

RESEARCH PAPER

TEMPERED RELAXATION EQUATION AND RELATED GENERALIZED STABLE PROCESSES

Luisa Beghin ¹, Janusz Gajda ²

Abstract

Fractional relaxation equations, as well as relaxation functions time-changed by independent stochastic processes have been widely studied (see, for example, [21], [33] and [11]). We start here by proving that the upper-incomplete Gamma function satisfies the tempered-relaxation equation (of index $\rho \in (0, 1)$); thanks to this explicit form of the solution, we can then derive its spectral distribution, which extends the stable law. Accordingly, we define a new class of selfsimilar processes (by means of the n -times Laplace transform of its density) which is indexed by the parameter ρ : in the special case where $\rho = 1$, it reduces to the stable subordinator.

Therefore the parameter ρ can be seen as a measure of the local deviation from the temporal dependence structure displayed in the standard stable case.

MSC 2010: Primary 26A33; Secondary 34A08, 33B20, 60G52, 60G18

Key Words and Phrases: incomplete gamma function; tempered derivative; stable laws; relaxation function; G - and H -functions; self-similar processes

1. Introduction

The Cole-Cole law, i.e. $\varphi(t) = e^{-\lambda t}$, which satisfies the ordinary first-order differential equation

$$\frac{d}{dt}\varphi(t) = -\lambda\varphi(t), \quad \varphi(0) = 1, \quad (1.1)$$

(for $\lambda, t \in \mathbb{R}^+$), is suited to model relaxation phenomena with an exponential decay for large values of t . In many real situations, such as, for example, when considering the electromagnetic properties of some materials, as well as the rheological models for viscoelastic materials, fractional relaxation functions, with long-memory, instead of exponential, decay, have been preferred. In particular, by replacing in (1.1) the first derivative with the fractional Caputo derivative of order $\alpha \in (0, 1)$, the fractional relaxation equation

$${}^C\mathcal{D}_t^\alpha \varphi(t) = -\lambda^\alpha \varphi(t), \quad \varphi(0) = 1, \quad (1.2)$$

has been studied. Its solution is proved in [21] to coincide with $\varphi(t) = E_{\alpha,1}(-\lambda^\alpha t^\alpha)$ (see the definitions of Mittag-Leffler function and Caputo derivative below): compared to the standard relaxation function, the fractional relaxation exhibits, for small values of t , a much faster decay (the derivative tends to $-\infty$ instead of -1), while, for large values of t , it shows a much slower decay (an algebraic decay instead of an exponential decay). In view of its slow decay, the phenomenon of fractional relaxation is usually referred to as a super-slow process (see [21]). For further generalizations of fractional relaxation equations, see [11].

Relaxation driven by the inverse of a tempered-stable subordinator has been considered in [33]. The frequency-domain expression of the tempered relaxation function is given there, together with its asymptotic estimates: this is proved to be an intermediate case between the superslow relaxation and the exponential one.

We recall that tempered fractional derivatives are defined and studied in [24], while a different definition, in terms of shifted derivative, is given in [1]. For some details on tempered and inverse tempered subordinators see also [36], [37], [10].

We start by considering here the tempered relaxation equation

$$\mathcal{D}_t^{\lambda,\rho} \varphi(t) = -\lambda^\rho \varphi(t), \quad \varphi(0) = 1, \quad (1.3)$$

for $\lambda > 0$, $t \geq 0$ and $\rho \in (0, 1]$, where $\mathcal{D}_t^{\lambda,\rho}$ is defined as particular case of the so-called convolution-type derivatives (see definitions (1.15) and (2.1) below). We prove that it is satisfied by the following relaxation function

$$\varphi_{\lambda,\rho}(t) = \frac{\Gamma(\rho; \lambda t)}{\Gamma(\rho)}, \quad (1.4)$$

where $\Gamma(\rho; x) \doteq \int_x^{+\infty} e^{-w} w^{\rho-1} dw$ is the so-called upper-incomplete gamma function with parameter $\rho \in (0, 1]$. Note that $\Gamma(\rho; x)$ is well-defined for any $\rho \in \mathbb{R}$ and it is real-valued when $x \in \mathbb{R}^+$.

In the special case where $\rho = 1$, equation (1.3) reduces to the standard relaxation equation (1.1). The asymptotic behavior of the solution coincides with that of the model in [33], but thanks to the explicit form of the solution, we can obtain here further results: we can prove the completely monotonicity and integrability of $\varphi_{\lambda,\rho}$ and obtain its spectral distribution, which, as we will explain below, can be considered a generalization of stable laws.

We recall that the Laplace transform of the α -stable subordinator, i.e. $\Phi(\eta) := \mathbb{E}e^{-\eta S_\alpha(t)} = e^{-\eta^\alpha t}$, $t \geq 0$, satisfies equation (1.1), with $\lambda = \eta^\alpha$. We then generalize this result and define a new class of stochastic processes by means of the n -times Laplace transform of its density, as follows

$$\mathbb{E}e^{-\sum_{k=1}^n \eta_k \mathcal{X}_{\alpha,\rho}(t_k)} = \frac{\Gamma\left(\rho; \sum_{k=1}^n \Xi_{k,n}^\alpha(t_k - t_{k-1})\right)}{\Gamma(\rho)}, \quad \eta_1, \dots, \eta_n \geq 0, \quad n \in \mathbb{N}, \quad (1.5)$$

where $\Xi_{k,n}^\alpha := (\sum_{i=k}^n \eta_i)^\alpha$, for $0 < t_1 < \dots < t_n$, for $\alpha \in (0, 1)$ and $\rho \in (0, 1]$. In the one-dimensional case (for $n = 1$), the Laplace transform in (1.5) coincides with (1.4) for $\lambda = \eta^\alpha$, i.e.

$$\mathbb{E}e^{-\eta \mathcal{X}_{\alpha,\rho}(t)} = \frac{\Gamma(\rho; \eta^\alpha t)}{\Gamma(\rho)}, \quad \eta > 0. \quad (1.6)$$

Moreover, in the special case $\rho = 1$, we obtain from (1.5) the Laplace transform of the α -stable subordinator. As we will see in the next section, both (1.5) and (1.6) are completely monotone functions (with respect to $\eta_1, \dots, \eta_n$ and η , respectively), under the conditions that $\alpha, \rho \leq 1$ and $\alpha\rho \leq 1$, respectively. Thus we will assume hereafter $\rho \in (0, 1/\alpha]$ when treating the one-dimensional distribution, whereas we will restrict to the case $\rho \in (0, 1]$ in the n -dimensional case.

Moment generating functions expressed as power series of incomplete gamma functions can be also found in [32], where they are used to generalize the gamma distribution.

The process $\mathcal{X}_{\alpha,\rho} := \{\mathcal{X}_{\alpha,\rho}(t), t \geq 0\}$ defined in (1.5) displays a self-similar property of order $1/\alpha$, for any ρ . On the other hand, it is a Lévy process only in the stable case (i.e. for $\rho = 1$), since its distribution is not, in general, infinitely divisible. The main feature of self-similar processes is the invariance in distribution under suitable scaling of time and space. Self-similarity is usually associated with long-range dependence (see, for example, [28]). The main tools used to describe the dependence structure of a stochastic processes are both the autocorrelation coefficient and the spectral analysis. However, in the case of heavy-tailed distributions, where the moments are not finite, we must resort to other tools, such as the autocodifference function. In particular, referring to the definition given in [29]

and [38], we are able to interpret ρ as a measure of the local deviation from the temporal dependence structure displayed in the standard α -stable case.

In order to derive the differential equation satisfied by the transition function $h_{\alpha,\rho}(x,t)$ of $\mathcal{X}_{\alpha,\rho}$, we distinguish the two cases $0 < \rho \leq 1$ and $1 < \rho \leq 1/\alpha$: indeed, in the first one, we prove that $h_{\alpha,\rho}(x,t)$ satisfies an integro-differential equation where a space-dependent time-fractional derivative (of convolution-type) and a Caputo space-fractional derivative appear. In the case where $\rho > 1$, the differential equation satisfied by the transition density is obtained only in the special case $\rho = 2$ and it involves the so-called logarithmic differential operator introduced in [2] and defined in (3.9) below.

We are able to write the explicit expression of $h_{\alpha,\rho}(x,t)$, in terms of H -functions, which for $\rho = 1$ and $t = 1$, reduces to the stable law formula given in (2.28) of [23]. Finally we present some properties of the generalized stable r.v. $\mathcal{X}_{\alpha,\rho}$, such as the tails' behavior and the moments. In particular, we prove that it possesses finite moments of order $\delta < \alpha\rho$, which can be written as

$$\mathbb{E}(\mathcal{X}_{\alpha,\rho})^\delta = \frac{c_\alpha^{\frac{\delta}{\alpha}}}{\Gamma(\rho)} \frac{\Gamma(\rho - \frac{\delta}{\alpha})}{\Gamma(1 - \delta)}.$$

We now recall some well-known definitions and results which will be used throughout the paper.

Let $H_{p,q}^{m,n}$ denote the Fox H -function defined as (see [23] p.13):

$$H_{p,q}^{m,n} \left[z \middle| \begin{matrix} (a_1, A_1) & \dots & (a_p, A_p) \\ (b_1, B_1) & \dots & (b_q, B_q) \end{matrix} \right] \quad (1.7)$$

$$= \frac{1}{2\pi i} \int_{\mathcal{L}} \frac{\left\{ \prod_{j=1}^m \Gamma(b_j + B_j s) \right\} \left\{ \prod_{j=1}^n \Gamma(1 - a_j - A_j s) \right\} z^{-s} ds}{\left\{ \prod_{j=m+1}^q \Gamma(1 - b_j - B_j s) \right\} \left\{ \prod_{j=n+1}^p \Gamma(a_j + A_j s) \right\}},$$

with $z \neq 0$, $m, n, p, q \in \mathbb{N}_0$, for $1 \leq m \leq q$, $0 \leq n \leq p$, $a_j, b_j \in \mathbb{R}$, $A_j, B_j \in \mathbb{R}^+$, for $i = 1, \dots, p$, $j = 1, \dots, q$ and $\mathcal{L}$ is a contour such that the following condition is satisfied

$$A_l(b_j + \alpha) \neq B_j(a_l - k - 1), \quad j = 1, \dots, m, \quad l = 1, \dots, n, \quad \alpha, k = 0, 1, \dots \quad (1.8)$$

Let us recall the following particular case of the H -function which we will use later, i.e. the Mittag-Leffler function, defined as

$$E_{\alpha,\beta}(z) := \sum_{j=0}^{+\infty} \frac{z^j}{\Gamma(\alpha j + \beta)} = H_{1,2}^{1,1} \left[-z \middle| \begin{matrix} (0, 1) \\ (0, 1) \end{matrix} \begin{matrix} (1 - \beta, \alpha) \end{matrix} \right], \quad (1.9)$$

for $\alpha > 0$ (see formula (1.136) in [23]). Moreover, we recall the other special case of (1.7) given by the Meijer G -function, which is obtained for $A_i = 1$, for any $i = 1, \dots, p$ and for $B_j = 1$, for any $j = 1, \dots, q$:

$$G_{p,q}^{m,n} \left[z \mid \begin{matrix} (a_1, \dots, a_p) \\ (b_1, \dots, b_q) \end{matrix} \right] := H_{p,q}^{m,n} \left[z \mid \begin{matrix} (a_1, 1) & \dots & (a_p, 1) \\ (b_1, 1) & \dots & (b_q, 1) \end{matrix} \right]. \quad (1.10)$$

In particular, we will use (1.10) in the case where $n = 0$ and $p = q = m$, for which the following result holds (see properties 3 and 9 in [14]):

$$G_{p,p}^{p,0} \left[x \mid \begin{matrix} (a_1, \dots, a_p) \\ (b_1, \dots, b_p) \end{matrix} \right] = 0, \quad |x| > 1, \quad (1.11)$$

while, under the assumption that $\sum_{j=1}^p (t^{b_j} - t^{a_j}) \geq 0$, it is

$$G_{p,p}^{p,0} \left[x \mid \begin{matrix} (a_1, \dots, a_p) \\ (b_1, \dots, b_p) \end{matrix} \right] \geq 0, \quad 0 < |x| < 1. \quad (1.12)$$

More details on the properties of the function (1.10), together with its applications in fractional calculus, can be found in [17]. In particular, for $p = q = 1$, the function $G_{1,1}^{1,0}$ is proved to be the kernel of the Erdélyi-Kober fractional integral (see also [15]).

