

A pde-based analysis of the spectrogram image for instantaneous frequency estimation

Vittoria Bruni ^{1,2,†}, Michela Tartaglione ^{1,†} and Domenico Vitulano ^{1,2,†,*}

¹ Department of Basic and Applied Sciences for Engineering, Sapienza University of Rome, via Antonio Scarpa 16, 00161 Rome, Italy; {vittoria.bruni;michela.tartaglione;domenico.vitulano}@uniroma1.it

² Institute for the Applications of Calculus, National Research Council, via dei Taurini 19, 00185 Rome, Italy

* Correspondence: domenico.vitulano@uniroma1.it

† These authors contributed equally to this work.

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Abstract: Instantaneous frequency (IF) is a fundamental feature in multicomponent signals analysis and its estimation is required in many practical applications. This goal can be successfully reached for well separated components, while it still is an open problem in case of interfering modes. Most of the methods addressing this issue are parametric, i.e. they apply to a specific IF class. Alternative approaches consist of non-parametric time filtering-based procedures, which do not show robustness to destructive interference — the most critical scenario in crossing modes. In this paper, a method for IF curves estimation is proposed. The case of amplitude and frequency modulated two-component signals is addressed by introducing a spectrogram time-frequency evolution law, whose coefficients depend on signal IFs time derivatives, i.e. the chirp rates. The problem is then turned into the resolution of a two-dimensional linear system which provides signal chirp rates; IF curves are then obtained by a simple integration. The method is non-parametric and it results quite robust to destructive interference. An estimate of the estimation error, as well as a numerical study concerning method sensitivity and robustness to noise are also provided in the paper.

Keywords: Partial differential equations; Multicomponent signals; Instantaneous Frequency estimation; Chirp rate estimation; Ridge curves recovery; Interfering AM-FM signals, Non-separable modes; Overlapping components

1. Introduction

Instantaneous frequency (IF) estimation is of great interest in many applications dealing with non-stationary signals, such as radar and Micro Doppler systems [1–3], seismic signals [4], some kinds of gravitational waves [5,6], audio [7] and human speech signals [8], animal sounds [9] and biomedical signals [10]. Practical applications deal with multicomponent signals (MCS), i.e. the superposition of individual waveforms, characterized by specific time-dependent frequency content, i.e. IF, as well as time-dependent amplitude, namely the instantaneous amplitude (IA) [11]. The correct analysis of MCS requires signal modes separation, that also allows for IFs estimation. Conversely, in many decomposition schemes, IFs estimation is a required step for the recovery of the individual modes by the observed mixture.

It is worth pointing out that IFs detection becomes a very challenging task if signal modes interfere with each other. That is why most of IF estimators in the literature are limited to non interfering components. The existing strategies addressing this issue can be grouped into two approaches: the former analyzes the target signal directly in the time domain, such as Empirical Mode Decomposition and its improvements [12–14], or in the frequency domain [15,16]. As an example, de-chirping technique aims at removing the “non-stationary term” of a signal so that a narrowband filter can be

used for extracting the target component [17–19]. In brief, the first approach consists of less or more advanced filtering procedures, performed in the time or the frequency domain. Conversely, the second approach analyzes the signal in the joint time-frequency (TF) plane, i.e. it is based on TF analysis.

Even though methods belonging to the first class do not suffer from eventual distortions introduced by the adopted device (TF transform), they commonly show some sensitivity to noise and result computationally demanding.

TF analysis provides efficient and adaptive tools for processing amplitude and frequency modulated (AM-FM) noisy signals [11,20,21]. In addition, the combination with energy-based transforms, such as Hough, Radon and Inverse Radon transform, has led to efficient IF detectors, designed for linear FM [22–24], polynomial FM [25], sinusoidal FM signals [26] and, recently, also for non linear FM MCS [27]. In case of monocomponent signals, as well as MCS with separable components, signal TF distributions (TFDs), for instance spectrogram, Smoothed Wigner Ville Distribution or Adaptive Directional TFD [28,29], reveal IFs as their maxima curve (ridge points). As a consequence, many approaches derive IFs from TFDs ridges [30–33].

For the same reason, in some methods TFD is regarded as an image, whose peaks are tracked and their local connectivity is exploited to estimate IF [34,35]. This is also the basic idea of Viterbi algorithm-based methods [36,37]. As main advantage, the formulation of these methods is simple but they are generally computationally expensive and sensitive to noise. Furthermore, peaks tracking-based procedures suffer from the so-called *switch problem*, i.e. the assignment of a detected ridge point to the wrong component. It limits their application to well separated modes (i.e. non interfering).

In this framework, some recently published works address this issue by the introduction of a generalized Viterbi algorithm. The latter provides an optimization step aimed at promoting the direction of the estimated IF curve [38,39], or its local monotonicity [40]. A similar peaks-detection and tracking procedure can be found in [41]. Following the same idea, Ridge Path Regrouping Method (RPRM) offers a less expensive alternative to Viterbi algorithm, although more sensitive to noise [42,43]. It is worth noticing that interference can highly affect the observed peaks' position, resulting in incorrect estimation. For this reason, more advanced procedures try to compensate for interference effects by means of some optimization techniques [32,42].

A powerful IF estimator suitable for wideband non linearly FM signals has been proposed in [44]. In this work, the decomposition problem (and thus IFs estimation) is turned into best signal demodulation (i.e. the removal of signal non-stationarity), by minimizing the bandwidth of the demodulated signal. This method does not require TF analysis (it belongs to the first class of approaches) and it is suitable for close or even crossing modes. In addition, it is non parametric, i.e. not limited to a specific signal class, as commonly done in the literature, where IFs separation is achieved by estimating the parameters of a predetermined IF model [3,9,18,26,45–54]. Since parametric methods usually involve multi-dimensional searches in the parameter space, they result computationally expensive, that is why a non-parametric approach is more advantageous. The method introduced in [44] is very efficient in dealing with crossing modes that interfere with each other in additive way, i.e. when the observed energy of the mixture increases. Unfortunately, it shows some sensitivity to the presence of destructive interference, i.e. when the observed energy decreases where the two modes overlap.

It is worth observing that the above-mentioned limitation is shared by many others approaches, such as reassignment method [55], synchrosqueezing [56,57] and synchroextracting transform [58,59]. This happens because the greater the loss of resolution in a TFD, the worst the accuracy. Methods introduced in [60,61] aim at overcoming this limitation. The same shortcoming is expected in RPRM, as it assumes that the observed peaks in signal TFD reflect IF shape as a whole. Unfortunately, this is not always the case. It is also worth pointing out that interference kind (additive or destructive) depends on signal modes characteristics. The latter can be more or less emphasized in a TFD, but it

remains signal dependent. As a matter of fact, the variational method proposed in [44], although TF analysis-free, suffers from the same limitation, as it will be shown later.

This paper aims at proposing an effective IF detector suitable for AM-FM two-component signals with close, overlapped and even crossing modes. In this last critical case, a certain robustness to destructive interference is achieved. First, a spectrogram evolution law, whose coefficients depend on signal amplitudes and phases derivatives, is introduced. The latter allows for the definition of a two-dimensional linear system, whose solution provides IFs derivatives, namely the chirp rates. Finally, IFs are achieved by integration, up to a constant.

Method sensitivity to the involved points needed to define the linear system, as well as robustness to moderate noise, are studied. A comparison to the methods introduced in [27,44] is also provided. The presented method is designed for AM-FM signals and it does not require additional assumptions concerning the signal class, resulting non parametric. It can be used as initialization of the method in [44], as well as a guide line for all peaks tracking-based procedures addressing overlapping components [42]. Finally, the proposed method may be extended to MCS with $N > 2$ modes if the interference regions are known in advance, as it will be discussed in the sequel.

The paper is organized as follows. Section 2 presents the mathematical background useful to introduce the problem and the proposed method; some experimental results and comparative studies are provided in Section 3 and, finally, Section 4 draws the conclusions.

2. Materials and Methods

2.1. Spectrogram of AM-FM signals

According to [11], AM-FM MCS are defined by eq.(1):

$$f(t) = \sum_{k=1}^N f_k(t) = \sum_{k=1}^N a_k(t) e^{i\phi_k(t)}, \quad (1)$$

where N is the number of components, $f_k \in L^2(\mathbb{R})$ denotes the k -th mode, characterized by its phase ϕ_k , its amplitude a_k (IA) and phase time-derivatives $\phi'_k(t)$, i.e. mode IF. Signal phases, IAs and IFs are assumed to be smooth time-varying functions.

The Short-Time Fourier Transform (STFT) of f , with respect to a real and symmetric analysis window $g \in L^2(\mathbb{R})$, is

$$S_f^g(u, \xi) = \int_{-\infty}^{+\infty} f(t) g(t-u) e^{-i\xi t} dt, \quad \forall (u, \xi) \in \mathbb{R} \times \mathbb{R}^+. \quad (2)$$

Spectrogram is defined as STFT squared modulus, i.e. $P(u, \xi) = |S_f^g(u, \xi)|^2$.

According to [62], if the STFT of a single mode $f(t) = a(t) \cos \phi(t)$ is computed by a normalized window g whose support is $[-\frac{1}{2}, \frac{1}{2}]$, then the STFT computed by the dilated window $g_s(t) = \frac{1}{\sqrt{s}} g(\frac{t}{s})$, with $s > 0$, can be expressed by eq.(3):

$$S_f^g(u, \xi) = \frac{\sqrt{s}}{2} a(u) e^{i(\phi(u) - \xi \cdot u)} [\hat{g}(s(\xi - \phi'(u)) + \epsilon(u, \xi))], \quad (3)$$

where $\hat{*}$ stands for the Fourier Transform of $*$ and $\epsilon(u, \xi)$ denotes a corrective term, which is negligible if $a(t)$ and $\phi'(t)$ have small relative variations over the support of the window g . Under this assumption, a simple spectrogram expression can be derived from eq.(3), i.e.

$$P(u, \xi) = \frac{s}{4} a^2(u) \hat{g}^2(s(\xi - \phi'(u))). \quad (4)$$

Eq. (4) shows that spectrogram reaches its maximum along the curve $(u, \phi'(u))$, $u \in \text{supp}\{f\}$, defined as the *ridge curve* or *IF curve*.

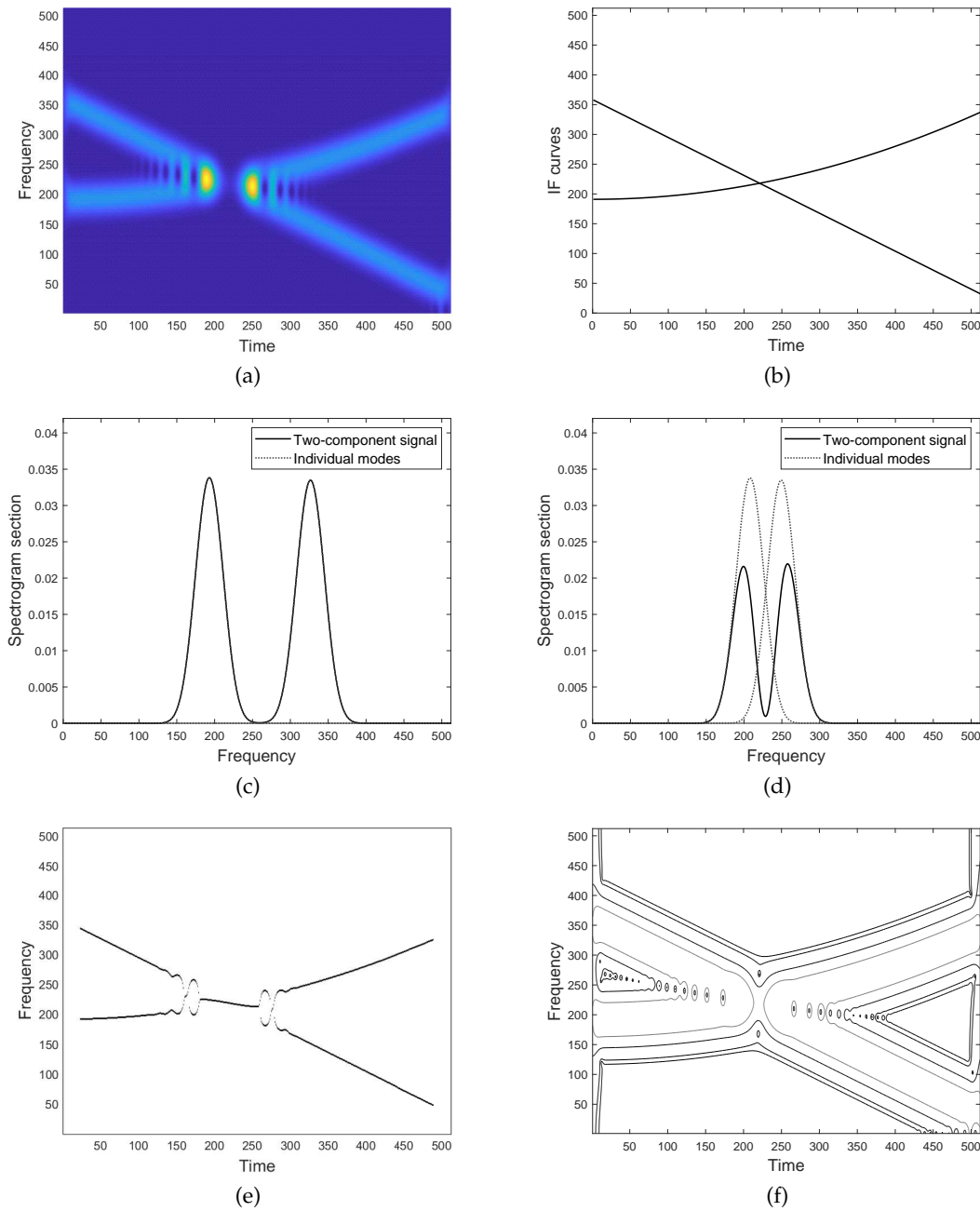


Figure 1. (a) Spectrogram of a two-component FM signal with constant amplitudes. (b) IFs curves. (c) Spectrogram section at a time location u belonging to the separability region: spectrogram profile (solid line) perfectly matches the profiles of spectrograms concerning the individual modes (dotted lines). (d) Spectrogram section at a time location u in the non-separability region: spectrogram profile (solid line) does not match the profiles of spectrograms concerning the individual modes (dotted lines) on the whole support. Maxima points locations are deviated. (e) Maxima points extracted from (a). (f) Spectrogram isolevel curves.

