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***Abstract*— The paper concerns the Linear Quadratic non-Gaussian (LQnG) sub-optimal control problem when the input and output signals travel through an unreliable network, namely Gilbert-Elliot channels. In particular, the input/output packet losses are modeled by Bernoulli sequences, and we assume that the moments of the non-Gaussian noises up to the fourth order are known. By mean of a suitable rewriting of the system through an intermittent output injection term, and by considering an augmented system with the second-order Kronecker power of the measurements, a simple solution is provided by substituting the Kalman predictor with intermittent observations of the LQG control law with a quadratic optimal predictor. Numerical simulations show the effectiveness of the proposed method.**

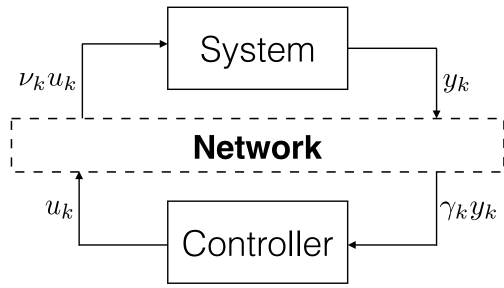

Fig. 1. Network & System diagram.

I. INTRODUCTION

Remote estimation and control of plants over unreliable networks have been hot topics in the last decades *e.g.* [25], [17], [23], [4], [22], [27], [7]. In this domain, temporary failures are an important issue, due to power constraints, communication delay, multi-path fading, data loss, background noise time synchronization or external attacks.

Besides, in many real engineering applications, the widely used Gaussian assumption cannot be accepted as a realistic statistical description of the random quantities involved, in particular in the case of heavy-tailed distributions. Recent developments on non-Gaussian systems in control engineering include the robust approach of [33] and [3], [5], [10] where polynomial suboptimal solution are exploited. The Maximum Correntropy Criterion for Kalman filtering, adopted firstly in the paper [14], has been exploited also in many works (*e.g.* [34], [21]) and in numerous applications (*e.g.* [20], [15], [28]). Furthermore, non-Gaussian problems often arise in digital communications when the noise interference includes noise components that are essentially non-Gaussian [29], in problems concerning fault estimation, sensor or actuator faults [26], multiplicative noises and bilinear systems [13]. In monopulse radars, heavy tailed non-Gaussian behavior is present in the angle tracking signals because of target glint [8]. Also, under some conditions, Gaussian systems with nonlinear measurements could be transformed, through a suitable rewriting of the output map, into systems with linear measurements and non-Gaussian output noise [9]. In this framework, suboptimality is something essential to design practical computable filters and/or control laws. Monte-Carlo methods [2], sum of Gaussian densities [1], and weighted sigma points [19] are some of the approaches employed to approximate the conditional distribution of the underlying non-Gaussian process, and they generally have high computational cost.

In this paper we will focus on the Linear Quadratic non-Gaussian (LQnG) suboptimal control problem when the input and output signals are subjected to packet losses, modeled as Bernoulli sequences. In this framework, the hypothesis of perfect acknowledgment of packet drops is done (sometimes known in the literature as the TCP-like case). Our aim is to extend the work [31] by removing the Gaussian assumption of the noise sequences. In order to cope with these non-Gaussian noise sequences, an effective alternative solution to the aforementioned methods, in the minimum variance sense, is to look for predictors that make use of quadratic (or generally polynomial) transformations of the measurements in order to enhance the estimation accuracy, maintaining simple computability and recursion. We note that the prediction provided by a quadratic predictor has been exploited in the paper [6] where Markovian input packet losses only are considered.

We finally remark that when either no acknowledgment or only imperfect acknowledgment occurs, then the separation principle does not hold true, and the joint design of estimator and controller becomes a non-convex problem, as shown in [32]. The authors of [24] investigate the latter case. Conversely, since we assume that a perfect acknowledgment mechanism is available, we shall see that the separation result, even in the non-Gaussian framework, is still valid. The resulting quadratic optimal controller yields better performance in terms of the standard quadratic cost function with respect to the standard linear optimal controller, namely the

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one of [31], and it is simply obtained by replacing the linear optimal prediction provided by the Kalman predictor with intermittent observations, with the quadratic optimal one, in virtue of the proved separation principle.

Notation. The Kronecker product of two matrices $A \in \mathbb{R}^{n\times m}$ and $B \in \mathbb{R}^{p\times q}$ is denoted by $A\otimes B$. The i-th Kronecker power of A is $A^{[i]}$, where $A^{[i]} = A\otimes A^{[i-1]}$, with $A^{[1]} = A$. The vectorization (or stack) function is denoted by $\mathrm{st}\{A\}$, and $\mathrm{st}^{-1}\{\cdot\}$ is its inverse function (we omit to specify the column size when it is clear from the context). The trace of a square matrix A is $\mathrm{tr}\{A\}$ and $v = \mathrm{col}(v_1,\dots,v_n)$ denotes the column vector $v = [v_1,\dots,v_n]^\top$, where $v_1,\dots,v_n$ are the entries of the vector v. The Moore-Penrose pseudo-inverse of a matrix A is denoted by $A^\dagger$. Moreover, given a vector $v\in\mathbb{R}^n$, then $v^{1:m}$, with $m < n$, denotes the vector of the first m entries of v.

