

HOMOTOPY ABELIANITY OF THE DG-LIE ALGEBRA CONTROLLING DEFORMATIONS OF PAIRS (VARIETY WITH TRIVIAL CANONICAL BUNDLE, LINE BUNDLE)

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ABSTRACT. We investigate the deformations of pairs (X, L) , where L is a line bundle on a smooth projective variety X , defined over an algebraically closed field \mathbb{K} of characteristic 0. In particular, we prove that the DG-Lie algebra controlling the deformations of the pair (X, L) is homotopy abelian whenever X has trivial canonical bundle, and so these deformations are unobstructed.

1. INTRODUCTION

Let X be a smooth projective variety with trivial canonical bundle defined over an algebraically closed field \mathbb{K} of characteristic 0. It is well known that the deformations of X are unobstructed by the Bogomolov-Tian-Todorov (BTT) Theorem. This was first proved over the field of complex numbers by Bogomolov [Bo79], under some additional assumptions, and then independently by Tian and Todorov [Ti88, To89]. The first algebraic proof was given by Ran and Kawamata, by using (and introducing) the nowadays called T^1 -lifting method [FM99, Ka92, Ra92]. The same method easily applies to prove that the deformations of pairs (X, L) are also unobstructed, whenever X is a smooth projective variety with trivial canonical bundle and L is a line bundle on X (see next Remark 2.6).

An improvement of the BTT Theorem consists in showing that the differential graded Lie algebra controlling deformations of X as above is quasi-isomorphic to an abelian DG-Lie algebra: this was proved by Goldman and Millson [GM90] in the Kähler case and then by Iacono and Manetti [IM10] over any algebraically closed field \mathbb{K} of characteristic 0.

The aim of this paper is to use the methods of [IM19] in order to prove that the DG-Lie algebra controlling deformations of pairs (X, L) is also quasi-isomorphic to an abelian DG-Lie algebra. By homotopy invariance of deformation functors, this implies that the associated deformation functor is smooth.

Theorem 1.1. *Let L be a line bundle on a smooth projective variety X defined over an algebraically closed field \mathbb{K} of characteristic 0. If X has trivial canonical bundle, then the DG-Lie algebra controlling the deformations of the pair (X, L) is homotopy abelian.*

It is well known (see e.g. [Hu95, IM19]) that the DG-Lie algebra controlling the deformations of the pair (X, L) is the algebra $R\Gamma(X, \mathcal{D}^1(L))$ of the derived sections of the sheaf of first-order differential operators on L : this is an object in the homotopy category of DG-Lie algebras and then it is represented by a DG-Lie algebra up to quasi-isomorphism. Over the complex numbers, a possible representative of $R\Gamma(X, \mathcal{D}^1(L))$ is given by the Dolbeault resolution of $\mathcal{D}^1(L)$ [Mar12, Example 2.12].

In this paper, we work over any algebraically closed field \mathbb{K} of characteristic 0 and so we adopt the purely algebraic construction of the Thom-Whitney-Sullivan totalization with respect to any affine open cover, described in [IM19, Sections 6 and 7].

Date: July 9, 2021.

2010 Mathematics Subject Classification. 14D15, 17B70, 13D10, 32G08.

Key words and phrases. Deformations of manifold and line bundle, differential graded Lie algebras.

The main idea behind the proof of Theorem 1.1 is the following. Given a pair (X, L) , we construct a new pair (Y, Δ) , where Y is a \mathbb{P}^1 -bundle on X and Δ is a smooth divisor in Y . Whenever X has trivial canonical bundle, the pair (Y, Δ) is a log Calabi-Yau pair. Then, we conclude the proof showing that there exists a quasi isomorphism between the DG-Lie algebra controlling the deformations of the pair (X, L) and the homotopy abelian DG-Lie algebra controlling the deformations of the pair (Y, Δ) (Lemma 2.2).