Eq.(3) also provides a model for the spectrogram of MCS, as shown in the sequel. Without loss of generality, let us consider the spectrogram of a two-component signal $f(t) = a_1(t)f_1(t) + a_2(t)f_2(t)$, as the one depicted in Fig.1(a). By STFT linearity and taking into account eqs.(3) and (4), spectrogram can be written as

$$P(u, \xi) = |S_{f_1}^g(u, \xi) + S_{f_2}^g(u, \xi)|^2 = P_1(u, \xi) + P_2(u, \xi) + 2\sqrt{P_1 P_2} \cos(\phi_2(u) - \phi_1(u)), \quad (5)$$

with $P_k = P_k(u, \xi) = |S_{f_k}^g(u, \xi)|^2 = \frac{s_{a_k}^2(u)}{4} \hat{g}^2(s(\xi - \phi'_k(u)))$, $k = 1, 2$.

For a fixed u , modes are well separated if the following separability condition holds true, i.e.

$$|\phi'_1(u) - \phi'_2(u)| \geq \Delta\omega, \quad (6)$$

where $\Delta\omega$ is the window frequency bandwidth [62]. In this case, for each ξ belonging to P_1 support, i.e. $\xi : |\xi - \phi'_1(u)| < \frac{\Delta\omega}{2}$, it follows $|\xi - \phi'_2(u)| > \frac{\Delta\omega}{2}$. Bandwidth definition thus yields to $\hat{g}(\xi - \phi'_2(u)) \approx 0$ and then eq.(5) reduces to

$$P(u, \xi) \approx P_1(u, \xi), \quad \forall \xi : |\xi - \phi'_1(u)| < \frac{\Delta\omega}{2}, \quad (7)$$

i.e. mode f_1 does not affect f_2 in the TF domain, and viceversa, as in the case shown in Fig.1(c).

Unfortunately, if the condition in eq.(6) is not met, the two modes have close or even overlapping supports both in time and frequency, resulting in non-negligible cross-terms. In particular, for a fixed u belonging to the non-separability region, spectrogram may reach its maximum value at a frequency point different from both the ridge points of the single modes, as in the case shown in Fig.1(d). For this reason, IF detectors/enhancers relying on spectrogram maxima are unreliable, especially in case of crossing IFs such that their TFDs show a lack of resolution at the non-separability region, as for the spectrogram shown in Fig.1. As it can be observed, the detected maxima in (e) do not well approximate IF curves in (b) at the region where they get closer and intersect.

If spectrogram energy increases at the non separability region, with respect to the overall energy of the two isolated contributions, i.e. P_1 and P_2 , interference is said to be additive. It is referred as destructive in the opposite case. It is worth noticing that, although spectrogram is accounted for, this taxonomy depends on the cosine sign in eq.(5), i.e. on signal characteristics.

Regardless of the interference effect, there are some curves other than the ridge one that convey the same information and that are less affected by cross-terms [45,60]. As proven in [61,63], the spectrogram $P(u, \xi)$ of a monocomponent signal $f(t) = a(t) \cos \phi(t)$ satisfies the following advection equation

$$\frac{\partial P(u, \xi)}{\partial u} + \phi''(u) \frac{\partial P(u, \xi)}{\partial \xi} - \frac{2a'(u)}{a(u)} P(u, \xi) = 0, \quad \forall u \in \text{supp}\{f\}, \quad (8)$$

whose characteristic curves $\mathcal{C}_{c,\phi}$ are

$$\xi(u) = \phi'(u) + c, \quad (9)$$

with $c = \xi_0 - \phi'(u_0)$ and (u_0, ξ_0) is a point in the TF plane.

It is worth observing that eq.(8) is a linear advection equation, whose characteristic curves, as in eq.(9), are shifted copies of the ridge curve, which is also a characteristic curve, i.e. $\mathcal{C}_{0,\phi}$. It means that IF information is not limited to the ridge curve, but it can be found in any characteristic curve. In addition, the introduced curves allow to formally characterize those points less influenced by interference, by means of the *weakened separability condition* (WSC), which is recalled below.

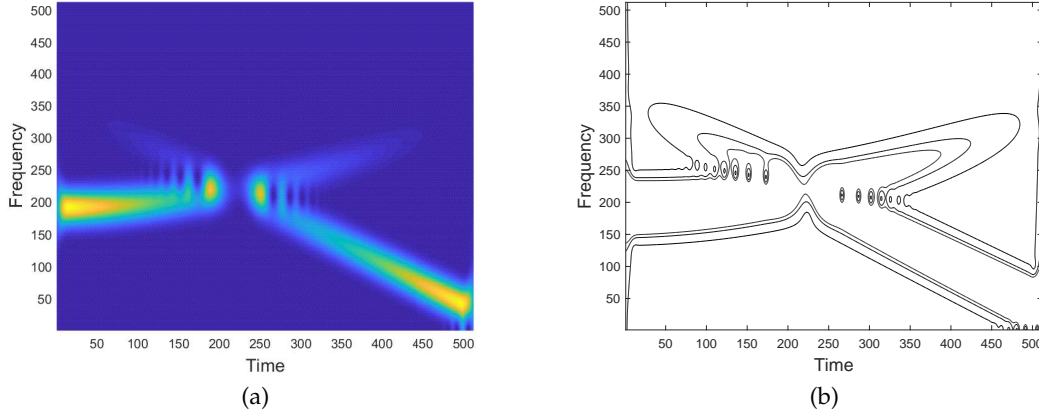


Figure 2. (a) Spectrogram of a two-component FM signal with time-varying amplitudes. (b) Isolevel curves.

Definition 1. [60] Two modes with IFs $\phi'_1(u)$ and $\phi'_2(u)$ are separated at time location u if there exists at least one curve in $\mathcal{C}_{c_1, \phi_1}$, i.e., $\xi_1(u) = \phi'_1(u) + c_1$, such that $|\xi_1(u) - \phi'_2(u)| \geq \Delta\omega$; or viceversa.

Remark 1. Two modes not separated at time location u in the classical sense, i.e. $|\phi'_1(u) - \phi'_2(u)| = \varepsilon \leq \Delta\omega$, can be weakly separated in the sense specified by Definition 1, see spectrogram external sides in Fig.1(d). Indeed, by assuming, for instance, $\xi < \phi'_1(u) < \phi'_2(u)$, then $\exists c_1 \in \mathbb{R}$, $c_1 < 0$ such that $\xi = \phi'_1(u) + c_1$, i.e. $\xi \in \xi_1(u) \in \mathcal{C}_{c_1, \phi_1}$. As a consequence, $|\xi - \phi'_2(u)| = |\phi'_1(u) + c_1 - \phi'_2(u)| = |-\varepsilon + c_1| \geq \Delta\omega \Leftrightarrow -c_1 \geq \Delta\omega - \varepsilon$. By looking again at eq.(5) and by the same argument which led to eq.(7), it follows that

$$P(u, \xi) \approx P_1(u, \xi), \quad \forall \xi \in \mathcal{C}_{c_1, \phi_1}. \quad (10)$$

In case of a constant amplitude signal, i.e. $a'(u) = 0$, $\forall u \in \text{supp}\{f\}$, from eq.(8) it follows that spectrogram is a constant function along its characteristic curves. As a result, spectrogram isolevel curves lie on the characteristic curves and, operatively, any distribution thresholding provides IF, up to a constant. Fig.1(f) depicts some characteristic curves obtained by thresholding the spectrogram shown in Fig.1(a). As it can be observed, higher levels are more influenced by cross-terms, while lower levels reveal IF shapes. Unfortunately, this argument can not be applied to time-varying signals, as the term $\frac{2a'(u)}{a(u)}P(u, \xi)$ in eq.(8) is non zero, as in the example provided in Fig.2.

2.2. The proposed method

This paper aims at estimating the IFs of AM-FM two-component signals. For constant amplitude signals, the latter may be estimated, up to a constant, from the characteristic curves of eq.(8), that can be easily derived from spectrogram isolevel curves. Unfortunately, in case of time-varying amplitudes, the characteristic curves of eq.(8) can not be obtained by a simple spectrogram thresholding. However, eq.(8) still is informative for our purposes. Indeed, it can be regarded as a linear equation into the unknown $x_1 = \phi''(u)$ and $x_2 = \frac{2a'(u)}{a(u)}$, $u \in \text{supp}\{f\}$. It is worth noticing that the latter do not depend on variable ξ . By evaluating eq.(8) at $\xi : \frac{\partial P(u, \xi)}{\partial \xi} = 0$, (i.e. the ridge point, as a single mode is now considered), one can easily obtain the ratio $\frac{2a'(u)}{a(u)}$. More in general, the following linear system can be defined:

$$\begin{pmatrix} P_\xi(u, \xi_1) & -P(u, \xi_1) \\ P_\xi(u, \xi_2) & -P(u, \xi_2) \end{pmatrix} \begin{pmatrix} \phi''(u) \\ \frac{2a'(u)}{a(u)} \end{pmatrix} = \begin{pmatrix} -P_u(u, \xi_1) \\ -P_u(u, \xi_2) \end{pmatrix}, \quad (11)$$

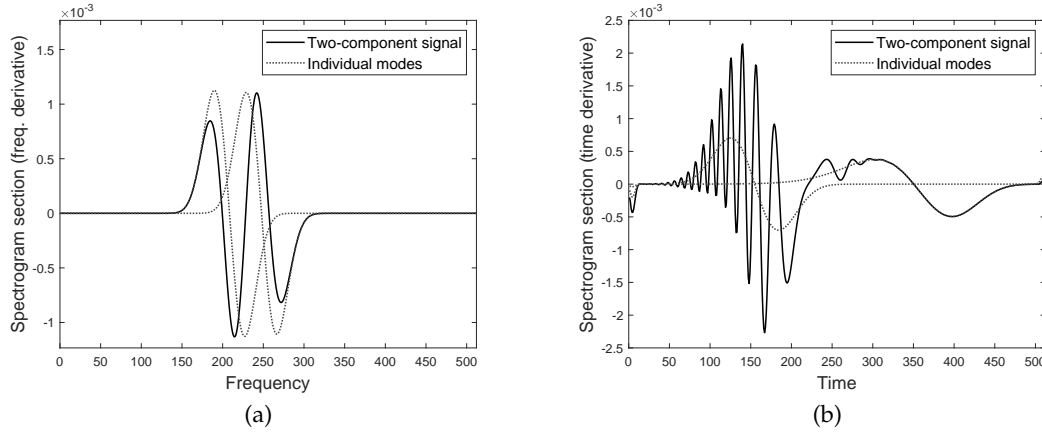


Figure 3. (a) Frequency derivative (estimated by central finite differences) at fixed u of spectrogram section in Fig.1(d) (solid line) compared to the derivatives of the single profiles (dotted lines). Matching is observed for those points lying on the external sides, i.e. where the WSC is satisfied. (b) Time derivative (estimated by central finite differences) of the spectrogram in Fig.1(a) (solid line) at fixed ξ , compared to the derivative of the single profiles (dotted lines). Profiles significantly differ on the whole support.

where $P_{\xi}(u, \xi_j) := \frac{\partial P(u, \xi)}{\partial \xi} \Big|_{\xi=\xi_j}$ and $P_u(u, \xi_j) := \frac{\partial P(u, \xi)}{\partial u} \Big|_{\xi=\xi_j}$ denote spectrogram partial derivatives evaluated at points ξ_j , $j = 1, 2$.

For each fixed $u \in \text{supp}\{f\}$, if ξ_1 and ξ_2 are selected so that the corresponding vectors $(P_{\xi}(u, \xi_j), -P_u(u, \xi_j))$, $j = 1, 2$, are linearly independent, then the linear system in eq.(11) has a unique solution, whose first entry provides $\phi''(u)$, i.e. signal chirp rate (CR). The corresponding IF can be then estimated by integration, up to a constant.

Once the case of a single mode has been presented, let us address the two-component one in the following Proposition.