II. Problem Formulation and Preliminaries

The control problem we aim to solve concerns the class of linear, detectable and stabilizable systems driven by non-Gaussian additive noise described by the following equations:

$$x_{k+1} = Ax_k + \nu_k Bu_k + f_k, \tag{1}$$

$$y_k = \gamma_k(Cx_k + g_k), \tag{2}$$

with the associated cost functional

$$J_{\mathsf{N}} = \mathbb{E}\left[x_{\mathsf{N}}^\top W_{\mathsf{N}} x_{\mathsf{N}} + \sum_{k=0}^{\mathsf{N}-1} x_k^\top W_k x_k + \nu_k u_k^\top U_k u_k\right] \tag{3}$$

where $\mathsf{N}\in\mathbb{N}$ is the time-horizon. For $k\geq 0$, $x_k\in\mathbb{R}^n$ is the state, $f_k\in\mathbb{R}^n$ and $g_k\in\mathbb{R}^q$ are not necessarily Gaussian noise sequences with strictly positive covariances, $ut_k\in\mathbb{R}^p$ is the control signal, $y_k\in\mathbb{R}^q$ is the measurement output and the matrices A, B, C, W_k, U_k are of appropriate dimensions. As usual, the matrices W_k, and U_k are symmetric non-negative definite (strictly positive definite in the case of U_k). Furthermore, we consider the case when the control signal u_k and the output signal y_k travel along an unreliable network, namely a Gilbert-Elliot channel model, which is modeled by Bernoulli sequences. Thus, let $\{\nu_k\}$ and $\{\gamma_k\}$ be Bernoulli sequences taking values in the set $\{0,1\}$, modeling the presence of packet losses in the actuators and missing observations. We note that if $\nu_k = 1$ ($\gamma_k = 1$), then the actuators (sensors) receive the input u_k (the measurement y_k), namely no failures have occurred, $\nu_k = 0$ ($\gamma_k = 0$) otherwise.

More precisely, denoting $\bar{x}_0 = \mathbb{E}[x_0]$, the expectation of the initial state x_0, we have the following conditions for $k\geq 0$:

(i) $\{f_k\}$ and $\{g_k\}$ are a zero mean i.i.d. sequence;

(ii) $\{f_k\}$, $\{g_k\}$ and x_0 have uncorrelated moments up to the fourth order;

(iii) for $i = 1,2,3,4$ there exist finite and known vectors $\Psi_{f,i} \doteq \mathbb{E}\left[f_k^{[i]}\right]$, $\Psi_{g,i} \doteq \mathbb{E}\left[g_k^{[i]}\right]$, and $\Psi_{x_0,i} \doteq \mathbb{E}\left[(x_0-\bar{x}_0)^{[i]}\right]$.

(iv) $\{\nu_k\}$ and $\{\gamma_k\}$ are independent Bernoulli sequences with $\mathbb{P}(\nu_k = 1) = 1-\mathbb{P}(\nu_k = 0) = \bar{\nu}$ and $\mathbb{P}(\gamma_k = 1) = 1-\mathbb{P}(\gamma_k = 0) = \bar{\gamma}$;

Clearly $\Psi_{x_0,1} = 0$ and (i) implies $\Psi_{f,1} = \Psi_{g,1} = 0$. Note that, when the sequences $\{f_k\}$, $\{g_k\}$ and x_0 are mutually independent, then assumption (ii) is satisfied. We set $\Psi_{x_0} = \mathrm{st}^{-1}\{\Psi_{x_0,2}\}$, $\Psi_f = \mathrm{st}^{-1}\{\Psi_{f,2}\}$, $\Psi_g = \mathrm{st}^{-1}\{\Psi_{g,2}\}$ the covariance matrices of the initial state, state noise and output noise, respectively. It is clear that in the non-Gaussian framework, the knowledge of the first four moments of the state and output noise sequences is weaker than assuming the knowledge of the whole probability distributions.

Finally, we shall consider control sequences $\{u_k\}_k$ measurable with respect to the σ-Algebra $\mathscr{F}_k = \sigma(y_\ell,\nu_\ell,\gamma_\ell \quad j\leq k-1)$. Thus, we note that the quantities ν_{k-1}, γ_k is available at time $k\geq 0$, which means that a reliable acknowledgment protocol is implemented (the so-called TCP-like case [18]). As pointed out in [16], this assumption is reasonable in several practical applications.

