WELL-POSEDNESS FOR A MODIFIED BIDOMAIN MODEL DESCRIBING BIOELECTRIC ACTIVITY IN DAMAGED HEART TISSUES

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Abstract. We prove the existence and the uniqueness of a solution for a modified bidomain model, describing the electrical behaviour of the cardiac tissue in pathological situations. The main idea is to reduce the problem to an abstract parabolic setting, which requires to introduce several auxiliary differential systems and a non-standard bilinear form. The main difficulties are due to the degeneracy of the bidomain system and to its non-standard coupling with the diffusion equation.

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:introduction

1. Introduction

In the last years, the mathematical modeling of the electrical activity of the heart was a topic of major interest in biomedical research. A better understanding of the complex bioelectrical processes involved in the activity of the heart is a key issue in order to find new drugs and diagnostic techniques, being well-known that a huge part of the heart diseases is produced by some disorders of its electrical activity.
One of the most well-known mathematical models in cardiac electrophysiology is the so-called *bidomain model* (see, e.g., [13, 22, 23] and the references therein; see, also, the references quoted in [14, Introduction]). In this model, at a macroscopic scale,

the electric activity of the heart is governed by a system of two degenerate reaction-diffusion partial differential equations for the averaged intra-cellular and, respectively, extra-cellular electric potentials, along with the transmembrane potential, coupled in a nonlinear manner to ordinary differential equations describing the dynamics of the ion channels. In such a model, the cardiac tissue is represented, at a macroscopic scale, despite its discrete cellular structure, as the superposition of two continuous media, called the *intra-cellular* and, respectively, the *extra-cellular domain*, coexisting at each point of the heart tissue and connected through a distributed continuous cellular membrane.
Several ionic models are considered in the literature for describing the cellular membrane dynamics, starting with the famous Hodgkin-Huxley formalism and continuing with more and more complex models (see, for instance, [14, 17, 18, 22]). The well-posedness of the bidomain model has been studied, for different nonlinear ionic models and by using different techniques, by several authors (see, for instance, [6, 11, 24, 26, 28]).
The bidomain model can be obtained from a corresponding appropriate microscopic one by homogenization techniques (see, for instance, [2, 3, 5, 11, 14, 19, 23, 24, 27]) Despite the fact that the bidomain model is widely recognized as being the standard model used in cardiac electrophysiology, it has some limitations (see [11, 16]). In particular, the bidomain model is suitable for describing the propagation of the action potential in a perfectly healthy cardiac tissue, but it is no longer valid (even if one tries to *ad-hoc* modify some of its relevant modeling parameters) in pathological situations, in which the heart contains electrically passive zones of fibrotic tissue, collagen or fat, as observed for instance in scars, inflammations, ischemic or rheumatic heart diseases, etc. Thus, it is important to find a suitable mathematical model that accounts for the presence of pathological zones in the heart. Such a model was proposed in [7, 9, 15, 16, 30]; it takes into account the presence in the cardiac tissue of damaged zones, called *diffusive inclusions* and assumed to be passive electrical conductors.
From a mathematical point of view, we have a bidomain system coupled with a diffusion equation. More precisely, the model consists of a degenerate reaction-diffusion system of partial differential equations modeling the intra-cellular and, respectively, the extra-cellular electric potentials of the healthy cardiac tissue, coupled with an elliptic equation for the passive regions and with an ordinary differential equation describing the cellular membrane dynamics. Such a model arises also in coupling the torso and the heart (see, e.g., [7, 10, 28]).
We point out that in all the above mentioned papers a perfect electrical coupling between the healthy part of the heart and the damaged tissue was assumed. More general coupling conditions were proposed in [10] and investigated through numerical simulations in [8, 30], in order to take into account the possible capacitive and resistive effects of the surface of the diffusive inclusions . However, up to our knowledge, there are no rigorous proofs in the literature covering such results.
The goal of the present paper is to study the well-posedness of such a *modified bidomain model*. We include the structural defects of the heart tissue in this model, by coupling a standard bidomain system in the healthy zone with a diffusion equation

posed in the damaged part of the heart, through non-standard imperfect transmission conditions (see equations (2.16)–(2.21)).
In order to describe the dynamic of the membrane, one can use a physiological ionic model or a phenomenological one (see, for instance, [14]). In this paper, the dynamic of the gating variable modeling the ionic transport through the cell membrane is described with the aid of a Hodgkin-Huxley type formalism. We can also include in our analysis the modified Mitchell-Schaeffer formalism proposed in [16] (see Remark 2.2).
We point out again that our model generalizes the modified bidomain model with diffusive inclusions and perfect transmission conditions considered in [16, 17], the original model being recovered by suitably rearranging the parameters appearing in equation (2.21).
The mathematical problem we address here is rather non-standard and, up to our knowledge, the proof of its well-posedness is new in the literature and generates difficulties due to the degeneracy of the bidomain system and its non-standard coupling with the diffusion equation.
Our main result is contained in Theorems 3.4 and 3.6, where the leading idea is to reduce the problem to an abstract parabolic setting (see [12, 25]). This requires to introduce several auxiliary differential systems and a non-standard bilinear form (see Proposition 3.3).
The problem proposed here can be seen as a mesoscopic model which will be analyzed in the homogenization limit in a forthcoming paper.

The paper is organized as follows: in Section 2, we introduce the mathematical description of our modified bidomain model, together with its geometrical and functional setting. In Section 3, we state and prove our main result.

2. The model

2.1. **Geometrical setting.** Let $N \geq 3$. Let Ω be an open connected bounded subset of $\mathbb{R}^N$; we assume that $\partial\Omega$ is of class $\mathcal{C}^\infty$, though this assumption can be weakened. Moreover, for $T > 0$, we set $\Omega_T = \Omega \times (0, T)$. We assume that $\Omega = \Omega^D \cup \Omega^B \cup \Gamma$, where Ω^D and Ω^B are two disjoint open subsets of Ω, and $\Gamma = \partial\Omega^D \cap \Omega = \partial\Omega^B \cap \Omega$. The domain Ω is occupied by the cardiac tissue, Ω^B represents the healthy part of the heart tissue, modeled with the aid of a standard bidomain system, Ω^D represents the diffusive region, accounting for the damaged part of the heart, and Γ is the common boundary of these two regions, assumed to be Lipschitz. From a geometrical point of view, we assume that Ω^B is connected, while Ω^D might be connected or disconnected. Indeed, we will consider two different cases: in the first one (to which we will refer as the *connected/disconnected case*, see Fig.1 on the left), we will assume $\Omega^D \subset\subset \Omega$ and Ω^D is made by a finite number of connected components. In this case, $\Gamma = \partial\Omega^D$ and $\partial\Omega^B \cap \partial\Omega \neq \emptyset$.
In the second case (to which we will refer as the *connected/connected case*, see Fig.1 on the right), we will assume that both Ω^D and Ω^B are connected, with $\partial\Omega^B \cap \partial\Omega \neq \emptyset$

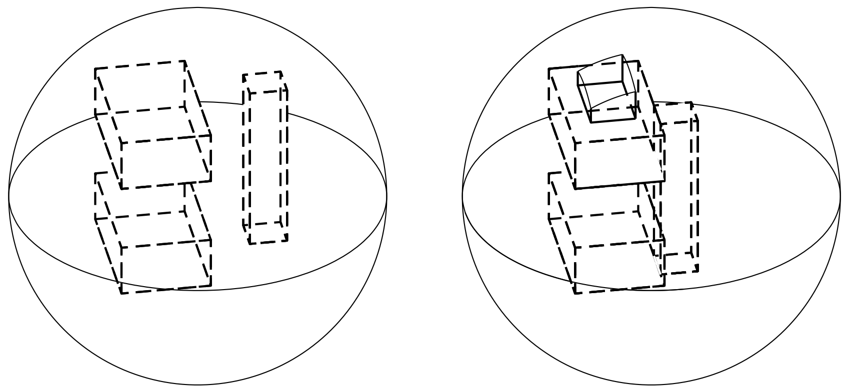

FIGURE 1. *On the left:* the connected/disconnected case. *On the right:* the connected/connected case. fig:mb

and $\partial\Omega^D \cap \partial\Omega \neq \emptyset$. Finally, let ν denote the normal unit vector to Γ pointing into Ω^B.
In the following, by γ we shall denote a strictly positive constant, which may depend on the geometry and vary from line to line.

