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ABSTRACT

In this paper, we consider the porous medium equation and establish a relationship between its Kompanets–Zel’dovich–Barenblatt solution $u(\mathbf{x}_d, t)$, $\mathbf{x}_d \in \mathbb{R}^d$, $t > 0$ and random flights. The time-rescaled version of $u(\mathbf{x}_d, t)$ is the fundamental solution of the Euler–Poisson–Darboux equation, which governs the distribution of random flights performed by a particle whose displacements have a Dirichlet probability distribution and choosing directions uniformly on a d -dimensional sphere. We consider the space-fractional version of the Euler–Poisson–Darboux equation and present the solution of the related Cauchy problem in terms of the probability distributions of random flights governed by the classical Euler–Poisson–Darboux equation. Furthermore, this research is also aimed at studying the relationship between the solutions of a fractional porous medium equation and the fractional Euler–Poisson–Darboux equation. A considerable part of this paper is devoted to the analysis of the probabilistic tools of the solutions of the fractional equations. The extension to the higher-order Euler–Poisson–Darboux equation is considered, and the solutions are interpreted as compositions of laws of pseudoprocesses.

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I. INTRODUCTION

The starting point of our paper is the Porous Medium Equation (PME),

$$\frac{\partial u}{\partial t} = \Delta(u^m), \quad m > 1, d \geq 1, t > 0, \quad (1.1)$$

with $u := u(\mathbf{x}_d, t)$, $\mathbf{x}_d := (x_1, x_2, \dots, x_d) \in \mathbb{R}^d$, $t > 0$, $\Delta := \sum_{j=1}^d \frac{\partial^2}{\partial x_j^2}$ (Laplace operator). Sometimes, the PME is written as

$$\frac{\partial u}{\partial t} = \nabla(mu^{m-1}\nabla u), \quad (1.2)$$

where $\nabla := (\frac{\partial}{\partial x_1}, \dots, \frac{\partial}{\partial x_d})$ is the gradient operator.

Equation (1.1) is one of the simplest examples of a nonlinear evolution equation of parabolic type. It appears in the description of different phenomena, and its theory and properties are substantially far from those of the heat equation, $\frac{\partial u}{\partial t} = \Delta u$. Hence, this explains the interest of its study, both for the pure mathematician and for the applied scientist. There are numerous generalizations of (1.1) (see, e.g., Ref. 1), for example, in the form $\frac{\partial u}{\partial t} = \Delta f(u)$ with a suitable function f . There is also the hyperbolic version of the PME in the form $\frac{\partial^2 u}{\partial t^2} = \Delta(u^m)$.

The PME was introduced with the aim of overcoming the paradox of an infinite velocity of the heat flow of the classical Fourier equation. An alternative approach circumventing this paradox of propagation at an infinite velocity of the heat flow is the Cattaneo–Maxwell equation (see Ref. 2), which is substantially a telegraph equation.

The PME was also considered in the hyperbolic half-space $\mathbb{H}_d := \{\mathbf{x}_d \in \mathbb{R}^d : x_d > 0\}$ (see Ref. 3), where Δ is replaced by the hyperbolic Laplacian $x_d^2 \Delta + (2-d)x_d$, also written in hyperbolic coordinates as

$$\frac{1}{(\sinh r)^{d-1}} \frac{\partial}{\partial r} \left(\frac{\partial}{\partial r} (\sinh r)^{d-1} \frac{\partial}{\partial r} \right) u + \frac{1}{(\sinh r)^2} \Delta_{\mathbb{S}_1^{d-1}} u,$$

where $\Delta_{\mathbb{S}_1^{d-1}}$ is the spherical Laplace–Beltrami operator. Space-fractional versions of the PME have been studied, for instance, in Refs. 4 and 5.

Under the initial condition $u(\mathbf{x}_d, 0) = \delta(\mathbf{x}_d) := \prod_{j=1}^d \delta(x_j)$, the source-type solution to (1.1) (also called sometimes Barenblatt solution) was found by Zel'dovich, Kompanets,⁶ Barenblatt,⁷ and later by Pattle⁸ and has the form

$$u(\mathbf{x}_d, t) = Ct^{-\alpha} \left(1 - B \frac{\|\mathbf{x}_d\|^2}{t^{2\beta}} \right)_+^{\frac{1}{m-1}}, \quad (1.3)$$

where $(x)_+ := \max(0, x)$,

$$\alpha := \frac{d}{2 + d(m-1)}, \quad \beta := \frac{\alpha}{d}, \quad B := \frac{\alpha(m-1)}{2md}.$$

The positive constant C is a normalizing factor that will be given below.

The aim of this paper is to present probabilistic interpretations of the solution (1.3). In particular, we show that the PME is associated with random flights of different forms. This must be meant in the sense that the solutions (1.3) of Eq. (1.1), coincide with suitable choices of the parameters with the probability distribution of random flights for a fixed time $t > 0$ and a fixed number of changes in the direction. These types of stochastic processes have been studied over the years by different authors; see, e.g., Refs. 9–20. They represent the stochastic motion of a particle moving in $\mathbb{R}^d$ with a constant speed $c > 0$ and a direction changing at Poisson-paced epochs and uniformly oriented displacements (see Refs. 10, 11, and 13) or with Dirichlet joint distributed steps (see Refs. 14 and 17).

The fundamental solution [which will be given in (A1)] of the PME has the same structure of the fundamental solution of the Euler–Poisson–Darboux (EPD) equation (studied, for instance, in Ref. 21),

$$\frac{\partial^2 u}{\partial t^2} + \frac{2\gamma + d - 1}{t} \frac{\partial u}{\partial t} = c^2 \Delta u, \quad \gamma > 0, \quad (1.4)$$

that is,

$$u(\mathbf{x}_d, t) = \frac{\Gamma(\gamma + \frac{d}{2})}{\pi^{d/2} \Gamma(\gamma)} \frac{1}{(ct)^d} \left(1 - \frac{\|\mathbf{x}_d\|^2}{c^2 t^2} \right)_+^{\gamma-1}. \quad (1.5)$$

By suitably changing the time scale, that is, $t' = t^\beta$, the solution of (1.3) takes the form (1.5).

The EPD equation (1.4) for $d = 1$ represents a generalization of the telegraph equation,

$$\frac{\partial^2 u}{\partial t^2} + 2\lambda(t) \frac{\partial u}{\partial t} = c^2 \frac{\partial^2 u}{\partial x^2},$$

which is the governing equation of the probability distribution of a telegraph process where the reversals of velocity are paced by a non-homogeneous Poisson process with rate $\lambda(t) = \frac{\gamma}{t}$ (see Ref. 21). The projection of (1.3) on the one-dimensional space has, instead, a density function, which is also governed by the EPD equation (4.5). The conditional probability density functions of the position of the random flights (when the number of changes in the direction is fixed) are solutions of EPD equations with suitable coefficients.

In this paper, we deal also with the fractional version of the EPD equation,

$$\frac{\partial^2 u_v}{\partial t^2} + \frac{2\gamma + d - 1}{t} \frac{\partial u_v}{\partial t} = -c^2 (-\Delta)^{v/2} u_v, \quad \gamma > 0, v \in (0, 2], \quad (1.6)$$

where $u_v := u_v(\xi_d, t)$ and $-(-\Delta)^{v/2}$ is a pseudo-differential operator called the fractional Laplace operator defined as follows for f belonging to the class of rapidly decreasing functions:

$$(-\Delta)^{v/2} f(\mathbf{x}_d) := \frac{1}{(2\pi)^d} \int_{\mathbb{R}^d} e^{-i\langle \xi_d, \mathbf{x}_d \rangle} \|\xi_d\|^{v} \hat{f}(\xi_d) d\xi_d,$$

where $\hat{f}$ is the Fourier transform of f and $\xi_d \in \mathbb{R}^d$ (see Ref. 22 for various equivalent definitions of the fractional Laplacian). In Ref. 23, the space-time generalized fractional telegraph equation is analyzed and its solution is interpreted as a time-changed isotropic stable process. The random time is represented as the inverse of a suitable combination of independent stable subordinators.

Let us consider

$$p_1^v(\mathbf{x}_d, w) = \frac{1}{(2\pi)^d} \int_{\mathbb{R}^d} e^{-i\langle \xi_d, \mathbf{x}_d \rangle} e^{i\|\xi_d\|^{v/2} w} \hat{\phi}(\xi_d) d\xi_d \quad (1.7)$$

and

$$p_2^v(\mathbf{x}_d, w) = \frac{1}{(2\pi)^d} \int_{\mathbb{R}^d} e^{-i\langle \xi_d, \mathbf{x}_d \rangle} e^{-i\|\xi_d\|^{1/2} w} \hat{\phi}(\xi_d) d\xi_d, \quad (1.8)$$

where $\hat{\phi}$ represents the Fourier transform of a sufficiently regular function ϕ . We obtain that the solution of the Cauchy problem associated with the fractional EPD equation is given by

$$u_v(\mathbf{x}_d, t) = \int_{-ct}^{ct} g(w, t) \left(\frac{p_1^v(\mathbf{x}_d, w) + p_2^v(\mathbf{x}_d, w)}{2} \right) dw, \quad (1.9)$$

where

$$g(w, t) = \frac{\Gamma(\gamma + \frac{d}{2})}{\sqrt{\pi} \Gamma(\frac{d}{2} + \gamma - \frac{1}{2}) ct} \left(1 - \frac{w^2}{c^2 t^2} \right)_+^{\frac{d}{2} + \gamma - \frac{1}{2} - 1}$$

satisfies (1.4) in the one-dimensional case. The functions (1.7) and (1.8) are solutions of fractional Schrödinger-type equations treated in the papers^{24,25} for $v = 2$, where the authors constructed probabilistically based solutions. Furthermore, we show that the solution of the fractional EPD equation satisfies the following equation:

$$\frac{\partial u_v(\mathbf{x}_d, t^\beta)}{\partial t} = -(-\Delta)^{v/2} \int_{-\frac{t^\beta}{\sqrt{\beta}}}^{\frac{t^\beta}{\sqrt{\beta}}} (g(w, t^\beta))^m \left(\frac{p_1^v(\mathbf{x}_d, w) + p_2^v(\mathbf{x}_d, w)}{2} \right) dw.$$

For the case of the higher-order EPD equations $\frac{\partial^2 u}{\partial t^2} + \frac{2\lambda}{t} \frac{\partial u}{\partial t} = c_n \frac{\partial^{2n} u}{\partial x^{2n}}$, $\lambda > 0, x \in \mathbb{R}, c_n > 0$, we study in the last part of this paper the solutions obtained as compositions of laws of pseudoprocesses (see Ref. 26) with the source-type solutions of (4.5).

II. PRELIMINARIES ON THE POROUS MEDIUM EQUATION

The standard d -dimensional PME is the following non-linear heat equation:

$$\frac{\partial u}{\partial t} = \Delta(u^m), \quad m > 1, d \geq 1, t > 0, \quad (2.1)$$

where $u := u(\mathbf{x}_d, t)$ is a non-negative scalar function defined on the space $\mathbb{R}^d \times (0, \infty)$ with $u(\mathbf{x}_d, 0) = \delta(\mathbf{x}_d)$ (which means $u(\mathbf{x}_d, t) \rightarrow \delta(\mathbf{x}_d)$ as $t \rightarrow 0$). Equation (2.1) is usually adopted to model the flow of a gas through a porous medium. The PME emerges also in the study of fluid mechanics where it models the filtration of an incompressible fluid through a porous stratum. Another important application of the PME concerns the heat radiation in plasmas, developed by Zel'dovich and collaborators in the early 1950s. Other applications have been proposed in mathematical biology. See Ref. 1 for an overall presentation of the physical and mathematical background of (2.1).

