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Relations between ageing and dependence for exchangeable lifetimes with an extension for the IFRA/DFRA property

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Abstract: We first review an approach that had been developed in the past years to introduce concepts of “bivariate ageing” for exchangeable lifetimes and to analyze mutual relations among stochastic dependence, univariate ageing, and bivariate ageing.

A specific feature of such an approach dwells on the concept of semi-copula and in the extension, from copulas to semi-copulas, of properties of stochastic dependence. In this perspective, we aim to discuss some intricate aspects of conceptual character and to provide the readers with pertinent remarks from a Bayesian Statistics standpoint. In particular we will discuss the role of extensions of dependence properties. “Archimedean” models have an important role in the present framework.

In the second part of the paper, the definitions of Kendall distribution and of Kendall equivalence classes will be extended to semi-copulas and related properties will be analyzed. On such a basis, we will consider the notion of “Pseudo-Archimedean” models and extend to them the analysis of the relations between the ageing notions of IFRA/DFRA-type and the dependence concepts of PKD/NKD.

Keywords: bivariate ageing, semi-copulas, generalized Kendall distributions, positive Kendall dependence, pseudo-Archimedean semi-copulas, positive dependence orderings, Schur-costant models

MSC: 60K10, 60E15, 62E10, 62H05, 60G09, 91B30

1 Introduction

Let $\mathbf{X} = (X_1, \dots, X_n)$ be a vector of non-negative random variables and denote by $\bar{F}_{\mathbf{X}} : \mathbb{R}_+^n \rightarrow [0, 1]$ the *joint survival function* of \mathbf{X} :

$$\bar{F}_{\mathbf{X}}(x_1, \dots, x_n) := \mathbb{P}(X_1 > x_1, \dots, X_n > x_n).$$

All along this note X_1, \dots, X_n are considered to be exchangeable and such that $\mathbb{P}(X_i = 0) = 0$, $i = 1, \dots, n$. By \bar{G} we denote the one dimensional marginal survival function:

$$\bar{G}(x) := \mathbb{P}(X_j > x), \quad \text{for } j = 1, \dots, n \text{ and } x \geq 0.$$

For simplicity's sake, $\bar{G}(\cdot)$ will be assumed continuous, strictly decreasing and positive over the half-line $[0, +\infty)$.

As it is well known, for random variables, any property of dependence is generally compatible with any arbitrary choice of a one-dimensional probability distribution. This is guaranteed by the Sklar Theorem (see, e.g., [23, 44]). However, compatibility may fail if we impose some extra condition on the joint probability distribution.

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When X_1, \dots, X_n are interpreted as *lifetimes* of different individuals, attention is often concentrated on the two different phenomena of *stochastic dependence* and of *stochastic ageing*. A very rich literature in applied probability has been devoted to this field and, from a technical viewpoint, we remind that several different notions of dependence and of ageing have been considered. Typically, such properties are described in terms of inequalities involving comparisons between probabilities of different events or conditional probabilities of a fixed event, given different information-states. Furthermore dependence and ageing are strictly related one another and, at a time, they are heavily affected by the actual state of information about (X_1, \dots, X_n) . See, e.g., references [1–3, 7, 31, 34, 36, 44, 49, 51, 53]. Concerning the relations among such notions, in this paper we first review an approach that had been developed in the past years, for the case of exchangeable lifetimes. This approach is based on the role of the family of the level curves of joint survival functions in the description of ageing for inter-dependent lifetimes. See [8, 10, 12, 51]. The earliest motivation for such an approach had been the effort to extend the property of *lack of memory* of exponential distributions to the case of a vector of dependent variables. Some of the results and other motivations are briefly recalled in Sections 2, 3, and 5. See also [41] and [18] for further developments. In such a context, the specific notion of semi-copula emerges as a rather natural extension of the notion of copula and several derivations hinge on the extension to semi-copulas of concepts of stochastic dependence. In [12] a conceptual method was introduced to single out the appropriate notions of dependence and bivariate ageing, “corresponding” to a fixed property of one-dimensional ageing. This method arises in a Bayesian framework. Actually the relations among dependence, univariate ageing, and bivariate ageing provide us with necessary conditions for reciprocal compatibility among these notions. At a same time, those relations are relevant to understand the effects of those changes of information which do not destroy exchangeability. One aim of our work amounts to illustrating this method and clarifying some intriguing aspects of it. Furthermore we propose a natural meaning of the term “corresponding” for the special class of dependence properties defined in terms of Positive Dependence Orderings (see Section 6, step g)).