We will use the definition and properties of completely monotone functions, both in the univariate and multivariate cases; thus we recall that, for Theorem 4.2.1 p.87 in [5], a function $f(z_1, \dots, z_n)$ in an octant $0 < z_j < \infty$, $j = 1, \dots, n$ can be represented as

$$f(z_1, \dots, z_n) = \int_0^{+\infty} \dots \int_0^{+\infty} e^{-z_1 x_1 - \dots - z_n x_n} d\rho(x_1, \dots, x_n) \quad (1.13)$$

for a measure ρ such that $\rho(A) \geq 0$, for any A , if and only if it is C^∞ and

$$(-1)^{k_1 + \dots + k_n} \frac{\partial^{k_1 + \dots + k_n}}{\partial z_1^{k_1} \dots \partial z_n^{k_n}} f(z_1, \dots, z_n) \geq 0, \quad (1.14)$$

for all combinations $k_1 \geq 0, \dots, k_n \geq 0$ (the representation being unique). The measure ρ is called spectral distribution of f and the function satisfying (1.14) is said to be completely monotone (hereafter CM). By the way, we note that the function (1.13) is linked to the kernel of a particular case of the Obrechhoff transform (see, for example, [7]).

We also recall the definition of the convolution-type derivative on the positive half-axes, in the sense of Caputo (see [34], Def.2.4, for $b = 0$): let $\overline{\nu}(ds)$ be a non-negative Lévy measure (i.e. satisfying the condition $\int_0^{+\infty} (z \wedge 1) \overline{\nu}(dz) < \infty$) and let $\nu(s) = \int_s^{+\infty} \overline{\nu}(dz)$ be its tail, under the assumption that it is absolutely continuous. Let, moreover, $g : \mathbb{R}^+ \rightarrow \mathbb{R}^+$,

be a Bernstein function, i.e. with the following representation $g(x) = a + bx + \int_0^{+\infty} (1 - e^{-sx}) \bar{\nu}(ds)$, for $a, b \geq 0$, then

$$\mathcal{D}_t^g u(t) := \int_0^t \frac{d}{ds} u(t-s) \nu(s) ds, \quad t > 0, \quad (1.15)$$

for an absolutely continuous function such that $|u(t)| \leq Me^{\lambda t}$, for some $M, \lambda > 0$ and for any t .

Convolution-type derivatives (or derivatives defined as integrals with memory kernels) have been treated recently by many authors: see, among the others, [18], [19], [33], [9], [34].

The Laplace transform of $\mathcal{D}_t^g$ is given by

$$\int_0^{+\infty} e^{-\theta t} \mathcal{D}_t^g u(t) dt = g(\theta) \tilde{u}(\theta) - \frac{g(\theta)}{\theta} u(0), \quad \mathcal{R}(\theta) > \theta_0, \quad (1.16)$$

(see [34], Lemma 2.5). It is easy to check that, in the trivial case where $g(\theta) = \theta$, the convolution-type derivative coincides with the standard first-order derivative. When $g(\theta) = \theta^\alpha$, for $\alpha \in (0, 1)$, this Bernstein function coincides with the Laplace exponent of the α -stable subordinator $\mathcal{S}_\alpha(t)$ (distributed, for any t , as a stable r.v. $\mathcal{S}_\alpha(\sigma, \beta, \mu)$, with $\sigma = (\cos(\pi\alpha/2)t)^{1/\alpha}$, $\beta = 1$ and $\mu = 0$), since

$$\mathbb{E} e^{-\theta \mathcal{S}_\alpha(t)} = \exp\{-\theta^\alpha t\}, \quad t, \theta > 0. \quad (1.17)$$

In this case, the Lévy measure reduces to $\bar{\nu}(ds) = \alpha s^{-\alpha-1}/\Gamma(1-\alpha)$ and thus (1.15) coincides with the fractional Caputo derivative of order α , i.e.

$${}^C \mathcal{D}_t^\alpha u(t) := \frac{1}{\Gamma(1-\alpha)} \int_0^t \frac{d}{ds} u(s) (t-s)^{-\alpha} ds, \quad t > 0, \quad \alpha \in (0, 1). \quad (1.18)$$

Finally, we will use the Riesz-Feller (RF) fractional derivative, which is defined, by means of its Fourier transform. Let $L^c(\mathcal{I})$ of functions for which the Riemann improper integral on any open interval $\mathcal{I}$ absolutely converges (see [22]). For any $f \in L^c(\mathcal{I})$, the RF fractional derivative is defined as

$$\mathcal{F}\{\mathcal{D}_{x,\theta}^\alpha f(x); \xi\} = -\psi_\theta^\alpha(\xi) \mathcal{F}\{f(x); \xi\}, \quad \alpha \in (0, 2], \quad |\theta| \leq \min\{\alpha, 2-\alpha\}, \quad (1.19)$$

with the symbol

$$\psi_\theta^\alpha(\xi) := |\xi|^\alpha e^{i \operatorname{sign}(\xi) \theta \pi/2}, \quad \xi \in \mathbb{R} \quad (1.20)$$

(see [16], p.131). Since we are interested in the special case where the support of the density is the positive half-line, we can take $\xi > 0$ and $\theta = -\alpha$ (which corresponds to the stable law totally skewed to the right).

2. Tempered relaxation function

Under the assumption that $\rho \leq 1$, we prove that the (normalized) upper-incomplete Gamma function given in (1.4) is the solution of a tempered relaxation equation, which extends the standard one, i.e. (1.1), by means of the tempered fractional derivative, in the Caputo sense. The latter can be obtained from the definition given in (1.15), by choosing the Bernstein function $g(x) = (\lambda + x)^\rho - \lambda^\rho$, for $\lambda, x \in \mathbb{R}^+$ and the Lévy measure $\overline{\nu}(ds) = \frac{\rho e^{-\lambda s} s^{-\rho-1}}{\Gamma(1-\rho)} ds$. Thus the tail Lévy measure is taken equal to $\nu(ds) = \frac{\rho \lambda^\rho \Gamma(-\rho; \lambda s)}{\Gamma(1-\rho)} ds$ (see [34] and [33], for details). For $\rho = 1$, we have that $g(x) = x$ and thus the tempered derivative reduces to the first-order derivative.

LEMMA 2.1. *Let $\mathcal{D}_t^{\lambda, \rho}$ be defined, for any absolutely continuous function $u : \mathbb{R}^+ \rightarrow \mathbb{R}^+$, such that $|u(t)| \leq ce^{kt}$, for some $c, k > 0$ and for any $t \geq 0$, as*

$$\mathcal{D}_t^{\lambda, \rho} u(t) = \frac{\rho \lambda^\rho}{\Gamma(1-\rho)} \int_0^t \frac{d}{dt} u(t-s) \Gamma(-\rho; \lambda s) ds, \quad \lambda > 0, \rho \in (0, 1], \quad (2.1)$$

then the solution of the following tempered relaxation equation

$$\mathcal{D}_t^{\lambda, \rho} u(t) = -\lambda^\rho u(t), \quad (2.2)$$

with initial condition $u(0) = 1$, is given by

$$\varphi_{\lambda, \rho}(t) = \frac{\Gamma(\rho; \lambda t)}{\Gamma(\rho)}. \quad (2.3)$$

P r o o f. The Laplace transform of the l.h.s. of (2.2) can be obtained from (1.16), by choosing $g(x) = (\lambda + x)^\rho - \lambda^\rho$ and considering the initial condition:

$$\int_0^{+\infty} e^{-\theta t} \mathcal{D}_t^{\lambda, \rho} u(t) dt = [(\lambda + \theta)^\rho - \lambda^\rho] \tilde{u}(\theta) - \frac{(\lambda + \theta)^\rho - \lambda^\rho}{\theta}. \quad (2.4)$$

Then, by taking into account the r.h.s. of (2.2), we get

$$\tilde{u}(\theta) = \frac{(\lambda + \theta)^\rho - \lambda^\rho}{\theta(\lambda + \theta)^\rho} \quad (2.5)$$

which coincides with the Laplace transform of (2.3), since

$$\begin{aligned} \frac{1}{\Gamma(\rho)} \int_0^{+\infty} e^{-\theta t} \Gamma(\rho; \lambda t) dt &= \frac{1}{\Gamma(\rho)} \int_0^{+\infty} e^{-\theta t} \int_{\lambda t}^{+\infty} e^{-w} w^{\rho-1} dw dt \\ &= \frac{\lambda^\rho}{\Gamma(\rho)} \int_0^{+\infty} e^{-\lambda w} w^{\rho-1} \int_0^w e^{-\theta t} dt dw \\ &= \frac{\lambda^\rho}{\theta \Gamma(\rho)} \int_0^{+\infty} e^{-\lambda w} (1 - e^{-\theta w}) w^{\rho-1} dw. \end{aligned}$$

□

We now derive some properties of $\varphi_{\lambda,\rho}(\cdot)$, which are usually required to a relaxation function, and obtain its spectral distribution in terms of H -function. The latter will be proved to coincide with the density of the process $\mathcal{X}_{\alpha,\rho}$ defined in the next section (in the special case $\alpha = 1$).

THEOREM 2.1. *The tempered relaxation function $\varphi_{\lambda,\rho}(\cdot)$, defined in (2.3) is $\mathcal{C}^\infty$, completely monotone and locally integrable. The spectral distribution of $\varphi_{\lambda,\rho}(\cdot)$ is given by*

$$K_\lambda(z) = \frac{\lambda 1_{z>\lambda}}{\Gamma(\rho)} G_{1,1}^{1,0} \left[\frac{\lambda}{z} \middle| \begin{matrix} 2 \\ 1+\rho \end{matrix} \right], \quad \rho \in (0, 1), \quad (2.6)$$

where $G_{p,q}^{m,n}[\cdot]$ is the Meijer G -function defined in (1.10).