In this framework, we consider the finite-horizon suboptimal control problem for non-Gaussian discrete-time linear systems with partial state information and Input/Output (I/O) Bernoulli packet losses. More precisely, our aim is to compute the control law in the class of recursively computable quadratic output feedback which minimizes (3).

We recall the important result of [31].

Proposition 1 ([31]): For the finite-horizon LQG regulator problem with I/O Bernoulli packet losses, the $\mathscr{F}_k$-measurable optimal output feedback control u_k is given by

$$u_k = -M_k\check{x}_{k|k-1}, \tag{4}$$

with

$$M_k = (U_k + B^\top S_{k+1}B)^{-1}B^\top S_{k+1}A, \tag{5}$$

where S_k is the solution of the backward Riccati equation

$$\begin{aligned} S_k = {} & W_k + A^\top S_{k+1}A + \\ & -\bar{\nu}A^\top S_{k+1}B(U_k + B^\top S_{k+1}B)^{-1}B^\top S_{k+1}A \end{aligned} \tag{6}$$

with final conditions $S_{\mathsf{N}} = W_{\mathsf{N}}$, and $\check{x}_{k|k-1}$ is the optimal prediction of x_k provided by the Kalman predictor (see [30]).

The next corollary is a straightforward consequence of the separation principle proved by the previous theorem.

Corollary 1: For the finite-horizon LQ non-Gaussian regulator problem (1)-(2)-(3) with I/O Bernoulli packet losses, the $\mathscr{F}_k$-measurable output feedback *linear* optimal control u_k is given by (4).

In other words, Corollary 1 states that the control input provided by (4) remains optimal in the class of *linear* transformations of the output: a direct consequence of the fact that, if the state and output noise sequences are non-Gaussian, then the KF (KP) is the optimal estimator (predictor) in the class of *linear* transformations of the output.

III. Quadratic Filtering and Prediction

A. Output injection

In this section we rewrite the system by mean of an output injection term, which is crucial to ensure some important properties of the quadratic predictor we shall see in Section

III-C. Thus, the state equation (1) is transformed using the intermittent measurements equation (2):

$$\begin{aligned} x_{k+1} &= Ax_k + \nu_k Bu_k + f_k \\ &= Ax_k + \nu_k Bu_k + f_k + L_k y_k - \gamma_k L_k C x_k - \gamma_k L_k g_k \\ &= \tilde{A}_k x_k + \nu_k Bu_k + L_k y_k + h_k, \end{aligned} \tag{7}$$

where $\tilde{A}_k = A - \gamma_k L_k C$, $L_k \in \mathbb{R}^{n\times q}$, and $h_k = f_k - \gamma_k L_k g_k$. Moreover, by noticing that $\gamma_k^p = \gamma_k$ for any $p \in \mathbb{N}$, the moments $\psi_{h,k}^{(i)} := \mathbb{E}\left[h_k^{[i]} \,|\, \gamma_k\right]$, $i = 2,3,4$, can be computed as functions of $\psi_f^{(i)}$, $\psi_g^{(i)}$ and γ_k as follows

$$\psi_{h,k}^{(2)} = \psi_f^{(2)} + \gamma_k L_k^{[2]} \psi_g^{(2)}, \tag{8}$$

$$\psi_{h,k}^{(3)} = \psi_f^{(3)} - \gamma_k L_k^{[3]} \psi_g^{(3)}, \tag{9}$$

$$\psi_{h,k}^{(4)} = \psi_f^{(4)} + \gamma_k M_2^4 \left(\psi_f^{(2)} \otimes L_k^{[2]} \psi_g^{(2)}\right) + \gamma_k L_k^{[4]} \psi_g^{(4)}, \tag{10}$$

where M_2^4 is the matrix coefficient of the binomial power formula (see Theorem 2.2.5 of [12]). Thus, since the sequence $\{\gamma_k\}$ is known online, the moments $\psi_{h,k}^{(i)}$ are known at each $k \geq 0$.

Furthermore, we split the state process into two sequences: a predictable sequence $\{x_k^p\}$ and a stochastic sequence $\{x_k^s\}$. The predictable component x_k^p satisfies

$$x_{k+1}^p = \tilde{A}_k x_k^p + \nu_k Bu_k + L_k y_k, \quad x_0^p = \bar{x}_0, \tag{11}$$

while the stochastic component x_k^s is the solution of

$$x_{k+1}^s = \tilde{A}_k x_k^s + h_k, \quad x_0^s = x_0 - \bar{x}_0, \tag{12}$$

and therefore $\psi_{x_0^s}^{(i)} := E\left[x_0^{s\,[i]}\right] = \psi_{x_0}^{(i)}$. From (11) and (12), for any $k \geq 0$, it follows $x_k = x_k^p + x_k^s$. Moreover, note that at time $k \geq 0$, since the quantities y_{k-1}, γ_{k-1}, and ν_{k-1} are available, the predictable component x_k^p is known. Subsequently, we can define the output map of the stochastic component (12) as

$$y_k^s = y_k - \gamma_k C x_k^d = \gamma_k (C x_k^s + g_k), \tag{13}$$

where y_k^s is an available quantity at time $k \geq 0$.