Some issues, concerning with properties of copulas and semi-copulas, still require further analysis and developments. Specifically the analysis of the dependence concept of PKD/NKD and the related ageing concepts of IFRA/DFRA had been carried out only for Archimedean models (i.e., models with Archimedean survival copulas). In this respect the further aim of this paper amounts to extending the results in [12] from the Archimedean models to a larger class of models. On this purpose, we will introduce new concepts such as generalized Kendall distributions for semi-copulas and related Kendall equivalence classes of semi-copulas. Of special relevance for our aims are the Pseudo-Archimedean semi-copulas (i.e., Kendall equivalent to Archimedean semi-copulas). Such semi-copulas emerge when looking for necessary conditions for compatibility between PKD/NKD and IFRA/DFRA. Actually, our extension (see Theorem 7.19) concerns the subclass of Pseudo-Archimedean models to be defined by suitable regularity conditions (see in particular properties **(P1)**, **(P2)** in Subsection 7.2).

Our arguments and results will be demonstrated by means of some examples and remarks. For our discussion an important class of models is the Archimax one (i.e., models with Archimax survival copulas). As a relevant property, we exhibit a large class of Pseudo-Archimedean Archimax copulas (and semi-copulas) (see Example 7.11) and give necessary conditions for compatibility between PKD/NKD and IFRA/DFRA for Archimax models (see Example 7.20).

In the literature about stochastic dependence, positive dependence properties are often supposed to imply PQD, whereas it has been pointed out that PKD does not generally imply PQD, though such an implication does hold in the Archimedean case. This circumstance suggested us to introduce a stronger concept of positive dependence, defined by requiring both PKD and PQD and denoted by PKQD. The family of PKQD copulas strictly contains PKD Archimedean copulas. Indeed we give sufficient conditions under which Archimax copulas are PKQD. Also for this dependence concept some necessary conditions for compatibility with IFRA/DFRA are found.

The paper has a structure as follows. In Section 2, we concentrate attention on some simple aspects of the basic notions of IFR/DFR for one-dimensional probability distributions. In particular we focus on the circumstance that the IFR property for a family of conditional distributions can go lost under the operation of

unconditioning. In Section 3 we recall the specific bivariate IFR/DFR notion of our interest and introduce the issue of the relations between ageing and dependence notions. Section 4 is divided into three subsections, and is devoted to reviewing necessary notions about bivariate copulas, semi-copulas, dependence properties, Kendall distributions and bivariate ageing functions. The relevant case of Archimedean copulas and semi-copulas is treated in details in Subsection 4.3. In Section 5 we concentrate attention on the univariate and bivariate ageing notions of IFR/DFR and review some specific results concerning with relations between them and the corresponding concept of positive dependence, namely Super/Submigrativity. In Section 6 we explain in details the conceptual method introduced in [12] and leading to appropriate extensions of the arguments reviewed in the previous Section 5. Such extensions concern other notions of dependence and univariate/bivariate ageing. Furthermore we pave the way to the extension, to be developed in Section 7, of the results about the relations among stochastic dependence and IFRA/DFRA-type concepts of ageing. On this purpose, we introduce the concept of generalized Kendall distributions for semi-copulas. Furthermore we present definitions and results about equivalence classes of semi-copulas, which are defined in terms of generalized Kendall distributions. Subsection 7.3 is devoted to PKQD dependence property. In Section 8, we present a short discussion containing some comments, concluding remarks, and open problems. In order to let the paper be self-contained, in the Appendix we present arguments and proofs for some already known results, adapting them to our context and notation.