Proposition 1. Let $f(t) = a_1(t) \cos \phi_1(t) + a_2(t) \cos \phi_2(t)$ be a two-components signal and let us set $\hat{g}_k = \hat{g}(s(\xi - \phi_k'(u)))$, where g is the analysis window with length $s > 0$, $a_k = a_k(u)$ and $\phi_k = \phi_k(u)$, $k = 1, 2$, $\Delta\phi = \phi_1 - \phi_2$, and $(*)'$ denotes the time derivative of $(*)$. Then, the spectrogram $P(u, \xi)$ satisfies the following evolution law

$$\begin{aligned} \frac{\partial P(u, \xi)}{\partial u} + \phi_1'' \frac{\partial P(u, \xi)}{\partial \xi} - \frac{s}{2} [a_1 a_1' \hat{g}_1^2 + a_2 a_2' \hat{g}_2^2 + \hat{g}_1 \hat{g}_2 (a_1' a_2 + a_2' a_1) \cos \Delta\phi] \\ + \frac{s}{2} a_1 a_2 \hat{g}_1 \hat{g}_2 \Delta\phi' \sin \Delta\phi + \frac{s^2}{2} \Delta\phi'' \hat{g}_2' [a_2 \hat{g}_2^2 - a_1 a_2 \hat{g}_1 \cos \Delta\phi] = 0. \end{aligned} \quad (12)$$

Proof. The proof can be found in the Appendix. \square

For each fixed u , eq.(12) is a non-linear equation in the unknown variables $a_k^{(d_1)}(u), \phi_k^{(d_2)}(u), d_1 = 0, 1, d_2 = 0, 1, 2$ and $k = 1, 2$, i.e. the signal amplitudes, phases and their derivatives. If enough frequency points are considered, the latter can be estimated by solving a non-linear system. However, this is not the strategy of the presented paper. The concept of weakened separability is exploited, instead. Precisely, if evaluated along the characteristic curves satisfying the WSC, eq.(12) reduces to eq.(11) (see Remark 1 and [60,61]), which is thus accounted for, also in the two-component case.

For the single case, by selecting two different ξ values, namely ξ_1, ξ_2 at a fixed u , according to eq.(11), the following linear system allows for the estimation of both $\phi_1''(u)$ and $\frac{2a_1'(u)}{a_1(u)}$, i.e.

$$\begin{cases} \phi_1''(u)P_{1\bar{\xi}}(u, \xi_1) - 2\frac{a_1'(u)}{a_1(u)}P_1(u, \xi_1) = -P_{1u}(u, \xi_1) \\ \phi_1''(u)P_{1\bar{\xi}}(u, \xi_2) - 2\frac{a_1'(u)}{a_1(u)}P_1(u, \xi_2) = -P_{1u}(u, \xi_2) \end{cases} \quad (13)$$

where $P_{1\bar{\xi}} := \frac{\partial P_1(u, \xi_j)}{\partial \bar{\xi}}$ and $P_{1u} := \frac{\partial P_1(u, \xi_j)}{\partial u}$.

A similar system can be written for $\phi_2''(u)$ and $\frac{2a_2'(u)}{a_2(u)}$. In particular, it holds

$$\phi_1''(u) = -\frac{P_{1u}(u, \xi_1)P_1(u, \xi_2) - P_{1u}(u, \xi_2)P_1(u, \xi_1)}{P_{1\bar{\xi}}(u, \xi_1)P_1(u, \xi_2) - P_{1\bar{\xi}}(u, \xi_2)P_1(u, \xi_1)}. \quad (14)$$

By using the same equation even for two-component signals, the following estimation for $\phi_1''(u)$ would be provided

$$\tilde{\phi}_1''(u) = -\frac{P_u(u, \xi_1)P(u, \xi_2) - P_u(u, \xi_2)P(u, \xi_1)}{P_{\bar{\xi}}(u, \xi_1)P(u, \xi_2) - P_{\bar{\xi}}(u, \xi_2)P(u, \xi_1)}, \quad (15)$$

where P has been considered in place of P_1 . It is worth observing that the latter assumption is valid for points (u, ξ_j) belonging to the weak separability region, as in Definition 1. Unfortunately, it does not hold true if, at a fixed time location u , that region results empty or whenever finite difference approximations are used for the partial derivatives of the spectrogram. As a result, the estimation error has to be computed in dependence on the frequency points ξ_1 and ξ_2 , as done in the next section.

2.2.1. Estimation error

In this section we are interested in the estimation error for $\phi_1''(u)$ in the region Ω where even both the separability condition and the WSC do not hold. As it will be pointed out later, the same estimation still hold in the region of weakened separability whenever numerical approximation is used for spectrogram derivatives. On the contrary, as mentioned in the previous section, outside Ω the problem resembles the case of a monocomponent signal and then the error is negligible or zero.

The estimation errors for $\phi_1''(u)$ and IF computation are provided in the following Proposition under proper conditions concerning the selection of frequency points ξ_1 and ξ_2 .

Proposition 2. Let $\tilde{\phi}_1''(u)$ be the estimation for $\phi_1''(u)$ using the method in eq.(15), with $|\xi_1 - \xi_2| \leq \rho$, with $\rho \in \mathbb{R}^+$ and small enough. Let $\tilde{\phi}_1'(u)$ be the estimation for $\phi_1'(u)$ that is obtained by integrating $\tilde{\phi}_1''(u)$ and let set $\varepsilon(u, \xi) = \frac{P(u, \xi) - P_1(u, \xi)}{P_1(u, \xi)}$. Hence, by neglecting the second order terms in the central finite differences approximation for $P_{\bar{\xi}}(u, \xi)$, $P_u(u, \xi)$, $P_{1\bar{\xi}}(u, \xi)$, and $P_{1u}(u, \xi)$, for each $u \in \Omega$ it holds

$$\phi_1''(u) - \tilde{\phi}_1''(u) \approx \frac{\varepsilon_u(u, \xi_1)}{1 + \varepsilon(u, \xi_1)} \frac{P_1(u, \xi_1)}{P_{1\bar{\xi}}(u, \xi_1)} \quad (16)$$

with $\xi_1 : \varepsilon_{\bar{\xi}}(u, \xi_1) = 0$, while

$$\phi_1'(u) - \tilde{\phi}_1'(u) \approx \int_{u_0}^u \frac{\varepsilon_u(\tau, \xi_1(\tau))}{1 + \varepsilon(\tau, \xi_1(\tau))} \frac{P_1(\tau, \xi_1(\tau))}{P_{1\bar{\xi}}(\tau, \xi_1(\tau))} d\tau, \quad (17)$$

with $u_0 : \varepsilon(u_0, \xi_1) = 0$, and where $P_{1\bar{\xi}} := \frac{\partial P_1(u, \xi)}{\partial \bar{\xi}}$.

Proof. The proof is in the Appendix. \square

It is worth observing that the error on the chirp rate, i.e. $\phi_1''(u)$, strongly and directly depends on the error on $P_1(u, \xi)$ estimation through $P(u, \xi)$, i.e. $\varepsilon(u, \xi)$ due to the presence of cross terms. In particular, the latter can present an oscillating behaviour, due to the presence of the term $\cos(\Delta\phi(u))$ in $\varepsilon(u, \xi)$ definition.

Remark 2. Spectrogram partial derivatives can be approximated from the observed data by using central finite differences, i.e.

$$P_u(u, \xi) = \frac{P(u + h_u, \xi) - P(u - h_u, \xi)}{2h_u}, \quad (18)$$

$$P_\xi(u, \xi) = \frac{P(u, \xi + h_\xi) - P(u, \xi - h_\xi)}{2h_\xi}, \quad \forall u \in \text{supp}\{f\}, \forall \xi > 0, \quad (19)$$

where h_u and h_ξ denote the discretization steps, respectively in time and frequency.

For each fixed u , two frequency points ξ_j , $j = 1, 2$ are needed to define the linear system as in eq.(11). In the case of a single mode, as well as separated components, any pair of frequency points can be employed. On the contrary, a more careful selection is required for close and overlapping components. In order to prevent instabilities due to the presence of cross-terms, it is convenient to uniformly select ξ_j , $j = 1, 2$ (i.e. they don't depend on the specific u) such that they satisfy the WSC, as in Definition 1. Taking into account eq.(10), $P(u, \xi)$ as a function of variable ξ is close to the spectrogram section of the single component (cross-terms are negligible). As a result, the approximation of $P_{k\xi}$, $k = 1, 2$ by P_ξ is more accurate, as in the case shown in Fig.3(a). Conversely, it generally holds

$$P(u \pm \delta, \xi) \not\approx P_k(u \pm \delta, \xi), \quad k = 1, 2, \forall \delta > 0, \quad (20)$$

i.e. $P(u, \xi)$, as a function of variable u , does not satisfy any separability condition, as shown in Fig.3(b). Then, the approximation of P_{ku} , $k = 1, 2$, through finite differences is less robust (cross-terms are not negligible). It is worth pointing out that the lack of separability in eq.(20) may affect any derivative estimates, even in the WSC region and Proposition 2 still holds.

2.2.2. Chirp Rate regularization

As discussed in the previous section and shown in Proposition 2, $\phi_k''(u)$, $k = 1, 2$ (signal CRs) estimation at the non-separability region is more delicate as the cross terms cannot be completely neglected. However, the estimation error can be reduced by considering the local mean of the estimated CRs $\tilde{\phi}_k''(u)$, $k = 1, 2$ by taking advantage of its oscillating behaviour. More precisely, CR can be defined as

$$CR_{k,mean}(u) = \frac{1}{2s} \int_{u-s}^{u+s} \tilde{\phi}_k''(\tau) d\tau, \quad k = 1, 2, \quad (21)$$

where $s > 0$ denotes the window size. On the contrary, when modes are well separated, CR estimate is more stable. Therefore, by denoting the crossing point as u_{cross} , the following function is considered as global CR approximation, i.e.

$$\overline{CR}_k(u) = \begin{cases} CR_{k,mean}(u), & u \in I = [u_{cross} - s, u_{cross} + s] \\ \tilde{\phi}_k''(u), & \text{otherwise,} \end{cases} \quad (22)$$

that is regularized by convolution, as follows

$$CR_k(u) = \overline{CR}_k * \rho(u) = \int_{-\infty}^{+\infty} \rho(\tau) \overline{CR}_k(u - \tau) d\tau, \quad k = 1, 2, \quad (23)$$

with $\rho(u) = \frac{1}{\sqrt{2\pi\epsilon^2}} \exp(-\frac{u^2}{2\epsilon^2})$, $\epsilon > 0$.

Finally, IFs are estimated by integrating CR_k , $k = 1, 2$, up to a constant. The first order rectangle quadrature scheme is adopted in this paper. The selection of the crossing point is discussed in the next section.

2.2.3. Crossing point detection

The so-called *switch problem*, i.e. the assignment of an estimated IF point to the wrong IF curve can be prevented by the knowledge of crossing point position. The latter allows us to correctly switch modes role when estimating the chirp rates. The crossing point can be estimated by considering to characteristic curves at low level and by computing the distance between those curves for increasing time. More precisely, let us assume $\phi'_1(u_0) > \phi'_2(u_0)$, where u_0 is an initial point and let us consider the set $\{(u, \xi) : P(u, \xi) = K, \min P(u, \xi) < K < \max P(u, \xi)\}$, that provides

$$\xi_1(u) = \phi'_1(u) + c_1(u), \quad c_1(u) > 0, \quad (24)$$

$$\xi_2(u) = \phi'_2(u) - c_2(u), \quad c_2(u) > 0, \quad \forall u \geq u_0. \quad (25)$$

If signal amplitudes are slow-varying and K is sufficiently low, the term $\frac{2a'(u)}{a(u)}P(u, \xi)$ in eq.(8) can be neglected and c_1, c_2 in eqs.(24)–(25) do not depend on the time variable. As a result, eqs.(24)–(25) indicate two characteristic curves on spectrogram lateral sides, thus less affected by eventual cross terms. Their distance is

$$d(u) = \phi'_1(u) - \phi'_2(u) + c_1 + c_2, \quad (26)$$

which reaches its minimum when $\phi'_1(u) = \phi'_2(u)$, i.e. at the crossing point.

3. Results and Discussion

The proposed method has been tested on several two-component signals with different frequency and amplitude modulations. This section presents the more significant results.

Synthetic signals of length $n = 512$ are considered and a Chebyshev analysis window of length $s = 44$ and side-lobe magnitude factor $r \in [80, 100]$ dB is used for STFT computation, unless otherwise specified. In general, an analysis window with smooth decay to zero is recommended in order to prevent instabilities in the estimations.

As presented in the previous section, the proposed procedure allows for IF estimation, up to a constant. The latter can be derived directly from spectrogram, by selecting a peak located at the separability region. To this aim, the thresholded spectrogram is processed columnwise and the number of detected peaks, as a function of time, is computed. The initial point u_0 is chosen as the first instant providing two peaks and sufficiently apart from TF boundary, in order to prevent instabilities.

