h \rfloor}{h} \Psi_\lambda^{nl}(Q, b, t, 1, r, 1).$$

Let β be admissible for $\Psi_\lambda^{nl}(Q, b, t, 1, r, 1)$. Then, by the growth condition on W and the rigidity estimate on cylinders, Corollary 3.7, there exist $c > 0$ (independent of the parameters) and $\tilde{Q} \in SO(3)$ such that

$$\int_{T_1^{r,1}} W(\beta) dx \geq c \int_{T_1^{r,1}} |\beta - \tilde{Q}|^2 dx,$$

and by Jensen's inequality and using the condition on the curl of β we obtain

$$\begin{aligned} \int_{T_1^{r,1}} W(\beta) dx &\geq c \int_0^1 \int_r^1 \frac{1}{2\pi\rho} \left| \int_0^{2\pi} (\beta - \tilde{Q}) \cdot e_{\theta\rho} d\theta \right|^2 d\rho dz \\ &= c \int_0^1 \int_r^1 \frac{1}{2\pi\rho} \left| \int_0^{2\pi} \beta \cdot e_{\theta\rho} d\theta \right|^2 d\rho dz \\ &= c \int_0^1 \int_r^1 \frac{r^2}{2\pi\rho} |\beta|^2 d\rho dz = c |b|^2 r^2 \log \frac{1}{r}. \end{aligned}$$

Analogously we obtain

$$\int_{T_1^{r,1}} \lambda |\beta - Q|^2 dx \geq \lambda c |b|^2 r^2 \log \frac{1}{r}.$$

Thus from (3.76) we conclude that

$$\Psi_\lambda^{nl}(Q, b, t, h, r, 1) \geq \frac{\lfloor h \rfloor}{h} (1 + \lambda) c |b|^2 \geq \left(1 - \frac{1}{h}\right) c |b|^2. \quad \square$$

4. A uniform estimate of the quadratic energy. This section is devoted to the analysis of the asymptotic behavior of the quadratic energy associated to a sequence of dilute dislocations. Indeed, as already pointed out, our approach exploits the fact that the energy asymptotically linearizes near a suitable rotation. We will provide in Proposition 4.4 an L^2 estimate of the strains corresponding to a sequence of dilute dislocations $\mu_\varepsilon \in \mathcal{M}_B^{h,\alpha_\varepsilon}(\bar{\Omega})$. This result will be crucial for the proof of compactness in section 5.

It is indeed known (see [9]) that for any measure $\mu \in \mathcal{M}(\mathbb{R}^3)$ there exists a unique $\eta \in L^{3/2}(\mathbb{R}^3; \mathbb{R}^{3 \times 3})$ which is a distributional solution of

$$(4.1) \quad \begin{cases} \operatorname{div} \eta = 0 & \text{in } \mathbb{R}^3, \\ \operatorname{curl} \eta = \mu & \text{in } \mathbb{R}^3. \end{cases}$$

In general we cannot expect a better summability for η ; in particular it might not belong to L^p if $p > 3/2$ (see [16, Lemma 4.1]). Nevertheless, if we assume that the measure is polyhedral, it is possible to show that η is in L^p for any $p < 2$. In particular, one can show that if $\mu \in \mathcal{M}_B(\mathbb{R}^3)$ is a dislocation measure of the form

$$(4.2) \quad \mu := \sum_i b_i \otimes t_i \mathcal{H}^1 \llcorner \gamma_i,$$

where γ_i are straight segments, then there is a constant $C > 0$, independent of μ , such that the following estimates hold:

$$(4.3) \quad |\eta(x)| \leq \frac{C|\mu|(\mathbb{R}^3)}{\operatorname{dist}(x, \operatorname{supp} \mu)^2},$$

$$(4.4) \quad |\eta(x)| \leq C \sum_i \frac{|b_i|}{\operatorname{dist}(x, \gamma_i)}.$$

This is, for instance, proved in [16, Lemma 4.1] in a more general context.

Note that in these estimates μ needs to be defined in the whole of \mathbb{R}^3 and divergence-free. Therefore, in order to use them we will extend any measure in $\mathcal{M}_B^{h,\alpha}(\bar{\Omega})$ to a measure in $\mathcal{M}_B(\mathbb{R}^3)$. In what follows we will first prove that similar estimates hold true for a larger class of measures, which we will call deformed polyhedral, namely measures of the form (4.2) where γ_i are segments deformed under suitably regular maps. Second, we will extend a dilute measure μ in $\bar{\Omega}$ to a deformed polyhedral one in the whole of \mathbb{R}^3 .

Precisely, a dislocation measure $\mu \in \mathcal{M}_B(\mathbb{R}^3)$ is said to be a *deformed polyhedral measure* if it is of the form (4.2) where now $\gamma_i := \Phi_i(L_i)$ for some L_i straight segments and $\Phi_i : \mathbb{R}^3 \rightarrow \mathbb{R}^3$ uniformly bi-Lipschitz functions; i.e., there is a positive constant $\ell > 1$ such that, for all i

$$(4.5) \quad \frac{1}{\ell}|x - y| \leq |\Phi_i(x) - \Phi_i(y)| \leq \ell|x - y| \quad \text{for all } x, y \in \mathbb{R}^3.$$

LEMMA 4.1. *Let $\mu \in \mathcal{M}_B(\mathbb{R}^3)$ be a deformed polyhedral dislocation measure. Let $\eta \in L^{3/2}(\mathbb{R}^3; \mathbb{R}^{3 \times 3})$ be the solution to (4.1). Then there is a constant $C > 0$ such that, for all $x \notin \operatorname{supp} \mu$, (4.3) and (4.4) hold.*

Proof. The proof follows the argument in Lemma 4.1 in [16], by means of a change of variable and exploiting the uniform bi-Lipschitz condition in (4.5). For the reader's convenience we sketch the main points.

From (4.1), since $\text{curl curl } \eta = \nabla \text{div } \eta - \Delta \eta$, we obtain that $\eta = (-\Delta)^{-1}(\text{curl } \mu)$, and therefore for all $x \notin \text{supp } \mu$

$$(4.6) \quad \eta_{ij}(x) = \int_{\mathbb{R}^3} K_{jk}(x-y) d\mu_{ik}(y),$$

where $K_{lm}(x) = -\varepsilon_{lmk} \frac{x_k}{4\pi|x|^3}$. Then we have

$$(4.7) \quad |\eta(x)| \leq C \sum_i |b_i| \int_{\gamma_i} \frac{1}{|x-y|^2} d\mathcal{H}^1(y).$$

Estimate (4.3) is easily achieved. To obtain (4.4) we use the property of Φ_i , (4.5), and with a change of variables we write

$$(4.8) \quad \int_{\gamma_i} \frac{1}{|x-y|^2} d\mathcal{H}^1(y) \leq \ell \int_{L_i} \frac{1}{|x'-y'|^2} d\mathcal{H}^1(y') \leq C \frac{\ell}{\text{dist}(x', L_i)},$$

where $x' = \Phi_i^{-1}(x)$, $y' = \Phi_i^{-1}(y)$, and the last inequality can be obtained by a direct estimate (see [16]). Again by assumption (4.5) we have $|x' - y'| \geq \frac{1}{\ell} |x - y|$, and hence

$$(4.9) \quad \text{dist}(x', L_i) \geq \ell \text{dist}(x, \gamma_i),$$

which in turn implies

$$\int_{\gamma_i} \frac{1}{|x-y|^2} d\mathcal{H}^1(y) \leq C \frac{1}{\text{dist}(x, \gamma_i)}$$

for some constant $C > 0$ independent of i . Therefore, together with (4.7), we readily conclude. \square

We then state and prove an extension lemma for a polyhedral measure $\mu \in \mathcal{M}_{\mathcal{B}}(\Omega)$, to a deformed polyhedral measure $\tilde{\mu} \in \mathcal{M}_{\mathcal{B}}(\mathbb{R}^3)$, which is a refinement of Lemma 2.3 proved in [12]. We recall that we are working with Ω a C^2 domain.

LEMMA 4.2. *Let $\mu \in \mathcal{M}_{\mathcal{B}}(\Omega)$ be a polyhedral dislocation measure, $\mu := \sum_{i \in I} b_i \otimes t_i \mathcal{H}^1 \llcorner \gamma_i$, with γ_i straight segments. Then there is a measure $\tilde{\mu} \in \mathcal{M}_{\mathcal{B}}(\mathbb{R}^3)$ that satisfies the following properties:*

- (i) $\tilde{\mu}$ extends μ , that is $\tilde{\mu} \llcorner \Omega = \mu$;
- (ii) $\tilde{\mu}$ is deformed polyhedral,

$$\tilde{\mu} := \sum_{i \in \tilde{I}} \tilde{b}_i \otimes \tilde{t}_i \mathcal{H}^1 \llcorner \tilde{\gamma}_i;$$

- (iii) *there is a constant $C > 0$ depending only on the domain Ω such that the following estimates hold:*

$$(4.10) \quad |\tilde{\mu}|(\mathbb{R}^3) \leq C |\mu|(\Omega),$$

$$(4.11) \quad \sum_{i \in \tilde{I}} |\tilde{b}_i|^2 \mathcal{H}^1(\tilde{\gamma}_i) \leq C \sum_{i \in I} |b_i|^2 \mathcal{H}^1(\gamma_i),$$

$$(4.12) \quad \#(\tilde{I}) \leq \#(I) + C |\mu|(\Omega).$$

Proof. The proof of Lemma 4.2 can be straightforwardly achieved by following the lines of the proof of Lemma 2.3 in [12], with the addition of minimal changes.

We discuss it without much detail. We can first define the extension $\tilde{\mu}$ in an outer neighborhood of $\partial\Omega$ by a reflection argument. If the boundary of Ω is of class C^2 , we can use the reflection Φ through $\partial\Omega$ that is a bi-Lipschitz transformation which sends the set

$$\partial\Omega_s^i := \{x \in \Omega : \text{dist}(x, \partial\Omega) = s\}$$

onto

$$\partial\Omega_s^e := \{x \in \mathbb{R}^3 \setminus \Omega : \text{dist}(x, \partial\Omega) = s\}$$

for any $s \leq s_0$, where $s_0 > 0$ is small enough and depends only on the geometry of Ω . Since by slicing it holds that

$$\sum_{i \in I} |b_i| \int_{s_0/2}^{s_0} \mathcal{H}^0(\gamma_i \cap \partial\Omega_s) ds \leq |\mu|(\Omega),$$

and

$$\sum_{i \in I} |b_i|^2 \int_{s_0/2}^{s_0} \mathcal{H}^0(\gamma_i \cap \partial\Omega_s) ds \leq C \sum_{i \in I} |b_i|^2 \mathcal{H}^1(\gamma_i),$$

we can find $\bar{s} \in (s_0/2, s_0)$ such that the slice of the measure μ on $\partial\Omega_{\bar{s}}^i$ satisfies

$$(4.13) \quad \sum_{i \in I} |b_i| \mathcal{H}^0(\gamma_i \cap \partial\Omega_{\bar{s}}^i) \leq C |\mu|(\Omega),$$

$$(4.14) \quad \sum_{i \in I} |b_i|^2 \mathcal{H}^0(\gamma_i \cap \partial\Omega_{\bar{s}}^i) \leq C \sum_{i \in I} |b_i|^2 \mathcal{H}^1(\gamma_i)$$

for some constant $C > 0$ independent of μ . In particular, $\partial\Omega_{\bar{s}}^i$ intersects the support of μ in a finite number of points, i.e.,

$$\mathcal{H}^0(\partial\Omega_{\bar{s}}^i \cap (\text{supp } \mu)) \leq |\mu|(\Omega).$$

Thus we set $\hat{\mu} := \Phi_{\#}(\mu \llcorner (\partial\Omega)_{\bar{s}}^i)$, where $(\partial\Omega)_{\bar{s}}^i = \cup_{s \in (0, \bar{s})} \partial\Omega_s^i$ and $\Phi_{\#}$ denotes the push forward of a measure by Φ . Notice that $\hat{\mu}$ is of the form

$$(4.15) \quad \hat{\mu} = \sum_{j \in \bar{I}} b_j \otimes \hat{t}_j \mathcal{H}^1 \llcorner \hat{\gamma}_j,$$

where $\hat{\gamma}_j$ is the image by Φ of some straight segment γ_j belonging to the support of μ and \hat{t}_j the unit tangent to $\hat{\gamma}_j$.

We then extend $\hat{\mu}$ outside Ω by connecting the endpoints of $\hat{\gamma}_j$ to each other using piecewise affine curves and associating to them a suitable multiplicity (for details we refer the reader to Step 3 of [12, Lemma 2.3]). The final extension is obtained as

$$(4.16) \quad \tilde{\mu} := \hat{\mu} + \mu.$$

Property (4.12) easily follows by the construction. □

REMARK 4.3.