Finally, we remark that the space of quadratic transformations of the output depends on the sequence $\{L_k\}$ (see [11]).

We shall see in the next section how to compute the optimal recursive quadratic estimate, *i.e.* the optimal estimate (in the minimum variance sense) among the quadratic transformations of the output, in the case of intermittent observations, and how to select the output injection gain matrices $\{L_k\}$ in order to guarantee the stability of the algorithm and the improvement of the performance with respect to the standard Kalman filter.

B. Optimal Recursive Quadratic Control with I/O Packet Losses

We prove in this section that the structure of the optimal recursive quadratic controller for the LQ non-Gaussian regulator problem with I/O packet losses remains unchanged, namely the gain (5) with the backward Riccati equation (6). For, we define the following vectors

$$\mathscr{X}_k = \mathrm{col}\left(x_k^p, x_k^s, x_k^{s\,[2]}\right), \tag{14}$$

$$\mathscr{Y}_k = \mathrm{col}\left(y_k^s, y_k^{s\,[2]}\right), \tag{15}$$

where x_k^p and x_k^s are defined in (11)–(12).

Lemma 1: The augmented state and output sequences $\{\mathscr{X}_k\}$ and $\{\mathscr{Y}_k\}$ defined in (14)–(15) obey to the following equations

$$\mathscr{X}_{k+1} = \mathscr{A}_k \mathscr{X}_k + \nu_k \mathscr{B} u_k + \phi_{h,k} + \mathscr{V}_k \tag{16}$$

$$\mathscr{Y}_k = \gamma_k \left(\mathscr{C} \mathscr{X}_k + \varphi_g + \mathscr{G}_k\right), \tag{17}$$

with

$$\mathscr{A}_k = \begin{bmatrix} A & L_k C & 0 \\ 0 & \tilde{A}_k & 0 \\ 0 & 0 & \tilde{A}_k^{[2]} \end{bmatrix} \mathscr{B} = \begin{bmatrix} B \\ 0 \\ 0 \end{bmatrix} \mathscr{C} = \begin{bmatrix} 0 & C & 0 \\ 0 & 0 & C^{[2]} \end{bmatrix}, \tag{18}$$

$$\phi_{h,k} = \begin{bmatrix} 0 \\ 0 \\ \psi_{h,k}^{(2)} \end{bmatrix} \mathscr{V}_k = \begin{bmatrix} Lg_k \\ h_k \\ h_k^{(2)} \end{bmatrix} \varphi_g = \begin{bmatrix} 0 \\ \psi_g^{(2)} \end{bmatrix} \mathscr{G}_k = \begin{bmatrix} g_k \\ g_k^{(2)} \end{bmatrix}, \tag{19}$$

where

$$h_k^{(2)} = \tilde{A}_k x_k^s \otimes h_k + h_k \otimes \tilde{A}_k x_k^s + h_k^{[2]} - \psi_{h,k}^{(2)}, \tag{20}$$

$$g_k^{(2)} = C x_k^s \otimes g_k + g_k \otimes C x_k^s + g_k^{[2]} - \psi_g^{(2)}. \tag{21}$$

Proof. By noticing that $x_k^{s\,[2]}$ and $y_k^{s\,[2]}$ satisfy

$$\begin{aligned} x_{k+1}^{s\,[2]} &= \tilde{A}_k^{[2]} x_k^{s\,[2]} + \psi_{h,k}^{(2)} + h_k^{(2)}, \\ y_k^{s\,[2]} &= \gamma_k \left(C^{[2]} x_k^{s\,[2]} + \psi_g^{(2)} + g_k^{(2)}\right), \end{aligned}$$

where $h_k^{(2)}$ and $g_k^{(2)}$ are defined in (20)–(21), it is straightforward to obtain (16)–(17). □

Remark 1: We note that the vector $\phi_{h,k}$ and φ_g are known online, whilst the stochastic sequences $\{\mathscr{V}_k\}$ and $\{\mathscr{G}_k\}$ are zero-mean, mutually correlated, white, and uncorrelated with the initial state x_0.

The following lemma provides the algorithm to compute the recursive quadratic estimate and prediction.

