THE VANISHING DISCOUNT PROBLEM FOR HAMILTON–JACOBI EQUATIONS IN THE EUCLIDEAN SPACE

HITOSHI ISHII* AND ANTONIO SICONOLFI

Abstract. We study the asymptotic behavior of the solutions to a family of discounted Hamilton-Jacobi equations, posed in $\mathbb{R}^N$, when the discount factor goes to zero. The ambient space being noncompact, we introduce an assumption implying that the Aubry set is compact and there is no degeneracy at infinity. Our approach is to deal not with a single Hamiltonian and Lagrangian but with the whole space of generalized Lagrangians, and then to define via duality minimizing measures associated with both the corresponding ergodic and discounted equations. The asymptotic result follows from the convergence properties of these measures concerning the narrow topology. We use as duality tool a separation theorem in locally convex Hausdorff spaces, and we use the strict topology in the space of the bounded generalized Lagrangians as well.

Contents

Key words. vanishing discount problem, Hamilton–Jacobi equations, ergodic problem.
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* Corresponding author.

1. Introduction

We study the asymptotic behavior, as the discount factor $\lambda > 0$ goes to 0, of the viscosity solutions to the Hamilton–Jacobi equations

$$\lambda\, u + H(x, Du) = c$$

posed in the Euclidean space $\mathbb{R}^N$. Here c is the so–called critical value defined as

$$c = \inf\{a \mid H = a \text{ admits subsolutions in } \mathbb{R}^N\}.$$

Under our assumption, this quantity is actually finite and is a minimum.

Our output provides an extension to the noncompact setting of the selection principle, first established in the compact case in [10]. It asserts that the whole family of solutions of the discounted problems, which are uniquely solved if the ambient space is compact, converges to a distinguished solution of the ergodic limit equation

$$H(x, Du) = c.$$

The latter has instead multiple solutions, parametrized by the Aubry set, denoted by $\mathcal{A}$, which is, roughly speaking, the set of points where is concentrated the obstruction of getting subsolutions to $H = a$, for $a < c$, see Appendix A.

We assume the Hamiltonian $H(x,p)$ from $\mathbb{R}^N \times \mathbb{R}^N$ to $\mathbb{R}$ to be continuous in both arguments, and convex, coercive in the momentum variable, locally uniformly in x. Since H can be modified for p of large norm, without affecting the analysis, a superlinear growth of H as $|p| \to +\infty$, can be in addition postulated without loss of generality. See Proposition 3.6. A Lagrangian, denoted by L, can be then defined via the Legendre-Fenchel transform.

We have one more key condition, see (A3)/(A3'), to specifically deal with the lack of compactness of the ambient space. It implies that the Aubry set is nonempty, compact and that the intrinsic distance associated with $H = c$, see Appendix A, is equivalent to the Euclidean one at infinity. Loosely speaking, the latter condition means that there is no Aubry set at infinity. In the case where the Hamiltonian is of the form

$$H(x,p) = |p| - f(x), \qquad \text{with } f \text{ continuous potential},$$

this corresponds to requiring the infimum of f to be not attained at infinity.

Under our assumption, due to the noncompactness, either of the discounted equations does not anymore single out a unique solution, see example in Section 3, and the Aubry set fails to be a uniqueness set for the critical equation.

This fact leads to single out a special type of solutions to the critical equation in $\mathbb{R}^N$, named weak KAM solutions, and defined as the functions u for which

$$u(x) = \min\{u(y) + S_0(y,x) \mid y \in \mathcal{A}\} \qquad \text{for any } x,$$

where S_0 is the intrinsic (semi)distance associated with the critical equation. In our setting, they are characterized among all the critical solutions, see Theorem 4.7, by the property of being bounded from below. By definition, $\mathcal{A}$ is then a uniqueness set for the weak KAM solutions.

Regarding the discounted equations, we consider the maximal solution obtained as the pointwise supremum of the family of all subsolutions. they possess, like the weak KAM solutions, the crucial property of being bounded from below, see Section 3.

Our main result can, therefore, be stated as follows:

Theorem. *The whole family of maximal solutions to the discounted equations converges, as the discount factor λ goes to zero, locally uniformly to a distinguished weak KAM solution of the limit ergodic equation.*

As in [10], we derive the asymptotic behavior of solutions from weak convergence of suitable associated measures. Our method, however, is rather different. The relevant measures are not defined as occupational measures on curves, and we seldom employ representation formulae for solutions or properties of curves in the space of the state variable.

Our approach instead relies on some functional analysis and appropriate duality principles between spaces of generalized Lagrangians and spaces of measures. We define in this way minimizing measures, named after Mather, associated with both the ergodic and discounted equations.

For the ergodic equation, they coincide with the classical Mather measures given when the Hamiltonian is, in addition, Tonelli and the ambient space compact. We also recover the relevant property that the closure of the union of the supports of such measures is a uniqueness set for the weak KAM solutions, see Section 11.

Our procedure is close in spirit to Evans's interpretation of Mather theory in terms of complementarity problems, see [11], [12], and also [16]. We think that this alternative approach is interesting per se and can handle to extend the asymptotic result to a more general setting, for instance in the case of fully nonlinear second-order equations (see [18, 19] and also [20] for such generalizations).

The idea of performing some duality between generalized Lagrangians and measures, to study the asymptotic of the solution to discounted equations, has been introduced in [18, 19]. The authors, however, use as duality tool the Sion minimax Theorem, while we instead employ a separation result for convex subsets in locally convex Hausdorff space, see Appendix B.

It implies that the normal cone at any element of the boundary of a convex set with nonempty interior has nonzero elements. We actually find the Mather measures as elements, up to change of sign, of the normal cone at L of suitable convex sets in the space of generalized Lagrangians. We need for this an appropriate topological frame.

We consider the space of bounded continuous functions from $\mathbb{R}^{2N}$ to $\mathbb{R}^N$ equipped with the so–called strict topology, see Appendix B. In this case a nice generalization of Riesz representation theorem holds true, namely the topological dual is the space of signed Borel measures with bounded variation with the narrow topology as corresponding weak star topology.

To implement our method, some effort has been put into constructing convex subsets of the space of bounded generalized Lagrangians with nonempty interior and $L \wedge M$ as boundary point, for suitable constants M. To this aim, we have preliminarily proved some localization results for both the ergodic and discounted equations, see Section 5 and Propositions 7.3, 8.2.

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2. Setting

Given $R > 0$, we denote by B_R the open ball of $\mathbb{R}^N$ or $\mathbb{R}^N \times \mathbb{R}^N$ centered at 0 with radius R, we write insead $B(x_0, R)$ if the center is at x_0. Given two elements x, y of $\mathbb{R}^N$, we write $x \cdot y$ to indicate their scalar product. For any subset E of a topological space, we denote by $\overline{E}$, $\operatorname{int} E$, ∂E its closure interior and boundary, respectively. If u is a locally Lipschitz continuous function from $\mathbb{R}^N$ to $\mathbb{R}$ we define its (Clarke) generalized gradient at some point x via

$$\partial u(x) = \operatorname{co}\{\lim_i Du(x_i) \mid x_i \to x,\ x_i \text{ differentiability points of } u\}$$

where co stands for convex hull.

We consider a Hamiltonian $H : \mathbb{R}^N \times \mathbb{R}^N \to \mathbb{R}$ satisfying the following conditions

(A1) $\qquad H \in C(\mathbb{R}^N \times \mathbb{R}^N)$.

(A2) $\qquad H$ is convex and coercive, that is, for any $x \in \mathbb{R}^N$, the function $H(x, \cdot)$ is convex in $\mathbb{R}^N$ and for any $R > 0$,

$$\lim_{R\to\infty} \inf\{H(x,p) \mid x \in B_R,\ p \in \mathbb{R}^N \setminus B_R\} = +\infty.$$

(A3) $\qquad$ There exists $\varepsilon > 0$ such that

$$\limsup_{|x|\to+\infty} \max_{p\in B_\varepsilon} H(x,p) < \max_{x\in\mathbb{R}^N} \min_{p\in\mathbb{R}^N} H(x,p)$$

We further consider the discount problem for the Hamilton-Jacobi equation

$$\lambda u + H(x, Du) = c \quad \text{in} \ \ \mathbb{R}^N, \tag{DP}$$

and the associated ergodic problem

$$H(x, Du) = c \quad \text{in} \ \ \mathbb{R}^N,$$

where $\lambda > 0$ is a given constant, and

$$c = \inf\{a \mid H(x, Du) = a \text{ admits subsolutions in } \mathbb{R}^N\} \tag{1}$$

is the so–called critical value of H. We will show that in our setting it is finite and is actually a minimum. Here and in what follows, the terms solutions, subsolutions, and supersolutions of Hamilton-Jacobi equations must be understood in the viscosity sense. We henceforth suppress the adjective word *viscosity*. We record for later use a weaker version of (A3):

(A3') $\qquad$ There exists $\varepsilon > 0$ such that

$$\limsup_{|x|\to+\infty} \max_{p\in B_\varepsilon} H(x,p) < c.$$

It is clear that the critical value is greater than or equal to the right hand–side of (A3). The advantage of the formulation (A3) is that that the quantity in the right hand–side is *observable* for any given Hamiltonian while the critical value could be in general not easy to compute.

We can assume by normalization that $c = 0$ and, consequently, the ergodic problem is stated as

$$\text{(EP)} \qquad H[u] = 0 \quad \text{in } \mathbb{R}^N.$$

Here we mean that the problems involving the Hamiltonian H are normalized so as to c when H is replaced by $H - c$. Note that if H satisfies (A1)–(A3) and $c \in \mathbb{R}$, then $H - c$ satisfies (A1)–(A3) as well. We interpret (DP) as an approximation procedure for (EP) when λ is sent to 0.

The first author studied, under assumptions to be compared with (A1)–(A3), the large time behavior of the Hamilton-Jacobi equation in $\mathbb{R}^N$ in [17].

Condition (A3) implies:

Proposition 2.1. *Assume that u is a subsolution of $H[u] = a$ for some $a \in \mathbb{R}$ and K a compact subset of $\mathbb{R}^N$. Then there exists a subsolution of the same equation, constant outside some compact subset, coinciding with u on K.*

Proof. Due to (A3) and $H = a$ admitting subsolutions, we have

$$a > \limsup_{|x| \to +\infty} \max_{p \in B_\varepsilon} H(x,p) \quad \text{for some } \varepsilon > 0,$$

we can therefore take a compact subset C with

$$\operatorname{int} C \supset K \cup \left\{ x \mid \max_{p \in B_\varepsilon} H(x,p) \geq a \right\}.$$

We set $\phi(x) = -\frac{\varepsilon}{2} |x|$ and select $b > 0$ with $\min_C (\phi + b) > \max_C u$. Because of the definition of C, the function

$$v = \min\{\phi + b, u\}$$

is a subsolution of $H = a$ with $v = u$ on K and

$$\lim_{|x| \to +\infty} v = -\infty.$$

The function

$$w_0 = \max\{v, \min_C u\},$$

satisfies the assertion. $\square$

3. Maximal subsolutions of (DP)

The first result of the section is

Proposition 3.1. *The family of subsolutions to* (DP) *is locally equibounded from above, when λ varies in* $(0, +\infty)$.

A lemma is preliminary

Lemma 3.2. *For each $R > 0$, there exist a constant $C_R > 0$ and a function $\psi_R \in C^1(B_R)$ such that*

$$H[\psi_R] > -C_R \quad \text{in } B_R, \quad \text{and} \quad \lim_{|x| \to R^-} \psi_R(x) = +\infty.$$

Proof. Fix $R > 0$ and choose a function $\psi_R \in C^1(\mathbb{R}^N)$ so that

$$\lim_{|x|\to R^-} \psi_R(x) = +\infty \quad \text{and} \quad \lim_{|x|\to R^-} |D\psi_R(x)| = +\infty.$$

Observe that

$$x \mapsto H(x, D\psi_R(x))$$

is continuous on B_R and that

$$\lim_{|x|\to R^-} H(x, D\psi_R(x)) = +\infty.$$

It is now obvious that

$$x \mapsto H(x, D\psi_R(x))$$

has a minimum in B_R. Thus, for some constant $C_R > 0$,

$$H(x, D\psi_R(x)) \geq -C_R \qquad \text{in} \;\; B_R.$$

□

Proof of Proposition 3.1. Let u be any subsolution of (DP), for some $\lambda > 0$. Fix $R > 0$. According to Lemma 3.2, there are a function $\psi \in C^1(B_R)$ and a constant $b > 0$ such that

$$H(x, D\psi(x)) \geq -b \quad \text{for} \;\; x \in B_R \quad \text{and} \quad \lim_{|x|\to R^-} \psi(x) = +\infty.$$

By adding a constant to ψ if necessary, we may assume that $\psi \geq 0$ in B_R.

Set

$$v(x) = \psi(x) + \lambda^{-1} b \qquad \text{for} \;\; x \in B_R,$$

and note that

$$\lambda v(x) + H(x, Dv(x)) \geq \lambda \, \lambda^{-1} b - b = 0 \qquad \text{for} \;\; x \in B_R. \tag{2}$$

We prove that

$$u \leq v \qquad \text{in} \;\; B_R. \tag{3}$$

By contradiction, we suppose that $\sup_{B_R}(u - v) > 0$. Since

$$\lim_{|x|\to R^-} (u - v)(x) = -\infty,$$

the function $u - v$ has a maximum point at some $x_0 \in B_R$ and hence, by the viscosity property of u

$$\lambda u(x_0) + H(x_0, D\psi(x_0)) \leq 0,$$

which yields, since $u(x_0) > v(x_0)$

$$\lambda v(x_0) + H(x_0, D\psi(x_0)) \leq 0.$$

contradicting (2).

