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Existence through convexity for the truncated Laplacians

I. Birindelli[1] · G. Galise[1] · H. Ishii[2]

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Abstract

We study the Dirichlet problem on a bounded convex domain of $\mathbb{R}^N$, with zero boundary data, for truncated Laplacians $\mathcal{P}_k^{\pm}$, which are degenerate elliptic operators, for $k < N$, defined by the upper and respectively lower partial sum of k eigenvalues of the Hessian matrix. We establish a necessary and sufficient condition (Theorem 1) in terms of the "flatness" of domains for existence of a solution for general inhomogeneous term. This result, in particular, shows that the strict convexity of the domain is sufficient for the solvability of the Dirichlet problem. The result and related ideas are applied to the solvability of the Dirichlet problem for the operator $\mathcal{P}_k^+$ with lower order term when the domain is strictly convex and the existence of principal eigenfunctions for the operator $\mathcal{P}_1^+$. An existence theorem is presented with regard to the principal eigenvalue for the Dirichlet problem with zero-th order term for the operator $\mathcal{P}_1^+$. A nonexistence result is established for the operator $\mathcal{P}_k^+$ with first order term when the domain has a boundary portion which is nearly flat. Furthermore, when the domain is a ball, we study the Dirichlet problem, with a constant inhomogeneous term and a possibly sign-changing first order term, and the associated eigenvalue problem.

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✉ I. Birindelli
isabeau@mat.uniroma1.it

G. Galise
galise@mat.uniroma1.it

H. Ishii
hitoshi.ishii@waseda.jp

[1] Dipartimento di Matematica, Sapienza Università di Roma, P.le Aldo Moro 2, 00185 Rome, Italy

[2] Institute for Mathematics and Computer Science, Tsuda University, 2-1-1 Tsuda-machi, Kodaira-shi, Tokyo 187-8577, Japan

1 Introduction

For any $N \times N$ symmetric matrix X, let

$$\lambda_1(X) \leq \lambda_2(X) \leq \cdots \leq \lambda_N(X) \tag{1.1}$$

be the ordered eigenvalues of X. For $k \in [1, N]$, k integer, let

$$\mathcal{P}_k^-(D^2u) = \sum_{i=1}^{k} \lambda_i(D^2u) \quad \text{and} \quad \mathcal{P}_k^+(D^2u) = \sum_{i=1}^{k} \lambda_{N+1-i}(D^2u). \tag{1.2}$$

For $k = N$ these operators coincide with the Laplacian, hence we will always consider $k < N$.

In the whole paper Ω will be a bounded domain of $\mathbb{R}^N$. The scope of the paper is to study existence of solutions for the following Dirichlet problem

$$\begin{cases} \mathcal{P}_k^+(D^2u) + H(x, Du) = f(x) & \text{in } \Omega \\ u = 0 & \text{on } \partial\Omega. \end{cases} \tag{1.3}$$

Throughout this paper, the Dirichlet boundary condition is understood in the classical pointwise sense. Before describing the results of this paper, let us mention that the operators $\mathcal{P}_k^+$ and $\mathcal{P}_k^-$ come out naturally in geometrical problems in particular when considering manifolds of partially positive curvature, see [19,20], or mean curvature flow in arbitrary codimension, see [2]. Lately the interest has been from a pure PDE theoretical point of view, starting from the works of Harvey and Lawson [15,16] and Caffarelli, Li and Nirenberg [10] continuing with [18] by Oberman and L. Silvestre on convex envelope. See also [1,6,9,13,14] for further contributions.

Some analogies can be found in the work of Blanc and Rossi [7] but we will be more explicit about their work at the end of the introduction.

In [4], when Ω is uniformly convex, i.e. when there exists $R > 0$ and $Y \subseteq \mathbb{R}^N$ such that

$$\Omega = \bigcap_{y \in Y} B_R(y) \tag{1.4}$$

we called these domains hula hoop domains and, in these domains we proved existence of solutions for any bounded f as long as $|H(x, p) - H(x, q)| \leq b|p-q|$ and $bR < k$.

On the other hand, in [5], if Ω is only convex, i.e. an intersection of half spaces or cubes, $k = 1$ and $H \equiv 0$, existence was established under some sign condition on f near the boundary of Ω.

In a general sense we wish to understand up to which point these conditions are optimal. We will see how these degenerate elliptic operators are extremely sensitive to the "convexity" of the domain and are strongly influenced by the presence of the first order term.

In fact, in order to concentrate on the domain, we shall treat first the case where $H(x, Du) \equiv 0$.

In a first step we shall see that convexity alone, does not allow to prove existence of supersolutions for any f. In order to solve the Dirichlet problem with general right hand side f we should impose that $\partial\Omega$ has at least $N-k$ directions of strict convexity. We are now going to be more precise.

We can introduce a sort of "classification" of strict convexity.

Consider for $j = 1, \dots, N$

$$C_j = \left\{ C \subset \mathbb{R}^N \ : \ C = \omega \times \mathbb{R}^{N-j}, \ \omega \subset \mathbb{R}^j \text{ bounded and strictly convex} \right\}. \tag{1.5}$$

Henceforth we denote by $\mathcal{C}_j$ the class of all convex and bounded domains $\Omega \subset \mathbb{R}^N$ which are intersection (up to rotations) of cylinders belonging to C_j. More precisely $\Omega \in \mathcal{C}_j$ if, and only if, for each $x \in \partial\Omega$, there exist $O \in \mathcal{O}^N$, with $\mathcal{O}^N$ being the class of orthogonal $N \times N$ matrices, and $C \in C_j$ such that

$$\Omega \subset OC \quad \text{and} \quad x \in \partial(OC). \tag{1.6}$$

We denote by $S_j = S_j(\Omega)$ the set of all $(O, C) \in \mathcal{O}^N \times C_j$ such that for some $x \in \partial\Omega$, (1.6) is satisfied. One has

$$\Omega = \bigcap_{(O,C)\in S_j} OC \quad \text{if } \Omega \in \mathcal{C}_j, \tag{1.7}$$

and

$$\mathcal{C}_1 \supset \mathcal{C}_2 \supset \cdots \supset \mathcal{C}_N.$$

Note that $\mathcal{C}_1$ and $\mathcal{C}_N$ correspond respectively to the class of bounded convex and strictly convex domains. It may be useful to note that if $\omega \subset \mathbb{R}^j$, $C \subset \mathbb{R}^N$, and $O \in \mathcal{O}^N$, then

$$\partial(\omega \times \mathbb{R}^{N-j}) = \partial\omega \times \mathbb{R}^{N-j} \quad \text{and} \quad \partial(OC) = O\partial C.$$

It might be remarked at this point that, when Ω is given by (1.4), one can find $y \in \overline{Y}$ for each $x \in \partial\Omega$ such that

$$\Omega \subset B_R(y) \quad \text{and} \quad x \in \partial B_R(y).$$

(To check this, one may choose a sequence $z_j \in \mathbb{R}^N \setminus \overline{\Omega}$ converging to x, then choose a sequence $y_j \in Y$ so that $z_j \notin B_R(y_j)$, and send $j \to \infty$ along a subsequence so that the subsequence converges to a point $y \in \overline{Y}$. It is clear that $\Omega \subset B_R(y)$ and $x \in \partial B_R(y)$.)

This is the relationship between existence of solutions and "strict convexity" of the domain.

Theorem 1 *Let Ω be a convex domain. The Dirichlet problem*

$$\begin{cases} \mathcal{P}_k^+(D^2u) = f(x) & \text{in } \Omega \\ u = 0 & \text{on } \partial\Omega \end{cases} \tag{1.8}$$

has a unique solution for any bounded $f \in \mathrm{C}(\Omega)$ if and only if $\Omega \in \mathcal{C}_{N-k+1}$.

Hence we have a sort of optimal condition for existence. In fact we have better, in the sense that we prove nonexistence of *supersolutions* when the domain is not in $\mathcal{C}_{N-k+1}$. For the part concerning existence, the construction of supersolutions is given in a constructive and elegant way. When $k = 1$, i.e. when the domain is strictly convex, this result will lead to the construction of the so called eigenfunction corresponding to the principal demi-eigenvalue, so generalizing the existence of eigenfunctions provided in [4] under the uniform convexity assumption.

As mentioned above if the forcing term f is positive or at least not too negative near the boundary, a solution of (1.8) exists as soon as Ω is convex, strict convexity is needed in order to allow f to be negative at the boundary. So the real question is to obtain existence e.g. for $f \equiv -1$.

Interestingly, the presence of the first order term changes dramatically the dependence of the existence of solutions on the convexity of the domain. In fact it worsens the situation in the sense that "strict convexity" in general is not enough for existence in the presence of the first order term. The problem can be of "local" type, i.e. if there is a point P of the boundary where the principal curvatures are zero, even if the domain is strictly convex, then, for $b > 0$ there are no positive supersolutions of

$$\mathcal{P}_k^+(D^2u) + b|Du| = -1 \tag{1.9}$$

which are zero at that point P, see Theorem 14.

Or the problem can be of a global nature, i.e. if Ω is too large, independently of its shape, there are no solutions. More precisely, if $B_R \subset \Omega$ and $bR \geq k$ there are no supersolutions of (1.9). Other cases with nonconstant b are also considered in Sect. 4.

Due to the relevance of the condition $\mathcal{C}_j$, we now give a characterization in term of flatness of the boundary, which will play a role in the proof of Theorem 1.

Given a bounded convex domain Ω and $x \in \partial\Omega$, we consider the maximal dimension $d_x(\Omega)$ of linear subspaces V of the tangent space of $\partial\Omega$ at x such that $(x+V)\cap\partial\Omega$ is a neighborhood of x in the relative topology of $x+V$. That is, $d_x(\Omega)$ is the maximum of $m \in \{0, 1, \dots, N-1\}$ such that there exist an m-dimensional linear subspace V in $\mathbb{R}^N$ and $\delta > 0$ such that $x + V \cap B_\delta \subset \partial\Omega$. We set $d(\Omega) = \max_{x\in\partial\Omega} d_x(\Omega)$.

Theorem 2 *Let Ω be a bounded convex domain. We have $\Omega \in \mathcal{C}_j$ if and only if $d(\Omega) \leq N - j$.*

Finally we wish to somehow compare our results with some results of Blanc and Rossi. In [7] they consider the problem

$$\begin{cases} \lambda_j(D^2u) = 0 & \text{in } \Omega \\ u = g & \text{on } \partial\Omega \end{cases}$$

and they prove that if $\Omega \in \mathcal{G}_j \cap \mathcal{G}_{N-j}$ then the above Dirichlet problem is solvable for any g while, if Ω is not in $\mathcal{G}_j \cap \mathcal{G}_{N-j}$ then there should be some g for which the problem is not solvable. The precise definition of $\mathcal{G}_j$ is recalled in the last section. Let us mention that these operators, as well as the truncated Laplacians treated here, are fully nonlinear operators and hence it is not possible to pass immediately from a Dirichlet problem with homogeneous boundary data to a Dirichlet problem with homogeneous forcing term. Nonetheless it is clear that both problems are related.

The definition of these $\mathcal{G}_j$ domains is different from the way we describe the "strict convexity" of our domains. In the sense that we use domains that are intersection of rotations and translations of "$(N-j+1)$-dimensional cylinders" in $\mathcal{C}_{N-j+1}$.

In fact these notions are, in general, different since $\mathcal{G}_j \cap \mathcal{G}_{N-j}$ contains domain that may not even be convex. On the other hand, if the domain is convex then the two notions are equivalent as it is proved in the last section together with the proof of Theorem 2.

2 Dirichlet problem

2.1 Nonexistence

We begin by proving that convexity alone is not enough to solve Dirichlet problems for $\mathcal{P}_k^+$ even for very regular forcing term.

Proposition 3 *Let $\Omega \subset \mathbb{R}^N$ be a convex domain and assume that up to a rigid motion there exists $\delta > 0$ such that the k-dimensional ball*

$$B_{k,\delta} = \left\{ x = (x_1, \dots, x_k, 0, \dots, 0) \in \mathbb{R}^N \ : \ |x| < \delta \right\} \subset \partial\Omega. \tag{2.1}$$

Then there are no supersolutions $u \in \mathrm{LSC}(\overline{\Omega})$ of

$$\begin{cases} \mathcal{P}_k^+(D^2u) = -1 & \text{in } \Omega \\ \quad u = 0 & \text{on } \partial\Omega \end{cases} \tag{2.2}$$

such that

$$\lim_{x\to 0} u(x) = 0. \tag{2.3}$$

We note that condition (2.1) implies that $d_0(\Omega) \geq k$. We recall that, for $x \in \partial\Omega$, $d_x(\Omega)$ is defined by

$$\begin{aligned} d_x(\Omega) = \max \Big\{ & m \in \{1, \dots, N-1\} \ : \ \exists V \ \ m - \text{dimensional linear subspace on } \mathbb{R}^N \\ & \text{and } \delta > 0 \text{ s.t. } x + V \cap B_\delta \subset \partial\Omega \Big\}. \end{aligned} \tag{2.4}$$

Proof of Proposition 3 Let us suppose by contradiction that there exists a supersolution u of (2.2) satisfying (2.3). It cannot achieve the minimum at an interior point x, since otherwise we would have $\mathcal{P}_k^+(D^2u(x)) \geq 0$. Hence u is positive in Ω. In view of (2.3), there exists a positive number r smaller than δ such that

$$u(x) < \frac{\delta^2}{16k} \quad \text{for any } x \in B_r \cap \Omega. \tag{2.5}$$

Claim: There exists a point $z \in \Omega$ and $\varepsilon < \frac{\delta}{2}$ such that $z \in \{0\} \times \mathbb{R}^{N-k} \subset \mathbb{R}^N$, $|z| < r$ and the cylinder

$$C = \left\{ x \in \mathbb{R}^N \ : \ \sum_{i=1}^k x_i^2 < \frac{\delta^2}{4}, \ \sum_{i=k+1}^N (x_i - z_i)^2 < \varepsilon^2 \right\} \subset \Omega.$$

We suppose that the claim is proved and we go on with the rest of the proof.

Since $z \in B_r$, (2.5) yields

$$u(z) < \frac{\delta^2}{16k}. \tag{2.6}$$

Let

$$\varphi(x) = -\alpha \sum_{i=1}^k x_i^2 - \beta \left[\sum_{i=k+1}^N (x_i - z_i)^2 - \varepsilon^2 \right], \tag{2.7}$$

where

$$\alpha = \frac{8u(z)}{\delta^2}, \quad \beta = \frac{2u(z)}{\varepsilon^2}. \tag{2.8}$$

We claim that $\min\limits_{\overline{C}}(u - \varphi)$ is attained at some point $\xi \in C$.

