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A CLASS OF NOWHERE DIFFERENTIABLE FUNCTIONS SATISFYING SOME CONCAVITY-TYPE ESTIMATE

Y. FUJITA[1,*,†], N. HAMAMUKI[2,‡], A. SICONOLFI[3] and N. YAMAGUCHI[4]

[1]Department of Mathematics, University of Toyama, 3190 Gofuku, Toyama-shi, Toyama 930-8555, Japan
e-mail: yfujita@sci.u-toyama.ac.jp

[2]Department of Mathematics, Hokkaido University, Kita 10, Nishi 8, Kita-Ku, Sapporo, Hokkaido, 060-0810, Japan
e-mail: hnao@math.sci.hokudai.ac.jp

[3]Department of Mathematics, Sapienza Università di Roma, Piazzale Aldo Moro 5, 00185 Roma, Italy
e-mail: siconolf@mat.uniroma1.it

[4]Faculty of Human Development, University of Toyama, 3190 Gofuku, Toyama-shi, Toyama 930-8555, Japan
e-mail: norikazu@edu.u-toyama.ac.jp

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Abstract. We introduce and investigate a class $\mathcal{P}$ of continuous and periodic functions on $\mathbb{R}$. The class $\mathcal{P}$ is defined so that second-order central differences of a function satisfy some concavity-type estimate. Although this definition seems to be independent of nowhere differentiable character, it turns out that each function in $\mathcal{P}$ is nowhere differentiable. The class $\mathcal{P}$ naturally appears from both a geometrical viewpoint and an analytic viewpoint. In fact, we prove that a function belongs to $\mathcal{P}$ if and only if some geometrical inequality holds for a family of parabolas with vertexes on this function. As its application, we study the behavior of the Hamilton–Jacobi flow starting from a function in $\mathcal{P}$. A connection between $\mathcal{P}$ and some functional series is also investigated. In terms of second-order central differences, we give a necessary and sufficient condition so that a function given by the series belongs to $\mathcal{P}$. This enables us to construct a large number of examples of functions in $\mathcal{P}$ through an explicit formula.

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1. Introduction

Let us denote by $C_p(\mathbb{R})$ the set of all continuous and periodic functions $f\colon \mathbb{R} \to \mathbb{R}$ with period 1 and $f(0) = 0$. Throughout this paper, we assume that r is an integer such that $r \geq 2$. Let $\mathbb{N}_0 := \mathbb{N} \cup \{0\}$.

Our aim in this paper is to introduce and investigate the class $\mathcal{P}$ of functions in $C_p(\mathbb{R})$ defined as follows: Given a function $f \in C_p(\mathbb{R})$, we consider, for each $(n, k, y) \in \mathbb{N}_0 \times \mathbb{Z} \times (0, 1)$, the first-order forward and backward differences of f at $\frac{k+y}{r^n}$ defined, respectively, by

$$\delta^+_{n,k}(y; f) = \frac{f(\frac{k+1}{r^n}) - f(\frac{k+y}{r^n})}{\frac{1-y}{r^n}}, \quad \delta^-_{n,k}(y; f) = \frac{f(\frac{k+y}{r^n}) - f(\frac{k}{r^n})}{\frac{y}{r^n}}. \tag{1.1}$$

Definition 1.1. Let $c > 0$ be a given constant. A function $f \in C_p(\mathbb{R})$ belongs to $\mathcal{P}_c$ if

$$\delta^+_{n,k}(y; f) - \delta^-_{n,k}(y; f) \leq -c \tag{1.2}$$

for all $(n, k, y) \in \mathbb{N}_0 \times \mathbb{Z} \times (0, 1)$. We use the notation $\mathcal{P} = \bigcup_{c>0} \mathcal{P}_c$. Note that both $\mathcal{P}_c$ and $\mathcal{P}$ depend on the choice of r though we omit it in our notation.

Inequality (1.2) can be written equivalently as

$$\Delta_{n,k}(y; f) \leq -2cr^n, \tag{1.3}$$

where $\Delta_{n,k}(y; f)$ is the second-order central difference defined by

$$\Delta_{n,k}(y; f) = 2r^n(\delta^+_{n,k}(y; f) - \delta^-_{n,k}(y; f)). \tag{1.4}$$

It is well-known that if a function $f\colon \mathbb{R} \to \mathbb{R}$ is concave and has the second derivative in some interval I, then $f'' \leq 0$ in I. Even if f is not twice differentiable, a discrete version of the estimate $\Delta_{n,k}(y, f) \leq 0$ still holds. Thus, the condition (1.3) can be regarded as a concavity-type estimate for f. Our definition of $\mathcal{P}$ requires a function to have the second-order differences which tend to $-\infty$ in the prescribed rate as $n \to \infty$.

Although Definition 1.1 seems to be independent of nowhere differentiable character, it turns out that each function in $\mathcal{P}$ is nowhere differentiable. This shows that our concavity-type estimate (1.3) is significantly different from a usual concavity since any concave function is twice differentiable almost everywhere.

We have two reasons to introduce and investigate the class $\mathcal{P}$. The first reason comes from a geometrical viewpoint. We show that each function in $\mathcal{P}$ has a geometrical characterization stated as follows: For any given function $f \in C_p(\mathbb{R})$, let $\{q_f(t,x;z)\}_{z\in\mathbb{R}}$ be the family of parabolas defined by

$$q_f(t,x;z) = f(z) + \frac{1}{2t}(x-z)^2, \quad (t,x,z) \in (0,\infty)\times\mathbb{R}\times\mathbb{R}. \tag{1.5}$$

Then, we show that a function f in $C_p(\mathbb{R})$ belongs to $\mathcal{P}_c$ if and only if f satisfies

(F1)$_c$ For all $(n,k,y) \in \mathbb{N}_0 \times \mathbb{Z} \times (0,1)$ and $t \geq \frac{1}{2cr^n}$,

$$q_f\Big(t,x;\frac{k+y}{r^n}\Big) \geq \min\Big\{q_f\Big(t,x;\frac{k}{r^n}\Big),\, q_f\Big(t,x;\frac{k+1}{r^n}\Big)\Big\}, \quad x \in \mathbb{R}. \tag{1.6}$$

Inequality (1.6) is a geometrical one related to position of the three parabolas; see Fig. 1.