From (3), we get

$$u(x) \leq v(x) \leq \lambda^{-1} b + \|\psi\|_{\infty, B_{R/2}} \qquad \text{for all} \;\; x \in B_{R/2}.$$

This gives the assertion. □

In view of the Perron method and (A2), we directly derive:

Theorem 3.3. *There exists, for each $\lambda > 0$, a maximal viscosity solution u_λ of* (DP), *which is locally Lipschitz continuous.*

From now on, we denote by u_λ, for any $\lambda > 0$, the maximal (sub)solution of (DP).

Lemma 3.4. *The functions u_λ are equibounded from below in $\mathbb{R}^N$ for $\lambda > 0$.*

Proof. By Proposition 2.1 there exists a compactly supported subsolution w of (EP). Let $b > 0$ an upper bound of $|w(x)|$ in $\mathbb{R}^N$, then the nonpositive function $w - b$ is a subsolution of (DP), for any $\lambda > 0$. By the maximality of u_λ among the subsolutions of (DP), we conclude that $u_\lambda \geq w - b \geq -2\,b$ in $\mathbb{R}^N$. □

Here we digress slightly from the streamline and consider an example where $N = 1$ and $H(x,p) = |p| - |x|$ for $(x,p) \in \mathbb{R}^2$. By solving the equations

$$\lambda u(x) + u'(x) = x \quad \text{and} \qquad \lambda u(x) - u'(x) = x \qquad \text{for } x > 0,$$

where $\lambda > 0$, we easily see that the functions

$$u_+(x) := \frac{|x|}{\lambda} + \frac{1}{\lambda^2}(-1 + e^{-\lambda|x|}),$$

and

$$u_C(x) := \frac{|x|}{\lambda} + \frac{1}{\lambda^2}(1 - Ce^{\lambda|x|}),$$

with $C \geq 1$, are solutions of

$$\lambda u(x) + |u'(x)| = |x| \quad \text{in } \mathbb{R}. \tag{4}$$

We can prove the following uniqueness claim: if u is a solution of (4) that satisfies

$$\liminf_{|x|\to\infty}(u(x) + \delta e^{\lambda|x|}) > 0 \quad \text{for all } \delta > 0, \tag{5}$$

then $u = u_+$. In particular, we have $u_\lambda = u_+$ in this example. That is, the maximal solution u_λ of (4) is characterized as the unique solution of (4) that satisfies (5). This example tempts us to conjecture that, in our standing assumptions, the maximal solution u_λ is a "unique" solution of (DP) that is bounded from below. The authors are not able to show the uniqueness of those solutions of (DP) that are bounded from below.

A brief idea to check the uniqueness claim above is that if u is a solution of (4) and (5), then consider the function

$$w(x) := (1-\delta)u_+(x) - \delta e^{\lambda|x|} \quad \text{on } \mathbb{R}$$

for small $\delta \in (0,\,1)$, observe that w is a subsolution of (4) and

$$\limsup_{|x|\to\infty}(w(x) - u(x)) = -\infty,$$

and apply a standard comparison theorem in a large interval $[-R,\,R]$, to see that $w \leq u$ in $\mathbb{R}$, which implies in the limit as $\delta \to 0$ that $u_+ \leq u$. Observing by (4) that $u(x) \leq |x|/\lambda$ for all $x \in \mathbb{R}$, we may repeat an argument, parallel to the above, with w and u replaced by $(1-\delta)u - \delta e^{\lambda|x|}$ and u_+, respectively, to conclude that $u \leq u_+$.

Proposition 3.5. *The family u_λ, for $\lambda > 0$, is relatively compact in $C(\mathbb{R}^N)$.*

Proof. We already know from Proposition 3.1 and Lemma 3.4 that the u_λ are locally equibounded. This implies that for any $R > 0$ there exists a constant b_R with

$$H[u_\lambda] \leq b_R \qquad \text{in } B_R, \text{ for any } \lambda > 0.$$

Taking into account the coercivity condition (A2), we derive from the above inequality that the u_λ are equiLipschitz–continuous in B_R, for any $R > 0$. This concludes the proof. □

We derive from the previous results on maximal solutions of (DP):

Proposition 3.6. *There exists a Hamiltonian $\widetilde{H}$ satisfying* (A1), (A2), (A3) *plus*

$$\lim_{|p|\to+\infty} \frac{\widetilde{H}(x,p)}{|p|} = +\infty \qquad \text{for any } x \in \mathbb{R}^N \tag{6}$$

such that in addition the subsolutions of the equations (EP) *and* $\widetilde{H}[u] = 0$ *are the same, and* u_λ *is the maximal subsolution to* $\lambda\, u + \widetilde{H}[u] = 0$ *for any* $\lambda > 0$.

Proof. We set

$$b = \max_{x\in\mathbb{R}^N} H(x,0) \geq 0,$$

this maximum does exist in force of (A3). The function $u \equiv -\frac{b}{\lambda}$ is subsolution to (DP) for any $\lambda > 0$, so that $\lambda\, u_\lambda \geq -b$, and accordingly

$$0 \geq \lambda\, u_\lambda + H[u_\lambda] \geq -b + H[u_\lambda]. \tag{7}$$

We define

$$\widetilde{H}(x,p) = H(x,p) + \big(0 \vee (H(x,p) - b)\big)^2$$

by exploiting the property that the square of any coercive nonnegative convex function from $\mathbb{R}^N$ to $\mathbb{R}$ is convex with superquadratic growth at infinity, we see that $\widetilde{H}$ satisfies (A1), (A2), (6).

Regarding property (A3), we have that

$$\min_p H(x,p) \leq H(x,0) \leq b \qquad \text{for any } x \in \mathbb{R}^N$$

which implies

$$\max_{x\in\mathbb{R}^N} \min_{p\in\mathbb{R}^N} H(x,p) = \max_{x\in\mathbb{R}^N} \min_{p\in\mathbb{R}^N} \widetilde{H}(x,p) \tag{8}$$

Moreover

$$\max_{p\in B_\varepsilon} H(x,p) \leq 0 \leq b \qquad \text{when } |x| \text{ is large enough}$$

so that

$$\limsup_{|x|\to+\infty} \max_{p\in B_\varepsilon} H(x,p) = \limsup_{|x|\to+\infty} \max_{p\in B_\varepsilon} \widetilde{H}(x,p) \tag{9}$$

where ε is the same constant appearing in (A3). We deduce from (8), (9) that condition (A3) holds for $\widetilde{H}$. Since $b \geq 0$, we have that

$$\{(x,p) \mid H(x,p) \leq 0\} = \{(x,p) \mid \widetilde{H}(x,p) \leq 0\}.$$

This implies that 0 is the critical value for $\widetilde{H}$ and the equations $H[u] = 0$, $\widetilde{H}[u] = 0$ have the same subsolutions. Further, due to $\widetilde{H} \geq H$, any subsolution of $\lambda\, u + \widetilde{H}[u]$ is also subsolution to (DP), which implies that the maximal subsolution to $\lambda\, u + \widetilde{H}[u] = 0$ is less than or equal to u_λ. On the other side, since by (7)

$$H(x, Du_\lambda(x)) = \widetilde{H}(x, Du_\lambda(x)) \qquad \text{for a.e. } x,$$

the function u_λ itself is subsolution to $\lambda\, u + \widetilde{H}[u] = 0$. This implies that u_λ is indeed the maximal subsolution to $\lambda\, u + \widetilde{H}[u] = 0$, ending the proof.

□

The above result allows us to assume, without any loss of generality, that superlinear growth property in (6) holds true for H. We can therefore define via Fenchel transform the corresponding Lagrangian

$$L(x,q) = \max_p p \cdot q - H(x,p),$$

which is convex and coercive in q. In addition, we have for any $R > 0$, $x \in B_R$

$$L(x,q) \geq R\,|q| - H(x, R\,|q|^{-1}\,q) \geq R\,|q| - \sup_{(x,p)\in B_R \times B_R} H(x,p),$$

which shows that

$$\lim_{|q|\to+\infty} \inf_{x\in B_R} \frac{L(x,q)}{|q|} = +\infty \qquad \text{for any } R > 0. \tag{10}$$

We moreover deduce from (A3) that there is a compact subset $K \subset \mathbb{R}^N$ and positive constants δ_0, M_0 such that

$$L(x,q) \geq \delta_0\,|q| - H(x, q\,|q|^{-1}\,\delta_0) \geq \delta_0\,|q|, \tag{11}$$

$$L(x,q) \geq -H(x,0) \geq M_0 > 0 \tag{12}$$

for $x \notin K$, any $q \in \mathbb{R}^N$.

4. Ergodic equation

Lemma 4.1. *The definition of critical value in* (1) *is well posed, the critical value is finite and is actually a minimum.*

Proof. By assumption (A3), $H(\cdot,0)$ attains a maximum in $\mathbb{R}^N$. If $a \geq \max_{\mathbb{R}^N} H(x,0)$, then $H[u] = a$ admits any constant function as subsolutions. This implies that the set in the right hand side of (1) is nonempty. On the other side, if $a < \min_p H(x,p)$ for some $x \in \mathbb{R}^N$, then $H[u] = a$ does not admit any subsolution, which shows that the critical value is finite. Finally it is a minimum by standard stability properties of viscosity subsolutions. □

We recall that we assume throughout the paper that the critical value is 0.

Proposition 4.2. *There exists a solution to*

$$H[u] = 0 \qquad \text{in } \mathbb{R}^N.$$

Proof. As already pointed out, there exists a subsolution to $H[u] = 0$ in $\mathbb{R}^N$. This implies that the intrinsic distance S_0 is finite. We use the usual covering argument, see [14, Theorem 3.3] plus existence of subsolution going to $-\infty$ and Proposition 2.1 to show there exists $y \in \mathbb{R}^N$ such that $S_0(\cdot,y)$ is a solution to $H[u] = 0$ in $\mathbb{R}^N$. □

Proposition 4.3. *The Aubry set $\mathcal{A}$ is a nonempty compact subset of $\mathbb{R}^N$.*

Proof. The argument of Proposition 4.2 shows that $\mathcal{A}$ is nonempty, it is in addition closed by stability properties of viscosity solutions. By Proposition 2.1, there exists a subsolution of $H[u] = 0$ which is strict outside a compact subset $C_0 \subset \mathbb{R}^N$. This implies by Proposition A.2 that $\mathcal{A} \subset C_0$.

□

We recall that if the ambient space is compact the ergodic equation admits solutions only at the critical level. This is not any more the case in the noncompact setting since a solutions can be found at any supercritical value as well.

Definition 4.4. *We say that a solution v to the critical equation is a weak KAM solution if it can be written in the form*

$$v(x) = \min\{v(y) + S_0(y,x) \mid y \in \mathcal{A}\}. \tag{13}$$

We directly derive from the definition of intrinsic distance:

Lemma 4.5. *A function v is weak KAM solution if and only if*

$$v(x) = \max\{u(x) \mid u \text{ subsolution to (EP) with } u = v \text{ on } \mathcal{A}\}.$$

We recall the following result, see for the proof [17], [14].

Lemma 4.6. *Let B_0 be a ball containing $\mathcal{A}$ than any solution v of (EP) in B_0 satisfies*

$$v(x) = \min\{v(y) + S_0(y,x) \mid y \in \partial B_0 \cup \mathcal{A}\}.$$

The following characterization holds:

Theorem 4.7. *A solution v is weak KAM if and only if it is bounded from below.*

Proof. Exploiting (A3), we can find $\varepsilon > 0$ and $R > 0$ with

$$\sup_{x \in \mathbb{R}^N \setminus B_R} \max_{p \in B_\varepsilon} H(x,p) < 0,$$

we can further assume that $B_R \supset \mathcal{A}$. We consequently have

$$\ell_0(\xi) \geq \varepsilon\, \ell(\xi) \tag{14}$$

for any curve ξ with support contained in $\mathbb{R}^N \setminus B_R$. Here, to repeat, ℓ stands for the Euclidean length, and ℓ_0 for the intrinsic length of a curve. Given $x_0 \in B_R$, we define

$$m = \min_{x \in \overline{B_R}} S_0(x, x_0). \tag{15}$$

Assume that v is not a weak KAM solution, then by applying Lemma 4.6 to a sequence of balls with diverging radii, we find that there exist x_n with $|x_n| \to +\infty$ such that

$$v(x_0) = v(x_n) + S_0(x_n, x_0). \tag{16}$$

We may assume that $|x_n| > R$ for any n. Let ξ_n be a sequence of curves, parametrized in $[0,1]$, linking x_n to x_0 such that

$$\ell_0(\xi_n) \leq S_0(x_n, x_0) + \frac{1}{n}. \tag{17}$$

Let t_n be the first entrance time of ξ_n in B_R. This means

$$\xi_n([0,t_n)) \cap B_R = \emptyset \quad \text{and} \quad y_n := \xi_n(t_n) \in \partial B_R.$$