The first test is aimed at evaluating method effectiveness in case of well separated components. A two-component signal with polynomial phases and gaussian amplitudes is considered in Fig.4. The corresponding spectrogram is shown in (a) and the initial time point for integration is depicted in (b). (c) shows a generic spectrogram section and the points, on the external sides, selected to define the linear system in eq.(11). The estimated CRs are plotted in (d) and the corresponding estimated IFs can be found in (e). Finally, (f) shows the IFs curves achieved by the classical peak detector algorithm. As it can be observed, the two methods provide similar results.

The second test, illustrated in Fig.5, concerns a FM two-component signal with intersecting modes, that interfere with each other additively. The spectrogram is shown in (a). As it can be observed, some boundary effects occur, therefore the very initial times have not been employed in the procedure, in order to prevent instabilities in the final estimate. The crossing point, detected by minimizing the distance function as in eq.(26), is depicted in (b). A generic spectrogram section is shown in (c), with emphasized points used for defining the linear system. The estimated CRs are depicted in (d) and the final result is shown in (e). As it can be noticed by comparison with (f), the proposed method outperforms the classical peak detector algorithm at the non-separability region. Fig.6 aims at evaluating the sensibility of the proposed method to the frequency points ξ_1, ξ_2 selection. With reference to the spectrogram shown in Fig.5(a), (a) emphasizes two points lying on a lower

characteristic curve. This choice results in IFs estimation shown in (b). As it can be observed, the result is comparable to the one in Fig.5(e), except for some instabilities at the left-hand boundary side. Indeed, the lower the characteristic the more the boundary effects. Fig.6(c) shows two frequency points on the lateral spectrogram sides which are closer to spectrogram maxima. The corresponding IFs estimation is provided in (d). As it can be observed, the result is less accurate.

Fig.7 refers to a FM two-component signal with intersecting modes and destructive interference. As shown in (a), spectrogram resolution is significantly reduced at the non-separability region. The distance function, as defined in eq.(26), and the crossing point estimated as its minimum are depicted in (b). A generic spectrogram section with emphasized points for defining the linear system is shown in (c). The estimated CRs are depicted in (d) and the final IFs estimation, provided by the proposed approach, is shown in (e). The latter can be compared to the result given by the classical peaks detector method in (d). As it can be observed, the proposed method better improves IF curves resolution at the non separability region.

Fig.8 and Fig.9 concern different combinations of the following modes:

$$\begin{aligned} f_1(t) &= \cos(\pi/3 n(t + 0.1)^2 + (\pi + 30)t), \\ f_2(t) &= \sin(0.26 \pi n(0.9 - t)^4 - 150t), \\ f_3(t) &= \sin(0.25 \pi n(0.9 - t)^4 - 150t), \end{aligned}$$

modulated by the amplitudes

$$\begin{aligned} a_{11}(t) &= 0.5(t + 1), & a_{21}(t) &= \sqrt{1 - t}, \\ a_{12}(t) &= 2\sqrt{t}, & a_{22}(t) &= t + 0.5, \quad t \in [0, 1]. \end{aligned}$$

Fig.8 refers to the signal

$$f_{12}(t) = a_{11}(t)f_1(t) + a_{21}(t)f_2(t), \quad (27)$$

whose modes interfere with each other additively. The corresponding result provided by the proposed method is shown in (b). As it can be noticed, IFs estimation is quite accurate, although some instabilities occur at the right-hand boundary side. This is due to the presence of spectrogram values close to zero, which affects derivatives approximation and thus the final result. Anyway, the achieved result outperforms the one given by the peak detector (c), as well as the one provided by the Radon Spectrogram-based method introduced in [27] — it is worth observing that method in [27] is designed for constant amplitude FM signals.

In Fig.9, destructive interference is addressed by considering the signal

$$f_{13}(t) = a_{12}(t)f_1(t) + a_{22}(t)f_3(t). \quad (28)$$

In this case, the achieved IFs estimation is less accurate, but still better than the ones given by the state-of-the-art methods shown in (c) and (d), especially if one focuses on the non-separability region, i.e. close to the crossing point.

Fig.10 considers different combination of the modes

$$\begin{aligned} h_1(t) &= \cos(0.5 n t^4 + 500t), \\ h_2(t) &= \cos(n(1 - t)^2 + 120(1 - t)), \end{aligned}$$

modulated by the amplitudes

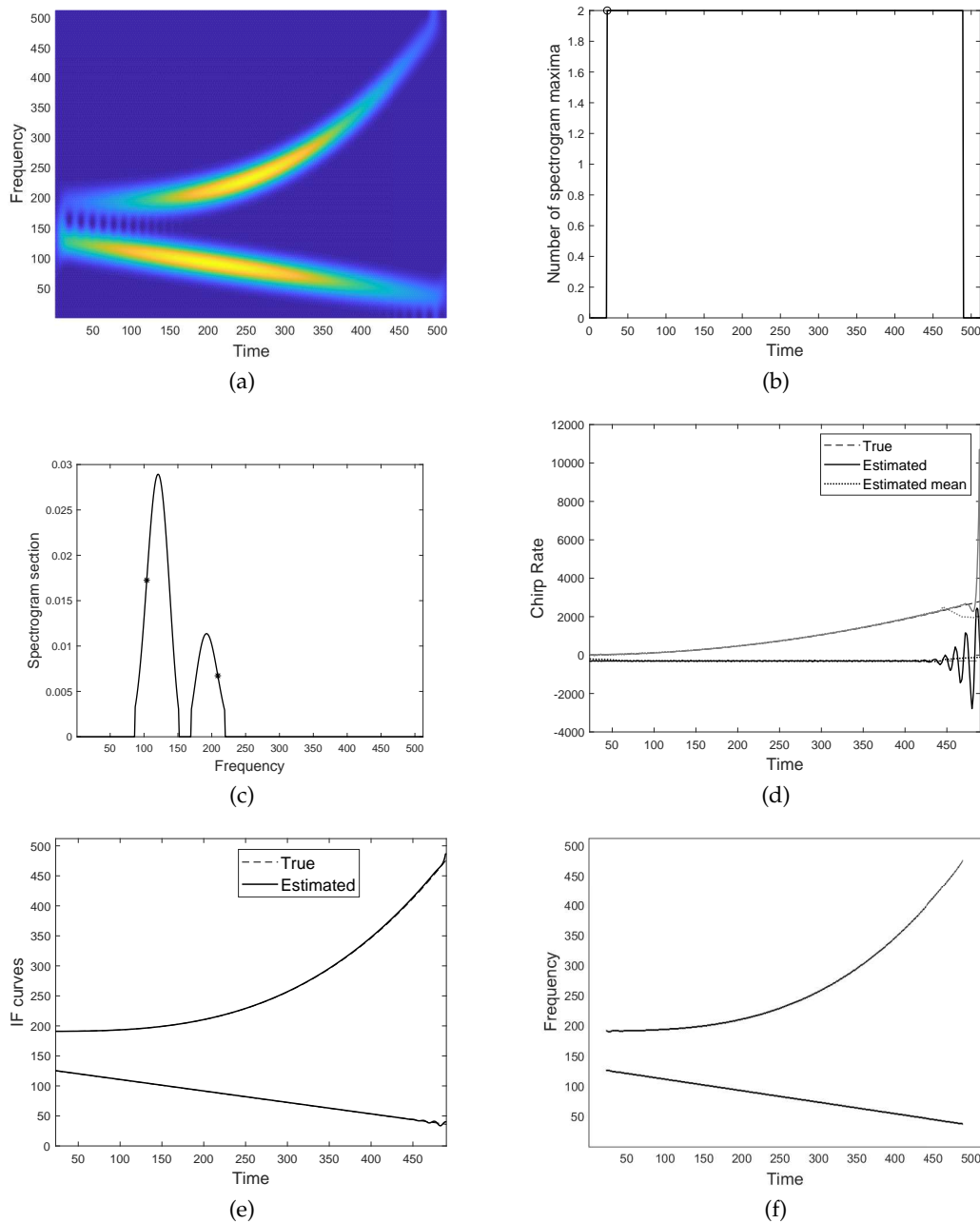


Figure 4. (a) Spectrogram of a two-component AM-FM signal whose modes are well separated, with gaussian amplitudes. (b) Number of detected maxima from (a) and initial point for integration (round marker). (c) Spectrogram section. The star markers indicate spectrogram values at the frequency points ζ_1, ζ_2 employed for defining the linear system, as in eq.(11). (d) Estimated chirp rates (solid line) and their local mean (dotted line). (e) Estimated IFs. (f) Ridge curves estimated from spectrogram maxima.

$$\begin{aligned} b_{11}(t) &= 0.1, & b_{21} &= 0.25t, \\ b_{12}(t) &= \sqrt{t}, & b_{22} &= \exp(-2(t - 0.4)^2), \quad t \in [0, 1]. \end{aligned}$$

With reference to the same Figure, the left-hand side refers to

$$h_{11}(t) = b_{11}(t)h_1(t) + b_{21}(t)h_2(t), \quad (29)$$

while the right-hand side refers to

$$h_{12}(t) = b_{12}(t)h_1(t) + b_{22}(t)h_2(t). \quad (30)$$

As it can be noticed, the proposed method is able to recover IF curves at the non-separability region. Some instabilities occur at TF boundaries, due to the presence of spectrogram value close to zero.

Some results concerning noisy signals are shown in Fig.11. The latter can be compared respectively to Fig.8(b), Fig.9(c) and Fig.10(d). As it can be observed, the proposed method is robust to moderate noise.

The following FM signals

$$g_1(t) = \cos(0.5n t^3 + 1000t) + \cos(0.8n(1-t)^2 + 500(1-t)) \quad (31)$$

$$g_2(t) = \cos(0.5n t^4 + 600t) + \cos(1.5n(1-t)^2 + 100(1-t)), \quad (32)$$

with $t \in [0, 1]$, sampled at frequency $n = 2000$ are accounted for in Fig.12. A Chebyshev analysis window of length $s = 116$ and side-lobe magnitude factor $r = 400$ dB is used for STFT computation. Our results are compared to the ones obtained by applying the method introduced in [44,64]. As it can be observed, the proposed approach is clearly advantageous in dealing with destructive interference.

Finally, Fig.13 refers to the complex signal

$$g(t) = \exp[i(20\pi \cos(4\pi t) - 1500t)] + \exp\left[i\left(0.95\pi n t^2 + 170t\right)\right], \quad (33)$$

where i denotes the imaginary unit. The signal spectrogram computed with a window of length $s = 44$ is shown in Fig.13(a), while $s = 22$ has been used for Fig.13(c). As it can be noticed, a narrow window in time is preferable for our purposes, as it provides a more spread spectrogram and then IFs estimation is more accurate.

3.1. Some remarks

1. The proposed method requires two-component signals and prior knowledge of the presence of a non-separability region. A method for detecting the non-separability region, as the one proposed in [65], can be used as a preprocessing step, in case of constant amplitude signals. Furthermore, weakly amplitude modulation is assumed, as in most approaches.
2. It is worth underlying that the proposed method could be easily generalized to MCS with $N > 2$ modes if the TF non-separability regions are known and they are sufficiently separated. Indeed, under these assumptions, the analysis of a more complex signal reduces to the analysis of a two-component signal locally. This point will be investigated in-depth in future studies.
3. The proposed model allows for IFs estimation, up to an integration constant, whose accuracy can affect the final result, as previously shown. However, it is worth pointing out that this error often causes the recovery of a characteristic different from the ridge, but still a characteristic. As a result, it contains significant information concerning IF. The integration constant can be directly