ss:spaces

2.2. **Functional spaces.** Let us introduce the following functional spaces:

$$\begin{aligned}
&H^1_{null}(\Omega^B) := \{w \in H^1(\Omega^B) : w = 0 \text{ on } \partial\Omega^B \cap \partial\Omega, \text{ in the sense of traces}\};\\
&H^1_{null}(\Omega^D) := \{w \in H^1(\Omega^D) : w = 0 \text{ on } \partial\Omega^D \cap \partial\Omega, \text{ in the sense of traces}\};\\
&H^{1/2}_0(\Gamma, \Omega) := \{r \in H^{1/2}(\Gamma) : r = \widetilde{r}_{|_\Gamma}, \text{ with } \widetilde{r} \in H^1_0(\Omega)\}.
\end{aligned} \tag{2.1}$$

eq:space1

Notice that $H^{1/2}_0(\Gamma, \Omega)$ is a Hilbert space and, in the connected/disconnected case, $H^{1/2}_0(\Gamma, \Omega) = H^{1/2}(\Gamma)$ and $H^1_{null}(\Omega^D) = H^1(\Omega^D)$.
We also set

$$W := H^1_{null}(\Omega^B) \times H^{1/2}_0(\Gamma, \Omega); \qquad H := L^2(\Omega^B) \times L^2(\Gamma), \tag{2.2}$$

eq:space2

where H is endowed with the scalar product

$$\big((w, r), (\overline{w}, s)\big)_H := \int\limits_{\Omega^B} w\overline{w} \,\mathrm{d}x + \alpha \int\limits_{\Gamma} rs \,\mathrm{d}\sigma\,, \tag{2.3}$$

eq:scalar1

where $\alpha > 0$ will be the constant appearing later in (2.21), and W is endowed with the scalar product

$$\big((w, r), (\overline{w}, s)\big)_W := \int\limits_{\Omega^B} \nabla w \cdot \nabla \overline{w} \,\mathrm{d}x + (r, s)_{1/2}\,, \tag{2.4}$$

eq:scalar2

where $(\cdot, \cdot)_{1/2}$ is the standard scalar product on $H^{1/2}(\Gamma)$.
Moreover, we define the space

$$\mathcal{X}^1_0(\Omega) := \{\mathcal{W} : \Omega \to \mathbb{R} \;:\; \mathcal{W}_{|_{\Omega^B}} \in H^1_{null}(\Omega^B),\; \mathcal{W}_{|_{\Omega^D}} \in H^1_{null}(\Omega^D)\}\,, \tag{2.5}$$

eq:a1

endowed with the norm

$$\|\mathcal{W}\|^2_{\mathcal{X}^1_0(\Omega)} := \|\nabla\mathcal{W}\|^2_{L^2(\Omega^B)} + \|\mathcal{W}\|^2_{H^1(\Omega^D)}. \tag{2.6}$$ eq:a3

We recall that $\partial\Omega^B \cap \partial\Omega$ is always non-empty, while $\partial\Omega^D$ can intersect or not the boundary of Ω, depending on the geometry. For $\mathcal{W} \in \mathcal{X}^1_0(\Omega)$, we have the following Poincaré inequality (see [1, Proposition 2]):

$$\|\mathcal{W}\|^2_{L^2(\Omega)} \le \gamma \left(\|\nabla\mathcal{W}\|^2_{L^2(\Omega^B)} + \|\nabla\mathcal{W}\|^2_{L^2(\Omega^D)} + \|[\mathcal{W}]\|^2_{L^2(\Gamma)} \right), \tag{2.7}$$ eq:poincare

where $[\mathcal{W}] = \mathcal{W}_{|\Omega^B} - \mathcal{W}_{|\Omega^D}$ and the last term is not necessary in the connected/connected case. Therefore, an equivalent norm on $\mathcal{X}^1_0(\Omega)$ is given by

$$\|\mathcal{W}\|^2_{\mathcal{X}^1_0(\Omega)} \sim \|\nabla\mathcal{W}\|^2_{L^2(\Omega)} + \|[\mathcal{W}]\|^2_{L^2(\Gamma)}; \tag{2.8}$$ eq:a7

again, the last term can be dropped in the connected/connected case.

ss:position

2.3. **Position of the problem.** Let α, β be strictly positive constants and $\sigma^B_1, \sigma^B_2, \sigma^D$ be measurable functions such that $\gamma_0 \le \sigma^B_1(x), \sigma^B_2(x), \sigma^D(x) \le \widetilde{\gamma}_0$, a.e. in Ω, for suitable strictly positive constants $\gamma_0, \widetilde{\gamma}_0$. The assumption that $\sigma^B_1, \sigma^B_2, \sigma^D$ are scalar functions is used only in Section 3. Removing this assumption is not trivial. If we want to consider general bounded and symmetric matrices satisfying

$$\begin{aligned} &\gamma_0|\zeta|^2 \le \sigma^B_1(x)\zeta\cdot\zeta \le \widetilde{\gamma}_0|\zeta|^2, &&\text{for every } \zeta \in \mathbb{R}^N \text{ and a.e. } x \in \Omega^B;\\ &\gamma_0|\zeta|^2 \le \sigma^B_2(x)\zeta\cdot\zeta \le \widetilde{\gamma}_0|\zeta|^2, &&\text{for every } \zeta \in \mathbb{R}^N \text{ and a.e. } x \in \Omega^B;\\ &\gamma_0|\zeta|^2 \le \sigma^D(x)\zeta\cdot\zeta \le \widetilde{\gamma}_0|\zeta|^2, &&\text{for every } \zeta \in \mathbb{R}^N \text{ and a.e. } x \in \Omega^D, \end{aligned} \tag{2.9}$$ eq:matrix

we have to require some other structural hypotheses as in [11, Lemma 1] and [21, Formula (1)] (see, also, [7] and [17]).

Let us consider a globally Lipschitz continuous function $g : \mathbb{R}^2 \to \mathbb{R}$, such that $g(p, 1) \ge 0$ and $g(p, 0) \le 0$. The example we have in mind here is a function of the form

$$g(p, q) = a(p)(q-1) + b(p)q, \tag{2.10}$$ eq:gating3

where $a, b : \mathbb{R} \to \mathbb{R}$ are positive, bounded and Lipschitz functions. Notice that the form of g in (2.10) is classical in this framework (see, for instance, [28]) and that g is Lipschitz continuous with respect to p and affine with respect to q. Let $I_{\text{ion}} : \mathbb{R}^2 \to \mathbb{R}$ be given by

$$I_{\text{ion}}(p, q) = h_1(p) + h_2(p)q, \tag{2.11}$$ eq:ion3

where h_1, h_2 are Lipschitz continuous functions and h_2 is bounded. Clearly, there exists a positive constant γ_I such that

$$|I_{ion}(p_1, q_1) - I_{ion}(p_2, q_2)| \le \gamma_I(|p_1 - p_2| + |q_1 - q_2|), \qquad \forall (p_1, q_1), (p_2, q_2) \in \mathbb{R}^2. \tag{2.12}$$ eq:stime1

Let $\mathrm{w}_o \in L^\infty(\Omega^B)$, with $0 \le \mathrm{w}_o(x) \le 1$ a.e. in Ω^B and $p \in L^2(\Omega^B_T)$. Consider the gating equation

$$\partial_t\widetilde{w}_p + g(p, \widetilde{w}_p) = 0, \qquad \text{in } \Omega^B_T; \tag{2.13}$$ eq:gating1

$$\widetilde{w}_p(x, 0) = \mathrm{w}_o(x), \qquad \text{in } \Omega^B. \tag{2.14}$$ eq:gating2

Notice that, by classical results, the previous problem admits a unique solution $\widetilde{w}_p \in H^1(0,T;L^\infty(\Omega^B))$ and, from our assumptions, $0 \le \widetilde{w}_p(x,t) \le 1$ a.e. in Ω_T^B, since $0 \le \mathrm{w}_o(x) \le 1$ a.e. in Ω^B (see [20]).
Moreover, from the previous assumptions, we can prove that there exists a strictly positive constant γ_I such that

$$\left\|I_{\text{ion}}(p_1,\widetilde{w}_{p_1}) - I_{\text{ion}}(p_2,\widetilde{w}_{p_2})\right\|_{L^2(\Omega_T^B)} \le \gamma_I \|p_1 - p_2\|_{L^2(\Omega_T^B)}\,, \tag{2.15}$$

eq:ion1

due to the Lipschitz dependence of $\widetilde{w}_p$ on p.

We give here a complete formulation of the problem we shall address in this paper. The operators div and ∇ act only with respect to the space variable x.
Let $f_1, f_2 \in H^1(\Omega_T^B)$, $\overline{v}_0 \in L^2(\Omega^B)$, $s_0 \in L^2(\Gamma)$ and consider the problem for $u_1^B, u_2^B \in L^2(0,T;H^1_{null}(\Omega^B))$, $u^D \in L^2(0,T;H^1_{null}(\Omega^D))$ and $\widetilde{w} \in H^1(0,T;L^\infty(\Omega^B))$, given by

$$\frac{\partial}{\partial t}(u_1^B - u_2^B) - \operatorname{div}(\sigma_1^B \nabla u_1^B) + I_{\text{ion}}(u_1^B - u_2^B, \widetilde{w}) = f_1, \qquad \text{in } \Omega_T^B; \tag{2.16}$$

eq:PDEin

$$\frac{\partial}{\partial t}(u_1^B - u_2^B) + \operatorname{div}(\sigma_2^B \nabla u_2^B) + I_{\text{ion}}(u_1^B - u_2^B, \widetilde{w}) = f_2, \qquad \text{in } \Omega_T^B; \tag{2.17}$$

eq:PDEout

$$-\operatorname{div}(\sigma^D \nabla u^D) = 0, \qquad \text{in } \Omega_T^D; \tag{2.18}$$

eq:PDEdis

$$\sigma_1^B \nabla u_1^B \cdot \nu = 0, \qquad \text{on } \Gamma_T; \tag{2.19}$$

eq:flux1

$$\sigma_2^B \nabla u_2^B \cdot \nu = \sigma^D \nabla u^D \cdot \nu, \qquad \text{on } \Gamma_T; \tag{2.20}$$

eq:flux2

$$\alpha \frac{\partial}{\partial t}(u_2^B - u^D) + \beta(u_2^B - u^D) = \sigma_2^B \nabla u_2^B \cdot \nu, \qquad \text{on } \Gamma_T; \tag{2.21}$$

eq:Circuit

$$u_1^B(x,t), u_2^B(x,t), u^D(x,t) = 0, \qquad \text{on } \partial\Omega \times (0,T); \tag{2.22}$$

eq:BoundDat

$$u_1^B(x,0) - u_2^B(x,0) = \overline{v}_0(x), \qquad \text{in } \Omega^B; \tag{2.23}$$

eq:InitData

$$u_2^B(x,0) - u^D(x,0) = s_0(x), \qquad \text{on } \Gamma, \tag{2.24}$$

eq:InitData

where $\widetilde{w}$ is the solution of the gating equation (2.13), (2.14), with $p = u_1^B - u_2^B$.