Lemma 2: The recursive quadratic estimate and the recursive quadratic prediction are given by the following algorithm

$$x_0^p = \bar{x}_0, \quad \widehat{\mathscr{X}}^s_{0|-1} = \mathrm{col}(\Psi_{x_0,1}, \Psi_{x_0,2}) \tag{22}$$

$$P_{0|-1} = \begin{bmatrix} \mathrm{st}^{-1}\{\Psi_{x_0,2}\} & \mathrm{st}^{-1}\{\Psi_{x_0,3}\} \\ \mathrm{st}^{-1}\{\Psi_{x_0,3}\}^\top & \mathrm{st}^{-1}\{\Psi_{x_0,4}\} - \Psi_{x_0,2}\Psi_{x_0,2}^\top \end{bmatrix}$$

$$\hat{x}_{k|k-1} = x_k^p + \hat{x}^s_{k|k-1}, \quad \hat{x}^s_{k|k-1} = \widehat{\mathscr{X}}^{s,1:n}_{k|k-1} \tag{23}$$

$$K_k = P_{k|k-1}\overline{\mathscr{C}}^\top \left(\overline{\mathscr{C}} P_{k|k-1}\overline{\mathscr{C}}^\top + \Psi_k\right)^\dagger \tag{24}$$

$$P_k = P_{k|k-1} - \gamma_k K_k \overline{\mathscr{C}} P_{k|k-1} \tag{25}$$

$$\widehat{\mathscr{X}}^s_k = \widehat{\mathscr{X}}^s_{k|k-1} + \gamma_k K_k \left(\mathscr{Y}_k - \overline{\mathscr{C}}\,\widehat{\mathscr{X}}^s_{k|k-1} - \varphi_g\right) \tag{26}$$

$$\hat{x}_k = x_k^p + \hat{x}_k^s, \quad \hat{x}_k^s = \widehat{\mathscr{X}}^{s,1:n}_k \tag{27}$$

$$x^p_{k+1} = \tilde{A}_k x_k^p + \nu_k B u_k + L y_k \tag{28}$$

$$\Gamma_k = \gamma_k \Upsilon_k \Psi_k^\dagger \tag{29}$$

$$\widehat{\mathscr{X}}^s_{k+1|k} = \bar{\mathscr{A}}_k \widehat{\mathscr{X}}^s_k + \Gamma_k\left(\mathscr{Y}_k - \overline{C}\widehat{\mathscr{X}}^s_k\right) + \varphi_{h,k} \tag{30}$$

$$P_{k+1|k} = \left(\bar{\mathscr{A}}_k - \Gamma_k \overline{C}\right) P_k \left(\bar{\mathscr{A}}_k - \Gamma_k\overline{C}\right)^\top + \Xi_k - \Gamma_k \Upsilon_k \tag{31}$$

where $\mathscr{X}^s_k = \mathrm{col}(x_k^s, x_k^{s\,[2]})$, $\varphi_h = \mathrm{col}(0, \psi_h^{(2)})$,

$$\bar{\mathscr{A}} = \begin{bmatrix} \tilde{A}_k & 0 \\ 0 & \tilde{A}_k^{[2]} \end{bmatrix} \quad \overline{\mathscr{C}} = \begin{bmatrix} C & 0 \\ 0 & C^{[2]} \end{bmatrix},$$

and the covariance matrices $\Xi_k = \mathbb{E}\left[\mathscr{H}_k \mathscr{H}_k^\top \,|\, \gamma_k\right]$ with $\mathscr{H}_k = \mathrm{col}\left(h_k, h_k^{(2)}\right)$, $\Psi_k = \mathbb{E}\left[\mathscr{G}_k\mathscr{G}_k^\top\right]$ and $\Upsilon_k = \mathbb{E}\left[\mathscr{H}_k\mathscr{G}_k^\top\right]$.

Proof. By Lemma 1, it is possible to consider the stochastic augmented sub-system

$$\mathscr{X}^s_{k+1} = \bar{\mathscr{A}}\,\mathscr{X}^s_k + \varphi_{h,k} + \mathscr{H}_k \tag{32}$$

$$\mathscr{Y}_k = \gamma_k\left(\overline{\mathscr{C}}\,\mathscr{X}^s_k + \varphi_g + \mathscr{G}_k\right). \tag{33}$$

Note that the noise sequences $\{\mathscr{H}_k\}$ and $\{\mathscr{G}_k\}$ are zero-mean, mutually correlated, white, and uncorrelated with the initial state x_0. Thus, the KF algorithm for mutually correlated state and output noise sequences provides the optimal estimate and prediction in the class of linear functions of $\{\mathscr{Y}_k\}$. Moreover, it is clear that the optimal filter in the class of linear functions of $\{\mathscr{Y}_k\}$ corresponds to the optimal filter in the class of the quadratic transformations, and the proof is completed. □

We are now able to state the main result of this section.