Let $x \in \partial C$. If $\sum_{i=1}^k x_i^2 = \frac{\delta^2}{4}$ then

$$u(x) - \varphi(x) \geq -\varphi(x) \geq \alpha \frac{\delta^2}{4} - \beta\varepsilon^2 = 0$$

in view of (2.8).
Otherwise $\sum_{i=k+1}^N (x_i - z_i)^2 = \varepsilon^2$ and

$$u(x) - \varphi(x) \geq -\varphi(x) = \alpha \sum_{i=1}^k x_i^2 \geq 0.$$

Since

$$u(z) - \varphi(z) = u(z) - \beta\varepsilon^2 < 0,$$

then necessarily $u - \varphi$ has a minimum at an interior point, say $\xi \in C$, and

$$\mathcal{P}_k^+(D^2\varphi(\xi)) \leq -1. \tag{2.9}$$

On the other hand

$$D^2\varphi = \mathrm{diag}(\underbrace{-2\beta, \ldots, -2\beta}_{N-k \text{ times}}, \underbrace{-2\alpha, \ldots, -2\alpha}_{k \text{ times}})$$

with $\alpha < \beta$. Then using (2.6) and (2.8) one has

$$\mathcal{P}_k^+(D^2\varphi(\xi)) = -2\alpha k > -1$$

in contradiction to (2.9).

We now give the proof of the claim. Since the origin is on $\partial\Omega$, we may choose a $y \in \Omega$ so that $|y| < r$. Set

$$y^{(1)} = (y_1, \ldots, y_k, 0, \ldots, 0), \quad y^{(2)} = (0, \ldots, 0, y_{k+1}, \ldots, y_N) \in \mathbb{R}^N.$$

By assumption (2.1) $-y^{(1)} \in B_{k,\delta}$ and using the convexity of Ω

$$\frac{1}{2}y^{(2)} = \frac{1}{2}y - \frac{1}{2}y^{(1)} \in \frac{1}{2}\Omega + \frac{1}{2}\partial\Omega \subset \Omega.$$

Set $2z = \frac{1}{2}y^{(2)}$ and note that $2z \in \{0\} \times \mathbb{R}^{N-k} \subset \mathbb{R}^N$. Select a positive constant $\varepsilon < \frac{\delta}{2}$ so that $B_{2\varepsilon}(2z) \subset \Omega$ and note that

$$B_{k,\frac{\delta}{2}} + B_\varepsilon(z) = \frac{1}{2}B_{k,\delta} + \frac{1}{2}B_{2\varepsilon}(2z) \subset \Omega.$$

Then we have the inclusion for the cylinder

$$C = \left\{ x \in \mathbb{R}^N \ : \ \sum_{i=1}^{k} x_i^2 < \frac{\delta^2}{4}, \ \sum_{i=k+1}^{N} (x_i - z_i)^2 < \varepsilon^2 \right\} \subset \Omega.$$

□

2.2 Existence

In order to solve the Dirichlet problem with general right hand side f we should impose that $\partial\Omega$ has at least $N-k$ directions of strict convexity, as anticipated in the Introduction, see (1.5)–(1.6).

Theorem 4 *Let $\Omega \in \mathcal{C}_{N-k+1}$ and let $f \in C(\Omega)$ be bounded. Then the Dirichlet problem*

$$\begin{cases} \mathcal{P}_k^+(D^2u) = f(x) & \text{in } \Omega \\ u = 0 & \text{on } \partial\Omega \end{cases} \tag{2.10}$$

has a unique solution.

Before discussing the Dirichlet problem (2.10), for a basis of our discussion, we state a proposition concerning the comparison principle.

Proposition 5 *Let $\Omega \subset \mathbb{R}^N$ be a bounded domain and $b, f \in \mathrm{C}(\Omega)$. Let F denote either $\mathcal{P}_k^+$ of $\mathcal{P}_k^-$. Let $v \in \mathrm{USC}(\overline{\Omega})$ and $w \in \mathrm{LSC}(\overline{\Omega})$ be a sub and supersolution of*

$$F(D^2u) + b(x)|Du| = f(x) \quad in\ \Omega \tag{2.11}$$

and satisfy $v \le w$ on $\partial\Omega$. Moreover, assume that either of b, v or w is locally Lipschitz in Ω. Then, under one of the following conditions, we have $v \le w$ in Ω.

(i) *There exists a ball B_R such that $\Omega \subset B_R$ and that $\|b\|_\infty R \le k$.*
(ii) *$f > 0$ in Ω or $f < 0$ in Ω.*

A comparison theorem under the condition (i) above (without equality) can be found in [14, Proposition 4.1], where it is also shown by a counterexample that the assumption $\|b\|_\infty R \le k$ cannot be improved in general.

It should be noted that $\mathrm{USC}(X)$ (resp., $\mathrm{LSC}(X)$) denotes here the set of *real-valued* upper (resp., lower) semicontinuous functions on X.

Outline of proof of Proposition 5 We consider only the case $F = \mathcal{P}_k^+$. Fix a small $\varepsilon > 0$ and consider the function $v_\varepsilon = v - \varepsilon$, which is still a subsolution of (2.11). Since $v_\varepsilon < w$ on $\partial\Omega$ and $v_\varepsilon - w \in \mathrm{USC}(\overline{\Omega})$, there exists $\delta \in (0, \varepsilon)$ so that for $\Omega_\delta = \{x \in \Omega : \mathrm{dist}(x, \partial\Omega) > \delta\}$, we have $v_\varepsilon < w$ on $\partial\Omega_\delta$. Note that either b, v_ε or w is Lipschitz continuous in Ω_δ.

The next step is to replace either v_ε or w by its small modification, which is, respectively, a strict subsolution or strict supersolution of (2.11) in Ω_δ.

Let $0 < \gamma < 1$ and first consider the case (i). By translation, we may assume that $\Omega_\delta \subset B_r$ for some $0 < r < R$ and consider the function $v_{\varepsilon,\gamma}(x) := v_\varepsilon(x) + \gamma|x|^2/2$ with $\gamma > 0$. This function $v_{\varepsilon,\gamma}$ is a subsolution of

$$\mathcal{P}_k^+(D^2u - \gamma I) + b(x)|Du - \gamma x| = f(x) \quad \text{in } \Omega,$$

where I denotes the $N \times N$ unit matrix. From this, it is easily seen $v_{\varepsilon,\gamma}$ is a subsolution of

$$\mathcal{P}_k^+(D^2u) + b(x)|Du| = f(x) + \gamma(k - \|b\|_\infty|x|) \quad \text{in } \Omega.$$

Note that, since $\gamma(k - \|b\|_\infty|x|) > 0$, $v_{\varepsilon,\gamma}$ is a strict subsolution of (2.11) in Ω_δ and that $v_{\varepsilon,\gamma} < w$ on $\partial\Omega_\delta$ for γ sufficiently small.

Next, consider the case (ii). If $f > 0$ in Ω, then, by the homogeneity of the operator $F(D^2\cdot) + b|\nabla\cdot|$, the function $v_{\varepsilon,\gamma} = (1+\gamma)v_\varepsilon$ is a subsolution of

$$\mathcal{P}_k^+(D^2u) + b|Du| = (1+\gamma)f \quad \text{in } \Omega_\delta,$$

which means that $v_{\varepsilon,\gamma}$ is a strict subsolution of (2.11) in Ω_δ. Similarly, if $f < 0$, the function $v_{\varepsilon,\gamma} = (1-\gamma)v_\varepsilon$ is a strict subsolution of (2.11) in Ω_δ. We may take $\gamma > 0$ small enough so that $v_{\varepsilon,\gamma} \le w$ on $\partial\Omega_\delta$

We may now apply [12, Theorem 3.3 and Sects. 5.A, 5.C], to conclude that $v_{\varepsilon,\gamma} \le w$ in Ω_δ. Sending $\gamma \to 0$ first and then $\varepsilon \to 0$ completes the proof.

Here are two remarks. For use of [12, Sect. 5.A], we observe that, if we set $G(x, p, X) = -\mathcal{P}_k^+(X) - b(x)|p|$ and two $N \times N$ matrices satisfy

$$\begin{pmatrix} X & 0 \\ 0 & -Y \end{pmatrix} \le 3\alpha \begin{pmatrix} I & -I \\ -I & I \end{pmatrix} \quad \text{for some } \alpha > 0, \tag{2.12}$$

then we have $X \le Y$ and therefore

$$\begin{aligned} G(y, \alpha(x-y), Y) - G(x, \alpha(x-y), X) &\le G(y, \alpha(x-y), Y) - G(x, \alpha(x-y), Y) \\ &\le \alpha|b(x) - b(y)||x-y| \quad \text{for } x, y \in \Omega_\delta. \end{aligned}$$

This shows that, taking limit under the condition that X and Y satisfy (2.12) and $\alpha|x-y| \le C$ for a fixed constant $C > 0$, we have

$$\limsup_{|x-y|\to 0} [G(y, \alpha(x-y), Y) - G(x, \alpha(x-y), X)] \le 0.$$

This observation is not enough for a direct application of [12, Sect. 5.A], but, in fact, a slight modification of the argument in [12, Sect. 5.A] yields $v_\varepsilon \le w$ in Ω_δ when either v and w is in $\mathrm{Lip}(\Omega_\delta)$.

Secondly, it is not trivial to see in the case of (ii) that if $\gamma > 0$ is small enough, then $v_{\varepsilon,\gamma} \le w$ on $\partial\Omega_\delta$. In fact, since $v_\varepsilon, -w \in \mathrm{USC}(\overline{\Omega}_\delta)$, we infer that $\max_{\partial\Omega_\delta}(v_\varepsilon - w) < 0$. Also, by the semicontinuity, there is a constant $M > 0$ such that $v_\varepsilon, -w \le M$ on $\partial\Omega_\delta$. For $x \in \partial\Omega_\delta$ and $\gamma > 0$ sufficiently small, if $v_\varepsilon(x) \le -2M$, then

$$v_{\varepsilon,\gamma}(x) = (1 \pm \gamma)v_\varepsilon(x) \le -2(1 \pm \gamma)M \le -M \le w(x),$$

and otherwise, we have $-2M < v_\varepsilon(x) \le M$ and

$$v_{\varepsilon,\gamma}(x) \le v_\varepsilon(x) + \gamma|v_\varepsilon(x)| < v_\varepsilon(x) + 2\gamma M \le w(x).$$

This way, one gets $v_{\varepsilon,\gamma} \le w$ on $\partial\Omega_\delta$ for small $\gamma > 0$. □

The proof of Theorem 4 is carried out by means of Perron method. It is worth pointing out that the standard procedure to construct subsolutions which are null on $\partial\Omega$ (see e.g. [11, Sect. 9]) works for $\mathcal{P}_k^+$ which is in fact a sup operator. On the other hand it fails for supersolutions owing to the strong degeneracy of $\mathcal{P}_k^+$ with respect to inf-type operations. The geometry of Ω plays here a crucial role.

Hence we will start by recalling a property concerning strict convex domains. Let Ω be a convex domain of $\mathbb{R}^N$ and $z \in \partial\Omega$. The set $N(z) = N_\Omega(z)$ of outward normal unit vectors at z is defined by

$$N(z) = \{p \in \mathbb{R}^N \;:\; |p| = 1, \;\; (x - z) \cdot p \le 0 \;\text{ for all } x \in \Omega\}.$$

It is well-known (a consequence of the Hahn-Banach theorem) that $N(z) \neq \emptyset$ for every $z \in \partial\Omega$.

Definition 6 A domain $\Omega \subset \mathbb{R}^N$ is strictly convex if

$$(1-t)x + ty \in \Omega \quad \text{for all } \ x, y \in \overline{\Omega}, \ \text{with } x \neq y, \ 0 < t < 1.$$

Lemma 7 *If a domain Ω is strictly convex, then*

$$N(x) \cap N(y) = \emptyset \quad \textit{for all } \ x, y \in \partial\Omega, \ \textit{with } x \neq y.$$

For the convenience of the reader, we give a proof.

Proof of Lemma 7 Let $x, y \in \partial\Omega, \ x \neq y$. Suppose that there is $p \in N(x) \cap N(y)$. It follows that

$$(z-x)\cdot p \leq 0, \quad (z-y)\cdot p \leq 0 \ \ \forall z \in \overline{\Omega}.$$

Adding these two yields

$$\left(z - \frac{x+y}{2}\right) \cdot p \leq 0 \ \forall z \in \overline{\Omega}. \tag{2.13}$$

Since Ω is strictly convex, we have

$$\frac{x+y}{2} \in \Omega,$$

and, therefore, there exists $\delta > 0$ such that

$$B_\delta\left(\frac{x+y}{2}\right) \subset \Omega,$$

and, in particular,

$$z := \frac{x+y}{2} + \delta p \in \overline{\Omega},$$

which shows that

$$\left(z - \frac{x+y}{2}\right) \cdot p = \delta|p|^2 = \delta > 0,$$

contradicting (2.13). □

Let ω be a bounded strictly convex domain in $\mathbb{R}^j$. Let $\mathcal{F}$ be the collection of functions

$$f(x) = \frac{1}{2}(R^2 - |x - x_0|^2),$$

where $R > 0, x,\ x_0 \in \mathbb{R}^j$, and, moreover,

$$f(x) > 0 \quad \text{on } \omega.$$

It is clear that $\mathcal{F} \neq \emptyset$. We set

$$\psi(x) = \inf_{f \in \mathcal{F}} f(x) \quad \text{for } \ x \in \overline{\omega}. \tag{2.14}$$

It is clear that ψ is concave, since it is infimum of concave functions. Hence $\psi \in \mathrm{Lip}_{\mathrm{loc}}(\omega)$ and one has $\mathcal{P}_1^+(D^2\psi) \leq -1$ in ω. Moreover

$$\psi \in \mathrm{USC}(\overline{\omega}), \quad \psi \geq 0 \ \text{ on } \ \overline{\omega}.$$

Theorem 8 $\psi(x) = 0$ *for all* $x \in \partial\omega$. *In particular,* $\psi \in \mathrm{C}(\overline{\omega})$.