2. PROOF OF THEOREM 1.1

Let L be a line bundle on a smooth algebraic variety X of dimension n over an algebraically closed field \mathbb{K} of characteristic 0 and denote by $\mathcal{L} = \mathcal{O}_X(L)$ the invertible sheaf of its sections. According to [IM19, Section 5], we denote by $\mathcal{D}(X, \mathcal{L})$ the sheaf of the derivations of pairs which is the subsheaf of $\mathcal{D}er_{\mathbb{K}}(\mathcal{O}_X, \mathcal{O}_X) \times \mathcal{H}om_{\mathbb{K}}(\mathcal{L}, \mathcal{L})$ consisting of pairs (h, u) such that $u(ax) = h(a)x + au(x)$ for every $a \in \mathcal{O}_X$ and $x \in \mathcal{L}$, i.e.,

$$\mathcal{D}(X, \mathcal{L}) = \{(h, u) \in \mathcal{D}er_{\mathbb{K}}(\mathcal{O}_X, \mathcal{O}_X) \times \mathcal{H}om_{\mathbb{K}}(\mathcal{L}, \mathcal{L}) \mid u(ae) - au(e) = h(a)e, \forall a \in \mathcal{O}_X, e \in \mathcal{L}\}.$$

It is almost immediate to see that $\mathcal{D}(X, \mathcal{L})$ is a sheaf of Lie algebras over \mathbb{K} and that the projection on the second factor $(h, u) \mapsto u$ induces an isomorphism with the sheaf $\mathcal{D}^1(L)$ of first-order differential operators [IM19, Example 5.2]. In particular $\mathcal{D}(X, \mathcal{L})$ is locally free of rank $n + 1$ and there exists the following exact sequence

$$0 \rightarrow \mathcal{H}om_{\mathcal{O}_X}(\mathcal{L}, \mathcal{L}) \rightarrow \mathcal{D}(X, \mathcal{L}) \rightarrow \Theta_X \rightarrow 0,$$

where $\Theta_X = \mathcal{D}er_{\mathbb{K}}(\mathcal{O}_X, \mathcal{O}_X)$ denotes the tangent sheaf of X . Consider the \mathbb{P}^1 -bundle

$$p: Y = \mathbb{P}(\mathcal{O}_X \oplus \mathcal{L}) = \mathbb{P}(\mathcal{L}^{-1} \oplus \mathcal{O}_X) \rightarrow X$$

together with the two distinguished sections Δ_0 and Δ_∞ corresponding to the two direct summands, namely:

$$Y = \Delta_\infty \cup \text{Spec}_X(\oplus_{n \geq 0} \mathcal{L}^n) = \Delta_0 \cup \text{Spec}_X(\oplus_{n \leq 0} \mathcal{L}^n).$$

If $\Delta = \Delta_0 + \Delta_\infty$, then we have the adjunction formula $p^*K_X = K_Y + \Delta$: this follows from the relative Euler exact sequence [Ha77, Exercise III.8.4]. It can be also proved by noticing that if ω is a rational n -form on X then $p^*\omega \wedge \frac{dt}{t}$, where t is a local coordinate frame on the fibres of L , is a well defined rational $n + 1$ -form on Y . Note that if X has trivial canonical bundle, then Δ is an anticanonical divisor in Y , i.e., (Y, Δ) is a log Calabi-Yau pair.

We denote by Θ_Y the tangent sheaf of Y and by $\Theta_Y(-\log \Delta)$ the subsheaf of vector fields that are tangent to the smooth divisor Δ . Note that $\Theta_Y(-\log \Delta)$ is the subsheaf of the derivations of the sheaf \mathcal{O}_Y preserving the ideal sheaf of Δ . Moreover, since $\Delta \subset Y$ is smooth, there exists the following exact sequence

$$0 \rightarrow \Theta_Y(-\log \Delta) \rightarrow \Theta_Y \rightarrow N_{\Delta/Y} \rightarrow 0.$$

Lemma 2.1. *In the above notation $R^i p_* \Theta_Y(-\log \Delta) = 0$ for every $i > 0$ and there exists a natural \mathcal{O}_X -linear isomorphism of sheaves of Lie algebras*

$$\Psi: \mathcal{D}(X, \mathcal{L}) \xrightarrow{\sim} p_* \Theta_Y(-\log \Delta).$$