P r o o f. It is well-known that the upper incomplete gamma function is $\mathcal{C}^\infty$ and thus the same holds true for $\varphi_{\lambda,\rho}(\cdot)$. Moreover, we can derive complete monotonicity by repeatedly differentiating (2.3) and showing that $(-1)^k \frac{d^k}{dt^k} \varphi_{\lambda,\rho}(t) \geq 0$, for any $k \in \mathbb{N}$ and $t \geq 0$:

$$\begin{aligned} -\frac{d}{dt} \varphi_{\lambda,\rho}(t) &= \lambda^\rho e^{-\lambda t} t^{\rho-1} \geq 0 \\ \frac{d^2}{dt^2} \varphi_{\lambda,\rho}(t) &= \lambda^{\rho+1} e^{-\lambda t} t^{\rho-1} - (\rho-1) \lambda^\rho e^{-\lambda t} t^{\rho-2} \geq 0 \\ -\frac{d^3}{dt^3} \varphi_{\lambda,\rho}(t) &= \lambda^{\rho+2} e^{-\lambda t} t^{\rho-1} - 2(\rho-1) \lambda^{\rho+1} e^{-\lambda t} t^{\rho-2} + \\ &\quad + (\rho-1)(\rho-2) \lambda^\rho e^{-\lambda t} t^{\rho-3} \geq 0, \end{aligned}$$

and so on. The sign of the k -derivative depends only on $\rho - j + 1$, for $j = 2, \dots, k$, which is non-positive, for any $k \in \mathbb{N}$ and for $\rho \leq 1$. As far as the integrability, we first derive an alternative expression of (2.3) in terms of Mittag-Leffler function: recalling formula (3.7) p.316 in [25] and formula

(4.2.8) in [12], we can write

$$\begin{aligned}\varphi_{\lambda,\rho}(t) &= 1 - \frac{\int_0^{\lambda t} e^{-w} w^{\rho-1} dw}{\Gamma(\rho)} \\ &= 1 - e^{-\lambda t} \lambda^\rho \mathcal{E}_t(\rho, \lambda) = 1 - e^{-\lambda t} (\lambda t)^\rho E_{1,\rho+1}(\lambda t),\end{aligned}\quad (2.7)$$

where $\mathcal{E}_x(\nu, \lambda) = \frac{e^{\lambda x}}{\Gamma(\nu)} \int_0^x w^{\nu-1} e^{-w} dw$ is the so-called Miller-Ross function.

By applying formula (6.5.31) in [36], we easily obtain the following limiting behavior of (2.3)

$$\varphi_{\lambda,\rho}(t) \simeq \frac{e^{-\lambda t} \lambda^{\rho-1} t^{\rho-1}}{\Gamma(\rho)}, \quad t \rightarrow +\infty. \quad (2.8)$$

Since $\varphi_{\lambda,\rho}(0) = 1$ and the function $\varphi_{\lambda,\rho}(\cdot)$ is bounded by one on any compact set, (2.8) is enough to prove its integrability. Finally, we check that

$$\varphi_{\lambda,\rho}(t) = \int_0^{+\infty} e^{-tz} K_\lambda(z) dz, \quad (2.9)$$

for $K(z)$ given in (2.6). Indeed we can write (2.6), by (1.10), in terms of H -functions, as

$$\begin{aligned}K_\lambda(z) &= \frac{\lambda}{\Gamma(\rho)} H_{1,1}^{1,0} \left[\frac{\lambda}{z} \middle| \begin{matrix} (2, 1) \\ (1 + \rho, 1) \end{matrix} \right] \\ &= [\text{by (1.60) in [23]}] = \frac{\lambda}{\Gamma(\rho)} H_{1,1}^{0,1} \left[\frac{z}{\lambda} \middle| \begin{matrix} (-\rho, 1) \\ (-1, 1) \end{matrix} \right],\end{aligned}$$

so that we can apply formula (2.19) in [23] and get

$$\begin{aligned}& \frac{1}{\lambda \Gamma(\rho)} \int_0^{+\infty} e^{-tz} H_{1,1}^{0,1} \left[\frac{z}{\lambda} \middle| \begin{matrix} (-\rho, 1) \\ (-1, 1) \end{matrix} \right] dz \\ &= \frac{1}{\Gamma(\rho) \lambda t} H_{2,1}^{0,2} \left[\frac{1}{\lambda t} \middle| \begin{matrix} (0, 1)(-\rho, 1) \\ (-1, 1) \end{matrix} \right] \\ &= [\text{by (1.58) and (1.60) in [23]}] = \frac{1}{\Gamma(\rho)} H_{1,2}^{2,0} \left[\lambda t \middle| \begin{matrix} (1, 1) \\ (0, 1)(\rho, 1) \end{matrix} \right] \\ &= \frac{1}{\Gamma(\rho)} \frac{1}{2\pi i} \int_{\mathcal{L}} (\lambda t)^{-s} \frac{\Gamma(s) \Gamma(\rho + s)}{\Gamma(1 + s)} ds = \varphi_{\lambda,\rho}(t),\end{aligned}$$

by means of the Mellin-Barnes integral form of the incomplete gamma function:

$$\Gamma(\rho; x) = \frac{1}{2\pi i} \int_{\mathcal{L}} \frac{\Gamma(s + \rho) x^{-s}}{\Gamma(\rho) s} ds, \quad (2.10)$$

where the contour $\mathcal{L}$ avoids all the poles of the gamma functions (see [27] for more details). Note that the indicator function in (2.6) comes from (1.11). $\square$

REMARK 2.1. We note that, as $t \rightarrow +\infty$, the function $\varphi_{\lambda,\rho}(\cdot)$ decays exponentially fast and thus faster than in the fractional case. On the other hand, for small values of t , the tempered relaxation function exhibits the same fast decay of the fractional relaxation (see [21]): indeed we can obtain the behavior of $\varphi_{\lambda,\rho}(\cdot)$, for $t \rightarrow 0^+$, by considering (2.7) together with

$$E_{\alpha,\beta}(z) \simeq \frac{1}{\Gamma(\beta)} + \frac{z}{\Gamma(\beta + \alpha)}, \quad z \rightarrow 0^+$$

(see [12], p.64): thus we have that

$$\varphi_{\lambda,\rho}(t) \simeq 1 - \frac{e^{-\lambda t} \lambda^\rho t^\rho}{\Gamma(\rho + 1)} \simeq 1 - \frac{t^\rho}{\Gamma(\rho + 1)}, \quad t \rightarrow 0^+. \quad (2.11)$$

We thus obtain asymptotic behaviors similar to those of the relaxation function introduced in [33] (by time-changing with inverse tempered subordinators), but, in this case, we have also a closed expression of the solution, in terms of well-known functions.

REMARK 2.2. As far as the spectral distribution $K_\lambda(z)$, $z \geq 0$, given in (2.6), we can check, by (1.12), that it is a proper probability distribution: indeed it is non-negative, since $t^{1+\rho} - t^2 \geq 0$, for $\rho < 1$ and $t \in (0, 1)$. Furthermore, $\int_0^{+\infty} K_\lambda(z) dz = 1$, by using (2.9) and considering that $\varphi_{\lambda,\rho}(0) = 1$. From (2.6) we can draw the conclusion that the process governed by $\varphi_{\lambda,\rho}(t)$ can be expressed in terms of a continuous distribution of standard (exponential) relaxation processes with frequencies on the range $(\lambda, +\infty)$ instead of the whole real positive semi-axes. In the special case $\rho = 1/2$, it is well-known that

$$G_{1,1}^{0,1} \left[\begin{matrix} x & -\frac{1}{2} \\ & -1 \end{matrix} \right] = \frac{1_{x>1}}{\sqrt{\pi x} \sqrt{x-1}},$$

(see (1.143) in [23]), so that we get

$$K_\lambda(z) = \frac{\lambda^{5/2} 1_{z>\lambda}}{\pi z \sqrt{z-\lambda}}. \quad (2.12)$$

REMARK 2.3. Let $S_{\lambda,\rho}(t)$, $t \geq 0$, be the tempered subordinator, for $\lambda \geq 0$ and $\rho \in (0, 1]$, with transition density $h_{\lambda,\rho}(z, t)$. Let moreover $L_{\lambda,\rho}(t)$, $t \geq 0$, be its inverse, i.e. $L_{\lambda,\rho}(t) := \inf\{s \geq 0, S_{\lambda,\rho}(s) > t\}$ (see [1], [10], [24] and [37], for details). As a consequence of Lemma 2.1 we can derive the following relationship between the tempered relaxation and the density of $L_{\lambda,\rho}$, i.e. $l_{\lambda,\rho}(x, t) := P\{L_{\lambda,\rho}(t) \in dx\}$:

$$\varphi_{\lambda,\rho}(t) = \frac{\Gamma(\rho; \lambda t)}{\Gamma(\rho)} = \int_0^{+\infty} e^{-\lambda^\rho z} l_{\lambda,\rho}(z, t) dz. \quad (2.13)$$

The previous integral expression of the incomplete gamma function can be obtained, by applying (19) of [36]. Let us denote by $h_\rho(u, z)$ the density of the ρ -stable subordinator (i.e. for $\lambda = 0$), then

$$\begin{aligned} & \int_0^{+\infty} e^{-\lambda^\rho z} l_{\lambda, \rho}(z, t) dz \\ &= - \int_0^{+\infty} e^{-\lambda^\rho z} \frac{\partial}{\partial z} \int_0^t h_{\lambda, \rho}(u, z) du dz \\ &= -\lambda^\rho \int_0^t e^{-\lambda u} \int_0^{+\infty} e^{-\lambda^\rho z + \lambda^\rho z} h_\rho(u, z) dz du + \int_0^t h_{\lambda, \rho}(u, 0) du \\ &= -\lambda^\rho \int_0^t e^{-\lambda u} \int_0^{+\infty} h_\rho(u, z) dz du + 1, \end{aligned}$$

for $t > 0$, since $h_{\lambda, \rho}(u, 0) = \delta(u)$. We now notice that $\int_0^{+\infty} h_\rho(u, z) dz = u^{\rho-1}/\Gamma(\rho)$ (as can be checked by the Laplace transform) and the result (2.13) easily follows.

The relationship (2.13) can be also checked by considering that both sides satisfy (2.2): by taking the tempered derivative of the r.h.s. of (2.13):

$$\begin{aligned} & \mathcal{D}_t^{\lambda, \rho} \left[\Gamma(\rho) \int_0^{+\infty} e^{-\lambda^\rho z} l_{\lambda, \rho}(z, t) dz \right] = \Gamma(\rho) \int_0^{+\infty} e^{-\lambda^\rho z} \mathcal{D}_t^{\lambda, \rho} l_{\lambda, \rho}(z, t) dz \\ &= [\text{by (6.13) and (2.33) in [34]}] \\ &= -\Gamma(\rho) \int_0^{+\infty} e^{-\lambda^\rho z} \frac{\partial}{\partial z} l_{\lambda, \rho}(z, t) dz - \Gamma(\rho) \nu(t) \int_0^{+\infty} e^{-\lambda^\rho z} l_{\lambda, \rho}(z, 0) dz \\ &= -\left[\Gamma(\rho) e^{-\lambda^\rho z} l_{\lambda, \rho}(z, t) \right]_{z=0}^{+\infty} - \Gamma(\rho) \nu(t) - \Gamma(\rho) \lambda^\rho \int_0^{+\infty} e^{-\lambda^\rho z} l_{\lambda, \rho}(z, t) dz \\ &= -\Gamma(\rho) \lambda^\rho \int_0^{+\infty} e^{-\lambda^\rho z} l_{\lambda, \rho}(z, t) dz, \end{aligned}$$

where we have considered the initial conditions $l_{\lambda, \rho}(0, t) = \frac{\rho \lambda^\rho \Gamma(-\rho, \lambda t)}{\Gamma(1-\rho)}$ and $l_{\lambda, \rho}(x, 0) = \delta(x)$ (see [34], for details).