We claim that

$$\lim_n \ell_0(\overline{\xi}_n) - S_0(x_n, y_n) \;=\; 0, \tag{18}$$

$$\lim_n S_0(x_n, x_0) - S_0(x_n, y_n) - S_0(y_n, x_0) \;=\; 0 \tag{19}$$

where $\overline{\xi}_n = \xi_n\big|_{[0,t_n)}$. We in fact have by (17) and the triangle inequality

$$\begin{aligned} \ell_0(\xi_n) &\leq S_0(x_n,x_0) + \frac{1}{n} \leq S_0(x_n,y_n) + S_0(y_n,x_0) + \frac{1}{n} \\ &\leq \ell_0(\xi_n\big|_{[t_n,1]}) + S_0(x_n,y_n) + \frac{1}{n} \leq \ell_0(\overline{\xi}_n) + \ell_0(\xi_n\big|_{[t_n,1]}) + \frac{1}{n} \\ &= \ell_0(\xi_n) + \frac{1}{n} \end{aligned}$$

which implies

$$\begin{aligned} 0 &\leq \ell_0(\overline{\xi}_n) - S_0(x_n,y_n) \leq \frac{1}{n}, \\ 0 &\leq S_0(x_n,y_n) + S_0(y_n,x_0) - S_0(x_n,x_0) \leq \frac{1}{n} \end{aligned}$$

and gives in the end (18), (19) sending n to $+\infty$. We further have by (16), (14), (18), (19) that

$$\begin{aligned} v(x_0) &= v(x_n) + S_0(x_n,x_0) \geq v(x_n) + S_0(x_n,y_n) + S_0(y_n,x_0) - \frac{1}{n} \\ &\geq v(x_n) + \ell_0(\overline{\xi}_n) + S_0(y_n,x_0) - 2\,\frac{1}{n} \\ &\geq v(x_n) + \varepsilon\,(|x_n| - R) + m - 2\,\frac{1}{n}, \end{aligned}$$

where m is defined as in (15), and we finally obtain

$$v(x_n) \leq v(x_0) - m - \varepsilon\,(|x_n| - R) + 2\,\frac{1}{n}.$$

Since $|R|$ can be sent to infinity, this proves that u is unbounded from below. Conversely, let v be a weak KAM solution. Let x_1 be a point with $|x_1| > R$. We denote by y_0 an element of the Aubry set with

$$v(x_1) = v(y_0) + S_0(y_0,x_0)$$

and by ξ a curve, parametrized in $[0,1]$, linking y_0 to x_1 such that

$$\ell_0(\xi) \leq S_0(y_0,x_1) + 1. \tag{20}$$

Let t_0 be the last exit time of ξ from B_R. This means

$$\xi((t_0,1)) \cap B_R = \emptyset \quad \text{and} \quad z_0 := \xi(t_0) \in \partial B_R.$$

We have by (20) and the triangle inequality

$$\begin{aligned} \ell_0(\xi) &\leq S_0(y_0,x_1) + 1 \leq S_0(y_0,z_0) + S_0(z_0,x_1) + 1 \\ &\leq \ell(\xi\big|_{[0,t_0]}) + S_0(z_0,x_1) + 1 \\ &\leq \ell_0(\xi\big|_{(t_0,1]}) + \ell_0(\xi\big|_{[0,t_0]}) + 1 \\ &= \ell_0(\xi) + 1, \end{aligned}$$

which implies

$$0 \leq \ell_0(\xi\big|_{[t_0,1]}) - S_0(z_0,x_1) \leq 1, \tag{21}$$

$$0 \leq S_0(y_0,z_0) + S_0(z_0,x_1) - S_0(y_0,x_1) \leq 1. \tag{22}$$

We further have by (16), (21), (22) and the fact that v is a weak KAM solution

$$\begin{aligned} v(x_1) &= v(y_0)+S_0(y_0,x_1) \geq v(y_0)+S_0(y_0,z_0)+S_0(z_0,x_1)-1 \\ &\geq v(y_0)+\ell_0(\xi_{\big|[t_0,1]})+S_0(y_0,z_0)-2 \\ &\geq v(z_0)-2 \geq \min_{\partial B_R} v-2, \end{aligned}$$

which gives that v is bounded from below since x_1 has been arbitrarily chosen in $\mathbb{R}^N \setminus B_R$. □

5. Localization results

The results of this section will be crucial to prove that some subsets of the space of bounded functions $\Phi : \mathbb{R}^N \times \mathbb{R}^N$ are open in a suitable topology. This in turn will allow showing the existence of minimizing measures for (EP), (DP).

Proposition 5.1. *Any open ball $B_0 \supset \mathcal{A}$ satisfies*

$$0 = \inf\{a \mid H = a \text{ admits subsolutions in } B_0\}.$$

Proof. Let B_0 be an open ball containing the Aubry set. Assume by contradiction that there is a strict subsolution to $H[u]=0$ in B_0. Then by standard comparison principles, there exists one and only one solution u_0 to $H[u]=0$ equal to a given trace g on ∂B_0 with

$$g(x)-g(y) \leq S_0(y,x) \qquad \text{for any } x,\, y \text{ in } \partial B_0$$

and it is given by

$$u_0(x) = \min\{g(y)+S_0(y,x) \mid y \in \partial B_0\}.$$

We fix $z \in \mathcal{A} \subset B_0$, since $S_0(z,\cdot)$ is solution of $H[u]=0$ in $\mathbb{R}^N$, we apply the comparison principle with $g = S_0(z,\cdot)$ on ∂B_0, and deduce that

$$0 = S_0(z,z) = S_0(z,y_0)+S_0(y_0,z) \qquad \text{for some } y_0 \in \partial B_0.$$

This implies that we can find a sequence ξ_n of cycles passing through z and y_0 with

$$\ell_0(\xi_n) \to 0 \qquad \text{and} \qquad \inf_n \ell(\xi_n) > 0.$$

This in turn implies that $y_0 \in \mathcal{A}$ by Proposition A.1 , against the assumption that $\mathcal{A}$ is contained in the interior of B_0. □

We proceed proving a localization property for the discounted equation.

Proposition 5.2. *Given $z \in \mathbb{R}^N$ there exist $\lambda_z > 0$ such that u_λ and the maximal subsolution of* (DP) *in $C_{z,\lambda}$ coincide at z for $\lambda < \lambda_z$ and some ball $C_{z,\lambda}$.*

Remark 5.3. *We recall that the maximal subsolution to* (DP) *in some ball B is nothing but the state constraint solution. If $u_\lambda(z)$ coincide with this solution then*

$$u_\lambda(z) = \inf\left\{\int_{-\infty}^0 e^{\lambda s}\, L(\xi,\dot\xi)\, ds \mid \xi(s) \in \overline{B}\; \forall\, s, \xi(0)=z\right\}.$$

If, on the contrary, $u_\lambda(z)$ is strictly less to the maximal subsolution in B then

$$u_\lambda(z) = \inf\left\{e^{-\lambda T}u(\xi(-T)) + \int_{-T}^0 e^{\lambda s}\, L(\xi,\dot\xi)\, ds \mid \xi(s) \in \overline{B}\, \forall\; s, \xi(0)=z,\, \xi(-T) \in \partial B\right\}.$$

We need two preliminary lemmata.

Lemma 5.4. *Let $\lambda > 0$, x be an arbitrary element of $\mathbb{R}^N$, ξ a curve defined in $[-t, 0]$, for some $t > 0$, with $\xi(0) = x$ then*

$$u_\lambda(x) \le e^{-\lambda t}\, u_\lambda(\xi(-t))) + \int_{-t}^{0} e^{\lambda s}\, L(\xi, \dot\xi)\, ds.$$

If $\eta : (-\infty, 0] \to \mathbb{R}^N$ with $\eta(0) = x$ then

$$u_\lambda(x) \le \int_{-\infty}^{0} e^{\lambda s}\, L(\eta, \dot\eta)\, ds + \liminf_{t\to+\infty} e^{-\lambda t}\, u_\lambda(\eta(-t)).$$

Note that L and the u_λ are bounded from below, the indeterminate form $+\infty - \infty$ cannot therefore appear in the above formula.

Proof. We just prove the first part of the assertion, the second part can be obtained sending t to infinity. We have

$$\begin{aligned} u_\lambda(x) - e^{-\lambda t}\, u_\lambda(\xi(-t)) &= \int_{-t}^{0} \frac{d}{ds}[e^{\lambda s}\, u_\lambda(\xi(s))]\, ds \\ &= \int_{-t}^{0} e^{\lambda s}\, (\lambda\, u_\lambda(\xi(s)) + p(s) \cdot \dot\xi(s))\, ds, \end{aligned}$$

where $p(s)$ is a suitable element of $\partial u_\lambda(\xi(s))$. Taking into account that u_λ is a subsolution, we further obtain

$$\begin{aligned} u_\lambda(x) - e^{-\lambda t}\, u_\lambda(\xi(-t)) &\le \int_{-t}^{0} e^{\lambda s}\, (\lambda\, u_\lambda(\xi(s)) + L(\xi(s), \dot\xi(s)) + H(\xi(s), p(s)))\, ds \\ &\le \int_{-t}^{0} e^{\lambda s}\, L(\xi, \dot\xi)\, ds. \end{aligned}$$

This concludes the proof. □

Lemma 5.5. *Given $\varepsilon > 0$, $\lambda > 0$, $x_0 \in \mathbb{R}^N$, let $\xi : (-T, 0] \to \mathbb{R}^N$, $T \in \mathbb{R} \cup \{+\infty\}$, be a curve with $\xi(0) = x_0$ and*

$$u_\lambda(x_0) \ge \int_{-T}^{0} e^{\lambda s}\, L(\xi, \dot\xi)\, ds + \liminf_{t\to T} e^{-\lambda t}\, u_\lambda(\xi(-t)) - \varepsilon. \tag{23}$$

Then

$$u_\lambda(\xi(-t_1)) \ge e^{-\lambda(t_2 - t_1)}\, u_\lambda(\xi(-t_2)) + \int_{t_1 - t_2}^{0} e^{\lambda s}\, L(\xi(\cdot - t_1)\, \dot\xi(\cdot - t_1))\, ds - \varepsilon \tag{24}$$

for any $T > t_2 > t_1 \ge 0$.

Proof. By (23) and Lemma 5.4 we have

$$\begin{aligned} u_\lambda(x_0) &\geq \int_{-T}^{-t_2} e^{\lambda s} L(\xi, \dot\xi)\, ds + \int_{-t_2}^{-t_1} e^{\lambda s} L(\xi, \dot\xi)\, ds \\ &+ \int_{-t_1}^{0} e^{\lambda s} L(\xi, \dot\xi)\, ds + \liminf_{t \to T} e^{-\lambda t}\, u_\lambda(\xi(-t)) - \varepsilon \\ &\geq e^{-\lambda t_2} u_\lambda(\xi(-t_2)) - \liminf_{t \to T} e^{-\lambda t}\, u_\lambda(\xi(-t)) \\ &+ e^{-\lambda t_1} \int_{t_1 - t_2}^{0} e^{\lambda s}\, L(\xi(\cdot - t_1)\, \dot\xi(\cdot - t_1))\, ds \\ &+ u_\lambda(x_0) - e^{-\lambda t_1}\, u_\lambda(\xi(-t_1)) + \liminf_{t \to T} e^{-\lambda t}\, u_\lambda(\xi(-t)) - \varepsilon, \end{aligned}$$

which gives (24). □

Proof of Proposition 5.2. Given $t > 0$ and a curve ξ defined in $[-t, 0]$, we set to ease notations

$$\rho_\lambda(t, \xi) = \int_{-t}^{0} e^{\lambda s}\, L(\xi, \dot\xi)\, ds.$$

Let B be a ball containing z such that there are positive constants M_0, δ_0, a, with

$$L(x, q) \geq \delta_0\, |q| \quad \text{for } x \notin B, \text{ any } q, \tag{25}$$

$$L(x, q) \geq M_0 \quad \text{for } x \notin B, \text{ any } q, \tag{26}$$

$$u_\lambda(x) \geq -a \quad \text{for any } \lambda > 0,\ x \in \mathbb{R}^N. \tag{27}$$

See (11), (12), and Lemma 3.4 to check out that this is possible. Further, we take λ_z such that

$$\max\{u_\lambda(x) \mid x \in \partial B,\ \lambda < \lambda_z\} < \frac{M_0}{\lambda} - a - 1. \tag{28}$$

We fix $\lambda < \lambda_z$, were the assertion not true for such a λ , we would find, see Remark 5.3, $T_n > 0$, $x_n \in \mathbb{R}^N$ with $|x_n| \to +\infty$ and curves $\xi_n : [-T_n, 0] \to \mathbb{R}^N$ joining x_n to z such that

$$u_\lambda(z) \geq e^{-\lambda T_n}\, u_\lambda(x_n) + \rho_\lambda(T_n, \xi_n) - \frac{1}{n}. \tag{29}$$