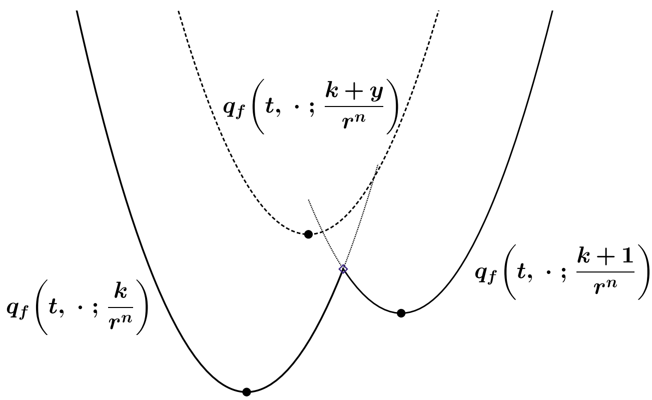

Fig. 1: The broken line and the solid line indicate, respectively, the function on the left- and right-hand side of (1.6)

Another interpretation of (1.6) is that the function $q_f(t,x;\cdot)$ takes a minimum over the interval $[\frac{k}{r^n}, \frac{k+1}{r^n}]$ at the endpoints.

The second reason comes from an analytic viewpoint. We consider the operator $U\colon C_p(\mathbb{R}) \ni \psi \mapsto U_\psi \in C_p(\mathbb{R})$ defined by the series

$$U_\psi(x) = \sum_{j=0}^{\infty} \frac{1}{r^j}\,\psi(r^j x), \quad x \in \mathbb{R}. \tag{1.7}$$

Such a series is known to generate nowhere differentiable functions under a suitable condition on ψ. We prove that the condition $U_\psi \in \mathcal{P}$ can be equivalently rephrased by the condition including the second-order differences of ψ. In fact, we establish

$$\Delta_{n,k}(y;U_\psi)=\sum_{j=0}^{n-1} r^j \Delta_{n-j,k}(y;\psi)-\frac{2r^n}{y(1-y)}U_\psi(y), \tag{1.8}$$

whenever $\psi \in C_p(\mathbb{R})$ and $(n,k,y) \in \mathbb{N}_0 \times \mathbb{Z} \times (0,1)$. When $n=0$, the first term of the right-hand side of (1.8) is interpreted as 0. Thus, for a given $c>0$, we see that $U_\psi \in \mathcal{P}_c$ if and only if the right-hand side of (1.8) is less than or equal to $-2cr^n$ for all $(n,k,y) \in \mathbb{N}_0 \times \mathbb{Z} \times (0,1)$. In other words, the class $\mathcal{P}$ is characterized via the operator U. Besides, making use of (1.8), we give some sufficient conditions on ψ in order that $U_\psi \in \mathcal{P}$. We show that U_ψ belongs to $\mathcal{P}$ if ψ is concave on $[0,1]$. Also, even if ψ is not concave on $[0,1]$, there is the case where U_ψ belongs to $\mathcal{P}$ provided that ψ is semiconcave on $[0,1]$ and satisfies some additional assumption. These simple sufficient conditions enable us to systematically construct a large number of examples of functions in the class $\mathcal{P}$ through the explicit formula (1.7).

A typical example of functions constructed by this procedure is the generalized Takagi function $\tau_r \in C_p(\mathbb{R})$ defined by

$$\tau_r(x)=U_d(x)=\sum_{j=0}^{\infty}\frac{1}{r^j}d(r^j x), \quad x \in \mathbb{R}, \tag{1.9}$$

where $d \in C_p(\mathbb{R})$ is the distance function to the set $\mathbb{Z}$, that is,

$$d(x)=\min\bigl\{\,|x-z| \mid z \in \mathbb{Z}\bigr\}, \quad x \in \mathbb{R}. \tag{1.10}$$

The celebrated Takagi function is given by τ_2. The function τ_2 is equivalent to the one first constructed by T. Takagi in 1903, who showed that τ_2 is nowhere differentiable (see [17]). Its relevance in analysis, probability theory and number theory has been widely illustrated by many contributions, see for instance [1,15,17,18]. Since d is concave on $[0,1]$, we can show that τ_r belongs to $\mathcal{P}$ for any integer $r \geq 2$.

In connection with $(\mathrm{F1})_c$, we also study the behavior of the Hamilton–Jacobi flow $\{H_t f\}_{t>0}$ starting from $f \in \mathcal{P}$, where

$$H_t f(x)=\inf_{z\in\mathbb{R}} q_f(t,x;z), \quad (t,x) \in (0,\infty)\times\mathbb{R}. \tag{1.11}$$

This formula is widely used in the theory of viscosity solutions, and $H_t f$ is also referred to as an *inf-convolution* of f.

There are several papers related to our work. In [12], Hata and Yamaguti proposed a different generalization of the Tagaki function, the so-called Tagaki class, which includes not only nowhere differentiable functions, but also differentiable and even smooth ones. To analyze this class, they used some functional equations containing second-order central differences. Although we also use the second-order central difference $\Delta_{n,k}(y;f)$ of a function $f \in C_p(\mathbb{R})$, the frame and the purpose of the investigation of [12] are however rather different from ours. In [3,13,16], an inequality for approximate midconvexity of the Takagi function was investigated. A precise behavior of the flow $\{H_t\tau\}_{t>0}$ starting from the Takagi function is studied in [7].

The function U_ψ of (1.7) has been considered by many authors. Cater [5] showed that if $\psi \in C_p(\mathbb{R})$ is concave on the interval $[0,1]$ and ψ takes its positive maximum over $[0,1]$ at $x = \frac{1}{2}$, then U_ψ is nowhere differentiable. Although the connection between the concavity of ψ and U_ψ was already explored in [5], we show in addition that the formula (1.7) provides examples of functions in the class $\mathcal{P}$. Furthermore, we show that U_ψ can belong to $\mathcal{P}$ even if $\psi \in C_p(\mathbb{R})$ is not concave on $[0,1]$. Heurteaux [14] gave another sufficient conditions on $\psi \in C_p(\mathbb{R})$ such that U_ψ is nowhere differentiable. The set of maximum points in $[0,1]$ of the function U_ψ was studied in [8] for $r = 2$. However, all of the above papers neither characterize a class of nowhere differentiable functions nor introduce a class like $\mathcal{P}$.

The structure of the paper is as follows. In Section 2 we prove nowhere differentiability and the geometrical characterization of a function in $\mathcal{P}$. Section 3 is devoted to the formula (1.8). We derive some sufficient conditions on $\psi \in \mathcal{P}$ in order that $U_\psi \in \mathcal{P}$. In Section 4, we study how the Hamilton–Jacobi flow $\{H_t f\}_{t>0}$ starting from $f \in \mathcal{P}$ behaves. Section 5 contains concluding remarks.