The Kompanets–Zel'dovich–Barenblatt solution (also called source-type solution since $u(\mathbf{x}_d, 0) = \delta(\mathbf{x}_d)$) represents a special solution of (2.1) and has the form (see the Appendix),

$$u(\mathbf{x}_d, t) = Ct^{-\alpha} \left(1 - B \frac{\|\mathbf{x}_d\|^2}{t^{2\beta}} \right)_+^{\frac{1}{m-1}}, \quad (2.2)$$

where $(x)_+ := \max(0, x)$,

$$\alpha := \frac{d}{2 + d(m-1)}, \quad \beta := \frac{\alpha}{d} = \frac{1}{2 + d(m-1)}, \quad B := \frac{\alpha(m-1)}{2md} = \frac{m-1}{2m(2 + d(m-1))}.$$

The normalizing constant C is chosen in such a way that

$$\int_{\mathbb{R}^d} u(\mathbf{x}_d, t) d\mathbf{x}_d = \text{area}(\mathbb{S}_1^{d-1}) \int_0^{t^\beta/\sqrt{\beta}} Ct^{-\alpha} \rho^{d-1} \left(1 - B \frac{\rho^2}{t^{2\beta}} \right)_+^{\frac{1}{m-1}} d\rho = 1,$$

and this yields

$$C := \frac{\Gamma(\frac{d}{2} + \frac{m}{m-1}) B^{\frac{d}{2}}}{\Gamma(\frac{m}{m-1}) \pi^{\frac{d}{2}}} = \frac{\Gamma(\frac{d}{2} + \frac{m}{m-1}) (\frac{m-1}{2m(2+d(m-1))})^{\frac{d}{2}}}{\Gamma(\frac{m}{m-1}) \pi^{\frac{d}{2}}}, \quad (2.3)$$

where the area $(\mathbb{S}_1^{d-1}) = \frac{2\pi^{d/2}}{\Gamma(d/2)}$. We observe that the solution considerably simplifies for $m = 2$. For $m \rightarrow 1$, (2.2) becomes the Gaussian kernel $(4\pi t)^{-d/2} \exp\{-x^2/4t\}$ representing the fundamental solution to the heat equation $\frac{\partial u}{\partial t} = \Delta u$.

The PME has the property of finite speed of propagation of disturbances from the rest level $u = 0$. We are able to explain this property as follows: If we take as initial data a density distribution given by a nonnegative, bounded, and compactly supported function, the physical

solution of the PME for these data is a continuous function $u(\mathbf{x}_d, t)$ such that for any $t > 0$, the profile $u(\cdot, t)$ is still nonnegative, bounded, and compactly supported. Hence, the support expands eventually to penetrate the whole space but is bounded at any fixed time. Therefore, for fixed $t > 0$, the support of (2.2) is given by the closed ball

$$D = \{\mathbf{x}_d \in \mathbb{R}^d : \|\mathbf{x}_d\|^2 \leq t^{2\beta}/B\},$$

while the free boundary (that is, the set separating the region where the solution is positive) is given by the sphere $\mathbb{S}_{t^\beta/\sqrt{B}}^{d-1} = \{\mathbf{x}_d \in \mathbb{R}^d : \|\mathbf{x}_d\|^2 = t^{2\beta}/B\}$. This implies that the Kompanets–Zel’dovich–Barenblatt solution spreads in space as t^β , that is, the radius of its spherical support increases as t^β .

The finite speed of propagation for the PME is in contrast with the infinite speed of propagation of the classical heat equation, that is, a nonnegative solution of the heat equation is positive everywhere in $\mathbb{R}^d$.

The solution (2.2) to the PME satisfies the following auto-similarity relationship:

$$u(\mathbf{x}_d, t) = A^\alpha u(A^\beta \|\mathbf{x}_d\|, At),$$

for all real positive numbers A .

Remark 2.1. For $d = 1$, the wave equation also has a non-linear counterpart of the form

$$\frac{\partial^2 u}{\partial t^2} = \frac{\partial^2}{\partial x^2}(u^m), \quad m > 1, t > 0,$$

which admits solutions

$$u(x, t) = \left(\frac{|x| + B}{\sqrt{mt} + D} \right)^{\frac{2}{m-1}}, \quad x \in \mathbb{R} \setminus \{0\},$$

where B and D are arbitrary constants. This statement can be checked by easy calculations.

III. STOCHASTIC PROCESSES RELATED TO THE PME

In this section, we discuss the connection between the solutions of the PME and random flights representing random motions with a finite velocity. The same approach developed in this section has been applied to the probabilistic interpretation of a fractional version of the PME (see Ref. 27). Therefore, we start by introducing this class of stochastic processes.

A random flight is a random motion in $\mathbb{R}^d$ described by a particle starting at the origin with a randomly chosen direction and with speed $c > 0$. The direction of the particle changes at each collision with some scattered obstacles where a new orientation of motion is taken. For $d \geq 2$, all the directions are independent and have the same probability distribution. The directions are chosen uniformly on the unit-radius sphere $\mathbb{S}_1^{d-1} := \{\mathbf{x}_d \in \mathbb{R}^d : \|\mathbf{x}_d\| = 1\}$, that is, for $d \geq 3$, they possess density

$$p(\theta_1, \dots, \theta_{d-2}, \phi) := \frac{\sin^{d-2}\theta_1 \sin^{d-3}\theta_2 \cdots \sin \theta_{d-2} \cos \phi}{\text{area}(\mathbb{S}_1^{d-1})}, \quad (3.1)$$

where $0 \leq \phi \leq 2\pi, 0 \leq \theta_j \leq \pi, j = 1, \dots, d-2$. For $d = 2$, (3.1) reduces to $p(\phi) = \frac{1}{2\pi}$, while for $d = 1$, we have two possible directions alternatively taken by the moving particle.

Let $(T_k, k \in \mathbb{N}_0)$ be the sequence of the instants where the random flight changes direction with $T_0 := 0$. For the case where the times T_k are governed by a homogeneous Poisson process $(N(t))_{t \geq 0}$, we have that the intertimes $\tau_{k+1} := T_{k+1} - T_k$, with $\tau_0 := 0$ and $\tau_{n+1} := t - \sum_{j=1}^n \tau_j$, have joint conditional distribution given by (see, e.g., Ref. 13)

$$f_1(t_1, \dots, t_n) = \frac{P(\tau_1 \in dt_1, \dots, \tau_n \in dt_n | N(t) = n)}{dt_1 \cdots dt_n} = \frac{n!}{t^n} 1_{S_n}(t_1, \dots, t_n),$$

where

$$S_n := \left\{ (t_1, \dots, t_n) : 0 < t_j < t - \sum_{k=0}^{j-1} t_k, 1 \leq j \leq n, t_0 = 0, t_{n+1} = t - \sum_{j=1}^n t_j \right\}.$$

For the case where the random vector $(\tau_1, \dots, \tau_n)$ has conditional density function,

$$f_2(t_1, \dots, t_n) = \frac{\Gamma((n+1)(d-1))}{(\Gamma(d-1))^{n+1}} \frac{\prod_{j=1}^{n+1} t_j^{d-2}}{t^{(n+1)(d-1)-1}} 1_{S_n}(t_1, \dots, t_n), \quad \text{for } d \geq 2 \quad (3.2)$$

or

$$f_3(t_1, \dots, t_n) = \frac{\Gamma((n+1)(\frac{d}{2}-1))}{(\Gamma(\frac{d}{2}-1))^{n+1}} \frac{\prod_{j=1}^{n+1} t_j^{\frac{d}{2}-2}}{t^{(n+1)(\frac{d}{2}-1)-1}} 1_{S_n}(t_1, \dots, t_n), \quad \text{for } d \geq 3, \quad (3.3)$$

we have different random flights where for all values of n and for all Euclidean spaces $\mathbb{R}^d$ with $d \geq 2$ or $d \geq 3$, we know (see Refs. 14 and 17) the explicit form of the conditional probability distribution of the position reached by the particle. The distributions (3.2) and (3.3) are rescaled Dirichlet distributions, with parameters $(d-1, \dots, d-1)$, $d \geq 2$ and $(\frac{d}{2}-1, \dots, \frac{d}{2}-1)$, $d \geq 3$, respectively.

By assuming that in the interval $[0, t]$, the motion has changed $n \in \mathbb{N}$ times its direction, we can describe mathematically the position $\mathbf{X}_d^n(t) := (X_1^n(t), \dots, X_d^n(t))$ reached by the particle at time $t > 0$. For $d = 1$, we have the classical telegraph process given by

$$X_1^n(t) := c \sum_{k=0}^n V_0(-1)^k \tau_{k+1}, \quad (3.4)$$

where V_0 is a random variable (r.v.) with distribution $P(V_0 = +1) = P(V_0 = -1) = \frac{1}{2}$. The instants τ_k have joint probability law f_1 .

For $d > 1$, the random vector $\mathbf{X}_d^n(t)$ has components

$$\begin{aligned} X_d^n(t) &= c \sum_{k=0}^n \tau_{k+1} \sin \theta_{1,k} \sin \theta_{2,k} \cdots \sin \theta_{d-2,k} \sin \phi_k, \\ X_{d-1}^n(t) &= c \sum_{k=0}^n \tau_{k+1} \sin \theta_{1,k} \sin \theta_{2,k} \cdots \sin \theta_{d-2,k} \cos \phi_k, \\ &\vdots, \\ X_2^n(t) &= c \sum_{k=0}^n \tau_{k+1} \sin \theta_{1,k} \cos \theta_{2,k}, \\ X_1^n(t) &= c \sum_{k=0}^n \tau_{k+1} \cos \theta_{1,k}, \end{aligned} \quad (3.5)$$

where $c > 0$ represents the velocity intensity of the motion, the r.v. $(\theta_{1,k}, \dots, \theta_{d-2,k}, \phi_k)$, $k = 0, 1, \dots, n$ has density (3.1), while the instants τ_k have joint density functions (3.2), for $d \geq 2$, or (3.3), for $d \geq 3$.

Fixed $t > 0$ as well as the number of changes in velocity $n \in \mathbb{N}$, the telegraph process (3.4) admits conditional density given by (see Ref. 28)

$$\frac{P(X_1^n(t) \in dx_1)}{dx_1} = \begin{cases} \frac{\Gamma(n+1)}{(\Gamma(\frac{n+1}{2}))^2 2^n c t} \left(1 - \frac{x_1^2}{c^2 t^2}\right)_+^{\frac{n-1}{2}}, & n \text{ odd}, \\ \frac{\Gamma(n+1)}{\Gamma(\frac{n}{2}+1)\Gamma(\frac{n}{2}) 2^n c t} \left(1 - \frac{x_1^2}{c^2 t^2}\right)_+^{\frac{n}{2}-1}, & n \text{ even}. \end{cases} \quad (3.6)$$

We observe that for n odd, we have that

$$P(X_1^n(t) \in dx_1) = P(X_1^{n+1}(t) \in dx_1).$$

For $t > 0$, under the assumptions (3.2) and (3.3), Refs. 14 and 17 obtained the explicit density functions of the random flights $\mathbf{X}_d^n(t)$ given the number of changes in the direction, that is,

$$\frac{P(\mathbf{X}_d^n(t) \in d\mathbf{x}_d)}{d\mathbf{x}_d} = \begin{cases} \frac{\Gamma(\frac{n+1}{2}(d-1) + \frac{1}{2})}{\Gamma(\frac{n}{2}(d-1))\pi^{\frac{d}{2}}(ct)^d} \left(1 - \frac{\|\mathbf{x}_d\|^2}{c^2 t^2}\right)_+^{\frac{n}{2}(d-1)-1}, & \text{if (3.2) holds} \\ \frac{\Gamma((n+1)(\frac{d}{2}-1) + 1)}{\Gamma(n(\frac{d}{2}-1))\pi^{\frac{d}{2}}(ct)^d} \left(1 - \frac{\|\mathbf{x}_d\|^2}{c^2 t^2}\right)_+^{n(\frac{d}{2}-1)-1}, & \text{if (3.3) holds.} \end{cases} \quad (3.7)$$

Now, we are able to relate the PME with a time rescaled version of the conditioned random flight processes $(\mathbf{X}_d^n(t), t \geq 0)$ when some constraints on the parameters hold true. More precisely, in the next theorem, we establish relationships between the degree m of the PME and

the number n of changes in the direction of the random flights described above and the dimension d of the Euclidean space where the random flights develop.