Proof. Fix $z_0 \in \partial\omega$ and $p \in N(z_0)$. By rotation and translation, we may assume that $z_0 = 0$ and $p = (0, \ldots, 0, 1)$. For generic $z \in \mathbb{R}^j$, we write

$$z = (x, y), \quad x \in \mathbb{R}^{j-1},\ y \in \mathbb{R}.$$

We choose $R_0 > 0$ so that

$$\omega \subset \left\{ z = (x, y) \in \mathbb{R}^j \ : \ |x|^2 + y^2 < R_0^2 \right\}.$$

For any $R \geq R_0$, set

$$\rho \equiv \rho(R) := \sup \left\{ h \geq 0 \ : \ \omega \subset \{(x, y) \ : \ |x|^2 + (y + h)^2 < R^2\} \right\}.$$

It is clear by simple geometry that $0 < \rho < \infty$, $R \mapsto \rho(R)$ is increasing and

$$\lim_{R \to \infty} \rho(R) = \infty.$$

Indeed, since, for $R \geq R_0$,

$$B_R\big((0, -(R - R_0))\big) \supset B_{R_0}((0, 0)) \supset \omega,$$

we see that

$$\rho(R) \geq R - R_0,$$

which shows that

$$\lim_{R \to \infty} \rho(R) = \infty.$$

Note also that

$$\overline{\omega} \subset \{(x, y) \, : \, |x|^2 + (y + \rho(R))^2 \le R^2\},$$

which implies that the function

$$f_R(x, y) := \frac{1}{2}\left(R^2 - |x|^2 - (y + \rho(R))^2\right)$$

is positive in ω, that is, $f_R \in \mathcal{F}$.

Observe that, if $r > R$, then

$$(0, 0) \notin \overline{B}_R((0, -r)),$$

which implies that

$$\overline{\omega} \not\subset \overline{B}_R((0, -r)),$$

and hence

$$\rho(R) \le r, \quad \text{ and, consequently,} \quad \rho(R) \le R.$$

We need only to show that

$$\lim_{R \to \infty} f_R(0) = 0.$$

(Notice that this implies that $\psi(0) \le 0$ and, moreover, that $\limsup_{\overline{\omega} \ni x \to 0} \psi(x) \le \psi(0) \le 0$ while $\liminf_{\overline{\omega} \ni x \to 0} \psi(x) \ge 0$ since $\psi \ge 0$ in $\overline{\omega}$.)

By the definition of $\rho(R)$ and the compactness of $\overline{\omega}$, there exists a point $z_R = (x_R, y_R) \in \partial\omega$ such that $f_R(z_R) = 0$. That is,

$$|x_R|^2 + (y_R + \rho(R))^2 = R^2.$$

By simple geometry again, we see that

$$p_R \equiv (\alpha_R, \beta_R) := \frac{1}{\sqrt{|x_R|^2 + |y_R + \rho(R)|^2}}(x_R, y_R + \rho(R)) \in N(z_R).$$

By the compactness of $\partial\omega$, there is a sequence $R_j \to \infty$ such that

$$z_{R_j} \to z_\infty \in \partial\omega.$$

Observe that, since $\lim_{R \to \infty} \rho(R) = \infty$,

$$p_{R_j} \to (0, 1).$$

Passing to the limit in the inequality $(x - z_{R_j}) \cdot p_{R_j} \le 0$ for all $x \in \omega$, we see that

$$(0, 1) \in N(z_\infty).$$

However, since $(0, 1) \in N(0) = N((0, 0))$, by the strict convexity of ω (Lemma 7), we must have

$$z_\infty = 0.$$

The above argument implies that

$$\lim_{R\to\infty} z_R = 0.$$

Observe that, since $R - R_0 \le \rho(R) \le R$,

$$\left|\frac{R\alpha_R \cdot x_R}{\beta_R}\right| = \left|\frac{R x_R \cdot x_R}{y_R + \rho(R)}\right| = \left|\frac{R}{y_R + \rho(R)}\right| |x_R|^2 \to 0 \quad \text{as } R \to \infty. \tag{2.15}$$

Noting that p_R is an outward normal vector to $B_R((0, -\rho(R))$ at z_R and that $(0, R - \rho(R)) \in \partial B_R((0, -\rho(R))$, we have

$$0 \ge p_R \cdot ((0, R - \rho(R)) - z_R) = \beta_R(R - \rho(R)) - \alpha_R \cdot x_R - \beta_R y_R,$$

and, if $\beta_R > 0$, then

$$R - \rho(R) \le \frac{\alpha_R \cdot x_R}{\beta_R} + y_R.$$

Since $(0, 1) \in N((0, 0))$, we have

$$0 \ge (0, 1) \cdot ((x_R, y_R) - (0, 0)) = y_R.$$

Thus, if $\beta_R > 0$, then

$$R - \rho(R) \le \frac{\alpha_R \cdot x_R}{\beta_R}.$$

Combining this with (2.15), we see that, as $R \to \infty$,

$$0 \le R(R - \rho(R)) \le R\frac{\alpha_R \cdot x_R}{\beta_R} \to 0,$$

and, moreover,

$$\lim_{R\to\infty} (R + \rho(R))(R - \rho(R)) = 0.$$

Hence,

$$\lim_{R\to\infty} f_R(0,0) = \frac{1}{2}\lim_{R\to\infty}(R^2-\rho(R)^2) = 0.$$

□

Lemma 9 *Let ψ be the function defined by (2.14) and $\omega \subset \mathbb{R}^j$ be as in (2.14). Assume that $j \le N$ and set $C = \omega \times \mathbb{R}^{N-j}$. Define the function Ψ on $\overline{C} = \overline{\omega} \times \mathbb{R}^{N-j}$ by $\Psi(x) = \psi(x_1, \dots, x_j)$. Let $k \in \mathbb{N}$ be such that $N - j < k$. Then, Ψ is continuous on $\overline{C}$ and $\mathcal{P}_k^+(D^2\Psi) \le -1$ in $\omega \times \mathbb{R}^{N-j}$.*

By definition, a set $C \subset \mathbb{R}^N$ is in C_j if and only if $C = \omega \times \mathbb{R}^{N-j}$ for some bounded strictly convex $\omega \subset \mathbb{R}^j$. The function ψ depends only on ω and if $C = \omega \times \mathbb{R}^{N-j} \in C_j$, then the function Ψ, defined in the lemma above, is considered to depend only on C. Thus, for later reference, we write Ψ_C for this Ψ.

Proof of Lemma 9 The continuity of Ψ is obvious, since $\psi \in \mathrm{C}(\overline{\omega})$. Recalling (2.14), the function Ψ is given as the infimum of a family of functions f on $\mathbb{R}^n$ of the form

$$f(x) = \frac{1}{2}\left(R^2 - \sum_{i=1}^{j}(x_i - x_{0,i})^2\right),$$

for some $R > 0$ and $x_0 \in \mathbb{R}^N$. Observe that

$$D^2 f = \mathrm{diag}(\underbrace{-1, \dots, -1}_{j \text{ times}}, \underbrace{0, \dots, 0}_{N-j \text{ times}}),$$

and, since $k > N - j$, $\mathcal{P}_k^+(D^2 f) \le -1$ in $\omega \times \mathbb{R}^{N-j}$. By the stability of the supersolution property under inf-operation, we conclude that $\mathcal{P}_k^+(D^2\Psi) \le -1$. □

Proof of Theorem 4 Since Ω is a convex set, the uniform exterior sphere condition is satisfied. Then for $r = |x|$ let us consider the function $G(r) = r^{-\alpha} - 1$ where $\alpha = \max\{k-1, 1\}$. Observe that for $r > 1$

$$G(r) < 0,\ G'(r) = -\alpha r^{-(\alpha+1)} < 0,\ G''(r) = \alpha(\alpha+1)r^{-(\alpha+2)} > 0.$$

Let $\Phi(x) = \sup_{x_b\in\partial\Omega} G(|x - z_b|)$, where z_b is such that $|x_b - z_b| = 1$ and for any $x \in \Omega$ one has $|x - z_b| > 1$. Then $\Phi \in \mathrm{C}(\overline{\Omega})$ and $\Phi = 0$ on $\partial\Omega$. Moreover

$$\mathcal{P}_k^+(D^2\Phi(x)) \ge \frac{1}{(1+\mathrm{diam}(\Omega))^{\alpha+2}}.$$

Then $\underline{u} = M_1\Phi$, with $M_1 = M_1(\Omega, \alpha, \|f\|_\infty)$ sufficiently large, is a continuous subsolution of (2.10) which vanishes on $\partial\Omega$.

Now we provide a continuous supersolution $\overline{u}$ such that $\overline{u} = 0$ on $\partial\Omega$. By the definition of $\mathcal{C}_{N-k+1}$, since $\Omega \in \mathcal{C}_{N-k+1}$, the set S_{N-k+1} is given associated with Ω. In view of (1.5), define for $x \in \overline{\Omega}$

$$w(x) = \inf_{(O,C)\in S_{N-k+1}} \Psi_C\left(O^T x\right),$$

where Ψ_C is the function on $\overline{C}$ defined in Lemma 9 (see also a comment after the lemma). From the properties of the function ψ defined by (2.14) it follows that Ψ_C is concave and nonnegative in C. Theorem 8 ensures that $\Psi_C = 0$ on ∂C and $\Psi_C \in \mathrm{C}(\overline{C})$. It is now obvious that w is nonnegative, concave and upper semicontinuous on $\overline{\Omega}$ and that $\Psi_C(O^T x) = 0$ for $x \in O(\partial C) = \partial(OC)$. It follows from (1.6) that

$$\partial\Omega \subset \bigcup_{(O,C)\in S_{N-k+1}} \partial(OC),$$

which implies that $w \le 0$ on $\partial\Omega$. These properties of w guarantee that $w \in \mathrm{C}(\overline{\Omega})$ and $w = 0$ on $\partial\Omega$.

Noting that if we set $j = N-k+1$, then $N-j < k$, we see by Lemma 9 that for any $(O,C) \in S_{N-k+1}$, $\mathcal{P}_k^+(D^2\Psi_C) \le -1$ in C and moreover, by the invariance of the operator $\mathcal{P}_k^+$ under orthogonal transformation, that the function $v(x) := \Psi_C(O^T x)$ satisfies $\mathcal{P}_k^+(D^2 v) \le -1$ in OC. The stability of the subsolution property under inf-operation implies that $\mathcal{P}_k^+(D^2 v) \le -1$ in Ω. We set $\overline{u} = \|f\|_\infty w$ and note that $\overline{u} \in \mathrm{C}(\overline{\Omega})$, $\overline{u} = 0$ on $\partial\Omega$ and $\mathcal{P}_k^+(D^2\overline{u}) \le f$ in Ω.

Now, the Perron method yields a function u on $\overline{\Omega}$ such that the upper semicontinuous envelope u^* of u is a subsolution of $\mathcal{P}_k^+(D^2u) = f$ in Ω, the lower semicontinuous envelope u_* of u is a supersolution of $\mathcal{P}_k^+(D^2u) = f$ in Ω, and $\underline{u} \le u_* \le u \le u^* \le \overline{u}$ on $\overline{\Omega}$. The standard argument including comparison between u^* and u_* assures that $u \in \mathrm{C}(\overline{\Omega})$ and u is a solution of (2.10). □

Proof of Theorem 1 Sufficiency of the $\mathcal{C}_{N-k+1}$ property of Ω has been proved in Theorem 4. Its necessity follows from Proposition 3. Indeed, if Ω is not in $\mathcal{C}_{N-k+1}$, then, by Theorem 2, $d(\Omega) \ge k$, which means after translation and orthogonal transformation that $0 \in \partial\Omega$, $d_0(\Omega) \ge k$, and, moreover, condition (2.1) holds. Thus, Proposition 3 implies that problem (1.8), with $f = -1$, does not have a solution continuous up to the boundary $\partial\Omega$. □

2.3 Application: eigenfunctions for $\mathcal{P}_1^+$ in strictly convex domains

Following the Berestycki-Nirenberg-Varadhan approach concerning the validity of the Maximum Principle, see [3], we have defined in [4] as candidate for the principal eigenvalue the values

$$\mu_k^+ = \sup\left\{\mu \in \mathbb{R} \ :\ \exists\varphi \in \mathrm{LSC}(\Omega),\ \varphi > 0,\ \mathcal{P}_k^+(D^2\varphi) + \mu\varphi \le 0 \ \text{ in } \Omega\right\},$$

$$\mu_k^- = \sup\left\{\mu \in \mathbb{R} \,:\, \exists\, \varphi \in \mathrm{USC}(\Omega),\ \varphi < 0,\ \mathcal{P}_k^+(D^2\varphi) + \mu\varphi \geq 0 \ \text{ in } \Omega\right\}.$$

For the convenience of the reader it is worth pointing out the change of notation: here μ_k^+ corresponds to what in [4] was called μ_k^- and vice versa, since in the present paper we deal with the maximal operator $\mathcal{P}_k^+$, whereas in [4] we considered the minimal one $\mathcal{P}_k^-$. In particular we proved that $\mu_k^- = +\infty$ while $\mu_k^+ < \infty$, so we will concentrate on the latter.

Even in the degenerate framework of the operators $\mathcal{P}_k^+$, we showed that if Ω is uniformly convex, then μ_k^+ gives threshold for the Maximum Principle (see [4, Theorems 4.1, 4.4]), this is true also for more general equations depending on gradient terms. Moreover, when $k = 1$, there exists a positive principal eigenfunction.

One of the question raised in [4] concerned the necessity of the uniform convexity of the domain. In the next theorem we show that the strict convexity assumption of Ω is sufficient for the existence of a principal eigenfunction, at least when there are no first order terms.

Theorem 10 *Let Ω be a bounded strictly convex domain and let f be a continuous and bounded function in Ω. Then there exists a solution $u \in \mathrm{C}(\overline{\Omega})$ of*

$$\begin{cases} \mathcal{P}_1^+(D^2u) + \mu u = f & \text{in } \Omega \\ u = 0 & \text{on } \partial\Omega \end{cases} \tag{2.16}$$

in the following two cases:

- *for $\mu < \mu_1^+$;*
- *for any μ if $f \geq 0$.*

Moreover in the case $\mu < \mu_1^+$ the solution is unique.

The uniqueness part of Theorem 10 is an obvious consequence of the following lemma.