2. The class $\mathcal{P}$

In this section, we state and prove several results on the class $\mathcal{P}$. The first result of this section is Theorem 2.1, where we prove that each function in $\mathcal{P}$ is nowhere differentiable. The second result of this section is Theorem 2.3, which shows that a function f in $C_p(\mathbb{R})$ belongs to $\mathcal{P}_c$ if and only if f satisfies $(\mathrm{F}1)_c$.

Since we study periodic functions with period 1, we often choose three points $\frac{k}{r^n}$, $\frac{k+y}{r^n}$, $\frac{k+1}{r^n}$ lying in $[0,1]$. For this reason, we prepare the set $\mathbb{A}$ of admissible triplets (n,k,y) as

$$\mathbb{A} := \left\{ (n,k,y) \mid n \in \mathbb{N}_0,\ k \in \{0,1,2,3,\dots,r^n-1\},\ y \in (0,1) \right\}.$$

For any $(n,k,y) \in \mathbb{A}$ we have $\left[\frac{k}{r^n}, \frac{k+1}{r^n}\right] \subset [0,1]$. For a constant $c > 0$, note that $f \in C_p(\mathbb{R})$ belongs to $\mathcal{P}_c$ if and only if (1.2) is satisfied for all $(n,k,y) \in \mathbb{A}$.

We first derive a fundamental inequality for $f \in \mathcal{P}$. For $f \in C_p(\mathbb{R})$, we see by (1.4) that

$$\Delta_{0,0}(y; f) = \frac{-2f(y)}{y(1-y)}, \quad y \in (0,1). \tag{2.1}$$

Thus, for $c > 0$ and $y \in (0,1)$, we have $\Delta_{0,0}(y; f) \le -2c$ if and only if

$$cy(1-y) \le f(y). \tag{2.2}$$

Therefore we see that every $f \in \mathcal{P}_c$ satisfies (2.2) for any $y \in (0,1)$. In particular, when $f \in \mathcal{P}$, we have $f > 0$ in $(0,1)$.

Now, we show that each function in $\mathcal{P}$ is nowhere differentiable. In what follows we write $[z]$ for $z \in \mathbb{R}$ to indicate the largest integer not exceeding z. We denote by $\mathbb{Q}_r$ the set of all rational numbers that can be written as $\frac{k}{r^n}$ for some $n \in \mathbb{N}$ and $k \in \mathbb{Z}$.

Theorem 2.1. *Each function in $\mathcal{P}$ is nowhere differentiable in $\mathbb{R}$.*

Proof. Fix $c > 0$. Suppose that $f \in \mathcal{P}_c$ is differentiable at some point $x \in [0,1]$.

We set $k_n = [r^n x]$ for each $n \in \mathbb{N}$. Also, set $y_n = y$ if $x \in \mathbb{Q}_r$ and $y_n = r^n x - [r^n x]$ if $x \notin \mathbb{Q}_r$, where $y \in (0,1)$ is an arbitrary constant. We claim that $\delta^{\pm}_{n,k_n}(y_n; f) \to f'(x)$ as $n \to \infty$. This gives a contradiction since taking the limit $n \to \infty$ in (1.2) along these k_n and y_n implies that $0 \le -c$.

When $x \in \mathbb{Q}_r$, we have $[r^n x] = r^n x$ for $n \in \mathbb{N}$ large. In fact, since $x \in \mathbb{Q}_r$, there are $n_0 \in \mathbb{N}_0$ and $k_0 \in \mathbb{Z}$ such that $x = \frac{k_0}{r^{n_0}}$, so that $r^n x = k_0 r^{n-n_0} \in \mathbb{N}$ if $n \ge n_0$. For $n \ge n_0$ we find that

$$\begin{aligned}
\delta^{+}_{n,k_n}(y_n; f) &= \frac{f\left(x + \frac{1}{r^n}\right) - f\left(x + \frac{y}{r^n}\right)}{\frac{1-y}{r^n}} \\
&= \frac{f\left(x + \frac{1}{r^n}\right) - f(x)}{\frac{1}{r^n}(1-y)} - \frac{f\left(x + \frac{y}{r^n}\right) - f(x)}{\frac{y}{r^n}\frac{1-y}{y}} \\
&\to \frac{f'(x)}{1-y} - y\frac{f'(x)}{1-y} = f'(x) \quad (n \to \infty).
\end{aligned}$$

In the same manner, we deduce that $\delta^{-}_{n,k_n}(y_n; f) \to f'(x)$ as $n \to \infty$.

Next, let $x \notin \mathbb{Q}_r$. We then have $[r^n x] < r^n x < [r^n x] + 1$ for each $n \in \mathbb{N}$. This implies that $y_n \in (0,1)$ for each $n \in \mathbb{N}$ and that $\frac{[r^n x]}{r^n} \to x$ as $n \to \infty$. Thus,

$$\delta^{+}_{n,k_n}(y_n; f) = \frac{f\left(\frac{[r^n x]+1}{r^n}\right) - f(x)}{\frac{[r^n x]+1}{r^n} - x} \to f'(x) \quad (n \to \infty).$$

Similarly, it follows that $\delta^-_{n,k_n}(y_n; f) \to f'(x)$. This completes the proof. $\square$

Next, we show that a function f in $C_p(\mathbb{R})$ belongs to $\mathcal{P}_c$ if and only if f satisfies $(\mathrm{F1})_c$. To prove this, the following proposition is essential:

PROPOSITION 2.2. *Let* $(n, k, y) \in \mathbb{A}$ *and* $t \in (0, \infty)$. *Then, for any* $f \in C_p(\mathbb{R})$, *inequality* (1.6) *holds if and only if*

$$\Delta_{n,k}(y; f) \le -\frac{1}{t}. \tag{2.3}$$

PROOF. Fix $(n, k, y) \in \mathbb{A}$ and $t \in (0, \infty)$. Let $x_1(n, k, y, t)$ be the unique solution of the equation

$$q_f\Big(t, x; \frac{k + y}{r^n}\Big) = q_f\Big(t, x; \frac{k}{r^n}\Big).$$

By direct calculation,

$$x_1(n, k, y, t) = \frac{k}{r^n} + \frac{y}{2r^n} + t\delta^-_{n,k}(y; f). \tag{2.4}$$

Then, we have

$$\begin{cases} q_f\Big(t, x; \dfrac{k}{r^n}\Big) \le q_f\Big(t, x; \dfrac{k + y}{r^n}\Big), & x \le x_1(n, k, y, t), \\ q_f\Big(t, x; \dfrac{k}{r^n}\Big) > q_f\Big(t, x; \dfrac{k + y}{r^n}\Big), & x_1(n, k, y, t) < x. \end{cases}$$