3. Generalized stable process

We recall that the Laplace transform of the α -stable subordinator, i.e. $\Phi(\eta) := \mathbb{E} e^{-\eta S_\alpha(t)} = e^{-\eta^\alpha t}$, $t \geq 0$, satisfies the standard relaxation equation (1.1), with $\lambda = \eta^\alpha$. We then apply the previous results, in order to define a new stochastic process, which generalizes the α -stable subordinator.

We first prove that, under some assumptions, the function

$$\Phi(\eta_1, \dots, \eta_n; t_1, \dots, t_n) = \frac{\Gamma\left(\rho; \sum_{k=1}^n \Xi_{k,n}^\alpha(t_k - t_{k-1})\right)}{\Gamma(\rho)}, \quad \eta_1, \dots, \eta_n \geq 0, \quad n \in \mathbb{N}, \quad (3.1)$$

where $\Xi_{k,n}^\alpha := (\sum_{i=k}^n \eta_i)^\alpha$, can be uniquely represented as a Laplace transform of a probability measure in $\mathbb{R}^n$.

LEMMA 3.1. *Let $0 < t_1 < \dots < t_n$, $\alpha \in (0, 1)$ and $\rho \in (0, 1]$. The function $\Phi(\eta_1, \dots, \eta_n; t_1, \dots, t_n)$ in (3.1) is C^∞ , is such that $\Phi(0, \dots, 0; t_1, \dots, t_n) = 1$ and is completely monotone (CM) with respect to $\eta_1, \dots, \eta_n$, for any choice of $t_1 < \dots < t_n$. Thus the following integral (unique) representation holds*

$$\Phi(\eta_1, \dots, \eta_n; t_1, \dots, t_n) = \int_0^{+\infty} \dots \int_0^{+\infty} e^{-\eta_1 x_1 - \dots - \eta_n x_n} \pi(dx_1, \dots, dx_n; t_1, \dots, t_n),$$

for a probability measure $\pi(\cdot, \dots, \cdot; t_1, \dots, t_n)$.

P r o o f. We first check that the one-dimensional function $\Phi(\eta, t) = \mathbb{E}e^{-\eta \mathcal{X}_{\alpha, \rho}(t)}$ given in (3.3) is C^∞ , since the incomplete gamma function is analytic w.r.t. both its arguments. Moreover, it is such that $\Phi(0, t) = 1$ and is a CM function, i.e. is absolutely differentiable and $(-1)^k \frac{d^k}{d\eta^k} \Phi(\eta, t) \geq 0$, for $k = 0, 1, \dots$. Indeed, η^α is a Bernstein function and we have proved in Theorem 2.1 that $\Gamma(\rho; \cdot)$ is CM. Then it is well-known that the composition of a CM and a Bernstein function is again CM. Thus the statement is proved for $n = 1$. Moreover, we can check directly that this is true if and only if $\alpha\rho < 1$:

$$\begin{aligned} \frac{d}{d\eta} \Phi(\eta, t) &= -\alpha t^\rho \eta^{\alpha\rho-1} e^{-\eta^\alpha t} \leq 0, \\ \frac{d^2}{d\eta^2} \Phi(\eta, t) &= -\alpha(\alpha\rho - 1) t^\rho \eta^{\alpha\rho-2} e^{-\eta^\alpha t} + \alpha^2 t^{\rho+1} \eta^{\alpha\rho+\alpha-2} e^{-\eta^\alpha t} \geq 0, \\ \frac{d^3}{d\eta^3} \Phi(\eta, t) &= -\alpha(\alpha\rho - 1)(\alpha\rho - 2) t^\rho \eta^{\alpha\rho-3} e^{-\eta^\alpha t} \\ &\quad + (\alpha\rho - 1) \alpha^2 t^{\rho+1} \eta^{\alpha\rho+\alpha-3} e^{-\eta^\alpha t} \\ &\quad + \alpha^2 t^{\rho+1} (\alpha\rho + \alpha - 2) \eta^{\alpha\rho+\alpha-3} e^{-\eta^\alpha t} - \alpha^3 t^{\rho+2} \eta^{\alpha\rho+2\alpha-3} e^{-\eta^\alpha t} \leq 0, \end{aligned}$$

and so on. The sign of the k -th derivatives depends only on $\alpha\rho - j + 1$, for $j = 2, \dots, k$, and $\alpha\rho + (j-2)\alpha - j + 1$, for $j = 3, \dots, k$, which are all non-positive under the unique condition $\alpha\rho < 1$.

For $n > 1$, we start by proving that the condition (1.14) holds, for $n = 2$, i.e. that

$$\begin{aligned} & (-1)^{k_1+k_2} \frac{\partial^{k_1+k_2}}{\partial \eta_1^{k_1} \partial \eta_2^{k_2}} \Phi(\eta_1, \eta_2; t_1, t_2) \\ &= (-1)^{k_1+k_2} \frac{\partial^{k_1+k_2}}{\partial \eta_1^{k_1} \partial \eta_2^{k_2}} \frac{\Gamma(\rho; (\eta_1 + \eta_2)^\alpha t_1 + \eta_2^\alpha (t_2 - t_1))}{\Gamma(\rho)} \geq 0. \end{aligned}$$

Let us denote $A_{1,2} \doteq (\eta_1 + \eta_2)^\alpha t_1 + \eta_2^\alpha (t_2 - t_1)$. Let, moreover, $J_{k_1} = \{j_1, j_2, j_3 : j_1 + j_2 + j_3 = k_1 - 1\}$ and $L_{k_2} = \{l_1, l_2, l_3 : l_1 + l_2 + l_3 = k_2\}$, for $j_i = 0, 1, \dots, k_1 - 1$ and $l_i = 0, \dots, k_2$, for $i = 1, 2, 3$. Thus we can write

$$\begin{aligned} & (-1)^{k_1+k_2} \frac{\partial^{k_1+k_2}}{\partial \eta_1^{k_1} \partial \eta_2^{k_2}} \Phi(\eta_1, \eta_2; t_1, t_2) \\ &= \frac{(-1)^{k_1+k_2}}{\Gamma(\rho)} \frac{\partial^{k_2}}{\partial \eta_2^{k_2}} \frac{\partial^{k_1-1}}{\partial \eta_1^{k_1-1}} \frac{\partial}{\partial \eta_1} \Gamma(\rho; A_{1,2}) \\ &= \frac{(-1)^{k_1+k_2+1}}{\Gamma(\rho)} \frac{\partial^{k_2}}{\partial \eta_2^{k_2}} \frac{\partial^{k_1-1}}{\partial \eta_1^{k_1-1}} \left[e^{-A_{1,2}} A_{1,2}^{\rho-1} \frac{\partial}{\partial \eta_1} A_{1,2} \right] \\ &= [\text{by the general Leibniz formula}] \\ &= \frac{(-1)^{k_2}}{\Gamma(\rho)} \frac{\partial^{k_2}}{\partial \eta_2^{k_2}} \left[\sum_{j_1, j_2, j_3 \in J_{k_1}} \binom{k_1-1}{j_1, j_2, j_3} (-1)^{j_1} \frac{\partial^{j_1}}{\partial \eta_1^{j_1}} e^{-A_{1,2}} \right. \\ &\quad \times (-1)^{j_2} \frac{\partial^{j_2}}{\partial \eta_1^{j_2}} A_{1,2}^{\rho-1} (-1)^{j_3+2} \frac{\partial^{j_3+1}}{\partial \eta_1^{j_3+1}} A_{1,2} \left. \right] \\ &\quad \frac{1}{\Gamma(\rho)} \sum_{\substack{l_1, l_2, l_3 \in L_{k_2} \\ j_1, j_2, j_3 \in J_{k_1}}} \binom{k_2}{l_1, l_2, l_3} \binom{k_1-1}{j_1, j_2, j_3} \left[(-1)^{l_1+j_1} \frac{\partial^{l_1+j_1}}{\partial \eta_2^{l_1} \partial \eta_1^{j_1}} e^{-A_{1,2}} \right] \\ &\quad \times \left[(-1)^{l_2+j_2} \frac{\partial^{l_2+j_2}}{\partial \eta_2^{l_2} \partial \eta_1^{j_2}} A_{1,2}^{\rho-1} \right] \left[(-1)^{l_3+j_3} \frac{\partial^{l_3+j_3+1}}{\partial \eta_2^{l_3} \partial \eta_1^{j_3+1}} A_{1,2} \right]. \end{aligned}$$

Under the assumptions that $\alpha \in (0, 1)$ and $\rho \in (0, 1]$ and recalling that $t_1, t_2 - t_1 \geq 0$, it is easy to show that $A_{1,2}$ is a Bernstein function w.r.t. η_1 and η_2 . On the other hand, $A_{1,2}^{\rho-1}$ and $e^{-A_{1,2}}$ are both CM, being the composition of CM and Bernstein functions (see Theorem 3.7 in [30], p.27). Moreover it can be checked that, in the last step, the sign of the partial derivatives depends only on the order, regardless of the variables involved; thus the functions inside square brackets are all non-negative.

This procedure can be successively applied in order to prove that (1.14) is true for any integer $n \geq 3$. $\square$

DEFINITION 3.1. We define the process $\mathcal{X}_{\alpha,\rho} := \{\mathcal{X}_{\alpha,\rho}(t), t \geq 0\}$, under the assumption that, for $t_0 = 0$, $\mathcal{X}_{\alpha,\rho}(t_0) = 0$, almost surely, by the n -times Laplace transform of its finite dimensional distributions given in (3.1), i.e.

$$\mathbb{E} e^{-\sum_{k=1}^n \eta_k \mathcal{X}_{\alpha,\rho}(t_k)} := \frac{\Gamma\left(\rho; \sum_{k=1}^n \Xi_{k,n}^\alpha(t_k - t_{k-1})\right)}{\Gamma(\rho)}, \quad \eta_1, \dots, \eta_n \geq 0, n \in \mathbb{N},$$

for $\Xi_{k,n}^\alpha := (\sum_{i=k}^n \eta_i)^\alpha$ and $\alpha \in (0, 1)$, $\rho \in (0, 1]$.