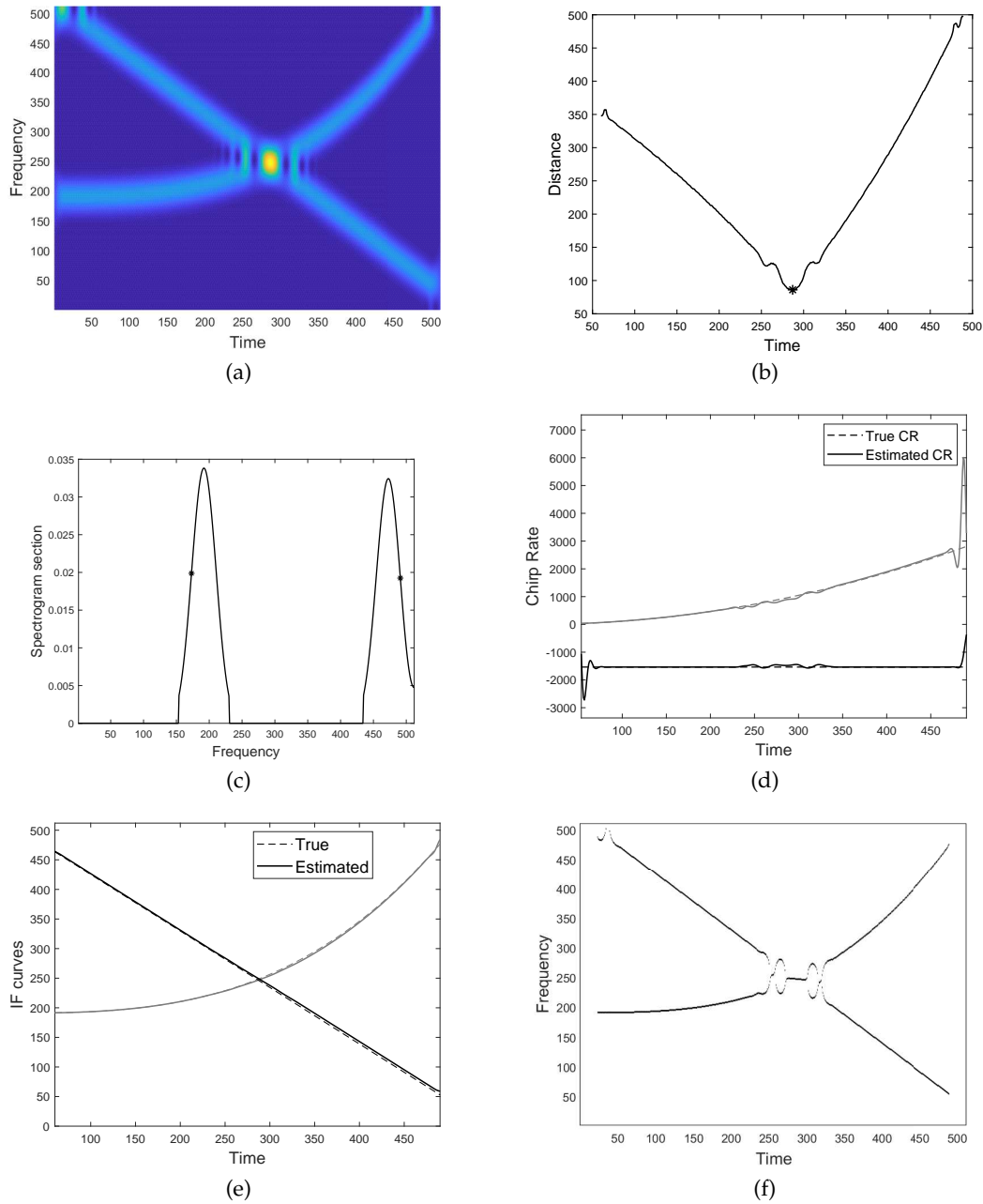


Figure 5. (a) Spectrogram of a two-component signal with crossing modes (additive interference). (b) Distance function as defined in eq.(26) and detected crossing point (star). (c) Spectrogram section. The star markers indicate spectrogram values at the frequency points ξ_1, ξ_2 employed for defining the linear system, as in eq.(11). (d) Estimated chirp rates compared to the true ones. (e) Estimated IFs. (f) Ridge curves estimated from spectrogram maxima.

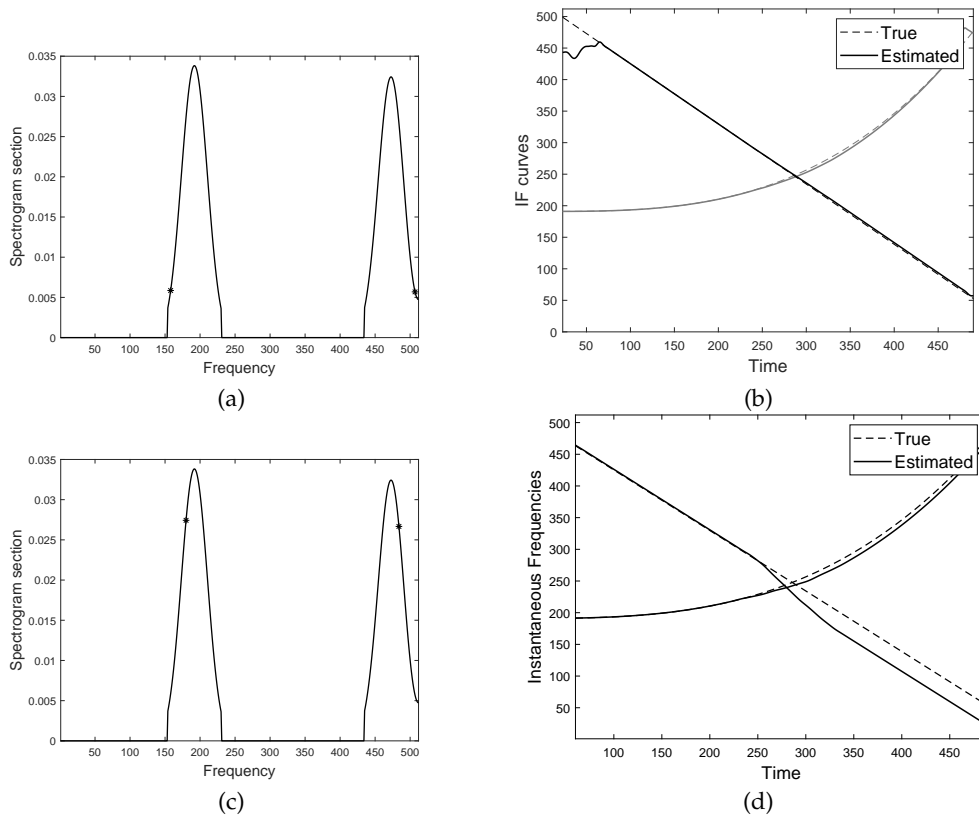


Figure 6. (a) Section of the spectrogram in Fig.5(a). The star markers indicate spectrogram values at the frequency points ζ_1, ζ_2 , close to the external bandwidth boundary, employed for defining the linear system, as in eq.(11), whose solution leads to IFs estimation (b). (c) The frequency points ζ_1, ζ_2 are selected close to spectrogram maxima, giving the estimated IFs (d).

- estimated from spectrogram ridges belonging to the separability region. Also in this case, a prior TF localization of the non-separability region would solve the problem.
4. The proposed procedure requires two frequency points for defining the linear system as in eq.(11), for each fixed u . The sensitivity to their selection has been numerically investigated. As shown in Figs.5 and 6, involving too low characteristic curves can affect the final accuracy due to boundary effects, while characteristic at higher levels are generally more subjected to interference, resulting in inaccurate IFs estimate. For this reason, a good compromise can be achieved by selecting two frequency points close to the one where spectrogram concavity changes, as done in the presented simulations.
 5. As shown in the experimental results, the proposed method is robust to additive interference and cross-terms. The presence of spectrogram values close to zero, which occurs in case of strong amplitude modulation as well as destructive interference, could result in instabilities in the derivatives approximation, that can affect the final IFs estimation. However, it is worth highlighting that the proposed approach outperforms the state-of-the-art method in dealing with the critical case of destructive interference, as shown in Figs.7 and 12. In addition, the proposed method has shown robustness to moderate noise.
 6. Spectrogram is widely used in practical applications because of its simplicity and computational benefit. It is well-known that, in case of MCS, spectrogram of close modes suffers from the presence of cross-terms that can affect each estimation concerning the signal. For this reason, more advanced and adaptive kernels with attenuated cross-terms, such as S-Transform or Locally Adaptive TF distribution, are often preferred in the literature. However, it is worth noticing that many real-life measurements, such as the ones concerning human gait classification and

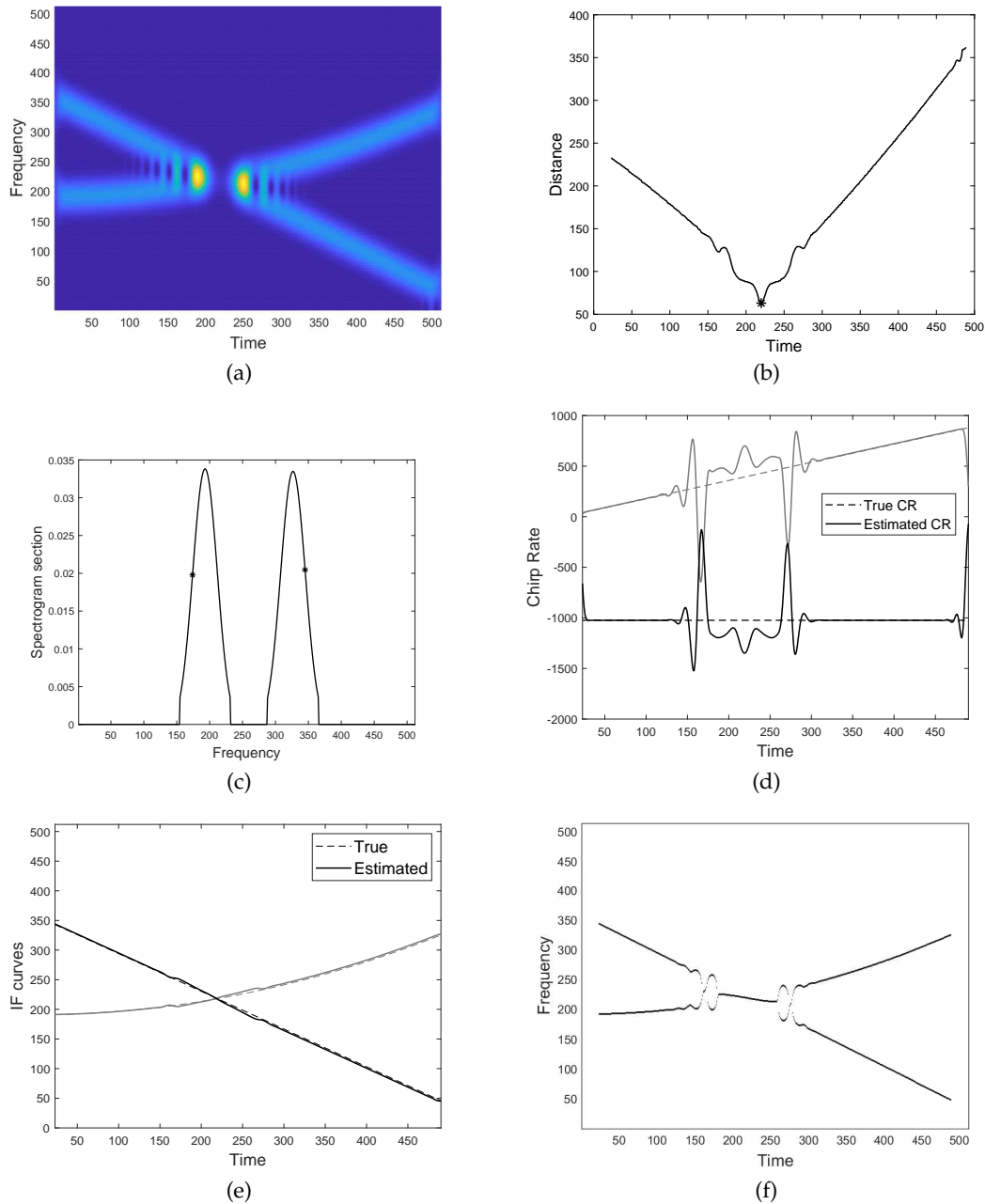


Figure 7. (a) Spectrogram of a two-component signal with crossing modes (destructive interference). (b) Distance function as defined in eq.(26) and detected crossing point (star). (c) Spectrogram section. The star markers indicate spectrogram values at the frequency points ξ_1, ξ_2 for defining the linear system, as in eq.(11). (d) Estimated chirp rates compared to the true ones. (e) Estimated IFs. (f) IF curves estimated from spectrogram maxima.

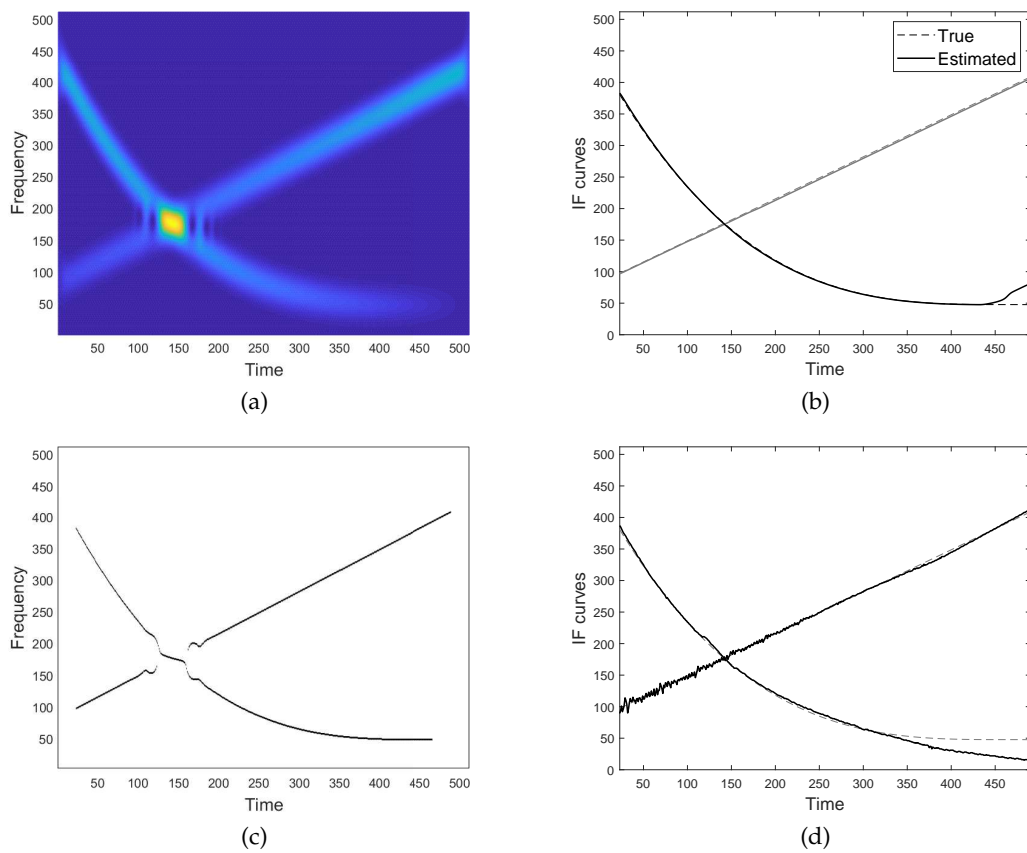


Figure 8. (a) Spectrogram of the AM-FM two-component signal as in eq.(27). (b) IF curves provided by the proposed method. (c) IF curves estimated from spectrogram maxima. (d) IF curves estimated by applying the Radon Spectrogram-based method in [27].