Similarly, the unique solution $x_2(n, k, y, t)$ of the equation

$$q_f\Big(t, x; \frac{k + y}{r^n}\Big) = q_f\Big(t, x; \frac{k + 1}{r^n}\Big)$$

is given by

$$x_2(n, k, y, t) = \frac{k}{r^n} + \frac{1 + y}{2r^n} + t\delta^+_{n,k}(y; f). \tag{2.5}$$

Furthermore,

$$\begin{cases} q_f\Big(t, x; \dfrac{k + y}{r^n}\Big) \ge q_f\Big(t, x; \dfrac{k + 1}{r^n}\Big), & x_2(n, k, y, t) \le x, \\ q_f\Big(t, x; \dfrac{k + y}{2r^n}\Big) < q_f\Big(t, x; \dfrac{k + 1}{r^n}\Big), & x < x_2(n, k, y, t). \end{cases}$$

Then, a geometrical investigation shows that inequality (1.6) holds if and only if

$$x_1(n, k, y, t) \ge x_2(n, k, y, t). \tag{2.6}$$

By (2.4) and (2.5), we see that inequality (2.6) holds if and only if

$$\delta^-_{n,k}(y;f) - \delta^+_{n,k}(y;f) \geq \frac{1}{2r^n t}.$$

The desired inequality follows immediately from (1.4). $\square$

Now, we state the second result of this section.

THEOREM 2.3. *Let* $f \in C_p(\mathbb{R})$ *and let* $c > 0$ *be a constant. Then,* f *satisfies* $(\mathrm{F1})_c$ *if and only if* $f \in \mathcal{P}_c$.

PROOF. Assume first that $f \in \mathcal{P}_c$. Fix $(n,k,y) \in \mathbb{A}$ and $t \geq \frac{1}{2cr^n}$ arbitrarily. By (1.3) and (1.4), we have

$$\Delta_{n,k}(y;f) \leq -2cr^n \leq -\frac{1}{t},$$

and so (1.6) holds by Proposition 2.2. Thus we see that f satisfies $(\mathrm{F1})_c$.

Next, assume that $(\mathrm{F1})_c$ holds. Then, by Proposition 2.2, we see that

$$\Delta_{n,k}(y;f) \leq -\frac{1}{t}$$

for all $(n,k,y) \in \mathbb{A}$ and $t \geq \frac{1}{2cr^n}$. Letting $t = \frac{1}{2cr^n}$, we conclude that $f \in \mathcal{P}_c$. $\square$

3. The functions U_ψ and $\mathcal{P}$

In this section, we give sufficient conditions on $\psi \in C_p(\mathbb{R})$ in order that $U_\psi \in \mathcal{P}$, where U is the operator defined by (1.7). The results enable us to generate a large number of functions in $\mathcal{P}$ through the explicit formula (1.7). We also give some examples of $\psi \in C_p(\mathbb{R})$ for which $U_\psi \notin \mathcal{P}$.

The following theorem provides a representation of $\Delta_{n,k}(U_\psi;y)$ in terms of $\Delta_{n,k}(\psi;y)$, which plays a crucial role to study if $U_\psi \in \mathcal{P}$. Note that, for every $\psi \in C_p(\mathbb{R})$, we have $U_\psi \in C_p(\mathbb{R})$ and $U_\psi(0) = 0$ by the definition of U_ψ.

THEOREM 3.1. *Let* $\psi \in C_p(\mathbb{R})$. *Then,* (1.8) *holds for each* $(n,k,y) \in \mathbb{A}$. *When* $n = 0$, *the first term of the right-hand side of* (1.8) *is interpreted as* 0.

PROOF. Let $(n,k,y) \in \mathbb{A}$. When $n = 0$, we have $k = 0$, so that (1.8) follows from (2.1) since $U_\psi(0) = 0$. If $n \geq 1$, then

$$U_\psi\Big(\frac{k+y}{r^n}\Big) - \sum_{j=0}^{n-1} \frac{1}{r^j}\psi\Big(\frac{k+y}{r^{n-j}}\Big)$$

$$= \sum_{j=n}^{\infty} \frac{1}{r^j}\psi\big(r^{j-n}(k+y)\big) = \sum_{j=n}^{\infty} \frac{1}{r^j}\psi\big(r^{j-n}y\big) = \frac{1}{r^n}U_\psi(y).$$

This is valid even for $y=0$ and $y=1$. Since $U_\psi(0)=U_\psi(1)=0$, we have

$$U_\psi\Big(\frac{k}{r^n}\Big) = \sum_{j=0}^{n-1}\frac{1}{r^j}\psi\Big(\frac{k}{r^{n-j}}\Big), \quad U_\psi\Big(\frac{k+1}{r^n}\Big) = \sum_{j=0}^{n-1}\frac{1}{r^j}\psi\Big(\frac{k+1}{r^{n-j}}\Big).$$

We therefore have

$$\Delta_{n,k}(y;U_\psi) = 2r^n\Bigg[\frac{U_\psi\big(\frac{k+1}{r^n}\big) - U_\psi\big(\frac{k+y}{r^n}\big)}{\frac{1-y}{r^n}} - \frac{U_\psi\big(\frac{k+y}{r^n}\big) - U_\psi\big(\frac{k}{r^n}\big)}{\frac{y}{r^n}}\Bigg]$$

$$= 2r^n\Bigg[\frac{\sum_{j=0}^{n-1}\frac{1}{r^j}\big(\psi\big(\frac{k+1}{r^{n-j}}\big) - \psi\big(\frac{k+y}{r^{n-j}}\big)\big) - \frac{1}{r^n}U_\psi(y)}{\frac{1-y}{r^n}}$$

$$- \frac{\sum_{j=0}^{n-1}\frac{1}{r^j}\big(\psi\big(\frac{k+y}{r^{n-j}}\big) - \psi\big(\frac{k}{r^{n-j}}\big)\big) + \frac{1}{r^n}U_\psi(y)}{\frac{y}{r^n}}\Bigg]$$

$$= \sum_{j=0}^{n-1} r^j\, 2\, r^{n-j}\Bigg[\frac{\psi\big(\frac{k+1}{r^{n-j}}\big) - \psi\big(\frac{k+y}{r^{n-j}}\big)}{\frac{1-y}{r^{n-j}}} - \frac{\psi\big(\frac{k+y}{r^{n-j}}\big) - \psi\big(\frac{k}{r^{n-j}}\big)}{\frac{y}{r^{n-j}}}\Bigg] - \frac{2r^n}{y(1-y)}U_\psi(y)$$

$$= \sum_{j=0}^{n-1} r^j \Delta_{n-j,k}(y;\psi) - \frac{2r^n}{y(1-y)}U_\psi(y).$$

This implies (1.8). $\square$

Applying Theorem 3.1, we derive some sufficient conditions on $\psi \in C_p(\mathbb{R})$ that guarantee $U_\psi \in \mathcal{P}$. As a typical result, it turns out that $U_\psi \in \mathcal{P}$ if ψ is concave in $[0,1]$ and positive in $(0,1)$.