Proof. In the sequel, we shall denote by $U = Y - \Delta_\infty$ the total space of the dual bundle of L . Assume first that $X = \text{Spec } A$ is an affine scheme and that L is the trivial line bundle. Thus $Y = \mathbb{P}^1 \times X$, $\Delta = \{0, \infty\} \times X$ and then

$$\Theta_Y(-\log \Delta) = p^* \Theta_X \oplus q^* \Theta_{\mathbb{P}^1}(-0 - \infty),$$

where q is the projection onto \mathbb{P}^1 . Since $\Theta_{\mathbb{P}^1}(-0 - \infty)$ is trivial, we have that $\Theta_Y(-\log \Delta) = p^* \Theta_X \oplus \mathcal{O}_Y$. Since $p_* \mathcal{O}_Y = \mathcal{O}_X$ and $R^i p_* \mathcal{O}_Y = 0$ for $i > 0$ [Ha77, Exercise III.8.4], by the projection formula [Ha77, Exercise III.8.3] we have

$$p_* \Theta_Y(-\log \Delta) = \Theta_X \oplus \mathcal{O}_X, \quad R^i p_* \Theta_Y(-\log \Delta) = 0, \quad i > 0.$$

We point out that $p_*\Theta_Y(-\log \Delta)$ is a locally free sheaf of rank $n+1$, whose sections are of type $\chi + at\frac{d}{dt}$, where $\chi \in \Theta_X$, $a \in \mathcal{O}_X$ and t is a linear coordinate on the fibres of L .

We have $U = \text{Spec } R$, $R = \Gamma(X, \oplus_{n \geq 0} \mathcal{L}^n)$; the choice of an isomorphism $z: \mathcal{O}_X \rightarrow \mathcal{L}$ provides an isomorphism of A -algebras $R = A[z]$. In this setting, there exists a unique A -linear morphism of Lie algebras

$$\Psi: \Gamma(X, \mathcal{D}(X, \mathcal{L})) \rightarrow \Gamma(X, p_*\Theta_U) = \Gamma(U, \Theta_U) = \text{Der}_{\mathbb{K}}(R, R),$$

such that $\Psi(h, u)(a) = h(a)$ and $\Psi(h, u)(m) = u(m)$ for every $a \in A \subset R$ and every $m \in \mathcal{L}$. The unicity is clear by Leibniz formula: for the existence, using the isomorphism $R = A[z]$ it is sufficient to define

$$\Psi(h, u) = h + u(z)\frac{d}{dz}.$$

We have $\left(h + u(z)\frac{d}{dz}\right)(a) = h(a)$ for every $a \in A$. Every section of \mathcal{L} is of type az for some $a \in A$ and then

$$\left(h + u(z)\frac{d}{dz}\right)(az) = h(a)z + u(z)a = u(az).$$

Notice that, since $u(z) = zk$ for some $k \in A$, the vector field $\Psi(h, u)$ is tangent to Δ and then belongs to $\Gamma(X, p_*\Theta_Y(-\log \Delta))$.

The local unicity allows to glue the morphisms Ψ on open affine subsets to a morphism of quasi-coherent sheaves $\mathcal{D}(X, \mathcal{L}) \rightarrow p_*\Theta_U$ whose image is contained in $p_*\Theta_Y(-\log \Delta)$. Moreover, the explicit local description of Ψ implies that $\Psi: \mathcal{D}(X, \mathcal{L}) \rightarrow p_*\Theta_Y(-\log \Delta)$ is an isomorphism of locally free sheaves of rank $n+1$. \square