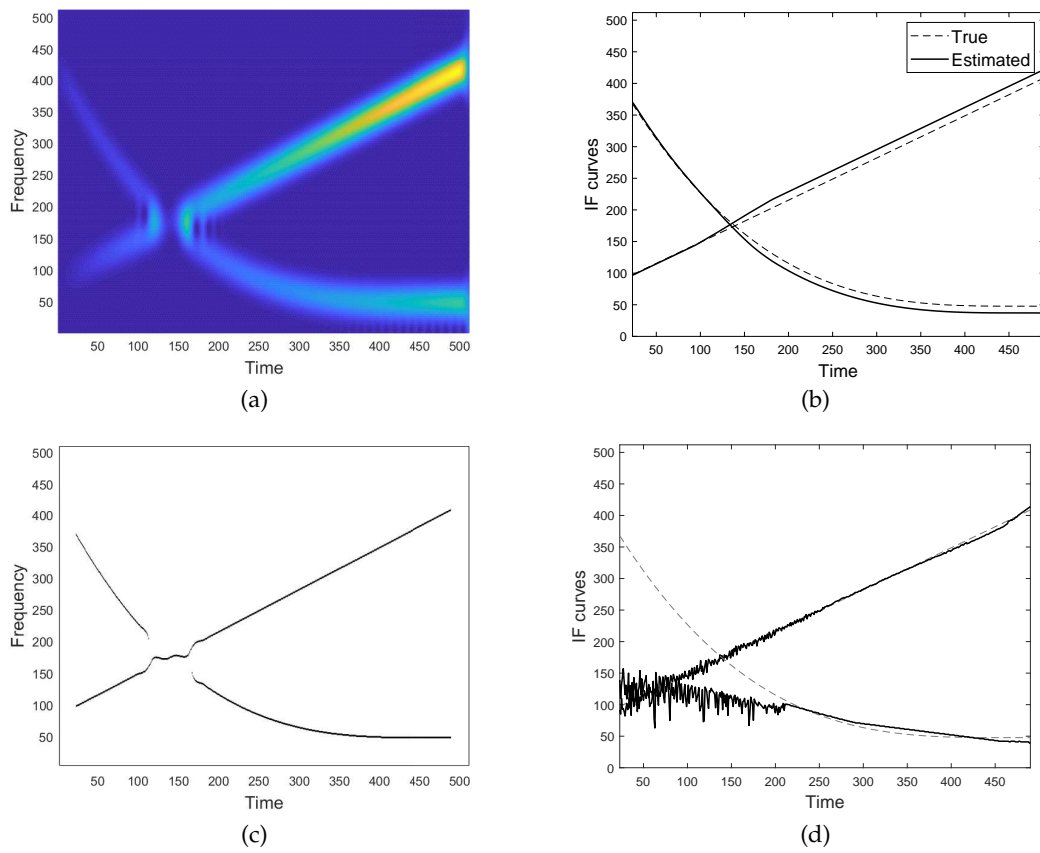


Figure 9. (a) Spectrogram of the AM-FM two-component signal as in eq.(28). (b) IF curves provided by the proposed method. (c) IF curves estimated from spectrogram maxima. (d) IF curves estimated by applying the Radon Spectrogram-based method in [27].

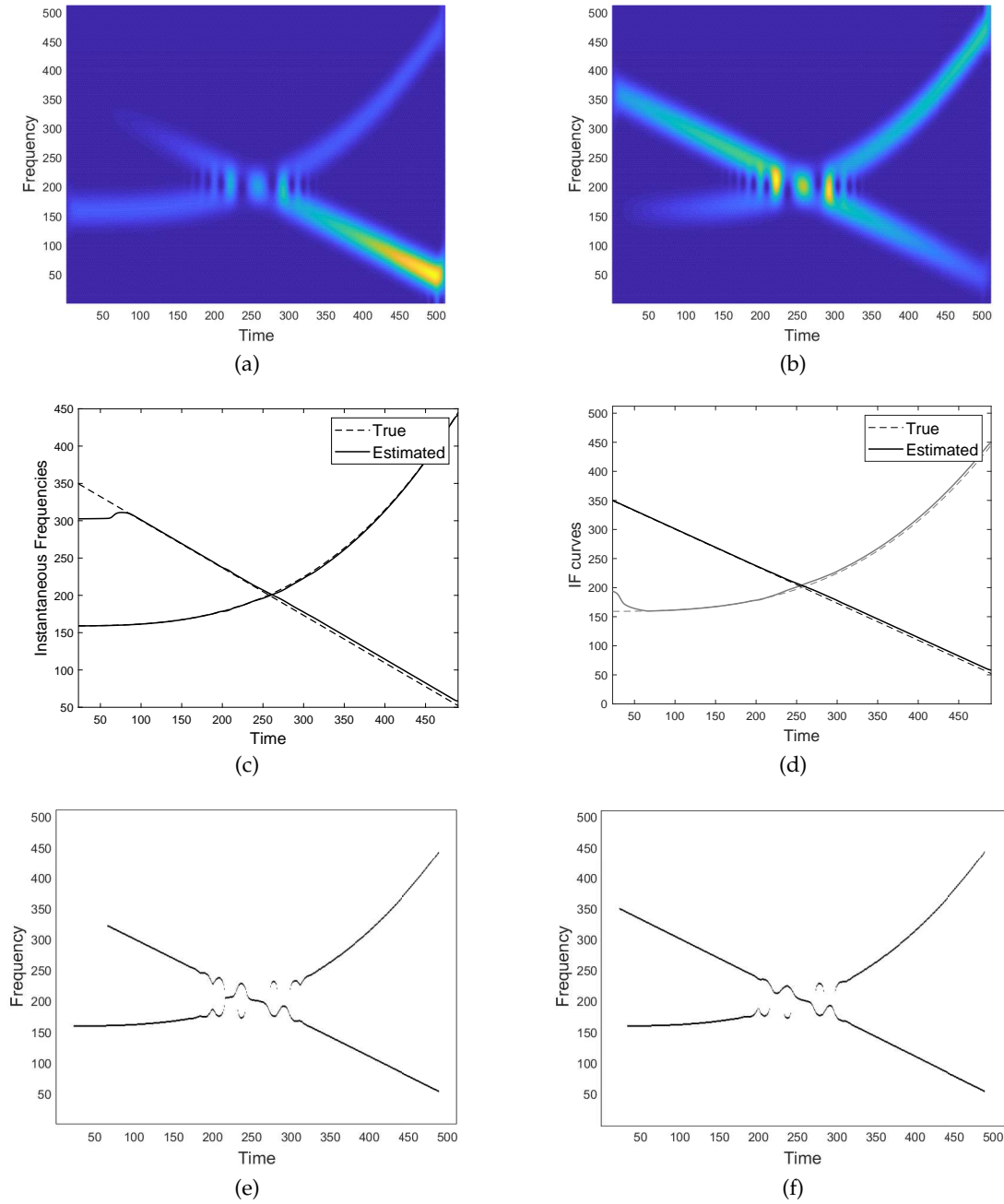


Figure 10. (a) Spectrogram of the AM-FM two-component signal as in eq.(29). (b) Spectrogram of the AM-FM two-component signal as in eq.(30). (c) IF curves of (a) provided by the proposed method. (d) IF curves of (b) provided by the proposed method. (e) IF curves estimated from spectrogram maxima in (a). (f) IF curves estimated from spectrogram maxima in (b).

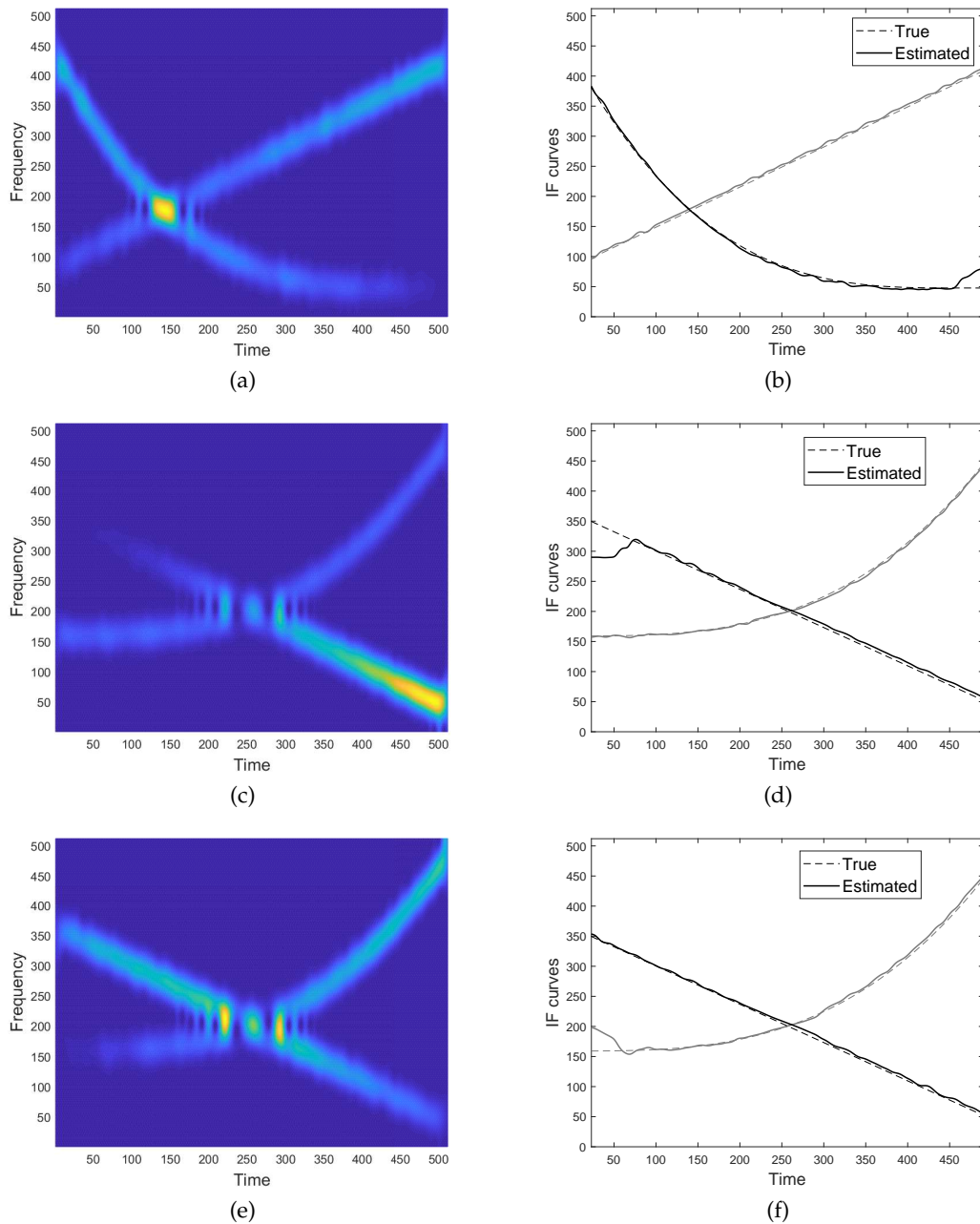


Figure 11. (a) Spectrogram of the signal in eq.(27), embedded in AWG noise at $SNR = 16\text{ dB}$ and (b) IF curves provided by the proposed method. (c) Spectrogram of the signal in eq.(29), embedded in AWG noise at $SNR = 20\text{ dB}$ and (d) IF curves provided by the proposed method. (e) Spectrogram of the signal in eq.(30), embedded in AWG noise at $SNR = 18\text{ dB}$ and (f) IF curves provided by the proposed method.