Let us recall a notion of concavity. A function $g : [0,1] \to \mathbb{R}$ is said to be concave on $[0,1]$ if the inequality

$$\lambda g(x) + (1-\lambda)g(y) \le g(\lambda x + (1-\lambda)y)$$

holds for all $x, y \in [0,1]$ and $\lambda \in [0,1]$. If the reverse inequality holds, then g is said to be convex. For a constant $\alpha \ge 0$, a function g on $[0,1]$ is said to be α-semiconcave on $[0,1]$ if $g(x) + \frac{\alpha}{2}x(1-x)$ is concave on $[0,1]$. This is equivalent to the condition that $g(x) - \frac{\alpha}{2}x^2$ is concave on $[0,1]$.

Remark 3.2. (i) Let $\psi \in C_p(\mathbb{R})$ and assume that ψ is concave on some interval I. Then it is easy to see that $\Delta_{n,k}(y;\psi) \le 0$ for all $(n,k,y) \in \mathbb{N}_0 \times \mathbb{Z} \times (0,1)$ such that $\big[\frac{k}{r^n}, \frac{k+1}{r^n}\big] \subset I$. More generally, if $\psi \in C_p(\mathbb{R})$

is α-semiconcave on I, then we have $\Delta_{n,k}(y;\psi) \le \alpha$ for all $(n,k,y) \in \mathbb{N}_0 \times \mathbb{Z} \times (0,1)$ such that $\left[\frac{k}{r^n}, \frac{k+1}{r^n}\right] \subset I$. The reverse inequalities hold for (α-semi)convex functions.

(ii) If $\psi \in C_p(\mathbb{R})$ is concave on $[0,1]$, then we have $\Delta_{n,k}(y,\psi) \le 0$ for all $(n,k,y) \in \mathbb{A}$ by (i). However, the converse is not true in general: that is, even if $\Delta_{n,k}(y,\psi) \le 0$ for all $(n,k,y) \in \mathbb{A}$, we cannot say that ψ is concave on $[0,1]$. Every $f \in \mathcal{P}$ gives a counterexample to this. In fact, $\Delta_{n,k}(y,f) \le 0$ for all $(n,k,y) \in \mathbb{A}$, but f is never concave on $[0,1]$ by Theorem 2.1, since a concave function must be differentiable almost everywhere.

We first prepare inequalities involving U_ψ and the generalized Takagi function τ_r defined in (1.9). Recall that d is the distance function given by (1.10).

Lemma 3.3. *Let $\psi \in C_p(\mathbb{R})$. Assume that there exists a constant $m > 0$ such that $md(x) \le \psi(x)$ for all $x \in [0,1]$. Then, we have*

$$\frac{mr}{r-1} x(1-x) \le m\tau_r(x) \le U_\psi(x), \quad x \in [0,1]. \tag{3.1}$$

Proof. It follows from our assumption that $md(r^j x) \le \psi(r^j x)$ for all $x \in [0,1]$ and $j \in \mathbb{N}_0$. Thus, $m\tau_r(x) \le U_\psi(x)$ by taking the sum.

It remains to prove that

$$\frac{r}{r-1} x(1-x) \le \tau_r(x), \quad x \in [0,1]. \tag{3.2}$$

Let

$$F(x) = d(x) + \frac{1}{r} d(rx), \quad G(x) = \frac{r}{r-1} x(1-x), \quad x \in [0,1].$$

Since $F \le \tau_r$, it suffices to show that $G(x) \le F(x)$ for $x \in [0,1]$. As F and G are symmetric about $x = \frac{1}{2}$, we may assume that $x \in [0, \frac{1}{2}]$. Note that

$$F(x) = 2x \ \left(0 \le x \le \frac{1}{2r}\right), \ F(x) = \frac{1}{r} \ \left(\frac{1}{2r} \le x \le \frac{1}{r}\right), \ F(x) \ge x \ \left(\frac{1}{r} \le x \le \frac{1}{2}\right).$$

When $0 \le x \le \frac{1}{r}$, we have

$$G(x) \le G\Big(\frac{1}{r}\Big) = \frac{1}{r}, \quad G(x) \le \frac{r}{r-1} x(1-0) \le 2x.$$

Thus $G(x) \le F(x)$. Next, let $\frac{1}{r} \le x \le \frac{1}{2}$. Then,

$$G(x) \le \frac{r}{r-1} x\Big(1 - \frac{1}{r}\Big) = x \le F(x).$$

Hence, we conclude (3.2). $\square$

REMARK 3.4. Assume that $\psi \in C_p(\mathbb{R})$ is concave in $[0,1]$ and $\psi > 0$ in $(0,1)$. Then, we have

$$2\psi\Big(\frac{1}{2}\Big)d(x) \le \psi(x), \quad x \in [0,1], \tag{3.3}$$

and thus ψ satisfies the assumption in Lemma 3.3 for $m = 2\psi(\frac{1}{2})$. Indeed, by the concavity of ψ, its graph lies above the segment connecting $(0,\psi(0))$ and $(\frac{1}{2},\psi(\frac{1}{2}))$ and the segment connecting $(\frac{1}{2},\psi(\frac{1}{2}))$ and $(1,\psi(1))$. This shows (3.3) since $\psi(0) = \psi(1) = 0$.

Now, we state the main result of this section.

THEOREM 3.5. *Let $\psi \in C_p(\mathbb{R})$. Assume that there exist two constants $m > 0$ and $\alpha \ge 0$ such that*

(i) *$md(x) \le \psi(x)$ for all $x \in [0,1]$.*

(ii) *$\Delta_{n,k}(y;\psi) \le \alpha$ for all $(n,k,y) \in \mathbb{A}$.*

If $2mr > \alpha$, then $U_\psi \in \mathcal{P}_c$ with $c = \frac{2mr-\alpha}{2(r-1)}$.