Given a coherent sheaf of Lie algebra \mathcal{F} over X , we denote by $R\Gamma(X, \mathcal{F})$ the DG-Lie algebra of derived sections. Let $\mathcal{U} = \{U_i\}_{i \in I}$ be an open affine cover of X , we denote by $C^*(\mathcal{U}, \mathcal{F})$ the Čech complex of \mathcal{F} , i.e., the cochain complex associated with the semicosimplicial Lie algebra:

$$\mathcal{F}(\mathcal{U}): \quad \prod_i \mathcal{F}(U_i) \rightrightarrows \prod_{i,j} \mathcal{F}(U_{ij}) \rightrightarrows \prod_{i,j,k} \mathcal{F}(U_{ijk}) \rightrightarrows \cdots,$$

where the face operators $\partial_h: \prod_{i_0, \dots, i_{k-1}} \mathcal{F}(U_{i_0 \dots i_{k-1}}) \rightarrow \prod_{i_0, \dots, i_k} \mathcal{F}(U_{i_0 \dots i_k})$ are given by

$$\partial_h(x)_{i_0 \dots i_k} = x_{i_0 \dots \widehat{i_h} \dots i_k}|_{U_{i_0 \dots i_k}}, \quad \text{for } h = 0, \dots, k.$$

An explicit model of $R\Gamma(X, \mathcal{F})$ is given by the Thom-Whitney-Sullivan totalization $\text{Tot}(\mathcal{U}, \mathcal{F})$ associated with the semicosimplicial Lie algebra $\mathcal{F}(\mathcal{U})$, see e.g. [FMM12, IM19]. Note that the homotopy class of the DG-Lie algebra $\text{Tot}(\mathcal{U}, \mathcal{F})$ does not depend on the choice of the open affine cover and, by Whitney's theorem (see e.g. [IM10, Sec. 2]), there exists a canonical quasi-isomorphism of complexes

$$I: \text{Tot}(\mathcal{U}, \mathcal{F}) \rightarrow C^*(\mathcal{U}, \mathcal{F}).$$

As we already point out, the sheaf of Lie algebras $\mathcal{D}(X, \mathcal{L})$ is isomorphic to the sheaf $\mathcal{D}^1(L)$, and so the DG-Lie algebra $R\Gamma(X, \mathcal{D}(X, \mathcal{L}))$ controls the deformations of the pair (X, L) [IM19, Theorem 7.5]. As regard the deformations of the pair (Y, Δ) , these are controlled by the DG-Lie algebra $R\Gamma(Y, \Theta_Y(-\log \Delta))$ [KKP08, Section 4.3.3 (i)] or [Ia15, Theorem 4.3].

Lemma 2.2. *The morphism $\Psi: \mathcal{D}(X, \mathcal{L}) \rightarrow p_*\Theta_Y(-\log \Delta)$ induces a quasi-isomorphism of DG-Lie algebras*

$$\Psi: R\Gamma(X, \mathcal{D}(X, \mathcal{L})) \rightarrow R\Gamma(Y, \Theta_Y(-\log \Delta)).$$

Therefore the DG-Lie algebra controlling the deformations of the pair (X, L) is quasi-isomorphic to the DG-Lie algebra controlling the deformations of the pair (Y, Δ) .

Proof. Let $\mathcal{U} = \{U_i\}_{i \in I}$ be an open affine cover of X and take an open affine cover $\mathcal{V} = \{V_j\}_{j \in J}$ of Y together with a refining map $r: J \rightarrow I$ such that $p(V_j) \subset U_{r(j)}$ for every j . The above data give a morphism of Čech complexes

$$C^*(\mathcal{U}, p_*\Theta_Y(-\log \Delta)) \rightarrow C^*(\mathcal{V}, \Theta_Y(-\log \Delta)),$$

which is a quasi-isomorphism by Leray spectral sequence (see e.g., [Vo12, Theorem 16.11]). Therefore, the morphism Ψ of Lemma 2.1 gives a quasi-isomorphism of Čech complexes

$$\Psi: C^*(\mathcal{U}, \mathcal{D}(X, \mathcal{L})) \rightarrow C^*(\mathcal{V}, \Theta_Y(-\log \Delta)).$$