REMARK 3.1. It is easy to check that $\mathcal{X}_{\alpha,\rho}$ displays the self-similar property of order $1/\alpha$, for any ρ : indeed we have the following equality of all the finite-dimensional distributions $\left\{a^{\frac{1}{\alpha}} \mathcal{X}_{\alpha,\rho}(t), t \geq 0\right\} \stackrel{d}{=} \{\mathcal{X}_{\alpha,\rho}(at), t \geq 0\}$, for any $a \geq 0$.

On the other hand, it is a Lévy process only for $\rho = 1$, when its distribution becomes infinitely divisible. Moreover, only in this special case the process has stationary and independent increments, since formula (3.1) can be rewritten as

$$\begin{aligned} \Phi(\eta_1, \dots, \eta_n; t_1, \dots, t_n) &= \exp\left\{\sum_{k=1}^n \Xi_{k,n}^\alpha(t_k - t_{k-1})\right\} = \prod_{k=1}^n e^{-\Xi_{k,n}^\alpha(t_k - t_{k-1})} \\ &= \prod_{k=1}^n \mathbb{E} e^{-\Xi_{k,n}^\alpha \mathcal{X}_{\alpha,1}(t_k - t_{k-1})} = \prod_{k=1}^n \mathbb{E} e^{-\Xi_{k,n}^\alpha [\mathcal{X}_{\alpha,1}(t_k) - \mathcal{X}_{\alpha,1}(t_{k-1})]}. \end{aligned}$$

It is evident that, for $\rho = 1$, the increments have an α -stable distribution (see [29], p.113) and thus $\mathcal{X}_{\alpha,1}$ reduces to the stable process.

We now consider the one-dimensional distribution of the process and we prove that it can be written in terms of the Fox's H -function defined in (1.7).

THEOREM 3.1. The transition density $h_\alpha^\rho(x, t)$ of $\mathcal{X}_{\alpha,\rho}$, for any $\rho \in (0, 1/\alpha]$, is given by

$$h_\alpha^\rho(x, t) = \frac{\alpha}{x\Gamma(\rho)} H_{1,1}^{0,1} \left[\frac{x^\alpha}{t} \left| \begin{matrix} (1-\rho, 1) \\ (0, \alpha) \end{matrix} \right. \right], \quad x > 0, \quad t \geq 0. \quad (3.2)$$

P r o o f. We invert the one-dimensional Laplace transform of the process $\mathcal{X}_{\alpha,\rho}$, i.e.

$$\mathbb{E}e^{-\eta\mathcal{X}_{\alpha,\rho}(t)} = \frac{\Gamma(\rho; \eta^\alpha t)}{\Gamma(\rho)}, \quad \eta > 0, \quad (3.3)$$

by using (2.10), as follows

$$\begin{aligned} & h_\alpha^\rho(x, t) \\ &= \frac{1}{2\pi i} \int_{\gamma-i\infty}^{\gamma+i\infty} \frac{\Gamma(s+\rho)(tx^{-\alpha})^{-s}}{x\Gamma(\rho)\Gamma(\alpha s)s} ds = \frac{\alpha}{2\pi i} \int_{\gamma-i\infty}^{\gamma+i\infty} \frac{\Gamma(s+\rho)(tx^{-\alpha})^{-s}}{x\Gamma(\rho)\Gamma(\alpha s+1)} ds \\ &= \frac{\alpha}{x\Gamma(\rho)} H_{1,1}^{1,0} \left[tx^{-\alpha} \middle| \begin{matrix} (1, \alpha) \\ (\rho, 1) \end{matrix} \right] = \frac{\alpha}{x\Gamma(\rho)} H_{1,1}^{0,1} \left[\frac{x^\alpha}{t} \middle| \begin{matrix} (1-\rho, 1) \\ (0, \alpha) \end{matrix} \right], \end{aligned} \quad (3.4)$$

which exists for any $x \neq 0$, since $\alpha - 1 < 0$ (see Theorem 1.1 in [23]). $\square$

REMARK 3.2. We can check that (3.2) is a proper density function, for any t , by applying the Mellin transform and recalling formula (2.8) in [23]:

$$\begin{aligned} \int_0^{+\infty} h_\alpha^\rho(x, t) dx &= \frac{\alpha}{\Gamma(\rho)} \int_0^{+\infty} \frac{1}{x} H_{1,1}^{0,1} \left[\frac{x^\alpha}{t} \middle| \begin{matrix} (1-\rho, 1) \\ (0, \alpha) \end{matrix} \right] dx \\ &= \frac{1}{\Gamma(\rho)} \int_0^{+\infty} \frac{1}{y} H_{1,1}^{0,1} \left[\frac{y}{t} \middle| \begin{matrix} (1-\rho, 1) \\ (0, \alpha) \end{matrix} \right] dy = \frac{1}{\Gamma(\rho)} \Gamma(\rho) = 1. \end{aligned}$$

The H -function distribution, which includes, among others, the gamma, beta, Weibull, chi-square, exponential and the half-normal distributions as particular cases, has been introduced by [6] and studied in many other papers (see, for example, [31], [35]).

For $\rho = 1$, formula (3.2) reduces to

$$\begin{aligned} & h_\alpha^1(x, t) \\ &= \alpha x^{-1} H_{1,1}^{0,1} \left[\frac{x^\alpha}{t} \middle| \begin{matrix} (0, 1) \\ (0, \alpha) \end{matrix} \right] = [\text{by (1.59) in [23]}] \\ &= x^{-1} H_{1,1}^{0,1} \left[\frac{x}{t^{1/\alpha}} \middle| \begin{matrix} (0, \frac{1}{\alpha}) \\ (0, 1) \end{matrix} \right] = [\text{by (1.60) in [23], for } \sigma = -1] \\ &= \frac{1}{t^{1/\alpha}} H_{1,1}^{0,1} \left[\frac{x}{t^{1/\alpha}} \middle| \begin{matrix} (-\frac{1}{\alpha}, \frac{1}{\alpha}) \\ (-1, 1) \end{matrix} \right] = \frac{1}{t^{1/\alpha}} \frac{1}{2\pi i} \int_{\mathcal{L}} \frac{\Gamma(1 + \frac{1}{\alpha} - \frac{s}{\alpha}) \left(\frac{x}{t^{1/\alpha}} \right)^{-s} ds}{\Gamma(2-s)} \\ &= \frac{1}{t^{1/\alpha}} \frac{1}{\alpha} H_{1,1}^{0,1} \left[\frac{x}{t^{1/\alpha}} \middle| \begin{matrix} (1 - \frac{1}{\alpha}, \frac{1}{\alpha}) \\ (0, 1) \end{matrix} \right] = [\text{by (1.58) in [23]}] \\ &= \frac{1}{t^{1/\alpha}} \frac{1}{\alpha} H_{1,1}^{1,0} \left[\frac{t^{1/\alpha}}{x} \middle| \begin{matrix} (1, 1) \\ (\frac{1}{\alpha}, \frac{1}{\alpha}) \end{matrix} \right], \end{aligned}$$

which coincides, for $t = 1$, with the expression of the stable law given in (2.28) of [23], in terms of H -functions.

On the other hand, for $\alpha = 1$, formula (3.2) reduces to (2.6) and thus Theorem 3.1 shows that the spectral density $K(\cdot)$ of the fractional relaxation function $\varphi_{1,\rho}(\cdot)$ coincides with the density of $\mathcal{X}_{1,\rho}$.

We now study the dependence structure of the process $\mathcal{X}_{\alpha,\rho}$, by means of the auto-codifference function: adapting the definition given in [29] and [38] to the Laplace transform (since our process is non-negative valued), we can write, from (3.1),

$$\begin{aligned} CD(\mathcal{X}_{\alpha,\rho}(t), \mathcal{X}_{\alpha,\rho}(s)) &= \ln \mathbb{E} e^{-\eta_1 \mathcal{X}_{\alpha,\rho}(t) - \eta_2 \mathcal{X}_{\alpha,\rho}(s)} - \ln \mathbb{E} e^{-\eta_1 \mathcal{X}_{\alpha,\rho}(t)} - \ln \mathbb{E} e^{-\eta_2 \mathcal{X}_{\alpha,\rho}(s)} \\ &= \ln \frac{\Gamma(\rho; (\eta_1 + \eta_2)^\alpha s) \Gamma(\rho; \eta_2^\alpha (t - s))}{\Gamma(\rho; \eta_1^\alpha s) \Gamma(\rho; \eta_2^\alpha s)}. \end{aligned} \quad (3.5)$$

By considering the asymptotic behavior (2.8) of the incomplete gamma function, we easily get

$$CD(\mathcal{X}_{\alpha,\rho}(t), \mathcal{X}_{\alpha,\rho}(s)) \simeq (\eta_1^\alpha - \eta_2^\alpha) t, \quad (3.6)$$

for $t \rightarrow +\infty$ and for fixed s . The limiting expression (3.6) coincides with the auto-codifference of the α -stable subordinator. Therefore, we can consider the parameter ρ as a measure of the local deviation from the temporal dependence structure displayed in the standard α -stable case.

In order to obtain the Cauchy problem satisfied by the one-dimensional density of $\mathcal{X}_{\alpha,\rho}$ given in (3.2), we distinguish the two cases: $0 < \rho \leq 1$ and $\rho \in (0, 1/\alpha]$.

3.1. Case $0 < \rho \leq 1$. When we restrict to the case $\rho \leq 1$, we can obtain the integro-differential equation satisfied by the density (3.2), in terms of a convolution-type derivative with space-dependent tail Lévy measure.

THEOREM 3.2. *Let $p_\alpha(\cdot)$ be the law of the α -stable r.v. $S_\alpha(1)$, for $\alpha \in (\frac{1}{2}, 1)$, then, for $\rho \in (0, 1)$,*

$$\overline{M}_z(dw) = \frac{\rho w^{-\frac{1}{\alpha}-\rho-1} p_\alpha(z/w^{1/\alpha})}{\Gamma(1-\rho)} dw, \quad w > 0, \quad (3.7)$$

is a Lévy measure, for any $z > 0$.

- (2) *If we define the convolution-type derivative (with space-dependent tail Lévy measure) ${}_z\mathcal{D}_t^\rho := \int_0^t \frac{d}{ds} u(t-s) M_z(ds)$, where $M_z(s) = \int_s^{+\infty} \overline{M}_z(dw)$, for any $z > 0$, then the transition density $h_\alpha^\rho(x, t)$ satisfies the following integro-differential equation:*

$$\int_0^x {}_z\mathcal{D}_t^\rho h(x-z, t) dz = - {}^C\mathcal{D}_x^{\alpha\rho} h(x, t), \quad (3.8)$$

with the following conditions:

$$\begin{cases} h(x, 0) = \delta(x) \\ \lim_{x \downarrow 0^+} h(x, t) = 0, \end{cases}$$

for any $x \geq 0$ and $t \geq 0$, respectively.