r:r3 *Remark* 2.1 (Biological interpretation). The previous system of equations represents the coupling of a standard bidomain model in Ω^B, for the intra and the extra-cellular potentials u_1^B and u_2^B of the healthy zone, with a Poisson equation in the diffusive part Ω^D, for the electrical potential u^D of the damaged zone. The function $u_2^B - u_1^B$ is the so-called transmembrane potential. The sources f_1 and f_2 are the internal and the external current stimulus, respectively. The coefficients σ_1^B, σ_2^B and σ^D are the conductivities of the two healthy phases and of the damaged one, while α and β are given parameters related to the capacitive and the resistive behaviour of the interface Γ. We point out that for the intra-cellular potential u_1^B we assume no flux condition on Γ (see (2.19)), while the extra-cellular potential u_2^B is coupled with the electrical potential u^D of the damaged zone through non-standard imperfect transmission conditions (see (2.20) and (2.21)). Our system is completed with suitable initial and boundary conditions. The variable $\widetilde{w}$, called the *gating variable*, describes the ionic transport through the cell membrane. The terms g and I_{ion} are nonlinear functions, modeling the membrane ionic currents.

For simplicity, we consider only one gating variable, but our results hold true also for the case in which the gating variable is vector valued. □

r:r2 *Remark* 2.2. Different examples of functions I_{ion} and g are considered in the literature. We consider here a Hodgkin-Huxley type model (see (2.10), (2.11)), as in [4, 14, 28]. However, we point out that the results obtained in this paper are also valid for a regularized version of the Mitchell-Schaeffer model proposed in [16] (see, also, [15, 17]). □

By standard approximation procedure, multiplying (2.16) by u_1^B, (2.17) by u_2^B, (2.18) by u^D, subtracting (2.17) from (2.16), adding (2.18), integrating by parts, using (2.19)–(2.24), (2.11), (2.12), (2.15) and moving the integral containing I_{ion} to the right-hand side, we get

$$
\begin{aligned}
&\frac{1}{2}\int\limits_{\Omega^B}(u_1^B-u_2^B)^2(x,T)\,\mathrm{d}x+\int\limits_{\Omega_T^B}\sigma_1^B|\nabla u_1^B|^2\,\mathrm{d}x\,\mathrm{d}t+\int\limits_{\Omega_T^B}\sigma_2^B|\nabla u_2^B|^2\,\mathrm{d}x\,\mathrm{d}t\\
&+\int\limits_{\Omega_T^D}\sigma^D|\nabla u^D|^2\,\mathrm{d}x\,\mathrm{d}t+\frac{\alpha}{2}\int\limits_{\Gamma}(u_2^B-u^D)^2(x,T)\,\mathrm{d}\sigma(x)+\beta\int\limits_{\Gamma_T}(u_2^B-u^D)^2(x,t)\,\mathrm{d}\sigma(x)\,\mathrm{d}t\\
&\qquad=\int\limits_{\Omega_T^B}(f_1u_1^B-f_2u_2^B)\,\mathrm{d}x\,\mathrm{d}t+\frac{1}{2}\int\limits_{\Omega^B}\overline{v}_0^2(x)\,\mathrm{d}x+\frac{\alpha}{2}\int\limits_{\Gamma}s_0^2(x)\,\mathrm{d}\sigma(x)\\
&\qquad\qquad-\int\limits_{\Omega_T^B}I_{\text{ion}}(u_1^B-u_2^B,\widetilde{w})(u_1^B-u_2^B)\,\mathrm{d}x\,\mathrm{d}t\\
&\qquad\leq\int\limits_{\Omega_T^B}(f_1u_1^B-f_2u_2^B)\,\mathrm{d}x\,\mathrm{d}t+\gamma(\|\overline{v}_0\|^2_{L^2(\Omega^B)}+\|\overline{s}_0\|^2_{L^2(\Gamma)})\\
&\qquad\quad-\int\limits_{\Omega_T^B}\Big(I_{\text{ion}}(u_1^B-u_2^B,\widetilde{w})-I_{\text{ion}}(0,w_0)\Big)(u_1^B-u_2^B)\,\mathrm{d}x\,\mathrm{d}t\\
&\qquad\qquad-\int\limits_{\Omega_T^B}I_{\text{ion}}(0,w_0)(u_1^B-u_2^B)\,\mathrm{d}x\,\mathrm{d}t\\
&\qquad\leq\gamma(\|f_1\|^2_{L^2(\Omega_T^B)}+\|f_2\|^2_{L^2(\Omega_T^B)}+\|\overline{v}_0\|^2_{L^2(\Omega^B)}+\|\overline{s}_0\|^2_{L^2(\Gamma)}+1)\\
&\qquad\quad+\frac{\delta}{2}(\|\nabla u_1^B\|^2_{L^2(\Omega_T^B)}+\|\nabla u_2^B\|^2_{L^2(\Omega_T^B)})+\gamma\|u_1^B-u_2^B\|^2_{L^2(\Omega_T^B)}\,,
\end{aligned}
\tag{2.25}
$$
eq:energy1

where γ and δ are positive constants depending on $\gamma_0,\alpha,\gamma_I;\gamma_H,\gamma_F,\gamma_a$, and the geometry, δ can be chosen smaller than $\min(\sigma_1^B,\sigma_2^B)$, and we have also applied Poincaré's inequality to u_1^B and u_2^B. By absorbing into the left-hand side the first two terms in the last line of (2.25) and using Gronwall's inequality, from the previous estimate, we obtain

$$\sup_{t\in(0,T)}\int\limits_{\Omega^B}(u_1^B-u_2^B)^2(x,t)\,\mathrm{d}x+\int\limits_{\Omega_T^B}|\nabla u_1^B|^2\,\mathrm{d}x\,\mathrm{d}t+\int\limits_{\Omega_T^B}|\nabla u_2^B|^2\,\mathrm{d}x\,\mathrm{d}t+\int\limits_{\Omega_T^D}|\nabla u^D|^2\,\mathrm{d}x\,\mathrm{d}t$$
$$+\sup_{t\in(0,T)}\int\limits_{\Gamma}(u_2^B-u^D)^2(x,t)\,\mathrm{d}\sigma(x)+\int\limits_{\Gamma_T}(u_2^B-u^D)^2(x,t)\,\mathrm{d}\sigma(x)\,\mathrm{d}t$$
$$\le\gamma(\|f_1\|^2_{L^2(\Omega_T^B)}+\|f_2\|^2_{L^2(\Omega_T^B)}+\|\overline{v}_0\|^2_{L^2(\Omega^B)}+\|\overline{s}_0\|^2_{L^2(\Gamma)}+1)\,. \quad (2.26)$$ eq:energy5

Proposition 2.3. *Under the assumptions stated above, problem* (2.16)–(2.24) *and* (2.13), (2.14) *admits at most one solution.*

Proof. Assume that $(u_1^B,u_2^B,u^D,\widetilde{w})$ and $(\overline{u}_1^B,\overline{u}_2^B,\overline{u}^D,\overline{\widetilde{w}})$ are two different solutions of problem (2.16)-(2.23), with $\overline{\widetilde{w}}$ being the solution of (2.13) and (2.14), corresponding to $p=\overline{u}_1^B-\overline{u}_2^B$. Setting $\upsilon_1^B:=u_1^B-\overline{u}_1^B,\upsilon_2^B:=u_2^B-\overline{u}_2^B$ and $\upsilon^D:=u^D-\overline{u}^D$, we obtain that $\upsilon_1^B,\upsilon_2^B,\upsilon^D,\widetilde{w}$ and $\overline{\widetilde{w}}$ solve the system

$$\begin{aligned}
&\frac{\partial}{\partial t}(\upsilon_1^B-\upsilon_2^B)-\mathrm{div}(\sigma_1^B\nabla\upsilon_1^B)+I_{\mathrm{ion}}(u_1^B-u_2^B,\widetilde{w})-I_{\mathrm{ion}}(\overline{u}_1^B-\overline{u}_2^B,\overline{\widetilde{w}})=0, && \text{in } \Omega_T^B;\\
&\frac{\partial}{\partial t}(\upsilon_1^B-\upsilon_2^B)+\mathrm{div}(\sigma_2^B\nabla\upsilon_2^B)+I_{\mathrm{ion}}(u_1^B-u_2^B,\widetilde{w})-I_{\mathrm{ion}}(\overline{u}_1^B-\overline{u}_2^B,\overline{\widetilde{w}})=0, && \text{in } \Omega_T^D;\\
&-\mathrm{div}(\sigma^D\nabla\upsilon^D)=0, && \text{in } \Omega_T^D;\\
&\sigma_1^B\nabla\upsilon_1^B\cdot\nu=0, && \text{on } \Gamma_T;\\
&\sigma_2^B\nabla\upsilon_2^B\cdot\nu=\sigma^D\nabla\upsilon^D\cdot\nu, && \text{on } \Gamma_T;\\
&\alpha\frac{\partial}{\partial t}(\upsilon_2^B-\upsilon^D)+\beta(\upsilon_2^B-\upsilon^D)=\sigma_2^B\nabla\upsilon_2^B\cdot\nu, && \text{on } \Gamma_T;\\
&\upsilon_1^B(x,t),\upsilon_2^B(x,t),\upsilon^D(x,t)=0, && \text{on } \partial\Omega\times(0,T);\\
&\upsilon_1^B(x,0)-\upsilon_2^B(x,0)=0, && \text{in } \Omega^B;\\
&\upsilon_2^B(x,0)-\upsilon^D(x,0)=0, && \text{on } \Gamma.
\end{aligned}$$