- (a) *The support of $\tilde{\mu}$ is given by curves consisting of straight segments, with the only exception being the part of $\text{supp } \tilde{\mu}$ which intersects the set $(\partial\Omega)_{\bar{s}}^e = \cup_{s \in (0, \bar{s})} \partial\Omega_s^e$ where the curves γ_i of the support of $\tilde{\mu}$ are obtained by reflecting by Φ the segments of μ contained in $(\partial\Omega)_{\bar{s}}^i$. We denote by $I_{\bar{s}}^e$ the set indices corresponding to the curves $\gamma_i \subseteq (\partial\Omega)_{\bar{s}}^e$.*

- (b) *The bi-Lipschitz map Φ defined in the proof of Lemma 4.2 is not a global map in the whole of \mathbb{R}^3 (as is required in the hypothesis of Lemma 4.1). However, with a covering argument and using the regularity of $\partial\Omega$ the map Φ can be extended locally in order to fulfill the assumptions in the lemma. Precisely by the fact that $\partial\Omega$ is C^1 we can choose s_0 such that for every curve $\gamma_i \subseteq (\partial\Omega)_{\bar{s}}^e$, with $i \in I_{\bar{s}}^e$, we can construct a Φ_i satisfying (4.5) which coincides with Φ in a neighborhood of γ_i of diameter s_0 . Therefore, if $\tilde{\eta} \in L^{3/2}(\mathbb{R}^3; \mathbb{R}^{3 \times 3})$ is the unique solution to (4.1) with μ replaced by $\tilde{\mu}$, then $\tilde{\eta}$ still satisfies estimates (4.3) and (4.4).*
- (c) *Since the piecewise straight lines built outside $\Omega \cup (\partial\Omega)_{\bar{s}}^e$ are arbitrary, we can assume they consist of segments which have minimal length $h > 0$. This will be useful when we extend a dilute measure μ in Ω .*
- (d) *From the proof of Lemma 4.2 we obtain that the number of points in $\text{supp}(\gamma) \cap \partial\Omega_{\bar{s}}$ weighted with the norm of their Burgers vectors (and their squares) is controlled, namely*

$$(4.17) \quad \sum_{i \in I} |b_i| \mathcal{H}^0(\gamma_i \cap \partial\Omega_{\bar{s}}^i) \leq C |\mu|(\Omega), \quad \sum_{i \in I} |b_i|^2 \mathcal{H}^0(\gamma_i \cap \partial\Omega_{\bar{s}}^i) \leq C \sum_{i \in I} |b_i|^2 \mathcal{H}^1(\gamma_i).$$

We recall that $\varphi_\varepsilon(x) = \varepsilon^{-3} \varphi(\frac{x}{\varepsilon})$ is a mollifier with $\varphi_\varepsilon \leq c \frac{\chi_{B_\varepsilon(0)}}{|B_\varepsilon(0)|}$ for $c > 0$.

PROPOSITION 4.4. *Let $\mu_\varepsilon := \sum_{i=1}^{M_\varepsilon} b_\varepsilon^i \otimes t_\varepsilon^i \mathcal{H}^1 \llcorner \gamma_\varepsilon^i$ be a sequence in $\mathcal{M}_B^{h_\varepsilon, \alpha_\varepsilon}(\bar{\Omega})$ such that*

$$(4.18) \quad \sum_{i=1}^{M_\varepsilon} |b_\varepsilon^i|^2 \mathcal{H}^1(\gamma_\varepsilon^i) \leq C < \infty.$$

Then there exists a sequence $\tilde{\eta}_\varepsilon \in L^{3/2}(\mathbb{R}^3; \mathbb{R}^{3 \times 3})$ with the following properties:

- (i) *$\text{curl } \tilde{\eta}_\varepsilon = \mu_\varepsilon$ distributionally in Ω ;*
- (ii) *there exists a constant $c > 0$ (independent of ε) such that*

$$\int_{\Omega_\varepsilon(\mu_\varepsilon)} |\tilde{\eta}_\varepsilon|^2 dx \leq c |\log \varepsilon|,$$

where $\Omega_\varepsilon(\mu_\varepsilon) := \{x \in \Omega : \text{dist}(x, \text{supp } \mu_\varepsilon) > \varepsilon\}$.

Furthermore, there exist a sequence $\hat{\eta}_\varepsilon \in L^{3/2}(\mathbb{R}^3; \mathbb{R}^{3 \times 3}) \cap L^2(\Omega; \mathbb{R}^{3 \times 3})$ and a constant $c > 0$ (independent of ε) such that for every $\Omega' \subset\subset \Omega$

- (i') *$\text{curl } \hat{\eta}_\varepsilon = \mu_\varepsilon * \varphi_\varepsilon$ distributionally in Ω' ;*
- (ii')

$$\int_{\Omega} |\hat{\eta}_\varepsilon|^2 dx \leq c |\log \varepsilon|.$$

Before proving the proposition, we fix some notation and give a preliminary result. If γ is a segment, for every $\rho > 0$ and $\delta \geq 0$, we denote by $T_{\rho, \delta}(\gamma)$ the cylinder with vertical axis γ , radius ρ , and length $\mathcal{H}^1(\gamma) - 2\delta$, namely

$$(4.19) \quad T_{\rho, \delta}(\gamma) := A(B'_\rho \times S_\delta),$$

where B'_ρ is the ball in \mathbb{R}^2 of radius ρ centered at the origin, $S_\delta \subset \mathbb{R}$ is a segment of length $\mathcal{H}^1(\gamma) - 2\delta$, and A is an affine transformation that maps S_δ in γ and the midpoint of S_δ into the midpoint of γ . To shorten the notation, if γ is fixed, we will simply write $T_{\rho, \delta}$.

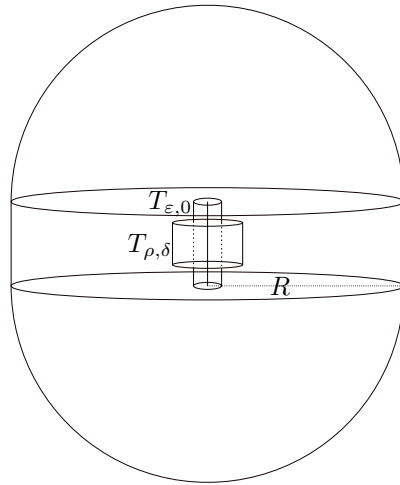


FIG. 1. Large domain of diameter $R + \mathcal{H}^1(\gamma)$ containing $T_{\rho, \delta}$ and $T_{\varepsilon, 0}$.

LEMMA 4.5. Let $\Phi: \mathbb{R}^3 \rightarrow \mathbb{R}^3$ be a bi-Lipschitz map and $\Omega \subseteq \mathbb{R}^3$. There exists a constant $C > 0$ such that given a segment γ , a number $\delta \geq 0$, and positive parameters $\rho > \varepsilon$, it holds that

$$(4.20) \quad \int_{\Omega \setminus (U \cup V)} \frac{1}{\text{dist}^2(x, \tilde{\gamma})} dx \leq C \left(\mathcal{H}^1(\tilde{\gamma}) \log \frac{C}{\rho} + \delta \log \frac{C}{\varepsilon} + 1 \right),$$

where $\tilde{\gamma} = \Phi(\gamma)$, $U = \Phi(T_{\rho, \delta})$, $V = \Phi(T_{\varepsilon, 0})$.

Proof. Let $\tilde{\gamma}$, U , and V be as in the statement. We observe that, for every $X \in \mathbb{R}^3$, $Y \in \gamma$ it holds that

$$(4.21) \quad \text{dist}(X, \gamma) \leq |X - Y| \leq \ell |\Phi(X) - \Phi(Y)|,$$

where $\ell > 0$ is the Lipschitz constant associated to Φ^{-1} and $\Phi(Y) \in \tilde{\gamma}$. If now we take the infimum over all $Y \in \gamma$ in (4.21), we get

$$(4.22) \quad \text{dist}(X, \gamma) \leq \ell \text{dist}(\Phi(X), \tilde{\gamma}) \quad \text{for all } X \in \mathbb{R}^3.$$

By a change of variable and property (4.22) we have that

$$\begin{aligned} \int_{\Omega \setminus (U \cup V)} \frac{1}{\text{dist}^2(x, \tilde{\gamma})} dx &\leq C \int_{\Phi^{-1}(\Omega) \setminus (T_{\rho, \delta} \cup T_{\varepsilon, 0})} \frac{1}{\text{dist}^2(\Phi(X), \tilde{\gamma})} dX \\ &\leq \frac{C}{\ell^2} \int_{\Phi^{-1}(\Omega) \setminus (T_{\rho, \delta} \cup T_{\varepsilon, 0})} \frac{1}{\text{dist}^2(X, \gamma)} dX. \end{aligned}$$

Next, we set $R = 3 \text{diam}(\Phi^{-1}(\Omega)) \leq 3C \text{diam}(\Omega)$ and assume $\text{dist}(\gamma, \Phi^{-1}(\Omega)) \leq R/2$. Up to a change of coordinates we can assume that γ is centered at the origin, namely

$$\gamma = \{(0, 0)\} \times (-\mathcal{H}^1(\gamma)/2, \mathcal{H}^1(\gamma)/2),$$

and $T_{\rho,\delta} T_{\varepsilon,0}$ are contained in the large domain of diameter $R + \mathcal{H}^1(\gamma)$ as represented in Figure 1. Thus, setting $X = (X', X_3)$, we obtain

$$\begin{aligned}
 (4.23) \quad & \int_{\Phi^{-1}(\Omega) \setminus (T_{\rho,\delta} \cup T_{\varepsilon,0})} \frac{1}{\text{dist}^2(X, \gamma)} dX \\
 & \leq |S_\delta| \int_{B'_R \setminus B'_\rho} \frac{1}{|X'|^2} dX' + 2\delta \int_{B'_R \setminus B'_\varepsilon} \frac{1}{|X'|^2} dX' + \int_{B_R} \frac{1}{|X|^2} dX \\
 & \leq 2\pi \mathcal{H}^1(\gamma) \log \frac{R}{\rho} + 4\pi\delta \log \frac{R}{\varepsilon} + 4\pi R.
 \end{aligned}$$

If instead $\text{dist}(\gamma, \Phi^{-1}(\Omega)) > R/2$, then we estimate as follows:

$$(4.24) \quad \int_{\Phi^{-1}(\Omega) \setminus (T_{\rho,\delta} \cup T_{\varepsilon,0})} \frac{1}{\text{dist}^2(X, \gamma)} dX \leq |\Omega| \frac{4}{R^2} \leq C. \quad \square$$

Proof of Proposition 4.4. Let $\mu_\varepsilon \in \mathcal{M}_B^{h_\varepsilon, \alpha_\varepsilon}(\bar{\Omega})$ be as in the statement

$$\mu_\varepsilon = \sum_{i=1}^{M_\varepsilon} b_\varepsilon^i \otimes t_\varepsilon^i \mathcal{H}^1 \llcorner \gamma_\varepsilon^i,$$

and consider the sequence of deformed polyhedral extended measures $\tilde{\mu}_\varepsilon \in \mathcal{M}_B(\mathbb{R}^3)$ given by Lemma 4.2 so that

$$\tilde{\mu}_\varepsilon = \sum_{i=1}^{N_\varepsilon} b_\varepsilon^i \otimes t_\varepsilon^i \mathcal{H}^1 \llcorner \gamma_\varepsilon^i \quad \text{for } N_\varepsilon \geq M_\varepsilon.$$

In particular, by (4.10) and (4.11) we have

$$(4.25) \quad |\tilde{\mu}_\varepsilon|(\mathbb{R}^3) \leq C, \quad \sum_{i=1}^{N_\varepsilon} |b_\varepsilon^i|^2 \mathcal{H}^1(\gamma_\varepsilon^i) \leq C,$$

and therefore, from (2.1) and (4.18) we infer that

$$(4.26) \quad M_\varepsilon h_\varepsilon \leq \sum_{i=1}^{M_\varepsilon} |b_\varepsilon^i| h_\varepsilon \leq \sum_{i=1}^{M_\varepsilon} |b_\varepsilon^i|^2 h_\varepsilon \leq C.$$

Analogously by (2.1), (4.25), and recalling Remark 4.3 (c), (d),

$$(4.27) \quad N_\varepsilon h_\varepsilon \leq \sum_{i=1}^{N_\varepsilon} |b_\varepsilon^i| h_\varepsilon \leq \sum_{i=1}^{N_\varepsilon} |b_\varepsilon^i|^2 h_\varepsilon \leq C.$$

Now let $\eta_\varepsilon \in L^{3/2}(\mathbb{R}^3, \mathbb{R}^{3 \times 3})$ be the distributional solution to (4.1) with μ replaced by $\tilde{\mu}_\varepsilon$, which, thanks to Lemma 4.1, satisfies

$$(4.28) \quad |\eta_\varepsilon(x)| \leq C \sum_{i=1}^{N_\varepsilon} \frac{|b_\varepsilon^i|}{\text{dist}(x, \gamma_\varepsilon^i)} \quad \text{for } x \notin \text{supp } \tilde{\mu}_\varepsilon.$$

To construct $\tilde{\eta}_\varepsilon$ (and consequently $\hat{\eta}_\varepsilon$) we will modify η_ε close to the support of μ_ε . We start by fixing two parameters $\rho_\varepsilon < h_\varepsilon^2$ and δ_ε and, to denote by $U_\varepsilon^i, V_\varepsilon^i$ (for $i = 1, \dots, N_\varepsilon$), the cylinders defined as follows:

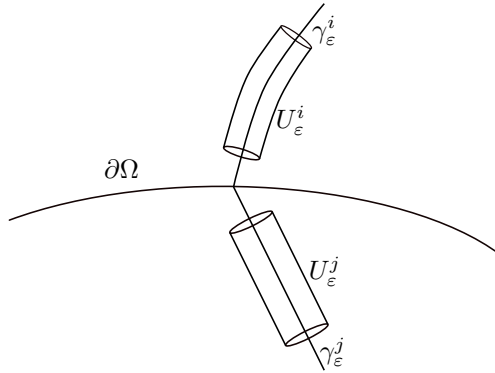


FIG. 2. Deformed cylinder U_ϵ^i obtained by reflection.