P r o o f . 1. We apply the Property 1.2.15 on the tail's behavior of the stable law, given in [29]: $\lim_{x \rightarrow +\infty} x^\alpha P(X > x) = C_\alpha \frac{1+\beta}{2} \sigma^\alpha$, for C_α given in (1.2.9), $\beta = 1$ (since the α -stable considered here is totally skewed to the right) and for $\sigma = (\cos(\pi\alpha/2)w)^{1/\alpha}$. Thus, by taking the first derivative, we get that $p_\alpha(x) = O(C'_\alpha w x^{-\alpha-1})$, for $x \rightarrow +\infty$ and a constant C'_α . Thus we can conclude that $p_\alpha(z/w^{1/\alpha}) = O(K w^{\frac{1}{\alpha}+2})$, for $w \rightarrow 0$, for a positive constant K . We now verify that the following finiteness condition is satisfied by $\overline{M}_z$:

$$\begin{aligned} & \int_0^{+\infty} (z \wedge 1) \overline{M}_z(dz) \\ &= \int_0^1 w^{-\frac{1}{\alpha}-\rho} p_\alpha(z/w^{1/\alpha}) dw + \int_1^{+\infty} w^{-\frac{1}{\alpha}-\rho-1} p_\alpha(z/w^{1/\alpha}) dw < \infty. \end{aligned}$$

The second integral is finite, since the stable law is finite in the origin, while for the first integral we can consider that the integrand function $w^{-\frac{1}{\alpha}-\rho} p_\alpha(z/w^{1/\alpha}) = O(K w^{2-\rho})$, for $w \rightarrow 0$.

2. We notice that the tail measure $M_z(s)$ is finite, for any $s, z > 0$, since $\int_s^{+\infty} \frac{\rho w^{-\frac{1}{\alpha}-\rho-1} p_\alpha(z/w^{1/\alpha})}{\Gamma(1-\rho)} dw < \int_s^{+\infty} \frac{\rho w^{-\frac{1}{\alpha}-\rho-1}}{\Gamma(1-\rho)} dw < \infty$. Then, by applying Lemma 2.1, with $\lambda = \eta^\alpha$, we have that

$$\frac{\rho \eta^{\alpha\rho}}{\Gamma(1-\rho)} \int_0^t \frac{d}{dt} \varphi_{\eta^\alpha, \rho}(t-s) \Gamma(-\rho; \eta^\alpha s) ds = -\eta^{\alpha\rho} \varphi_{\eta^\alpha, \rho}(t),$$

with $\varphi_{\eta^\alpha, \rho}(0) = 1$, is satisfied by (3.3). Let now $\mu_\eta(s) := \frac{\rho \Gamma(-\rho; \eta^\alpha s)}{\Gamma(1-\rho)} = \eta^{-\alpha\rho} \nu(s)$, then it is the tail of a new Lévy measure denoted by $\overline{\mu}_\eta(s) = \eta^{-\alpha\rho} \overline{\nu}(s)$ (since it differs only by a constant from the original one). Then we get

$$\begin{aligned} & \frac{\rho}{\Gamma(1-\rho)} \int_0^t \frac{d}{dt} \varphi_{\eta^\alpha, \rho}(t-s) \Gamma(-\rho; \eta^\alpha s) ds = \int_0^t \frac{d}{dt} \varphi_{\eta^\alpha, \rho}(t-s) \mu_\eta(s) ds \\ &= \eta^{-\alpha\rho} \int_0^t \frac{d}{dt} \varphi_{\eta^\alpha, \rho}(t-s) \nu_\eta(s) ds = -\varphi_{\eta^\alpha, \rho}(t), \end{aligned}$$

which moreover gives, by considering the expression of $\nu_\eta(s)$,

$$\frac{\rho}{\Gamma(1-\rho)} \int_0^t \frac{d}{dt} \varphi_{\eta^\alpha, \rho}(t-s) \int_s^{+\infty} e^{-\eta^\alpha w} w^{-\rho-1} dw ds = -\eta^{\alpha\rho} \varphi_{\eta^\alpha, \rho}(t).$$

We can now invert the Laplace transform (in the variable η) in the l.h.s. since $\varphi_{\eta^\alpha, \rho}(t)e^{-\eta^\alpha w} = \mathcal{L}\{h_\alpha^\rho(\cdot, t)\} \mathcal{L}\{p_\alpha(\cdot, w)\} = \mathcal{L}\{h_\alpha^\rho(\cdot, t) * p_\alpha(\cdot, w)\}$, where $*$ denotes the convolution operator:

$$\begin{aligned} & \frac{\rho}{\Gamma(1-\rho)} \int_0^t dz \int_0^x \frac{d}{dt} h_\alpha^\rho(x-z, t-s) \int_s^{+\infty} p_\alpha(z, w) w^{-\rho-1} dw ds \\ &= [\text{by the self-similarity property}] \\ &= \frac{\rho}{\Gamma(1-\rho)} \int_0^x dz \int_0^t \frac{d}{dt} h_\alpha^\rho(x-z, t-s) \int_s^{+\infty} p_\alpha(z/w^{1/\alpha}) w^{-\frac{1}{\alpha}-\rho-1} dw ds, \end{aligned}$$

which coincides with the l.h.s. of (3.8). The r.h.s. follows by taking into account the Laplace transform of the Caputo derivative (1.18), of order $\alpha\rho < 1$, since $\alpha, \rho < 1$, and the second initial condition, which will be proved below to hold for $h_\alpha^\rho(x, t)$, together with the first one.

Finally, $h_\alpha^\rho(x, 0) = \delta(x)$, by the assumption that $\mathcal{X}_{\alpha, \rho}(0) = 0$, almost surely. On the other hand, the limit for $x \rightarrow 0^+$ follows from the following considerations: the density (3.2) can be rewritten (by (1.60) in [23], with $\sigma = -1/\alpha$) as

$$h_\alpha^\rho(x, t) = \frac{\alpha}{t^{1/\alpha}\Gamma(\rho)} H_{1,1}^{0,1} \left[\frac{x^\alpha}{t} \left| \begin{matrix} (1-\rho-\frac{1}{\alpha}, 1) \\ (-1, \alpha) \end{matrix} \right. \right].$$

The limit for $x \rightarrow 0^+$ is then obtained by considering the asymptotic behavior of the previous expression (see Theorem 1.3.(ii) in [23]) and since, in this case, $\mu = \alpha - 1 < 0$, $\delta = \rho + \frac{1}{\alpha} - 2$ and since $A_1 - B_1 = 1 - \alpha > 0$: thus we get that

$$h_\alpha^\rho(x, t) = O \left(x^{-\frac{\alpha\rho+1-\frac{3\alpha}{2}}{1-\alpha}} \right) \exp \left\{ -(1-\alpha)\alpha^{\frac{1}{1-\alpha}} x^{-\frac{\alpha}{1-\alpha}} \right\}, \quad x \rightarrow 0^+.$$

□

REMARK 3.3. We recall that Lévy measures expressed in terms of α -stable densities have been already treated in [2], in connection with stable processes subordinated by independent Poisson processes. Moreover, pseudo-differential operators defined by means of time-dependent Lévy measures are used in [26], [3] and [4], in connection with time-inhomogeneous subordinators.

3.2. Case $1 < \rho \leq 1/\alpha$. For ρ greater than 1, we derive the fractional differential equation satisfied by the transition density, at least in the special case where $\rho = 2$. To this aim we recall the definition of the logarithmic differential operator introduced in [2]: let $\widehat{f}$ denote the Fourier transform,

then $\mathcal{P}_{x,c}^\alpha$ is the operator with the following symbol

$$(\widehat{\mathcal{P}_{x,c}^\alpha f})(\xi) = -\ln(1 + c\psi_\theta^\alpha(\xi))\widehat{f}(\xi), \quad c > 0, \quad |\xi| < 1/c, \quad (3.9)$$

for any $f \in \mathcal{L}^c$.

THEOREM 3.3. *Let $\alpha \in (0, 1)$ and let $h_\alpha^2(x, t)$ denote the transition density of the process defined in (3.2), for $\rho = 2$, then*

$$h_\alpha^2(x, t) = \frac{\alpha}{x} H_{1,1}^{0,1} \left[\frac{x^\alpha}{t} \middle| \begin{matrix} (-1, 1) \\ (0, \alpha) \end{matrix} \right] \quad (3.10)$$

satisfies the following equation

$$\frac{\partial}{\partial t} u(x, t) = \mathcal{D}_{x,-\alpha}^\alpha u(x, t) - \frac{\partial}{\partial t} \mathcal{P}_{x,t}^\alpha u(x, t) + \mathcal{P}_{x,t}^\alpha \left[\frac{\partial}{\partial t} u(x, t) \right], \quad (3.11)$$

with initial condition $u(x, 0) = \delta(x)$.

P r o o f. By taking the Fourier transform of (3.11) and considering (1.19)-(1.20) (for $\theta = -\alpha$) and (3.9), we get

$$\begin{aligned} & \frac{\partial}{\partial t} \widehat{u}(\xi, t) \quad (3.12) \\ &= -\psi_{-\alpha}^\alpha(\xi) \widehat{u}(\xi, t) + \frac{\partial}{\partial t} [\ln(1 + t\psi_{-\alpha}^\alpha(\xi)) \widehat{u}(\xi, t)] - \ln(1 + t\psi_{-\alpha}^\alpha(\xi)) \frac{\partial}{\partial t} \widehat{u}(\xi, t) \\ &= -\psi_{-\alpha}^\alpha(\xi) \widehat{u}(\xi, t) + \frac{\psi_{-\alpha}^\alpha(\xi)}{1 + t\psi_{-\alpha}^\alpha(\xi)} \widehat{u}(\xi, t) + \ln(1 + t\psi_{-\alpha}^\alpha(\xi)) \frac{\partial}{\partial t} \widehat{u}(\xi, t) \\ &\quad - \ln(1 + t\psi_{-\alpha}^\alpha(\xi)) \frac{\partial}{\partial t} \widehat{u}(\xi, t) \\ &= -\psi_{-\alpha}^\alpha(\xi) \widehat{u}(\xi, t) + \frac{\psi_{-\alpha}^\alpha(\xi)}{1 + t\psi_{-\alpha}^\alpha(\xi)} \widehat{u}(\xi, t). \end{aligned}$$