Reasoning in a similar way as done for (2.25), i.e. by multiplying the first equation by υ_1^B, the second one by υ_2^B, the third one by υ^D, subtracting the second equation from the first one adding the third one, integrating by parts, using the remaining equation of the previous system moving the integral containing I_{ion} in the right-hand side and using Hölder inequality, we get

$$\sup_{t\in(0,T)}\int\limits_{\Omega^B}(\upsilon_1^B-\upsilon_2^B)^2(x,t)\,\mathrm{d}x+\int\limits_{\Omega_T^B}|\nabla\upsilon_1^B|^2\,\mathrm{d}x\,\mathrm{d}t+\int\limits_{\Omega_T^B}|\nabla\upsilon_2^B|^2\,\mathrm{d}x\,\mathrm{d}t+\int\limits_{\Omega_T^D}|\nabla\upsilon^D|^2\,\mathrm{d}x\,\mathrm{d}t$$
$$+\sup_{t\in(0,T)}\int\limits_{\Gamma}(\upsilon_2^B-\upsilon^D)^2(x,t)\,\mathrm{d}\sigma(x)+\int\limits_{\Gamma_T}(\upsilon_2^B-\upsilon^D)^2(x,t)\,\mathrm{d}\sigma(x)\,\mathrm{d}t$$
$$\le\|I_{\mathrm{ion}}(u_1^B-u_2^B,\widetilde{w})-I_{\mathrm{ion}}(\overline{u}_1^B-\overline{u}_2^B,\overline{\widetilde{w}})\|_{L^2(\Omega_T^B)}\cdot\|\upsilon_1^B-\upsilon_2^B\|_{L^2(\Omega_T^B)}\le\gamma\|\upsilon_1^B-\upsilon_2^B\|^2_{L^2(\Omega_T^B)}\,,$$

where, in the last inequality, we used (2.15). We can conclude by using Gronwall inequality. $\square$

Notice that, by setting $V = u_1^B - u_2^B$, $U = u_2^B$ a.e. in Ω^B, $U = u^D$ a.e. in Ω^D, and denoting by $[\cdot]$ the jump across Γ of the quantity in the square brackets, i.e. $[U] = u_2^B - u^D$ and $[\sigma \nabla U \cdot \nu] = (\sigma_2^B \nabla u_2^B - \sigma^D \nabla u^D) \cdot \nu$, the previous system can be written in the more convenient form

$$
\begin{aligned}
&\frac{\partial V}{\partial t} - \operatorname{div}(\sigma_1^B \nabla V) + I_{\text{ion}}(V, \widetilde{w}) = f_1 + \operatorname{div}(\sigma_1^B \nabla U), && \text{in } \Omega_T^B; && (2.27) \quad \boxed{\texttt{eq:PDEinc}}\\
&- \operatorname{div}((\sigma_1^B + \sigma_2^B)\nabla U) = f_1 - f_2 + \operatorname{div}(\sigma_1^B \nabla V), && \text{in } \Omega_T^B; && (2.28) \quad \boxed{\texttt{eq:PDEoutc}}\\
&- \operatorname{div}(\sigma^D \nabla U) = 0, && \text{in } \Omega_T^D; && (2.29) \quad \boxed{\texttt{eq:PDEdisc}}\\
&\sigma_1^B \nabla (V + U) \cdot \nu = 0, && \text{on } \Gamma_T; && (2.30) \quad \boxed{\texttt{eq:flux1c}}\\
&[\sigma \nabla U \cdot \nu] = 0, && \text{on } \Gamma_T; && (2.31) \quad \boxed{\texttt{eq:flux2c}}\\
&\alpha \frac{\partial}{\partial t}[U] + \beta [U] = \sigma_2^B \nabla U \cdot \nu, && \text{on } \Gamma_T; && (2.32) \quad \boxed{\texttt{eq:Circuitc}}\\
&V, U = 0, && \text{on } \partial\Omega \times (0,T); && (2.33) \quad \boxed{\texttt{eq:BoundDat}}\\
&V(x,0) = \overline{v}_0(x), && \text{in } \Omega^B; && (2.34) \quad \boxed{\texttt{eq:InitData}}\\
&[U](x,0) = s_0(x), && \text{on } \Gamma, && (2.35) \quad \boxed{\texttt{eq:InitData}}
\end{aligned}
$$

complemented with the gating problem (2.13), (2.14), where again $u_1^B - u_2^B$ is replaced by V. Clearly, $V \in L^2(0,T; H^1_{null}(\Omega^B))$ and $U \in L^2(0,T; \mathcal{X}_0^1(\Omega))$. We remark that, since F is Lipschitz, the composed function $I_{\text{ion}}(V, \widetilde{w})$ is also a Lipschitz function with respect to V.
The weak formulation of the previous problem is given by

$$
\begin{aligned}
&- \int\limits_{\Omega_T^B} V \partial_t \varphi_B \,\mathrm{d}x\,\mathrm{d}t + \int\limits_{\Omega_T^B} \sigma_1^B \nabla V \cdot \nabla \varphi_B \,\mathrm{d}x\,\mathrm{d}t + \int\limits_{\Omega_T^B} \sigma_1^B \nabla U \cdot \nabla \varphi_B + \int\limits_{\Omega_T^B} I_{\text{ion}}(V, \widetilde{w}) \varphi_B \,\mathrm{d}x\,\mathrm{d}t \\
&\qquad + \int\limits_{\Omega_T^B} (\sigma_1^B + \sigma_2^B) \nabla U \cdot \nabla \varphi_D^1 \,\mathrm{d}x\,\mathrm{d}t + \int\limits_{\Omega_T^B} \sigma_1^B \nabla V \cdot \nabla \varphi_D^1 \,\mathrm{d}x\,\mathrm{d}t \\
&\qquad + \int\limits_{\Omega_T^D} \sigma^D \nabla U \cdot \nabla \varphi_D^2 \,\mathrm{d}x\,\mathrm{d}t - \alpha \int\limits_{\Gamma_T} [U] \partial_t [\varphi_D] \,\mathrm{d}\sigma\,\mathrm{d}t + \beta \int\limits_{\Gamma_T} [U][\varphi_D] \,\mathrm{d}\sigma\,\mathrm{d}t \\
&= \int\limits_{\Omega_T^B} f_1 \varphi_B \,\mathrm{d}x\,\mathrm{d}t + \int\limits_{\Omega_T^B} (f_1 - f_2) \varphi_D^1 \,\mathrm{d}x\,\mathrm{d}t \,, + \int\limits_{\Omega^B} \overline{v}_0 \varphi_B(0) \,\mathrm{d}x + \alpha \int\limits_{\Gamma} s_0 [\varphi_D](0) \,\mathrm{d}\sigma \qquad (2.36) \quad \boxed{\texttt{eq:weak5}}
\end{aligned}
$$

for every $\varphi_B \in L^2(0,T; H^1_{null}(\Omega^B)) \cap H^1(0,T; L^2(\Omega^B))$, $\varphi_D^1 \in L^2(0,T; H^1_{null}(\Omega^B))$, $\varphi_D^2 \in L^2(0,T; H^1_{null}(\Omega^D))$, and $[\varphi_D] \in H^1(0,T; L^2(\Gamma))$, with $\varphi_B(T) = 0$ and $[\varphi_D](T) = 0$. Here, $[\varphi_D] = \varphi_D^1 - \varphi_D^2$ on Γ.