- If γ_ϵ^i is a straight segment, then

$$U_\epsilon^i := T_{\rho_\epsilon, \delta_\epsilon}(\gamma_\epsilon^i), \quad V_\epsilon^i := T_{\epsilon, 0}(\gamma_\epsilon^i);$$

- If γ_ϵ^i is obtained by reflecting some γ_ϵ^j ($i \neq j$), then U_ϵ^i and V_ϵ^i are the reflections of U_ϵ^j and V_ϵ^j (see Figure 2).

By the diluteness assumption, if

$$(4.29) \quad h_\epsilon \alpha_\epsilon \gg \alpha_\epsilon \delta_\epsilon \gg \rho_\epsilon,$$

then the cylinders U_ϵ^i for $i \in \{1, \dots, M_\epsilon\}$ are contained inside Ω , while all the U_ϵ^i with $i \in \{M_\epsilon + 1, \dots, N_\epsilon\}$ lie outside Ω . Furthermore,

$$U_\epsilon^i \cap U_\epsilon^k = \emptyset, \quad U_\epsilon^i \cap \gamma_\epsilon^k = \emptyset \quad \text{for } i, k = 1, \dots, N_\epsilon, i \neq k.$$

For the convenience of the reader we divide the rest of the proof into three steps.

Step 1. Estimate of the L^2 norm of η_ϵ in $D^\epsilon := \tilde{\Omega}_\epsilon(\mu_\epsilon) \setminus \cup_i U_\epsilon^i$, with $\tilde{\Omega}_\epsilon(\mu_\epsilon) := \{x \in (\Omega)_\epsilon : \text{dist}(x, \text{supp } \mu_\epsilon) > \epsilon\}$ and $(\Omega)_\epsilon$ the ϵ -neighborhood of Ω . Estimate (4.28) gives

$$(4.30) \quad \begin{aligned} \int_{D^\epsilon} |\eta_\epsilon|^2 dx &\leq C \int_{D^\epsilon} \left(\sum_{i=1}^{N_\epsilon} \frac{|b_\epsilon^i|}{\text{dist}(x, \gamma_\epsilon^i)} \right)^2 dx \\ &\leq CN_\epsilon \sum_{i=1}^{N_\epsilon} \int_{D^\epsilon} \frac{|b_\epsilon^i|^2}{\text{dist}^2(x, \gamma_\epsilon^i)} dx \\ &\leq CN_\epsilon \sum_{i=1}^{N_\epsilon} \int_{(\Omega)_\epsilon \setminus (U_\epsilon^i \cup V_\epsilon^i)} \frac{|b_\epsilon^i|^2}{\text{dist}^2(x, \gamma_\epsilon^i)} dx. \end{aligned}$$

We then use Lemma 4.5 replacing Ω with $\Omega' \supset \supset \Omega$ and get

$$(4.31) \quad \begin{aligned} \int_{(\Omega)_\epsilon \setminus (U_\epsilon^i \cup V_\epsilon^i)} \frac{1}{\text{dist}^2(x, \gamma_\epsilon^i)} dx &\leq \int_{\Omega' \setminus (U_\epsilon^i \cup V_\epsilon^i)} \frac{1}{\text{dist}^2(x, \gamma_\epsilon^i)} dx \\ &\leq C \left(\mathcal{H}^1(\gamma_\epsilon^i) \log \frac{C}{\rho_\epsilon} + 4\pi\delta_\epsilon \log \frac{C}{\epsilon} + 1 \right) \end{aligned}$$

for some positive constant depending only on the domain Ω' . Then from (4.25), (4.27), (4.30), and (4.31), it follows that

$$\begin{aligned}
 \int_{D_\varepsilon} |\eta_\varepsilon|^2 dx &\leq CN_\varepsilon \sum_{i=1}^{N_\varepsilon} |b_\varepsilon^i|^2 \left(\mathcal{H}^1(\gamma_\varepsilon^i) \log \frac{C}{\rho_\varepsilon} + \delta_\varepsilon \log \frac{C}{\varepsilon} + 1 \right) \\
 (4.32) \qquad &\leq C \left(\frac{1}{h_\varepsilon} \log \frac{C}{\rho_\varepsilon} + \frac{\delta_\varepsilon}{h_\varepsilon^2} \log \frac{C}{\varepsilon} + \frac{1}{h_\varepsilon^2} \right).
 \end{aligned}$$

Step 2. Interpolation with the radial solution in U_ε^i for all $i \in \{1, \dots, M_\varepsilon\}$. We construct for every $i \in \{1, \dots, M_\varepsilon\}$ a function $\tilde{\eta}_\varepsilon^i \in L^1(U_\varepsilon^i, \mathbb{R}^{3 \times 3})$ such that

$$\begin{aligned}
 \tilde{\eta}_\varepsilon^i &= \eta_\varepsilon \quad \text{in a neighborhood of } \partial U_\varepsilon^i, \\
 \text{curl } \tilde{\eta}_\varepsilon^i &= \text{curl } \eta_\varepsilon \quad \text{in } U_\varepsilon^i,
 \end{aligned}$$

and

$$\begin{aligned}
 \int_{U_\varepsilon^i \setminus V_\varepsilon^i} |\tilde{\eta}_\varepsilon^i|^2 dx &\leq C |b_\varepsilon^i|^2 \mathcal{H}^1(\gamma_\varepsilon^i) \log \frac{\rho_\varepsilon}{\varepsilon} \\
 (4.33) \qquad &+ \frac{C}{h_\varepsilon^2} |b_\varepsilon^i|^2 \left(\frac{(\mathcal{H}^1(\gamma_\varepsilon^i))^2}{h_\varepsilon^2} + \frac{(\mathcal{H}^1(\gamma_\varepsilon^i))^3}{\rho_\varepsilon^2} \right).
 \end{aligned}$$

We first observe that by (4.27) and (4.28) it follows that

$$(4.34) \qquad |\eta_\varepsilon(x)| \leq \frac{C}{h_\varepsilon} \frac{|b_\varepsilon^i|}{\text{dist}(x, \gamma_\varepsilon^i)} \quad \forall x \in U_\varepsilon^i, i \in \{1, \dots, M_\varepsilon\}$$

(recall (2.1)). Furthermore, it holds that

$$\text{curl } \eta_\varepsilon = b_\varepsilon^i \otimes t_\varepsilon^i \mathcal{H}^1 \llcorner \gamma_\varepsilon^i \quad \text{in } U_\varepsilon^i, i \in \{1, \dots, M_\varepsilon\}.$$

To simplify the notation we fix $i \in \{1, \dots, M_\varepsilon\}$ and assume without loss of generality that $b_\varepsilon^i = b$, $t_\varepsilon^i = e_3$, $\mathcal{H}^1(\gamma_\varepsilon^i) = 2\ell$, and $\gamma_\varepsilon^i = [0, 2\ell]e_3$, so that $U_\varepsilon^i = U_\varepsilon = T_{2(\ell-\delta_\varepsilon)}^{\rho_\varepsilon} + \delta_\varepsilon e_3$ and $V_\varepsilon^i = V_\varepsilon = T_{2\ell}^\varepsilon$. Then we let $\eta_{b,e_3} \in L^{3/2}(\mathbb{R}^3; \mathbb{R}^{3 \times 3})$ be the solution to

$$\begin{cases} \text{div } \mathbb{C}\xi = 0 & \text{in } \mathbb{R}^3, \\ \text{curl } \xi = b \otimes e_3 \mathcal{H}^1 \llcorner \mathbb{R}e_3 & \text{in } \mathbb{R}^3. \end{cases}$$

From (3.7) we have that

$$(4.35) \qquad |\eta_{b,e_3}(x)| \leq C \frac{|b|}{\text{dist}(x, \mathbb{R}e_3)} = C \frac{|b|}{|x'|},$$

with $x = (x', x_3)$ and $x' = (x_1, x_2)$. From (4.35) we deduce that

$$(4.36) \qquad \int_{T_{2\ell}^{\varepsilon, \rho_\varepsilon}} |\eta_{b,e_3}|^2 dx \leq C |b|^2 \ell \log \frac{\rho_\varepsilon}{\varepsilon}.$$

Hence there exists $v_\varepsilon \in W^{1,1}(U_\varepsilon; \mathbb{R}^3)$ such that $\eta_{b,e_3} = \eta_\varepsilon + \nabla v_\varepsilon$ in U_ε . Then (4.34) and (4.35) imply that

$$(4.37) \qquad |\nabla v_\varepsilon(x)| \leq \frac{C}{h_\varepsilon} \frac{|b|}{|x'|} \quad \forall x \in U_\varepsilon,$$

and in particular

$$(4.38) \quad |v_\varepsilon(x)| \leq \frac{C}{h_\varepsilon} |b| \left(\frac{2\ell}{\rho_\varepsilon} + \log \frac{\rho_\varepsilon}{|x'|} \right) \quad \forall x \in U_\varepsilon.$$

Next we take a cut-off function $\psi_\varepsilon \in C_c^\infty(\mathbb{R}^3; [0, 1])$ defined as follows:

$$\psi_\varepsilon := \psi_\varepsilon^1 \psi_\varepsilon^2, \quad \psi_\varepsilon^1(x) := \psi(|x'|/\rho_\varepsilon), \quad \psi_\varepsilon^2(x) := \phi_\varepsilon(x_3),$$

where $\psi, \phi_\varepsilon \in C_c^\infty(\mathbb{R}; [0, 1])$ are such that

$$\psi(t) \equiv 1 \quad \text{if } |t| < \frac{1}{2}, \quad \psi(t) \equiv 0 \quad \text{if } |t| \geq 1, \quad |\psi'| \leq 3,$$

$$\phi_\varepsilon(t) \equiv 1 \quad \text{if } t \in [\delta_\varepsilon + \sigma_\varepsilon, \ell - (\delta_\varepsilon + \sigma_\varepsilon)],$$

$$\phi_\varepsilon(t) \equiv 0 \quad \text{if } t \in [0, \delta_\varepsilon] \cap [\ell - \delta_\varepsilon, \ell], \quad |\phi'_\varepsilon| \leq \frac{C}{\sigma_\varepsilon}.$$

Here σ_ε is a small parameter to be chosen later, satisfying $h_\varepsilon > \sigma_\varepsilon > \rho_\varepsilon$. Finally, we define $u_\varepsilon := v_\varepsilon \psi_\varepsilon$ and (recall that $i \in \{1, \dots, M_\varepsilon\}$ is fixed)

$$(4.39) \quad \tilde{\eta}_\varepsilon^i = \tilde{\eta}_\varepsilon := \eta_\varepsilon + \nabla u_\varepsilon \quad \text{in } U_\varepsilon.$$

In this way $\tilde{\eta}_\varepsilon \equiv \eta_{b, e_3}$ in $\{\psi_\varepsilon = 1\} = T_{2(\ell - \delta_\varepsilon - \sigma_\varepsilon)}^{\frac{\rho_\varepsilon}{2}} + (\delta_\varepsilon + \sigma_\varepsilon)e_3$ and $\tilde{\eta}_\varepsilon = \eta_\varepsilon$ on ∂U_ε .