On the other hand, we evaluate the Fourier transform of (3.2) as follows (by considering formulae (2.49) and (2.59) of the sine and cosine transform of the H -function):

$$\begin{aligned}
& \int_{-\infty}^{+\infty} e^{i\xi x} h_{\alpha}^{\rho}(x, t) dx = \int_{-\infty}^{+\infty} e^{i\xi x} \frac{\alpha}{x} H_{1,1}^{0,1} \left[\frac{x^{\alpha}}{t} \middle| \begin{matrix} (1-\rho, 1) \\ (0, \alpha) \end{matrix} \right] dx \\
&= \alpha \int_{-\infty}^{+\infty} \cos(\xi x) \frac{1}{x} H_{1,1}^{0,1} \left[\frac{x^{\alpha}}{t} \middle| \begin{matrix} (1-\rho, 1) \\ (0, \alpha) \end{matrix} \right] dx \\
&\quad + i\alpha \int_{-\infty}^{+\infty} \sin(\xi x) \frac{1}{x} H_{1,1}^{0,1} \left[\frac{x^{\alpha}}{t} \middle| \begin{matrix} (1-\rho, 1) \\ (0, \alpha) \end{matrix} \right] dx \\
&= \frac{\alpha\sqrt{\pi}}{2} H_{3,1}^{0,2} \left[\frac{2^{\alpha}}{t\xi^{\alpha}} \middle| \begin{matrix} (1, \frac{\alpha}{2}) & (1-\rho, 1) & (\frac{1}{2}, \frac{\alpha}{2}) \\ (0, \alpha) \end{matrix} \right] \\
&\quad + \frac{i\alpha\sqrt{\pi}}{2} H_{3,1}^{0,2} \left[\frac{2^{\alpha}}{t\xi^{\alpha}} \middle| \begin{matrix} (\frac{1}{2}, \frac{\alpha}{2}) & (1-\rho, 1) & (1, \frac{\alpha}{2}) \\ (0, \alpha) \end{matrix} \right] \\
&= \frac{\alpha\sqrt{\pi}}{2} \frac{1}{2\pi i} \int_{\mathcal{L}} \left(\frac{2^{\alpha}}{t\xi^{\alpha}} \right)^{-s} \frac{\Gamma(-\frac{\alpha s}{2})\Gamma(\rho-s)}{\Gamma(1-\alpha s)\Gamma(\frac{1}{2}+\frac{\alpha s}{2})} ds \\
&\quad + \frac{i\alpha\sqrt{\pi}}{2} \frac{1}{2\pi i} \int_{\mathcal{L}} \left(\frac{2^{\alpha}}{t\xi^{\alpha}} \right)^{-s} \frac{\Gamma(\frac{1}{2}-\frac{\alpha s}{2})\Gamma(\rho-s)}{\Gamma(1-\alpha s)\Gamma(1+\frac{\alpha s}{2})} ds.
\end{aligned} \tag{3.13}$$

By the duplication formula of the Gamma function, we get

$$\begin{aligned}
& \int_{-\infty}^{+\infty} e^{i\xi x} h_{\alpha}^{\rho}(x, t) dx \\
&= \frac{\alpha}{4} \frac{1}{2\pi i} \int_{\mathcal{L}} \left(\frac{1}{t\xi^{\alpha}} \right)^{-s} \frac{\Gamma(-\frac{\alpha s}{2})\Gamma(\rho-s)\Gamma(\frac{\alpha s}{2})}{\Gamma(1-\alpha s)\Gamma(\alpha s)} ds \\
&\quad + i\alpha\pi \frac{1}{2\pi i} \int_{\mathcal{L}} \left(\frac{1}{t\xi^{\alpha}} \right)^{-s} \frac{\Gamma(-\alpha s)\Gamma(\rho-s)}{\Gamma(-\frac{\alpha s}{2})\Gamma(1-\alpha s)\Gamma(1+\frac{\alpha s}{2})} ds \\
&= [\text{by the reflection formula}] \\
&= -\frac{1}{2} \frac{1}{2\pi i} \int_{\mathcal{L}} \left(\frac{1}{t\xi^{\alpha}} \right)^{-s} \frac{\sin(\pi\alpha s)\Gamma(\rho-s)}{\sin(\pi\alpha s/2)s} ds \\
&\quad + i\frac{1}{2\pi i} \int_{\mathcal{L}} \left(\frac{1}{t\xi^{\alpha}} \right)^{-s} \frac{\sin(\pi\alpha s/2)\Gamma(\rho-s)}{s} ds \\
&= \frac{1}{2\pi i} \int_{\mathcal{L}} (t\xi^{\alpha})^{-s} \frac{\cos(\pi\alpha s/2)\Gamma(\rho+s)}{s} ds \\
&\quad + \frac{i}{2\pi i} \int_{\mathcal{L}} (t\xi^{\alpha})^{-s} \frac{\sin(\pi\alpha s/2)\Gamma(\rho+s)}{s} ds \\
&= \frac{1}{2\pi i} \int_{\mathcal{L}} \left(e^{-\frac{i\pi\alpha}{2}} t\xi^{\alpha} \right)^{-s} \frac{\Gamma(\rho+s)}{s} ds.
\end{aligned}$$

In this special case $\rho = 2$, we can write (3.13) as follows

$$\begin{aligned}
 & \int_{-\infty}^{+\infty} e^{i\xi x} \frac{\alpha}{x} H_{1,1}^{0,1} \left[\frac{x^\alpha}{t} \middle| \begin{matrix} (1-\rho, 1) \\ (0, \alpha) \end{matrix} \right] dx \\
 &= \frac{1}{2\pi i} \int_{\mathcal{L}} \left(e^{-\frac{i\pi\alpha}{2}} t\xi^\alpha \right)^{-s} (1+s)\Gamma(s) ds \\
 &= \frac{1}{2\pi i} \int_{\mathcal{L}} \left(e^{-\frac{i\pi\alpha}{2}} t\xi^\alpha \right)^{-s} \Gamma(s) ds + \frac{1}{2\pi i} \int_{\mathcal{L}} \left(e^{-\frac{i\pi\alpha}{2}} t\xi^\alpha \right)^{-s} \Gamma(s+1) ds \\
 &= [\text{by formula (1.38) in [23]}] \\
 &= e^{-t\xi^\alpha} e^{-\frac{i\pi\alpha}{2}} \left(1 + t\xi^\alpha e^{-\frac{i\pi\alpha}{2}} \right) = e^{-\psi_{-\alpha}^\alpha(\xi)t} (1 + t\psi_{-\alpha}^\alpha(\xi)).
 \end{aligned} \tag{3.14}$$

It is easy to check that (3.14) satisfies (3.12) as well as the initial condition. $\square$

4. Properties of the generalized stable law

Let now consider a fix time argument and define $\mathcal{X}_{\alpha,\rho}$ as the r.v. with Laplace transform

$$\mathbb{E}e^{-\eta\mathcal{X}_{\alpha,\rho}} = \frac{\int_{c\eta^\alpha}^{+\infty} e^{-w} w^{\rho-1} dw}{\Gamma(\rho)} = \frac{\Gamma(\rho; c\eta^\alpha)}{\Gamma(\rho)}, \quad \eta > 0, \tag{4.1}$$

for $\alpha \in (0, 1]$, $\rho \leq 1/\alpha$ and scale parameter $\sigma = c^{1/\alpha} \in \mathbb{R}^+$. Then an interesting particular case can be obtained for $\rho = n \in \mathbb{N}$: by successive integrating by parts of (4.1), it can be checked that

$$\mathbb{E}e^{-\eta\mathcal{X}_{\alpha,\rho}} = \frac{\int_{c\eta^\alpha}^{+\infty} e^{-w} w^{n-1} dw}{\Gamma(n)} = e^{-c\eta^\alpha} \sum_{j=0}^{n-1} \frac{(c\eta^\alpha)^j}{j!}, \tag{4.2}$$

thus coinciding with the cumulative distribution function of a Poisson r.v. $Z_{c\eta^\alpha}$ with parameter $c\eta^\alpha$ (i.e. $P(Z_{c\eta^\alpha} < n)$). By inverting (4.2), the density of $\mathcal{X}_{\alpha,\rho}$ can be written, for $\rho = n > 1$, as follows

$$h_\alpha^n(x) = \sum_{j=0}^{n-1} \frac{c^j}{j!} \frac{1}{\Gamma(-\alpha j)} \int_0^x \frac{p_\alpha(z)}{(x-z)^{\alpha j+1}} dz, \tag{4.3}$$

i.e. as a finite sum of convolutions of the standard α -stable density $p_\alpha(\cdot)$.

In the case $\rho = 2$, we can also rewrite (3.4) as

$$\begin{aligned}
 h_\alpha^2(x) &= \frac{\alpha}{x} H_{1,1}^{0,1} \left[\frac{x^\alpha}{c} \middle| \begin{matrix} (-1, 1) \\ (0, \alpha) \end{matrix} \right] = \frac{\alpha}{x} \frac{1}{2\pi i} \int_{\mathcal{L}} \frac{\Gamma(2-s) \left(\frac{x^\alpha}{c}\right)^{-s}}{\Gamma(1-\alpha s)} ds \\
 &= \frac{\alpha}{x} \frac{1}{2\pi i} \int_{\mathcal{L}} \frac{\Gamma(z) \left(\frac{x^\alpha}{c}\right)^{z-2}}{\Gamma(1+\alpha z-2\alpha)} dz = \frac{\alpha c^2}{x^{2\alpha+1}} W_{-\alpha, 1-2\alpha} \left(-\frac{c}{x^\alpha}\right).
 \end{aligned}$$

Moreover, when $\rho = 2$ and $\alpha = 1/2$, it reduces to

$$\begin{aligned}
 h_{1/2}^2(x) &= \frac{c^2}{2x^2} \frac{1}{2\pi i} \int_{\mathcal{L}} \frac{\Gamma(s) \left(\frac{c}{\sqrt{x}}\right)^{-s}}{\Gamma(s/2)} ds \\
 &= \text{by the duplication formula of the gamma function} \\
 &= \frac{c^2}{4x^2\sqrt{\pi}} \frac{1}{2\pi i} \int_{\mathcal{L}} \Gamma\left(\frac{s}{2} + \frac{1}{2}\right) \left(\frac{c}{2\sqrt{x}}\right)^{-s} ds \\
 &= \frac{c^2}{4x^2\sqrt{\pi}} H_{0,1}^{1,0} \left[\frac{c}{2\sqrt{x}} \middle| \begin{matrix} \\ (\frac{1}{2}, \frac{1}{2}) \end{matrix} \right] = [\text{by (1.125) in [23]}] \\
 &= \frac{c^3}{4x\sqrt{\pi}x^3} e^{-\frac{c^2}{4x}},
 \end{aligned}$$

which is linked to a Lévy distribution Y_a with scale parameter $a = c^2/2$ by the following relationship:

$$h_{1/2}^2(x) = \frac{a}{x} P(Y_a \in dx). \quad (4.4)$$

which is linked to a Lévy distribution Y_a with scale parameter $a = c^2/2$ by the following relationship:

$$h_{1/2}^2(x) = \frac{a}{x} P(Y_a \in dx). \quad (4.5)$$

The Laplace transform can be checked to coincide with (4.1), indeed

$$\int_{c\sqrt{\eta}}^{+\infty} e^{-w} w dw = e^{-c\sqrt{\eta}} (1 + c\sqrt{\eta})$$

and

$$\frac{d}{d\eta} \int_{c\sqrt{\eta}}^{+\infty} e^{-w} w dw = \frac{d}{d\eta} e^{-c\sqrt{\eta}} (1 + c\sqrt{\eta}) = -\frac{c^2}{2} e^{-c\sqrt{\eta}}$$

On the other hand,

$$\frac{d}{d\eta} \int_0^{+\infty} e^{-\eta x} \frac{a}{x} P(Y_a \in dx) = -a \int_0^{+\infty} e^{-\eta x} P(Y_a \in dx) = -a e^{-\sqrt{2a\eta}}.$$

Moreover, from (4.5) it is evident that the first moment is finite and reads

$$\mathbb{E}\mathcal{X}_{\frac{1}{2},2} = \frac{c^2}{2}.$$

The tail probabilities of the r.v. $\mathcal{X}_{\alpha,\rho}$ can be obtained, by means of the Tauberian theorem (see e.g.[8], Theorem XIII-5-4, p.446), as follows:

$$\begin{aligned} \int_0^{+\infty} e^{-ux} P(\mathcal{X}_{\alpha,\rho} > x) dx &= \frac{1 - \mathbb{E}e^{-u\mathcal{X}_{\alpha,\rho}}}{u} \\ &= \frac{\int_0^{cu^\alpha} e^{-w} w^{\rho-1} dw}{u\Gamma(\rho)} = [\text{for } u \rightarrow 0] \simeq \frac{u^{\alpha\rho-1}}{\Gamma(\rho+1)}, \end{aligned}$$

by considering the well-known asymptotics of the (lower) incomplete gamma function. Under the condition $\rho < 1/\alpha$ we get, from (5.17) in [8], for a constant $L(\cdot)$,

$$\lim_{x \rightarrow +\infty} x^{\alpha\rho} P(\mathcal{X}_{\alpha,\rho} > x) = \frac{1}{\Gamma(\rho+1)\Gamma(1-\alpha\rho)}. \quad (4.6)$$

For $\rho = 1$ (α -stable case), formula (4.6) reduces with the well-known result (see formula (1.2.10) in [29]).