Theorem 1: For the finite-horizon LQG regulator problem with I/O Bernoulli packet losses, the $\mathscr{F}_k$-measurable output feedback *quadratic* optimal control u_k is given by

$$u_k = -M_k \hat{x}_{k|k-1}, \tag{34}$$

where M_k is defined in (5) with the backward Riccati equation (6), and $\hat{x}_{k|k-1} = x_k^p + \hat{x}^s_{k|k-1}$, with the recursive quadratic prediction $\hat{x}^s_{k|k-1} = \Pi[x_k^s | \mathscr{Q}^{k-1}_{y_s}]$ given in Lemma 2.

Proof. Firstly, we can rewrite the cost index J_{N} as follows

$$J_{\mathsf{N}} = \mathbb{E}\left[\mathscr{X}_{\mathsf{N}}^\top \mathscr{W}_{\mathsf{N}} \mathscr{X}_{\mathsf{N}} + \sum_{k=0}^{\mathsf{N}-1} \mathscr{X}_k^\top \mathscr{W}_k \mathscr{X}_k + \nu_k u_k^\top U_k u_k\right], \tag{35}$$

where $\mathscr{X}_k$ is the extended state vector defined in (14) and

$$\mathscr{W}_k := \begin{bmatrix} W_k & W_k & 0 \\ W_k & W_k & 0 \\ 0 & 0 & 0 \end{bmatrix}.$$

By Corollary 1, the LQG solution applied to the extended system (16)–(17) with the cost index (35), yields the optimal linear controller, namely the optimal recursive quadratic control of the original system. In particular, by noticing that the contribute of the known forcing term $\phi_{h,k}$ in (16) vanishes because of its zero-block structure, by Proposition 1, the above sub-optimal control input is given by

$$u_k = -\mathscr{M}_k \widehat{\mathscr{X}}_{k|k-1}, \tag{36}$$

with

$$\mathscr{M}_k = (U_k + \mathscr{B}^\top \mathscr{S}_{k+1}\mathscr{B})^{-1}\mathscr{B}^\top \mathscr{S}_{k+1}\mathscr{A},$$

where $\mathscr{S}_k$ is the solution of the backward Riccati equation

$$\mathscr{S}_k = \mathscr{W}_k + A^\top S_{k+1}\mathscr{A}_k + \\ - \bar{\nu}\mathscr{A}_k^\top \mathscr{S}_{k+1}\mathscr{B}(U_k + \mathscr{B}^\top\mathscr{S}_{k+1}\mathscr{B})^{-1}\mathscr{B}^\top\mathscr{S}_{k+1}\mathscr{A}_k \tag{37}$$

with final conditions $\mathscr{S}_{\mathsf{N}} = \mathscr{W}_{\mathsf{N}}$. It is easy to see that, by backward induction, the matrix

$$\mathscr{S}_k = \begin{bmatrix} S_k & S_k & 0 \\ S_k & S_k & 0 \\ 0 & 0 & 0 \end{bmatrix}, \tag{38}$$

with S_k given by (6) with its final condition, is the solutions to (37). Thus, by equations (38), the control law (36) simplifies in

$$u_k = -M_k\left(\widehat{\mathscr{X}}^{1:n}_{k|k-1} + \widehat{\mathscr{X}}^{s,1:n}_{k|k-1}\right),$$

where M_k is defined in (5), $\widehat{\mathscr{X}}^{1:n}_{k|k-1} = x_k^p$ is known and, by Lemma 2, $\widehat{\mathscr{X}}^{s,1:n}_{k|k-1} = \hat{x}^s_{k|k-1}$ is the optimal recursive quadratic prediction provided by (23). □

C. Some remarks

The previous theorem shows that the optimal controller in the class of quadratic transformation of the output is a linear function of the quadratic prediction. Thus, the separation principle continues to hold even in the non-Gaussian case with known I/O Bernoulli packet losses, since estimation and control can be designed separately.

In [30] an analysis of the statistical convergence properties of the estimation error covariance shows the existence of a critical value for the arrival rate of the observations, *i.e.* $\lambda_c \in [0,1)$, beyond which a transition to an unbounded mean state error covariance occurs.
In particular, Theorem 2 of [30] shows that if $\lambda \in (\lambda_c, 1]$, then for any initial covariance matrix P_0, we have $\mathbb{E}[P_{k|k-1}] \leq M$ for some matrix M depending on P_0, where $P_{k|k-1}$ is the covariance of the prediction error of the Kalman filter given in [30]. We note that the error dynamics $e_k = x_k - \check{x}_{k|k-1}$ of the prediction error of the filter in [30] is given by