It remains to show the validity of (4.33). In fact we have

$$(4.40) \quad \int_{U_\varepsilon \setminus V_\varepsilon} |\tilde{\eta}_\varepsilon|^2 dx \leq \int_{T_{2\ell}^{\varepsilon, \rho_\varepsilon}} |\eta_{b, e_3}|^2 dx + \int_{\{\psi_\varepsilon \neq 1\} \cap (U_\varepsilon \setminus V_\varepsilon)} |\tilde{\eta}_\varepsilon|^2 dx.$$

Moreover, in $\{\psi_\varepsilon \neq 1\}$ there holds

$$(4.41) \quad \begin{aligned} |\nabla u_\varepsilon(x)| &\leq |\nabla v_\varepsilon| + |v_\varepsilon(x)| (|\nabla \psi_\varepsilon^1(x)| + |\nabla \psi_\varepsilon^2(x)|) \\ &\leq \frac{C}{h_\varepsilon} |b| \left(\frac{1}{|x'|} + \left(\frac{2\ell}{\rho_\varepsilon} + \log \frac{\rho_\varepsilon}{|x'|} \right) \left(\frac{1}{\rho_\varepsilon} + \frac{1}{\sigma_\varepsilon} \right) \right). \end{aligned}$$

We take the partition of the set $\{\psi_\varepsilon \neq 1\} \cap (U_\varepsilon \setminus V_\varepsilon)$ given by

$$S_1 := T_{2(\ell - \delta_\varepsilon)}^{\frac{\rho_\varepsilon}{2}, \rho_\varepsilon} + \delta_\varepsilon e_3, \quad S_2 := (T_{2(\ell - \delta_\varepsilon)}^{\varepsilon, \rho_\varepsilon} + \delta_\varepsilon e_3) \setminus (T_{2(\ell - \delta_\varepsilon)}^{\frac{\rho_\varepsilon}{2}} + \delta_\varepsilon e_3).$$

Gathering together (4.34) and (4.41) and using that $h_\varepsilon > \sigma_\varepsilon > \rho_\varepsilon$ in the L^∞ estimate of $\tilde{\eta}_\varepsilon$, we get

$$(4.42) \quad \int_{S_1} |\tilde{\eta}_\varepsilon|^2 dx \leq C \rho_\varepsilon^2 \ell \|\tilde{\eta}_\varepsilon\|_{L^\infty(S_1)}^2 \leq \frac{C|b|^2 \ell^3}{h_\varepsilon^2 \rho_\varepsilon^2}.$$

Moreover, recalling that $\psi_\varepsilon^1 \equiv 1$ in S_2 by (4.34), (4.41), we can estimate

$$(4.43) \quad \begin{aligned} \int_{S_2} |\tilde{\eta}_\varepsilon|^2 dx &\leq \frac{C|b|^2 \sigma_\varepsilon}{h_\varepsilon^2} \int_{\{\varepsilon \leq |x'| \leq \frac{\rho_\varepsilon}{2}\}} \frac{1}{|x'|^2} + \frac{1}{\sigma_\varepsilon^2} \left(\frac{\ell}{\rho_\varepsilon} + \log \frac{\rho_\varepsilon}{|x'|} \right)^2 dx' \\ &\leq \frac{C|b|^2 \sigma_\varepsilon}{h_\varepsilon^2} \left(\log \frac{\rho_\varepsilon}{\varepsilon} + \frac{\ell^2}{\sigma_\varepsilon^2} + \frac{\rho_\varepsilon^2}{\sigma_\varepsilon^2} \right) \\ &\leq \frac{C|b|^2 \sigma_\varepsilon}{h_\varepsilon^2} \left(\log \frac{\rho_\varepsilon}{\varepsilon} + \frac{\ell^2}{\sigma_\varepsilon^2} \right), \end{aligned}$$

where we also used that $\ell > h_\varepsilon$. Plugging (4.36), (4.42), and (4.43) into (4.40), we obtain

$$\int_{U_\varepsilon \setminus V_\varepsilon} |\tilde{\eta}_\varepsilon|^2 dx \leq C|b|^2 \ell \log \frac{\rho_\varepsilon}{\varepsilon} + \frac{C}{h_\varepsilon^2} |b|^2 \left(\frac{\ell^3}{\rho_\varepsilon^2} + \sigma_\varepsilon \log \frac{\rho_\varepsilon}{\varepsilon} + \frac{\ell^2}{\sigma_\varepsilon} \right).$$

Finally, by choosing $\sigma_\varepsilon = h_\varepsilon^2 > \rho_\varepsilon$ we derive (4.33).

We remark that with our choice of the parameters we have that

$$(4.44) \quad \left(\frac{2\ell}{\rho_\varepsilon} + \log \frac{\rho_\varepsilon}{|x'|} \right) \left(\frac{1}{\rho_\varepsilon} + \frac{1}{\sigma_\varepsilon} \right) \leq \left(\frac{2\ell}{\rho_\varepsilon} + \log \frac{\rho_\varepsilon}{|x'|} \right) \frac{2}{\rho_\varepsilon} \leq C \frac{1}{3\varepsilon} + C \frac{1}{|x'|} \leq C \frac{1}{|x'|}$$

for all $x \in U_\varepsilon$ such that $|x'| \leq 3\varepsilon$. In particular, this shows, combined with (4.34), (4.39), and (4.41), that the field $\tilde{\eta}_\varepsilon$ constructed in this step also satisfies

$$(4.45) \quad |\tilde{\eta}_\varepsilon(x)| \leq C|b| \frac{1}{h_\varepsilon} \frac{1}{|x'|}$$

for all $x \in U_\varepsilon$ such that $|x'| \leq 3\varepsilon$. This will be used in order to show property (ii') for the mollified field $\hat{\eta}_\varepsilon$.

Step 3. Construction of $\tilde{\eta}_\varepsilon$. We define $\tilde{\eta}_\varepsilon$ as follows:

$$(4.46) \quad \tilde{\eta}_\varepsilon(x) := \begin{cases} \tilde{\eta}_\varepsilon^i(x) & \text{if } x \in U_\varepsilon^i \text{ for } i = 1, \dots, M_\varepsilon, \\ \eta_\varepsilon(x) & \text{otherwise in } \mathbb{R}^3. \end{cases}$$

Then by estimate (4.32) in Step 1 and (4.33) we have that for $h_\varepsilon^2 > \rho_\varepsilon$

$$\begin{aligned} \int_{\tilde{\Omega}_\varepsilon(\mu_\varepsilon)} |\tilde{\eta}_\varepsilon|^2 dx &= \int_{D_\varepsilon} |\eta_\varepsilon|^2 dx + \sum_{i=1}^{M_\varepsilon} \int_{U_\varepsilon^i \setminus V_\varepsilon^i} |\tilde{\eta}_\varepsilon^i|^2 dx \\ &\leq C \left(\frac{1}{h_\varepsilon} \log \frac{C}{\rho_\varepsilon} + \frac{\delta_\varepsilon}{h_\varepsilon^2} \log \frac{C}{\varepsilon} + \frac{1}{h_\varepsilon^2} + \log \frac{\rho_\varepsilon}{\varepsilon} + \frac{1}{h_\varepsilon^4} + \frac{1}{h_\varepsilon^2 \rho_\varepsilon^2} \right) \\ &\leq C \left(\frac{\delta_\varepsilon}{h_\varepsilon^2} \log \frac{C}{\varepsilon} + \log \frac{\rho_\varepsilon}{\varepsilon} + \frac{1}{h_\varepsilon^4} + \frac{1}{h_\varepsilon^2 \rho_\varepsilon^2} \right), \end{aligned}$$

where $\tilde{\Omega}_\varepsilon(\mu_\varepsilon)$ is defined as in Step 1. Eventually by choosing $\delta_\varepsilon = h_\varepsilon^2$ and $\rho_\varepsilon = (\alpha_\varepsilon h_\varepsilon)^2$, which is compatible with (4.29), we immediately obtain, using (2.9),

$$(4.47) \quad \int_{\tilde{\Omega}_\varepsilon(\mu_\varepsilon)} |\tilde{\eta}_\varepsilon|^2 dx \leq c |\log \varepsilon|.$$

Step 4. Construction of $\hat{\eta}_\varepsilon$. We define $\hat{\eta}_\varepsilon := \tilde{\eta}_\varepsilon * \varphi_\varepsilon$. Clearly

$$\text{curl}(\hat{\eta}_\varepsilon) = \tilde{\mu}_\varepsilon * \varphi_\varepsilon \quad \text{in } \mathbb{R}^3,$$

which in particular gives (i'). We need to show (ii'). First we notice that

$$(4.48) \quad \int_{\Omega_{2\varepsilon}(\mu_\varepsilon)} |\hat{\eta}_\varepsilon|^2 dx = \int_{\Omega_{2\varepsilon}(\mu_\varepsilon)} |\tilde{\eta}_\varepsilon * \varphi_\varepsilon|^2 dx \leq \int_{\tilde{\Omega}_\varepsilon(\mu_\varepsilon)} |\tilde{\eta}_\varepsilon|^2 dx \leq c |\log \varepsilon|,$$

where we have used (4.47). In order to conclude, it remains to show the estimate of the L^2 norm of $\hat{\eta}_\varepsilon$ in a 2ε neighborhood of the support of μ_ε . We define $\tilde{T}_{2\varepsilon} = \{x \in$

$(\Omega)_\varepsilon : \text{dist}(x, \text{supp } \tilde{\mu}_\varepsilon) \leq 2\varepsilon\}$, and for each $i = 1, \dots, M_\varepsilon$ we define $(\gamma_\varepsilon^i)_{2\varepsilon} := \{x \in \mathbb{R}^3 : 0 < \text{dist}(x, \gamma_\varepsilon^i) < 2\varepsilon\}$ and let x_ε^j be the endpoints of the segments γ_ε^i . From the construction of the field $\tilde{\eta}_\varepsilon$, recalling (4.45) and (4.28), we have in particular that

$$(4.49) \quad |\tilde{\eta}_\varepsilon(x)| \leq C \frac{|b_\varepsilon^i|}{h_\varepsilon} \frac{1}{\text{dist}(x, \gamma_\varepsilon^i)} \quad \forall x \in (\gamma_\varepsilon^i)_{3\varepsilon} \setminus \cup_j B_{2\varepsilon/\alpha_\varepsilon}(x_\varepsilon^j).$$

Setting $f_i(x) = \frac{|b_\varepsilon^i|}{\text{dist}(x, \gamma_\varepsilon^i)}$, one can see that

$$(4.50) \quad |f_i * \varphi_\varepsilon(x)| \leq \frac{|b_\varepsilon^i|}{\text{dist}(x, \gamma_\varepsilon^i) + \varepsilon} \quad \forall x \in \mathbb{R}^3.$$

Therefore, from (4.49) for all $i \in \{1, \dots, M_\varepsilon\}$

$$(4.51) \quad |\tilde{\eta}_\varepsilon * \varphi_\varepsilon(x)| \leq |\tilde{\eta}_\varepsilon| * \varphi_\varepsilon(x) \leq \frac{C}{h_\varepsilon} \frac{|b_\varepsilon^i|}{\text{dist}(x, \gamma_\varepsilon^i) + \varepsilon} \quad \forall x \in (\gamma_\varepsilon^i)_{2\varepsilon} \setminus \cup_j B_{3\varepsilon/\alpha_\varepsilon}(x_\varepsilon^j).$$

Moreover, from (4.28) we have

$$(4.52) \quad |\tilde{\eta}_\varepsilon * \varphi_\varepsilon(x)| \leq C \sum_{i=1}^{N_\varepsilon} |f_i * \varphi_\varepsilon(x)| \leq C \sum_{i=1}^{N_\varepsilon} \frac{|b_\varepsilon^i|}{\text{dist}(x, \gamma_\varepsilon^i) + \varepsilon} \leq C \frac{1}{h_\varepsilon \varepsilon} \quad \text{for } x \notin \text{supp } \tilde{\mu}_\varepsilon;$$

for the last inequality we have used (4.27). From this, setting $\hat{B} := \cup_j B_{3\varepsilon/\alpha_\varepsilon}(x_\varepsilon^j)$, we obtain

$$(4.53) \quad \begin{aligned} \int_{\hat{T}_{2\varepsilon}} |\hat{\eta}_\varepsilon|^2 dx &\leq \sum_{i=1}^{M_\varepsilon} \int_{(\gamma_\varepsilon^i)_{2\varepsilon} \setminus \hat{B}} |\hat{\eta}_\varepsilon|^2 dx + \int_{\hat{B}} |\hat{\eta}_\varepsilon|^2 dx \\ &\leq \frac{C}{h_\varepsilon^2} \sum_{i=1}^{M_\varepsilon} \int_{(\gamma_\varepsilon^i)_{2\varepsilon}} |f_i * \varphi_\varepsilon|^2 dx + \int_{\hat{B}} |\tilde{\eta}_\varepsilon * \varphi_\varepsilon|^2 dx \\ &\leq \frac{C}{h_\varepsilon^2} \sum_{i=1}^{M_\varepsilon} \int_{(\gamma_\varepsilon^i)_{2\varepsilon}} \frac{|b_\varepsilon^i|^2}{(\text{dist}(x, \gamma_\varepsilon^i) + \varepsilon)^2} dx + C |\hat{B}| \frac{1}{h_\varepsilon^2 \varepsilon^2} \\ &\leq \frac{C}{h_\varepsilon^2} \sum_{i=1}^{M_\varepsilon} \mathcal{H}^1(\gamma_\varepsilon^i) |b_\varepsilon^i|^2 + 2M_\varepsilon C \frac{\varepsilon}{\alpha_\varepsilon^3 h_\varepsilon^2} \leq \frac{C}{h_\varepsilon^2}, \end{aligned}$$

where in the last inequality we have used (4.25). The conclusion follows from (4.48) and the above estimate considering (2.9). \square