2 A brief review about one-dimensional IFR and DFR properties

Let T be a non-negative random variable and let the symbol \bar{G} to be again used to denote the survival function of T .

As a basic concept and a paradigmatic notion of univariate ageing we recall that T is *Increasing Failure Rate* (IFR) when, for any fixed $s \geq 0$, the function

$$t \geq 0 \mapsto \mathbb{P}(T > s + t | T > t)$$

turns out to be a non-increasing function. Similarly, T is *Decreasing Failure Rate* (DFR) when it is a non-decreasing function of t . According to a common language, the notion of IFR defines a concept of *positive ageing*, whereas DFR defines a concept of *negative ageing*. If T is exponentially distributed then it is IFR and DFR, at a time, and the lack of memory property of univariate exponential distributions is also seen as a property of *no-ageing*.

The arguments in this section will be concentrated on the notions of IFR and DFR. This will constitute a basis for the brief discussion concerning bivariate ageing, that will be presented in the next Sections 3 and 5.

In the following result, attention is preliminarily focused on different characterizations of the concept of IFR distribution. The proof is almost immediate and will be omitted. Details can be found, e.g., in [51]. Before stating it, we recall the notion of Schur-Concavity (Schur-Convexity): A function $W : \mathbb{R}_+ \times \mathbb{R}_+ \rightarrow \mathbb{R}_+$ is Schur-concave (Schur-convex) iff

$$W(x, y) = W(y, x) \text{ and}$$

$$\text{for } 0 \leq x \leq y, \quad t \geq 0, \quad W(x + t, y) \geq (\leq) W(x, y + t). \quad (1)$$

It is crucial in our discussion that Schur-Concavity (Schur-Convexity) for a function $W(x, y)$ is a property of the level curves $\{(x, y) \text{ s.t. } W(x, y) = c\}$. For this and other properties of Schur-concave functions, see, e.g., [39].

Proposition 2.1. *The following conditions are equivalent*

- (i) T is IFR.
- (ii) $\bar{G}(\cdot)$ is log-concave.
- (iii) The function $(t_1, t_2) \mapsto \bar{G}(t_1) \cdot \bar{G}(t_2)$ is Schur-concave.

(iv) For two i.i.d. random variables T_1, T_2 , distributed according to $\bar{G}(\cdot)$ one has, for given $s > 0$ and for given $0 \leq t_1 < t_2$,

$$\mathbb{P}(T_1 > t_1 + s | T_1 > t_1, T_2 > t_2) \geq \mathbb{P}(T_2 > t_2 + s | T_1 > t_1, T_2 > t_2). \quad (2)$$

Remark 2.2. In the case when a non-negative random variable T has a probability density function denoted by $g(t)$, we can consider the failure rate function

$$r(t) := -\frac{d}{dt} \log \bar{G}(t) = \frac{g(t)}{\bar{G}(t)}.$$

By item (ii) of Proposition 2.1 the latter is, of course, non-decreasing when T is IFR (whence the present terminology just arises).

Remark 2.3. Item (iii) of Proposition 2.1 is a property of the level curves of the survival function $\bar{G}(t_1)\bar{G}(t_2)$. We notice furthermore that the characterizations given in both items (iii) and (iv) allow us to express the univariate positive ageing notion of IFR in terms of conditions for the bivariate joint distribution of two i.i.d. random variables. These observations will turn out to be relevant in our discussion.

By suitably modifying the statement of Proposition 2.1, and also the above two Remarks, one can directly obtain corresponding statements that are valid for the univariate, negative-ageing, concept of DFR. So far there is in fact a complete symmetry between the two notions of IFR and DFR. On the contrary, the following result points out an aspect of lack of symmetry between the two notions.