Clearly, by (2.26), we get the following energy inequality:

$$\begin{aligned}\sup_{t\in(0,T)}\int\limits_{\Omega^B} V^2(x,t)\,\mathrm{d}x+\int\limits_{\Omega^B_T}|\nabla V+\nabla U|^2\,\mathrm{d}x\,\mathrm{d}t+\int\limits_{\Omega^B_T}|\nabla U|^2\,\mathrm{d}x\,\mathrm{d}t\\ +\int\limits_{\Omega^D_T}|\nabla U|^2\,\mathrm{d}x\,\mathrm{d}t+\sup_{t\in(0,T)}\int\limits_{\Gamma}[U]^2(x,t)\,\mathrm{d}\sigma+\int\limits_{\Gamma_T}[U]^2\,\mathrm{d}\sigma\,\mathrm{d}t\\ \le\gamma\left(\|f_1\|^2_{L^2(\Omega^B_T)}+\|f_2\|^2_{L^2(\Omega^B_T)}+\|\overline{v}_0\|^2_{L^2(\Omega^B)}+\|s_0\|^2_{L^2(\Gamma)}+1\right),\end{aligned}\tag{2.37}$$

eq:energy3

where γ depends on $\gamma_0,\gamma_I,\alpha,\beta$, and the geometry. Notice that, by (2.37), it follows also that

$$\int\limits_{\Omega^B_T}|\nabla V|^2\,\mathrm{d}x\,\mathrm{d}t\le\gamma\left(\|f_1\|^2_{L^2(\Omega^B_T)}+\|f_2\|^2_{L^2(\Omega^B_T)}+\|\overline{v}_0\|^2_{L^2(\Omega^B)}+\|s_0\|^2_{L^2(\Gamma)}+1\right).\tag{2.38}$$

eq:energy4

s:exist

3. Well-posedness

In this section, we first prove the well-posedness of the problem (2.27)–(2.35) corresponding to $I_{\mathrm{ion}}\equiv 0$; then, the complete problem will be treated as a nonlinear perturbation of this case (see, for instance, [12, 16, 20, 25]).
In order to prove existence and uniqueness for the linear problem, we will consider it in an abstract setting and to this purpose we need "to move" the source f_1-f_2 from (2.28) to (2.27), (2.32), and (2.35). This will be done by considering, for a.e. $t\in(0,T)$, the following auxiliary problem:

$$-\operatorname{div}\left((\sigma_1^B+\sigma_2^B)\nabla\widetilde{u}\right)=f_1-f_2,\qquad \text{in } \Omega^B;\tag{3.1}$$

eq:pde1

$$\sigma_1^B\nabla\widetilde{u}\cdot\nu=0,\qquad \text{on } \Gamma;\tag{3.2}$$

eq:pde2

$$\widetilde{u}=0,\qquad \text{on } \partial\Omega^B\cap\partial\Omega.\tag{3.3}$$

eq:pde3

Clearly, problem (3.1)–(3.3) is classical and admits a unique solution $\widetilde{u}\in H^1(0,T;H^1_{null}(\Omega^B))$. Moreover, we extend $\widetilde{u}$ inside Ω^D by zero, so that it has a nonzero jump on Γ; i.e., $\widetilde{u}\in\mathcal{X}^1_0(\Omega)$. On Γ, let us define

$$q:=-\alpha\frac{\partial[\widetilde{u}]}{\partial t}-\beta[\widetilde{u}]\,,\tag{3.4}$$

eq:a8

and consider the problem for $(v,u) \in L^2(0,T;H^1_{null}(\Omega^B)) \times L^2(0,T;\mathcal{X}^1_0(\Omega))$ given by

$$\frac{\partial v}{\partial t} - \operatorname{div}(\sigma_1^B \nabla v) = f_1 + \operatorname{div}(\sigma_1^B \nabla u) + \operatorname{div}(\sigma_1^B \nabla \widetilde{u}), \qquad \text{in } \Omega_T^B; \tag{3.5}$$
eq:PDEins

$$-\operatorname{div}((\sigma_1^B + \sigma_2^B)\nabla u) = \operatorname{div}(\sigma_1^B \nabla v), \qquad \text{in } \Omega_T^B; \tag{3.6}$$
eq:PDEouts

$$-\operatorname{div}(\sigma^D \nabla u) = 0, \qquad \text{in } \Omega_T^D; \tag{3.7}$$
eq:PDEdiss

$$\sigma_1^B \nabla(v+u)\cdot\nu = 0, \qquad \text{on } \Gamma_T; \tag{3.8}$$
eq:flux1s

$$\sigma_2^B \nabla u \cdot \nu = \sigma^D \nabla u \cdot \nu, \qquad \text{on } \Gamma_T; \tag{3.9}$$
eq:flux2s

$$\alpha \frac{\partial}{\partial t}[u] + \beta [u] = \sigma_2^B \nabla u \cdot \nu + q, \qquad \text{on } \Gamma_T; \tag{3.10}$$
eq:Circuits

$$v, u = 0, \qquad \text{on } \partial\Omega \times (0,T); \tag{3.11}$$
eq:BoundDat

$$v(x,0) = \overline{v}_0(x), \qquad \text{on } \Omega^B; \tag{3.12}$$
eq:InitData

$$[u](x,0) = s_0(x) - [\widetilde{u}](x,0), \qquad \text{on } \Gamma. \tag{3.13}$$
eq:InitData

The weak formulation of previous problem is given by

$$\begin{aligned}
&-\int_{\Omega_T^B} v \partial_t \varphi_B \,\mathrm{d}x\,\mathrm{d}t + \int_{\Omega_T^B} \sigma_1^B \nabla v \cdot \nabla \varphi_B \,\mathrm{d}x\,\mathrm{d}t + \int_{\Omega_T^B} \sigma_1^B \nabla u \cdot \nabla \varphi_B \,\mathrm{d}x\,\mathrm{d}t \\
&\qquad + \int_{\Omega_T^B} (\sigma_1^B + \sigma_2^B) \nabla u \cdot \nabla \varphi_D^1 \,\mathrm{d}x\,\mathrm{d}t + \int_{\Omega_T^B} \sigma_1^B \nabla v \cdot \nabla \varphi_D^1 \,\mathrm{d}x\,\mathrm{d}t \\
&\qquad + \int_{\Omega_T^D} \sigma^D \nabla u \cdot \nabla \varphi_D^2 \,\mathrm{d}x\,\mathrm{d}t - \alpha \int_{\Gamma_T} [u] \partial_t [\varphi_D] \,\mathrm{d}\sigma\,\mathrm{d}t + \beta \int_{\Gamma_T} [u][\varphi_D] \,\mathrm{d}\sigma\,\mathrm{d}t \\
&\qquad = \int_{\Omega_T^B} f_1 \varphi_B \,\mathrm{d}x\,\mathrm{d}t - \int_{\Omega_T^B} \sigma_1^B \nabla \widetilde{u} \cdot \nabla \varphi_B \,\mathrm{d}x\,\mathrm{d}t + \int_{\Gamma_T} q [\varphi_D] \,\mathrm{d}\sigma\,\mathrm{d}t \,,
\end{aligned} \tag{3.14}$$
eq:weak1

for every $\varphi_B \in L^2(0,T;H^1_{null}(\Omega^B)) \cap H^1_0(0,T;L^2(\Omega^B))$, $\varphi_D \in L^2(0,T;\mathcal{X}^1_0(\Omega))$, where, as before, $[\varphi_D] = \varphi_D^1 - \varphi_D^2$ and $[\varphi_D] \in H^1_0(0,T;L^2(\Gamma))$. The weak formulation (3.14) shall be complemented with the initial conditions. Indeed, as it will be proved in Theorem 3.4, we have $v \in \mathcal{C}^0([0,T];L^2(\Omega^B))$ and $[u] \in \mathcal{C}^0([0,T];L^2(\Gamma))$.

The next step is to define a suitable bilinear form on W, which is continuous and coercive. To this purpose, we need the following result.

p:p1 **Proposition 3.1.** *Let $(w,r) \in W$ be assigned. Then, there exists a unique solution $\mathcal{W} \in \mathcal{X}^1_0(\Omega)$ of the problem*

$$-\operatorname{div}((\sigma_1^B + \sigma_2^B)\nabla \mathcal{W}) = \operatorname{div}(\sigma_1^B \nabla w), \qquad \textit{in } \Omega^B; \tag{3.15}$$
eq:PDEuno

$$-\operatorname{div}(\sigma^D \nabla \mathcal{W}) = 0, \qquad \textit{in } \Omega^D; \tag{3.16}$$
eq:PDEdue

$$(\sigma_1^B + \sigma_2^B)\nabla \mathcal{W} \cdot \nu = \sigma^D \nabla \mathcal{W} \cdot \nu - \sigma_1^B \nabla w \cdot \nu, \qquad \textit{on } \Gamma; \tag{3.17}$$
eq:fluxuno

$$[\mathcal{W}] = r, \qquad \textit{on } \Gamma; \tag{3.18}$$
eq:Circuitu

$$\mathcal{W} = 0, \qquad \textit{on } \partial\Omega. \tag{3.19}$$
eq:Bound

Moreover, there exists a constant $\gamma > 0$*, depending on* $\sigma_1^B, \sigma_2^B, \sigma^D$*, and the geometry, such that*

$$\|\mathcal{W}\|_{\mathcal{X}_0^1(\Omega)} \leq \gamma(\|w\|_{H^1(\Omega^B)} + \|r\|_{H^{1/2}(\Gamma)}) \,. \tag{3.20}$$

eq:a4

Proof. Uniqueness for problem (3.15)–(3.19) is a straightforward consequence of its linearity. In order to prove that a solution does exist, we first consider the following auxiliary problem:

$$\begin{aligned}
&-\operatorname{div}((\sigma_1^B + \sigma_2^B)\nabla\overline{\mathcal{W}}_1) = \operatorname{div}(\sigma_1^B \nabla w), && \text{in } \Omega^B; && (3.21)\\
&\overline{\mathcal{W}}_1 = r, && \text{on } \Gamma; && (3.22)\\
&\overline{\mathcal{W}}_1 = 0, && \text{in } \Omega^D; && (3.23)\\
&\overline{\mathcal{W}}_1 = 0, && \text{on } \partial\Omega. && (3.24)
\end{aligned}$$

eq:PDEtre
eq:datauno
eq:PDEtreb
eq:datadue

Clearly, the previous problem admits a unique solution $\overline{\mathcal{W}}_1 \in \mathcal{X}_0^1(\Omega)$. Moreover, there exists a constant $\gamma > 0$, depending on σ_1^B, σ_2^B, such that