Similarly, Ψ and the refining map induce a morphism of semicosimplicial Lie algebra

$$\mathcal{D}(X, \mathcal{L})(\mathcal{U}) \rightarrow \Theta_Y(-\log \Delta)(\mathcal{V})$$

and so a DG-Lie algebras morphism of the Thom-Whitney-Sullivan totalizations

$$\Psi: \text{Tot}(\mathcal{U}, \mathcal{D}(X, \mathcal{L})) \rightarrow \text{Tot}(\mathcal{V}, \Theta_Y(-\log \Delta)),$$

which is a quasi-isomorphism by Whitney's Theorem. \square

If X has trivial canonical bundle then (Y, Δ) is a log Calabi-Yau pair, thus Theorem 1.1 is an immediate consequence of Lemma 2.2 and of the following theorem [Ia15, Corollary 5.4] or [KKP08, Lemma 4.19], cf. [Ia17, Sec. 4.2].

Theorem 2.3. *Let Y be a smooth projective variety defined over an algebraically closed field of characteristic 0 and $\Delta \subset Y$ a smooth divisor. If (Y, Δ) is a log Calabi-Yau pair, then the DG-Lie algebra $R\Gamma(Y, \Theta_Y(-\log \Delta))$ is homotopy abelian.*

Finally, we can conclude the unobstructedness of deformations.

Corollary 2.4. *Let L be a line bundle on a smooth projective variety X defined over an algebraically closed field \mathbb{K} of characteristic 0. If X has trivial canonical bundle, then the pair (X, L) has unobstructed deformations.*

Proof. It is sufficient to recall that every deformation problem controlled by a homotopy abelian DG-Lie algebra is unobstructed (see e.g. [Ma04]). \square

Remark 2.5. Over the field of complex numbers, the unobstructedness of the pair (X, L) was also proved using a geometric approach in the fifth version of [LP19].

Remark 2.6. It is also possible to prove Corollary 2.4 by using the T^1 -lifting theorem in view of the following observation (for simplicity of exposition we assume here $\mathbb{K} = \mathbb{C}$). The short exact sequence of sheaves

$$0 \rightarrow \mathcal{O}_X \rightarrow \mathcal{D}(X, \mathcal{L}) \rightarrow \Theta_X \rightarrow 0,$$

gives a cohomology exact sequence

$$H^0(\Theta_X) \xrightarrow{\alpha_0} H^1(\mathcal{O}_X) \rightarrow H^1(\mathcal{D}(X, \mathcal{L})) \rightarrow H^1(\Theta_X) \xrightarrow{\alpha_1} H^2(\mathcal{O}_X)$$

where α_0 and α_1 are given by contraction with the first Chern class of L . Then, the T^1 -lifting theorem applies if the corank of α_0 and the nullity of α_1 are invariant under deformations of the pair (X, L) over $\mathbb{C}[t]/(t^n)$, for every $n > 0$.

Let n be the dimension of X , every choice of a holomorphic volume form gives two isomorphisms $\mathcal{O}_X \simeq \Omega_X^n$, $\Theta_X \simeq \Omega_X^{n-1}$, and the maps

$$H^0(\Omega_X^{n-1}) \xrightarrow{\alpha_0} H^1(\Omega_X^n), \quad H^1(\Omega_X^{n-1}) \xrightarrow{\alpha_1} H^2(\Omega_X^n)$$

are given by the cup product with $c_1(L)$. Since the ranks of the two maps

$$H^{n-1}(X, \mathbb{C}) \xrightarrow{c_1(L)} H^{n+1}(X, \mathbb{C}), \quad H^n(X, \mathbb{C}) \xrightarrow{c_1(L)} H^{n+2}(X, \mathbb{C})$$

are clearly invariant under deformations of the pair (X, L) , the conclusion follows immediately from the Hodge decomposition in cohomology.