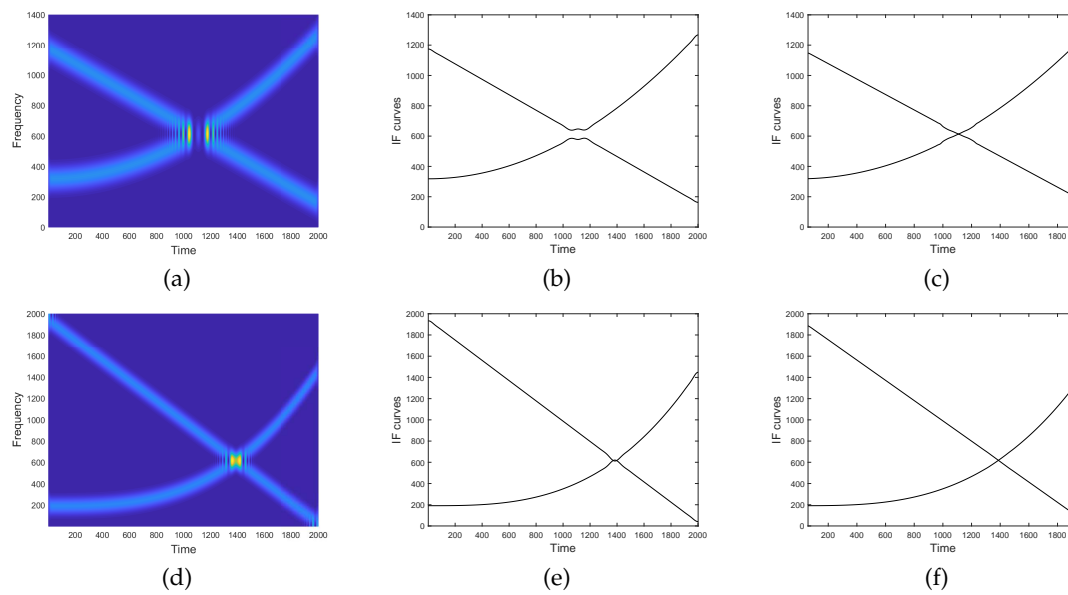


Figure 12. (a) Spectrogram of the two-component FM signal as in eq.(31) (destructive interference), (b) IF curves provided by the method proposed in [44] and (c) IF curves provided by the proposed method. (d) Spectrogram of the two-component signal as in eq.(32) (additive interference), (e) IF curves provided by the method proposed in [44] and (f) IF curves provided by the proposed method.

detection, precisely deal with spectrograms. That is why spectrogram processing is still of interest, today. In addition, cross-terms do not represent a limitation for the presented method, but a tool for estimating IF.

4. Conclusions

This paper has proposed a non-linear time-frequency evolution law for the spectrogram of a frequency and amplitude modulated two-component signal, which is employed for instantaneous frequencies (IFs) estimation. Based on the concept of weakened separability, the latter equation can be reduced to the one referred to a single mode, whose coefficients linearly depend on signal IFs time derivatives, namely the chirp rates (CRs). Approximating spectrogram derivatives and replacing them in the evolution law yields to a linear system with signals CRs as unknown variables. As a result, CRs can be detected by simply solving a two-dimensional linear system and IFs can be estimated by numerical integration. The experimental results presented in the paper have confirmed method effectiveness in dealing with amplitude modulated two-component signals with interfering modes. A comparative study with some of the state-of-the-art methods has been also provided, showing the benefits of the proposed method in processing modes affected by destructive interference.

IF curves estimation provided by the proposed method can be adopted as starting point for all variational methods dealing with interfering modes and requiring an initialization step. In addition, since it provides CRs estimation, the introduced procedure can be useful in reallocation techniques that involve high-order phase time derivatives.

As main advantage, the presented approach is non-parametric. However, IF law may be obtained by fixing the IF class and by interpolating the result provided by the proposed method.

As future developments, the method could be employed also for instantaneous amplitudes estimation. In addition, future studies will be aimed at defining an automatic procedure for interference region detection, in order to extend the proposed method to general multicomponent signals, having more than two modes. The applicability of the presented approach to time-frequency distributions different from the spectrogram will be also investigated.

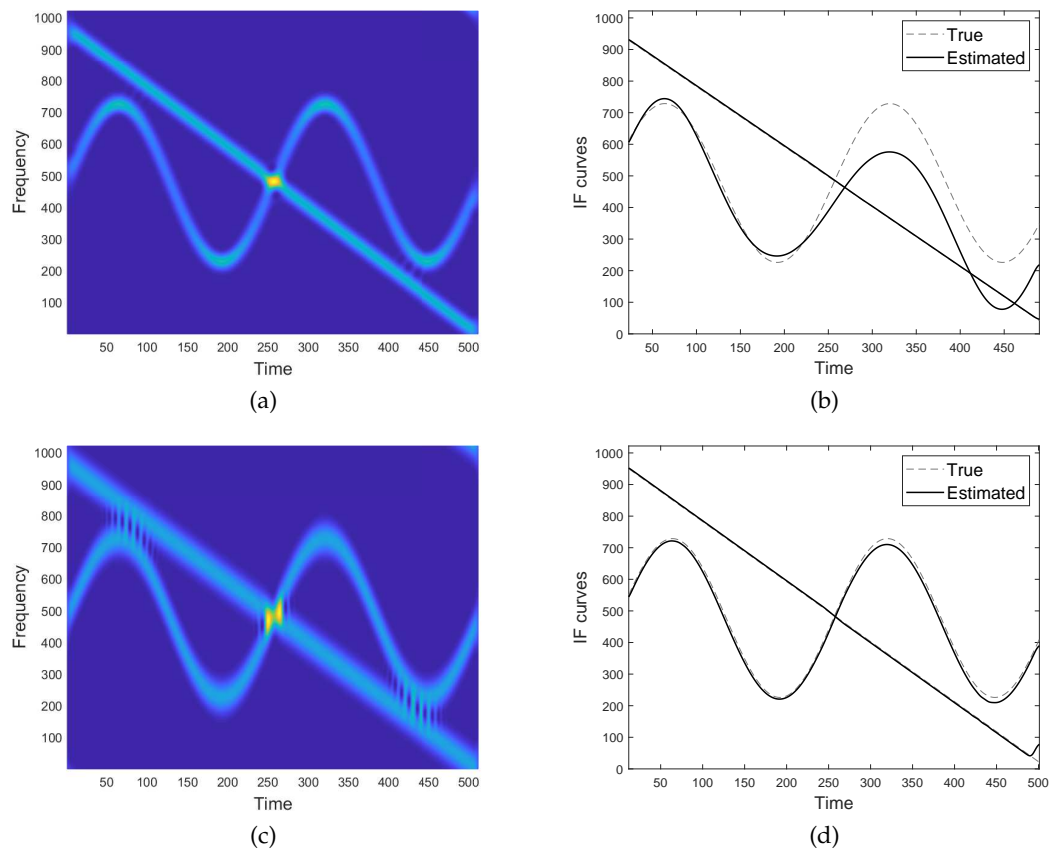


Figure 13. (a) Spectrogram of the complex two-component FM signal as in eq.(33), computed with a window of length $s = 44$. (b) IF curves provided by the proposed method applied to (a). (c) Spectrogram of the complex two-component FM signal as in eq.(33), computed with a window of length $s = 22$. (d) IF curves provided by the proposed method applied to (c).

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Abbreviations

The following abbreviations are used in this manuscript:

AM	Amplitude Modulated
CR	Chirp Rate
FM	Frequency Modulated
IA	Instantaneous Amplitude
IF	Instantaneous Frequency
MCS	Multicomponent Signal(s)
RPRM	Ridge Path Regrouping Method
STFT	Short-Time Fourier Transform
TF	Time-Frequency
TFD	Time-Frequency Distribution
WSC	Weakened Separability Condition

Appendix A

Proof of Proposition 1. By eq.(3) with negligible corrective terms and by STFT linearity, it follows

$$\begin{aligned}
 |S_f^g(u, \xi)| &= |S_{f_1}^g(u, \xi) + S_{f_2}^g(u, \xi)| \\
 &= \left| \frac{\sqrt{s}}{2} a_1 e^{i(\phi_1(u) - \xi u)} \hat{g}(s(\xi - \phi'_1(u))) + \frac{\sqrt{s}}{2} a_2 e^{i(\phi_2(u) - \xi u)} \hat{g}(s(\xi - \phi'_2(u))) \right| \\
 &= \frac{\sqrt{s}}{2} \left| a_1 \hat{g}(s(\xi - \phi'_1(u))) + a_2 \hat{g}(s(\xi - \phi'_2(u))) e^{i(\phi_2(u) - \phi_1(u))} \right| \\
 &= \frac{\sqrt{s}}{2} \left[a_1^2 \hat{g}_1^2 + a_2^2 \hat{g}_2^2 \cos^2 \Delta\phi + 2a_1 a_2 \hat{g}_1 \hat{g}_2 \cos \Delta\phi + a_2^2 \hat{g}_2^2 \sin^2 \Delta\phi \right]^{\frac{1}{2}} \\
 &= \frac{\sqrt{s}}{2} \left[a_1^2 \hat{g}_1^2 + a_2^2 \hat{g}_2^2 + 2a_1 a_2 \hat{g}_1 \hat{g}_2 \cos \Delta\phi \right]^{\frac{1}{2}},
 \end{aligned}$$

where $a_i = a_i(u)$, $\hat{g}_i = \hat{g}_i(s(\xi - \phi'_i(u)))$, $i = 1, 2$ and $\Delta\phi = \phi_2(u) - \phi_1(u)$. As a result, the spectrogram is

$$P(u, \xi) = \frac{s}{4} \left[a_1^2 \hat{g}_1^2 + a_2^2 \hat{g}_2^2 + 2a_1 a_2 \hat{g}_1 \hat{g}_2 \cos \Delta\phi \right],$$

and then its derivatives are

$$\begin{aligned}
 \frac{\partial P}{\partial \xi} &= \frac{s^2}{2} \left(a_1^2 \hat{g}_1 \hat{g}'_1 + a_2^2 \hat{g}_2 \hat{g}'_2 + a_2 a_1 \cos(\phi_2 - \phi_1) (\hat{g}'_1 \hat{g}_2 + \hat{g}_1 \hat{g}'_2) \right), \\
 \frac{\partial P}{\partial u} &= \frac{s^2}{2} \left(-\phi''_1 a_1^2 \hat{g}_1 \hat{g}'_1 - \phi''_2 a_2^2 \hat{g}_2 \hat{g}'_2 + a_2 a_1 \cos(\phi_2 - \phi_1) (-\phi''_1 \hat{g}'_1 \hat{g}_2 - \phi''_2 \hat{g}_2 \hat{g}'_1) \right. \\
 &\quad \left. - \frac{a_1 a_2}{s} \hat{g}_1 \hat{g}_2 \sin \Delta\phi \Delta\phi' \right) + \frac{s}{2} \left[a_1 a'_1 \hat{g}_1^2 + a_2 a'_2 \hat{g}_2^2 + \hat{g}_1 \hat{g}_2 \cos \Delta\phi (a'_1 a_2 + a'_2 a_1) \right].
 \end{aligned}$$

It follows

$$\begin{aligned}
 \frac{\partial P}{\partial u} &= -\phi''_1 \frac{\partial P}{\partial \xi} + \frac{s^2}{2} \phi''_1 \left(a_2^2 \hat{g}_2 \hat{g}'_2 + a_2 a_1 \cos(\phi_2 - \phi_1) (\hat{g}'_1 \hat{g}_2 + \hat{g}_1 \hat{g}'_2) \right) + \\
 &\quad \frac{s^2}{2} \left(-\phi''_2 a_2^2 \hat{g}_2 \hat{g}'_2 + a_2 a_1 \cos(\phi_2 - \phi_1) (-\phi''_1 \hat{g}'_1 \hat{g}_2 - \phi''_2 \hat{g}_2 \hat{g}'_1) \right. \\
 &\quad \left. - \frac{a_1 a_2}{s} \hat{g}_1 \hat{g}_2 \sin \Delta\phi \Delta\phi' \right) + \frac{s}{2} \left[a_1 a'_1 \hat{g}_1^2 + a_2 a'_2 \hat{g}_2^2 + \hat{g}_1 \hat{g}_2 \cos \Delta\phi (a'_1 a_2 + a'_2 a_1) \right] \\
 &= -\phi''_1 \frac{\partial P}{\partial \xi} - s \frac{a_1 a_2}{2} \hat{g}_1 \hat{g}_2 \sin \Delta\phi \Delta\phi' - s^2 \frac{\Delta\phi''}{2} \hat{g}'_2 \left[a_2 \hat{g}_2^2 - a_1 a_2 \cos \Delta\phi \hat{g}_1 \right] \\
 &\quad + \frac{s}{2} \left[a_1 a'_1 \hat{g}_1^2 + a_2 a'_2 \hat{g}_2^2 + \hat{g}_1 \hat{g}_2 \cos \Delta\phi (a'_1 a_2 + a'_2 a_1) \right],
 \end{aligned}$$

then eq.(12) holds true.

Proof of Proposition 2. In the following, eq.(16) is proved. Eq.(17) is then obtained by a straightforward integration. Without loss of generality, we set $s = 1$ in eqs.(4) and (5).