$$e_{k+1} = (A - \gamma_k A K_k C)e_k + f_k - \gamma_k A K_k g_k, \tag{39}$$

where K_k is the standard Kalman filter gain.
If $\lambda \in (\lambda_c, 1]$, Theorem 2 of [30] implies that the dynamics (39) is uniformly bounded except for a set of measure zero. Thus, it is enough to set $L_k = AK_k$, with K_k the Kalman gain filter applied to (1)–(2) to let the matrix $\tilde{A}_k$ coincides with the dynamical matrix of the prediction error (39), which implies by similar arguments of Lemma 1 of [10] the uniformly boundedness of a system with dynamical matrix $\mathscr{A}_k$ defined in (18). Finally, by Theorem 3 of [10] we are able to guarantee that the processes $\{\mathscr{X}_k\}$, $\{\mathscr{H}_k\}$ and $\{\mathscr{G}_k\}$ are second-order uniformly bounded, which is essential to ensure that the quadratic filter (22)-(31) has a trace of the covariance of the estimation and prediction error not greater than the Kalman filter of [30] (see [5]).

The following proposition is a direct consequence of Theorem 1 of [31] and the Theorem 1 of this section.

Proposition 2: For the finite-horizon LQG regulator problem with I/O Bernoulli packet losses, with the control input (34), the cost J_N is given by

$$J_\mathsf{N} = \bar{x}_0^\top S_0 \bar{x}_0 + \mathrm{tr}\Bigg\{\Psi_{x_0} S_0 + \sum_{k=0}^{\mathsf{N}-1}\big(S_{k+1}Q + \\ + \bar{\nu} A^\top R_{k+1} B M_k \mathbb{E}[\bar{P}_{k|k-1}]\big)\Bigg\}, \qquad (40)$$

where M_k and S_k are defined in (5)-(6), and $\mathbb{E}[\bar{P}_{k|k-1}]$ is the expectation of the first $n \times n$ block of the matrix $P_{k|k-1}$ given by (31), namely the covariance of the prediction error.

The sequence of matrices $\{P_k\}$ is stochastic since the matrices are nonlinear functions of the sequence γ_k. The exact expected value of these matrices cannot be computed analytically, as shown in [30]. However, it can be upper bounded as $\mathbb{E}[\bar{P}_{k|k-1}] \leq \hat{P}_{k|k-1}$, where $\hat{P}_{k|k-1}$ is the first $n \times n$ block of the computable deterministic sequence (31) with $\bar{\gamma}$ instead of γ_k.

Finally, as pointed out at the end of Section III-A, we note that, since the space of quadratic transformation of the output depends on L, so it is for the matrix $P_{k|k-1}$. Therefore, a reasonable alternative static choice of the output injection matrices, *i.e.* $L_k = L$, is the one of setting the gain L such that it minimizes the upper bound of the cost

$$\bar{L} = \underset{\substack{\lambda \in \sigma(\tilde{A}) \\ |\lambda| \leq 1}}{\arg\min}\ \mathrm{tr}\Bigg\{\sum_{k=0}^{\mathsf{N}-1}\big(A^\top R_{k+1} B M_k \hat{P}_{k|k-1}\big)\Bigg\}. \qquad (41)$$

We notice that, with this latter choice of $\bar{L}$, we can not resort to the stability result of [30] for the critical threshold and further analysis would be required.

IV. Simulation Example

In this section we show the effectiveness of the proposed approach. We compare the linear optimal solution of [31] (Corollary 1), namely the control law (4) (we call *Kalman predictor (KP) controller*), with the one proposed in this paper, *i.e.* the control law (34) (we call *Quadratic predictor (QP) controller*). We consider an academic example where

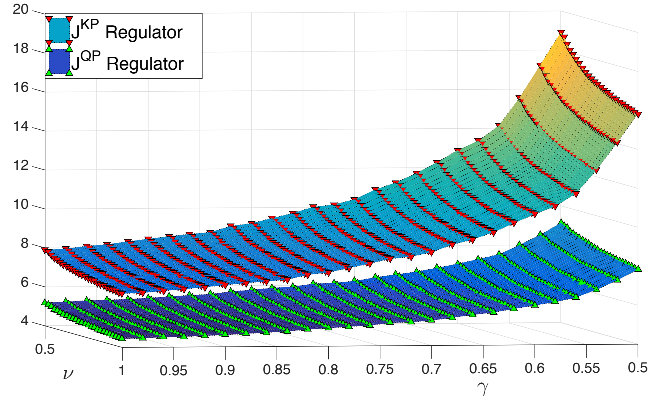

Fig. 2. Empirical cost of the KP controller of [31] and the proposed QP controller.