We set for any n

$$-T'_n = \min\{t > -T_n, \mid \xi_n(t) \in \partial B\},$$

so that the support of ξ_n is outside B in the time interval $(-T_n, -T'_n)$ and $\xi_n(-T'_n) \in \partial B$. We have by (29) and Lemma 5.5

$$\begin{aligned} u_\lambda(\xi_n(-T'_n)) &\geq e^{-\lambda(T'_n - T_n)}\, u_\lambda(\xi_n(-T_n)) + \rho_\lambda(T_n - T'_n, \xi_n(\cdot - T'_n)) - \frac{1}{n} \\ &\geq -a + \rho_\lambda(T_n - T'_n, \xi_n(\cdot - T'_n)) - \frac{1}{n}, \end{aligned} \tag{30}$$

for any n. We first assume that $\lim_n T_n - T'_n = +\infty$, then

$$\liminf_n \rho_\lambda(T_n - T'_n, \xi_n(\cdot - T'_n)) \geq \frac{M_0}{\lambda}$$

and consequently

$$\liminf_n u_\lambda(\xi_n(-T'_n)) \geq -a + \frac{M_0}{\lambda}, \tag{31}$$

in contradiction with (28) since $\xi_n(-T'_n) \in \partial B$. If instead $T_n - T'_n < T$ for any n, some $T > 0$, we integrate by parts, bearing in mind that $L(x,q) > 0$ for $x \notin B$, any q, to get

$$\begin{aligned}
\rho_\lambda(T_n - T'_n, \xi_n(\cdot - T'_n)) &= \left[-e^{\lambda t} \int_t^0 L(\xi_n(\cdot - T'_n), \dot\xi_n(\cdot - T'_n))\, ds \right]_{T'_n - T_n}^0 \\
&+ \int_{T'_n - T_n}^0 \lambda\, e^{\lambda t} \left(\int_t^0 L(\xi_n(\cdot - T'_n), \dot\xi_n(\cdot - T'_n))\, ds \right) dt \\
&\geq e^{-\lambda(T_n - T'_n)} \int_{T'n - T_n}^0 L(\xi(\cdot - T'_n), \dot\xi(\cdot - T'_n))\, ds \\
&\geq e^{-\lambda T}\, \delta_0\, |\xi_n(-T_n) - \xi_n(-T'_n)|.
\end{aligned}$$

Taking into account that $\xi_n(-T'_n) \in \partial B$ and $|\xi_n(-T_n)| = |x_n| \to +\infty$, we get

$$\lim_n \rho_\lambda(T_n - T'_n, \xi(\cdot - T'_n)) = +\infty, \tag{32}$$

which contradicts (30). □

6. Generalized Lagrangians and narrow convergence of measures

We consider the space $C(\mathbb{R}^{2N})$ of the continuous functions from $\mathbb{R}^{2N}$ to $\mathbb{R}$. Given such a function $\Phi(x,q)$, we say that a locally Lipschitz continuous function u is a subsolution for Φ if

$$Du(x) \cdot q \leq \Phi(x,q) \qquad \text{for a.e. } x \in \mathbb{R}^N, \text{ any } q \in \mathbb{R}^N.$$

We further say that u is a strict subsolution if

$$Du(x) \cdot q \leq \Phi(x,q) - \varepsilon \qquad \text{for a.e. } x, \text{ any } q, \text{ some } \varepsilon > 0.$$

A real number c is called critical value of Φ if $\Phi + c$ admits subsolutions but not strict subsolutions.

Given a discount factor λ, we similarly say that a locally Lipschitz continuous function u is a λ–discounted subsolution for Φ if

$$\lambda\, u(x) + Du(x) \cdot q \leq \Phi(x,q) \qquad \text{for a.e. } x \in \mathbb{R}^N, \text{ any } q \in \mathbb{R}^N.$$

Remark 6.1. *If Φ has superlinear growth when $|q|$ goes to $+\infty$, for any x, then we can apply Fenchel transform to define a Hamiltonian, denoted by H_Φ, which is convex in p and satisfies* (A1), (A2)*. A subsolution corresponding to $\Phi + a$, for some $a \in \mathbb{R}$, is nothing but a subsolution of the equation $H_\Phi[u] = a$. Finally, the critical value of Φ is equal to the critical value of H_Φ.*

We will denote by $\mathfrak{P}$ the space of Radon probability measure on $\mathbb{R}^{2N}$. Given $\mu \in \mathfrak{P}$ and $\Phi \in C(\mathbb{R}^{2N})$, integrable with respect to μ, we will write from now on, to ease notations, $\langle \mu, \Phi \rangle$ in place of $\int \Phi\, d\mu$.

We state and prove some convergence lemmata with respect to the narrow topology we will use in what follows.

Lemma 6.2. *Let $\Phi \in C(\mathbb{R}^N \times \mathbb{R}^N)$ be bounded from below, $\mu \in \mathfrak{P}$, then*

$$a := \liminf_n \langle \mu_n, \Phi \rangle \geq \langle \mu, \Phi \rangle$$

for any sequence μ_n narrowly converging to μ.

Proof. We can assume that a is finite, otherwise the assertion is trivial. Given $\varepsilon > 0$, $R > 0$, we find a subsequence μ_{n_k} of μ_n with

$$a + \varepsilon \geq \langle \mu_{n_k}, \Phi \rangle \geq \langle \mu_{n_k}, \Phi \wedge R \rangle$$

for n_k large enough. Since the functions $\Phi \wedge R$, for $R > 0$, are bounded continuous by the assumption, we get

$$a + \varepsilon \geq \lim_{n_k} \langle \mu_{n_k}, \Phi \wedge R \rangle = \langle \mu, \Phi \wedge R \rangle$$

and consequently, letting ε going to 0

$$\langle \mu, \Phi \wedge R \rangle \leq a \qquad \text{for any } R.$$

Since $\Phi \wedge R$ converges monotonically to Φ as $R \longrightarrow +\infty$, we finally obtain by the monotone convergence theorem

$$\langle \mu, \Phi \rangle = \lim_{R \to +\infty} \langle \mu, \Phi \wedge R \rangle \leq a.$$

This ends the proof. □

We recall that L is bounded from below in force of (10), (11). We set

$$m = \min_{\mathbb{R}^N \times \mathbb{R}^N} L.$$

Lemma 6.3. *Given any real number a the sublevel*

$$\mathcal{V}_a := \{\mu \in \mathfrak{P} \mid \langle \mu, L \rangle \leq a\}$$

is compact in the narrow topology, provided that it is not empty.

Proof. Given $\varepsilon > 0$, we find by (11) a compact subset K_ε of $\mathbb{R}^N \times \mathbb{R}^N$ with

$$L(x, q) \geq \frac{1}{\varepsilon} \qquad \text{whenever } (x, q) \notin K_\varepsilon.$$

Given $\mu \in \mathcal{V}_a$, we have

$$a \geq \langle \mu, L \rangle \geq m\, \mu(K_\varepsilon) + \frac{1}{\varepsilon}\, \mu(K_\varepsilon^c) \geq -|m| + \frac{1}{\varepsilon}\, \mu(K_\varepsilon^c),$$

where A^c indicates the complement of A. This implies

$$(a + |m|)\, \varepsilon \geq \mu(K_\varepsilon^c) \qquad \text{for } \mu \in \mathcal{V}_a$$

and shows that the measures in $\mathcal{V}_a$ are uniformly tight, so that $\mathcal{V}_a$ is conditionally narrowly compact. Finally if $\mu_n \in \mathcal{V}_a$ narrowly converges to some μ, we have by Lemma 6.2

$$a \geq \liminf_n \langle \mu_n, L \rangle \geq \langle \mu, L \rangle,$$

which concludes the proof. □

Lemma 6.4. *Let μ_n a sequence in $\mathfrak{P}$ narrowly converging to some μ, with $\langle \mu_n, L \rangle$ bounded, then*

$$\langle \mu_n, \Phi \rangle \to \langle \mu, \Phi \rangle$$

whenever Φ is continuous, supported in $K \times \mathbb{R}^N$, for some compact subset K of $\mathbb{R}^N$, and with linear growth as $|q|$ goes to infinity.

Proof. We take Φ as in the statement. The function

$$R \mapsto \mu(B_R \times \mathbb{R}^N)$$

is nondecreasing and so possesses countably many discontinuities. For any of its continuity points R we have $\mu(\partial B_R \times \mathbb{R}^N) = 0$ and consequently by the Portmanteau Theorem

$$\mu_n(B_R \times \mathbb{R}^N) \to \mu(B_R \times \mathbb{R}^N). \tag{33}$$

We can therefore find $R > 0$ satisfying (33) such that $\operatorname{supp} \Phi \subset B_R \times \mathbb{R}^N$, $\mu(B_R \times \mathbb{R}^N) > 0$, $L > 0$ in $B_R^c \times \mathbb{R}^N$, see (11). We consider the conditional probabilities

$$\overline{\mu}_n(\cdot) = \mu_n(\cdot \mid B_R \times \mathbb{R}^N), \qquad \overline{\mu}(\cdot) = \mu(\cdot \mid B_R \times \mathbb{R}^N).$$

Note that the $\langle \overline{\mu}_n, L \rangle$ are bounded and $\overline{\mu}_n$ narrowly converge to $\overline{\mu}$. Given $\varepsilon > 0$, we find by (10) a compact subset $C \subset \mathbb{R}^N$ with

$$L(x,q) \geq \frac{1}{\varepsilon}\,|q| \qquad \text{whenever } x \in B_R,\ q \notin C.$$

We set $m = \min_{\mathbb{R}^N \times \mathbb{R}^N} L$ and denote by a an upper bound of $\langle \overline{\mu}_n, L \rangle$. Bearing in mind that the the $\overline{\mu}_n$ are supported in $B_R \times \mathbb{R}^N$, we have

$$\begin{aligned} a &\geq \langle \overline{\mu}_n, L \rangle \geq m\, \overline{\mu}_n(B_R \times C) + \int_{(B_R \times C)^c} L \, d\overline{\mu}_n \\ &\geq -|m| + \int_{B_R \times (\mathbb{R}^N \setminus C)} L \, d\overline{\mu}_n \\ &\geq -|m| + \frac{1}{\varepsilon} \int_{B_R \times (\mathbb{R}^N \setminus C)} |q| \, d\overline{\mu}_n, \end{aligned}$$

which implies

$$(a + |m|)\, \varepsilon \geq \int_{B_R \times (\mathbb{R}^N \setminus C)} |q| \, d\overline{\mu}_n = \int_{(B_R \times C)^c} |q| \, d\overline{\mu}_n.$$

We derive that the $\overline{\mu}_n$ are 1–uniformly integrable and so conditionally compact with respect to the Wasserstein distance of order 1, denoted by W^1. Since $\overline{\mu}_n$ narrowly converges to $\overline{\mu}$, we deduce that the convergence actually holds with respect to W^1. Such a convergence can be equivalently expressed in duality with continuous function with linear growth at infinity, and we thus get, taking into account that $\operatorname{supp} \Phi \subset B_R \times \mathbb{R}^N$,

$$\frac{1}{\mu_n(B_R \times \mathbb{R}^N)} \langle \mu_n, \Phi \rangle = \langle \overline{\mu}_n, \Phi \rangle \to \langle \overline{\mu}, \Phi \rangle = \frac{1}{\mu(B_R \times \mathbb{R}^N)} \langle \mu, \Phi \rangle,$$

which gives the assertion. □

7. Mather measures for the discounted equation

The aim of this section is to show the existence of minimizing measures related to (DP). More precisely, we will prove:

Theorem 7.1. *Given $z \in \mathbb{R}^N$, $0 < \lambda < \lambda_z$ (λ_z as in Proposition 5.2), there exists a probability measure $\mu \in \mathfrak{P}$ with*

$$\langle \mu, L \rangle = \lambda\, u_\lambda(z), \quad \langle \mu, \Phi \rangle \geq \lambda\, u(z) \tag{34}$$

for any $\Phi \in C(\mathbb{R}^{2N})$ bounded from below admitting a λ–discounted subsolution and any λ–discounted subsolution u of Φ.

Our strategy is to construct a suitable convex subset of $C_b(\mathbb{R}^N)$, with $L \wedge M$ for $M > 0$ large in its boundary, possessing nonempty interior, and then to apply Proposition C.1 about the existence of nonzero elements in normal cones. The nonzero elements in the normal cone at $L \wedge M$ are, up to change of sign and normalization, the probability measure appearing in the previous statement.

We proceed to prove some preliminary results.

Lemma 7.2. *Given $z \in \mathbb{R}^N$, $a \in \mathbb{R}$, $\lambda > 0$, an open ball B containing z, any Lipschitz function u in B satisfying $u(z) \geq a$ and*

$$\lambda\, u(x) + Du(x) \cdot q \leq L(x, q) \qquad \text{for a.e. } x \in B,\ q \text{ with } |q| = 1 \tag{35}$$

has Lipschitz constant in B bounded from above by a quantity solely depending on a, λ, L and the diameter of B.