Theorem 1. The random flights $(Y_d^n(t), t \geq 0)$, where $Y_d^n(t) := X_d^n(t^\beta)$, admit the probability density function, for a fixed time $t > 0$ and $n \in \mathbb{N}$, given by

$$\frac{P(Y_d^n(t) \in d\mathbf{x}_d)}{d\mathbf{x}_d} = u(\mathbf{x}_d, t), \quad (3.8)$$

where $u(\mathbf{x}_d, t)$ is the solution (2.2). The equality (3.8) is true when the following relations between the number n of velocity changes in the random motion and the parameter m of the PME fulfill:

(i) for $d = 1$,

$$m = \begin{cases} \frac{n+1}{n-1} = 1 + \frac{1}{k}, & n = 2k+1, \\ \frac{n}{n-2} = 1 + \frac{1}{k}, & n = 2k+2, \end{cases} \quad k \geq 1,$$

(ii) when (3.2) holds, $m = \frac{n(d-1)}{n(d-1)-2}$ with $d > \frac{2}{n} + 1$, and

(iii) when (3.3) holds, $m = \frac{n(d-2)}{n(d-2)-2}$ with $d > \frac{2}{n} + 2$.

Proof. For $d = 1$, we deal with a telegraph process defined by (3.5) with time scale $t' = t^\beta$ and speed $c' = 1/\sqrt{B}$. By exploiting the duplication formula for the Gamma function, we can write the solution (4.7) for $d = 1$ as follows:

$$u(x_1, t') = \frac{\Gamma(\frac{2m}{m-1})2^{1-\frac{2m}{m-1}}}{(\Gamma(\frac{m}{m-1}))^2} \frac{1}{c't'} \left(1 - \frac{x_1^2}{(c't')^2}\right)_+^{\frac{1}{m-1}}. \quad (3.9)$$

For

$$\frac{1}{m-1} = \frac{n-1}{2}, \quad \text{that is,} \quad m = \frac{n+1}{n-1},$$

the solution (2.2) coincides with the first one of (3.6), while for

$$\frac{1}{m-1} = \frac{n}{2} - 1, \quad \text{that is} \quad m = \frac{n}{n-2},$$

the solution (2.2) coincides with the second one of (3.6). For $n > 2$, in both cases, $1 < m < \infty$.

Now, let us consider a random flight defined in $\mathbb{R}^d$, $d \geq 2$ by (3.5) with time scale $t' = t^\beta$ and speed $c' = 1/\sqrt{B}$. Under the assumption (3.2), for

$$\frac{m}{m-1} = \frac{n}{2}(d-1), \quad \text{that is} \quad m = \frac{n(d-1)}{n(d-1)-2},$$

the function (4.7) coincides with the first one of (3.7). Since $m \in (1, \infty)$, we infer that

$$d > \frac{2}{n} + 1. \quad (3.10)$$

For $d = 2$, the inequality (3.10) holds for $n \geq 3$; for $d = 3$, it holds for $n \geq 2$; and for $d > 3$, (3.10) holds for all $n \geq 1$. Therefore, under the condition (3.10), we can write

$$P(X_d^n(t') \in d\mathbf{x}_d) = u(\mathbf{x}_d, t')d\mathbf{x}_d.$$

Analogously, under the assumption (3.3), for

$$\frac{m}{m-1} = n\left(\frac{d}{2} - 1\right), \quad \text{that is,} \quad m = \frac{n(d-2)}{n(d-2)-2},$$

the function (4.7) coincides with the second one of (3.7). Since $m \in (1, \infty)$, we infer that

$$d > \frac{2}{n} + 2. \quad (3.11)$$

For $d = 3$, the inequality (3.10) holds for $n \geq 3$; for $d = 4$, it holds for $n \geq 2$; and for $d > 4$, (3.10) holds for all $n \geq 1$. Therefore, under the condition (3.11), it turns out that

$$P(X_d^n(t') \in d\mathbf{x}_d) = u(\mathbf{x}_d, t')d\mathbf{x}_d.$$

□

Remark 3.1. From Theorem 1, it follows that $(\mathbf{Y}_d^n(t), t \geq 0)$ is rotationally invariant, that is, if $O(d)$ is the group of $d \times d$ orthogonal matrices acting in $\mathbb{R}^d$, we have $u(M^T \mathbf{x}_d, t) = u(\mathbf{x}_d, t) = u(\|\mathbf{x}_d\|, t)$, where $M \in O(d)$.

Remark 3.2. The study of stochastic processes associated with the PME has been also developed in Refs. 29–32. They proved that there exists a non-linear diffusion process $\{(\mathbf{Z}_d(t) = (Z_1(t), \dots, Z_d(t))), t \geq 0, P\}$, satisfying the following system of stochastic differential equations:

$$Z_j(t) = \int_0^t u(Z_j(s), s)^{(m-1)/2} dB_j(s), \quad j = 1, 2, \dots, d, \quad (3.12)$$

where $(B_1(t), \dots, B_d(t))$ is a standard d -dimensional Brownian motion such that

$$P(\mathbf{Z}_d(t) \in d\mathbf{x}_d) = u(\mathbf{x}_d, t) d\mathbf{x}_d,$$

with $u(\mathbf{x}_d, t)$ given by (2.2).

The following result concerning the Fourier transform of $u(\mathbf{x}_d, t)$ has also been proved in Ref. 5:

Theorem 2. The Fourier transform of the probability law $u(\mathbf{x}_d, t)$ given by (2.2), denoted by $\hat{u}(\xi_d, t)$, is equal to

$$\hat{u}(\xi_d, t) = \left(\frac{2\sqrt{B}}{t^\beta \|\xi_d\|} \right)^{\frac{d}{2\alpha(m-1)}} \Gamma\left(\frac{d}{2} + \frac{m}{m-1}\right) J_{\frac{d}{2\alpha(m-1)}}\left(\frac{\|\xi_d\| t^\beta}{\sqrt{B}}\right), \quad (3.13)$$

where $\xi_d \in \mathbb{R}^d$, $\alpha = \frac{d}{2+d(m-1)}$, $\beta = \frac{1}{2+d(m-1)}$ and $J_\mu(x) = \sum_{k=0}^{\infty} (-1)^k \frac{(x/2)^{2k+\mu}}{k! \Gamma(k+\mu+1)}$, with $\mu \in \mathbb{R}$ being the Bessel function.

Proof. Let σ be the uniform measure on $\mathbb{S}_1^{d-1}$. We recall that [see (2.12), p. 690, Ref. 17]

$$\int_{\mathbb{S}_1^{d-1}} e^{ip\langle \xi_d, \theta_d \rangle} d\sigma(\theta_d) = (2\pi)^{d/2} \frac{J_{\frac{d}{2}-1}(\rho \|\xi_d\|)}{(\rho \|\xi_d\|)^{\frac{d}{2}-1}}. \quad (3.14)$$

One has

$$\begin{aligned} \hat{u}(\xi_d, t) &= \int_{\mathbb{R}^d} e^{i\langle \xi_d, \mathbf{x}_d \rangle} u(\mathbf{x}_d, t) d\mathbf{x}_d \\ &= (\text{by Remark 3.1}) \\ &= \int_0^{\frac{\rho}{\sqrt{B}}} \rho^{d-1} C t^{-\alpha} \left(1 - \frac{B\rho^2}{t^{2\beta}}\right)^{\frac{1}{m-1}} d\rho \int_{\mathbb{S}_1^{d-1}} e^{ip\langle \xi_d, \theta_d \rangle} d\sigma(\theta_d) \\ &= (\text{by (3.14)}) \\ &= (2\pi)^{d/2} \int_0^{\frac{\rho}{\sqrt{B}}} \rho^{d-1} C t^{-\alpha} \left(1 - \frac{B\rho^2}{t^{2\beta}}\right)^{\frac{1}{m-1}} \frac{J_{\frac{d}{2}-1}(\rho \|\xi_d\|)}{(\rho \|\xi_d\|)^{\frac{d}{2}-1}} d\rho \\ &= \frac{(2\pi)^{d/2} C t^{-\alpha+\beta(\frac{d}{2}+1)}}{(\sqrt{B})^{\frac{d}{2}+1} \|\xi_d\|^{\frac{d}{2}-1}} \int_0^1 (1-w^2)^{\frac{1}{m-1}} w^{d/2} J_{\frac{d}{2}-1}\left(\frac{\|\xi_d\| t^\beta}{\sqrt{B}} w\right) dw. \end{aligned}$$

In view of formula 6.567(1) of p. 688 of Ref. 33,

$$\int_0^1 x^{\nu+1} (1-x^2)^\mu J_\nu(bx) dx = 2^\mu \Gamma(\mu+1) b^{-(\mu+1)} J_{\nu+\mu+1}(b), \quad (3.15)$$

where $b > 0$, $\text{Re } \nu > -1$, and $\text{Re } \mu > -1$; we obtain (3.13). $\square$

The previous discussion entails that (3.13) is the characteristic function of $(\mathbf{Y}_d^n(t), t \geq 0)$ under the conditions of Theorem 1.

Proposition 1. For $d = 1$, the p th moment of $(\mathbf{Y}_1^n(t), t \geq 0)$ [under the condition (i) of Theorem 1 relating m and n] is equal to

$$E[\mathbf{Y}_1^n(t)]^p = \begin{cases} 0, & p \text{ odd} \\ \frac{\Gamma(\frac{p+1}{2}) \Gamma(\frac{1}{2} + \frac{m}{m-1})}{\sqrt{\pi} \Gamma(\frac{p+1}{2} + \frac{m}{m-1})} \left(\frac{t^\alpha}{\sqrt{B}}\right)^p, & p \text{ even}, \end{cases} \quad (3.16)$$

with $\alpha = \frac{1}{1+m}$ and $B = \frac{m-1}{2m(m+1)}$.

Proof. We have to compute

$$E[Y_1^n(t)]^p = Ct^{-\alpha} \int_{-t^{\beta}/\sqrt{B}}^{t^{\beta}/\sqrt{B}} x_1^p \left(1 - B \frac{x_1^2}{t^{2\beta}}\right)^{\frac{1}{m-1}} dx_1. \quad (3.17)$$

Simple calculations lead to (3.16). $\square$

From the above proposition, we derive

$$\text{Var}[Y_1^n(t)] = \frac{2m(m+1)}{3m-1} t^{\frac{2}{1+m}} = \frac{2n(n+1)}{(n-1)(n+2)} t^{\frac{n-1}{n}}. \quad (3.18)$$

Since $\frac{n-1}{n} < 1$, from (3.18), we can conclude that the process $(Y_1^n(t), t \geq 0)$ spreads like a sub-diffusion.