Lemma 11 *Under the hypothesis of Theorem 10, let $\mu < \mu_1^+$, let $u \in \mathrm{USC}(\Omega)$ and $v \in \mathrm{LSC}(\Omega)$ be sub and supersolution of*

$$\mathcal{P}_1^+(D^2u) + \mu u = f \ \text{ in } \Omega,$$

respectively, and assume that

$$\lim_{\Omega \ni x \to \partial\Omega} (u(x) - v(x)) \leq 0.$$

Then, $u \leq v$ in Ω.

Proof Set $w = u - v$ and observe (see [14, Lemma 3.1] and also [8, Theorem 5.8], [17, Proposition 4.1]) that w is a subsolution of

$$\mathcal{P}_1^+(D^2w) + \mu w = 0 \ \text{ in } \Omega. \tag{2.17}$$

The maximum principle ([4, Theorem 4.1 and Remark 4.8]) yields $w \leq 0$ in Ω, which concludes the proof. □

For the reader's convenience, we recall here [4, Proposition 3.2] stated for $\mathcal{P}_1^+$.

Lemma 12 *Let $u \in \mathrm{LSC}(\overline{\Omega})$ be a supersolution of*

$$\begin{cases} \mathcal{P}_1^+(D^2u) = f(x) \ \text{in } \Omega \\ \qquad u = 0 \qquad \text{on } \partial\Omega \end{cases} \tag{2.18}$$

Then for each $\varepsilon > 0$ there exists a positive constant $L_\varepsilon = L_\varepsilon(\|u\|_\infty, \|f\|_\infty)$ such that

$$|u(x) - u(y)| \leq L_\varepsilon |x - y| \quad \text{for } x, y \in \Omega_\varepsilon,$$

where $\Omega_\varepsilon := \{x \in \Omega \ : \ \mathrm{dist}(x, \partial\Omega) > \varepsilon\}$. Furthermore, if for some constant $C > 0$

$$u(x) \leq C\,\mathrm{dist}(x, \partial\Omega) \quad \text{for } x \in \overline{\Omega},$$

then there exists a positive constant $L = L(C, \|u\|_\infty, \|f\|_\infty)$ such that

$$|u(x) - u(y)| \leq L|x - y| \quad \text{for } x, y \in \overline{\Omega}.$$

Proof of Theorem 10 We need only to prove the existence part of the theorem.

The Dirichlet problem (2.18) is uniquely solvable by means of Theorem 4. We henceforth assume that $\mu > 0$. We shall first prove Theorem 10 for $f := h \leq 0$ then for $f := g \geq 0$ and any μ, and, finally, for the general case.

Let $h = -f^- \leq 0$. Let $(w_n)_{n \in \mathbb{N}} \subset \mathrm{C}(\overline{\Omega})$ be the sequence defined in the following way:
set $w_1 = 0$ and, given w_n, define w_{n+1} as the unique solution of

$$\begin{cases} \mathcal{P}_1^+(D^2 w_{n+1}) = h - \mu w_n \ \text{in } \Omega \\ \qquad\quad w_{n+1} = 0 \qquad\quad \text{on } \partial\Omega. \end{cases} \tag{2.19}$$

Note that $(w_n)_{n\in\mathbb{N}}$ is nondecreasing, in particular $w_n \geq 0$ in Ω for any $n \in \mathbb{N}$. At each step the existence is done by using zero as a subsolution and $(\|h\|_\infty + \mu\|w_n\|_\infty)\psi$ as a supersolution, where ψ is the function defined by (2.14) in the case $\omega = \Omega$, see Theorem 8. We need to prove that the sequence $(\|w_n\|_\infty)_{n\in\mathbb{N}}$ is bounded.

Suppose that it is not, hence up to some subsequence $\lim_{n\to+\infty} \|w_n\|_\infty = +\infty$. Then consider $v_n = \frac{w_n}{\|w_n\|_\infty}$. Then $\|v_n\|_\infty = 1$ and v_n satisfies

$$\mathcal{P}_1^+(D^2 v_{n+1}) = \frac{h}{\|w_{n+1}\|_\infty} - \mu v_n \frac{\|w_n\|_\infty}{\|w_{n+1}\|_\infty}.$$

By construction v_n is a sequence of bounded functions. We want to prove that they are equicontinuous. Observe that,

$$\frac{h}{\|w_{n+1}\|_\infty} - \mu v_n \frac{\|w_n\|_\infty}{\|w_{n+1}\|_\infty} \geq -\|h\|_\infty - \mu.$$

Hence $0 \leq v_n \leq (\|h\|_\infty + \mu)\psi$ for any $n \in \mathbb{N}$.

For any $\delta > 0$, in $\Omega_\delta := \{x \in \overline{\Omega} : \operatorname{dist}(x, \partial\Omega) > \delta\}$, the functions v_n are uniformly Lipschitz continuous by Lemma 12. For any $\varepsilon > 0$, choose $\delta > 0$ such that $(\|h\|_\infty + \mu)\psi \leq \frac{\varepsilon}{2}$ for any $x \in \Omega \setminus \Omega_\delta$. Hence for any x, y in $\Omega \setminus \Omega_\delta$:

$$|v_n(x) - v_n(y)| \leq v_n(x) + v_n(y) \leq (\|h\|_\infty + \mu)(\psi(x) + \psi(y)) \leq \varepsilon.$$

Hence the sequence $(v_n)_{n\in\mathbb{N}}$ is equicontinuous in $\overline{\Omega}$ and up to a subsequence, for some $k \leq 1$, v_n converges to v_∞ solution of

$$\mathcal{P}_1^+(D^2 v_\infty) + k\mu v_\infty = 0, \quad \text{in } \Omega, \; v_\infty = 0 \quad \text{on } \partial\Omega.$$

By maximum principle, since $k\mu < \mu_1^+$ this implies that $v_\infty = 0$. This is a contradiction since $\|v_\infty\|_\infty = 1$.

We have just proved that there exists some constant K such that $\|w_n\|_\infty \leq K$ and clearly

$$0 \leq w_n \leq (\|h\|_\infty + K\mu)\psi. \tag{2.20}$$

Hence, reasoning as above the sequence is also equicontinuous in $\overline{\Omega}$ and then, up to a subsequence it converges to a solution $\overline{w}$ of (2.16) with f replaced by $h = -f^-$.

Let us consider now the case $f = f^+$. As above let us define the sequence $(w_n)_{n\in\mathbb{N}}$ by setting $w_1 = 0$ and, once w_n is given, solving (2.19) with f^+ in place of h. In particular $w_{n+1} \leq w_n \leq 0$. Arguing by contradiction as above and applying the global Lipschitz regularity result (Lemma 12) to negative functions $v_n := w_n/\|w_n\|_\infty$, we observe that the sequence $(w_n)_{n\in\mathbb{N}}$ is bounded in $\mathrm{C}(\overline{\Omega})$. Using again the same global Lipschitz estimates to w_n, we infer that the sequence $(w_n)_{n\in\mathbb{N}}$ is equi-Lipschitz. Then there is a subsequence converging to a solution $\underline{w}$ of

$$\begin{cases} \mathcal{P}_1^+(D^2\underline{w}) + \mu\underline{w} = f^+ & \text{in } \Omega \\ \underline{w} = 0 & \text{on } \partial\Omega. \end{cases}$$

Now we assume $\mu < \mu_1^+$ and consider general f. The above functions $\underline{w}$ and $\overline{w}$ are respectively sub and supersolution of (2.16). To apply the Perron method, we introduce

$$\mathcal{W} = \left\{ w \in \mathrm{C}(\overline{\Omega}) : \; \underline{w} \leq w \leq \overline{w} \quad \text{and } w \text{ supersolution of (2.16)} \right\}.$$

and, arguing as in proving the equi-continuity of (w_n) in the case $h = -f^-$, observe by the local estimates of Lemma 12 that $\mathcal{W}$ is equi-continuous on $\overline{\Omega}$.

Setting

$$u(x) = \inf\{w(x) : \ w \in \mathcal{W}\},$$

we get a continuous function on $\overline{\Omega}$, which solves (2.16) due to the Perron method. □

Theorem 13 *Let Ω be a strictly convex domain. Then there exists a function $\psi_1 \in C(\overline{\Omega})$ such that*

$$\begin{cases} \mathcal{P}_1^+(D^2\psi_1) + \mu_1^+\psi_1 = 0, \ \ \psi_1 > 0 & \textit{in } \Omega \\ \qquad\qquad \psi_1 = 0 & \textit{on } \partial\Omega. \end{cases} \tag{2.21}$$

Proof Let $\mu_n \nearrow \mu_1^+$ and use Theorem 10 to build $u_n \in C(\overline{\Omega})$ the solution of

$$\begin{cases} \mathcal{P}_1^+(D^2u_n) + \mu_n u_n = -1 & \text{in } \Omega \\ u_n = 0 & \text{on } \partial\Omega. \end{cases} \tag{2.22}$$

We claim that, up to some subsequence, $\lim_{n\to+\infty} \|u_n\|_\infty = +\infty$. Assume by contradiction that $\sup_{n\in\mathbb{N}} \|u_n\|_\infty < +\infty$. Reasoning as in the proof of Theorem 10 the sequence $(u_n)_{n\in\mathbb{N}}$ is bounded and equicontinuous. Hence, up to a subsequence, it converges to a nonnegative solution u of

$$\begin{cases} \mathcal{P}_1^+(D^2u) + \mu_1^+ u = -1 & \text{in } \Omega \\ u = 0 & \text{on } \partial\Omega. \end{cases}$$

The function u is positive in Ω, otherwise if $\min_{x\in\overline{\Omega}} u = u(x_0) = 0$ and $x_0 \in \Omega$, then $\varphi(x) = 0$ should be a test function touching u from below in x_0 and therefore should satisfy $0 \leq -1$, a contradiction.
Hence, for small positive ε, we have

$$\mathcal{P}_1^+(D^2u) + (\mu_1^+ + \varepsilon)u \leq 0 \quad \text{in } \Omega$$

contradicting the maximality of μ_1^+.

For $n \in \mathbb{N}$ the functions $v_n = \frac{u_n}{\|u_n\|_\infty}$ satisfy

$$\begin{cases} \mathcal{P}_1^+(D^2v_n) + \mu_n v_n = \frac{-1}{\|u_n\|_\infty} & \text{in } \Omega \\ \qquad\qquad v_n = 0 & \text{on } \partial\Omega \end{cases} \tag{2.23}$$

and $\sup_{n\in\mathbb{N}} \|v_n\|_\infty = 1$. Again by equicontinuity, extracting a subsequence if necessary, $(v_n)_{n\in\mathbb{N}}$ converges uniformly to a nonnegative function ψ_1 such that $\|\psi_1\|_\infty = 1$. Taking the limit as $n \to \infty$ in (2.23) we have

$$\begin{cases} \mathcal{P}_1^+(D^2\psi_1) + \mu_1^+\psi_1 = 0 & \text{in } \Omega \\ \qquad\qquad \psi_1 = 0 & \text{on } \partial\Omega. \end{cases}$$

By the strong minimum principle ([4, Remark 2.6]), we conclude $\psi_1 > 0$ in Ω as we wanted to show. □

3 Influence of the first order term

We shall see that, in the presence of the first order term $H(Du) = b|Du|$ with $b > 0$, strict convexity may not be enough to have existence. And even in the uniformly convex case, if the coefficient b is "too large" with respect the principal curvatures of $\partial\Omega$ there may not be existence of solutions. This phenomenon is a striking feature connected with the highly degeneracy of the operators $\mathcal{P}_k^{\pm}$. Roughly speaking, this can be viewed as a consequence that the first order term competes with the principal part.

3.1 Nonexistence results for strictly convex domain

Let Ω be a bounded convex domain in $\mathbb{R}^N$ and $k < N$. Assume that

$$\Omega \subset \{z = (x, y) \in \mathbb{R}^{N-1} \times \mathbb{R} \, : \, y > 0\}, \quad 0 = (0, 0) \in \partial\Omega,$$

and, as $(x, y) \in \partial\Omega$ and $x \to 0$,

$$y = o(|x|^2). \tag{3.1}$$

Theorem 14 *Under the hypotheses above, there are no positive supersolutions* $u \in \mathrm{LSC}(\Omega)$ *of*

$$\mathcal{P}_k^+(D^2u) + b|Du| \leq 0 \quad in\ \Omega,$$

where b is a positive constant, with the property $\lim_{\Omega \ni z \to 0} u(z) = 0$.

Remark 15 It is worth to point out, as a consequence of Theorem 14, that there are no positive eigenfunctions (with Dirichlet boundary) associated to $\mathcal{P}_k^+(D^2 \cdot) + b|D \cdot|$ if $b > 0$. This striking feature is further emphasized by the positivity of the so called "generalized principal eigenvalue" μ_k^+, at least if $\Omega \subseteq B_R$ and $bR < k$. In fact $\mu_k^+ \geq \frac{2(k-bR)}{R^2}$. This inequality can be easily deduced by considering $v(x) = R^2 - |x|^2$ in the definition of μ_k^+.

Proof of Theorem 14 By contradiction we suppose that there is a supersolution $u \in \mathrm{LSC}(\overline{\Omega})$ of

$$\mathcal{P}_k^+(D^2u) + b|Du| \leq 0 \quad \text{in } \Omega,$$

with $b > 0$, such that $\lim_{z \to 0} u(z) = 0$ and

$$u > 0 \quad \text{in } \Omega. \tag{3.2}$$

We may choose, in view of (3.1), a constant $R > 0$ and a function $g \in \mathrm{C}^2(\mathbb{R}^{N-1})$ such that

$$g(0) = 0, \quad Dg(0) = 0, \quad D^2g(0) = 0,$$

and

$$\{(x, y) \in \overline{B}_R((0,0))\backslash\{(0,0)\} \ : \ y \geq g(x)\} \ \subset \ \Omega. \tag{3.3}$$

We may moreover assume that

$$k|D^2 g(x)| < b \quad \text{for all } \ x \in \mathbb{R}^{N-1}, \quad \text{with} |x| < R, \tag{3.4}$$

where $|D^2 g(x)| = \max_i |\lambda_i(D^2 g(x))|$.