$$\|\overline{\mathcal{W}}_1\|_{\mathcal{X}_0^1(\Omega)} \leq \gamma(\|w\|_{H^1(\Omega^B)} + \|r\|_{H^{1/2}(\Gamma)}) \,. \tag{3.25}$$

eq:a5

Indeed, let us denote by $\widetilde{r} \in H^1_{null}(\Omega^B)$ an extension of r from Γ to the whole Ω^B, such that $\|\widetilde{r}\|_{H^1(\Omega^B)} \leq \gamma \|r\|_{H^{1/2}(\Gamma)}$, and set $\overline{\mathcal{W}}_1^r := \overline{\mathcal{W}}_1 - \widetilde{r}$. Clearly, $\overline{\mathcal{W}}_1^r$ satisfies the problem

$$-\operatorname{div}((\sigma_1^B + \sigma_2^B)\nabla\overline{\mathcal{W}}_1^r) = \operatorname{div}(\sigma_1^B \nabla w) + \operatorname{div}((\sigma_1^B + \sigma_2^B)\nabla\widetilde{r}), \quad \text{in } \Omega^B; \tag{3.26}$$

eq:PDEquatt

$$\overline{\mathcal{W}}_1^r = 0, \qquad \text{on } \partial\Omega^B. \tag{3.27}$$

eq:dataquat

Therefore, by the standard energy inequality, we get

$$\int\limits_{\Omega^B} |\nabla\overline{\mathcal{W}}_1^r|^2 \,\mathrm{d}x \leq \gamma \left(\|\nabla w\|^2_{L^2(\Omega^B)} + \|r\|^2_{H^{1/2}(\Gamma)} \right), \tag{3.28}$$

eq:energy2

which implies (3.25), with γ depending only on σ_1^B, σ_2^B.
Now, let us consider the second auxiliary problem for $\overline{\mathcal{W}}_2 \in H_0^1(\Omega)$ given by

$$\begin{aligned}
&-\operatorname{div}((\sigma_1^B + \sigma_2^B)\nabla\overline{\mathcal{W}}_2) = 0, && \text{in } \Omega^B; && (3.29)\\
&-\operatorname{div}(\sigma^D \nabla\overline{\mathcal{W}}_2) = 0, && \text{in } \Omega^D; && (3.30)\\
&[\overline{\mathcal{W}}_2] = 0, && \text{on } \Gamma; && (3.31)\\
&\big((\sigma_1^B + \sigma_2^B)\nabla\overline{\mathcal{W}}_2 - \sigma^D \nabla\overline{\mathcal{W}}_2\big) \cdot \nu = -\big((\sigma_1^B + \sigma_2^B)\nabla\overline{\mathcal{W}}_1 + \sigma_1^B \nabla w\big) \cdot \nu, && \text{on } \Gamma; && (3.32)\\
&\overline{\mathcal{W}}_2 = 0, && \text{on } \partial\Omega. && (3.33)
\end{aligned}$$

eq:PDEcinqu
eq:PDEsei
eq:jumpcinc
eq:circuit
eq:datacinc

Existence and uniqueness for the previous problem is guaranteed by [1, Lemma 5]; moreover, the weak formulation of (3.29)–(3.33) is given by

$$
\begin{aligned}
0 = \int_{\Omega^B} (\sigma_1^B + \sigma_2^B)\nabla\overline{\mathcal{W}}_2 \cdot \nabla\varphi \,\mathrm{d}x + \int_{\Gamma} (\sigma_1^B + \sigma_2^B)\nabla\overline{\mathcal{W}}_2 \cdot \nu\, \varphi \,\mathrm{d}\sigma \\
+ \int_{\Omega^D} \sigma^D \nabla\overline{\mathcal{W}}_2 \cdot \nabla\varphi \,\mathrm{d}x - \int_{\Gamma} \sigma^D \nabla\overline{\mathcal{W}}_2 \cdot \nu\, \varphi \,\mathrm{d}\sigma \\
= \int_{\Omega^B} (\sigma_1^B + \sigma_2^B)\nabla\overline{\mathcal{W}}_2 \cdot \nabla\varphi \,\mathrm{d}x + \int_{\Omega^D} \sigma^D \nabla\overline{\mathcal{W}}_2 \cdot \nabla\varphi \,\mathrm{d}x \\
- \int_{\Gamma} (\sigma_1^B + \sigma_2^B)\nabla\overline{\mathcal{W}}_1 \cdot \nu\, \varphi \,\mathrm{d}\sigma - \int_{\Gamma} \sigma_1^B \nabla w \cdot \nu\, \varphi \,\mathrm{d}\sigma\,, \quad (3.34)
\end{aligned}
$$

eq:weak2

for every $\varphi \in H_0^1(\Omega)$. From (3.21), we obtain

$$
\begin{aligned}
- \int_{\Gamma} (\sigma_1^B + \sigma_2^B)\nabla\overline{\mathcal{W}}_1 \cdot \nu\, \varphi \,\mathrm{d}\sigma - \int_{\Gamma} \sigma_1^B \nabla w \cdot \nu\, \varphi \,\mathrm{d}\sigma \\
= \int_{\Omega^B} (\sigma_1^B + \sigma_2^B)\nabla\overline{\mathcal{W}}_1 \cdot \nabla\varphi \,\mathrm{d}x + \int_{\Omega^B} \sigma_1^B \nabla w \cdot \nabla\varphi \,\mathrm{d}x\,.
\end{aligned}
$$

which, replaced in (3.34), provides

$$
\begin{aligned}
0 = \int_{\Omega^B} (\sigma_1^B + \sigma_2^B)\nabla\overline{\mathcal{W}}_2 \cdot \nabla\varphi \,\mathrm{d}x + \int_{\Omega^D} \sigma^D \nabla\overline{\mathcal{W}}_2 \cdot \nabla\varphi \,\mathrm{d}x \\
+ \int_{\Omega^B} (\sigma_1^B + \sigma_2^B)\nabla\overline{\mathcal{W}}_1 \cdot \nabla\varphi \,\mathrm{d}x + \int_{\Omega^B} \sigma_1^B \nabla w \cdot \nabla\varphi \,\mathrm{d}x\,. \quad (3.35)
\end{aligned}
$$

eq:weak3

By taking $\varphi = \overline{\mathcal{W}}_2$ in (3.35), we get

$$
\begin{aligned}
\int_{\Omega^B} (\sigma_1^B + \sigma_2^B)|\nabla\overline{\mathcal{W}}_2|^2 \,\mathrm{d}x + \int_{\Omega^D} \sigma^D |\nabla\overline{\mathcal{W}}_2|^2 \,\mathrm{d}x \\
= - \int_{\Omega^B} (\sigma_1^B + \sigma_2^B)\nabla\overline{\mathcal{W}}_1 \cdot \nabla\overline{\mathcal{W}}_2 \,\mathrm{d}x - \int_{\Omega^B} \sigma_1^B \nabla w \cdot \nabla\overline{\mathcal{W}}_2 \,\mathrm{d}x\,,
\end{aligned}
$$

which, taking into account (3.25), implies

$$
\|\nabla\overline{\mathcal{W}}_2\|_{L^2(\Omega)} \le \gamma(\|\nabla\overline{\mathcal{W}}_1\|_{L^2(\Omega^B)} + \|\nabla w\|_{L^2(\Omega^B)}) \le \gamma(\|w\|_{H^1(\Omega^B)} + \|r\|_{H^{1/2}(\Gamma)})\,, \quad (3.36)
$$

eq:a2

with γ depending only on $\sigma_1^B, \sigma_2^B, \sigma^D$ and the geometry.
Finally, setting $\mathcal{W} = \overline{\mathcal{W}}_1 + \overline{\mathcal{W}}_2$, it is easy to see that $\mathcal{W} \in \mathcal{X}_0^1(\Omega)$ and satisfies (3.15)–(3.19) and (3.20). □

r:r1 *Remark* 3.2. We point out that, taking $\widetilde{r} \in H^1_{null}(\Omega^B)$ as in the proof of Proposition 3.1 and defining

$$\mathcal{W}^r = \begin{cases} \mathcal{W} - \widetilde{r}\,, & \text{in } \Omega^B; \\ \mathcal{W}\,, & \text{in } \Omega^D, \end{cases} \tag{3.37}$$ eq:a10

we get that $\mathcal{W}^r \in H^1_0(\Omega)$ and it satisfies

$$-\operatorname{div}((\sigma_1^B + \sigma_2^B)\nabla \mathcal{W}^r) = \operatorname{div}(\sigma_1^B \nabla w) + \operatorname{div}((\sigma_1^B + \sigma_2^B)\nabla \widetilde{r}), \quad \text{in } \Omega^B; \tag{3.38}$$ eq:PDEunod

$$-\operatorname{div}(\sigma^D \nabla \mathcal{W}^r) = 0, \quad \text{in } \Omega^D; \tag{3.39}$$ eq:PDEdued

$$(\sigma_1^B + \sigma_2^B)\nabla \mathcal{W}^r \cdot \nu = \sigma^D \nabla \mathcal{W}^r \cdot \nu - \sigma_1^B \nabla w \cdot \nu - (\sigma_1^B + \sigma_2^B)\nabla \widetilde{r} \cdot \nu, \quad \text{on } \Gamma; \tag{3.40}$$ eq:fluxunod