the system in the form (1)–(2) is characterized by

$$A = \begin{bmatrix} 1 & 0.25 \\ 0.2 & 1 \end{bmatrix}, \quad B = \begin{bmatrix} 0 \\ 1 \end{bmatrix}, \quad C = \begin{bmatrix} 1 & 0 \end{bmatrix},$$

with $x_0 \sim \mathscr{N}(0, I_2)$ with I_2 the identity matrix of dimension 2. Moreover, the signals are transmitted through unreliable channels. The cost index (3) to be minimized is defined by $W_k = I_2$, and $U_k = 1$ for all $k \geq 0$ and the time horizon is $\mathsf{N} = 120$. Furthermore, the system is driven by the zero-mean i.i.d. non-Gaussian noise sequences $f_k = \mathrm{col}(f_{1,k}, f_{2,k})$ and g_k, where for any $k \geq 0$ we have $\mathrm{P}(f_{1,k} = 0.05) = 1 - \mathrm{P}(f_{1,k} = -0.2) = 0.8$, $\mathrm{P}(f_{2,k} = -0.01) = 1 - \mathrm{P}(f_{2,k} = 0.09) = 0.9$, and $\mathrm{P}(g_k = 0.01) = 1 - \mathrm{P}(g_k = 0.0025) = 0.2$. Finally, the output injection gain L_k is chosen according to (41).

Figure 2 shows the empirical cost across 100 Monte Carlo runs of the KP controller and QP controller, *i.e.* the mean of the cost $x_\mathsf{N}^\top W_\mathsf{N} x_\mathsf{N} + \sum_{k=0}^{\mathsf{N}-1} x_k^\top W_k x_k + \bar{u}_k^\top U_k \bar{u}_k$ obtained for the 100 Monte Carlo runs. The 3D plot has on the x-axis and y-axis the values $\bar{\gamma} \in [0.5, 1]$ and $\bar{\nu} \in [0.5, 1]$ respectively as independent variables, and on the z-axis the value of the empirical cost. We see that, in the case $\bar{\gamma} = \bar{\nu} = 1$, the averaged cost $\bar{J}_\mathsf{N}^{\mathrm{KP}}$ of the KP controller of [31], *i.e.* the cost obtained with the control law (4), is $\bar{J}_\mathsf{N}^{\mathrm{KP}} = 6.76$, whilst the averaged cost $\bar{J}_\mathsf{N}^{\mathrm{QP}}$ of the proposed QP controller, *i.e.* the cost obtained with the control law (34), is $\bar{J}_\mathsf{N}^{\mathrm{QP}} = 4.44$. It can be noticed that the function of the empirical cost in Figure 2 is monotonically increasing when $\bar{\gamma}$ or $\bar{\nu}$ decrease. As described in Section III, the superiority of the proposed method descends from the fact that the proposed sub-optimal solution is optimal in the larger class of quadratic transformation output feedback controller, whilst the solution of [31] is the optimal linear solution. Finally, Table I summarizes the results of the empirical cost for both KP and QP Regulator for selected values of $\bar{\gamma}$ and $\bar{\nu}$.

V. Conclusions

In this paper we propose a sub-optimal solution for the Linear Quadratic non-Gaussian (LQnG) Regulator problem in the presence of known input/output packet losses. We show that an optimal controller in the class of recursive

$\overline{J}_{\mathrm{N}}^{\mathrm{KP}}$ / $\overline{J}_{\mathrm{N}}^{\mathrm{QP}}$	$\bar{\nu}=1$	$\bar{\nu}=0.9$	$\bar{\nu}=0.8$	$\bar{\nu}=0.7$	$\bar{\nu}=0.6$
$\bar{\gamma}=1$	6.76	6.87	7.03	7.22	7.47
	4.44	4.53	6.63	4.75	4.91
$\bar{\gamma}=0.9$	7.17	7.29	7.48	7.69	7.98
	4.61	4.69	4.80	4.92	5.10
$\bar{\gamma}=0.8$	7.91	7.91	8.11	8.35	8.68
	4.92	5.01	5.12	5.27	5.46
$\bar{\gamma}=0.7$	8.79	8.92	9.14	9.40	9.80
	5.28	5.38	5.50	5.66	5.88
$\bar{\gamma}=0.6$	10.72	10.90	11.17	11.52	12.07
	6.03	6.14	6.24	6.44	6.71

TABLE I

EMPIRICAL COST $\overline{J}^{\mathrm{KP}}$ OF THE LINEAR OPTIMAL SOLUTION OF [31], *i.e.* THE KALMAN PREDICTOR (KP) CONTROLLER, AND OF THE PROPOSED QUADRATIC OPTIMAL SOLUTION $\overline{J}^{\mathrm{QP}}$, *i.e.* THE QUADRATIC PREDICTOR (QP) CONTROLLER FOR SELECTED VALUES OF $\bar{\gamma}$ AND $\bar{\nu}$.

quadratic transformation of the output is obtained as a linear map of a quadratic prediction of the state. As a consequence, the separation principle continue to be true even in this case since estimation and control can be designed separately. Numerical results validate the proposed approach that outperforms the linear optimal solution. Further developments can include: extension to the polynomial filtering, partial packet losses, Semi-Markov packet losses, intermittent and/ or probabilistic acknowledgment of packet drops.

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