Let \mathcal{E} be a set of indexes and, for the sake of notational simplicity, let us consider a family $\{\bar{G}_\theta; \theta \in \mathcal{E}\}$ of absolutely continuous survival functions over the half-line $[0, +\infty)$ and let \bar{G} be given by a mixture of the \bar{G}_θ 's.

Proposition 2.4 ([7], see also [37] and [5] for a Bayesian interpretation). *If \bar{G}_θ is DFR, $\forall \theta \in \mathcal{E}$, then also any mixture \bar{G} is DFR.*

Remark 2.5. Generally a mixture \bar{G} will not be IFR, under the condition that \bar{G}_θ is IFR, $\forall \theta \in \mathcal{E}$.

Remark 2.6. It is easy to prove Proposition 2.4. As an immediate consequence, since the exponential distributions are DFR, one obtains that a mixture of exponential distributions is DFR. Since the exponential distributions are IFR as well, this property is a counterexample related with the above Remark 2.5.

In the Appendix we present an argument that allows one to understand both the Proposition 2.4 and the above Remarks from the point of view of Bayesian Statistics. The circumstance, that the IFR property can go lost under the operation of making mixtures, can also be understood as a *Simpson-type paradox*. (See Remark 3.1 below).

3 A bivariate concept of IFR for exchangeable variables

In the items (iii) and (iv) of Proposition 2.1, the case of two i.i.d. random times was considered. Focusing attention on the vector $\mathbf{X} \equiv (X_1, \dots, X_n)$ of non-negative *exchangeable* (non-independent) random variables, we notice that it still makes sense to consider for X_1, \dots, X_n a condition such as in (2): namely, for $i \neq j$, for $0 \leq x < y$, and $t \geq 0$,

$$\mathbb{P}(X_i > x + t | X_i > x, X_j > y) \geq \mathbb{P}(X_j > y + t | X_i > x, X_j > y). \quad (3)$$

Actually, in a subjective-probability or Bayesian framework, the latter can be interpreted as a *bivariate condition of positive ageing* (see [6, 8, 9, 34, 50, 51]). As remarked above, (see (2) in Proposition 2.1), such a condition is equivalent to the IFR property of the marginal survival function \bar{G} , when X_1, \dots, X_n are i.i.d. variables. The opposite inequality

$$\mathbb{P}(X_i > x + t | X_i > x, X_j > y) \leq \mathbb{P}(X_j > y + t | X_i > x, X_j > y). \quad (4)$$

is, on the contrary, equivalent to the DFR property of \bar{G} and can therefore be interpreted as a *bivariate condition of negative ageing*.

The above inequalities (3) and (4) are conditions on the joint bivariate survival function of any pair (X_i, X_j) , with $1 \leq i \neq j \leq n$, and we will denote the latter by

$$\bar{F}^{(2)}(x, y) = \mathbb{P}(X_i > x, X_j > y); \quad x, y \geq 0. \quad (5)$$

In order to highlight that (3) and (4) are properties of positive/negative of bivariate ageing, it will be convenient to say that $\bar{F}^{(2)}(x, y)$ is *Bayesian bivariate Increasing/Decreasing Failure Rate*, abbreviated to Bayesian biv-IFR/Bayesian biv-DFR.

Let us now consider the case of conditionally independent, identically distributed, variables X_1, \dots, X_n . What can be said for this case?

Let Θ be a \mathcal{E} -valued random parameter (with $\mathcal{E} \subseteq \mathbb{R}^d$, say), with probability distribution Π_Θ and let X_1, \dots, X_n be conditionally independent given Θ , with a conditional one-dimensional survival function $\bar{G}(\cdot|\theta)$ for $\theta \in \mathcal{E}$, namely

$$\bar{F}_X(x_1, \dots, x_n) = \int_{\mathcal{E}} \bar{G}(x_1|\theta) \cdot \dots \cdot \bar{G}(x_n|\theta) d\Pi_\Theta(\theta).$$