In Theorem 3, in Ref. 17, the moments of order p of $(\|Y_d^n(t)\|, t \geq 0)$, with $d \geq 2$ (with suitable changes in the parameters) have been evaluated; then, also the multidimensional random flights $(Y_d^n(t), t \geq 0)$ represent anomalous diffusions.

IV. FRACTIONAL EPD EQUATION AND RELATED SOLUTIONS

We now consider the space-fractional version of the EPD equation, that is,

$$\begin{cases} \left(\frac{\partial^2}{\partial t^2} + \frac{d+2\gamma-1}{t} \frac{\partial}{\partial t} \right) u_v = -c^2 (-\Delta)^{\frac{\nu}{2}} u_v, \\ u(\mathbf{x}_d, 0) = \varphi(\mathbf{x}_d), \\ \left. \frac{\partial u(\mathbf{x}_d, t)}{\partial t} \right|_{t=0} = 0, \end{cases} \quad (4.1)$$

where $u_v := u_v(\mathbf{x}_d, t)$ and $0 < \nu < 2$. We assume that $\varphi \in \mathcal{S}(\mathbb{R}^d)$, representing the Schwartz space of rapidly decreasing functions on $\mathbb{R}^d$. The fractional Laplace operator $(-\Delta)^{\nu/2}$ is a non-local pseudo-differential operator, defined via Fourier multipliers as follows:

$$(-\Delta)^{\nu/2} f(\mathbf{x}_d) := \frac{1}{(2\pi)^d} \int_{\mathbb{R}^d} e^{-i\langle \xi_d, \mathbf{x}_d \rangle} \|\xi_d\|^\nu \hat{f}(\xi_d) d\xi_d. \quad (4.2)$$

Formally, let $L^p(\mathbb{R}^d)$, $p \in [1, 2]$; we say that $f \in \text{Dom}((-\Delta)^{\nu/2}, L^p(\mathbb{R}^d))$ whenever $f \in L^p(\mathbb{R}^d)$ and there is $(-\Delta)^{\nu/2} f(\mathbf{x}_d) \in L^p(\mathbb{R}^d)$ such that (4.2) holds (see Definition 2.1 in Ref. 22). For $d = 1$, the operator $(-\Delta)^{\nu/2}$ coincides with the Riesz fractional derivative

$$\frac{\partial^\nu f(x)}{\partial |x|^\nu} = -\frac{1}{2 \cos(\pi\nu/2)} \frac{1}{\Gamma(m-\nu)} \frac{d^m}{dx^m} \int_{\mathbb{R}} \frac{f(y)}{|x-y|^{\nu+1-m}} dy, \quad m-1 < \nu < m, m \in \mathbb{N}.$$

An alternative definition of the fractional Laplace operator is the following one: Let $[S_\nu(t), t \geq 0]$, with $0 < \nu < 2$, be an isotropic, d -dimensional, ν -stable process with

$$E\left[e^{i\langle \xi_d, S_\nu(t) \rangle}\right] = e^{-t\|\xi_d\|^\nu}. \quad (4.3)$$

Let $(T_t f)(\mathbf{x}_d) = E[f(S_\nu(t) + \mathbf{x}_d)]$ be the semigroup associated with $S_\nu(t)$, $t \geq 0$. The infinitesimal generator of the semigroup $T_t f$ is given by the limit in norm, if it exists, $\mathcal{L}f = \lim_{t \rightarrow 0^+} \frac{T_t f - f}{t}$, where $f \in \text{Dom}(\mathcal{L})$. It is well-known that $\mathcal{L} = -(-\Delta)^{\nu/2}$ and its domain becomes $\text{Dom}(\mathcal{L}) = \left\{ f \in L^2(\mathbb{R}^d) : \int_{\mathbb{R}^d} (1 + \|\xi_d\|^\nu) |\hat{f}(\xi_d)|^2 d\xi_d < \infty \right\}$ (see, e.g., Ref. 34). On the definition of the fractional Laplace operator based on fractional Sobolev spaces, see Ref. 35.

The reader can consult Ref. 22 for a discussion on the definition of the fractional Laplace operator. Moreover, if $\nu_1 > 0$, $\nu_2 > 0$, and $0 < \nu_1 + \nu_2 \leq 2$, for the fractional Laplacian, the following semigroup property holds:

$$(-\Delta)^{\frac{\nu_1}{2}} (-\Delta)^{\frac{\nu_2}{2}} = (-\Delta)^{\frac{\nu_1 + \nu_2}{2}}. \quad (4.4)$$

Indeed, by definition (4.2), under the previous constraints on ν_1 and ν_2 , it is easy to verify that

$$\int_{\mathbb{R}^d} e^{i\langle \xi_d, \mathbf{x}_d \rangle} (-\Delta)^{\frac{\nu_1}{2}} (-\Delta)^{\frac{\nu_2}{2}} f(\mathbf{x}_d) d\mathbf{x}_d = \|\xi_d\|^{\frac{\nu_1}{2} + \frac{\nu_2}{2}} \hat{f}(\xi_d),$$

and then, the equality (4.4) is proved. For this reason, hereafter, we adopt the definition (4.2).

Remark 4.1. For $\nu = 2$, Eq. (4.1) coincides with the d -dimensional EPD equation, which is itself a special case of the multidimensional telegraph equation,

$$\frac{\partial^2 u}{\partial t^2} + 2\lambda(t) \frac{\partial u}{\partial t} = c^2 \Delta u, \quad (4.5)$$

with $\lambda(t) = \frac{d+2\gamma-1}{2t}$. The previous equation has the fundamental solution (see Ref. 21),

$$p(\mathbf{x}_d, t) = \frac{\Gamma(\gamma + \frac{d}{2})}{\pi^{d/2} \Gamma(\gamma)} \frac{1}{(ct)^d} \left(1 - \frac{\|\mathbf{x}_d\|^2}{(ct)^2} \right)_+^{\gamma-1}, \quad (4.6)$$

where $\gamma > 0$. The EPD equation has different forms and emerged in different contexts (e.g., fluid dynamics and geometry). For $d = 1$, a probabilistic derivation of the EPD equation based on the integrated telegraph process where the reversals of velocity are paced by a non-homogeneous Poisson process with rate $\lambda(t) = \frac{\gamma}{t}$ is given in Refs. 36 and 37. By rescaling the time coordinate,

$$t' := t^\beta,$$

the solution (2.2) of the PME coincides with (4.6). Indeed, in the frame $(\mathbf{x}_d, t')$, the Kompanets–Zel'dovich–Barenblatt solution (2.2) can be written as

$$u(\mathbf{x}_d, t') = p(\mathbf{x}_d, t') = \frac{\Gamma(\frac{d}{2} + \frac{m}{m-1})}{\Gamma(\frac{m}{m-1}) \pi^{\frac{d}{2}}} \frac{1}{(c't')^d} \left(1 - \frac{\|\mathbf{x}_d\|^2}{(c't')^2} \right)_+^{\frac{1}{m-1}}, \quad (4.7)$$

where

$$c' := \frac{1}{\sqrt{B}} = \sqrt{\frac{2m(2+d(m-1))}{m-1}},$$

and solves the EPD equation (4.5) with $\gamma = \frac{m}{m-1}$, that is,

$$\left(\frac{\partial^2}{\partial t'^2} + \frac{d + \frac{m+1}{m-1}}{t'} \frac{\partial}{\partial t'} \right) u(\mathbf{x}_d, t') = c'^2 \Delta u(\mathbf{x}_d, t'). \quad (4.8)$$

We observe that in the original time variable $t = (t')^{1/\beta}$, the solution (4.7), that is (2.2), satisfies the EPD equation

$$\left[\frac{\partial^2}{\partial t^2} + \frac{\beta}{t} \left(d + \frac{m+1}{m-1} - \beta(\beta-1)t^{2\beta-1} \right) \frac{\partial}{\partial t} \right] u(\mathbf{x}_d, t) = \frac{\beta^2 t^{2\beta-2}}{B} \Delta u(\mathbf{x}_d, t).$$

Remark 4.2. The k -th one-dimensional marginals of (4.7) have the form

$$p_k(x_k, t') = \frac{\Gamma(\frac{d}{2} + \frac{m}{m-1})}{\Gamma(\frac{m}{m-1} + \frac{d-1}{2}) \pi^{\frac{1}{2}}} \frac{1}{c't'} \left(1 - \frac{x_k^2}{(c't')^2} \right)_+^{\frac{m}{m-1} + \frac{d-1}{2}} \quad (4.9)$$

and satisfy the EPD equation

$$\left(\frac{\partial^2}{\partial t'^2} + \frac{d + \frac{m+1}{m-1}}{t'} \frac{\partial}{\partial t'} \right) p_k(x_k, t') = c'^2 \frac{\partial^2 p_k(x_k, t')}{\partial x_k^2}$$

(on this point, the reader can consult Ref. 21). For $t' = t^\beta$ and $c' = \frac{1}{\sqrt{B}}$, the probability density (4.9) coincides with the Barenblatt solution of the PME. Therefore, for $d = 1$, we can interpret (2.2) as the law of a time-rescaled telegraph process on the real line observed on $(0, t]$, where the particle alternates the two directions at epochs of a non-homogeneous Poisson process with rate $\lambda(t) = \frac{d + \frac{m+1}{m-1}}{2t^\beta}$. Unlike the classical telegraph process, the process with probability distribution (4.9) has only the absolutely continuous component. This is due to the fact that the rate function of the Poisson process is $\lambda(t) = \frac{\gamma}{t}$ so that in the neighborhood of the initial time interval, the particle changes the direction infinitely often. It is important to underline that this interpretation also holds for all $m > 1$.

Lemma 1. If we deal with $\varphi(\mathbf{x}_d) = \delta(\mathbf{x}_d)$, the characteristic function of u_ν becomes

$$\hat{u}_\nu(\xi_d, t) = \left(\frac{2}{ct \|\xi_d\|^{v/2}} \right)^{\gamma + \frac{d}{2} - 1} \Gamma\left(\gamma + \frac{d}{2}\right) J_{\gamma + \frac{d}{2} - 1}(ct \|\xi_d\|^{v/2}). \quad (4.10)$$

Proof. By means of the same arguments adopted in Proof of Theorem 2, we are able to prove that the Fourier transform of (4.6) becomes

$$\hat{u}(\xi_d, t) = \left(\frac{2}{ct\|\xi_d\|} \right)^{\gamma + \frac{d}{2} - 1} \Gamma\left(\gamma + \frac{d}{2}\right) J_{\gamma + \frac{d}{2} - 1}(ct\|\xi_d\|).$$

The Fourier transform $\hat{u} := \hat{u}(\xi_d, t)$ satisfies

$$\frac{\partial^2 \hat{u}}{\partial t^2} + \frac{d + 2\gamma - 1}{t} \frac{\partial \hat{u}}{\partial t} = -c^2 \|\xi_d\|^2 \hat{u}, \quad (4.11)$$

while the Fourier transform $\hat{u}_v := \hat{u}_v(\xi_d, t)$ of the fundamental solution to (4.1) is solution to

$$\frac{\partial^2 \hat{u}_v}{\partial t^2} + \frac{d + 2\gamma - 1}{t} \frac{\partial \hat{u}_v}{\partial t} = -c^2 \|\xi_d\|^v \hat{u}_v. \quad (4.12)$$

Since $\hat{u}(\xi_d, t)$ is solution to (4.11), it follows that (4.10) solves (4.12). $\square$

We are able to present two different forms of the inverse Fourier transform $u_v(\mathbf{x}_d, t)$ of (4.10).