PROOF. Let us derive $\Delta_{n,k}(y;U_\psi) \le -2cr^n$ for a fixed $(n,k,y) \in \mathbb{A}$. From Lemma 3.3 it follows that

$$-\frac{2r^n}{y(1-y)}U_\psi(y) \le -\frac{2mr^{n+1}}{r-1}.$$

If $n = 0$, we see by (2.1) that $\Delta_{0,0}(y;U_\psi) \le -\frac{2mr}{r-1} < -2c$. For $n \ge 1$ we have

$$\sum_{j=0}^{n-1} r^j \Delta_{n-j,k}(y;\psi) \le \sum_{j=0}^{n-1} r^j \alpha = \alpha \cdot \frac{r^n - 1}{r-1} < \alpha \cdot \frac{r^n}{r-1}.$$

Thus, by (1.8)

$$\Delta_{n,k}(y;U_\psi) \le \alpha \cdot \frac{r^n}{r-1} - \frac{2mr^{n+1}}{r-1} = -2cr^n,$$

which proves the theorem. $\square$

Let us denote by E the set of $\psi \in C_p(\mathbb{R})$ satisfying (i) and (ii) in Theorem 3.5 for some $m > 0$ and $\alpha \ge 0$ with $2mr > \alpha$. Theorem 3.5 asserts that $U_\psi \in \mathcal{P}$ for every $\psi \in E$. We give typical classes that are included in E.

PROPOSITION 3.6. *The set E includes the following two sets*:

(1) $SC_0 := \big\{\psi \in C_p(\mathbb{R}) \mid \psi$ *is concave in* $[0,1]$ *and* $\psi > 0$ *in* $(0,1)\big\}$.

(2) $\mathcal{P}$.

PROOF. (1) Let $\psi \in SC_0$. It follows from Remark 3.4 that ψ satisfies Theorem 3.5(i) for $m = 2\psi(\frac{1}{2})$, while we can take $\alpha = 0$ in Theorem 3.5(ii) by Remark 3.2(i). Since $2mr > \alpha$, we have $\psi \in E$ and $U_\psi \in \mathcal{P}_c$ with $c = \frac{2r}{r-1}\psi(\frac{1}{2})$.

(2) Let $\psi \in \mathcal{P}_c$ for some $c > 0$. By (2.2), we can take $m = c$ in Theorem 3.5(i). We also take $\alpha = 0$ in Theorem 3.5(ii) by the definition of $\mathcal{P}_c$. Since $2mr > \alpha$, we conclude that $\psi \in E$ and $U_\psi \in \mathcal{P}_{c'}$ with $c' = \frac{cr}{r-1}$. $\square$

Note that the two sets SC_0 and $\mathcal{P}$ above are mutually disjoint, since a concave function is differentiable almost everywhere. Also, if ψ belongs to $\mathcal{P}$, then U_ψ also belongs to $\mathcal{P}$ since $\mathcal{P} \subset E$ by Proposition 3.6(2). Thus, $\mathcal{P}$ is an invariant set under the operator U.

Remark 3.7. By Proposition 3.6(1) and its proof, we see that the generalized Takagi function τ_r belongs to $\mathcal{P}_c$ with $c = \frac{r}{r-1}$ since $d \in C_p(\mathbb{R})$ is concave in $[0,1]$ and $d(\frac{1}{2}) = \frac{1}{2}$. In particular, the Takagi function τ_2 is in $\mathcal{P}_2$ for $r = 2$.

If $\psi \in C_p(\mathbb{R})$ is α-semiconcave in $[0,1]$, then (ii) in Theorem 3.5 is fulfilled by Remark 3.2(i). However, (i) does not hold in general even if $\psi > 0$ in $(0,1)$. One may then wonder if U_ψ belongs to $\mathcal{P}$ for ψ in

$$SC_\alpha := \left\{\psi \in C_p(\mathbb{R}) \mid \psi \text{ is } \alpha\text{-semiconcave in } [0,1] \text{ and } \psi > 0 \text{ in } (0,1)\right\}$$

with $\alpha > 0$. The answer is no. Besides, U_ψ for $\psi \in SC_\alpha$ does not necessarily possess nowhere differentiable character. Namely, for every $\alpha > 0$ there are the following three examples of $\psi \in SC_\alpha$:

(A) $U_\psi \in \mathcal{P}$ and $\psi \notin SC_0$.

(B) $U_\psi \notin \mathcal{P}$ and U_ψ is nowhere differentiable in $[0,1]$.

(C) $U_\psi \notin \mathcal{P}$ and $U_\psi \in C^\infty((0,1))$.

Let us give an example of $\psi \in SC_\alpha$ satisfying each (A)–(C).

Example 3.8. For constants $a, b > 0$, let $\psi_0 = ad + bd^2 \in C_p(\mathbb{R})$. Then, ψ_0 is not concave on $[0,1]$ but $2b$-semiconcave on $[0,1]$. In addition, when $ar > b$, $U_{\psi_0} \in \mathcal{P}$. We thus obtain a function satisfying (A).

Indeed, since $\psi_0(x) = ax + bx^2$ on $[0, \frac{1}{2}]$, ψ_0 is not concave on $[0,1]$. Also, we have $\psi_0(x) + bx(1-x) = (a+b)d(x)$ on $[0,1]$, and so ψ_0 is $2b$-semiconcave on $[0,1]$. Finally, since $\psi_0 \geq ad$ on $[0,1]$, we can take $m = a$ and $\alpha = 2b$ in Theorem 3.5. Thus, $\psi_0 \in E$ and so $U_{\psi_0} \in \mathcal{P}$.

This example also shows that $SC_0 \cup \mathcal{P} \subsetneq E$.

Let us next discuss the example of (B). Let $\theta \in C_p(\mathbb{R})$ be a function such that

$$\theta(x) = x^2 \text{ for } x \in \left[0, \frac{1}{r}\right], \quad \theta \in C^2(\mathbb{R}), \quad \theta > 0 \text{ in } (0,1).$$

We now apply [14, Theorem 3.1], which asserts that, if $\psi \in C_p(\mathbb{R}) \cap C^1(\mathbb{R})$ and ψ' is Hölder continuous in $\mathbb{R}$, then U_ψ is nowhere differentiable in $\mathbb{R}$.

Since θ satisfies these conditions, we deduce that U_θ is nowhere differentiable in $\mathbb{R}$. However, U_θ does not belong to $\mathcal{P}$ as shown below.