In such a case, the condition that $\bar{G}(\cdot|\theta)$ is IFR for all $\theta \in \mathcal{E}$ implies that the bivariate condition of positive ageing (3) holds true. However (see Remarks 2.5, 2.6) such a condition does not imply the IFR property of the marginal survival function $\bar{G}(x) = \int_{\mathcal{E}} \bar{G}(x|\theta) d\Pi_\Theta(\theta)$, which would result, for $t > 0$ and $0 \leq x < y$, in the inequality

$$\mathbb{P}(X_i > x + t | X_i > x) \geq \mathbb{P}(X_j > y + t | X_j > y). \quad (6)$$

The condition that $\bar{G}(\cdot|\theta)$ is DFR for all $\theta \in \mathcal{E}$ implies that both the condition that \bar{G} is DFR and the condition of bivariate negative ageing (4) hold true.

Remark 3.1. Notice that the comparison in (2) is established between the conditional probabilities of two different events given a same conditioning event. In (6), on the contrary, we compare two conditional probabilities containing two different conditioning events. The inequality (6) is not implied then by the assumption

$$\mathbb{P}(X_i > x + t | X_i > x; \theta) \geq \mathbb{P}(X_j > y + t | X_j > y; \theta)$$

as it happens when, e.g., $\mathcal{E} = \mathbb{R}_+$ and $\Theta | (X_1 > x)$ is stochastically larger than $\Theta | (X_1 > y)$, for $0 \leq x < y$ (see the Appendix). This conclusion can be looked at as a Simpson-type Paradox (see [48]).

As it is well known, dating back to the original work by de Finetti (see, e.g., [17]), the other cases of exchangeability different from those of conditional independent and identical distribution, are those of *finite exchangeability*.

What about the relation between the univariate and bivariate properties of ageing in the case when, alternatively to conditional independence, we assume X_1, \dots, X_n to be *finitely exchangeable*?

Such relations are generally influenced by the type of stochastic dependence among X_1, \dots, X_n . The property of conditional independence does, in any case, imply some sort of positive dependence among X_1, \dots, X_n . At least, positive correlation between X_i, X_j for $1 \leq i \neq j \leq n$, as is very well-known.

In the case when X_1, \dots, X_n are finitely exchangeable, the relations between the ageing properties of \bar{G} and the conditions (3), (4) may be a bit more involved. Actually, in such a case, one can meet different types of (positive or negative) stochastic dependence (see, e.g., [51]) and the marginal survival function \bar{G} can still be a mixture of given survival functions, but some of the coefficients of the mixture may be negative. (See, e.g., [30, 32, 35, 38]).

For our purposes, however, it is not really relevant to distinguish between finite or infinite exchangeability. Rather, we look at the actual properties of stochastic dependence of bivariate distributions, i.e., of the joint

survival function $\bar{F}^{(2)}$, defined in (5). The rest of the paper is devoted to showing some results relating stochastic dependence and ageing properties. Preliminarily, on this purpose it is convenient to present a brief review of technical definitions and related properties concerning bivariate survival models as in (5). This will be done in the next section, before continuing our analysis in the subsequent sections.

4 A review about bivariate copulas and ageing functions

For our convenience, and to fix the notation, we start this section by just recalling well-known facts about bivariate copulas. We recall that a *bivariate copula* is a function $C : [0, 1]^2 \rightarrow [0, 1]$ such that

$$C(0, v) = C(u, 0) = 0, \quad 0 \leq u, v \leq 1 \quad (7)$$

$$C(1, v) = v, \quad C(u, 1) = u, \quad 0 \leq u, v \leq 1; \quad (8)$$

$$C(u, v) \text{ is increasing in each variable;} \quad (9)$$

$$C(u, v) + C(u', v') - C(u, v') - C(u', v) \geq 0, \quad (10)$$

for all $0 \leq u \leq u' \leq 1, \quad 0 \leq v \leq v' \leq 1$.

In other words a bivariate copula is the restriction to $[0, 1]^2$ of a joint distribution function for a pair of random variables U, V , uniformly distributed in $[0, 1]$.

From now on the term bivariate will be generally dropped.