Theorem 3. We have that the inverse Fourier transform of (4.10) becomes

$$u_v(\mathbf{x}_d, t) = \left(\frac{2}{ct} \right)^\gamma \frac{\Gamma(\gamma + \frac{d}{2})}{v\pi^{d/2}(ct\|\mathbf{x}_d\|)^{\frac{d}{2}-1}} \int_0^\infty \rho^{d(\frac{2-v}{2v}) + \frac{2}{v} - \gamma} J_{\gamma + \frac{d}{2} - 1}(ct\rho) J_{\frac{d}{2}-1}(\rho^{2/v}\|\mathbf{x}_d\|) d\rho \quad (4.13)$$

or

$$u_v(\mathbf{x}_d, t) = \frac{2\Gamma(\gamma + \frac{d}{2})}{v\pi^{d/2}(ct\|\mathbf{x}_d\|)^{\frac{d}{2}-1}\Gamma(\gamma)} \quad (4.14)$$

$$\times \int_0^1 w^{\frac{d}{2}} (1 - w^2)^{\gamma-1} dw \int_0^\infty \rho^{d(\frac{2-v}{2v}) + \frac{2}{v} - \gamma} J_{\frac{d}{2}-1}(ct\rho w) J_{\frac{d}{2}-1}(\rho^{2/v}\|\mathbf{x}_d\|) d\rho.$$

Proof. We start from (4.10) and write [in view of formula (3.14)]

$$\begin{aligned} u_v(\mathbf{x}_d, t) &= \frac{1}{(2\pi)^d} \int_{\mathbb{R}^d} e^{-i\langle \xi_d, \mathbf{x}_d \rangle} \hat{u}_v(\xi_d, t) d\xi_d \\ &= \frac{1}{(2\pi)^d} \int_0^\infty \rho^{d-1} \left(\frac{2}{ct\rho^{v/2}} \right)^{\gamma + \frac{d}{2} - 1} \Gamma\left(\gamma + \frac{d}{2}\right) J_{\gamma + \frac{d}{2} - 1}(ct\rho^{v/2}) d\rho \int_{\mathbb{S}_1^{d-1}} e^{-i\rho\langle \theta_d, \mathbf{x}_d \rangle} d\sigma(\theta_d) \\ &= (by (3.14)) \\ &= \frac{1}{(2\pi)^{d/2}} \int_0^\infty \rho^{d-1} \left(\frac{2}{ct\rho^{v/2}} \right)^{\gamma + \frac{d}{2} - 1} \Gamma\left(\gamma + \frac{d}{2}\right) J_{\gamma + \frac{d}{2} - 1}(ct\rho^{v/2}) \frac{J_{\frac{d}{2}-1}(\rho\|\mathbf{x}_d\|)}{(\rho\|\mathbf{x}_d\|)^{\frac{d}{2}-1}} d\rho \\ &= \frac{1}{(2\pi)^{d/2}} \left(\frac{2}{ct} \right)^{\gamma + \frac{d}{2} - 1} \frac{\Gamma(\gamma + \frac{d}{2})}{\|\mathbf{x}_d\|^{\frac{d}{2}-1}} \int_0^\infty \rho^{\frac{d}{2} - \frac{v}{2}(\frac{d}{2}-1+\gamma)} J_{\gamma + \frac{d}{2} - 1}(ct\rho^{v/2}) J_{\frac{d}{2}-1}(\rho\|\mathbf{x}_d\|) d\rho. \end{aligned}$$

By a simple change in the variable in the above integral, we obtain the result (4.13).

In order to obtain the result (4.14), we apply (3.15),

$$\int_0^1 w^{\frac{d}{2}} (1 - w^2)^{\gamma-1} J_{\frac{d}{2}-1}(bw) dw = 2^{\gamma-1} \Gamma(\gamma) b^{-\gamma} J_{\gamma + \frac{d}{2} - 1}(b), \quad (4.15)$$

with $b = ct\rho^{v/2}$, $v = \frac{d}{2} - 1$ and $\mu = \gamma - 1$. By inserting (4.15) into (4.13), we get

$$\begin{aligned}
u_v(\mathbf{x}_d, t) &= \frac{1}{(2\pi)^{d/2}} \left(\frac{2}{ct}\right)^{\gamma+\frac{d}{2}-1} \frac{\Gamma(\gamma+\frac{d}{2})}{\|\mathbf{x}_d\|^{\frac{d}{2}-1}} \int_0^\infty \rho^{\frac{d}{2}-\frac{\gamma}{2}(\frac{d}{2}-1+\gamma)} J_{\gamma+\frac{d}{2}-1}(ct\rho^{v/2}) J_{\frac{d}{2}-1}(\rho\|\mathbf{x}_d\|) d\rho \\
&= \frac{1}{\pi^{d/2}(ct)^{\frac{d}{2}-1}} \frac{\Gamma(\gamma+\frac{d}{2})}{\Gamma(\gamma)\|\mathbf{x}_d\|^{\frac{d}{2}-1}} \\
&\quad \times \int_0^\infty \rho^{\frac{d}{2}+\frac{\gamma}{2}-\frac{vd}{4}} J_{\frac{d}{2}-1}(\rho\|\mathbf{x}_d\|) d\rho \int_0^1 w^{\frac{d}{2}}(1-w^2)^{\gamma-1} (ct\rho^{v/2}w) dw.
\end{aligned}$$

By a simple change in variable in the above integral, we obtain the result (4.14). $\square$

An alternative form of the characteristic function (4.10) of the solution to the fractional EPD equation is given in the next theorem, and this inspires an alternative representation of the solution u_v in terms of one-dimensional random flights arising from (4.5).

Theorem 4. The characteristic function (4.10) can be written as

$$\hat{u}_v(\xi_d, t) = \frac{\Gamma(\gamma+\frac{d}{2})}{\sqrt{\pi}\Gamma(\frac{d}{2}+\gamma-\frac{1}{2})ct} \int_{-ct}^{ct} \left(1 - \frac{w^2}{c^2t^2}\right)^{\frac{d}{2}+\gamma-\frac{1}{2}-1} \left(\frac{e^{i\|\xi_d\|^{v/2}w} + e^{-i\|\xi_d\|^{v/2}w}}{2}\right) dw. \quad (4.16)$$

Proof. The Poisson integral representation of the Bessel functions reads

$$J_\mu(z) = \frac{(z/2)^\mu}{\sqrt{\pi}\Gamma(\mu+\frac{1}{2})} \int_{-1}^{+1} (1-w^2)^{\mu-\frac{1}{2}} \cos(zw) dw, \quad (4.17)$$

valid for $\mu > -\frac{1}{2}$, $z \in \mathbb{R}$ [see Ref. 38, p. 114, formula (5.10.3)]. By inserting (4.17) into (4.10), we readily have that

$$\begin{aligned}
\hat{u}_v(\xi_d, t) &= \frac{\Gamma(\gamma+\frac{d}{2})}{\sqrt{\pi}\Gamma(\frac{d}{2}+\gamma-\frac{1}{2})} \int_{-1}^1 (1-w^2)^{\frac{d}{2}+\gamma-\frac{1}{2}-1} \cos(ct\|\xi_d\|^{v/2}w) dw \\
&= \frac{\Gamma(\gamma+\frac{d}{2})}{\sqrt{\pi}\Gamma(\frac{d}{2}+\gamma-\frac{1}{2})} \int_{-1}^1 (1-w^2)^{\frac{d}{2}+\gamma-\frac{1}{2}-1} \left(\frac{e^{ict\|\xi_d\|^{v/2}w} + e^{-ict\|\xi_d\|^{v/2}w}}{2}\right) dw \\
&= \frac{\Gamma(\gamma+\frac{d}{2})}{\sqrt{\pi}\Gamma(\frac{d}{2}+\gamma-\frac{1}{2})ct} \int_{-ct}^{ct} \left(1 - \frac{w^2}{c^2t^2}\right)^{\frac{d}{2}+\gamma-\frac{1}{2}-1} \left(\frac{e^{i\|\xi_d\|^{v/2}w} + e^{-i\|\xi_d\|^{v/2}w}}{2}\right) dw.
\end{aligned}$$

$\square$

Remark 4.3. Since the stable subordinator process has characteristic function (4.3), the function $e^{\pm i\|\xi_d\|^{v/2}w}$ seems to be related to the Fourier transform of $S_v(t)$. We observe that

$$\begin{aligned}
\frac{P(S_v(t) \in d\mathbf{x}_d)}{d\mathbf{x}_d} &= \frac{1}{(2\pi)^d} \int_{\mathbb{R}^d} e^{-i\langle \xi_d, \mathbf{x}_d \rangle} e^{-t\|\xi_d\|^v} d\xi_d \\
&= \frac{1}{(2\pi)^{d/2}} \int_0^\infty \rho^{d-1} \frac{J_{\frac{d}{2}-1}(\rho\|\mathbf{x}_d\|)}{(\rho\|\mathbf{x}_d\|)^{\frac{d}{2}-1}} e^{-t\rho^v} d\rho \\
&= \frac{1}{(2\pi)^{d/2}} \frac{1}{\|\mathbf{x}_d\|^{\frac{d}{2}-1}} \sum_{k=0}^\infty (-1)^k \left(\frac{\|\mathbf{x}_d\|}{2}\right)^{2k+\frac{d}{2}-1} \frac{1}{k!\Gamma(k+\frac{d}{2})} \int_0^\infty \rho^{d+2k-1} e^{-t\rho^v} d\rho \\
&= \frac{1}{(2\pi)^{d/2}} \sum_{k=0}^\infty (-1)^k \left(\frac{1}{2}\right)^{2k+\frac{d}{2}-1} \|\mathbf{x}_d\|^{2k} \frac{\Gamma(d+2k)}{k!\Gamma(k+\frac{d}{2}) t^{\frac{2k+d}{v}}}.
\end{aligned} \quad (4.18)$$

For $v = 1$, by inserting

$$\left(\frac{-\frac{d+1}{2}}{k}\right) = \frac{(-1)^k \Gamma(d+2k) \Gamma(\frac{d}{2}+1)}{\Gamma(\frac{d}{2}+k) 2^{2k-1} k! \Gamma(d+1)}$$

into (4.18), we obtain

$$\begin{aligned} \frac{P(\mathbf{S}_d(t) \in d\mathbf{x}_d)}{d\mathbf{x}_d} &= \frac{1}{(2\pi)^{d/2}} \frac{1}{2^{\frac{d}{2}} t^d} \sum_{k=0}^{\infty} \binom{-\frac{d+1}{2}}{k} \left(\frac{\|\mathbf{x}_d\|^2}{t^2} \right)^k \frac{\Gamma(d+1)}{\Gamma(\frac{d}{2}+1)} \\ &= \frac{1}{(2\pi)^{d/2}} \frac{1}{2^{\frac{d}{2}} t^d} \frac{\Gamma(d+1)}{\Gamma(\frac{d}{2}+1)} \frac{1}{\left(1 + \frac{\|\mathbf{x}_d\|^2}{t^2}\right)^{\frac{d+1}{2}}}. \end{aligned} \quad (4.19)$$

Thus, for $\nu = 1$, (4.3) becomes the characteristic function of a d -dimensional Cauchy process $[\mathbf{C}_d(t), t \geq 0]$ having probability distribution equal to (4.19). Therefore, the density function (4.19) can be interpreted as the distribution of the hitting point of a $(d+1)$ -dimensional Brownian motion on the subspace $\mathbb{S}_1^d$ with the starting point $(0, \dots, 0, t) \in \mathbb{R}^{d+1}$. For the sake of completeness, we remind that (4.19) solves the Laplace equation

$$\frac{\partial^2 p}{\partial t^2} + \Delta p = 0.$$

Remark 4.4. By Bochner's subordination, it is possible to show that

$$\mathbf{S}_\nu(t) \stackrel{(\text{law})}{=} \mathbf{B}_d(Y_{\nu/2}(t)), \quad 0 < \nu \leq 2, \quad (4.20)$$

where $[\mathbf{B}_d(t), t \geq 0]$ is a standard d -dimensional Brownian motion and $[Y_{\nu/2}(t), t \geq 0]$ is an independent $\nu/2$ -stable subordinator. Equivalently, we have that

$$\mathbf{S}_\nu(t) \stackrel{(\text{law})}{=} \mathbf{C}_d(Y_\nu(t)), \quad 0 < \nu < 1, \quad (4.21)$$

where $[\mathbf{C}_d(t), t \geq 0]$ and $[Y_\nu(t), t \geq 0]$ are independent. Indeed,

$$\begin{aligned} E\left[e^{i\langle \xi_d, \mathbf{C}_d(Y_\nu(t)) \rangle}\right] &= E\left[E\left[e^{i\langle \xi_d, \mathbf{C}_d(Y_\nu(t)) \rangle} \middle| \mathcal{F}_{Y_\nu}\right]\right] \\ &= E\left[e^{-Y_\nu(t)\|\xi_d\|}\right] = e^{-t\|\xi_d\|^\nu}, \end{aligned}$$

where $\mathcal{F}_{Y_\nu}$ is the natural filtration of $[\mathbf{C}_d(t), t \geq 0]$ stopped at $Y_\nu(t)$.