Proof. We set

$$R = \max\{L(x, q) \mid x \in \overline{B}, |q| = 1\}.$$

We have by (35)

$$Du(x) \cdot q \leq L(x, q) - \lambda \min_{\overline{B}} u \qquad \text{for a.e. } x \in B, \text{ any } |q| = 1$$

and consequently

$$\|Du\|_{\infty, B} \leq R - \lambda \min_{\overline{B}} u.$$

It is then enough to show that $\min_{\overline{B}} u$ is bounded from below, when u varies among the functions satisfying the assumptions. We set

$$v(x, t) = u(x) e^{\lambda t} \quad \text{for } (x, t) \in B \times \mathbb{R},$$

and observe that, for any $q \in \mathbb{R}^N$ with $|q| \leq 1$, v satisfies

$$Dv(x, t) \cdot q + (\partial/\partial t) v(x, t) \leq R e^{\lambda t} \quad \text{for } (x, t) \in B \times \mathbb{R}.$$

Fix any point $y \in B \setminus \{z\}$ and set $q = (z - y)/|z - y|$. Thanks to [9, Theorem I.14], we have

$$v(y + tq, t) - v(y, 0) \leq \int_0^t R e^{\lambda s}\, ds \quad \text{for } t \in [0, |z - y|\,],$$

and, in particular,

$$u(z) e^{\lambda |z - y|} \leq u(y) + \frac{R}{\lambda} \left(e^{\lambda |z - y|} - 1 \right).$$

Hence, if d denotes the diameter of B, then

$$u(y) \geq (a \wedge 0)\, e^{\lambda\, d} - \frac{R}{\lambda} \left(e^{\lambda\, d} - 1 \right). \qquad \square$$

We exploit the above lemma to prove:

Proposition 7.3. *Given $z \in \mathbb{R}^N$, $\lambda > 0$, $a \in \mathbb{R}$, a ball B containing z, there exists $R > 1$, such that any Lipschitz function u in B satisfying $u(z) \geq a$ and*

$$\lambda\, u(x) + Du(x) \cdot q \leq L(x,q) \qquad \text{for a.e. } x \in B,\ |q| \leq R \tag{36}$$

is a subsolution to (DP) *in B.*

Proof. If (36) holds true for $|q| = 1$ then we know from Lemma 7.2 that there exists an upper bound, denoted by M, of the Lipschitz constants of all functions satisfying the assumptions, and consequently

$$\max_B u \leq a + r\, M,$$

where r denotes the diameter of B. We take $R > 1$ such that

$$\inf_{x \in B,\, |q| > R} \frac{L(x,q)}{|q|} \geq M + \lambda\, (a + r\, M),$$

note that this choice is possible in force of (10). We get

$$\begin{aligned} L(x,q) &\geq |q|\, (M + \lambda\, (a + r\, M)) \\ &\geq Du(x) \cdot q + \lambda\, u(x) \end{aligned}$$

for a.e. $x \in B$, any q with $|q| > R$. This last relation, together with (36), gives the assertion. $\square$

We fix $z \in \mathbb{R}^N$, $\lambda < \lambda_z$, and set $B_0 = C_{z,\lambda}$, see Proposition 5.2. We further denote by R_0 the constant provided by Proposition 7.3 in correspondence to B_0, z, λ, $a = u_\lambda(z)$.

We define in the space $C_b(\mathbb{R}^{2n})$ the set $\mathcal{G}_{\lambda,z}$ made up by the $\Phi \in C_b(\mathbb{R}^{2n})$ for which there exist a positive constant ε and a Lipschitz continuous function u in B_0 with $u(z) \geq u_\lambda(z)$ and

$$\lambda u(x) + Du(x) \cdot q \leq \Phi(x,q) - \varepsilon \tag{37}$$

for a.e. $x \in B_0$, any q with $|q| \leq R_0$.

Proposition 7.4. *The set $\mathcal{G}_{\lambda,z}$ is a convex subset, open with respect to the strict topology (see Appendix B for the definition of strict topology).*

Proof. The convexity property is apparent. If Φ_0 satisfies (37), then any Φ with $\|\Phi - \Phi_0\|_{\infty, \overline{B}_0 \times \overline{B}_{R_0}} < \frac{\varepsilon}{2}$ satisfies

$$\lambda u(x) + Du(x) \cdot q \leq \Phi_0(x,q) - \varepsilon \leq \Phi(x,q) - \frac{\varepsilon}{2}$$

for a.e. $x \in B_0$, any q with $|q| \leq R_0$. This shows that any such Φ belong to $\mathcal{G}_{\lambda,z}$, in addition these elements make up an open neighborhood of Φ_0 in the compact–open topology and consequently in the strict topology. This shows that $\mathcal{G}_{\lambda,z}$ is open.

$\square$

Proposition 7.5. *Given $\Phi \in C(\mathbb{R}^{2n})$ bounded from below and possessing a λ–discounted subsolution u in $\mathbb{R}^N$ with $u(z) \geq u_\lambda(z)$, there exists M_0 such that $\Phi \wedge M \in \overline{\mathcal{G}_{\lambda,z}}$ for $M \geq M_0$.*

Proof. We have that

$$\lambda u(x) + Du(x) \cdot q \leq \Phi(x, q)$$

for a.e. $x \in B_0$, any q. Since the left hand–side of the above formula is bounded when $|q| \leq R_0$, we find M_0 such that

$$\lambda u(x) + Du(x) \cdot q \leq \Phi(x, q) \wedge M \qquad \text{for } M \geq M_0,\ |q| \leq R_0, \text{ a.e. } x \in B_0.$$

This shows that $\Phi(x, q) \wedge M + \varepsilon$ belongs to $\mathcal{G}_{\lambda,z}$ for any ε. We therefore get the assertion because $\Phi(x, q) \wedge M + \varepsilon$ strictly converges to $\Phi(x, q) \wedge M$ as ε goes to 0. □

Proposition 7.6. *There exists M_0 such that $L \wedge M \in \partial \mathcal{G}_{\lambda,z}$ for $M \geq M_0$.*

Proof. Since L satisfies the assumptions of Proposition 7.5, we know that $L \wedge M \in \overline{\mathcal{G}_{\lambda,z}}$ for M greater than or equal to some M_0. It is left to show that $L \wedge M$ cannot be in the interior of $\mathcal{G}_{\lambda,z}$. In fact, if this is the case, there are a Lipschitz continuous function u, with $u(z) \geq u_\lambda(z)$, and $\varepsilon > 0$ such that

$$L(x, q) - \varepsilon \geq (L(x, q) \wedge M) - \varepsilon \geq \lambda\, u(x) + Du(x) \cdot q$$

for a.e. $x \in B_0$, any $|q| \leq R_0$. We then deduce from Proposition 7.3 that u is strict subsolution to (DP) in B_0. This implies in view of Proposition 5.2 that $u(z) = u_\lambda(z)$, and u is subtangent to u_λ at z. This is impossible because

$$L(z, q) \geq p \cdot q + \lambda\, u(z) + \varepsilon \qquad \text{for any } p \in \partial u(z),\ q \in \mathbb{R}^N,$$

while at least for a $p_0 \in \partial u(z)$, any $q \in \mathbb{R}^N$, we must have by the subtangency condition

$$L(z, q) = p_0 \cdot q + \lambda\, u(z).$$

□

Lemma 7.7. *Let M such that $L \wedge M \in \overline{\mathcal{G}}_{\lambda,z}$, then any nonzero element μ in $-N_{\overline{\mathcal{G}}_{\lambda,z}}(L \wedge M)$ belongs to $\mathfrak{P}$, up to a normalization. For any such μ we have*

$$\langle \mu, L \wedge M \rangle = \lambda\, u_\lambda(z).$$

Proof. We know that $L \wedge M \in \partial \mathcal{G}_{\lambda,z}$ for M large enough thanks to Proposition 7.6. Since $\mathcal{G}_{\lambda,z}$ is a convex set with nonempty interior in force of Proposition 7.4, we deduce from Proposition C.1 that $N_{\overline{\mathcal{G}}_{\lambda,z}}(L \wedge M)$ contains nonzero elements.

If one of the elements μ of $-N_{\overline{\mathcal{G}}_{\lambda,z}}(L \wedge M)$ were not positive, we would find $\Phi \in C_b(\mathbb{R}^{2N})$, $\Phi \geq 0$ with $\langle \mu, \Phi \rangle < 0$. This implies that $L \wedge M + \Phi$ belongs to $\overline{\mathcal{G}}_{z,\lambda}$ and

$$\langle \mu, L \wedge M + \Phi \rangle < \langle \mu, L \rangle,$$

which is in contrast with $-\mu$ belonging to the normal cone at $L \wedge M$. This proves that μ is a probability measure, up to a normalization. Since $\Phi \equiv \lambda\, u_\lambda(z)$ belongs to $\overline{\mathcal{G}}_{\lambda,z}$, we get

$$\langle \mu, L \wedge M \rangle \leq \lambda\, u_\lambda(z). \tag{38}$$

Since

$$\rho\,(L \wedge M) + (1 - \rho)\lambda\, u_\lambda(z) \in \overline{\mathcal{G}}_{\lambda,z} \qquad \text{for } \rho > 1,$$

we further get

$$(1 - \rho)\,\big(\langle \mu, L \wedge M \rangle - \lambda\, u_\lambda(z)\big) \leq 0$$

and consequently

$$\langle \mu, L \wedge M \rangle \geq \lambda\, u_\lambda(z).$$

This inequality, together with (38), completely gives the assertion. □

Corollary 7.8. *Let M_0 be such that $L \wedge M_0 \in \overline{\mathcal{G}}_{\lambda,z}$, then*

$$N_{\overline{\mathcal{G}}_{\lambda,z}}(L \wedge M_0) \supset N_{\overline{\mathcal{G}}_{\lambda,z}}(L \wedge M) \qquad \text{for any } M > M_0.$$

Moreover

$$\langle \mu, L \rangle \leq M_0 \quad \text{for } \mu \in -N_{\overline{\mathcal{G}}_{\lambda,z}}(L \wedge M) \cap \mathfrak{P},\ M \geq M_0.$$

Proof. Let $M > M_0$. If $\mu \in -N_{\overline{\mathcal{G}}_{\lambda,z}}(L \wedge M) \cap \mathfrak{P}$ then by Lemma 7.7 and the very definition of normal cone, we have

$$\langle \mu, L \wedge M \rangle = \lambda\, u_\lambda(z) \quad \text{and} \quad \langle \mu, \Phi \rangle \geq \lambda\, u_\lambda(z) \quad \text{for } \Phi \in \mathcal{G}_{\lambda,z}.$$

This implies that $\langle \mu, L \wedge M_0 \rangle \geq \lambda\, u_\lambda(z)$. On the other side, since $L \wedge M \geq L \wedge M_0$, the opposite inequality holds true as well. We deduce that

$$\langle \mu, L \wedge M_0 \rangle = \lambda\, u_\lambda(z),$$

which in turn implies that $\mu \in -N_{\overline{\mathcal{G}}_{\lambda,z}}(L \wedge M_0) \cap \mathfrak{P}$ showing the first part of the assertion.

We claim that

$$\operatorname{supp} \mu \subset \{(x,q) \mid L(x,q) \leq M_0\} =: W.$$

If not, there should be $(x_0, q_0) \in \operatorname{supp} \mu$ with $L(x_0, q_0) > M_0$. There thus should exist a neighborhood U of (x_0, q_0) with $\mu(U) > 0$ and

$$M \geq L(x,q) > M_0 \qquad \text{for any } (x,q) \in U \text{ and some } M > M_0,$$

so that

$$\int_U L \wedge M \, d\mu = \int_U L \, d\mu > \int_U L \wedge M_0 \, d\mu$$

and consequently

$$\langle \mu, L \wedge M \rangle > \langle \mu, L \wedge M_0 \rangle$$

in contrast to what shown above. We finally have

$$\langle \mu, L \wedge M \rangle = \int_W L \, d\mu \leq M_0.$$

□

Proposition 7.9. *There is $\mu \in \mathfrak{P}$ such that*

$$\langle \mu, \Phi \rangle \geq \langle \mu, L \rangle = \lambda\, u_\lambda(z)$$

for any $\Phi \in C(\mathbb{R}^{2N})$ bounded from below and admitting a λ–discounted subsolution u with $u(z) \geq u_\lambda(z)$.

Proof. We consider an increasing positively diverging sequence M_n with

$$L \wedge M_n \in \partial \mathcal{G}_{\lambda,z} \qquad \text{for any } n,$$

and $\mu_n \in -N_{\overline{\mathcal{G}}_{\lambda,z}}(L \wedge M_n) \cap \mathfrak{P}$. According to Corollary 7.8, we have

$$\langle \mu, L \rangle \leq M_1.$$

This implies by Lemma 6.3 that μ_n narrowly converges to some $\mu \in \mathfrak{P}$, up to subsequences. For any fixed j, we have that

$$\mu_n \in -N_{\overline{\mathcal{G}}_{\lambda,z}}(L \wedge M_j) \qquad \text{for } n \geq j$$

and consequently

$$\lambda\, u_\lambda(z) = \langle \mu_n, L \wedge M_j \rangle \qquad \text{for } n \geq j.$$

We deduce that
$$\lambda\, u_\lambda(z) = \lim_n \langle \mu_n, L \wedge M_j\rangle = \langle \mu, L \wedge M_j\rangle$$
and we get by the monotone convergence theorem, sending j to infinity
$$\langle \mu, L\rangle = \lambda\, u_\lambda(z).$$
If Φ is an element of $\mathbb{R}^{2N}$ satisfying the properties in the statement, we have by Proposition 7.5 that
$$\Phi \wedge M_n \in \overline{\mathcal{G}}_{\lambda,z} \qquad \text{for } n \text{ large enough}$$
then
$$\lambda\, u_\lambda(z) \leq \lim_j \langle \mu_j, \Phi \wedge M_n\rangle = \langle \mu, \Phi \wedge M_n\rangle,$$
which implies
$$\langle \mu, \Phi\rangle \geq \lambda\, u_\lambda(z).$$
This ends the proof. □

Proof of Theorem 7.1. Given Φ, u as indicated in the statement, we have that
$$\Phi + \lambda\,(u_\lambda(z) - u(z))$$
has a subsolution coinciding with u_λ at z. We derive from Proposition 7.9 that there exists μ with
$$\langle \mu, \Phi\rangle + \lambda\,(u_\lambda(z) - u(z)) \geq \lambda\, u_\lambda(z)$$
and
$$\langle \mu, L\rangle = \lambda\, u_\lambda(z).$$
This gives the assertion. □

Given $z \in \mathbb{R}^N$, $\lambda < \lambda_z$, we call (λ, z)– Mather measure, any measure μ satisfying the statement of Theorem 7.1. We denote by $\mathfrak{M}_{z,\lambda}$ the set of such measures μ.