5. Proof of compactness and Γ -limit. We finally pass to the proof of the main results, namely Theorems 2.4 and 2.7.

Proof of Theorem 2.4. Compactness of μ_j . Let $(\mu_j, \beta_j) \in \mathcal{M}_{\mathbb{B}^3}^{h_{\varepsilon_j}, \alpha_{\varepsilon_j}}(\bar{\Omega}) \times \mathcal{AS}_{\varepsilon_j}(\mu_j)$ be as in the statement. Since μ_j is dilute, we can write $\mu_j = \sum_i b_j^i \otimes t_j^i \mathcal{H}^1 \llcorner \gamma_j^i$ where $\gamma_j^i \subset \Omega$ satisfy the conditions of Definition 2.3. We choose the parameters

$$(5.1) \quad \rho_j := (\alpha_{\varepsilon_j} h_{\varepsilon_j})^3, \quad \delta_j := (\alpha_{\varepsilon_j} h_{\varepsilon_j})^2$$

and define the cylinders

$$(5.2) \quad U_j^i := T_{\rho_j, \delta_j}(\gamma_j^i), \quad V_j^i := T_{\varepsilon_j, 0}(\gamma_j^i)$$

according to definition (4.19) given in section 4.

It turns out that, by the choice of ρ_j and δ_j , the cylinders U_j^i are pairwise disjoint and $U_j^i \cap \gamma_j^k = \emptyset$ for all $i \neq k$; therefore, we have

$$(5.3) \quad \mathcal{E}_{\varepsilon_j}(\mu_j, \beta_j) \geq \sum_i \int_{U_j^i \setminus V_j^i} W(\beta_j) dx.$$

Recalling estimate (3.66), we find a constant $c_* > 0$ such that

$$(5.4) \quad \frac{1}{|S_j^i| \varepsilon_j^2 \log \frac{\rho_j}{\varepsilon_j}} \int_{U_j^i \setminus V_j^i} W(\beta_j) dx \geq \left(1 - \frac{\rho_j}{|S_j^i|}\right) c_* |b_j^i|^2,$$

where $|S_j^i| = \mathcal{H}^1(\gamma_j^i) - 2\delta_j$. This and (5.3) give

$$(5.5) \quad \begin{aligned} \mathcal{F}_{\varepsilon_j}(\mu_j, \beta_j) &\geq c \left(1 - \frac{\rho_j}{|S_j^i|}\right) \frac{\log(\rho_j/\varepsilon_j)}{|\log \varepsilon_j|} \sum_i |b_j^i|^2 |S_j^i| \\ &\geq c \left(1 - \frac{\rho_j}{|S_j^i|}\right) (1 - 2h_{\varepsilon_j} \alpha_{\varepsilon_j}^2) \frac{\log(\rho_j/\varepsilon_j)}{|\log \varepsilon_j|} \sum_i |b_j^i|^2 \mathcal{H}^1(\gamma_j^i). \end{aligned}$$

By (2.8), (2.9), and the definition of ρ_ε we get

$$(5.6) \quad C \geq \sum_i |b_j^i|^2 \mathcal{H}^1(\gamma_j^i),$$

and in particular, by (2.1), we get

$$(5.7) \quad C \geq \sum_i |b_j^i| \mathcal{H}^1(\gamma_j^i) = |\mu_j|(\Omega).$$

By (5.7) and [12, Theorem 2.5], up to subsequences, we derive

$$\mu_{\varepsilon_j} \xrightarrow{*} \mu \in \mathcal{M}_B(\Omega).$$

Compactness of β_j . For every j let $\Omega_j \subset\subset \Omega$ be as in the statement, namely

$$\Omega_j := \Omega \setminus \{x \in \Omega : \text{dist}(x, \partial\Omega) \leq \varepsilon_j\}.$$

Notice that, since Ω is of class C^2 , any Ω_j can be obtained by a bi-Lipschitz transformation of Ω with Lipschitz constant $L_j \leq L$; moreover, the characteristic function χ_{Ω_j} converges to 1 in measure. Thanks to Proposition 4.4 there exist a constant $c > 0$, not depending on ε_j , and a sequence

$$\hat{\eta}_j \in L^{3/2}(\mathbb{R}^3; \mathbb{R}^{3 \times 3}) \cap L^2(\Omega; \mathbb{R}^{3 \times 3}),$$

such that

$$(5.8) \quad \int_{\Omega} |\hat{\eta}_j|^2 dx \leq c |\log \varepsilon_j|,$$

and

$$(5.9) \quad \text{curl } \hat{\eta}_j = \mu_j * \varphi_{\varepsilon_j} \quad \text{in } \Omega_j.$$

In particular,

$$\operatorname{curl}(\beta_j - \varepsilon_j \hat{\eta}_j) = 0 \quad \text{in } \Omega_j.$$

Therefore, there exists $u_j \in W^{1,2}(\Omega_j; \mathbb{R}^3)$ such that

$$(5.10) \quad \beta_j = \varepsilon_j \hat{\eta}_j + \nabla u_j \quad \text{in } \Omega_j.$$

Using the rigidity estimate on Ω_j , we find a constant $C_j > 0$ and a sequence $\{Q_j\} \subset SO(3)$ such that

$$\int_{\Omega_j} |\nabla u_j - Q_j|^2 dx \leq C_j \int_{\Omega_j} \operatorname{dist}^2(\nabla u_j, SO(3)) dx.$$

Now Theorem 5 in [22] and the hypothesis on Ω_j imply that $C_j \leq C$ for some constant $C > 0$ independent of j . This together with the growth conditions on W gives

$$\begin{aligned} C\varepsilon_j^2 |\log \varepsilon_j| &\geq \int_{\Omega} \operatorname{dist}^2(\beta_j, SO(3)) dx \\ &\geq \frac{1}{2} \int_{\Omega_j} \operatorname{dist}^2(\nabla u_j, SO(3)) dx - \int_{\Omega_j} \varepsilon_j^2 |\hat{\eta}_j|^2 dx \\ &\geq \frac{1}{2C} \int_{\Omega_j} |\nabla u_j - Q_j|^2 dx - \int_{\Omega_j} \varepsilon_j^2 |\hat{\eta}_j|^2 dx \\ &\geq \frac{1}{4C} \int_{\Omega_j} |\beta_j - Q_j|^2 dx - \frac{1}{2C} \int_{\Omega_j} \varepsilon_j^2 |\hat{\eta}_j|^2 dx, \end{aligned}$$

and by (5.8) we conclude that

$$(5.11) \quad \int_{\Omega_j} |\beta_j - Q_j|^2 dx \leq C\varepsilon_j^2 |\log \varepsilon_j|.$$

Hence, there exist $\xi \in L^2(\Omega, \mathbb{R}^{3 \times 3})$ with $\operatorname{curl} \xi = 0$ and $Q \in SO(3)$, such that, up to subsequences, it holds that

$$\frac{(Q_j)^T \beta_j - I}{\varepsilon_j \sqrt{|\log \varepsilon_j|}} \chi_{\Omega_j} \rightharpoonup \xi \quad \text{in } L^2(\Omega, \mathbb{R}^{3 \times 3}) \quad \text{and} \quad Q_j \rightarrow Q \in SO(3). \quad \square$$

REMARK 5.1. *The crucial point in order to obtain the compactness of the β 's is the decomposition (5.10) which is guaranteed by (5.9). In the case in which we fix the extension measure in the definition of admissible configurations $\mathcal{AS}_{\varepsilon_j}^*(\mu_j)$ for the functionals $\mathcal{F}_{\varepsilon_j}(\mu_j, \cdot)$ (see Remark 2.14), thanks to Proposition 4.4 we obtain the decomposition in the whole of Ω . Eventually we can proceed as above and obtain (5.11) in Ω .*

PROPOSITION 5.2 (lower bound). *For any sequence $\varepsilon_j \rightarrow 0$ and for any $(\mu_j, \beta_j) \in \mathcal{M}_{\mathcal{B}}^{h_{\varepsilon_j}, \alpha_{\varepsilon_j}}(\bar{\Omega}) \times \mathcal{AS}_{\varepsilon_j}(\mu_j)$ converging to $(\mu, \xi, Q) \in \mathcal{M}_{\mathcal{B}}(\Omega) \times L^2(\Omega; \mathbb{R}^{3 \times 3}) \times SO(3)$ in the sense of Definition 2.6 with $\operatorname{curl} \xi = 0$, we have*

$$(5.12) \quad \mathcal{F}_0(\mu, \xi, Q) \leq \liminf_{j \rightarrow \infty} \mathcal{F}_{\varepsilon_j}(\mu_j, \beta_j).$$

Proof. Let (μ_j, β_j) be a sequence with equibounded energy that converges to (μ, ξ, Q) with $\text{curl } \xi = 0$ as in the statement. Then by Definition 2.6 there exists a sequence $\{Q_j\} \subset SO(3)$ such that, up to subsequence, $Q_j \rightarrow Q$ and

$$(5.13) \quad \mu_j \xrightarrow{*} \mu \quad \text{in } \mathcal{M}_B(\Omega),$$

$$(5.14) \quad \frac{Q_j^T \beta_j - I}{\varepsilon_j \sqrt{|\log \varepsilon_j|}} \chi_{\Omega_j} \rightharpoonup \xi \quad \text{in } L^2(\Omega; \mathbb{R}^{3 \times 3}),$$

where $\Omega_j := \Omega \setminus \{x \in \Omega : \text{dist}(x, \partial\Omega) \leq \varepsilon_j\}$. Let $\rho_j > 0$ be as in (5.1) and define the sets

$$(5.15) \quad \Omega'_j := \{x \in \Omega : \text{dist}(x, \text{supp } \mu_j) \geq 2\rho_j\} \quad \text{and} \quad \Omega''_j := \Omega \setminus \Omega'_j.$$

Then

$$\mathcal{F}_{\varepsilon_j}(\mu_j, \beta_j) = \frac{1}{\varepsilon_j^2 |\log \varepsilon_j|} \int_{\Omega'_j} W(\beta_j) dx + \frac{1}{\varepsilon_j^2 |\log \varepsilon_j|} \int_{\Omega''_j} W(\beta_j) dx =: F'_j + F''_j.$$

Step 1: Lower bound for F'_j . We can perform a Taylor expansion of W near the identity as in (3.51) that yields

$$(5.16) \quad W(I + F) \geq \frac{1}{2} \mathbb{C} F : F - \omega(|F|),$$

with $\omega(t)/t^2 \rightarrow 0$ as $t \rightarrow 0$. We then set

$$G_j := \frac{Q_j^T \beta_j - I}{\varepsilon_j \sqrt{|\log \varepsilon_j|}}, \quad \tilde{G}_j := G_j \chi_{\Omega_j},$$

and

$$\chi_j := \begin{cases} 1 & \text{if } |\tilde{G}_j| \leq \varepsilon_j^{-1/2}, \\ 0 & \text{otherwise in } \Omega, \end{cases} \quad \tilde{\chi}_j := \chi_j \cdot \chi_{\Omega'_j}.$$

Then, using (5.16), we have that

$$\begin{aligned} F'_j &= \frac{1}{\varepsilon_j^2 |\log \varepsilon_j|} \int_{\Omega'_j} W(Q_j^T \beta_j) dx \geq \frac{1}{\varepsilon_j^2 |\log \varepsilon_j|} \int_{\Omega} W(I + \varepsilon_j \sqrt{|\log \varepsilon_j|} G_j) \tilde{\chi}_j \cdot \chi_{\Omega_j} dx \\ &\geq \int_{\Omega} \left(\frac{1}{2} \mathbb{C} \tilde{\chi}_j \tilde{G}_j : \tilde{\chi}_j \tilde{G}_j - \tilde{\chi}_j \cdot \chi_{\Omega_j} \frac{\omega(\varepsilon_j \sqrt{|\log \varepsilon_j|} |G_j|)}{\varepsilon_j^2 |\log \varepsilon_j|} \right) dx. \end{aligned}$$