Our task now is to obtain the inverse Fourier transform of (4.16). The term $e^{\pm i\|\xi_d\|^{v/2}w} \hat{\varphi}(\xi_d)$ is the Fourier transform of the solution to the following Cauchy problem involving the fractional Schrödinger equation:

$$\begin{cases} \pm i \frac{\partial u}{\partial w} = -(-\Delta)^{v/4} u, \\ u(\mathbf{x}_d, 0) = \varphi(\mathbf{x}_d), \end{cases} \quad (4.22)$$

where $\varphi(\mathbf{x}_d) \in \mathcal{S}(\mathbb{R}^d)$. Therefore, we can say that $\frac{e^{i\|\xi_d\|^{v/2}w} + e^{-i\|\xi_d\|^{v/2}w}}{2} \hat{\varphi}(\xi_d)$ is the Fourier transform of the solution to

$$\begin{cases} \left(\frac{\partial^2}{\partial w^2} + (-\Delta)^{v/2} \right) u = \left(i \frac{\partial}{\partial w} + (-\Delta)^{v/4} \right) \left(-i \frac{\partial}{\partial w} + (-\Delta)^{v/4} \right) u = 0, \\ u(\mathbf{x}_d, 0) = \varphi(\mathbf{x}_d), \\ \left. \frac{\partial u(\mathbf{x}_d, w)}{\partial w} \right|_{w=0} = 0. \end{cases} \quad (4.23)$$

Let us denote by $p_1^v := p_1^v(\mathbf{x}_d, w)$ and $p_2^v := p_2^v(\mathbf{x}_d, w)$ the solutions to the Cauchy problems (4.22). We have that

$$p_1^v(\mathbf{x}_d, w) = \frac{1}{(2\pi)^d} \int_{\mathbb{R}^d} e^{-i\langle \xi_d, \mathbf{x}_d \rangle} e^{i\|\xi_d\|^{v/2}w} \hat{\varphi}(\xi_d) d\xi_d. \quad (4.24)$$

Analogously, we get

$$p_2^v(\mathbf{x}_d, w) = \frac{1}{(2\pi)^d} \int_{\mathbb{R}^d} e^{-i\langle \xi_d, \mathbf{x}_d \rangle} e^{-i\|\xi_d\|^{v/2}w} \hat{\varphi}(\xi_d) d\xi_d. \quad (4.25)$$

We obtain the following result relating the EPD equation with the random flights:

Corollary 5. *The solution of the Cauchy problem (4.1) is given by*

$$u_v(\mathbf{x}_d, t) = \int_{-ct}^{ct} g(w, t) \left(\frac{p_1^v(\mathbf{x}_d, w) + p_2^v(\mathbf{x}_d, w)}{2} \right) dw, \quad (4.26)$$

where

$$g(w, t) = \frac{\Gamma(\gamma + \frac{d}{2})}{\sqrt{\pi} \Gamma(\frac{d}{2} + \gamma - \frac{1}{2}) ct} \left(1 - \frac{w^2}{c^2 t^2} \right)_+^{\frac{d}{2} + \gamma - \frac{1}{2} - 1}$$

is the solution to the one-dimensional EPD equation (4.5) with $\lambda(t) = \frac{\gamma + \frac{d}{2} - \frac{1}{2}}{2t}$.

Proof. The result (4.26) follows from (4.16), (4.24), and (4.25) when we consider $\varphi(\mathbf{x}_d)$ as the initial condition in the Cauchy problem (4.1). Furthermore, since $\varphi(\mathbf{x}_d) \in \mathcal{S}(\mathbb{R}^d)$, by Plancharel's theorem, we can conclude that $p_1^v(\mathbf{x}_d, w), p_2^v(\mathbf{x}_d, w) \in L^2(\mathbb{R}^d)$ and then $u_v(\mathbf{x}_d, t) \in L^2(\mathbb{R}^d)$. $\square$

Remark 4.5. *The results (4.16) and (4.26) show that the solution of the Cauchy problem of the fractional EPD equation represents the Erdélyi–Kober integral of the solution of the fractional wave Eq. (4.23). We recall that the Erdélyi–Kober integral is defined as*

$$(I_\alpha^m f)(x) = \frac{m}{\Gamma(\alpha)} \int_0^x (x^m - y^m)^{\alpha-1} y^{m-1} f(y) dy, \quad \alpha > 0, m > 0.$$

For $v = 2$, we obtain as a special case the classical result connecting the fundamental solution of the EPD equation with the D'Alembert solution of the wave equation (see, e.g., Ref. 21).

Moreover, we observe that the solutions of the EPD equation can be obtained by means of a general method based on the transmutation theory, as discussed in Ref. 39. This solution involves an operator containing as a particular case the Erdélyi–Kober integral.

Remark 4.6. *Constructing probabilistic solutions to the Schrödinger equation has been undertaken in Ref. 24. In this paper, the authors study a probabilistic interpretation of the solution to the Cauchy problem,*

$$\frac{\partial u}{\partial t} = \frac{\sigma^2}{2} \Delta u, \quad u(\mathbf{x}_d, 0) = \varphi(\mathbf{x}_d), \quad \mathbf{x}_d \in \mathbb{R}^d, t > 0, \quad (4.27)$$

where σ is a complex number with $\text{Re } \sigma^2 \geq 0$. For $\sigma^2 = i$, Eq. (4.27) becomes the Schrödinger equation. In Theorems 1–3, the authors show that the solution to (4.27) can be written as the mean value of a functional of the Brownian motion $[\mathbf{B}_d(t), t \geq 0]$, that is,

$$u(\mathbf{x}_d, t) = E\varphi(\mathbf{x}_d + \sigma \mathbf{B}_d(t)), \quad (4.28)$$

if φ belongs to a suitable class of functions on the d -dimensional complex space $\mathbb{C}^d$ (depending on $\text{Re } \sigma^2 > 0$ and $\text{Re } \sigma^2 = 0$; see Ref. 24).

In the same spirit of Ref. 24, we suggest a probabilistic interpretation to the solution of the following fractional Cauchy problem:

$$\frac{\partial u_v}{\partial t} = -\sigma^2 (-\Delta)^{v/4} u_v, \quad u_v(\mathbf{x}_d, 0) = \varphi(\mathbf{x}_d), \quad \mathbf{x}_d \in \mathbb{R}^d, t > 0, \quad (4.29)$$

where $\sigma \in \mathbb{C}$ with $\text{Re } \sigma^2 \geq 0$ and $0 < v < 2$. If we write the solution to (4.29) as

$$u_v(\mathbf{x}_d, t) = E\varphi(\mathbf{x}_d + \sigma \mathbf{S}_{v/2}(t)), \quad (4.30)$$

where $\mathbf{S}_{v/2}(t) \stackrel{(law)}{=} \mathbf{B}_d(Y_{v/4}(t))$, for $0 < v < 2$, or $\mathbf{S}_{v/2}(t) \stackrel{(law)}{=} \mathbf{C}_d(Y_{v/2}(t))$, for $0 < v < 1$, we obtain a probabilistic representation of the solution of (4.29) by considering a suitable class of functions φ on the complex space $\mathbb{C}^d$. For $\sigma^2 = \mp i$, (4.29) reduces to (4.22).

We note that there is a huge recent literature on the time and space fractional Schrödinger equation (including the non-linear case). See, for example, the recent papers.^{40,41} Furthermore, for some connections between the generalized Schrödinger equation and Lévy processes, see Ref. 42.

V. A RELATIONSHIP BETWEEN THE FRACTIONAL EPD EQUATION AND THE PME

The solution to the fractional EPD equation (4.1), after a suitable time-change, can be related to some form of fractionalized PME as we now show. By changing the time scale as $t = t'^\beta$ (and for the reader's convenience, we indicate the new time coordinate t' by t) and by setting $c = 1/\sqrt{B}$, we are able to write (4.26) as follows:

$$u_v(\mathbf{x}_d, t) := u_v(\mathbf{x}_d, t^\beta) = \int_{-\frac{t^\beta}{\sqrt{B}}}^{\frac{t^\beta}{\sqrt{B}}} g(w, t) \left(\frac{p_1^v(\mathbf{x}_d, w) + p_2^v(\mathbf{x}_d, w)}{2} \right) dw, \quad (5.1)$$

where

$$g(w, t) := g(w, t^\beta) = \frac{\Gamma(\gamma + \frac{d}{2})B}{\sqrt{\pi}\Gamma(\frac{d}{2} + \gamma - \frac{1}{2})t^\beta} \left(1 - \frac{Bw^2}{t^{2\beta}} \right)_+^{\frac{d}{2} + \gamma - \frac{1}{2} - 1} \quad (5.2)$$

is the Barenblatt solution to the one-dimensional PME with $m = 1 + \frac{2}{d+2\gamma-3}$, $\beta = \frac{1}{1+m}$ and $B = \frac{m-1}{2m(m+1)}$. The parameter m is strictly greater than 1 for all $d \geq 3$. For $d = 1$ and $d = 2$, $m > 1$ if $\gamma > 1$ and $\gamma > \frac{1}{2}$, respectively. As shown in Theorem 1, $g(w, t)$ is related to some type of random flights. Therefore, the next result allows us to connect the fractional EPD equation with the PME by means of the function $g(w, t)$.

We have now the following theorem:

Theorem 6. *The function $u_v(\mathbf{x}_d, t)$ solves the following fractional equation:*

$$\frac{\partial u_v(\mathbf{x}_d, t)}{\partial t} = -(-\Delta)^{v/2} \int_{-\frac{t^\beta}{\sqrt{B}}}^{\frac{t^\beta}{\sqrt{B}}} (g(w, t))^m \left(\frac{p_1^v(\mathbf{x}_d, w) + p_2^v(\mathbf{x}_d, w)}{2} \right) dw, \quad (5.3)$$

with $0 < v \leq 2$.