THEOREM 3.9. *$\Delta_{n,0}(\frac{1}{r}; U_\theta) = -\frac{2}{r-1}$ for each $n \in \mathbb{N}_0$. Thus, $U_\theta \notin \mathcal{P}$.*

PROOF. Let $n \in \mathbb{N}_0$. We have

$$U_\theta\Big(\frac{1}{r}\Big) = \sum_{j=0}^{\infty} \frac{1}{r^j}\theta(r^{j-1}) = \theta(r^{-1}) = \frac{1}{r^2}.$$

Thus,

$$\frac{2r^n}{y(1-y)}U_\theta(y)\Big|_{y=\frac{1}{r}} = \frac{2r^n}{r-1}.$$

When $n = 0$, this and (2.1) show that $\Delta_{0,0}(\frac{1}{r}; U_\theta) = -\frac{2}{r-1}$. Let $n \geq 1$. Since $\Delta_{m,0}\big(\frac{1}{r}, \theta\big) = 2$ for any $m \in \mathbb{N}$, it follows from Theorem 3.1 that

$$\begin{aligned}\Delta_{n,0}\Big(\frac{1}{r}; U_\theta\Big) &= \sum_{j=0}^{n-1} r^j \Delta_{n-j,0}\left(\frac{1}{r}; \theta\right) - \frac{2r^n}{y(1-y)}U_\theta(y)\Big|_{y=\frac{1}{r}} \\ &= 2\sum_{j=0}^{n-1} r^j - \frac{2r^n}{r-1} = -\frac{2}{r-1}. \quad \square\end{aligned}$$

Let $\alpha > 0$. Since $\theta \in C^2(\mathbb{R})$, we have $\varepsilon\theta \in SC_\alpha$ if $\varepsilon > 0$ is sufficiently small. Also, it is easy to see that $U_{\varepsilon\theta}$ is still nowhere differentiable and $U_{\varepsilon\theta} \notin \mathcal{P}$. We thus obtain a function satisfying (B).

EXAMPLE 3.10. Let us give an example of a function satisfying (C). Define

$$\psi(x) = |\sin(\pi x)| - \frac{1}{r}\,|\sin(\pi r x)| \in C_p(\mathbb{R}).$$

Then, by the definition of U_ψ, we easily see that $U_\psi(x) = |\sin(\pi x)| \in C_p(\mathbb{R})$. Thus $U_\psi \in C^\infty((0,1))$ and in particular $U_\psi \notin \mathcal{P}$ as required in (C).

Let us next check that $\psi \in SC_\alpha$ for some $\alpha > 0$. The positivity of ψ in $(0,1)$ follows from a straightforward calculation, so we omit the proof. Next, since functions $\frac{1}{r}\sin(\pi r x)$ and $-\frac{1}{r}\sin(\pi r x)$ are semiconcave, the minimum $-\frac{1}{r}\,|\sin(\pi r x)|$ of them is also semiconcave. Therefore, ψ being the sum of two semiconcave functions in $[0,1]$ is semiconcave in $[0,1]$.

Similarly to the previous example, for a given $\alpha > 0$, we have $\varepsilon\psi \in SC_\alpha$ if $\varepsilon > 0$ is sufficiently small. A function satisfying (C) has thus been obtained.

We conclude this section by studying if a Weierstrass type function belongs $\mathcal{P}$.

Example 3.11. The famous Weierstrass function W is given by

$$W(x) = \sum_{j=0}^{\infty} a^j \rho(b^j x), \quad \rho(x) = \cos(\pi x),$$

where $a \in (0,1)$ and b is an odd integer with $ab > 1 + \frac{3\pi}{2}$. Note that ρ is continuous and periodic on $\mathbb{R}$ with period 2 and $\rho(0) \neq 0$. Since we consider functions ψ in $C_p(\mathbb{R})$ with $\psi(0) = 0$, we study U_η for $\eta(x) = \sin(2\pi x) \in C_p(\mathbb{R})$ instead of W. By Hardy [11], it is shown that U_η is nowhere differentiable. We also remark that η possesses a balance of convexity and concavity properties, since it is concave on $[0, \frac{1}{2}]$ and convex on $[\frac{1}{2}, 1]$.

We claim that U_η does not belong to $\mathcal{P}$. In fact, noting that $\eta(\frac{r^j}{2}) = \sin(\pi r^j) = 0$ for all $j \in \mathbb{N}_0$, we see that $U_\eta(\frac{1}{2}) = 0$ by the definition of U_η. This implies that $U_\eta \notin \mathcal{P}$ since, if $U_\eta \in \mathcal{P}$, we have $U_\eta > 0$ in $(0,1)$ by (2.2).

4. The behavior of $\{H_t f\}_{t>0}$ for $f \in \mathcal{P}$

In this section we consider the behavior of the Hamilton–Jacobi flow $\{H_t f\}_{t>0}$ for $f \in \mathcal{P}$, where $H_t f$ is the function defined by (1.11). It is known that $H_t f$ belongs to $C_p(\mathbb{R})$ and uniformly approximates f as t goes to 0 (see [4, Ch. 3.5]). Also, $H_t f$ is a unique viscosity solution of the initial value problem of the Hamilton–Jacobi equation:

$$\begin{cases} u_t(t,x) + \dfrac{1}{2}(u_x(t,x))^2 = 0, & (t,x) \in (0,\infty) \times \mathbb{R}, \\ u(0,x) = f(x), & x \in \mathbb{R} \end{cases} \tag{4.1}$$

(cf. [6]). Here, $u_t(t,x) = \frac{\partial u}{\partial t}(t,x)$ and $u_x(t,x) = \frac{\partial u}{\partial x}(t,x)$.

First of all, we prove that the range of z in (1.11) can be reduced.

Lemma 4.1. *Let $f \in C_p(\mathbb{R})$. If $f(z) \geq 0$ for all $z \in [0,1]$, then*

$$H_t f(x) = \min_{z \in [0,1]} q_f(t,x;z), \quad (t,x) \in (0,\infty) \times [0,1]. \tag{4.2}$$

Proof. Fix $(t,x) \in (0,\infty) \times [0,1]$. We first let $z < 0$. Since $f(z) \geq 0$, the geometrical investigation implies that $q_f(t,x;z) > q_f(t,x;0)$. Thus, the minimum in (1.11) is never attained for $z < 0$. The same arguments show that $z > 1$ is not a minimizer of (1.11), and hence (4.2) holds. $\square$

Now, we state the main result of this section.