Three special copulas are the following ones:

$$\text{the independence copula: } \Pi(u, v) = uv;$$

$$\text{the maximal copula: } M(u, v) = \min(u, v);$$

$$\text{the minimal copula: } W(u, v) = \max(1 - (u + v), 0).$$

We remind that any bivariate distribution function $F(x, y)$, with marginal distribution functions $F_X(x)$, $F_Y(y)$ can be written as

$$F(x, y) = C(F_X(x), F_Y(y)),$$

where C is a copula. When F_X , F_Y and F are continuous, then the copula is unique and we refer to it as the corresponding *connecting copula*. The term *distributional copula* is also used in the literature.

Moreover, under the same conditions, the bivariate survival function $\bar{F}(x, y)$, with survival marginal distribution functions $\bar{F}_X(x)$, $\bar{F}_Y(y)$ can be written as

$$\bar{F}(x, y) = \hat{C}(\bar{F}_X(x), \bar{F}_Y(y)), \quad (11)$$

where \hat{C} is the corresponding *two-dimensional survival copula*. Furthermore looking at C and \hat{C} as joint distribution functions of the pairs (U, V) and (\hat{U}, \hat{V}) , respectively, one can write

$$U = F_X(X), \quad V = F_Y(Y), \quad \hat{U} = \bar{F}_X(X) = 1 - U, \quad \hat{V} = \bar{F}_Y(Y) = 1 - V. \quad (12)$$

The survival copula \hat{C} is linked to the connecting copula C by means of the following relation

$$\hat{C}(u, v) = u + v - 1 + C(1 - u, 1 - v).$$

The concept of copula is relevant in the description of dependence properties among random variables (see, e.g., [23, 31, 44]). Often a dependence property for a joint bivariate distribution, is equivalent to the same dependence property of the connecting copula. Obviously, properties of dependence may be assessed for the survival copula as well. It may happen that assessing a dependence property on the survival copula or on the connecting copula gives rise to different conditions for the joint distribution (see in particular Remarks 4.1

and 4.7, below).

In the paper [43] it is shown how the conditional distribution of Y , given different types of information on X , can be described in terms of appropriate distortions of the marginal distribution of Y . The specific forms of such distortions are obtained in terms of the connecting copula. On the other hand, several conditions of stochastic dependence between X and Y , such as SI and LTD (to be recalled in the next subsection), can be characterized in terms of corresponding conditions of stochastic monotonicity for Y given X (or viceversa). In particular, the role of the types of information on Y in determining the type of dependence is analyzed in [26]. Therefore, in view of [43], the effect of copulas in the analysis of these dependence properties may also be described by means of different forms of distortions of a marginal distribution.

4.1 A brief review of dependence properties

We now move to recalling a few relevant definitions of dependence properties. We start with the *Positive Quadrant Dependence* (PQD) property, i.e., the events $\{Y \leq y\}$ and $\{X \leq x\}$, are positively correlated for all x and y :

$$\begin{aligned} C(u, v) \geq \Pi(u, v) = uv &\Leftrightarrow F(x, y) \geq F_X(x) F_Y(y), \\ \Updownarrow &\Updownarrow \\ \widehat{C}(u, v) \geq \Pi(u, v) = uv &\Leftrightarrow \bar{F}(x, y) \geq \bar{F}_X(x) \bar{F}_Y(y). \end{aligned}$$

Another dependence property to be recalled is the *Stochastic Increasingness* (SI) in X :

$$x \mapsto \mathbb{P}(Y > y | X = x) \text{ is increasing,}$$

which is equivalent to the following properties of the two pairs of random variables U, V and \widehat{U}, \widehat{V} , i.e. of their respective copulas $C(u, v)$ and $\widehat{C}(u, v)$:

$$u \mapsto \mathbb{P}(V > v | U = u) \text{ is increasing,} \quad \Leftrightarrow \quad u \mapsto \mathbb{P}(\widehat{V} > v | \widehat{U} = u) \text{ is increasing.}$$