REFERENCES

- [Bo79] F. Bogomolov: *Hamiltonian Kählerian manifolds*, Dokl. Akad. Nauk SSSR **243**, (1978), 1101-1104. Soviet Math. Dokl. **19**, (1979), 1462-1465. [1](#)
- [FM99] B. Fantechi, M. Manetti: *On the T^1 -lifting theorem*, J. Algebraic Geom. **8**, (1999), 31-39. [1](#)
- [FMM12] D. Fiorenza, M. Manetti and E. Martinengo: *Cosimplicial DGLAs in deformation theory*, Communications in Algebra **40** (2012), 2243-2260; [arXiv:0803.0399](#). [3](#)
- [GM90] W.M. Goldman and J.J. Millson: *The homotopy invariance of the Kuranishi space*, Illinois J. Math., **34**, (1990), 337-367. [1](#)
- [Ha77] R. Hartshorne: *Algebraic geometry*. Graduate texts in mathematics **52**, Springer-Verlag, New York/Berlin, (1977). [2](#)
- [Hu95] L. Huang: *On joint moduli spaces*, Math. Ann. **302**, (1995), 61-79. [1](#)
- [Ia15] D. Iacono: *Deformations and obstructions of pairs (X, D)* , Internat. Math. Res. Notices (IMRN), Volume 2015, Issue 19, 9660-9695; [arXiv:1302.1149](#) [3](#), [4](#)
- [Ia17] D. Iacono: *On the abstract Bogomolov-Tian-Todorov theorem*, Rend. Mat. Appl. **38**, (2017), 175-198; [www1.mat.uniroma1.it/ricerca/rendiconti/](#) [4](#)
- [IM10] D. Iacono and M. Manetti: *An algebraic proof of Bogomolov-Tian-Todorov theorem*, Deformation Spaces, **39**, (2010), 113-133; [arXiv:0902.0732](#) [[math.AG](#)]. [1](#), [3](#)
- [IM19] D. Iacono and M. Manetti: *On deformations of pairs (manifold, coherent sheaf)*, Canad. J. Math. **71**, (2019), 1209-1241; [arXiv:1707.06612](#). [1](#), [2](#), [3](#)
- [KKP08] L. Katzarkov, M. Kontsevich and T. Pantev: *Hodge theoretic aspects of mirror symmetry*, From Hodge theory to integrability and TQFT tt*-geometry, Proc. Sympos. Pure Math., **78**, Amer. Math. Soc., Providence, (2008), 87-174; [arXiv:0806.0107](#). [3](#), [4](#)
- [Ka92] Y. Kawamata: *Unobstructed deformations - a remark on a paper of Z. Ran*, J. Algebraic Geom. **1**, (1992), 183-190. [1](#)
- [Ma04] M. Manetti: *Lectures on deformations of complex manifolds*, Rend. Mat. Appl. (7) **24**, (2004), 1-183; [www1.mat.uniroma1.it/ricerca/rendiconti/](#) [4](#)
- [Mar12] E. Martinengo: *Infinitesimal deformations of Hitchin pairs and Hitchin map*, Internat. J. Math. **23** no. 7, (2012), 30pp.; [arXiv:1003.5531](#). [1](#)
- [LP19] L. Shizhang and P. Xuanyu: *Unobstructedness of deformations of Calabi-Yau varieties with a line bundle*; preprint [arXiv:1310.7162v5](#) (2019). [4](#)
- [Ra92] Z. Ran: *Deformations of manifolds with torsion or negative canonical bundle*. J. Algebraic Geom. **1**, (1992), 279-291. [1](#)
- [Ti88] G. Tian: *Smoothness of the universal deformation space of compact Calabi-Yau manifolds and its Peterson-Weil metric*, In: S.T. Yau ed. Math. aspects of String Theory, 629-646, Singapore (1988). [1](#)
- [To89] A.N. Todorov: *The Weil-Peterson geometry of the moduli space of $SU(n \geq 3)$ (Calabi-Yau) manifolds I*. Commun. Math. Phys. **126**, (1989), 325-346. [1](#)
- [Vo12] C. Voisin: *Théorie de Hodge et géométrie algébrique complexe*, Société Mathématique de France, Paris (2002). [4](#)

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