$$[\mathcal{W}^r] = 0, \quad \text{on } \Gamma; \tag{3.41}$$ eq:Circuitu

$$\mathcal{W}^r = 0, \quad \text{on } \partial\Omega. \tag{3.42}$$ eq:Boundd

The weak formulation of problem (3.38)–(3.41) is given by

$$\int_{\Omega^B} (\sigma_1^B + \sigma_2^B)\nabla \mathcal{W}^r \cdot \nabla \phi \,\mathrm{d}x + \int_{\Omega^D} \sigma^D \nabla \mathcal{W}^r \cdot \nabla \phi$$
$$= -\int_{\Omega^B} \sigma_1^B \nabla w \cdot \nabla \phi \,\mathrm{d}x - \int_{\Omega^B} (\sigma_1^B + \sigma_2^B)\nabla \widetilde{r} \cdot \nabla \phi \,\mathrm{d}x\,, \tag{3.43}$$ eq:weak6

for every $\phi \in H^1_0(\Omega)$, which implies

$$\int_{\Omega^B} (\sigma_1^B + \sigma_2^B)\nabla \mathcal{W} \cdot \nabla \phi \,\mathrm{d}x + \int_{\Omega^D} \sigma^D \nabla \mathcal{W} \cdot \nabla \phi = -\int_{\Omega^B} \sigma_1^B \nabla w \cdot \nabla \phi \,\mathrm{d}x\,, \tag{3.44}$$ eq:weak7

for every $\phi \in H^1_0(\Omega)$. □

Now, we are in the position to define the bilinear form $a : W \times W \to \mathbb{R}$ as

$$a\big((w,r),(\overline{w},s)\big) := \int_{\Omega^B} \sigma_1^B \nabla w \cdot \nabla \overline{w} \,\mathrm{d}x + \int_{\Omega^B} \sigma_1^B \nabla \mathcal{W} \cdot \nabla \overline{w} \,\mathrm{d}x$$
$$+ \int_{\Omega^B} (\sigma_1^B + \sigma_2^B)\nabla \mathcal{W} \cdot \nabla \overline{\mathcal{W}} \,\mathrm{d}x + \int_{\Omega^B} \sigma_1^B \nabla w \cdot \nabla \overline{\mathcal{W}} \,\mathrm{d}x + \int_{\Omega^D} \sigma^D \nabla \mathcal{W} \cdot \nabla \overline{\mathcal{W}} \,\mathrm{d}x + \beta \int_{\Gamma} rs \,\mathrm{d}\sigma\,, \tag{3.45}$$ eq:bilin

where $\mathcal{W}$ and $\overline{\mathcal{W}}$ are the solutions of (3.15)–(3.19) corresponding to (w,r) and $(\overline{w},s)$, respectively.

p:p4 **Proposition 3.3.** *The bilinear form $a : W \times W \to \mathbb{R}$, defined in* (3.45)*, is symmetric, continuous, and coercive.*

Proof. Notice that the bilinear form a can be rewritten as

$$
\begin{aligned}
a\big((w,r),(\overline{w},s)\big) &= \int\limits_{\Omega^B} \sigma_1^B \nabla(w+\mathcal{W})\cdot\nabla\overline{w}\,\mathrm{d}x + \int\limits_{\Omega^B} \sigma_1^B \nabla(w+\mathcal{W})\cdot\nabla\overline{\mathcal{W}}\,\mathrm{d}x \\
&\quad + \int\limits_{\Omega^B} \sigma_2^B \nabla\mathcal{W}\cdot\nabla\overline{\mathcal{W}}\,\mathrm{d}x + \int\limits_{\Omega^D} \sigma^D \nabla\mathcal{W}\cdot\nabla\overline{\mathcal{W}}\,\mathrm{d}x + \beta\int\limits_{\Gamma} rs\,\mathrm{d}\sigma \\
&= \int\limits_{\Omega^B} \sigma_1^B \nabla(w+\mathcal{W})\cdot\nabla(\overline{w}+\overline{\mathcal{W}})\,\mathrm{d}x + \int\limits_{\Omega^B} \sigma_2^B \nabla\mathcal{W}\cdot\nabla\overline{\mathcal{W}}\,\mathrm{d}x + \int\limits_{\Omega^D} \sigma^D \nabla\mathcal{W}\cdot\nabla\overline{\mathcal{W}}\,\mathrm{d}x + \beta\int\limits_{\Gamma} rs\,\mathrm{d}\sigma\,,
\end{aligned}
$$

which immediately proves that it is symmetric. Moreover, from (3.20), it easily follows that a is continuous.
In order to prove that it is also coercive, we note that

$$
\begin{aligned}
a\big((w,r),(w,r)\big) &= \int\limits_{\Omega^B} \sigma_1^B \nabla(w+\mathcal{W})\cdot\nabla(w+\mathcal{W})\,\mathrm{d}x + \int\limits_{\Omega^B} \sigma_2^B \nabla\mathcal{W}\cdot\nabla\mathcal{W}\,\mathrm{d}x \\
&\quad + \int\limits_{\Omega^D} \sigma^D \nabla\mathcal{W}\cdot\nabla\mathcal{W}\,\mathrm{d}x + \beta\int\limits_{\Gamma} r^2\,\mathrm{d}\sigma \\
&\geq \gamma(\|\nabla w+\nabla\mathcal{W}\|^2_{L^2(\Omega^B)} + \|\nabla\mathcal{W}\|^2_{L^2(\Omega^B)} + \|\nabla\mathcal{W}\|^2_{L^2(\Omega^D)} + \|r\|^2_{L^2(\Gamma)}) \\
&\geq \gamma(\|w\|^2_{H^1(\Omega^B)} + \|r\|^2_{H^{1/2}(\Gamma)})\,, \qquad (3.46)
\end{aligned}
$$

eq:a6

where, in the last inequality, we take into account that $r=[\mathcal{W}]$ (see (3.18)) and we use the Poincaré inequality (2.7) and the classical trace inequality, which assure that

$$
\|r\|^2_{H^{1/2}(\Gamma)} \leq \gamma\|\mathcal{W}\|^2_{\mathcal{X}^1_0(\Omega^B)} \leq \gamma\left(\|\nabla\mathcal{W}\|^2_{L^2(\Omega^B)} + \|\nabla\mathcal{W}\|^2_{L^2(\Omega^D)} + \|r\|^2_{L^2(\Gamma)}\right).
$$

□

p:p2 **Theorem 3.4.** *Assume that $\sigma_1^B, \sigma_2^B, \sigma^D, \alpha, \beta, f_1, f_2, \overline{v}_0, s_0$ are as in Subsection 2.3. Let $\widetilde{u}$ be the unique solution of problem (3.1)–(3.3), extended by zero inside Ω^D, and q be defined in (3.4). Then, problem (3.5)–(3.13) admits a unique solution $(v,u) \in L^2(0,T;H^1_{null}(\Omega^B)) \times L^2(0,T;\mathcal{X}^1_0(\Omega))$, such that $v \in \mathcal{C}^0([0,T];L^2(\Omega^B))$ and $[u] \in \mathcal{C}^0([0,T];L^2(\Gamma))$.*

Proof. Let us denote by $\langle\cdot,\cdot\rangle_{W^*W}$ the duality pairing between W and its dual space W^* and define $B \in W^*$ as

$$
\langle B,(\varphi_B,[\varphi_D])\rangle_{W^*W} = \int\limits_{\Omega^B} f_1\varphi_B\,\mathrm{d}x - \int\limits_{\Omega^B} \sigma_1^B\nabla\widetilde{u}\cdot\nabla\varphi_B\,\mathrm{d}x + \int\limits_{\Gamma} q[\varphi_D]\,\mathrm{d}\sigma\,,
$$

for every $(\varphi_B, [\varphi_D]) \in W$. By using the bilinear form a introduced in (3.45), we can consider the following abstract problem:

$$\begin{aligned}
&\text{find } (v,[u]) \in L^2(0,T;W) \cap \mathcal{C}^0([0,T];H) \\
&\text{with } v(0) = \overline{v}_0 \text{ and } [u](0) = s_0 - [\widetilde{u}](0) \text{ such that} \\
&\frac{\mathrm{d}}{\mathrm{d}t}\big((v,[u]),(\varphi_B,[\varphi_D])\big)_H + a\big((v,[u]),(\varphi_B,[\varphi_D])\big) \\
&\quad = \langle B, (\varphi_B,[\varphi_D])\rangle_{W^*W}\,, \qquad \forall(\varphi_B,[\varphi_D]) \in W\,,
\end{aligned} \tag{3.47}$$

eq:problem

in the distributional sense. By [29, Theorem 23.A], the problem (3.47) is well posed and it is not difficult to see that its weak formulation is given by