In the separability region, $\Delta\phi'(u) \approx 0$, hence by using the Taylor expansion for P_2 , we get

$$P_2(u, \xi) = \frac{a_2^2}{4} \hat{g}^2(\xi - \phi'_2(u)) = \frac{a_2^2}{4} \hat{g}^2(\xi - \phi'_1(u) - \Delta\phi'(u)) =$$

$$\begin{aligned}
&= \frac{a_2^2}{4} \hat{g}^2(\xi - \phi'_1(u)) - 2 \frac{a_2^2}{4} \hat{g}(\xi - \phi'_1(u)) \hat{g}'(\xi - \phi'_1(u)) \Delta \phi'(u) + o(\Delta \phi'^2(u)) = \\
&= \frac{a_2^2}{a_1^2} P_1(u, \xi) - \frac{a_2^2}{a_1^2} P_{1\xi}(u, \xi) \Delta \phi' + o(\Delta \phi'^2(u)).
\end{aligned}$$

As a result, by neglecting the second order terms, we have

$$P_2(u, \xi) = \frac{a_2^2(u)}{a_1^2(u)} P_1(u, \xi) \eta(u, \xi),$$

with

$$\eta(u, \xi) = 1 - \frac{P_{1\xi}(u, \xi)}{P_1(u, \xi)} \Delta \phi'(u), \quad (\text{A1})$$

and then eq.(5) becomes

$$\begin{aligned}
P(u, \xi) &= P_1(u, \xi) + \frac{a_2^2}{a_1^2} P_1(u, \xi) \eta(u, \xi) + 2 \cos(\Delta \phi(u)) \sqrt{P_1(u, \xi) \frac{a_2^2}{a_1^2} P_1(u, \xi) \eta(u, \xi)} = \\
&= P_1(u, \xi) \left(1 + \frac{a_2^2}{a_1^2} \eta(u, \xi) + 2 \frac{a_2}{a_1} \sqrt{\eta(u, \xi) \cos(\Delta \phi(u))} \right) = P_1(u, \xi) (1 + \varepsilon(u, \xi)),
\end{aligned}$$

where

$$\varepsilon(u, \xi) = \frac{a_2^2}{a_1^2} \eta(u, \xi) + 2 \frac{a_2}{a_1} \sqrt{\eta(u, \xi) \cos(\Delta \phi(u))}. \quad (\text{A2})$$

It is worth observing that, for $u : \Delta \phi'(u) = 0$, $\varepsilon(u, \xi) = \varepsilon(u)$, i.e. it is independent of the variable ξ ; as a result

$$\begin{aligned}
P(u, \xi) &= P_1(u, \xi) (1 + \varepsilon(u)), \\
P_\xi(u, \xi) &= P_{1\xi}(u, \xi) (1 + \varepsilon(u)), \\
P_u(u, \xi) &= P_{1u}(u, \xi) (1 + \varepsilon(u)) + P_1(u, \xi) \varepsilon_u(u),
\end{aligned}$$

and, taking into account eq.(15),

$$\begin{aligned}
\tilde{\phi}_1'(u) &= - \frac{P_u(u, \xi_1) P(u, \xi_2) - P_u(u, \xi_2) P(u, \xi_1)}{P_\xi(u, \xi_1) P(u, \xi_2) - P_\xi(u, \xi_2) P(u, \xi_1)} = - \frac{P_{1u}(u, \xi_1) P_1(u, \xi_2) - P_{1u}(u, \xi_2) P_1(u, \xi_1)}{P_{1\xi}(u, \xi_1) P(u, \xi_2) - P_{1\xi}(u, \xi_2) P(u, \xi_1)} + \\
&\quad - \frac{1}{1 + \varepsilon(u)} \frac{P_1(u, \xi_1) \varepsilon_u(u) P_1(u, \xi_2) - P_1(u, \xi_2) \varepsilon_u(u) P_1(u, \xi_1)}{P_{1\xi}(u, \xi_1) P(u, \xi_2) - P_{1\xi}(u, \xi_2) P(u, \xi_1)} = \phi_1''(u).
\end{aligned}$$

The estimation still is correct, independently of ξ_1 and ξ_2 .

On the contrary, if u is such that $\Delta \phi'(u) \neq 0$, $\varepsilon(u, \xi)$ is not independent of ξ and then

$$\begin{aligned}
P(u, \xi) &\approx P_1(u, \xi) (1 + \varepsilon(u, \xi)), \\
P_\xi(u, \xi) &\approx P_{1\xi}(u, \xi) (1 + \varepsilon(u, \xi)) + P_1(u, \xi) \varepsilon_\xi(u, \xi), \\
P_u(u, \xi) &= P_{1u}(u, \xi) (1 + \varepsilon(u, \xi)) + P_1(u, \xi) \varepsilon_u(u, \xi).
\end{aligned}$$

4.31 However, it is possible to properly select ξ_1 and ξ_2 in order to make this contribution almost
 4.32 negligible. More precisely, ξ_1 and ξ_2 have to be such that $\eta(u, \xi) \approx 1$ or $\varepsilon_\xi(u, \xi) \approx 0$. In particular, if ξ_1
 4.33 and ξ_2 are such that $\varepsilon_\xi(u, \xi) \approx 0$, we have the following estimation for $\phi_1''(u)$, i.e.

$$-\tilde{\phi}_1''(u) = -\phi_1''(u) + \frac{P_1(u, \xi_1)P_1(u, \xi_2) \left(\frac{\varepsilon_u(u, \xi_1)}{1+\varepsilon(u, \xi_1)} - \frac{\varepsilon_u(u, \xi_2)}{1+\varepsilon(u, \xi_2)} \right)}{P_{1\xi}(u, \xi_1)P_1(u, \xi_2) - P_{1\xi}(u, \xi_2)P_1(u, \xi_1)}.$$

4.34 It is worth observing that,

$$\begin{aligned} \varepsilon_\xi(u, \xi) &= \frac{a_2^2(u)}{a_1^2(u)} \eta_\xi(u, \xi) + \frac{a_2(u)}{a_1(u)} \frac{\eta_\xi(u, \xi)}{\sqrt{\eta(u, \xi)}} \cos(\Delta\phi(u)) = 0 \Leftrightarrow \\ &\Leftrightarrow \frac{a_2^2(u)}{a_1^2(u)} \eta_\xi(u, \xi) \left(1 + \frac{a_1(u)}{a_2(u)} \frac{1}{\sqrt{\eta(u, \xi)}} \cos(\Delta\phi(u)) \right) = 0, \end{aligned}$$

4.35 i.e. $\eta_\xi(u, \xi) = 0$ or $1 + \frac{a_1(u)}{a_2(u)} \frac{1}{\sqrt{\eta(u, \xi)}} \cos(\Delta\phi(u)) = 0$.

4.36 In the first case, by observing that

$$\eta_\xi(u, \xi) = -\Delta\phi'(u) \frac{P_{1\xi\xi}(u, \xi)P_1(u, \xi) - P_{1\xi}^2(u, \xi)}{P_1^2(u, \xi)},$$

4.37 we have

$$\eta_\xi(u, \xi) = 0 \Leftrightarrow \xi : P_{1\xi\xi} = \frac{P_{1\xi}^2(u, \xi)}{P_1(u, \xi)}. \quad (\text{A3})$$

4.38 In the second case

$$1 + \frac{a_1(u)}{a_2(u)} \frac{1}{\sqrt{\eta(u, \xi)}} \cos(\Delta\phi(u)) = 0 \Leftrightarrow \sqrt{\eta(u, \xi)} = -\frac{a_1(u)}{a_2(u)} \cos(\Delta\phi(u)), \quad (\text{A4})$$

i.e.

$$\frac{P_1(u, \xi)}{P_{1\xi}(u, \xi)} = \frac{\Delta\phi'(u)}{1 - \frac{a_1^2(u)}{a_2^2(u)} \cos^2(\Delta\phi(u))}. \quad (\text{A5})$$

4.39 In particular, the condition in eq. (A4) provides the following form for $\varepsilon(u, \xi)$

$$\varepsilon(u, \xi) = \varepsilon(u) = \frac{a_2^2(u)}{a_1^2(u)} \frac{a_1^2(u)}{a_2^2(u)} \cos^2(\Delta\phi(u)) - 2 \frac{a_2(u)}{a_1(u)} \frac{a_1(u)}{a_2(u)} \cos^2(\Delta\phi(u)) = -\cos^2(\Delta\phi(u)). \quad (\text{A6})$$

4.40 Under these conditions, i.e. $\varepsilon(u, \xi) \approx \varepsilon(u)$, we can also write the estimation for $\phi_1''(u)$ whenever
 4.41 the finite difference approximation is used for P_ξ and P_u in eq. (14). More precisely, by denoting with
 4.42 δ and ρ the discretization steps respectively for u and ξ variables, we have

$$P_\xi(u, \xi) = \frac{P(u, \xi + \rho) - P(u, \xi - \rho)}{2\rho} + o(\rho^2) = (1 + \varepsilon(u)) \frac{P_1(u, \xi + \rho) - P_1(u, \xi - \rho)}{2\rho} + o(\rho^2). \quad (\text{A7})$$

4.43 while

$$\begin{aligned} P_u(u, \xi) &= \frac{P(u + \delta, \xi) - P(u - \delta, \xi)}{2\delta} + o(\delta^2) = \\ &= \frac{P_1(u + \delta, \xi)(1 + \varepsilon(u + \delta)) - P_1(u - \delta, \xi)(1 + \varepsilon(u - \delta))}{2\delta} + o(\delta^2) = \end{aligned}$$

$$= \frac{P_1(u + \delta, \xi) - P_1(u - \delta, \xi)}{2\delta} + \frac{P_1(u + \delta, \xi)\varepsilon(u + \delta) - P_1(u - \delta, \xi)\varepsilon(u - \delta)}{2\delta} + o(\delta^2).$$

Since $\varepsilon(u + \delta) \approx \varepsilon(u) + \varepsilon_u(u)\delta$ and $\varepsilon(u - \delta) \approx \varepsilon(u) - \varepsilon_u(u)\delta$, the last term in the previous section can be rewritten as

$$\varepsilon(u) \frac{P_1(u + \delta, \xi) - P_1(u - \delta, \xi)}{2\delta} + \varepsilon_u(u)\delta \frac{P_1(u + \delta, \xi) + P_1(u - \delta, \xi)}{2\delta} = \frac{P_1(u + \delta, \xi) - P_1(u - \delta, \xi)}{2\delta} + \varepsilon_u(u)P_1(\tilde{u}, \xi), \quad \tilde{u} \in [u - \delta, u + \delta],$$

where the intermediate value theorem has been applied to the last term. As a result,

$$P_u(u, \xi) = (1 + \varepsilon(u)) \frac{P_1(u + \delta, \xi) - P_1(u - \delta, \xi)}{2\delta} + \varepsilon_u(u)P_1(\tilde{u}, \xi) + o(\delta^2). \quad (\text{A8})$$

By using eqs.(A8) and (A7) and neglecting the second order error terms, eq.(15) can be rewritten as follows

$$\begin{aligned} -\tilde{\phi}_1''(u) &= \frac{(1 + \varepsilon(u))^2}{(1 + \varepsilon(u))^2} \frac{P_1(u + \delta, \xi_1) - P_1(u - \delta, \xi_1)}{2\delta} P_1(u, \xi_2) - \frac{P_1(u + \delta, \xi_2) - P_1(u - \delta, \xi_2)}{2\delta} P_1(u, \xi_1) + \\ &+ \frac{\varepsilon_u(u)}{1 + \varepsilon(u)} \frac{P_1(\tilde{u}, \xi_1)P_1(u, \xi_2) - P_1(\tilde{u}, \xi_2)P_1(u, \xi_1)}{\frac{P_1(u, \xi_1 + \rho) - P_1(u, \xi_1 - \rho)}{2\rho} P_1(u, \xi_2) - \frac{P_1(u, \xi_2 + \rho) - P_1(u, \xi_2 - \rho)}{2\rho} P_1(u, \xi_1)}. \end{aligned}$$

By setting $P_1(u, \xi_2) \approx P_1(u, \xi_1) + P_{1\xi}(u, \xi_1)(\xi_2 - \xi_1)$ and $\xi_2 - \xi_1 = \rho$, and reminding eq.(A7), we have

$$-\tilde{\phi}_1''(u) = -\phi_1''(u) + \frac{\varepsilon_u(u)}{1 + \varepsilon(u)} \frac{(P_1(\tilde{u}, \xi_1) - P_1(\tilde{u}, \xi_2))P_1(u, \xi_1) + P_1(\tilde{u}, \xi_1)P_{1\xi}(u, \xi_1)\rho}{(P_{1\xi}(u, \xi_1) - P_{1\xi}(u, \xi_2))P_1(u, \xi_1) + \rho P_{1\xi}^2(u, \xi_1)}.$$

By assuming $\tilde{u} \approx \tilde{\tilde{u}} \approx u$, we get

$$\begin{aligned} -\tilde{\phi}_1''(u) &= -\phi_1''(u) + \frac{\varepsilon_u(u)}{1 + \varepsilon(u)} \frac{\rho P_{1\xi}(u, \xi_1)P_1(u, \xi_1)}{(P_{1\xi}(u, \xi_1) - P_{1\xi}(u, \xi_2))P_1(u, \xi_1) + \rho P_{1\xi}^2(u, \xi_1)} = \\ &= -\phi_1''(u) + \frac{\varepsilon_u(u)}{1 + \varepsilon(u)} \frac{\rho}{1 - \frac{P_{1\xi}(u, \xi_2)}{P_{1\xi}(u, \xi_1)} + \rho \frac{P_{1\xi}(u, \xi_1)}{P_{1\xi}(u, \xi_1)}}. \end{aligned}$$

For ξ_2 close to ξ_1 it holds $P_{1\xi}(u, \xi_1) \approx P_{1\xi}(u, \xi_2)$ and then eq. (16) follows.

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