Proof. Since $g(\pm \frac{t^\beta}{\sqrt{B}}, t) = 0$, from (5.1), we have that

$$\frac{\partial u_v(\mathbf{x}_d, t)}{\partial t} = \int_{-\frac{t^\beta}{\sqrt{B}}}^{\frac{t^\beta}{\sqrt{B}}} \frac{\partial g(w, t)}{\partial t} \left(\frac{p_1^v(\mathbf{x}_d, w) + p_2^v(\mathbf{x}_d, w)}{2} \right) dw.$$

Therefore, bearing in mind that g is the fundamental solution of the PME (1.1) for $d = 1$, the following equality holds:

$$\begin{aligned} \frac{\partial u_v(\mathbf{x}_d, t)}{\partial t} &= \int_{-\frac{t^\beta}{\sqrt{B}}}^{\frac{t^\beta}{\sqrt{B}}} \frac{\partial^2 (g(w, t))^m}{\partial w^2} \left(\frac{p_1^v(\mathbf{x}_d, w) + p_2^v(\mathbf{x}_d, w)}{2} \right) dw \\ &= -\frac{1}{2} \int_{-\frac{t^\beta}{\sqrt{B}}}^{\frac{t^\beta}{\sqrt{B}}} \frac{\partial (g(w, t))^m}{\partial w} \left(\frac{\partial p_1^v(\mathbf{x}_d, w)}{\partial w} + \frac{\partial p_2^v(\mathbf{x}_d, w)}{\partial w} \right) dw, \end{aligned} \quad (5.4)$$

where in the last step, we have used the fact that $\frac{\partial g^m(\pm \frac{t^\beta}{\sqrt{B}}, t)}{\partial w} = 0$ and the quantities $p_1^v(\mathbf{x}_d, \pm \frac{t^\beta}{\sqrt{B}})$ and $p_2^v(\mathbf{x}_d, \pm \frac{t^\beta}{\sqrt{B}})$ are bounded. Since p_1^v and p_2^v satisfy Eqs. (4.24) and (4.25), respectively, we can write (5.4) as

$$\begin{aligned} \frac{\partial u_v(\mathbf{x}_d, t)}{\partial t} &= -\frac{1}{2} \int_{-\frac{t^\beta}{\sqrt{B}}}^{\frac{t^\beta}{\sqrt{B}}} \frac{\partial (g(w, t))^m}{\partial w} \left(i(-\Delta)^{v/4} p_1^v(\mathbf{x}_d, w) - i(-\Delta)^{v/4} p_2^v(\mathbf{x}_d, w) \right) dw \\ &= -\frac{i(-\Delta)^{v/4}}{2} \int_{-\frac{t^\beta}{\sqrt{B}}}^{\frac{t^\beta}{\sqrt{B}}} \frac{\partial (g(w, t))^m}{\partial w} (p_1^v(\mathbf{x}_d, w) - p_2^v(\mathbf{x}_d, w)) dw, \end{aligned} \quad (5.5)$$

where the last step follows from the following observation: since $p_1^v(\mathbf{x}_d, w), p_2^v(\mathbf{x}_d, w) \in L^2(\mathbb{R}^d)$, then the integral appearing in the second line of (5.5) belongs to $L^2(\mathbb{R}^d)$. A further integration by parts in (5.5) yields

$$\begin{aligned} \frac{\partial u_v(\mathbf{x}_d, t)}{\partial t} &= \frac{i(-\Delta)^{v/4}}{2} \int_{-\frac{t^\beta}{\sqrt{B}}}^{\frac{t^\beta}{\sqrt{B}}} (g(w, t))^m \left(\frac{\partial p_1^v(\mathbf{x}_d, w)}{\partial w} - \frac{\partial p_2^v(\mathbf{x}_d, w)}{\partial w} \right) dw \\ &= -(-\Delta)^{v/4} (-\Delta)^{v/4} \int_{-\frac{t^\beta}{\sqrt{B}}}^{\frac{t^\beta}{\sqrt{B}}} (g(w, t))^m \left(\frac{p_1^v(\mathbf{x}_d, w) + p_2^v(\mathbf{x}_d, w)}{2} \right) dw, \end{aligned}$$

where the last step is justified analogously to (5.5). Finally, by applying the semigroup property for the fractional Laplace operator, we obtain the result (5.3). $\square$

Remark 5.1. In the special case where $\nu = 2$ and $\varphi(\mathbf{x}_d) = \delta(\mathbf{x}_d)$, we obtain the PME. Indeed, in this case, $p_1^2(\mathbf{x}_d, w) = \delta(\|\mathbf{x}_d\| + w)$ and $p_2^2(\mathbf{x}_d, w) = \delta(\|\mathbf{x}_d\| - w)$, and then

$$\begin{aligned}\frac{\partial u_2(\mathbf{x}_d, t)}{\partial t} &= \Delta \int_{-\frac{\beta}{\sqrt{\beta}}}^{\frac{\beta}{\sqrt{\beta}}} (g(w, t))^m \left(\frac{\delta(\|\mathbf{x}_d\| - w) + \delta(\|\mathbf{x}_d\| + w)}{2} \right) dw \\ &= \Delta \left(\frac{(g(\|\mathbf{x}_d\|, t))^m + (g(-\|\mathbf{x}_d\|, t))^m}{2} \right) \\ &= \Delta (u_2(\mathbf{x}_d, t))^m.\end{aligned}$$

VI. SOLUTIONS TO THE HIGHER-ORDER EPD EQUATIONS

We now turn back to the solution (4.1) of the EPD equation and consider the case where $\nu > 2, d = 1$. Our starting point here is the fractional equation

$$\left(\frac{\partial}{\partial t} - i \frac{\partial^\nu}{\partial |\mathbf{x}|^\nu} \right) \left(\frac{\partial}{\partial t} + i \frac{\partial^\nu}{\partial |\mathbf{x}|^\nu} \right) u = 0, \quad (6.1)$$

which, for integer values of $\nu = n, n \in \mathbb{N}$ and $c_n > 0$, reduces to the form

$$\frac{\partial^2 u}{\partial t^2} + c_n \frac{\partial^{2n} u}{\partial x^{2n}} = 0. \quad (6.2)$$

Equation (6.2) for $n = 2, c_n = 1$ becomes the famous equation of vibrations of rods.

The solutions of

$$\begin{cases} \left(\frac{\partial}{\partial t} - c_n^{1/2} \frac{\partial^n}{\partial x^n} \right) u = 0, \\ \left(\frac{\partial}{\partial t} + c_n^{1/2} \frac{\partial^n}{\partial x^n} \right) u = 0, \end{cases} \quad (6.3)$$

are solutions of

$$\left(\frac{\partial}{\partial t} - c_n^{1/2} \frac{\partial^n}{\partial x^n} \right) \left(\frac{\partial}{\partial t} + c_n^{1/2} \frac{\partial^n}{\partial x^n} \right) u = 0, \quad (6.4)$$

which, for special values of c_n , yield higher-order Schrödinger equations. Clearly, the Fourier transforms of (6.3) become

$$\begin{cases} \frac{\partial \hat{u}}{\partial t} = c_n^{1/2} (-i\xi)^n \hat{u}, \\ \frac{\partial \hat{u}}{\partial t} = -c_n^{1/2} (-i\xi)^n \hat{u} \end{cases}$$

so that

$$\hat{u}(\xi, t) = \frac{1}{2} \left(e^{c_n^{1/2} (-i\xi)^n t} + e^{-c_n^{1/2} (-i\xi)^n t} \right) = \cos(c_n^{1/2} e^{-in\frac{\pi}{2}} |\xi|^n t)$$

solves the equation emerging from the Fourier transform of (6.4) with the initial condition $u(x, 0) = \delta(x)$. The solution u takes the form

$$\begin{aligned}u(x, t) &= \frac{1}{2\pi} \int_{-\infty}^{+\infty} e^{-i\xi x} \cos(c_n^{1/2} e^{-in\frac{\pi}{2}} |\xi|^n t) d\xi \\ &= \frac{1}{\pi} \int_0^{+\infty} \cos(\xi x) \cos(c_n^{1/2} e^{-in\frac{\pi}{2}} |\xi|^n t) d\xi.\end{aligned} \quad (6.5)$$

With the choice of $c_n^{1/2} = e^{in\pi/2} = (-1)^{n/2}$, the solution (6.5) reduces to

$$u(x, t) = \frac{1}{\pi} \int_0^{+\infty} \cos(\xi x) \cos(|\xi|^n t) d\xi. \quad (6.6)$$

For $n = 2$, the integral (6.6) yields

$$u(x, t) = \frac{1}{\pi} \int_0^{+\infty} \cos(\xi x) \cos(|\xi|^2 t) d\xi = \frac{1}{\sqrt{2\pi t}} \cos\left(\frac{x^2}{2t} - \frac{\pi}{4}\right) \quad (6.7)$$

and represents the fundamental solution to the equation of vibrations of rods.

Pseudoprocesses related to higher-order heat equations $\frac{\partial u}{\partial t} = c_n \frac{\partial^n u}{\partial x^n}$ have been introduced in several papers, mimicking the construction of the Wiener measure. Thus, a signed measure P is constructed on a set of real-valued functions $t : t \in [0, \infty) \rightarrow x(t)$ called the sample paths of the process, in a manner similar to that described, for instance, in Refs. 43 and 44. In particular, for cylinder sets C of the form

$$C := \{x : a_j \leq x(t) \leq b_j, j = 1, 2, \dots, n\}, \quad 0 < t_1 < \dots < t_j < \dots < t_n,$$

the measure P is defined by

$$P(C) = \int_{a_1}^{b_1} \dots \int_{a_n}^{b_n} \prod_{j=1}^n u(x_j - x_{j-1}, t_j - t_{j-1}) dx_j \quad (6.8)$$

where $t_0 = 0, x_0 = 0$, and $x_j := x(t_j)$. The measure P is countably additive on the field of sets generated by $x(t_j), j = 1, 2, \dots, n$, for fixed t_j and n finite. Pseudoprocesses related to (6.2) for $n = 2$ and $c_n = -1/4$ have been studied in Ref. 45, and the corresponding measure on cylinder sets becomes

$$P(C) = \int_{a_1}^{b_1} \dots \int_{a_n}^{b_n} \frac{(2\pi)^{-n/2}}{\prod_{j=1}^n \sqrt{t_j - t_{j-1}}} \cos\left(\sum_{j=1}^n \frac{(x_j - x_{j-1})^2}{2(t_j - t_{j-1})} - n\frac{\pi}{4}\right) dx_j.$$

For $n = 3, c_n = \mp 1$, from (6.6), we have that

$$\begin{aligned} u(x, t) &= \frac{1}{\pi} \int_0^{+\infty} \cos(\xi x) \cos(|\xi|^3 t) d\xi \\ &= \frac{1}{2\pi} \int_0^{+\infty} [\cos(\xi x + \xi^3 t) + \cos(\xi x - \xi^3 t)] d\xi \\ &= \frac{1}{2(3t)^{1/3}} \left[Ai\left(\frac{x}{(3t)^{1/3}}\right) + Ai\left(-\frac{x}{(3t)^{1/3}}\right) \right], \end{aligned}$$

where $Ai(x) := \frac{1}{\pi} \int_0^\infty \cos\left(\frac{t^3}{3} + xt\right) dt$ is the Airy function of first order.