By (3.2) and (3.3), we have

$$\rho := \min\{u(x, y) \ : \ (x, y) \in \partial B_R((0,0)), \ y \geq g(x)\} > 0.$$

Set

$$\Omega_R = \{(x, y) \in B_R((0,0)) \ : \ y > g(x)\},$$

and note that

$$\begin{aligned}
&\overline{\Omega}_R = \{(x, y) \in \overline{B}_R((0,0)) \ : \ y \geq g(x)\} \subset \overline{\Omega},\\
&\overline{\Omega}_R\backslash\{(0,0)\} \subset \Omega,\\
&\partial\Omega_R = \{(x, y) \in \partial B_R((0,0)) \ : \ y \geq g(x)\} \ \cup \ \{(x, y) \in \overline{B}_R((0,0)) \ : \ y = g(x)\}.
\end{aligned}$$

Using $\lim_{z\to 0} u(z) = 0$, we may select a point $z_0 = (x_0, y_0) \in \Omega_R$ (close to the origin) so that

$$u(z_0) < \rho.$$

We may as well choose a function $\theta \in C^2(\mathbb{R})$ so that

$$\theta(0) = 0, \quad \theta'(r) > 0 \ \ \forall r \in \mathbb{R}, \quad \text{and} \quad \lim_{r\to+\infty} \theta(r) = \rho.$$

Let $\varepsilon > 0$ and set

$$\theta_\varepsilon(r) = \theta(r/\varepsilon) \quad \text{for } \ r \in \mathbb{R},$$

and

$$\phi_\varepsilon(x, y) = \theta_\varepsilon(y - g(x)) \quad \text{for } \ (x, y) \in \mathbb{R}^N.$$

Consider the function

$$\overline{\Omega}_R \ni z \ \mapsto \ u(z) - \phi_\varepsilon(z),$$

and note that, for $z = (x, y) \in \partial\Omega_R$,

$$u(z) - \phi_\varepsilon(z) \geq \begin{cases} u(z) - \theta_\varepsilon(0) \geq 0 - 0 = 0 & \text{if } y = g(x), \\ \rho - \phi_\varepsilon(z) > \rho - \rho = 0 & \text{otherwise,} \end{cases}$$

and, as $\varepsilon \to 0$,

$$u(z_0) - \phi_\varepsilon(z_0) = u(z_0) - \theta\left(\frac{y_0 - g(x_0)}{\varepsilon}\right) \to u(z_0) - \rho < 0.$$

We fix $\varepsilon > 0$ so that

$$u(z_0) - \phi_\varepsilon(z_0) < 0,$$

choose a minimum point $z_\varepsilon = (x_\varepsilon, y_\varepsilon) \in \overline{\Omega}_R$ of $u - \phi_\varepsilon$ and note that $u(z_\varepsilon) - \phi_\varepsilon(z_\varepsilon) < 0$ and, hence, $z_\varepsilon \in \Omega_R$. Thus, by the viscosity property of u, we have

$$\mathcal{P}_k^+(D^2\phi_\varepsilon(z_\varepsilon)) + b|D\phi_\varepsilon(z_\varepsilon)| \leq 0,$$

where

$$D\phi_\varepsilon(x, y) = \theta_\varepsilon'(y - g(x))(-Dg(x),\ 1),$$

and

$$\begin{aligned} D^2\phi_\varepsilon(x, y) = \theta_\varepsilon'(y - g(x)) &\begin{pmatrix} -D^2g(x) & 0 \\ 0 & 0 \end{pmatrix} \\ &+ \theta_\varepsilon''(y - g(x))(-Dg(x),\ 1) \otimes (-Dg(x),\ 1). \end{aligned}$$

Let $\xi \in \mathbb{R}^{N-1}$ and $\eta \in \mathbb{R}$, and compute that

$$\begin{aligned} \langle D^2\phi_\varepsilon(x, y)(\xi, \eta), (\xi, \eta)\rangle = &-\theta_\varepsilon'(y - g(x))\langle D^2g(x)\xi,\ \xi\rangle \\ &+ \theta_\varepsilon''(y - g(x))(-Dg(x) \cdot \xi + \eta)^2. \end{aligned}$$

If $Dg(x) = 0$, then

$$\begin{aligned} \mathcal{P}_k^+(D^2\phi_\varepsilon(x, y)) = \sup\Bigg\{ &\sum_{i=1}^k \left(-\theta_\varepsilon'(y - g(x))\langle D^2g(x)\xi_i, \xi_i\rangle + \theta_\varepsilon''(y - g(x))\eta_i^2\right) \\ &\text{such that} \quad (\xi_i, \eta_i) \cdot (\xi_j, \eta_j) = \delta_{ij}\Bigg\}. \end{aligned}$$

Taking $\eta_i = 0$ and $\xi_i \in \mathbb{R}^{N-1}$ such that $\xi_i \cdot \xi_j = \delta_{ij}$ for any $i, j = 1, \ldots, k$ we get

$$\begin{aligned} \mathcal{P}_k^+(D^2\phi_\varepsilon(x, y)) &\geq \sum_{i=1}^k \left(-\theta_\varepsilon'(y - g(x))\langle D^2g(x)\xi_i, \xi_i\rangle\right) \\ &\geq -k\theta_\varepsilon'(y - g(x))|D^2g(x)|. \end{aligned}$$

Otherwise if $Dg(x) \neq 0$, choosing $(\xi_1, \eta_1) = (Dg(x), |Dg(x)|^2) / \sqrt{|Dg(x)|^2 + |Dg(x)|^4}$ and $(\xi_2, 0), \ldots, (\xi_k, 0)$ in such a way $\xi_i \cdot \xi_j = \delta_{ij}$ for all $i, j = 1, \ldots, k$, we get

$$\begin{aligned}\mathcal{P}_k^+(D^2\phi_\varepsilon(x,y)) &\geq \sum_{i=1}^{k} \Big(-\theta_\varepsilon'(y - g(x)) \langle D^2 g(x)\xi_i, \xi_i \rangle \Big) \\ &\geq -k\theta_\varepsilon'(y - g(x))|D^2 g(x)|.\end{aligned}$$

Since $|D\phi_\varepsilon(x,y)| = \theta_\varepsilon'(y-g(x))\sqrt{|Dg(x)|^2+1} \geq \theta_\varepsilon'(y-g(x))$ and $k|D^2 g(x_\varepsilon)| < b$ by (3.4), we obtain the following contradiction:

$$0 \geq \mathcal{P}_k^+(D^2\phi_\varepsilon(z_\varepsilon)) + b|D\phi_\varepsilon(z_\varepsilon)| \geq \theta_\varepsilon'(y_\varepsilon - g(x_\varepsilon))(b - k|D^2 g(x_\varepsilon)|) > 0.$$

□

Remark 16 The above nonexistence result can be generalized to the nonconstant coefficient case $b = b(x, y)$ and $b(0, 0) = 0$ by assuming

$$b(x, y) > k|D^2 g(x)|$$

in a neighbourhood of $(0, 0)$.

Even in the case where b is constant, we can replace condition (3.1) by the condition

$$b > k|D^2 g(x)|$$

in a neighbourhood of $(0, 0)$.

3.2 Uniformly convex domain with large Hamiltonian

Look at

$$\begin{cases} \mathcal{P}_k^+(D^2 u) + b\,|Du| = -1 & \text{in } B_R \\ u = 0 & \text{on } \partial B_R. \end{cases} \tag{3.5}$$

Proposition 17 *If $0 \leq bR < k$, then the problem (3.5) has a unique solution which is radial, while if $bR \geq k$ there are no supersolutions.*

The case $bR > k$ is included in Remark 16. In the radial setting the proof is in fact much easier and it includes the case $bR = k$. For the convenience of the reader we report the proof.

Proof of Proposition 17 First, thanks to Proposition 5 (ii), since the right hand side (3.5) is negative, the comparison principle always holds and solutions of (3.5) are unique.

We consider the case $b > 0$ and $bR < k$. For $r \in [0, R]$ let

$$g(r) = \frac{r - R}{b} + \frac{k}{b^2} \log \frac{k - br}{k - bR}. \tag{3.6}$$

By a straightforward computation one has

$$k\frac{g'(r)}{r}+b|g'(r)|=-1$$
$$\frac{g'(r)}{r}\geq g''(r)$$
$$g'(0)=g(R)=0.$$

Hence $u(x)=g(|x|)$ is the solution of

$$\begin{cases}\mathcal{P}_k^+(D^2u)+b\,|Du|=-1 & \text{in } B_R\\ u=0 & \text{on } \partial B_R.\end{cases} \tag{3.7}$$

Let us assume now that u is a supersolution of (3.5) and $bR\geq k$. In particular $u>0$ in B_R and it is a supersolution too in any ball $B_{\frac{k-\varepsilon}{b}}\subset B_R$ for $\varepsilon\in(0,\,k)$. Let $\varepsilon\in(0,\,k)$ and set

$$g_\varepsilon(|x|):=\frac{|x|-\frac{k-\varepsilon}{b}}{b}+\frac{k}{b^2}\log\frac{k-b|x|}{\varepsilon},$$

which, as we have seen above, is the solution of (3.7) in $B_{\frac{k-\varepsilon}{b}}$. Since

$$u\geq g_\varepsilon\quad\text{on } \partial B_{\frac{k-\varepsilon}{b}}\qquad\text{and}\qquad b\frac{k-\varepsilon}{b}<k$$

the comparison principle yields

$$u(x)\geq g_\varepsilon(|x|)\qquad\text{for } x\in B_{\frac{k-\varepsilon}{b}}.$$

This leads to a contradiction after letting $\varepsilon\to 0$, i.e.

$$u(x)=\infty\qquad\text{for all } x\in B_{\frac{k}{R}}.$$

The function $u(x)=\frac{R^2-|x|^2}{2k}$ is the solution of (3.5), with $b=0$. By a direct computation, one can see that the solution $g(|x|)$ of (3.5) with $b>0$, where g is defined in (3.6), converges to $u(x)=\frac{R^2-|x|^2}{2k}$ as $b\to 0$. □

Corollary 18 *Let Ω be a domain such that $B_R\subset\Omega$. Then*

- *if $bR\geq k$ there are no positive supersolutions of*

$$\mathcal{P}_k^+(D^2u)+b\,|Du|\leq -1\qquad\text{in }\ \Omega;$$

- *if $bR > k$ there are no $(\mu, \psi(x)) \in \mathbb{R}_+ \times \mathrm{LSC}(\Omega)$ such that*

$$\mathcal{P}_k^+(D^2\psi) + b|D\psi| + \mu\psi \leq 0, \quad \psi > 0 \quad in\ \Omega,$$

i.e. $\overline{\mu}_k^+ = \mu_k^+ = 0$, where

$$\mu_k^+ = \sup\Big\{\mu \in \mathbb{R}\ :\ \exists\varphi \in \mathrm{LSC}(\Omega),\ \varphi > 0,\ \mathcal{P}_k^+(D^2\varphi) + b|Du| \\ +\mu\varphi \leq 0 \ \ in\ \Omega\Big\}$$

$$\bar{\mu}_k^+ = \sup\Big\{\mu \in \mathbb{R}\ :\ \exists\varphi \in \mathrm{LSC}(\overline{\Omega}),\ \varphi > 0 \ \ in\ \overline{\Omega},\ \mathcal{P}_k^+(D^2\varphi) + b|Du| \\ +\mu\varphi \leq 0 \ \ in\ \Omega\Big\}.$$

Proof The first part directly follows from Proposition 17.
Assume now by contradiction that there exist $\mu > 0$, $\psi(x) > 0$ in Ω such that

$$\mathcal{P}_k^+(D^2\psi) + b|D\psi| + \mu\psi \leq 0 \quad \text{in } \Omega.$$

Let $\rho = \frac{k}{b} < R$. Then $B_\rho \Subset \Omega$ and $\min_{\overline{B}_\rho} \psi > 0$. Taking M sufficiently large we can guarantee that $u = M\psi$ satisfies

$$\mathcal{P}_k^+(D^2u) + b|Du| \leq -1 \quad \text{in } B_\rho$$

which is not possible since $b\rho = k$. □

Now we consider the equation

$$\mathcal{P}_k^+(D^2u) - b|Du| = -1 \quad \text{in } B_R \tag{3.8}$$

with any $b > 0$.

Proposition 19 *There exists a unique solution $u \in \mathrm{C}(\overline{B}_R)$ of the Dirichlet problem for (3.8), with boundary condition $u = 0$ on ∂B_R. The solution u is radial.*

Proof The uniqueness is a consequence of the comparison principle (Proposition 5 (ii)).

The presence of the sign minus in front of b leads us to look for radial solutions $u(x) = g(|x|)$ of (3.8) with $g = g(r)$ solution of

$$\begin{cases} g''(r) + (k-1)\frac{g'(r)}{r} + bg'(r) = -1 & \text{for } r \in (0, R] \\ g' \leq 0 & \text{for } r \in [0, R] \\ g''(r) \geq \frac{g'(r)}{r} & \text{for } r \in (0, R] \\ g'(0) = g(R) = 0. \end{cases} \tag{3.9}$$

For solving this, consider the first order problem

$$\begin{cases} h'(r) + (k-1)\frac{h(r)}{r} + bh(r) = -1 & \text{for } r \in (0, R] \\ h(0) = 0 \end{cases} \tag{3.10}$$

whose solution is

$$h(r) = -\frac{e^{-br}}{r^{k-1}} \int_0^r e^{bs} s^{k-1}\, ds.$$

It is clear that

$$h(r) < 0 \quad \text{for } r \in (0, R] \quad \text{and} \quad \lim_{r\to 0} h(r) = 0. \tag{3.11}$$

Moreover by (3.10) one has

$$h'(r) \geq \frac{h(r)}{r} \quad \Longleftrightarrow \quad a(r) := (k+br)\int_0^r e^{bs} s^{k-1}\, ds - r^k e^{br} \geq 0 \quad \text{for } r \in (0, R]. \tag{3.12}$$

Since $a'(r) = b\int_0^r e^{bs} s^{k-1}\, ds \geq 0$ and $a(0) = 0$, then the inequality on the left hand side of (3.12) holds true. Using now (3.10)–(3.12) we deduce that

$$g(r) = \int_r^R \frac{e^{-bs}}{s^{k-1}} \left(\int_0^s e^{bt} t^{k-1}\, dt\right) ds$$

is a solution of (3.9), and $u(x) = g(|x|)$ is in turn a solution of (3.8) such $u = 0$ on $\partial\Omega$. □

3.3 Case $bR = k$ with $\Omega = B_R$

For $\mu > 0$ consider

$$\begin{cases} \mathcal{P}_k^+(D^2u) + \frac{k}{R}|Du| + \mu u = 0, \quad u > 0 & \text{in } B_R \\ u = 0 & \text{on } \partial B_R. \end{cases} \tag{3.13}$$