$$\begin{aligned}
&-\int_{\Omega_T^B} v\varphi_B\partial_t\psi\,\mathrm{d}x\,\mathrm{d}t + \int_{\Omega_T^B} \sigma_1^B\nabla v\cdot\nabla\varphi_B\psi\,\mathrm{d}x\,\mathrm{d}t + \int_{\Omega_T^B} \sigma_1^B\nabla u\cdot\nabla\varphi_B\psi\,\mathrm{d}x\,\mathrm{d}t \\
&\quad + \int_{\Omega_T^B} (\sigma_1^B+\sigma_2^B)\nabla u\cdot\nabla\varphi_D^1\psi\,\mathrm{d}x\,\mathrm{d}t + \int_{\Omega_T^B} \sigma_1^B\nabla v\cdot\nabla\varphi_D^1\psi\,\mathrm{d}x\,\mathrm{d}t \\
&\quad + \int_{\Omega_T^D} \sigma^D\nabla u\cdot\nabla\varphi_D^2\psi\,\mathrm{d}x\,\mathrm{d}t - \alpha\int_{\Gamma_T}[u][\varphi_D]\partial_t\psi\,\mathrm{d}\sigma\,\mathrm{d}t + \beta\int_{\Gamma_T}[u][\varphi_D]\psi\,\mathrm{d}\sigma\,\mathrm{d}t \\
&\quad = \int_{\Omega_T^B} f_1\varphi_B\psi\,\mathrm{d}x\,\mathrm{d}t - \int_{\Omega_T^B}\sigma_1^B\nabla\widetilde{u}\cdot\nabla\varphi_B\psi\,\mathrm{d}x\,\mathrm{d}t + \int_{\Gamma_T} q[\varphi_D]\psi\,\mathrm{d}\sigma\,\mathrm{d}t\,,
\end{aligned} \tag{3.48}$$

eq:weak8

for every $(\varphi_B, [\varphi_D]) \in W$ and every $\psi \in \mathcal{C}_0^\infty(0,T)$, where φ_D inside Ω^B and Ω^D is defined as the solution of (3.15)–(3.19), starting from $\varphi_B \in H^1_{null}(\Omega^B)$ and $[\varphi_D] \in H_0^{1/2}(\Gamma,\Omega)$. Clearly, (3.48) shall be complemented with the initial conditions.
Notice that (3.48) formally coincides with (3.14); however, in (3.14) the test function φ_D is a generic function belonging to $L^2(0,T;\mathcal{X}_0^1(\Omega))$, while in the present case it is the solution of an assigned differential problem. Hence, in order to state that, actually, the two weak formulations are equivalent, we have to prove that we can replace the prescribed φ_D in (3.48) with a generic test function belonging to $\mathcal{X}_0^1(\Omega)$.
To this purpose, let us fix $[\varphi_D] \in H_0^{1/2}(\Gamma,\Omega)$ and choose two generic functions $\varphi^1 \in H^1_{null}(\Omega^B)$ and $\varphi^2 \in H^1_{null}(\Omega^D)$, such that $[\varphi] := \varphi^1 - \varphi^2 = [\varphi_D]$. By (3.44), with $\phi = \varphi_D - \varphi \in H_0^1(\Omega)$, we get

$$\begin{aligned}
&\int_{\Omega_T^B} (\sigma_1^B+\sigma_2^B)\nabla u\cdot\nabla\varphi_D^1\psi\,\mathrm{d}x\,\mathrm{d}t + \int_{\Omega_T^B} \sigma_1^B\nabla v\cdot\nabla\varphi_D^1\psi\,\mathrm{d}x\,\mathrm{d}t + \int_{\Omega_T^D} \sigma^D\nabla u\cdot\nabla\varphi_D^2\psi\,\mathrm{d}x\,\mathrm{d}t \\
&= \int_{\Omega_T^B} (\sigma_1^B+\sigma_2^B)\nabla u\cdot\nabla\varphi^1\psi\,\mathrm{d}x\,\mathrm{d}t + \int_{\Omega_T^B} \sigma_1^B\nabla v\cdot\nabla\varphi^1\psi\,\mathrm{d}x\,\mathrm{d}t + \int_{\Omega_T^D} \sigma^D\nabla u\cdot\nabla\varphi^2\psi\,\mathrm{d}x\,\mathrm{d}t\,.
\end{aligned} \tag{3.49}$$

eq:weak10

Hence, replacing (3.49) in (3.48), it follows that it is possible to take $\varphi_D \in \mathcal{X}_0^1(\Omega)$ arbitrarily in (3.48) and, thus, such a weak formulation coincides with (3.14), once we take into account the density of product functions in $L^2(0,T;H^1_{null}(\Omega^B))$ and in $L^2(0,T;\mathcal{X}^1_0(\Omega))$. $\square$

p:p3 **Proposition 3.5.** *Assume that $\sigma_1^B, \sigma_2^B, \sigma^D, \alpha, \beta, f_1, f_2, \overline{v}_0, s_0$ are as in Subsection 2.3. Assume that $I_{ion} \equiv 0$. Then, problem* (2.27)–(2.35) *admits a unique solution $(V,U) \in L^2(0,T;H^1_{null}(\Omega^B)) \times L^2(0,T;\mathcal{X}^1_0(\Omega))$, such that $V \in \mathcal{C}^0([0,T];L^2(\Omega^B))$ and $[U] \in \mathcal{C}^0([0,T];L^2(\Gamma))$.*

Proof. Uniqueness easily follows by the linearity of problem (2.27)–(2.35). In order to prove existence, set $V = v$ and $U = u + \widetilde{u}$, where $\widetilde{u} \in H^1(0,T;\mathcal{X}^1_0(\Omega))$ is the solution of problem (3.1)–(3.3) and the pair $(v,u) \in L^2(0,T;H^1_{null}(\Omega^B)) \times L^2(0,T;\mathcal{X}^1_0(\Omega))$ with $v \in \mathcal{C}^0([0,T];L^2(\Omega^B))$ and $[u] \in \mathcal{C}^0([0,T];L^2(\Gamma))$, is the solution of (3.5)–(3.13), whose existence is guaranteed by Proposition 3.4. Then, by (3.14), we get

$$
\begin{aligned}
& -\int\limits_{\Omega^B_T} V \partial_t \varphi_B \,\mathrm{d}x\,\mathrm{d}t + \int\limits_{\Omega^B_T} \sigma_1^B \nabla V \cdot \nabla \varphi_B \,\mathrm{d}x\,\mathrm{d}t + \int\limits_{\Omega^B_T} \sigma_1^B \nabla U \cdot \nabla \varphi_B \,\mathrm{d}x\,\mathrm{d}t \\
& + \int\limits_{\Omega^B_T} (\sigma_1^B + \sigma_2^B) \nabla U \cdot \nabla \varphi_D^1 \,\mathrm{d}x\,\mathrm{d}t - \int\limits_{\Omega^B_T} (\sigma_1^B + \sigma_2^B) \nabla \widetilde{u} \cdot \nabla \varphi_D^1 \,\mathrm{d}x\,\mathrm{d}t + \int\limits_{\Omega^B_T} \sigma_1^B \nabla V \cdot \nabla \varphi_D^1 \,\mathrm{d}x\,\mathrm{d}t \\
& \qquad + \int\limits_{\Omega^D_T} \sigma^D \nabla U \cdot \nabla \varphi_D^2 \,\mathrm{d}x\,\mathrm{d}t - \alpha \int\limits_{\Gamma_T} [U] \partial_t [\varphi_D] \,\mathrm{d}\sigma\,\mathrm{d}t + \beta \int\limits_{\Gamma_T} [U][\varphi_D] \,\mathrm{d}\sigma\,\mathrm{d}t \\
& \qquad = \int\limits_{\Omega^B_T} f_1 \varphi_B \,\mathrm{d}x\,\mathrm{d}t \,,
\end{aligned}
\tag{3.50}
$$
eq:weak4

for every $\varphi_B \in L^2(0,T;H^1_{null}(\Omega^B)) \cap H^1_0(0,T;L^2(\Omega^B))$, $\varphi_D \in L^2(0,T;\mathcal{X}^1_0(\Omega))$, where, as before, $[\varphi_D] = \varphi_D^1 - \varphi_D^2$ and $[\varphi_D] \in H^1_0(0,T;L^2(\Gamma))$. Recalling (2.36), the thesis is achieved, up to an integration in time, once we have taken into account that

$$
\int\limits_{\Omega^B_T} (\sigma_1^B + \sigma_2^B) \nabla \widetilde{u} \cdot \nabla \varphi_D^1 \,\mathrm{d}x\,\mathrm{d}t = \int\limits_{\Omega^B_T} (f_1 - f_2) \varphi_D^1 \,\mathrm{d}x\,\mathrm{d}t \,, \tag{3.51}
$$
eq:a9

as follows from (3.1)–(3.3). $\square$

As a consequence of the previous results, we finally get our main theorem.

t:t1 **Theorem 3.6.** *Assume that $\sigma_1^B, \sigma_2^B, \sigma^D, \alpha, \beta, f_1, f_2, \overline{v}_0, s_0,$ and I_{ion} satisfy the assumptions stated in Subsection 2.3.*
Then, problem (2.27)–(2.35) *admits a unique solution $(V,U) \in L^2(0,T;H^1_{null}(\Omega^B)) \times L^2(0,T;\mathcal{X}^1_0(\Omega))$, such that $V \in \mathcal{C}^0([0,T];L^2(\Omega^B))$ and $[U] \in \mathcal{C}^0([0,T];L^2(\Gamma))$.*

Proof. The proof can be obtained following the same approach as in [16] (see, also, [17, §2.4.1]). Indeed, recalling that the function g appearing in the gating equation (2.13) is affine with respect to its second entry, problem (2.13)–(2.14) can be explicitly solved in term of $u_1^B - u_2^B = V$. Therefore, denoting by $\widetilde{w}_V$ such a solution and by $h(V) := I_{\mathrm{ion}}(V, \widetilde{w}_V)$, we obtain that problem (2.27)–(2.35) is a nonlinear version of the problem considered in Proposition 3.5. Moreover, since the nonlinearity h satisfies the assumptions [12, Definition 4.3.1], the thesis follows by the results in [12, Section 4.3] (see, also, [25, Ch. 6, Theorem 1.2]). □

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