The formula (34) can be seen as an analog, to Hamilton-Jacobi equations, of the representation of solutions of linear elliptic PDE via Green's kernel or Poisson integral. In this regard, for $\mu \in \mathfrak{M}_{z,\lambda}$ one may call the measure $\lambda^{-1}\mu$ a Green–Poisson measure associated with (λ, z).

8. Mather measures for the ergodic equation

We perform in this section a construction parallel to that of Section 7 to show existence of Mather measures for the ergodic equation.

The main result is:

Theorem 8.1. *There is $\mu \in \mathfrak{P}$ such that*
$$\langle \mu, \Phi\rangle \geq \langle \mu, L\rangle = 0$$
for any $\Phi \in C(\mathbb{R}^{2N})$ bounded from below and admitting a subsolution.

We call Mather measure any measure satisfying the statement of Theorem 8.1. We denote by $\mathfrak{M}$ the set of Mather measures. In Propositions 9.2 and 9.6 we will actually show something more, namely that any measure $\mu \in \mathfrak{M}$ is compactly supported and that the inequality $\langle \mu, \Phi\rangle \geq 0$ holds for any Φ admitting subsolution.

We start by:

Proposition 8.2. *Given a ball B, there exists $R > 0$ such that if a locally Lipschitz function u satisfies*

$$Du(x) \cdot q \leq L(x, q) - \varepsilon \tag{39}$$

for some $\varepsilon > 0$, a.e. $x \in B$, any q with $|q| \leq R$, then u is strict subsolution of $H[u] = 0$ in B.

Proof. We set $M = \sup_{x \in B, |q|=1} L(x, q)$. Exploiting (10), we can select $R > 1$ with

$$\inf_{x \in B, |q| > R} \frac{L(x, q)}{|q|} > M. \tag{40}$$

If (39) holds true for such an R then

$$|Du(x)| = Du(x) \cdot \frac{Du(x)}{|Du(x)|} \leq L\left(x, \frac{Du(x)}{|Du(x)|}\right) - \varepsilon \qquad \text{for a.e. } x \in B,$$

which shows that $|Du(x)| \leq M - \varepsilon$ in B. This in turn implies, in combination with (40)

$$Du(x) \cdot q \leq (M - \varepsilon)\, |q| \leq L(x, q) - \varepsilon$$

for a.e. $x \in B$, any q with $|q| > R$. This last inequality, together with (39), gives the assertion. □

We consider the set $\mathcal{G}$ of elements $\Phi \in C_b(\mathbb{R}^{2N})$ such that there exist $\varepsilon > 0$ and a Lipschitz continuous function u in B_0 with

$$Du(x) \cdot q \leq \Phi(x, q) - \varepsilon \qquad \text{for a,e, } x \in B_0,\ |q| \leq R_0, \tag{41}$$

where B_0 is an open ball containing $\mathcal{A}$, and so satisfying Proposition 5.1, and R_0 is the constant provided by Proposition 8.2 in correspondence to B_0.

Proposition 8.3. *The set $\mathcal{G}$ is a convex cone with vertex at 0 open in the strict topology.*

Proof. The cone property of $\mathcal{G}$ is apparent. Given $\Phi_0 \in \mathcal{G}$ satisfying (41), we claim that

$$\{\Phi \mid \|\Phi - \Phi_0\|_{\infty, K} < \varepsilon/2\} \subset \mathcal{G},$$

where $K = \overline{B}_0 \times \overline{B}_R$. This will prove the assertion because the set in the left hand–side of the above formula is an open neighborhood of Φ_0 with respect to rhe compact–open topology, and consequently with respect to the strict topology. For ϕ belonging to it, we in fact have

$$\Phi(x, q) \geq \Phi_0(x, q) - \frac{\varepsilon}{2} \qquad \text{for } x \in \overline{B}_0,\ q \in \overline{B}_{R_0},$$

then

$$\Phi(x, q) - \frac{\varepsilon}{2} \geq \Phi_0(x, q) - \varepsilon \geq Du(x) \cdot q \qquad \text{for a.e. } x \in B_0, \text{ any } q \in B_R.$$

This shows that $\Phi \in \mathcal{G}$. □

Arguing as in Proposition 7.5, we also get

Proposition 8.4. *Given $\Phi \in C(\mathbb{R}^{2n})$ bounded from below and possessing a subsolution u in $\mathbb{R}^N$, there exists M_0 such that $u \wedge M \in \overline{\mathcal{G}}$ for $M \geq M_0$.*

Proposition 8.5. *There exists M_0 such that $L \wedge M \in \partial\mathcal{G}$ for $M \geq M_0$.*

Proof. By Propositions 4.2 and 8.4, we have that $L \wedge M \in \overline{\mathcal{G}}$ for M suitably large, on the other hand $L \wedge M$ cannot be in $\mathcal{G}$ otherwise by Proposition 8.2 $H[u] = 0$ should admit a strict subsolution in B_0, which is against Proposition 5.1. □

We derive arguing as in Corollary 7.8

Corollary 8.6. *Let M_0 be such that $L \wedge M_0 \in \overline{\mathcal{G}}$, then*

$$N_{\overline{\mathcal{G}}}(L \wedge M_0) \supset N_{\overline{\mathcal{G}}}(L \wedge M) \qquad \textit{for any } M > M_0.$$

Moreover

$$\langle \mu, L \rangle \leq M_0 \quad \textit{for } \mu \in -N_{\overline{\mathcal{G}}}(L \wedge M) \cap \mathfrak{P},\ M \geq M_0.$$

We finally get Theorem 8.1 with the same argument as in Proposition 7.9.

9. Properties of Mather measures

Proposition 9.1. *Given $z \in \mathbb{R}^N$, $\lambda < \lambda_z$ we have*

$$\langle \mu, \lambda\, \psi + D\psi \cdot q \rangle = \lambda\, \psi(z)$$

for any (λ, z)–Mather measure μ, $\psi \in C^1(\mathbb{R}^N)$, constant outside a compact subset.

Proof. We define

$$\begin{aligned} \overline{L}_n(x,q) &= \lambda\, \psi(x) + D\psi(x) \cdot q + \frac{1}{n}\,(L(x,q) \vee 0), \\ \underline{L}_n(x,q) &= -\lambda\, \psi(x) - D\psi(x) \cdot q + \frac{1}{n}\,(L(x,q) \vee 0). \end{aligned}$$

It is clear that both $\overline{L}_n(x,q)$, $\underline{L}_n(x,q)$ are bounded from below and the functions $\pm\psi$ are λ–discounted subsolution for $\overline{L}_n$, $\underline{L}_n$, respectively. We then derive from Theorem 7.1 that

$$\langle \mu, \overline{L}_n \rangle \geq \lambda\, \psi(z) \qquad \text{and} \qquad \langle \mu, \underline{L}_n \rangle \geq -\lambda\, \psi(z),$$

which implies

$$|\langle \mu, D\psi \cdot q \rangle + \lambda\, \langle \mu, \psi \rangle - \lambda\, \psi(z)| \leq \frac{1}{n}\, \langle \mu, (L(x,q) \vee 0) \rangle.$$

Taking into account that $\langle \mu, L \vee 0 \rangle$ is finite because $L \vee 0$ is a compact perturbation of L, we further deduce sending n to infinity.

$$\langle \mu, \lambda\, \psi + D\psi \cdot q \rangle = \lambda\, \psi(z).$$

□

Proposition 9.2. *Any $\mu \in \mathfrak{M}$ is compactly supported. More precisely there exists $M > 0$ such that satisfying*

$$\operatorname{supp} \mu \subset \mathcal{A} \times B_M \qquad \textit{for any } \mu \in \mathfrak{M}.$$

Proof. Let μ be a Mather measure, we first prove that the support of the first marginal of μ, denoted by μ_1, is contained in the Aubry set, which is compact in force of Proposition 4.3. Assume by contradiction that there exists $y \in \operatorname{supp}\, \mu_1 \setminus \mathcal{A}$. This means that

$$\mu_1(U) = \mu(U \times \mathbb{R}^N) > 0 \quad \text{for any neighborhood } U \text{ of } y.$$

By Proposition A.2 there exists $\varepsilon > 0$, a neighborhood U_0 of y in $\mathbb{R}^N$, and a locally Lipschitz continuous function $v : \mathbb{R}^N \to \mathbb{R}$ with

$$Dv(x) \cdot q \leq L(x,q) \qquad \text{a.e. in } \mathbb{R}^N, \tag{42}$$

$$Dv(x) \cdot q \leq L(x,q) - \varepsilon \qquad \text{a.e. in } U_0. \tag{43}$$

We define

$$\overline{L}(x,q) = L(x,q) - \rho(x),$$

where ρ is a continuous nonnegative function supported in U_0 with $\max \rho = \varepsilon$. We derive from (42), (43) that $\overline{L}$ admits v as subsolution and is in addition bounded from below. On the other side, we get

$$\langle \mu, \overline{L} \rangle = \langle \mu, L \rangle - \int_{U_0} \rho \, d\mu_1 < 0,$$

in contrast with the definition of Mather measure. We have therefore found that the projection of supp μ with respect to the first component is contained in $\mathcal{A}$. Let B a ball in $\mathbb{R}^N$ containing $\mathcal{A}$. We set

$$R = \sup\{|p| \mid H(x,p) \leq 0, \, x \in B\}, \tag{44}$$

then R is a Lipschitz constant in B for any subsolution to $H[u] = 0$. According to (10), we can further choose a positive constant M with

$$L(x,q) > R\,|q| \qquad \text{for any } x \in B, \, |q| > M - 1.$$

We claim that

$$\text{supp } \mu \subset \mathcal{A} \times B_M.$$

In fact, assume for purposes of contradiction that there is $(y_0, q_0) \in \text{supp } \mu$ with $y_0 \in \mathcal{A}$, $|q_0| \geq M$. We take a neighborhood W of (y_0, q_0) in $\mathbb{R}^N \times \mathbb{R}^N$ with $W \subset B \times \{|q| > M - 1\}$ such that

$$L(x,q) > R\,|q| + \varepsilon \qquad \text{for any } (x,q) \in W, \text{ some } \varepsilon > 0. \tag{45}$$

We proceed defining

$$\widetilde{L}(x,q) = L(x,q) - \widetilde{\rho}(x,q),$$

where $\widetilde{\rho}$ is a continuous nonnegative function supported in W with $\max \widetilde{\rho} = \varepsilon$. Due to (44), (45), we see that any subsolution for L is still a subsolution for $\widetilde{L}$, and $\widetilde{L}$ is bounded from below. With the same computations as in the first part of the proof, we find that

$$\langle \mu, \widetilde{L} \rangle < 0,$$

which is impossible.

□

Looking back to the proof of the previous proposition, we realize that the argument actually shows a more general property.

Corollary 9.3. *Let $\mu \in \mathfrak{P}$ such that $\langle \mu, L \rangle = 0$ and $\langle \mu, \Phi \rangle \geq 0$ for all Φ admitting subsolutions such that*

$$\Phi(x,q) = L(x,q) \qquad \text{in } (\mathbb{R}^N \times \mathbb{R}^N) \setminus K, \text{ with } K \subset \mathbb{R}^{2N} \text{ compact.} \tag{46}$$

Then μ is compactly supported.

Corollary 9.4. *The set $\mathfrak{M}$ is a nonempty compact subset of the space of Radon measures endowed with the narrow topology.*

Proof. This is a consequence of all Mather measure being supported in the same compact, according to Proposition 9.2. The same holds true for any narrow limit μ of sequences μ_n in $\mathfrak{M}$, therefore

$$\langle \mu_n, \Phi \rangle \to \langle \mu, \Phi \rangle \qquad \text{for any } \Phi \in C(\mathbb{R}^{2N}).$$

□

We say that a measure μ is closed if

$$\langle \mu, Du \cdot q \rangle = 0 \qquad \text{for any } C^1 \text{ function } u.$$

We say in addition that it is locally closed if the above equality holds true just for C^1 functions with compact support. For a compactly supported measure the properties of being closed or locally closed are equivalent.

Proposition 9.5. *All the measures $\mu \in \mathfrak{M}$ are closed.*

Proof. Given $\mu \in \mathfrak{M}$, we consider a C^1 function ψ on $\mathbb{R}^N$, and set for $\varepsilon > 0$

$$\overline{L}_\varepsilon(x,q) = D\psi(x) \cdot q + \varepsilon\,(L(x,q) \vee 0), \qquad \underline{L}_\varepsilon(x,q) = -D\psi(x) \cdot q + \varepsilon\,(L(x,q) \vee 0).$$

The argument goes along the same lines as in Proposition 9.1. The functions $\pm\psi$ are subsolutions corresponding to $\overline{L}_\varepsilon$, $\underline{L}_\varepsilon$, so that

$$-\varepsilon\langle \mu, (L(x,q) \vee 0)\rangle \leq \langle \mu, D\psi \cdot q\rangle \leq \varepsilon\langle \mu, (L(x,q) \vee 0)\rangle. \tag{47}$$

Since $\langle \mu, L \vee 0\rangle$ is finite, and ε is arbitrary, we derive from (47)

$$\langle \mu, D\psi \cdot q\rangle = 0.$$

□

We finally get a characterization of $\mathfrak{M}$.

Proposition 9.6. *The following conditions are equivalent:*

- **(i)** $\mu \in \mathfrak{M}$,
- **(ii)** *μ is locally closed and $\langle \mu, L\rangle = 0$,*
- **(iii)** *$\langle \mu, L\rangle = 0$ and any Φ admitting subsolution is integrable with respect to μ with $\langle \mu, \Phi\rangle \geq 0$.*

Proof. The implication (**i**) $\Rightarrow$ (**ii**) has been already proved in Proposition 9.5. We proceed proving (**ii**) $\Rightarrow$ (**iii**). Let μ be a measure satisfying **(ii)**.