Theorem 4.2. *Let $f \in \mathcal{P}_c$ for $c > 0$. Then, the following holds:*

(F2)$_c$ *For all* $n \in \mathbb{N}_0$,

$$H_t f(x) = \min_{k \in \{0,1,2,3,\dots,r^n\}} q_f\Big(t, x; \frac{k}{r^n}\Big), \quad (t,x) \in \Big[\frac{1}{2cr^n}, \infty\Big) \times [0,1]. \tag{4.3}$$

PROOF. This is a consequence of (4.2) and (F1)$_c$. In fact, since $f \in \mathcal{P}_c$ satisfies the inequality $f(z) \geq 0$ for $z \in [0,1]$ by (2.2), we have (4.2), while Theorem 2.3 guarantees that (F1)$_c$ holds. □

By Theorem 4.2 we see that $H_t f$ with $f \in \mathcal{P}_c$ is a piecewise quadratic function in $[0,1]$ for all $t > 0$ and that the x-coordinate of each vertex of the parabolas making up $H_t f$ always belongs to $\mathbb{Q}_r$. In general it is known that $H_t f$ for $f \in C_p(\mathbb{R})$ is $\frac{1}{2t}$-semiconcave in $[0,1]$ for all $t > 0$. For $f \in \mathcal{P}_c$ we deduce from (4.3) that

$$H_t f(x) - \frac{x^2}{2t} = \frac{1}{2t} \min_{k \in \{0,1,2,3,\dots,r^n\}} \Big[-\frac{2k}{r^n} x + \Big(\frac{k}{r^n}\Big)^2 + f\Big(\frac{k}{r^n}\Big)\Big]$$

for $(t,x) \in [\frac{1}{2cr^n}, \infty) \times [0,1]$. This shows that $H_t f(x) - \frac{x^2}{2t}$ is not only concave but also piecewise linear in $[0,1]$.

One may ask if, conversely, a function $f \in C_p(\mathbb{R})$ satisfying (F2)$_c$ for some $c > 0$ is nowhere differentiable. We have no complete answer to this question at the moment. However, we can prove that such an f is non-differentiable on a dense subset of $\mathbb{R}$. In general this is not enough to infer that it is nowhere differentiable, as is shown by the Riemann function. Indeed, let R be the Riemann function defined by

$$R(x) = \sum_{j=1}^{\infty} \frac{\sin(\pi j^2 x)}{j^2}, \quad x \in \mathbb{R}.$$

Set

$$F := \Big\{ \frac{2A+1}{2B+1} \,\Big|\, A, B \in \mathbb{Z} \Big\} \ (\subset \mathbb{Q}).$$

By Hardy [11] and Gerver [9,10], it is shown that R is differentiable on the set F and that R is non-differetiable on the set $(\mathbb{R} \setminus \mathbb{Q}) \cup (\mathbb{Q} \setminus F)$.

THEOREM 4.3. *Let* $f \in C_p(\mathbb{R})$ *and let* $c > 0$ *be a constant. Assume that* (F2)$_c$ *holds. Then, there exists a dense subset of the interval* $[0,1]$ *such that* f *is non-differentiable at each point of this subset.*

We denote by $D^- f(x)$ the *subdifferential* of f at x, that is, the set of $\phi'(x)$ such that $\phi \in C^1$ near x and $f - \phi$ has a local minimum at x. We list basic properties of the subdifferential used in the proof of Theorem 4.3. Let $f \in C_p(\mathbb{R})$ and $x \in \mathbb{R}$.

(I) If f is differentiable at x, then $D^-f(x) = \{f'(x)\}$ ([2, Lemma II.1.8(b)]);

(II) Let $t > 0$ and choose $z \in \mathbb{R}$ such that $H_t f(x) = q_f(t, x; z)$. Then $\frac{x-z}{t} \in D^-f(z)$ ([2, Lemma II.4.12(iii)]).

Proof of Theorem 4.3. Fix $x_0 \in (0,1)$ and $\varepsilon > 0$, and let $I = (x_0 - \varepsilon, x_0 + \varepsilon)$. We prove that there is some $z \in I$ such that f is not differentiable at z. We may assume that $\varepsilon < \min\{x_0,\, 1 - x_0\}$, so that $I \subset [0,1]$. Let $t \in (0, \frac{\varepsilon^2}{2M})$, with $M > 0$ the oscillation of f, that is, $M = \sup_{\mathbb{R}} f - \inf_{\mathbb{R}} f$. Since $H_t f$ is represented by (4.3) with n such that $t \geq \frac{1}{2cr^n}$, there exists some $\delta \in (0, \varepsilon)$ such that $H_t f = q_f(t, \cdot; z)$ in $J := [x_0 - \delta, x_0] \subset I$ with $z = \frac{k}{r^n}$ for some $k \in \{0, 1, 2, 3, \dots, r^n\}$. The choice of t then guarantees that $z \in I$. Indeed, we have

$$f(x_0) \geq H_t f(x_0) = f(z) + \frac{1}{2t}(x_0 - z)^2,$$

and hence $(x_0 - z)^2 \leq 2t(f(x_0) - f(z)) \leq 2Mt < \varepsilon^2$, that is, $z \in I$.

It follows from (II) that $\frac{x-z}{t} \in D^-f(z)$ for all $x \in J$. This implies that $\left[\frac{x_0-\delta-z}{t}, \frac{x_0-z}{t}\right] \subset D^-f(z)$: that is, $D^-f(z)$ is not a singleton. Hence we conclude by (I) that f is not differentiable at z. $\square$

Remark 4.4. The above proof actually shows that the dense set we found is a subset of $\mathbb{Q}_r$.

5. Concluding remark

We conclude this paper by mentioning another possible definition of $\mathcal{P}_c$. Let us define $\mathcal{P}'_c$ as the set of all $f \in C_p(\mathbb{R})$ such that there exists an infinite subset $\mathbb{N}' \subset \mathbb{N}_0$ such that f satisfies (1.2) for all $(n, k, y) \in \mathbb{A}$ with $n \in \mathbb{N}'$. In other words, we require (1.2) only for some subsequence of $n \in \mathbb{N}_0$. Even if this generalized class $\mathcal{P}'_c$ is used, one can easily see that Theorem 2.3 is obtained in a suitable sense. Namely, $f \in \mathcal{P}'_c$ if and only if f satisfies $(\mathrm{F1})_c$ with "for all $n \in \mathbb{N}'$" instead of "for all $n \in \mathbb{N}_0$". The proof is almost the same as before.

Moreover, Theorem 2.1 is true for a function in $\mathcal{P}' := \bigcup_{c>0} \mathcal{P}'_c$ since the proof still works when taking the limit along $\mathbb{N}'$. The formula (1.7) still gives many examples of functions in $\mathcal{P}'$. Though $\mathcal{P}'$ provides a more general class than $\mathcal{P}$, there are, however, no essential changes or difficulties in the proofs. For this reason, for simplicity of presentation, the authors decided to give results for $\mathcal{P}_c$ instead of $\mathcal{P}'_c$.

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