Now (5.14) implies that (\tilde{G}_j) is bounded in $L^2(\Omega; \mathbb{R}^{3 \times 3})$; then $\tilde{\chi}_j \rightarrow 1$ in Ω in measure and $\tilde{\chi}_j \tilde{G}_j \rightharpoonup \xi$ in $L^2(\Omega; \mathbb{R}^{3 \times 3})$. Therefore, by lower semicontinuity it follows that

$$\liminf_{j \rightarrow \infty} \int_{\Omega} \frac{1}{2} \mathbb{C} \tilde{\chi}_j \tilde{G}_j : \tilde{\chi}_j \tilde{G}_j dx \geq \int_{\Omega} \frac{1}{2} \mathbb{C} \xi : \xi dx.$$

On the other hand, we have that

$$|G_j|^2 \chi_{\Omega_j} \cdot \tilde{\chi}_j \frac{\omega(\varepsilon_j \sqrt{|\log \varepsilon_j|} |G_j|)}{\varepsilon_j^2 |\log \varepsilon_j| |G_j|^2} = |\tilde{G}_j|^2 \cdot \tilde{\chi}_j \frac{\omega(\varepsilon_j \sqrt{|\log \varepsilon_j|} |\tilde{G}_j|)}{\varepsilon_j^2 |\log \varepsilon_j| |\tilde{G}_j|^2}$$

is the product of a bounded sequence in $L^1(\Omega)$ and a sequence converging to zero in $L^\infty(\Omega)$ since $\varepsilon_j \sqrt{|\log \varepsilon_j|} |\tilde{G}_j| \leq \varepsilon_j^{1/2} \sqrt{|\log \varepsilon_j|}$ whenever $\tilde{\chi}_j \neq 0$. Then

$$\lim_{j \rightarrow \infty} \int_{\Omega} \tilde{\chi}_j \cdot \chi_{\Omega_j} \frac{\omega(\varepsilon_j \sqrt{|\log \varepsilon_j|} |G_j|)}{\varepsilon_j^2 |\log \varepsilon_j|} dx = 0,$$

which implies

$$(5.17) \quad \liminf_{j \rightarrow \infty} F'_j \geq \int_{\Omega} \frac{1}{2} \mathbb{C} \xi : \xi dx.$$

Step 2: Lower bound for F''_j . Notice that by (5.14) there exists a constant $C > 0$ such that

$$(5.18) \quad \int_{\Omega} |\beta_j - Q_j|^2 \chi_{\Omega_j} dx \leq C \varepsilon_j^2 |\log \varepsilon_j|.$$

Without loss of generality we can assume that $|\mu|(\Omega) \geq \tilde{C} > 0$ (otherwise there is nothing to prove), and then by lower semicontinuity we can also assume that $|\mu_j|(\Omega) \geq \tilde{C}$.

Moreover, diluteness of μ_j corresponds to

$$\mu_j = \sum_i b_j^i \otimes t_j^i \mathcal{H}^1 \llcorner \gamma_j^i,$$

with γ_j^i satisfying conditions of Definition 2.3. Consider the cylinders U_j^i, V_j^i defined as in (5.2); thus, recalling (5.4), we have that

$$\begin{aligned} \frac{1}{\varepsilon_j^2 |\log \varepsilon_j|} \int_{\Omega_j''} W(\beta_j) dx &\geq \frac{1}{\varepsilon_j^2 |\log \varepsilon_j|} \sum_i \int_{U_j^i \setminus V_j^i} W(\beta_j) dx \\ &\geq (1 + o(1)) C \sum_i |b_j^i| \mathcal{H}^1(\gamma_j^i) \geq (1 + o(1)) C |\mu_j|(\Omega) \geq (1 + o(1)) C, \end{aligned}$$

where $o(1) \rightarrow 0$ as $j \rightarrow \infty$. Let $\nu > 0$ and set $\tilde{\lambda} := \nu / (1 - \nu)$; then using the previous estimates and (5.18) it holds that

$$(5.19) \quad \begin{aligned} F''_j &= \frac{1 - \nu}{\varepsilon_j^2 |\log \varepsilon_j|} \left(\int_{\Omega_j''} W(\beta_j) dx + \tilde{\lambda} \int_{\Omega_j''} W(\beta_j) dx \right) \\ &\geq \frac{1 - \nu}{\varepsilon_j^2 |\log \varepsilon_j|} \left(\int_{\Omega_j''} W(\beta_j) dx + \tilde{\lambda} (1 + o(1)) C \varepsilon_j^2 |\log \varepsilon_j| \right) \\ &\geq \frac{1 - \nu}{\varepsilon_j^2 |\log \varepsilon_j|} \left(\int_{\Omega_j''} W(\beta_j) dx + \tilde{\lambda} C \int_{\Omega_j''} |\beta_j - Q_j|^2 \chi_{\Omega_j} dx \right). \end{aligned}$$

By (3.28) we get for any $\lambda > 0$

$$\begin{aligned} \int_{\Omega_j'' \cap \Omega_j} W(\beta_j) + \lambda |\beta_j - Q_j|^2 dx &\geq \sum_{i \in I(j)} \int_{T_j^i \setminus V_j^i} W(\beta_j) + \lambda |\beta_j - Q_j|^2 dx \\ &\geq \sum_{i \in I(j)} |S_j^i| \varepsilon_j^2 \log \frac{\rho_j}{\varepsilon_j} \Psi_\lambda^{nl}(Q_j, b_j^i, t_j^i, |S_j^i|, \varepsilon_j, \rho_j) \\ &= \sum_{i \in I(j)} |S_j^i| \varepsilon_j^2 \log \frac{\rho_j}{\varepsilon_j} \Psi_\lambda^{nl}(I, Q_j^T b_j^i, t_j^i, |S_j^i|, \varepsilon_j, \rho_j), \end{aligned}$$

where

$$I(j) := \{i : \gamma_j^i \subset \Omega_j\}.$$

Plugging this into (5.19) with $\lambda := \tilde{\lambda}C$, by (5.1) we infer that

$$(5.20) \quad F_j'' \geq (1 - \nu)(1 - 2h_{\varepsilon_j} \alpha_{\varepsilon_j}^2) \frac{\log(\rho_j/\varepsilon_j)}{|\log \varepsilon_j|} \sum_{i \in I(j)} \mathcal{H}^1(\gamma_j^i) \Psi_\lambda^{nl}(I, Q_j^T b_j^i, t_j^i, |S_j^i|, \varepsilon_j, \rho_j).$$

To conclude the proof, fix $M > 1$ and $K > 0$, and denote

$$I_j^1 := \{i \in I(j) : |b_j^i| \leq K\} \quad \text{and} \quad I_j^2 := \{i \in I(j) : |b_j^i| > K\}.$$

For sufficiently big j we have $|S_j^i| \geq \frac{1}{2}h_{\varepsilon_j} \geq M\rho_j$ for all $i \in I(j)$; thus if $\omega_{M,K}$ is the function given by Lemma 3.12, from estimate (3.64) it follows that

$$(5.21) \quad \begin{aligned} & \sum_{i \in I_j^1} \mathcal{H}^1(\gamma_j^i) \Psi_\lambda^{nl}(I, Q_j^T b_j^i, t_j^i, |S_j^i|, \varepsilon_j, \rho_j) \\ & \geq \sum_{i \in I_j^1} \mathcal{H}^1(\gamma_j^i) \left(\Psi_0(Q_j^T b_j^i, t_j^i) - \frac{cK^2}{M} - \omega_{M,K} \left(\frac{\varepsilon_j}{\rho_j} \right) \right) \\ & \geq \sum_{i \in I_j^1} \mathcal{H}^1(\gamma_j^i) \Psi_0(Q_j^T b_j^i, t_j^i) - C \left(\frac{cK^2}{M} + \omega_{M,K} \left(\frac{\varepsilon_j}{\rho_j} \right) \right). \end{aligned}$$

Moreover, from (3.65), using that $|b_j^i| \geq K$ if $i \in I_j^2$, we get

$$(5.22) \quad \begin{aligned} \sum_{i \in I_j^2} \mathcal{H}^1(\gamma_j^i) \Psi_\lambda^{nl}(I, Q_j^T b_j^i, t_j^i, |S_j^i|, \varepsilon_j, \rho_j) & \geq \left(1 - \frac{\rho_j}{|S_j^i|} \right) c_* \sum_{i \in I_j^2} \mathcal{H}^1(\gamma_j^i) |b_j^i|^2 \\ & \geq \left(1 - \frac{\rho_j}{|S_j^i|} \right) \sum_{i \in I_j^2} \mathcal{H}^1(\gamma_j^i) c_* K |b_j^i|. \end{aligned}$$

If now we choose K such that $c_*K \geq \tilde{c}_1$, where \tilde{c}_1 satisfies (3.12), and recalling that $\Psi_0(Q_j^T b_j^i, t_j^i) \geq \tilde{\Psi}_0(Q_j^T b_j^i, t_j^i)$, we have from (5.21) and (5.22)

$$\begin{aligned} & \sum_{i \in I(j)} \mathcal{H}^1(\gamma_j^i) \Psi_\lambda^{nl}(I, Q_j^T b_j^i, t_j^i, |S_j^i|, \varepsilon_j, \rho_j) \\ & \geq \left(1 - \frac{\rho_j}{|S_j^i|} \right) \sum_{i \in I(j)} \mathcal{H}^1(\gamma_j^i) \tilde{\Psi}_0(Q_j^T b_j^i, t_j^i) - C \left(\frac{cK^2}{M} + \omega_{M,K} \left(\frac{\varepsilon_j}{\rho_j} \right) \right); \end{aligned}$$

thus for every $\tilde{\Omega} \subset\subset \Omega$

$$(5.23) \quad \begin{aligned} \liminf_{j \rightarrow \infty} F_j'' & \geq \liminf_{j \rightarrow \infty} \frac{1}{\varepsilon_j^2 |\log \varepsilon_j|} \int_{\Omega_j'' \cap \tilde{\Omega}_j} W(\beta_j) dx \\ & \geq (1 - \nu) \liminf_{j \rightarrow \infty} \sum_{i \in I(j)} \mathcal{H}^1(\gamma_j^i) \tilde{\Psi}_0(Q_j^T b_j^i, t_j^i) - \frac{CK^2}{M} \\ & \geq (1 - \nu) \int_{\gamma \cap \tilde{\Omega}} \tilde{\Psi}_0(Q^T b(x), t(x)) d\mathcal{H}^1(x) - \frac{CK^2}{M}, \end{aligned}$$

where the last inequality follows by the \mathcal{H}^1 -ellipticity of $\tilde{\Psi}_0$ and then by the lower semicontinuity of the associated line tension energy, from the weak* convergence of $Q_j^T \mu_j$ to $Q^T \mu$, with $\mu = b \otimes t\mathcal{H}^1 \llcorner \gamma \in \mathcal{M}_{\mathcal{B}}(\Omega)$.

Conclusion. From Step 1 and Step 2, specifically from estimates (5.17) and (5.23), taking the limit as $M \rightarrow \infty$ and then as $\nu \rightarrow 0$, we obtain

$$\liminf_{j \rightarrow \infty} \mathcal{F}_{\varepsilon_j}(\mu_j, \beta_j) \geq \int_{\Omega} \frac{1}{2} \mathbb{C} \xi : \xi \, dx + \int_{\gamma \cap \tilde{\Omega}} \tilde{\Psi}_0(Q^T b, t) d\mathcal{H}^1.$$

Since $\tilde{\Omega} \subset \subset \Omega$ is arbitrary, this concludes the proof. □

For the upper bound we split the proof into three propositions, exploiting the fact that all measures in $\mathcal{M}_{\mathcal{B}}(\Omega)$ can be approximated by dilute measures.