Consider moreover the ODE

$$\begin{cases} k\left(\frac{1}{r} - \frac{1}{R}\right)\varphi'(r) + \mu\varphi(r) = 0 & \text{for } r \in (0, R) \\ \varphi(0) = a > 0, \;\; \varphi(R) = 0, \end{cases} \tag{3.14}$$

where a is a constant. By computations, the solution $\varphi = \varphi_{\mu,a}$ is given by

$$\varphi_{\mu,a}(r) = a\left(1 - \frac{r}{R}\right)^{\frac{\mu R^2}{k}} \exp\left(\frac{\mu R}{k} r\right)$$

and

$$\varphi'_{\mu,a}(r) < 0 \qquad \text{for any } r \in (0, R). \tag{3.15}$$

If in addition

$$\mu \le \frac{k}{R^2}$$

then

$$\frac{\varphi'_{\mu,a}(r)}{r} \ge \varphi''_{\mu,a}(r) \qquad \text{for all } r \in (0, R). \tag{3.16}$$

Combining (3.15)–(3.16) we deduce that $u_{\mu,a}(x) = \varphi_{\mu,a}(|x|)$ satisfies for any $\mu \le \frac{k}{R^2}$

$$\mathcal{P}_k^+(D^2 u_{\mu,a}) + \frac{k}{R}\left|Du_{\mu,a}\right| + \mu u_{\mu,a} = 0 \qquad \text{for } 0 < |x| < R.$$

Moreover by direct computation

$$Du_{\mu,a}(0) = 0, \quad D^2 u_{\mu,a}(0) = -\frac{\mu}{k} u_{\mu,a}(0) I,$$

hence

$$\begin{cases} \mathcal{P}_k^+(D^2 u_{\mu,a}(x)) + \frac{k}{R}\left|Du_{\mu,a}(x)\right| + \mu u_{\mu,a}(x) = 0, \;\; u_{\mu,a} > 0 & \text{in } B_R \\ u_{\mu,a} = 0 & \text{on } \partial B_R. \end{cases}$$

In particular

$$\mu_k^+ \ge \frac{k}{R^2},$$

while, since the maximum principle is violated, we deduce by [4, Theorem 4.1] that

$$\overline{\mu}_k^+ = 0.$$

This shows that the equality $\overline{\mu}_k^+ = \mu_k^+$, which holds when $bR < k$, see [4, Theorem 4.4], may fail as soon as $bR = k$.

Remark 20 Note that μ_k^+ is finite since it is bounded from above by the principal eigenvalue of the operator $\Delta \cdot + \frac{k}{R}|D \cdot|$.

4 More on the weight of the first order problem

Let us consider the problem

$$\begin{cases} \mathcal{P}_k^+(D^2 u) + b(r)\,|Du| = -1 & \text{in } B_R \\ u = 0 & \text{on } \partial B_R \end{cases} \tag{4.1}$$

where $b \in C([0, R]) \cap C^1(0, R)$ is a radial function. We aim to generalize the existence results of Sect. 3.2 to this setting and, at least in a model case, see (4.2), we shall analyze how the solutions of (4.1) are affected by the monotonicity changes of $b(r)$. Having in mind the case b constant, roughly speaking a transition from b negative to b positive force the solutions u to solve a second order initial value problem near the origin, then a first order boundary value problem.

Concerning $b(r)$ we assume that there exists $r_0 \in [0, R]$ such that

$$(r - r_0)(rb(r))' \geq 0 \quad \text{for } r \in (0, R). \tag{4.2}$$

Note that if $r_0 = 0$ or $r_0 = R$ then (4.2) reduces respectively to the cases $(rb(r))' \geq 0$ or $(rb(r))' \leq 0$ in $(0, R)$, i.e. the constant sign case of $b(r)$.

Definition of R_0. We define $R_0 \in (0, R]$ as follows.
If $rb(r) < k$ for any $r < R$ then $R_0 := R$.
If there exists $r \in (0, R)$ such that $rb(r) = k$ then $R_0 := \inf\{r < R : rb(r) = k\}$.

The above definition of R_0 makes sense for any $b \in C([0, R])$ since $rb(r) < k$ holds for $r = 0$.

Remark 21 If b is a positive constant then $R_0 = \min\left\{\frac{k}{b}, R\right\}$.

Proposition 22 *Assume condition* (4.2). *If*

$$\int^{R_0} \frac{r}{k - rb(r)}\, dr < +\infty \tag{4.3}$$

then $R_0 = R$ *and problem (4.1) has a unique solution, which is radial. On the other hand, if*

$$\int^{R_0} \frac{r}{k - rb(r)}\, dr = +\infty \tag{4.4}$$

then no supersolutions of (4.1) *exist in* B_{R_0}.

Proof First we assume condition (4.3). By contradiction let us assume that $R_0 < R$. Since $rb(r) \in C^1$ then there exists positive M such that

$$k + M(r - R_0) \leq rb(r) \quad \text{in } [R_0/2,\ R_0].$$

This would imply

$$\int^{R_0} \frac{r}{k - rb(r)}\, dr \geq \int^{R_0} \frac{r}{M(R_0 - r)}\, dr = +\infty,$$

contradiction. Hence $R_0 = R$.

As usual, the uniqueness follows from the comparison principle.

Case $r_0 \in (0, R)$.
We start by looking for a radial solution $u(x) = g_1(|x|)$ with $g_1 = g_1(r)$ solution of

$$\begin{cases} g_1''(r) + (k-1)\frac{g_1'(r)}{r} - bg_1'(r) = -1 \\ g_1' \le 0 \\ g_1''(r) \ge \frac{g_1'(r)}{r} \\ g_1'(0) = 0 \end{cases} \tag{4.5}$$

in a neighbourhood of zero. This leads us to consider the following first order problem, $h_1 = g_1'$,

$$\begin{cases} h_1'(r) + (k-1)\frac{h_1(r)}{r} - bh_1(r) = -1 \\ h_1'(r) \ge \frac{h_1(r)}{r} \\ h_1(r) \le 0 \\ h_1(0) = 0. \end{cases} \tag{4.6}$$

As in the proof of Proposition 19, the function

$$h_1(r) = -\frac{e^{B(r)}}{r^{k-1}} \int_0^r e^{-B(s)} s^{k-1}\, ds \tag{4.7}$$

where $B' = b$, satisfies (4.6) and $h_1' \ge \frac{h_1}{r}$ in an interval $[0,\ c]$ provided

$$a(r) := (k - rb(r)) \int_0^r e^{-B(s)} s^{k-1}\, ds - e^{-B(r)} r^k \ge 0 \quad \text{in } [0,\ c]. \tag{4.8}$$

Since $a(0) = 0$ and

$$a'(r) = -(rb(r))' \int_0^r e^{-B(s)} s^{k-1}\, ds \ge 0 \quad \text{in } [0,\ r_0],$$

then (4.8) holds for any $r \in [0,\ r_0]$.

Now if $a(r) \ge 0$ in $[0, R]$, then h_1 is a global solution of (4.6) and

$$g_1(r) = -\int_r^R h_1(s)\, ds \tag{4.9}$$

is the solution (4.5) in $[0, R]$ satisfying $g_1(R) = 0$.

If otherwise there exists $\bar{r} \in [r_0,\ R)$ such that

$$a(\bar{r}) = 0 \quad \text{and} \quad a(r) < 0 \quad \text{in} \quad [\bar{r}, R] \tag{4.10}$$

then the function

$$g_2(r) = \int_r^R \frac{s}{k - sb(s)}\, ds \tag{4.11}$$

is well defined by (4.3) and it is a solution of

$$\begin{cases} k\frac{g_2'(r)}{r} - b(r)g_2'(r) = -1 & \text{for } r \in (\bar{r}, R) \\ g_2'(r) \leq 0 & \text{for } r \in (\bar{r}, R) \\ g_2(R) = 0. \end{cases} \tag{4.12}$$

Moreover, using (4.2), one has

$$g_2''(r) = \frac{g_2'(r)}{r} - \frac{r}{(k - rb(r))^2}(rb(r))' \leq \frac{g_2'(r)}{r} \qquad \text{for } r \in (\bar{r}, R). \tag{4.13}$$

Let us define

$$g(r) = \begin{cases} g_1(r) & \text{for } r \in [0, \bar{r}] \\ g_2(r) & \text{for } r \in (\bar{r}, R], \end{cases} \tag{4.14}$$

where $g_1(r) = -\int_r^{\bar{r}} h_1(s)\,ds + g_2(\bar{r})$ and h_1 is defined by (4.7). By (4.10) $g(r) \in C^1([0, R]) \cap C^2([0, R]\setminus\{\bar{r}\})$. We claim that $u(x) = g(|x|)$ is solution of (4.1). Clearly it is a classical solution for any $x \in B_R$ such that $|x| \neq \bar{r}$. Moreover note that if $(rb(r))'\big|_{r=\bar{r}} = 0$ then $u(x)$ is in fact $C^2(B_R)$. So we may assume that $(rb(r))'\big|_{r=\bar{r}} > 0$, hence by construction the only points x that we need to consider are those for which $|x| = \bar{r}$. Fix such $x_0 \in B_R$ and let $\varphi \in C^2(B_R)$ touching u from above at x. First we note that, since the function $g_1(r)$ and $g_2(r)$ are both twice differentiable in a neighbourhood of $\bar{r}$, using (4.13) one has

$$(g_1 - g_2)''(\bar{r}) = -\frac{1}{k - \bar{r}b(\bar{r})} - g_2''(\bar{r}) > -\frac{1}{k - \bar{r}b(\bar{r})} - \frac{g_2'(\bar{r})}{\bar{r}} = 0$$

hence $g_1 \geq g_2$ around $\bar{r}$. In this way φ touches from above $g_2(|x|)$ at x_0 and

$$D\varphi(x_0) = Dg_2(\bar{r}), \quad D^2\varphi(x_0) \geq D^2 g_2(\bar{r}).$$

Then using (4.12)

$$\mathcal{P}_k^+(D^2\varphi(x_0)) \geq \mathcal{P}_k^+(D^2 g_2(\bar{r})) = k\frac{g_2'(\bar{r})}{\bar{r}} = 1 - b(\bar{r})|D\varphi(x_0)|,$$

which shows that u is a viscosity subsolution. The supersolution property of u can be proved in a similar way, using in particular that if φ touches u from below at x_0, then φ is in fact a test function for $g_1(|x|)$.

Cases $r_0 = 0$ or $r_0 = R$.

The solution is given by $u(x) = g_2(|x|)$ if $\bar{r} = 0$ where g_2 is defined in (4.11), while if $\bar{r} = R$ then $u(x) = g_1(|x|)$ with g_1 defined by (4.9).

This ends the proof of the first part of the proposition.

Now we assume (4.4). By contradiction we assume that there exists a supersolution of (4.1). By the definition of R_0 one has $\inf_{r\in[0,R_0-\varepsilon]} k - rb(r) > 0$ for any $\varepsilon \in (0,\ R_0)$.

Consider the function

$$u_\varepsilon(x) := (1-\varepsilon)g_\varepsilon(|x|) := (1-\varepsilon)\int_{|x|}^{R_0-\varepsilon} \frac{r}{k-rb(r)}\,dr.$$

It is a classical strict subsolution of (4.1), since

$$\mathcal{P}_k^+(D^2u_\varepsilon(x)) + b(|x|)|Du_\varepsilon(x)| \geq \left(k\frac{g_\varepsilon'(|x|)}{|x|} - b(|x|)g_\varepsilon'(|x|)\right) = -(1-\varepsilon).$$

By comparison $u(x) \geq u_\varepsilon(x)$ in $B_{R_0-\varepsilon}$ which leads to $u = +\infty$ in B_{R_0} by letting $\varepsilon \to 0$. □

Remark 23 We briefly discuss the effects of reversing the inequality (4.2) in Proposition 22. Assume

$$(r-r_0)(rb(r))' \leq 0 \quad \forall r \in (0, R). \tag{4.15}$$

Without loss of generality we may assume $r_0 \in (0, R)$.
In $[0,\ r_0]$ the function

$$g_1(r) = -\int_0^r \frac{s}{k-sb(s)}\,ds + c_1$$

is a solution of

$$\begin{cases} k\frac{g_1'(r)}{r} - b(r)g_1'(r) = -1 & \text{in } (0,\ r_0] \\ g_1'(0) = 0 \end{cases}$$

for any choice of the constant c_1. Moreover $g_1'(r) \leq 0$ and $g_1''(r) \leq \frac{g_1'(r)}{r}$ for any $r \in (0,\ r_0]$.
In $[r_0, R]$ where $g_1''(r) \geq \frac{g_1'(r)}{r}$ we look at the second order problem

$$\begin{cases} g_2''(r) + \frac{k-1}{r}g_2'(r) - b(r)g_2'(r) = -1 & \text{in } [r_0, R] \\ g_2''(r) \geq \frac{g_2'(r)}{r},\ \ g_2'(r) \leq 0 & \text{in } [r_0, R] \\ g_2(R) = 0. \end{cases}$$

By computations

$$g_2(r) = \int_r^R \frac{e^{B(s)}}{s^{k-1}}\left(\int_{r_0}^s e^{-B(\tau)}\tau^{k-1}\,d\tau + c_2\right)ds$$

where $B' = b$ and any $c_2 \geq \frac{r_0^k e^{-B(r_0)}}{k-r_0b(r_0)}$. If we fix

$$c_2 = \frac{r_0^k e^{-B(r_0)}}{k-r_0b(r_0)} \quad \text{and} \quad c_1 = \int_0^{r_0}\frac{s}{k-sb(s)}\,ds + g_2(r_0)$$

then the function

$$g(r) := \begin{cases} g_1(r) & r \in [0, r_0] \\ g_2(r) & r \in (r_0, R] \end{cases}$$

is in fact of class C^2. Then $u(x) = g(|x|)$ is a classical solution of (4.1). This is the main difference with respect to Proposition 22, where the solution was not in general C^2 in the set $\partial B_{\bar{r}}$, see (4.8)–(4.10) for the definition of $\bar{r}$. This is due to the fact that here we switch from a first order to a second order problem exactly at r_0, the point where the derivative of $rb(r)$ vanishes and so $g_1'' = g_2''$, while in Proposition 22 this happens at $\bar{r} > r_0$ where $g_1''(\bar{r}) \geq g_2''(\bar{r})$.

The uniqueness of solutions of (4.1) with (4.15) is due to the comparison principle, as usual.

The nonexistence of supersolutions under the assumption (4.4), can be obtained as in the proof Proposition 22.