By taking into account (4.6) and the fact that $\mathbb{E}X^\delta = \int_0^{+\infty} P(X^\delta > x) dx$, for any positive r.v., we can give the following condition for the finiteness of the δ -th moments:

$$\mathbb{E}(\mathcal{X}_{\alpha,\rho})^\delta < +\infty, \quad \text{iff} \quad 0 < \delta < \alpha\rho.$$

Therefore, in the case where $\rho < 1$, the r.v. $\mathcal{X}_{\alpha,\rho}$ has finite moments only of fractional order less than 1, as happens for the stable law of order $\alpha \in (0, 1)$. For any ρ , we can evaluate the moment of order $\delta \in (0, \alpha\rho)$, for $\delta \neq n \in \mathbb{N}$, by applying the Mellin transform formula for the H -function given in (2.8) of [23]:

$$\begin{aligned} \mathbb{E}(\mathcal{X}_{\alpha,\rho})^\delta &= \frac{\alpha}{\Gamma(\rho)} \int_0^{+\infty} x^{\delta-1} H_{1,1}^{0,1} \left[\frac{x^\alpha}{c} \middle| \begin{matrix} (1-\rho, 1) \\ (0, \alpha) \end{matrix} \right] dx \quad (4.7) \\ &= \frac{1}{\Gamma(\rho)} \int_0^{+\infty} z^{\frac{\delta}{\alpha}-1} H_{1,1}^{0,1} \left[\frac{z}{c} \middle| \begin{matrix} (1-\rho, 1) \\ (0, \alpha) \end{matrix} \right] dz \\ &= \frac{c^{\frac{\delta}{\alpha}}}{\Gamma(\rho)} \frac{\Gamma(\rho - \frac{\delta}{\alpha})}{\Gamma(1-\delta)}. \end{aligned}$$

Formula (4.7) reduces, for $\rho = 1$, to the well-known δ -th order moment of the stable law (see [39]).

Acknowledgements

The authors are grateful to the referees and the Editor for many useful comments on an earlier version and to Claudio Macci for insightful discussions and suggestions.

References

- [1] L. Beghin, On fractional tempered stable processes and their governing differential equations. *J. of Comput. Physics* **293** (2015), 29–39.
- [2] L. Beghin, Fractional diffusion-type equations with exponential and logarithmic differential operators. *Stoch. Proc. Appl.* **128** (2018), 2427–2447.
- [3] L. Beghin, C. Ricciuti, Time-inhomogeneous fractional Poisson processes defined by the multistable subordinator. *Stoch. Anal. Appl.* **37**, No 2 (2019), 171–188.
- [4] L. Beghin, C. Ricciuti, Pseudo-differential operators and related additive geometric stable processes. *Markov Processes and Related Fields* **25** (2019), 415–444.
- [5] S. Bochner, *Harmonic Analysis and the Theory of Probability*. University of California Press, California Monogr. Math. Sci., Berkeley (1955).
- [6] B.D. Carter, M.D. Springer, The distribution of products, quotients and powers of independent H -function variates. *SIAM J. Appl. Math.* **33**, No 4 (1977), 542–558.
- [7] I. Dimovski, V. Kiryakova, The Obrechhoff integral transform: Properties and relation to a generalized fractional calculus. *Numer. Funct. Anal. and Optimiz.* **21**, No 1-2 (2000), 121–144; DOI:10.1080/01630560008816944.
- [8] W. Feller, *An Introduction Probability Theory and its Applications*. Vol. 2 (2nd Ed.), Wiley, New York (1971).
- [9] J. Gajda, A. Wylomanska, Time-changed Ornstein–Uhlenbeck process. *J. Phys. A: Math. Theor.* **48** (2015), 1–19.
- [10] J. Gajda, A. Kumar, A. Wylomanska, Stable Lévy process delayed by tempered stable subordinator. *Statistics and Probability Letters* **145** (2019), 284–292.
- [11] R. Garra, A. Giusti, F. Mainardi, G. Pagnini, Fractional relaxation with time-varying coefficient. *Fract. Calc. Appl. Anal.* **17**, No 2 (2014), 424–439; DOI: 10.2478/s13540-014-0178-0; <https://www.degruyter.com/view/journals/fca/17/2/fca.17.issue-2.xml>.
- [12] R. Gorenflo, A.A. Kilbas, F. Mainardi, S.V. Rogosin, *Mittag-Leffler Functions, Related Topics and Applications*. Springer-Verlag, Berlin Heidelberg (2014).
- [13] G.J. Jameson, The incomplete gamma functions. *The Math. Gazette* **100** (548) (2016), 298–306.
- [14] D.B. Karp, J.L. Lopez, Representations of hypergeometric functions for arbitrary parameter values and their use. *J. of Approximation Theory* **218** (2017), 42–70.

- [15] D.B. Karp, J.L. Lopez, An extension of the multiple Erdélyi-Kober operator and representations of the generalized hypergeometric functions. *Fract. Calc. Appl. Anal.* **21**, No 5 (2018), 1360–1376; DOI: 10.1515/fca-2018-0071; <https://www.degruyter.com/view/journals/fca/21/5/fca.21.issue-5.xml>.
- [16] A.A. Kilbas, H.M. Srivastava, J.J. Trujillo, *Theory and Applications of Fractional Differential Equations*. Vol. 204 of North-Holland Math. Studies, Elsevier Science B.V., Amsterdam (2006).
- [17] V. Kiryakova, *Generalized Fractional Calculus and Applications*. Pitman Research Notes in Mathematics, Vol. 301, Longman (1994).
- [18] A.N. Kochubei, General fractional calculus, evolution equations and renewal processes. *Integr. Equat. Operator Theory* **71** (2011), 583–600.
- [19] V. Kolokoltsov, The probabilistic point of view on the generalized fractional partial differential equations. *Fract. Calc. Appl. Anal.* **22**, No 3 (2019), 543–600; DOI: 10.1515/fca-2019-0033; <https://www.degruyter.com/view/journals/fca/22/3/fca.22.issue-3.xml>.
- [20] A. Kumar, P. Vellaisamy, Inverse tempered stable subordinators. *Statistics and Probability Letters* **103** (2015), 134–141.
- [21] F. Mainardi, Fractional relaxation-oscillation and fractional diffusion-wave phenomena. *Chaos Solitons Fractals* **7**, No 9 (1996), 1461–1477.
- [22] O.I. Marichev, *Handbook of Integral Transforms of Higher Transcendental Functions, Theory and Algorithmic Tables*. Ellis Horwood, Chichester (1983).
- [23] A.M. Mathai, R.K. Saxena, H.J. Haubold, *The H-Function: Theory and Applications*. Springer, New York (2010).
- [24] M.M. Meerschaert, A. Sikorskii, *Stochastic Models for Fractional Calculus*. De Gruyter, Berlin/Boston (2012).
- [25] K.S. Miller, B. Ross, *An Introduction to the Fractional Calculus and Fractional Differential Equations*. Wiley, New York (1993).
- [26] E. Orsingher, C. Ricciuti, B. Toaldo, Time-inhomogeneous jump processes and variable order operators. *Potential Analysis* **45**, No 3 (2016), 435–461.
- [27] R.B. Paris, *Incomplete Gamma Function*. In: F.W. Olver, D.M. Lozier, R. Boisvert, C.W. Clark, *NIST Handbook of Mathematical Functions*, Cambridge University Press (2010).
- [28] V. Pipiras, M. Taqqu, *Long-Range Dependence and Self-Similarity*. Cambridge University Press (2017).
- [29] G. Samorodnitsky, M. Taqqu, *Stable Non-Gaussian Random Processes: Stochastic Models with Infinite Variance*. Chapman etc., New York (1994).

- [30] R.L. Schilling, R. Song, Z. Vondracek, *Bernstein Functions: Theory and Applications*. Vol. 37, De Gruyter Studies in Math. Ser., Berlin (2010).
- [31] M.D. Springer, W.E. Thompson, The distribution of products of beta, gamma and Gaussian random variables. *SIAM J. Appl. Math.* **18**, No 4 (1970), 721–737.
- [32] E.W. Stacy, A generalization of the gamma distribution. *The Annals of Mathematical Statistics*, **33**, No 3 (1962), 1187–1192.
- [33] A. Stanislavsky, K. Weron, A. Weron, Diffusion and relaxation controlled by tempered α -stable processes. *Physical Review E* **2008** (2008), # 051106, 1–6.
- [34] B. Toaldo, Convolution-type derivatives, hitting-times of subordinators and time-changed C_0 -semigroups. *Potential Analysis* **42**, No 1 (2015), 115–140.
- [35] P. Vellaisamy, K.K. Kataria, The I -function distribution and its extensions. *Teor. Veroyatnost. i Primenen.* **63**, No 2 (2018), 284–305.
- [36] A. Wyłomanska, Arithmetic Brownian motion subordinated by tempered stable and inverse tempered stable processes. *Physica A* **391** (2012), 5685–5696.
- [37] A. Wyłomanska, The tempered stable process with infinitely divisible inverse subordinators. *J. Stat. Mech. Theory Exp.* (2013), 1–18.
- [38] A. Wyłomanska, A. Chechkin, J. Gajda, I.M. Sokolov, Codifference as a practical tool to measure interdependence. *Physica A: Statistical Mechanics and its Applications* **421**, No 1 (2015), 412–429.
- [39] S.J. Wolfe, On moments of probability distribution functions. In: *Fractional Calculus and Its Applications*, B. Ross (Ed.), Lect. Notes in Math. 457, Springer, Berlin (1975), 306–316.

¹ *Sapienza University of Rome*

p. le A.Moro 5, 00185 Rome, ITALY

e-mail: luisa.beghin@uniroma1.it (Corresponding author)

Received: December 27, 2019, Revised: February 8, 2020

² *Faculty of Economic Sciences, University of Warsaw*

Długa 44/50, 00-241 Warsaw, POLAND

e-mail: jgajda@wne.uw.edu.pl

Please cite to this paper as published in:

Fract. Calc. Appl. Anal., Vol. **23**, No 5 (2020), pp. 1248–1273,
DOI: 10.1515/fca-2020-0063