PROPOSITION 5.3. *Let $\nu \in \mathcal{M}_{\mathcal{B}}(\mathbb{R}^3)$ be polyhedral, and fix $r > 0$. Then for any $\varepsilon > 0$, there exists $\theta_{\varepsilon}^{\nu} \in L^1(\Omega; \mathbb{R}^{3 \times 3})$ such that $\text{curl } \theta_{\varepsilon}^{\nu} = \nu$ in Ω and*

$$(5.24) \quad \limsup_{\varepsilon \rightarrow 0} \frac{1}{|\log \varepsilon|} \int_{\Omega_{\varepsilon}(\nu)} \mathbb{C} \theta_{\varepsilon}^{\nu} : \theta_{\varepsilon}^{\nu} \, dx \leq \int_{\gamma \cap (\Omega)_r} \Psi_0(b, t) d\mathcal{H}^1,$$

where $(\Omega)_r := \{x \in \mathbb{R}^3 : \text{dist}(x, \Omega) < r\}$ and the sequence $\theta_{\varepsilon}^{\nu}$ satisfies

$$(5.25) \quad |\theta_{\varepsilon}^{\nu}(x)| \leq \frac{C}{\text{dist}(x, \gamma)} \quad \forall x \in \mathbb{R}^3.$$

Moreover, the sequence $\hat{\theta}_{\varepsilon}^{\nu} = \theta_{\varepsilon}^{\nu} * \varphi_{\varepsilon} \in L^2(\Omega; \mathbb{R}^{3 \times 3})$, with $\text{curl } \hat{\theta}_{\varepsilon}^{\nu} = \nu * \varphi_{\varepsilon}$ in Ω , satisfies

$$(5.26) \quad \limsup_{\varepsilon \rightarrow 0} \frac{1}{|\log \varepsilon|} \int_{\Omega} \mathbb{C} \hat{\theta}_{\varepsilon}^{\nu} : \hat{\theta}_{\varepsilon}^{\nu} \, dx \leq \int_{\gamma \cap (\Omega)_r} \Psi_0(b, t) d\mathcal{H}^1$$

and

$$(5.27) \quad |\hat{\theta}_{\varepsilon}^{\nu}(x)| \leq \frac{C}{\text{dist}(x, \gamma) + \varepsilon} \quad \forall x \in \mathbb{R}^3.$$

Proof. The proof of this statement is given in [16, Proposition 6.7], where the Γ -limsup estimate for the linear problem is obtained. The explicit estimates (5.25) and (5.27) can be deduced from the construction of the recovery sequence. Indeed, in [16] the latter is obtained essentially by gluing the solution in the whole space given in (4.1) with $\mu = \nu$ together with the cell problem solution for each single segment in the support of ν . This is rigorously done by using Lemmas 5.10 and 5.11 in [16]. In particular this gives the estimate (5.25) as a combination of (4.4) and (3.7). Finally, (5.27) can be easily obtained by (5.25). □

PROPOSITION 5.4. *Let $r > 0$ and $(\mu, \xi, Q) \in \mathcal{M}_{\mathcal{B}}(\mathbb{R}^3) \times L^{\infty}(\Omega; \mathbb{R}^{3 \times 3}) \times SO(3)$ with $\mu = \sum_i b^i \otimes t^i \mathcal{H}^1 \llcorner \gamma^i$ polyhedral and $\text{curl } \xi = 0$.*

Then setting $\beta_{\varepsilon} = Q(I + \varepsilon \sqrt{|\log \varepsilon|} \xi + \varepsilon \hat{\theta}_{\varepsilon}^{\nu})$ with $\nu = Q^T \mu = \sum_i Q^T b^i \otimes t^i \mathcal{H}^1 \llcorner \gamma^i$ and $\hat{\theta}_{\varepsilon}^{\nu}$ given by Proposition 5.3, we have $\beta_{\varepsilon} \in \mathcal{AS}_{\varepsilon}(\mu \llcorner \Omega)$,

$$(5.28) \quad \frac{Q^T \beta_{\varepsilon} - I}{\varepsilon \sqrt{|\log \varepsilon|}} = \xi + \frac{\hat{\theta}_{\varepsilon}^{\nu}}{\sqrt{|\log \varepsilon|}} \rightarrow \xi \quad \text{in } L^2(\Omega; \mathbb{R}^{3 \times 3}),$$

and

$$\limsup_{\varepsilon \rightarrow 0} \mathcal{F}_{\varepsilon}(\mu, \beta_{\varepsilon}) \leq \int_{\Omega} \frac{1}{2} \mathbb{C} \xi : \xi \, dx + \int_{\gamma \cap (\Omega)_r} \Psi_0(Q^T b, t) d\mathcal{H}^1.$$

Proof. Let (μ, ξ, Q) and β_ε be as in the statement. Clearly $\beta_\varepsilon \in \mathcal{AS}_\varepsilon(\mu \llcorner \Omega)$; furthermore, it holds that

$$\frac{Q^T \beta_\varepsilon - I}{\varepsilon \sqrt{|\log \varepsilon|}} = \xi + \frac{\hat{\theta}_\varepsilon^\nu}{\sqrt{|\log \varepsilon|}} \rightharpoonup \xi \quad \text{in } L^2(\Omega; \mathbb{R}^{3 \times 3}).$$

Indeed, from (5.27) we obtain that $\hat{\theta}_\varepsilon^\nu / \sqrt{|\log \varepsilon|}$ is bounded in $L^2(\Omega; \mathbb{R}^{3 \times 3})$ and converges to zero strongly in $L^1(\Omega; \mathbb{R}^{3 \times 3})$. Then $(\mu \llcorner \Omega, \beta_\varepsilon)$ converges to $(\mu \llcorner \Omega, \xi, Q)$ in the sense of Definition 2.6 and also satisfies (5.28).

We define $\Omega_{\varepsilon^\alpha}(\mu) := \{x \in \Omega : \text{dist}(x, \text{supp } \mu) > \varepsilon^\alpha\}$ and $(\gamma)_{\varepsilon^\alpha} := \Omega \setminus \Omega_{\varepsilon^\alpha}(\mu)$ for $\alpha \in (0, 1)$. Then using the frame indifference and the Taylor expansion we get

$$\begin{aligned} & \frac{1}{\varepsilon^2 |\log \varepsilon|} \int_{\Omega_{\varepsilon^\alpha}(\mu)} W(I + \varepsilon \sqrt{|\log \varepsilon|} \xi + \varepsilon \hat{\theta}_\varepsilon^\nu) dx \\ &= \int_{\Omega_{\varepsilon^\alpha}(\mu)} \frac{1}{2} \mathbb{C} \xi : \xi dx + \frac{1}{|\log \varepsilon|} \int_{\Omega_{\varepsilon^\alpha}(\mu)} \frac{1}{2} \mathbb{C} \hat{\theta}_\varepsilon^\nu : \hat{\theta}_\varepsilon^\nu dx + \frac{1}{\sqrt{|\log \varepsilon|}} \int_{\Omega_{\varepsilon^\alpha}(\mu)} \mathbb{C} \xi : \hat{\theta}_\varepsilon^\nu dx \\ & \quad + \int_{\Omega_{\varepsilon^\alpha}(\mu)} \frac{\sigma(\varepsilon \sqrt{|\log \varepsilon|} \xi + \varepsilon \hat{\theta}_\varepsilon^\nu)}{\varepsilon^2 |\log \varepsilon|} dx, \end{aligned}$$

where $\sigma(F)/|F|^2 \rightarrow 0$ as $|F| \rightarrow 0$. By Proposition 5.3 we deduce

$$(5.29) \quad \limsup_{\varepsilon \rightarrow 0} \frac{1}{|\log \varepsilon|} \int_{\Omega} \frac{1}{2} \mathbb{C} \hat{\theta}_\varepsilon^\nu : \hat{\theta}_\varepsilon^\nu dx \leq \int_{\gamma \cap \Omega_r} \Psi_0(Q^T b, t) d\mathcal{H}^1.$$

Recalling that $\xi \in L^\infty(\Omega; \mathbb{R}^{3 \times 3})$ and, from (5.27), $\hat{\theta}_\varepsilon^\nu / \sqrt{|\log \varepsilon|}$ converges to zero strongly in $L^1(\Omega; \mathbb{R}^{3 \times 3})$, we have

$$\lim_{\varepsilon \rightarrow 0} \frac{1}{\sqrt{|\log \varepsilon|}} \int_{\Omega_{\varepsilon^\alpha}(\mu)} \mathbb{C} \xi : \hat{\theta}_\varepsilon^\nu dx = 0.$$

Finally, setting $\omega(t) := \sup_{|F| \leq t} |\sigma(F)|$, we find

$$\begin{aligned} & \lim_{\varepsilon \rightarrow 0} \left| \int_{\Omega_{\varepsilon^\alpha}(\mu)} \frac{\sigma(\varepsilon \sqrt{|\log \varepsilon|} \xi + \varepsilon \hat{\theta}_\varepsilon^\nu)}{\varepsilon^2 |\log \varepsilon|} dx \right| \\ & \leq \lim_{\varepsilon \rightarrow 0} \int_{\Omega} \chi_{\Omega_{\varepsilon^\alpha}(\mu)} \frac{\omega(|\varepsilon \sqrt{|\log \varepsilon|} \xi + \varepsilon \hat{\theta}_\varepsilon^\nu|)}{|\varepsilon \sqrt{|\log \varepsilon|} \xi + \varepsilon \hat{\theta}_\varepsilon^\nu|^2} \cdot \frac{|\varepsilon \sqrt{|\log \varepsilon|} \xi + \varepsilon \hat{\theta}_\varepsilon^\nu|^2}{\varepsilon^2 |\log \varepsilon|} dx = 0. \end{aligned}$$

Indeed, by (5.27), in $\Omega_{\varepsilon^\alpha}(\mu)$ we have $|\hat{\theta}_\varepsilon^\nu| \leq C\varepsilon^{-\alpha}$, and then the integrand

$$\frac{\omega(|\varepsilon \sqrt{|\log \varepsilon|} \xi + \varepsilon \hat{\theta}_\varepsilon^\nu|)}{|\varepsilon \sqrt{|\log \varepsilon|} \xi + \varepsilon \hat{\theta}_\varepsilon^\nu|^2} \cdot \frac{|\varepsilon \sqrt{|\log \varepsilon|} \xi + \varepsilon \hat{\theta}_\varepsilon^\nu|^2}{\varepsilon^2 |\log \varepsilon|}$$

is the product of a sequence converging to zero in $L^\infty(\Omega)$ and a bounded sequence in $L^1(\Omega)$.

Finally, it remains to estimate the energy in $(\gamma)_{\varepsilon^\alpha}$. Using the frame indifference and the estimate from above for W we get

$$\begin{aligned} \frac{1}{\varepsilon^2 |\log \varepsilon|} \int_{(\gamma)_{\varepsilon^\alpha}} W(\beta_\varepsilon) dx &\leq \frac{1}{\varepsilon^2 |\log \varepsilon|} \int_{(\gamma)_{\varepsilon^\alpha}} |\varepsilon \sqrt{|\log \varepsilon|} \xi + \varepsilon \hat{\theta}_\varepsilon^\nu|^2 dx \\ &\leq 2 \int_{(\gamma)_{\varepsilon^\alpha}} |\xi|^2 dx + \frac{2}{|\log \varepsilon|} \int_{(\gamma)_{\varepsilon^\alpha}} |\hat{\theta}_\varepsilon^\nu|^2 dx, \end{aligned}$$

where the first term of the right-hand side tends to zero as $\varepsilon \rightarrow 0$, while from (5.27) we have

$$\lim_{\varepsilon \rightarrow 0} \frac{1}{|\log \varepsilon|} \int_{(\gamma)_{\varepsilon^\alpha}} |\hat{\theta}_\varepsilon^\nu|^2 dx \leq C(1 - \alpha).$$

Thus we conclude that

$$\begin{aligned} \limsup_{\varepsilon \rightarrow 0} \mathcal{F}_\varepsilon(\mu, \beta_\varepsilon) &\leq \int_\Omega \frac{1}{2} \mathbb{C} \xi : \xi dx + \limsup_{\varepsilon \rightarrow 0} \frac{1}{|\log \varepsilon|} \int_\Omega \frac{1}{2} \mathbb{C} \hat{\theta}_\varepsilon^\nu : \hat{\theta}_\varepsilon^\nu dx \\ &\quad + \frac{2}{|\log \varepsilon|} \int_{(\gamma)_{\varepsilon^\alpha}} |\hat{\theta}_\varepsilon^\nu|^2 dx \\ &\leq \int_\Omega \frac{1}{2} \mathbb{C} \xi : \xi dx + \int_{\gamma \cap (\Omega)_r} \Psi_0(Q^T b, t) d\mathcal{H}^1 + C(1 - \alpha), \end{aligned}$$

which concludes the proof taking the limit as $\alpha \rightarrow 1$. □

PROPOSITION 5.5 (upper bound). *Let $(\mu, \xi, Q) \in \mathcal{M}_B(\Omega) \times L^2(\Omega; \mathbb{R}^{3 \times 3}) \times SO(3)$ with $\text{curl} \xi = 0$. Then, for every sequence $\varepsilon_k \rightarrow 0$ there exists a sequence $(\tilde{\mu}_k, \beta_k) \in \mathcal{M}_B(\mathbb{R}^3) \times \mathcal{AS}_{\varepsilon_k}(\tilde{\mu}_k \llcorner \Omega)$ such that $\tilde{\mu}_k \llcorner \Omega$ is $(h_{\varepsilon_k}, \alpha_{\varepsilon_k})$ -dilute in $\bar{\Omega}$, $\tilde{\mu}_k \xrightarrow{*} \mu$ in Ω ,*

$$\frac{Q^T \beta_k - I}{\varepsilon_k \sqrt{|\log \varepsilon_k|}} \rightharpoonup \xi \quad \text{in } L^2(\Omega; \mathbb{R}^{3 \times 3}),$$

and

$$\limsup_{k \rightarrow \infty} \mathcal{F}_{\varepsilon_k}(\tilde{\mu}_k \llcorner \Omega, \beta_k) \leq \mathcal{F}_0(\mu, \xi, Q).$$

Proof. By a standard density argument we can assume that $\xi \in L^\infty(\Omega; \mathbb{R}^{3 \times 3})$. The proof is based on a diagonal argument using Proposition 5.4, and it is analogous to the linear case [16, Proposition 6.8].