We take Φ admitting subsolution and coinciding with L outside a compact subset of $\mathbb{R}^{2N}$, namely satisfying (46), then the critical value of Φ is less than or equal 0, and the corresponding Hamiltonian H_Φ satisfies (A1)–(A3). We can therefore apply Proposition 2.1 to H_Φ and find that there is a compactly supported subsolution for Φ, say u.

Given $\varepsilon > 0$, we can regularize u obtaining a compactly supported smooth function $\overline{u}$ which is subsolution for $\Phi + \varepsilon$. Exploiting that μ is locally closed, we get

$$\langle \mu, \Phi + \varepsilon\rangle \geq \langle \mu, D\overline{u} \cdot q\rangle = 0,$$

and the positive quantity ε being arbitrary

$$\langle \mu, \Phi\rangle \geq 0.$$

This implies by Corollary 9.3 that μ is compactly supported, and consequently any function of $C(\mathbb{R}^{2N})$ is integrable with respect to μ. We proceed proving that $\langle \mu, \Phi\rangle \geq 0$ for any Φ admitting subsolution. We denote by B an open ball of $\mathbb{R}^N$ such that $\operatorname{supp}\mu \subset B \times \mathbb{R}^N$. Taken $\varepsilon > 0$ and Φ admitting a subsolution u, we can regularize u in some open ball containing B obtaining a function $\bar{u}$ of class C^1 in B such that

$$\Phi(x,q) + \varepsilon \geq D\bar{u}(x) \cdot q \qquad \text{for } (x,q) \in B \times \mathbb{R}^N.$$

Exploiting that $\operatorname{supp}\mu \subset B \times \mathbb{R}^N$ and that μ is closed, we therefore get

$$\langle \mu, \Phi + \varepsilon\rangle \geq \langle \mu, Du \cdot q\rangle = 0.$$

This proves the claim since ε has been arbitrarily chosen. The implication (**iii**) $\Rightarrow$ (**i**) is trivial. $\square$

10. Asymptotic results

The first asymptotic result is:

Theorem 10.1. *Given $z \in \mathbb{R}^N$ and an infinitesimal sequence $\lambda_j < \lambda_z$, we consider a sequence $\mu_j \in \mathfrak{M}_{\lambda_j,z}$, then μ_j narrowly converges, up to subsequences, to a probability measure $\mu \in \mathfrak{M}$.*

Proof. Since the sequence $\langle \mu_j, L\rangle = \lambda_j\, u_{\lambda_j}(z)$ is bounded by Proposition 3.1, Lemma 3.4, we get that μ_j narrowly converges to some measure μ, up to subsequences, in force of Lemma 6.3. Let $\psi \in C^1_c$, then by Proposition 9.1

$$\langle \mu_j, \lambda_j\, \psi + D\psi \cdot q\rangle = \lambda_j\, \psi(z). \tag{48}$$

Since ψ is compactly supported, then

$$\langle \mu_j, \psi\rangle \to \langle \mu, \psi\rangle \qquad \text{as } j \to +\infty$$

and by Lemma 6.4

$$\langle \mu_j, D\psi \cdot q\rangle \to \langle \mu, D\psi \cdot q\rangle \qquad \text{as } j \to +\infty.$$

Sending j to infinity, we thus derive from (48) that $\langle \mu, D\psi \cdot q\rangle = 0$, or in other terms that μ is locally closed. We further deduce via regularization of a compactly supported subsolution for L, which does exist by Proposition 2.1

$$\langle \mu, L\rangle \geq 0. \tag{49}$$

On the other side, we have by Lemma 6.2

$$0 = \lim_j \lambda_j\, u_{\lambda_j}(z) = \lim_j \langle \mu_j, L\rangle \geq \langle \mu, L\rangle,$$

so that

$$\langle \mu, L\rangle = 0.$$

This concludes the proof in force of the characterization of Mather measures provided in Proposition 9.6. $\square$

We define

$$w(x) = \max\{v(x) \mid v \text{ subsolution to (EP) with } \langle \mu, v\rangle \leq 0\ \forall \mu \in \mathfrak{M}.\} \tag{50}$$

Proposition 10.2. *The function w defined above is a weak KAM solution.*

Proof. As maximum of subsolutions, w is a subsolution to (EP). Since all the Mather measures are supported in $\mathcal{A} \times \mathbb{R}^N$, then w is the maximum of subsolutions with a given trace on $\mathcal{A}$. This implies the assertion by Lemma 4.5. $\square$

We give an alternative formula for w using the Peierls barrier.

Theorem 10.3. *The function w defined in* (50) *coincide with the function on $\mathbb{R}^N$ given by*

$$x \mapsto \min\{\langle \mu, P_0(\cdot, x)\rangle \mid \mu \in \mathfrak{M}\}. \tag{51}$$

Proof. We denote by u the function defined in (51). We know that the function $x \mapsto P_0(y,x)$ is a weak KAM solution for any y. By the convexity of $H(x,p)$ in the variables p, we deduce that the function $x \mapsto \langle \mu, P_0(\cdot,x)\rangle$ is a subsolution of (EP) and the same holds true for u.

Next, we show that $w \leq u$ in $\mathbb{R}^N$. Since w is a subsolution o (EP), we have

$$w(x) - w(y) \leq P_0(y,x) \quad \text{for all } x,y \in \mathbb{R}^N.$$

Integration of both sides of the above in y with respect to $\mu \in \mathfrak{M}$ yields

$$w(x) \leq \langle \mu, w\rangle + \langle \mu, P_0(\cdot,x)\rangle \leq \langle \mu, P_0(\cdot,x)\rangle.$$

This shows that $u \leq w$ in $\mathbb{R}^N$.

Since $-S_0(\cdot,z)$ is a subsolution of (EP), the function

$$y \mapsto -P_0(x,y) = \max_{z\in\mathcal{A}}(-S_0(x,z) - S_0(z,y))$$

is a subsolution as well. Thus, the function $y \mapsto -P_0(x,y) + u(x)$ is a subsolution of (EP) for all $x \in \mathbb{R}^N$. Integrating this function with respect to $\mu \in \mathfrak{M}$, we get

$$\langle \mu, -P_0(\cdot,x) + u(x)\rangle = -\langle \mu, P_0(\cdot,x)\rangle + \inf_{\nu\in\mathfrak{M}}\langle \nu, P_0(\cdot,x)\rangle \leq 0.$$

The definition of w in (50) guarantees that

$$w(y) \geq -P_0(y,x) + u(x) \quad \text{for all } x,y \in \mathbb{R}^N.$$

In particular, we have in view of Lemma A.3

$$w(z) \geq u(z) \quad \text{for all } z \in \mathcal{A}.$$

Since w is a weak KAM solution and u a subsolution, the inequality above ensures that $u \leq w$ in $\mathbb{R}^N$. Thus, we conclude that $u = w$ in $\mathbb{R}^N$. □

We proceed proving the main result:

Theorem 10.4. *The functions u_λ locally uniformly converge to w defined as in* (50)/(51).

A lemma is preliminary:

Lemma 10.5. *We have that*

$$\langle \mu, u_\lambda\rangle \leq 0 \qquad \textit{for any } \lambda > 0, \textit{ any Mather measure } \mu.$$

Proof. The function u_λ is a subsolution for $L - \lambda\, u_\lambda$. We then get by Proposition 9.6

$$0 \leq \langle \mu, L - \lambda\, u_\lambda\rangle = -\lambda\,\langle \mu, u_\lambda\rangle.$$

showing the assertion. □

Proof of Theorem 10.4. Let v be such that $u_{\lambda_j} \to v$ for some sequence λ_j converging to 0. We fix $z \in \mathbb{R}^N$ and assume $\lambda_j < \lambda_z$. We denote by μ_j a sequence of (λ_j, z)–Mather measures. Owing to Theorem 10.1, the μ_j converge, up to subsequences, to some probability measure $\mu \in \mathfrak{M}$.

We apply Proposition 2.1 to the function w defined in (50) with the compact subset $K = \mathcal{A} \cup \{z\}$. We obtain in this way a bounded subsolution $\bar{w}$ to $H[u] = 0$, coinciding with w on $\mathcal{A} \cup \{z\}$, which is at the same time a λ_j–discounted subsolution for $L + \lambda_j\,\bar{w}$. Since $L + \lambda_j\,\bar{w}$ is bounded from below, we get by Theorem 7.1

$$\langle \mu_j, L + \lambda_j\,\bar{w}\rangle \geq \lambda_j\, w(z)$$

and consequently

$$u_{\lambda_j}(z) + \langle \mu_j, \bar{w} \rangle \geq w(z). \tag{52}$$

The function $\bar{w}$ is a critical subsolution agreeing with w on the Aubry set, and so $\bar{w} \leq w$ on $\mathbb{R}^N$. We deduce from the definition of w in (50)

$$\langle \mu, \bar{w} \rangle \leq \langle \mu, w \rangle \leq 0,$$

and we get passing to the limit in (52) as $j \to +\infty$

$$v(z) \geq w(z). \tag{53}$$

On the other side, given any $\nu \in \mathfrak{M}$, we have by Lemma 10.5

$$\langle \nu, u_{\lambda_j} \rangle \leq 0$$

and, being ν compactly supported

$$\langle \nu, u_{\lambda_j} \rangle \longrightarrow \langle \nu, v \rangle,$$

which gives

$$\langle \nu, v \rangle \leq 0.$$

This last relation and (53) imply, by the maximality of w, $w(z) = v(z)$. This concludes the proof since z has been chosen arbitrarily. $\square$

11. Mather set

The (projected) Mather set $\mathcal{M}$ is defined as the image by the projection $((x,q) \mapsto x)$ of the set

$$\overline{\bigcup_{\mu \in \mathfrak{M}} \operatorname{supp} \mu}\,.$$

The main result of the section is:

Theorem 11.1. *Let u_0 be a weak KAM solution of* (EP)*. Then*

$$u_0(x) = \max\{v(x) \mid v \text{ weak KAM solution with } \langle \mu, w - u_0 \rangle \leq 0 \, \forall \, \mu \in \mathfrak{M}\}.$$

Note that by Lemma 4.5 the right hand–side of the above formula is equal to

$$\max\{v(x) \mid v \text{ subsolution to (EP) with } \langle \mu, w - u_0 \rangle \leq 0 \, \forall \, \mu \in \mathfrak{M}\}.$$

By the very definition of $\mathfrak{M}$, we have

$$\int_{\mathbb{R}^{2N}} \psi(x)\, d\mu = \int_{\mathcal{M} \times \mathbb{R}^N} \psi(x)\, d\mu \quad \text{for } \psi \in C(\mathbb{R}^N), \mu \in \mathfrak{M}.$$

Accordingly, Theorem 11.1 readily yields the following proposition.

Corollary 11.2. *Let v, w be weak KAM solutions of* (EP)*. Assume that $v \leq w$ in $\mathcal{M}$, then $v \leq w$ in $\mathbb{R}^N$.*

Remark by Proposition 9.2 that $\mathcal{M} \subset \mathcal{A}$. The corollary above claims that $\mathcal{M}$ is a uniqueness set of (EP), that is, if v, w are two weak KAM solutions of (EP) and $v = w$ in $\mathcal{M}$, then $v = w$ in $\mathbb{R}^N$. see [17], and [13], [21] for related results.

In our proof, we consider the following variation of the discount problem

$$\lambda v + H(x, Dv(x)) = \lambda u_0(x) \quad \text{in } \mathbb{R}^N, \tag{54}$$

where λ is a given positive constant and u_0 is a weak KAM solution as in Theorem 11.1. Here it is obvious that u_0 is a solution of (54).

Lemma 11.3. *Let u_0 be a weak KAM solution. Then, u_0 is a maximal subsolution of* (54).