Corollary 24 *Let Ω be a domain such that $B_R \subset \Omega$ and assume (4.2). Then*

- *if $\int^{R_0} \frac{r}{k - rb(r)}\, dr = +\infty$ there are no positive supersolutions of*

$$\mathcal{P}_k^+(D^2 u) + b(r)\,|Du| \leq -1 \qquad in \quad \Omega;$$

- *if $R_0 < R$ there are no $(\mu, \psi) \in \mathbb{R}_+ \times \mathrm{LSC}(\Omega)$ such that*

$$\mathcal{P}_k^+(D^2\psi) + b(r)|D\psi| + \mu\psi \leq 0, \quad \psi > 0 \quad in\ \Omega,$$

i.e. $\overline{\mu}_k^+ = \mu_k^+ = 0.$

Proof First part is a direct consequence of Proposition 22. Let us assume now that $R_0 < R$ and that ψ is a positive supersolution of

$$\mathcal{P}_k^+(D^2\psi) + b(r)|D\psi| + \mu\psi = 0 \quad \text{in } \Omega,$$

with $\mu > 0$. Then the function $u = \dfrac{\psi}{\mu \min_{\overline{B}_{R_0}} \psi}$ satisfies

$$\mathcal{P}_k^+(D^2 u) + b(r)|Du| = -1 \quad \text{in } B_{R_0},$$

so, by (4.4), $\int^{R_0} \frac{r}{k - rb(r)}\, dr < +\infty$. Hence (4.3) implies $R_0 = R$, a contradiction. □

4.1 Case $R = R_0$ and $\Omega = B_R$

In Sect. 3.3 we showed that the two notions of generalized principal eigenvalues, $\bar{\mu}_k^+$ and μ_k^+, does not coincide in the case $bR = k$. This fact still holds in the nonconstant case $b = b(r)$ under some additional assumptions.

First the condition $bR = k$ now reads as $Rb(R) = k$. Then we assume

$$l := \inf_{r\in(0,R)} \frac{(rb(r))'}{r} > 0, \tag{4.16}$$

which obviously holds if b is a positive constant. Note that (4.16) implies (4.2) with $r_0 = 0$.

For $\mu > 0$ let us consider the problem

$$\begin{cases} \mathcal{P}_k^+(D^2u) + b(r)|Du| + \mu u = 0, \quad u > 0 & \text{in } B_R \\ u = 0 & \text{on } \partial B_R. \end{cases} \tag{4.17}$$

By straightforward computation the functions

$$\varphi(r) = \varphi(0)\exp\left\{-\mu\int_0^r \frac{s}{k - sb(s)}\,ds\right\}$$

is a solution of the ODE

$$\begin{cases} \left(\frac{k}{r} - b(r)\right)\varphi'(r) + \mu\varphi(r) = 0 \quad r \in (0, R) \\ \varphi'(0) = 0,\ \varphi(0) > 0. \end{cases}$$

Using (4.16) it is easily seen that

$$\frac{\varphi'(r)}{r} \geq \varphi''(r) \quad \forall r \in (0, R)$$

for any $\mu \leq l$. Hence $u(x) = \varphi(|x|)$ are positive radial solution of the equation in (4.17). If in addition

$$\int_0^R \frac{s}{k - sb(s)}\,ds = +\infty$$

then $u = 0$ on ∂B_R, leading to

$$\overline{\mu}_k^+ = 0 < l \leq \mu_k^+.$$

5 Convex domains

In order to prove Theorem 2 we need the following lemmas.

Lemma 25 *Let K be a compact subset of $\mathbb{R}^m$. Assume that*

$$0 \in K \quad and \quad K\backslash\{0\} \subset \{x = (x_1, \ldots, x_m) \in \mathbb{R}^m : x_m < 0\}.$$

Then there exists a bounded, open, strictly convex set $\omega \subset \mathbb{R}^m$ *such that*

$$K \subset \overline{\omega} \quad and \quad \omega \subset \{x \in \mathbb{R}^m \,:\, x_m < 0\}.$$

Lemma 26 *Let* A *be a compact convex subset of* $\mathbb{R}^N$ *such that* $0 \in A$. *Let* V *be the linear span of* A *and set* $m = \dim V$. *Then there exist a basis* $\{a_1, \ldots, a_m\}$ *of* V *such that* $a_i \in A$ *for all* $i \in \{1, \ldots, m\}$.

Assuming the lemmas above, we first present the proof of Theorem 2. Henceforth $e_1, \ldots, e_N$ will denote the standard basis of $\mathbb{R}^N$.

Proof of Theorem 2 Assume that $\Omega \in \mathcal{C}_j$. Fix any $x \in \partial\Omega$ and prove that $d_x(\Omega) \leq N - j$. There is a $(O, C) \in S_j(\Omega)$, with $C = \omega \times \mathbb{R}^{N-j}$, such that $x \in \partial(OC)$ and $\Omega \subset OC$. Suppose by contradiction that $d_x(\Omega) > N - j$ and set $m = d_x(\Omega)$. There exist an m-dimensional linear subspace V in $\mathbb{R}^N$ and $\delta > 0$ such that $x + V \cap B_\delta \subset \partial\Omega$. Observe that

$$x + V \cap B_\delta \subset \partial\Omega \subset \overline{OC} = O\overline{C},$$

and hence

$$O^T x + O^T V \cap B_\delta \subset O^T O\overline{C} = \overline{C} = \overline{\omega} \times \mathbb{R}^{N-j}. \tag{5.1}$$

Since $x \in \partial(OC) = O\partial C = O(\partial\omega \times \mathbb{R}^{N-j})$, we have $O^T x \in \partial\omega \times \mathbb{R}^{N-j}$. Set $y = O^T x$ and $W = O^T V$ and note that $y \in \partial\omega \times \mathbb{R}^{N-j}$ and W is m-dimensional.

Since $m > N - j$, the m-dimensional ball $W \cap B_\delta$ is not contained in $\{0^j\} \times \mathbb{R}^{N-j}$, where $0^j := (0, \ldots, 0) \in \mathbb{R}^j$. Hence, there exists $w \in W \cap B_\delta \setminus \{0^j\} \times \mathbb{R}^{N-j}$, which also means that $-w \in W \cap B_\delta \setminus \{0^j\} \times \mathbb{R}^{N-j}$. We set $w^{(j)} = (w_1, \ldots, w_j, 0, \ldots, 0) \in \mathbb{R}^N$ and note that $w^{(j)} \neq 0$ since $w \notin \{0^j\} \times \mathbb{R}^{N-j}$. Moreover, we observe by (5.1) that

$$y \pm w \in y + W \cap B_\delta \subset \overline{\omega} \times \mathbb{R}^{N-j},$$

and hence, $y^{(j)} \pm w^{(j)} \in \overline{\omega}$. Since ω is strictly convex and $w^{(j)} \neq 0$, it is obvious that

$$y = \frac{1}{2}(y + w^j + y - w^j) \in \omega \times \mathbb{R}^{N-j}.$$

This contradicts that $y \in \partial\omega \times \mathbb{R}^{N-j}$. Thus, we have shown that $d(\Omega) \leq N - j$.

Next, we assume that $d(\Omega) \leq N - j$. Fix any $z \in \partial\Omega$ and $\nu \in N_\Omega(z)$. By translation, we may assume that $z = 0$. Set

$$A = \overline{\Omega} \cap \{x \in \mathbb{R}^N \,:\, \nu \cdot x \geq 0\},$$

and note that $0 \in A \subset \partial\Omega$ and A is a compact convex set. Consider the linear span V_0 of A. It follows that $\dim V_0 \leq d(\Omega)$. Indeed, by Lemma 26, there exists a linear basis

$\{a_1, \dots, a_m\} \subset A$ of V_0. Set $a = (a_1 + \cdots + a_m)/m$ and observe that $a \in A \subset \partial\Omega$ and, for $\delta > 0$ small enough,

$$a + B_\delta \cap V_0 = B_\delta(a) \cap V_0 \subset \left\{ \sum_{i=1}^m t_i a_i \,:\, t_i \geq 0, \, \sum_{i=1}^m t_i \leq 1 \right\} \subset A \subset \partial\Omega,$$

which ensures that $\dim V_0 \leq d(\Omega)$.

Since A is included in the supporting plane $\{x \in \mathbb{R}^N \,:\, \nu \cdot x = 0\}$ of Ω at 0, which is $(N-1)$-dimensional, we may choose a $(N-j)$-dimensional subspace V of $\{x \in \mathbb{R}^N \,:\, \nu \cdot x = 0\}$ such that $A \subset V_0 \subset V$.

Now, we observe that

$$\nu \cdot x < 0 \quad \text{for all } x \in \overline{\Omega} \setminus V. \tag{5.2}$$

Indeed, if $x \in \overline{\Omega} \setminus V$, then $x \in \overline{\Omega} \setminus A$ and, by the definition of A, $\nu \cdot x < 0$.

By orthogonal transformation, we may assume that

$$\nu = e_j \quad \text{and} \quad V = \{0^j\} \times \mathbb{R}^{N-j}.$$

Set

$$K = \{(x_1, \dots, x_j) \in \mathbb{R}^j \,:\, (x_1, \dots, x_N) \in \overline{\Omega} \text{ for some } (x_{j+1}, \dots, x_N) \in \mathbb{R}^{N-j}\}.$$

This K is the projection of $\overline{\Omega}$ onto $\mathbb{R}^j$ and is compact and convex. Clearly, we have $0 \in K$. Moreover, if $x \in K \setminus \{0^j\}$, then, by the definition of K, there is $y = (y_1, \dots, y_N) \in \overline{\Omega}$ such that $x = (y_1, \dots, y_j)$ and we have $y \notin V = \{0^j\} \times \mathbb{R}^{N-j}$ since $x \neq 0^j$, and, by (5.2), $\nu \cdot y < 0$, which reads $y_j < 0$. We may apply Lemma 25, to conclude that there is a bounded strictly convex domain $\omega \subset \mathbb{R}^j$ such that

$$K \subset \overline{\omega} \quad \text{and} \quad \omega \subset \{x = (x_1, \dots, x_j) \in \mathbb{R}^j \,:\, x_j < 0\}. \tag{5.3}$$

Hence, by the definition of K, we see that

$$\overline{\Omega} \subset \overline{\omega} \times \mathbb{R}^{N-j},$$

which implies, since Ω and ω are nonempty convex sets that

$$\Omega \subset \omega \times \mathbb{R}^{N-j}.$$

It is obvious from (5.3) that for the boundary point 0 of Ω, $0 \in \partial\omega \times \mathbb{R}^{N-j} = \partial(\omega \times \mathbb{R}^{N-j})$. $\square$

We need the following lemma for the proof of Lemma 25.

Lemma 27 *Let $B_R(z)$ be the open ball of $\mathbb{R}^N$ with radius $R > 0$ and center z. For any $x, y \in \overline{B}_R(z)$, with $x \neq y$, and $0 < t < 1$, there exists a positive constant $\delta = \delta(R, t, |x-y|)$ such that*

$$B_\delta(tx + (1-t)y) \subset B_R(z).$$

Moreover, $\delta(R, t, |x-y|)$ can be chosen depending only on R, t, and $|x-y|$ and decreasingly on R.

Proof Combine

$$|tx + (1-t)y - z|^2 = t^2|x-z|^2 + (1-t)^2|y-z|^2 + 2t(1-t)(x-z)\cdot(y-z),$$

and

$$|x-y|^2 = |x-z-(y-z)|^2 = |x-z|^2 + |y-z|^2 - 2(x-z)\cdot(y-z),$$

to get

$$\begin{aligned} |tx + (1-t)y - z|^2 &= t^2|x-z|^2 + (1-t)^2|y-z|^2 + t(1-t)(|x-z|^2 \\ &\quad + |y-z|^2 - |x-y|^2) \\ &= t|x-z|^2 + (1-t)|y-z|^2 - t(1-t)|x-y|^2 \\ &\le R^2 - t(1-t)|x-y|^2. \end{aligned}$$

Hence, if we set

$$\delta := \frac{1}{2}\left(R - \sqrt{R^2 - t(1-t)|x-y|^2}\right) = \frac{t(1-t)|x-y|^2}{2(R + \sqrt{R^2 - t(1-t)|x-y|^2})},$$

then we have $B_\delta(tx + (1-t)y) \subset B_R(z)$. The choice δ above has the required dependence on R, z and so on. □

Proof of Lemma 25 Fix an $R_0 > 0$ so that $K \subset B_{R_0}$ and for $R \ge R_0$, set

$$\rho(R) = \sup\{h \ge 0 \,:\, K \subset \overline{B}_R(-he_m)\},$$

where e_m is the unit vector in $\mathbb{R}^m$, with unity as the last (m-th) entry. Since $0 \in K$, we see that $\rho(R) \le R$. Also, since $K \subset B_{R_0} \subset B_R(-(R-R_0)e_m)$, we have $\rho(R) \ge R - R_0$. It is now clear that $\rho(R)$ is achieved. In particular, if $S > R \ge R_0$, then $K \subset \overline{B}_R(-\rho(R)e_m) \subset \overline{B}_S(-\rho(R)e_m)$ and hence, $\rho(S) \ge \rho(R)$. Thus, $\rho(R)$ depends on R nondecreasingly (in fact, increasingly).

We claim that

$$\lim_{R\to\infty} (R^2 - \rho(R)^2) = 0. \tag{5.4}$$

To prove this, fix first any $r > 0$. Since $K \backslash B_r$ is compact and $K \backslash B_r \subset \{x \in \mathbb{R}^m : x_m < 0\}$, we may choose $0 < \gamma_1 < R_1$ so that

$$K \backslash B_r \subset \left\{x \in \mathbb{R}^m \,:\, x_1^2 + \cdots + x_{m-1}^2 < R_1^2,\ -R_1 < x_m < -\gamma_1\right\}.$$

One can always replace R_1 and γ_1, without violating the above inclusion, by larger and smaller ones, respectively. In what follows, we may fix R_1 so that $R_1 > r$ and consider $0 < \gamma < \gamma_1$.