In light of Theorem 4, the solution to the higher-order EPD equation,

$$\frac{\partial^2 u}{\partial t^2} + \frac{2\lambda}{t} \frac{\partial u}{\partial t} = c_n \frac{\partial^{2n} u}{\partial x^{2n}}, \quad \lambda > 0, \quad (6.9)$$

is the law of the composition $X_{2n}(T(t))$, where $[X_{2n}(t), t \geq 0]$ is the pseudoprocess whose signed measure is

$$u(x, t) = \frac{1}{\pi} \int_0^{+\infty} \cos(\xi x) \cos(|\xi|^{2n} t) d\xi, \quad (6.10)$$

and $[T(t), t \geq 0]$ is the telegraph process having the probability density function given by the solution $g(x, t)$ to the one-dimensional EPD equation. For a more detailed description of pseudoprocesses, see, for example, Ref. 26. We were able to obtain explicitly (6.10) in the special cases $n = 2, 3$, as shown above.

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APPENDIX: DETAILS ON THE DERIVATION OF THE KOMPANETS-ZEL'DOVICH-BARENBLATT SOLUTION

In the derivation of the Kompanets–Zel'dovich–Barenblatt solution (2.2), one must consider a function

$$f(\mathbf{x}_d, t) = t^\delta \left(1 - B \frac{\|\mathbf{x}_d\|^2}{t^\eta} \right)_+^\gamma, \quad (A1)$$

where δ, η, γ , and B represent the real constant, and compare the following derivatives for $f := f(\mathbf{x}_d, t)$:

$$\begin{aligned}\frac{\partial f}{\partial t} &= t^{\delta-1}(\delta - \eta\gamma) \left(1 - B \frac{\|\mathbf{x}\|^2}{t^\eta}\right)_+^\gamma + \gamma\eta t^{\delta-1} \left(1 - B \frac{\|\mathbf{x}\|^2}{t^\eta}\right)_+^{\gamma-1}, \\ \Delta(f^m) &= -2Bt^{m\delta-\eta}\gamma m \left(1 - B \frac{\|\mathbf{x}\|^2}{t^\eta}\right)_+^{m-1} (d + 2(\gamma m - 1)) \\ &\quad + \gamma m(\gamma m - 1)4Bt^{m\delta-\eta} \left(1 - B \frac{\|\mathbf{x}\|^2}{t^\eta}\right)_+^{m-2}.\end{aligned}$$

In order to prove that (A1) satisfies (1.1), we have to impose that

$$\begin{cases} \gamma = \gamma m - 1, \\ \delta - 1 = m\delta - \eta, \\ (\delta - \eta\gamma) = -\gamma m 2B(d + 2(\gamma m - 1)), \\ \gamma\eta = \gamma m(\gamma m - 1)4B, \end{cases} \quad (\text{A2})$$

from which we derive

$$\gamma = \frac{1}{m-1}, \quad \delta = -\frac{d}{2+d(m-1)}, \quad \eta = \frac{2}{2+d(m-1)}, \quad B := \frac{\alpha(m-1)}{2md} = \frac{m-1}{2m(2+d(m-1))}.$$

REFERENCES

- ¹J. L. Vazquez, "The porous medium equation," in *Mathematical Theory*, Oxford Mathematical Monographs (Oxford Univ. Press, Oxford, 2007).
- ²C. R. Cattaneo, "Sur une forme de l' equation de la chaleur éliminant le paradoxe d'une propagation instantanée," *C. R. Acad. Sci.* **247**, 431–433 (1958).
- ³J. L. Vázquez, "Fundamental solution and long time behavior of the Porous Medium Equation in hyperbolic space," *J. Math. Pures Appl.* **104**, 454–484 (2015).
- ⁴J. Vázquez, "Barenblatt solutions and asymptotic behaviour for a nonlinear fractional heat equation of porous medium type," *J. Eur. Math. Soc.* **16**, 769–803 (2014).
- ⁵P. Biler, C. Imbert, and G. Karch, "The nonlocal porous medium equation: Barenblatt profiles and other weak solutions," *Arch. Ration. Mech. Anal.* **215**, 497–529 (2015).
- ⁶Ya. B. Zel'dovich and A. S. Kompanets, "Towards a theory of heat conduction with thermal conductivity depending on the temperature," in *Collection of Papers Dedicated to the 70th Birthday of A. F. Ioffe* (Izd. Akad. Nauk SSSR, Moscow, 1950), pp. 61–72.
- ⁷G. I. Barenblatt, "On some unsteady motions of a liquid or a gas in a porous medium," *Prikl. Mat. Mekh.* **16**, 67–78 (1952) (in Russian).
- ⁸R. E. Pattle, "Diffusion from an instantaneous point source with a concentration-dependent coefficient," *Q. J. Mech. Appl. Math.* **12**, 407–409 (1959).
- ⁹D. W. Stroock, "Some stochastic processes which arise from a model of the motion of a bacterium," *Z. Wahrscheinlichkeitstheorie Verw. Gebi.* **28**, 303–315 (1974).
- ¹⁰W. Stadje, "The exact probability distribution of a two-dimensional random walk," *J. Stat. Phys.* **46**, 207–216 (1987).
- ¹¹W. Stadje, "Exact probability distributions for noncorrelated random walk models," *J. Stat. Phys.* **56**, 415–435 (1989).
- ¹²A. Di Crescenzo, "Exact transient analysis of a planar random motion with three directions," *Stoch. Stoch. Rep.* **72**, 175–189 (2002).
- ¹³E. Orsingher and A. De Gregorio, "Random flights in higher spaces," *J. Theor. Probab.* **20**, 769–806 (2007).
- ¹⁴G. Le Caër, "A Pearson random walk with steps of uniform orientation and Dirichlet distributed lengths," *J. Stat. Phys.* **140**, 728–751 (2010).
- ¹⁵G. Le Caër, "A new family of solvable Pearson–Dirichlet random walks," *J. Stat. Phys.* **144**, 23–45 (2011).
- ¹⁶A. De Gregorio, "On random flights with non-uniformly distributed directions," *J. Stat. Phys.* **147**, 382–411 (2012).
- ¹⁷A. De Gregorio and E. Orsingher, "Flying randomly in $\mathbb{R}^d$ with Dirichlet displacements," *Stoch. Process. Appl.* **122**, 676–713 (2012).
- ¹⁸A. A. Pogorui and R. M. Rodríguez-Dagnino, "Random motion with gamma steps in higher dimensions," *Stat. Probab. Lett.* **83**, 1638–1643 (2013).
- ¹⁹A. P. Ghosh, R. Rastegar, and A. Roitershtein, "On a directionally reinforced random walk," *Proc. Am. Math. Soc.* **142**, 3269–3283 (2014).
- ²⁰R. Garra and E. Orsingher, "Random flights governed by Klein-Gordon-type partial differential equations," *Stoch. Process. Appl.* **124**, 2171–2187 (2014).
- ²¹R. Garra and E. Orsingher, "Random flights related to the Euler-Poisson-Darboux equation," *Markov Process. Rel. Fields* **22**, 87–110 (2016).
- ²²M. Kwasnicki, "Ten equivalent definitions of the fractional Laplace operator," *Fract. Calculus Appl. Anal.* **20**, 7–51 (2017).
- ²³E. Orsingher and B. Toaldo, "Space-time fractional equations and the related stable processes at random time," *J. Theor. Probab.* **30**, 1–26 (2017).
- ²⁴I. A. Ibragimov, N. V. Smorodina, and M. M. Faddeev, "A limit theorem on convergence of random walk functionals to a solution of the Cauchy problem for the equation $\frac{\partial u}{\partial t} = \frac{\sigma^2}{2}u$ with complex σ ," *J. Math. Sci.* **206**, 171–180 (2015).
- ²⁵I. A. Ibragimov, N. V. Smorodina, and M. M. Faddeev, "Limit theorems for symmetric random walks and probabilistic approximation of the Cauchy problem solution for Schrödinger type evolution equations," *Stoch. Process. Appl.* **125**, 4455–4472 (2015).
- ²⁶A. Lachal, "Distributions of sojourn time, maximum and minimum for pseudo-processes governed by higher-order heat-type equations," *Electron. J. Probab.* **20**, 53 (2003).
- ²⁷A. De Gregorio, "Stochastic models associated to a nonlocal porous medium equation," *Mod. Stoch.: Theory Appl.* **5**, 457–470 (2019).
- ²⁸A. De Gregorio, E. Orsingher, and L. Sakhno, "Motions with finite velocity analyzed with order statistics and differential equations," *Theory Probab. Math. Stat.* **71**, 63–79 (2005).

- ²⁹M. Inoue, "A Markov process associated with a porous medium equation," *Proc. Jpn. Acad. Ser. A* **60**, 157–160 (1989).
- ³⁰M. Inoue, "Construction of diffusion processes associated with a porous medium equation," *Hiroshima Math. J.* **19**, 281–297 (1989).
- ³¹M. Inoue, "Derivation of a porous medium equation from many Markovian particles and the propagation of chaos," *Hiroshima Math. J.* **21**, 85–110 (1991).
- ³²S. Benachour, P. Chassaing, B. Roynette, and P. Vallois, "Processus associés à l'équation des milieux poreux," *Ann. Scuola Super. Pisa* **4**, 793–832 (1996).
- ³³I. S. Gradshteyn and I. M. Ryzhik, *Tables of Integrals, Series, and Products*, 4th ed. (Academic Press, New York, 1980).
- ³⁴D. Applebaum, *Lévy Process and Stochastic Calculus*, Cambridge Studies in Advanced Mathematics Vol. 116, 2nd ed. (Cambridge University Press, Cambridge, 2009), p. 460.
- ³⁵E. Di Nezza, G. Palatucci, and E. Valdinoci, "Hitchhiker's guide to the fractional Sobolev spaces," *Bull. Sci. Math.* **136**, 521–573 (2012).
- ³⁶S. K. Foong and U. van Kolck, "Poisson random walk for solving wave equations," *Prog. Theor. Phys.* **87**, 285–292 (1992).
- ³⁷A. V. Glushak and E. Orsingher, "General solution of a hyperbolic equation that arises in the study of motion with random acceleration," *Theory Probab. Math. Stat.* **55**, 49–53 (1997).
- ³⁸N. N. Lebedev, *Special Functions and Their Applications* (Dover Publication, Inc., New York, 1972).
- ³⁹E. L. Shishkina and S. M. Sitnik, "General form of the Euler-Poisson-Darboux equation and application of the transmutation method," *Electron. J. Differ. Equations* **2017**(177), 1–20.
- ⁴⁰F. D. M. Bezerra, A. N. Carvalho, T. Dlotko, and M. J. D. Nascimento, "Fractional Schrödinger equation; solvability and connection with classical Schrödinger equation," *J. Math. Anal. Appl.* **457**, 336–360 (2018).
- ⁴¹X. Su, S. Zhao, and M. Li, "Dispersive estimates for time and space fractional Schrödinger equations," *Math. Methods Appl. Sci.* (unpublished).
- ⁴²N. Cufaro Petroni and M. Pusterla, "Lévy processes and Schrödinger equation," *Physica A* **388**, 824–836 (2009).
- ⁴³V. Yu. Krylov, "Some properties of the distribution corresponding to the equation $\frac{\partial u}{\partial t} = (-1)^{p+1} \frac{\partial^{2p} u}{\partial x^{2p}}$," *Sov. Math. Dokl.* **1**, 260–263 (1960).
- ⁴⁴Yu. L. Daletsky and S. V. Fomin, "Generalized measures in function spaces," *Theory Probab. Appl.* **10**, 304–316 (1965).
- ⁴⁵E. Orsingher and M. D'Ovidio, "Vibrations and fractional vibrations of rods, plates and Fresnel pseudo-processes," *J. Stat. Phys.* **145**, 143–174 (2011).