Proof. Assume by contradiction that there is an usc subsolution v of (54) with $v(x) > u_0(x)$ at some point x. Since the maximum of two subsolutions is still a subsolution, we can assume in addition that $v \geq u_0$ in $\mathbb{R}^N$. Therefore

$$H(x, Dv) \leq \lambda\,(u_0(x) - v(x)) \leq 0,$$

so that v is a subsolution to (EP) and is locally Lipschitz–continuous. By Lemma A.2 we further derive that $v = u_0$ on $\mathcal{A}$. Since u_0 is a weak KAM solution, this implies that $u_0 \geq v$ in $\mathbb{R}^N$, which is contradictory. $\square$

Proof of Theorem 11.1. By Proposition 2.1 there exists a subsolution $\bar{u}$ of (EP) coinciding with u on $\mathcal{A}$ and constant at infinity. By regularization we get for any $\varepsilon > 0$ a sequence $\bar{u}_\varepsilon$ of C^1 functions satisfying

$$\begin{aligned} |\bar{u}_\varepsilon(x) - \bar{u}(x)| &< \varepsilon \qquad \text{for } x \in \mathbb{R}^N, \\ |\bar{u}_\varepsilon(x) - u_0(x)| &< \varepsilon \qquad \text{for } x \in \mathcal{A}. \end{aligned}$$

Taking into account that u_0 is a weak KAM solution we derive from Lemma 4.5

$$\bar{u}_\varepsilon(x) \leq \bar{u}(x) + \varepsilon \leq u_0(x) + \varepsilon \qquad \text{for } x \in \mathbb{R}^N.$$

We consider the equations

$$\lambda\, u + H(x, Du) \;=\; \lambda\, \bar{u}_\varepsilon, \tag{55}$$

$$\lambda\, u + H(x, Du - D\bar{u}_\varepsilon(x)) \;=\; 0. \tag{56}$$

It is easy to check that u is a subsolution to (55) if and only if $u - \bar{u}_\varepsilon$ is a subsolution of (56). Since $u_0 + \varepsilon$ is the maximal solution of

$$\lambda\, u + H(x, Du) = \lambda\,(u_0 + \varepsilon)$$

by Lemma 11.3 and $u_0 + \varepsilon \geq \bar{u}_\varepsilon$ in $\mathbb{R}^N$, we deduce that

$$u_0 + \varepsilon \geq u \qquad \text{for any subsolution to (55).}$$

We define the Lagrangian

$$L_\varepsilon(x, q) = L(x, q) + D\bar{u}_\varepsilon(x) \cdot q$$

corresponding to the Hamiltonian $H(x, Du - D\bar{u}_\varepsilon(x))$. A function u is subsolution to $H[u] = a$, for any $a \in \mathbb{R}$, if and only if $u - \bar{u}_\varepsilon$ is subsolution to $H(x, Du + D\bar{u}_\varepsilon(x)) = a$, this implies that L and L_ε has both 0 as critical value. In addition, Mather measures being closed, we have that $\mathfrak{M} = \mathfrak{M}(L_\varepsilon)$, where $\mathfrak{M}(L_\varepsilon)$ indicates the Mather measures associated with L_ε. By applying Theorem 10.4 to L_ε, we see that the maximal subsolutions of (56) converge to

$$\begin{aligned} &\max\{u \text{ subsolution for } L_\varepsilon \text{ with } \langle \mu, u\rangle \leq 0\,\forall\,\mu \in \mathfrak{M}\} \\ =\;& \max\{u - \bar{u}_\varepsilon \mid\; u \text{ subsolution for } L \text{ with } \langle \mu, u\rangle \leq \langle \mu, \bar{u}_\varepsilon\rangle\,\forall\,\mu \in \mathfrak{M}\}. \end{aligned}$$

We derive that

$$\begin{aligned} u_0(x) + \varepsilon \;&\geq\; \max\{u(x) \;\text{ subsolution for } L \text{ with } \langle \mu, u\rangle \leq \langle \mu, \bar{u}_\varepsilon\rangle\,\forall\,\mu \in \mathfrak{M}\} \\ &\geq\; \max\{u(x) \;\text{ subsolution for } L \text{ with } \langle \mu, u\rangle \leq \langle \mu, u_0\rangle\,\forall\,\mu \in \mathfrak{M}\} - \varepsilon \\ &\geq\; u_0(x) - 2\,\varepsilon. \end{aligned}$$

We get the assertion sending ε to 0.

□

Appendix A. Weak KAM facts

We define an intrinsic (semi)distance $S_0(\cdot,\cdot)$ in $\mathbb{R}^N$ related to the ergodic equation (see [13, 17] and also [14]) via

$$S_0(x,y) = \sup\{u(y) - u(x) \mid u \text{ subsolution of (EP)}\} \quad \text{for } x, y \in \mathbb{R}^N.$$

Since the family of subsolution to (EP) vanishing at some point $y \in \mathbb{R}^N$, is locally equi-Lipschitz continuous and, hence, locally uniformly bounded in $\mathbb{R}^N$, the function $x \mapsto S_0(x,y)$ is well-defined as a locally Lipschitz continuous function in $\mathbb{R}^N$.

Moreover, because of the stability of the viscosity properties under locally uniform convergence, the function $x \mapsto S_0(x,y)$ is a subsolution of (EP) for any $y \in \mathbb{R}^N$. It is clear that $S_0(x,x) = 0$ for all $x \in \mathbb{R}^N$ and that $S_0(x,y) \le S_0(x,z) + S_0(z,y)$ for all $x, y, z \in \mathbb{R}^N$. In view of the Perron method, for any $y \in \mathbb{R}^N$, the function $x \mapsto S_0(x,y)$ is a solution of (EP) in $\mathbb{R}^N \setminus \{y\}$.

Due to the convexity of H in p, it turns out that S_0 is the geodesic distance related to a length functional of the curves in $\mathbb{R}^N$. We define

$$\sigma_0(x,q) = \max\{p \cdot q \mid H(x,p) \le 0\} \quad \text{for } (x,q) \in \mathbb{R}^N \times \mathbb{R}^N$$

and, given a (Lipschitz continuous) curve $\xi : [0,1] \to \mathbb{R}^N$, we set

$$\ell_0(\xi) = \int_0^1 \sigma_0(\xi, \dot\xi)\, dt.$$

Note that the above integral is invariant for orientation preserving change of parameter. We have

$$\begin{aligned} S_0(y,x) &= \inf\{\ell_0(\xi) \mid \xi : [0,1] \to \mathbb{R}^N \text{ with } \xi(0) = y,\, \xi(1) = x\} \\ &= \inf\left\{\int_0^T L(\zeta, \dot\zeta)\, dt \mid T > 0,\, \zeta : [0,T] \to \mathbb{R}^N \text{ with } \zeta(0) = y,\, \zeta(T) = x\right\}. \end{aligned}$$

We define the Aubry set $\mathcal{A}$ as

$$\mathcal{A} = \{y \in \mathbb{R}^N \mid S_0(y,\cdot) \text{ is a solution to } H = 0\}.$$

Proposition A.1. *An element $y \in \mathcal{A}$ if and only there is a sequence ξ_n of cycles based on y with*

$$\inf_n \ell(\xi_n) > 0, \qquad \lim_n \ell_0(\xi_n) = 0.$$

Proof. One can argue as in [14, Lemma Proposition 5.4 – Lemma 5.5].

□

Proposition A.2. *Given $x \in \mathbb{R}^N$, if there is a subsolution of* (EP) *which is strict in some neighborhood of x then $x \notin \mathcal{A}$, conversely if $x \notin \mathcal{A}$ there exists a subsolution of* (EP) *which is strict in some neighborhood of x.*

Proof. If such a subsolution u does exist for $x \in \mathcal{A}$, we find by maximality properties of $S_0(\cdot,x)$, that the function $u(x) + S_0(x,\cdot)$ is supertangent to u at x. Being $S_0(\cdot,x)$ solution, there is $p_0 \in \partial u(x)$ (the generalized gradient of u at x), with $H(x,p_0) \ge 0$, on the other side, being u strict subsolution, any $p \in \partial u(x)$ satisfies $H(x,p) < 0$, which is contradictory.

Conversely, if $x \notin \mathcal{A}$, then $S_0(x, \cdot)$ is not a solution to (EP) and there exists consequently a strict subtangent to $S_0(x, \cdot)$ at x with $H(x, D\psi(x)) < 0$, then the function

$$\min\{S_0(x, \cdot), \psi + a\}$$

is a subsolution of (EP) locally strict around x, for a suitable choice of $a > 0$. $\square$

The function P_0 in $\mathbb{R}^{2N}$ given by

$$P_0(x, y) = \min_{z \in \mathcal{A}} [S_0(x, z) + S_0(z, y)] \quad \text{for } x, y \in \mathbb{R}^N$$

is called the Peierls barrier. See [7, Proposition 3.7.2] and [1, 5, 10, 13].

Lemma A.3. *For any $z \in \mathbb{R}^N$, $P_0(z, z) = 0$ if and only if $z \in \mathcal{A}$.*

Proof. First of all, we examine some properties of the function P_0. Since

$$0 = S_0(x, x) \leq S_0(x, y) + S_0(y, x) \quad \text{for all } x, y \in \mathbb{R}^N,$$

we find that $P_0(x, x) \geq 0$ for all $x \in \mathbb{R}^N$. Note next that if $z \in \mathcal{A}$, then the function $x \mapsto S_0(z, x)$ is a weak KAM solution of (EP). Hence, the function $x \mapsto P_0(y, x)$ is a weak KAM solution of (EP) as well, for any $y \in \mathbb{R}^N$. We note by the triangle inequality for S_0 that for any $x, y \in \mathbb{R}^N$,

$$S_0(x, y) \leq \min_{z \in \mathcal{A}} [S_0(x, z) + S_0(z, y)] = P_0(x, y).$$

Now, we assume that $z \in \mathcal{A}$. We have

$$0 \leq P_0(z, z) \leq S_0(z, z) + S_0(z, z) = 0.$$

Hence, $P_0(z, z) = 0$.

Next, assume that $S_0(z, z) = 0$. We need to show that the function $x \mapsto S_0(z, x)$ is a solution of (EP). In fact, since the function $x \mapsto S_0(z, x)$ is a solution of (EP) in $\mathbb{R}^N \setminus \{z\}$, we only need to show that $H(z, D\psi(z)) \geq 0$ for all C^1 subtangent ψ to $S_0(z, \cdot)$ at z. Indeed, such a function is also subtangent to $P_0(z, \cdot)$ at z, and the sought inequality comes from $P_0(z, \cdot)$ being solution to (EP). This completes the proof. $\square$

Appendix B. Strict topology

We denote by $C_0(\mathbb{R}^{2N})$, $C_c(\mathbb{R}^{2N})$ the space of compactly supported and vanishing at infinity continuous functions, respectively. We endow the space of continuous bounded functions in $\mathbb{R}^{2N}$, denoted by $C_b(\mathbb{R}^{2N})$, with the strict topology. It is is the locally convex Hausdorff topology defined by the family of seminorms

$$\{\| \cdot \|_\Psi \mid \Psi \in C_0(\mathbb{R}^{2N})\},$$

where

$$\|\Phi\|_\Psi = \|\Phi \, \Psi\|_\infty \qquad \text{for any } \Psi \in C_b(\mathbb{R}^{2N}).$$

We recall that the compact open topology is instead given by the seminorms

$$\{\| \cdot \|_\Psi \mid \Psi \in C_c(\mathbb{R}^{2N})\}.$$

It induces the local uniform convergence and a base of neighborhoods at any given $\Phi_0 \in C_b(\mathbb{R}^{2N})$ is given by

$$\{\Phi \mid \|\Phi - \Phi_0\|_{\infty, K} < \varepsilon\} \qquad \text{with } K \text{ compact subset of } \mathbb{R}^N,\ \varepsilon > 0.$$

The strict topology is stronger than the compact–open topology since it has a larger class of defining seminorms. Any open set for the compact–open topology is consequently an open set for the strict one. Further, the strict topology is weaker than the topology induced by $\| \cdot \|_\infty$. Also recall that the completion of $C_c(\mathbb{R}^{2N})$ with respect to the norm topology is $C_0(\mathbb{R}^{2N})$, while it is $C_b(\mathbb{R}^{2N})$ in the strict topology.

The interest of introducing the strict topology is that we get in this frame a nice generalization of Riesz representation theorem, namely the topological dual of $C_b(\mathbb{R}^{2N})$ is the space of signed Radon measures with bounded variation, the normalized positive elements are then Radon probability measures, see [4]. The corresponding weak star topology on the dual, namely the weakest topology for which

$$\mu \mapsto \int \Phi \, d\mu$$

is continuous for any $\Phi \in C_b(\mathbb{R}^{2N})$ is called the narrow topology. Accordingly a sequence of measures μ_n narrow converges to some μ if

$$\int \Phi \, d\mu_n \to \int \Phi \, d\mu \qquad \text{for any } \Phi \in C_b(\mathbb{R}^{2N}).$$

The matter is slippery because the bounded signed Radon measures make up the topological dual of $C_0(\mathbb{R}^{2N})$ with the norm topology as well, but the induced weak star topology, the so–called vague topology, is strictly weaker than the narrow topology. Regarding the dual of $C_b(\mathbb{R}^{2N})$ with the norm topology, it is given by the bounded signed measures on the Stone–Cech compactification of $\mathbb{R}^{2N}$.

Appendix C. Separation theorem

Let X be a general locally convex Hausdorff space, we indicate by X^* its topological dual and by $(\cdot, \cdot)$ the pairing between X^* and X. Given a closed convex subset E and $x \in \partial E$, we denote by $N_E(x)$ the normal cone to E at X, defined as

$$N_E(x) = \{p \in X^* \mid (p, y - x) \leq 0 \text{ for any } y \in E\}.$$

Note that in contrast to what happens for finite dimensional spaces, in the infinite dimensional case $N_E(x)$ can reduce to $\{0\}$, see for instance [6]. However we have

Proposition C.1. *Let E be a closed convex subset of X with nonempty interior, then $N_E(x)$ contains nonzero elements for any $x \in \partial E$.*

This is actually a simple consequence of the hyperplane separation theorem in locally convex Hausdorff spaces, see [22], which can be stated as follows:

Theorem C.2. *Ler E be a convex subset of X with nonempty interior and $y \notin E$. There exists $0 \neq p \in X^*$ with*

$$(p, y) \geq (p, x) \qquad \textit{for any } x \in E.$$

To get Proposition C.1 it is enough to use the property that the interior of any convex set is convex, and to apply the hyperplane separation theorem to the interior of E and to any point in ∂E.

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Institute for Mathematics and Computer Science, Tsuda University, 2-1-1 Tsuda, Kodaira, Tokyo 187-8577, Japan
Email address: hitoshi.ishii@waseda.jp

Department of Mathematics, Università degli Studi di Roma "La Sapienza", Piazzale Aldo Moro 5, 00185 Roma, Italy.
Email address: siconolf@mat.uniroma1.it