We next choose $0 < h < R$ such that

$$\begin{aligned} &\overline{B}_R(-he_m) \cap \left\{x \in \mathbb{R}^m \,:\, x_m = 0\right\} \\ &\quad = \left\{x \in \mathbb{R}^m \,:\, x_1^2 + \cdots + x_{m-1}^2 \le r^2,\ x_m = 0\right\} \end{aligned}$$

and

$$\begin{aligned} &\overline{B}_R(-he_m) \cap \left\{x \in \mathbb{R}^m \,:\, x_m = -\gamma\right\} \\ &\quad = \left\{x \in \mathbb{R}^m \,:\, x_1^2 + \cdots + x_{m-1}^2 \le R_1^2,\ x_m = -\gamma\right\} \end{aligned}$$

i.e. we choose

$$h = \frac{R_1^2 + \gamma^2 - r^2}{2\gamma} \quad \text{and} \quad R = \sqrt{h^2 + r^2} = \sqrt{(h-\gamma)^2 + R_1^2}\,.$$

Reducing γ_1 if necessary, we can suppose that the function $g(\gamma) = \frac{R_1^2+\gamma^2-r^2}{2\gamma}$ is decreasing in $(0, \gamma_1]$. Let $h_1 = g(\gamma_1)$. Fix any $R > \sqrt{h_1^2 + r^2}$ and let $h = \sqrt{R^2 - r^2} > h_1$. Since $g(\gamma)$ is continuous and $\lim_{\gamma \to 0^+} g(\gamma) = +\infty$, there exists $\gamma \in (0, \gamma_1]$ such that

$$\frac{R_1^2 + \gamma^2 - r^2}{2\gamma} = h. \tag{5.5}$$

Simple geometry tells us that

$$\begin{aligned} K \,\subset\, B_r \cap \{x \in \mathbb{R}^m \,:\, x_m \le 0\} \,\cup\, \{x \in \mathbb{R}^m \,:\, x_1^2 + \cdots + x_{m-1}^2 < R_1^2, \\ - R_1 < x_m < -\gamma\} \,\subset\, B_R(-he_m). \end{aligned}$$

This inclusion ensures that $\rho(R) \ge h$ and hence $R^2 - \rho(R)^2 \le R^2 - h^2 = r^2$. Hence, we have $R^2 - \rho(R)^2 \le r^2$ for any $R > \sqrt{h_1^2 + r^2}$. Since $r > 0$ is arbitrary, we conclude (5.4).

In case when $\rho(R) = R$ for some $R \ge R_0$, we fix such an $R \ge R_0$ and set

$$\omega := B_R(-\rho(R)e_m) = B_R(-Re_m).$$

It is clear that ω is a bounded, open, strictly convex subset of $\mathbb{R}^m$ and that $K \subset \overline{\omega}$ and $\omega \subset \{x \in \mathbb{R}^m : x_m < 0\}$.

Now, we consider the general case. We set

$$\Delta = \bigcap_{R \geq R_0} \overline{B}_R(-\rho(R)e_m).$$

It is obvious that Δ is compact and convex and that $K \subset \Delta$. Since $K \subset B_{R_0}$, $K \setminus \{0\} \subset \{x \in \mathbb{R}^m : x_m < 0\}$ and K is compact, it is easily seen that $\rho(R_0) > 0$. Moreover, since $0 \in \overline{B}_R(-\rho(R)e_m)$ for all $R \geq R_0$ and $R \mapsto \rho(R)$ is nondecreasing, we find that $-\rho(R_0)e_m \in \Delta$.

We define ω as the interior $\operatorname{int}\Delta$ of Δ. We need only to show that $\omega \neq \emptyset$, which implies by the convexity of Δ that $\overline{\omega} = \Delta$, and also that ω is strictly convex and contained in $\{x \in \mathbb{R}^m : x_m < 0\}$.

For this, we first check that $\omega \subset \{x \in \mathbb{R}^m : x_m < 0\}$. It is enough to show that

$$\Delta \cap \{x \in \mathbb{R}^m : x_m \geq 0\} = \{0\}. \tag{5.6}$$

Fix any $x \in \Delta$, with $x_m \geq 0$ and note that for $R \geq R_0$,

$$R^2 \geq |x + \rho(R)e_m|^2 = |x|^2 + 2\rho(R)x_m + \rho(R)^2 \geq |x|^2 + \rho(R)^2,$$

and, accordingly,

$$R^2 - \rho(R)^2 \geq |x|^2.$$

Hence, (5.4) implies that $x = 0$.

Next, fix any $\varepsilon > 0$ and set

$$R_\varepsilon = R_0 + \frac{R_0^2 - \rho(R_0)^2 + 2\varepsilon\rho(R_0)}{2\varepsilon} \tag{5.7}$$

and

$$\Delta_\varepsilon = \bigcap_{R_0 \leq R \leq R_\varepsilon} \overline{B}_R(-\rho(R)e_m).$$

We observe that for any $x \in \Delta_\varepsilon \cap \{y \in \mathbb{R}^m : y_m < -\varepsilon\}$, if $R > R_\varepsilon$, then we have

$$\begin{aligned}
|x + \rho(R)e_m|^2 &= |x + \rho(R_0)e_m + (\rho(R) - \rho(R_0))e_m|^2 \\
&\leq R_0^2 + (\rho(R) - \rho(R_0))^2 + 2(\rho(R) - \rho(R_0))(x + \rho(R_0)e_m) \cdot e_m \\
&\leq R_0^2 + (\rho(R) - \rho(R_0))^2 + 2(\rho(R) - \rho(R_0))(-\varepsilon + \rho(R_0)) \\
&= R_0^2 + \rho(R)^2 - \rho(R_0)^2 - 2\varepsilon(\rho(R) - \rho(R_0)),
\end{aligned}$$

and, since

$$\rho(R) \geq R - R_0 > R_\varepsilon - R_0 = \frac{R_0^2 - \rho(R_0)^2 + 2\varepsilon\rho(R_0)}{2\varepsilon},$$
$$|x + \rho(R)e_m|^2 \leq R_0^2 + \rho(R)^2 - \rho(R_0)^2 - (R_0^2 - \rho(R_0)^2) = \rho(R)^2 \leq R^2.$$

Hence, we find that

$$\Delta_\varepsilon \cap \{x \in \mathbb{R}^m \,:\, x_m < -\varepsilon\} \subset \Delta \cap \{x \in \mathbb{R}^m \,:\, x_m < -\varepsilon\}.$$

The reverse inclusion is trivial and thus we have

$$\Delta_\varepsilon \cap \{x \in \mathbb{R}^m \,:\, x_m < -\varepsilon\} = \Delta \cap \{x \in \mathbb{R}^m \,:\, x_m < -\varepsilon\}. \tag{5.8}$$

Let $x, y \in \Delta\backslash\{0\}$, with $x \neq y$ and $0 < t < 1$. By (5.6), we may select $\varepsilon > 0$ so that $x_m, y_m < -2\varepsilon$. It follows that $tx_m + (1-t)y_m < -2\varepsilon$. Define $R_\varepsilon > R_0$ by (5.7). Since

$$x, y \in \bigcap_{R_0 \leq R \leq R_\varepsilon} B_R(-\rho(R)e_m),$$

thanks to Lemma 27, we can choose $\delta \in (0,\ \varepsilon/2)$ such that

$$B_\delta(tx + (1-t)y) \subset \bigcap_{R_0 \leq R \leq R_\varepsilon} \overline{B}_R(-\rho(R)e_m),$$

which readily yields

$$B_\delta(tx + (1-t)y) \subset \bigcap_{R_0 \leq R \leq R_\varepsilon} \overline{B}_R(-\rho(R)e_m) \cap \{z \in \mathbb{R}^m \,:\, z_m < -\varepsilon\}.$$

Using identity (5.8), we find that

$$B_\delta(tx + (1-t)y) \subset \Delta. \tag{5.9}$$

In particular, this, with $x = -\rho(R_0)e_m$ and $y = -(\rho(R_0)/2)e_m$, ensures that $\omega \neq \emptyset$.

Inclusion (5.9) implies the strict convexity of ω. Indeed, let $x, y \in \Delta$, with $x \neq y$, and $0 < t < 1$. If x, y are both not zero, then (5.9) shows that $tx + (1-t)y \in \omega$. Otherwise, we may assume that $y = 0$. Note that $z := (t/2)x \in \Delta$, $z \neq 0$ and

$$tx = \frac{t}{2-t}x + \left(1 - \frac{t}{2-t}\right)z$$

and apply (5.9), to conclude that $tx \in \omega$. Thus, we find that ω is strict convex and completes the proof. $\square$

Proof of Lemma 26 If $A = \{0\}$, then the conclusion of the lemma is obvious since $V = \{0\}$ and $\dim V = 0$. (As usual, we agree that the linear span of $\emptyset$ is $\{0\}$.) Assume that $A \neq \{0\}$. Consider all the collections $\{b_1, \dots, b_j\}$ of linearly independent vectors $b_i \in A$. Obviously, we have $1 \leq j \leq N$ for any such collection $\{b_1, \dots, b_j\}$. Select such a collection $\{a_1, \dots, a_k\}$, with maximum number of elements k. Since $\{a_1, \dots, a_k\} \subset A$, it follows that $\{a_1, \dots, a_k\} \subset V$, and $\mathrm{Span}\{a_1, \dots, a_k\} \subset V$. Suppose for the moment that $\mathrm{Span}\{a_1, \dots, a_k\} \neq V$, which implies that $A \setminus \mathrm{Span}\{a_1, \dots, a_k\} \neq \emptyset$. Then there exists $a_{k+1} \in A \setminus \mathrm{Span}\{a_1, \dots, a_k\}$, which means that $\{a_1, \dots, a_{k+1}\} \subset A$ is a collection of linearly independent vectors. This contradicts the choice of $\{a_1, \dots, a_k\}$ and proves that $\mathrm{Span}\{a_1, \dots, a_k\} = V$ (as well as $k = m$). $\square$

In a recent article [7], Blanc and Rossi have introduced the following notion. Here, unlike [7], we are only concerned with convex domains.

Definition 28 ($\mathcal{G}_j$ *condition*) Let $j \in \{1, \dots, N\}$ and let $\Omega \subset \mathbb{R}^N$ be a bounded convex domain. We say that $\Omega \in \mathcal{G}_j$ if for any $y \in \partial\Omega$ and any $r > 0$ there exists $\delta > 0$ such that for every $x \in B_\delta(y) \cap \Omega$ and $S \subset \mathbb{R}^N$ subspace with $\dim S = j$, then there exists a unit vector $v \in S$ such that

$$\{x + tv\}_{t\in\mathbb{R}} \cap B_r(y) \cap \partial\Omega \neq \emptyset. \tag{5.10}$$

In [7] they consider the problem

$$\begin{cases} \lambda_j(D^2u) = 0 & \text{in } \Omega \\ u = g & \text{on } \partial\Omega \end{cases}$$

and they prove that if $\Omega \in \mathcal{G}_j \cap \mathcal{G}_{N-j}$ then the above Dirichlet problem is solvable for any g while, if Ω is not in $\mathcal{G}_j \cap \mathcal{G}_{N-j}$ then there may be some g for which the problem is not solvable. This problem is very much related with the results in the present article hence we prove the following equivalence.

Proposition 29 *When Ω is bounded, open and convex, $\Omega \in \mathcal{G}_j$ if and only if $\Omega \in \mathcal{C}_{N-j+1}$.*

Proof The property that $\Omega \notin \mathcal{G}_j$ can be stated as follows: there exist $y \in \partial\Omega$ and $r > 0$ such that for any $\delta > 0$, there exist $x \in B_\delta(y) \cap \Omega$ and a linear subspace S of $\mathbb{R}^N$, with $\dim SV = j$, for which

$$\{x + tv\}_{t\in\mathbb{R}} \cap B_r(y) \cap \partial\Omega = \emptyset \quad \text{for any } v \in S, \text{ with } |v| = 1.$$

This equality reads

$$(x + S) \cap B_r(y) \cap \partial\Omega = \emptyset,$$

and moreover, since $x \in \Omega$,

$$(x + S) \cap B_r(y) \subset \Omega. \tag{5.11}$$

Now, we assume that $\Omega \notin \mathcal{G}_j$ and prove that $d(\Omega) \geq j$. By the above consideration, there exist $y \in \partial\Omega$ and $r > 0$ such that for each $k \in \mathbb{N}$, there exist $x \in B_{1/k}(y) \cap \Omega$ and a linear subspace $S_k \subset \mathbb{R}^N$, with $\dim S_k = j$, such that (5.11) holds with $x = x_k$ and $S = S_k$.

Noting that $\lim_{k\to\infty} x_k = y$ and taking limit as $k \to \infty$ along an appropriate subsequence, we can find a linear subspace $S \subset \mathbb{R}^N$, with $\dim S = j$, such that

$$(y + S) \cap B_r(y) \subset \overline{\Omega}. \tag{5.12}$$

(Here, regarding the convergence of S_k, one may fix an orthonormal basis $\{v_{k,1}, \ldots, v_{k,j}\}$ of S_k for each k and look for a subsequence of the k for which $\{v_{k,1}, \ldots, v_{k,j}\}$ converge in $\mathbb{R}^{N\times j}$.) Since $(y + S) \cap B_r(y)$ is a j-dimensional ball, with center $y \in \partial\Omega$ and Ω is convex, it is easily seen by (5.12) that

$$(y + S) \cap B_r(y) \subset \partial\Omega,$$

which shows that $d(\Omega) \geq j$.

Next, we assume that $d(\Omega) \geq j$ and prove that $\Omega \notin \mathcal{G}_j$. This assumption implies that there exist $y \in \partial\Omega$ and a linear subspace $S \subset \mathbb{R}^N$, with $\dim S \geq j$, such that

$$y + S \cap B_r \subset \partial\Omega.$$

We may assume by replacing S, by a subspace of S if necessary, that $\dim S = j$. Since $\partial\Omega \neq \emptyset$, there exists a point $z \in \Omega$. By the convexity of Ω, with nonempty interior, we see that

$$tz + (1-t)y + S \cap B_{(1-t)r} = tz + (1-t)(y + S \cap B_r) \subset \Omega \quad \text{for } t \in (0,\ 1).$$

Hence, for $t \in (0, 1/2)$, if we set $x_t = tz + (1-t)y$, then

$$(x_t + S) \cap B_{r/2}(x_t) = x_t + S \cap B_{r/2} \subset x_t + S \cap B_{(1-t)r} \subset \Omega,$$

and, also, $\lim_{t\to 0} x_t = y$. This shows that $\Omega \notin \mathcal{G}_j$. Thus, we see that $\Omega \in \mathcal{G}_j$ if and only if $d(\Omega) \leq j - 1$. This observation and Theorem 2 assure that $\Omega \in \mathcal{G}_j$ if and only if $\Omega \in \mathcal{C}_{N-j+1}$. □

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