Let $Q^T \mu = Q^T b \otimes t \mathcal{H}^1 \llcorner \gamma$; by [12, Theorem 3.1]

$$\int_\gamma \tilde{\Psi}_0(Q^T b, t) d\mathcal{H}^1,$$

with $\tilde{\Psi}_0$ as defined in (2.12), is the lower semicontinuous envelope of

$$\int_\gamma \Psi_0(Q^T b, t) d\mathcal{H}^1.$$

Then, we can find a sequence $\nu_j = b_j \otimes t_j \mathcal{H}^1 \llcorner \gamma_j \in \mathcal{M}_B(\Omega)$ converging weak* to μ such that

$$(5.30) \quad \limsup_{j \rightarrow \infty} \int_{\gamma_j} \Psi_0(Q^T b_j, t_j) d\mathcal{H}^1 \leq \int_\gamma \tilde{\Psi}_0(Q^T b, t) d\mathcal{H}^1.$$

Now we denote by \mathcal{F} the functional

$$(5.31) \quad \mathcal{F}(\mu, \xi, Q, r) := \int_\Omega \frac{1}{2} \mathbb{C} \xi : \xi dx + \int_{\gamma \cap (\Omega)_r} \Psi_0(Q^T b, t) d\mathcal{H}^1$$

for $r > 0$. Then for all j we apply [16, Lemma 6.4] and find a polyhedral measure $\mu_j := b_j \otimes t_j \mathcal{H}^1 \llcorner \gamma_j \in \mathcal{M}_B(\mathbb{R}^3)$ such that

$$(5.32) \quad \mathcal{F}(\mu_j, \xi, Q, \frac{1}{j}) \leq (1 + c \frac{1}{j}) \mathcal{F}(\nu_j, \xi, Q, 0) + C \frac{1}{j},$$

and μ_j is close to ν_j in the following sense: there exists a bi-Lipschitz map $f^j : \mathbb{R}^3 \rightarrow \mathbb{R}^3$, with

$$(5.33) \quad |f^j(x) - x| + |Df^j(x) - Id| < \frac{1}{j} \quad \forall x \in \mathbb{R}^3,$$

such that

$$(5.34) \quad |\mu_j - f^j_{\#} \nu_j|(\Omega) < \frac{1}{j}.$$

In particular, $\mu_j \xrightarrow{*} \mu$ in Ω as $j \rightarrow \infty$. Furthermore, since the restriction $\mu_j \llcorner \Omega$ is polyhedral, it is not restrictive to assume that the segments of the support of μ_j intersect the boundary of Ω with an angle at most $\alpha_j > 0$ (otherwise a small modification of the support of μ_j for segments that are tangent to $\partial\Omega$ will reduce to the latter case with arbitrarily small errors in the line tension energy and therefore in (5.32)). Thus μ_j is $(h_\varepsilon, \alpha_\varepsilon)$ -dilute in $\bar{\Omega}$ according to Definition 2.3 for sufficiently small ε . From Proposition 5.4 applied to (μ_j, ξ, Q) , for every j there is a sequence $\beta_k^j \in \mathcal{AS}_{\varepsilon_k}(\mu_j \llcorner \Omega)$ that satisfies

$$(5.35) \quad \frac{Q^T \beta_k^j - I}{\varepsilon_k \sqrt{|\log \varepsilon_k|}} = \xi + \frac{\hat{\theta}_{\varepsilon_k}^{\mu_j}}{\sqrt{|\log \varepsilon_k|}} \rightharpoonup \xi \quad \text{in } L^2(\Omega; \mathbb{R}^{3 \times 3}),$$

and

$$(5.36) \quad \limsup_{k \rightarrow \infty} \mathcal{F}_{\varepsilon_k}(\mu_j \llcorner \Omega, \beta_k^j) \leq \mathcal{F}(\mu_j, \xi, Q, \frac{1}{j}).$$

The function $\hat{\theta}_{\varepsilon_k}^{\mu_j}$ is given by Proposition 5.3 and by (5.26) satisfies

$$\left\| \frac{\hat{\theta}_{\varepsilon_k}^{\mu_j}}{\sqrt{|\log \varepsilon_k|}} \right\|_{L^2(\Omega; \mathbb{R}^{3 \times 3})} \leq M.$$

By (5.30) we finally obtain

$$(5.37) \quad \limsup_{j \rightarrow \infty} \limsup_{k \rightarrow \infty} \mathcal{F}_{\varepsilon_k}(\mu_j \llcorner \Omega, \beta_k^j) \leq \mathcal{F}_0(\mu, \xi, Q).$$

In order to construct a diagonal sequence which satisfies the thesis, we follow the same idea of [16], and we notice that the following properties are satisfied for k large enough:

- (1) The measures μ_j are $(h_{\varepsilon_k}, \alpha_{\varepsilon_k})$ -dilute in $\bar{\Omega}$;
- (2) $\mathcal{F}_{\varepsilon_k}(\mu_j \llcorner \Omega, \beta_k^j) \leq \mathcal{F}(\mu_j, \xi, Q, \frac{1}{j}) + \frac{1}{j}$;
- (3) $d(\hat{\theta}_{\varepsilon_k}^{\mu_j} / \sqrt{|\log \varepsilon_k|}, 0) \leq \frac{1}{j}$, where d denotes the distance that metrizes the weak convergence in $L^2 \cap \{f : \|f\|_{L^2(\Omega; \mathbb{R}^{3 \times 3})} \leq M\}$.

For every j we define an increasing sequence of indices $m(j)$ as follows:

$$(5.38) \quad m(j) := \min \{m \geq m(j-1) : \mu_j \text{ satisfies (1)–(3)} \quad \forall k \geq m\}.$$

Now for every $k > 0$ we define $\tilde{\mu}_k := \mu_j$ and $\beta_k := \beta_k^j$ if $k \in [m(j), m(j+1)) \cap \mathbb{N}$. By (5.34) we have

$$|\tilde{\mu}_k - f^j_{\#} \nu_j|(\Omega) < \frac{1}{j} \quad \forall k \in [m(j), m(j+1)),$$

and then from (5.33) and the fact that ν_j weak* converges to μ , we conclude that $\tilde{\mu}_k$ weak* converges to μ . In addition, by (3) and (2) we also have

$$\frac{Q^T \beta_k - I}{\varepsilon_k \sqrt{|\log \varepsilon_k|}} \rightharpoonup \xi \quad \text{in } L^2(\Omega; \mathbb{R}^{3 \times 3}),$$

$$\limsup_{k \rightarrow \infty} \mathcal{F}_{\varepsilon_k}(\tilde{\mu}_k \llcorner \Omega, \beta_k) \leq \mathcal{F}_0(\mu, \xi, Q),$$

and the proof is concluded. □

Proof of Theorem 2.7. The thesis is a direct consequence of Propositions 5.2 and 5.5. □

Appendix A. Notation. Here we collect the notation we adopted throughout the paper.

- C, c denote strictly positive constants which may vary from line to line.
- \mathcal{H}^1 denotes the one-dimensional Hausdorff measure.
- $\mathcal{H}^1 \llcorner E$ denotes the one-dimensional Hausdorff measure restricted to the set E .
- $u \otimes v$ denotes the tensor product for $u, v \in \mathbb{R}^3$.
- $\text{curl } A$ denotes the 3×3 matrix whose rows are the curl of the rows of A if $A \in \mathbb{R}^{3 \times 3}$, the vector $(\partial_1 a_{12} - \partial_2 a_{11}, \partial_1 a_{22} - \partial_2 a_{21})$ if $A \in \mathbb{R}^{2 \times 2}$ with $(A)_{ij} = a_{ij}$.
- $\text{div } A$ denotes the vector in \mathbb{R}^n whose components are the divergence of the rows of A with $A \in \mathbb{R}^{n \times n}$.
- $A : B$ denotes the Euclidean scalar product of matrices, i.e., $A : B = \sum_{i,j} A_{ij} B_{ij}$.
- For $n \in \mathbb{N}$, $SO(n)$ denotes the n -dimensional special orthogonal group.
- $\text{dist}(A, SO(n)) := \min_{Q \in SO(n)} |A - Q|$ for $n \in \mathbb{N}$ and $A \in \mathbb{R}^{n \times n}$.
- $\mathcal{M}(\Omega; \mathbb{R}^{3 \times 3})$ denotes the space of 3×3 matrix-valued measures.
- \mathcal{B} denotes a discrete lattice in \mathbb{R}^3 that represents the space of admissible renormalized Burgers vectors.
- For $\gamma \subset \mathbb{R}^3$ rectifiable, $L^1(\gamma; \mathcal{B}; \mathcal{H}^1 \llcorner \gamma)$ denotes the space of the admissible Burgers vector fields, i.e., the space of all measurable functions $b: \gamma \rightarrow \mathcal{B}$ that satisfy

$$\int_{\gamma} |b(x)| d\mathcal{H}^1(x) < +\infty.$$

- $\mathcal{M}_{\mathcal{B}}(\Omega)$ denotes the set of admissible dislocation densities, i.e.,

$$\mathcal{M}_{\mathcal{B}}(\Omega) := \left\{ \mu \in \mathcal{M}(\Omega; \mathbb{R}^{3 \times 3}) : \mu = b \otimes t\mathcal{H}^1 \llcorner \gamma, \right. \\ \left. \text{div } \mu = 0, b \in \mathcal{B}, \gamma \text{ 1-rectifiable} \right\}.$$

- $|\mu|(\Omega)$ denotes the total variation of a measure $\mu \in \mathcal{M}(\Omega; \mathbb{R}^{n \times n})$, i.e., the mass of $|\mu|$.
- For $\Omega \subset \mathbb{R}^3$, we set $\Omega_{\varepsilon}(\mu) := \{x \in \Omega : \text{dist}(x, \text{supp } \mu) > \varepsilon\}$.
- For $\varepsilon > 0$, we set $(\Omega)_{\varepsilon} := \{x \in \mathbb{R}^3 : \text{dist}(x, \Omega) < \varepsilon\}$.
- χ_E denotes the characteristic function of the set $E \in \mathbb{R}^n$.
- For $\varepsilon > 0$, $\varphi_{\varepsilon} := \varepsilon^{-3} \varphi(x/\varepsilon)$ denotes a mollification kernel with $\varphi \leq C \chi_{B_1(0)}$.

- For $t \in S^2$, $Q_t \in SO(3)$ is the matrix such that $Q_t e_3 = t$, and

$$\Phi_t(r, \theta, z) := Q_t(r \cos \theta, r \sin \theta, z)$$

is the change of variables to cylindrical coordinates with axis t .

- (e_r, e_θ, e_3) denotes the local basis in cylindrical coordinates of \mathbb{R}^3 .
- For $R > 0$ B'_R denotes the ball of radius R in \mathbb{R}^2 centered in the origin and B_R the ball of radius R in \mathbb{R}^3 centered in the origin.
- $\mu_{b,t} = b \otimes t \mathcal{H}^1 \llcorner \mathbb{R}t$ for $b \in \mathbb{R}^3$ and $t \in S^2$.
- For $b \in \mathbb{R}^3$, $t \in S^2$, $r, R, h \in (0, \infty)$ with $r < R \leq h$ we define the hollow cylinders,

$$(A.1) \quad T_h^{r,R} := (B'_R \setminus B'_r) \times (0, h), \quad Q_t T_h^{r,R} := Q_t((B'_R \setminus B'_r) \times (0, h)),$$

and the (full) cylinders,

$$(A.2) \quad T_h^R := T_h^{0,R} = B'_R \times (0, h), \quad Q_t T_h^R := Q_t T_h^{0,R} = Q_t(B'_R \times (0, h)).$$

- For the $\gamma \subset \mathbb{R}^3$ segment, $\rho > 0$, $\delta \geq 0$, we define, with a little abuse of notation, the cylinder

$$T_{\rho,\delta}(\gamma) := A(B'_\rho \times S_\delta),$$

where $S_\delta \subset \mathbb{R}$ is a segment of length $\mathcal{H}^1(\gamma) - 2\delta$ and A is an affine transformation that maps S_δ into γ and the midpoint of S_δ into the midpoint of γ . If γ is fixed, we simply write $T_{\rho,\delta}$ in place of $T_{\rho,\delta}(\gamma)$. The latter will be used only in section 4.

- $(\gamma)_\varepsilon = \{x \in \mathbb{R}^3 : \text{dist}(x, \gamma) < \varepsilon\}$.

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