A further dependence property is the condition: Y is *Left Tail Decreasing* (LTD) in X , i.e., the function $x \mapsto \mathbb{P}(Y \leq y | X \leq x)$ is decreasing:

$$\begin{aligned} \frac{F(x', y)}{F_X(x')} \leq \frac{F(x, y)}{F_X(x)} \quad \text{for all } x \leq x', \text{ and for all } y, \\ \Updownarrow \\ \frac{C(u', v)}{u'} \leq \frac{C(u, v)}{u}, \quad \text{for all } 0 < u \leq u' \leq 1, \text{ and for all } 0 \leq v \leq 1, \end{aligned}$$

i.e., V is LTD in U , where U and V have been defined in (12).

Remark 4.1. The corresponding property for the survival copula \widehat{C} , i.e., \widehat{V} is LTD in \widehat{U} , namely

$$\frac{\widehat{C}(u', v)}{u'} \leq \frac{\widehat{C}(u, v)}{u}, \quad \text{for all } 0 < u \leq u' \leq 1, \text{ and for all } 0 \leq v \leq 1,$$

is instead equivalent to the condition that Y is *Right Tail Increasing* (RTI) in X , i.e., for all $y \geq 0$, the function $x \mapsto \mathbb{P}(Y > y | X > x)$ is increasing.

We are now going to recall further dependence properties that are specially relevant for what follows.

We start with the so called *Supermigrativity* property: for an arbitrary copula D , i.e.,

$$D(us, v) \geq D(u, sv), \quad \text{for all } 0 \leq v \leq u \leq 1 \text{ and } s \in (0, 1). \quad (13)$$

This condition, applied to the survival copula \widehat{C} , has emerged to describe the Schur-concavity of \bar{F} (see [12], see also Proposition 5.2, below). The term *Supermigrativity* has been coined in [19]. In a sense, Supermigrativity can be seen as a property of positive dependence: in particular it implies the PQD property.

Example 4.2. *The extreme case of positive dependence is the one of “comonotone” dependence:*

$$\mathbb{P}(X = Y) = 1$$

In such a case the survival copula coincides with the maximal copula:

$$\widehat{C}(u, v) = M(u, v) = \min(u, v),$$

and is obviously supermigrative.

For each of these positive dependence notions (PQD, SI, LTD, Supermigrativity), one can consider the corresponding notions of negative dependence (NQD, SD, LTI, Submigrativity). The latter ones are defined in an obvious way, in the present bivariate case.

For what concerns Super/Submigrativity properties, in order to highlight that they are notions of positive/negative dependence, it will be convenient to use also the terms *Positive/Negative Migrativity Dependence*, abbreviated to PMD/NMD.

Remark 4.3. *As pointed out in [12] (see Proposition 6.1 therein), Supermigrativity (or equivalently PMD property) does coincide with the LTD property in the case of an Archimedean copula (a brief review of Archimedean copulas will be given next). Other features related with Supermigrativity have been analyzed in [19]).*

The last dependence property to be recalled here is the *Positive Kendall Dependence* (PKD) property (see Definiton 4.6, below). This property will have a relevant role for our results in Section 7 and is connected with the *Kendall distribution function associated to F* (see, e.g., [45])

$$K(t) := \mathbb{P}(F(X, Y) \leq t) = \mathbb{P}(C(U, V) \leq t). \quad (14)$$

We point out that $K(t)$ depends only on the connecting copula C , and we will also use the notation $K_C(t)$. We also recall that $K_C(t) \geq t$ in $[0, 1]$, as is easily checked.

In particular for the independence copula $\Pi(u, v) = uv$, one has

$$K_\Pi(t) = t - t \log(t).$$

In analogy with the Kendall distribution one can consider (see [41]) the *upper-orthant Kendall distribution* associated to \bar{F} , i.e.,

$$\widehat{K}(t) := \mathbb{P}(\bar{F}(X, Y) \leq t) = \mathbb{P}(\widehat{C}(\widehat{U}, \widehat{V}) \leq t). \quad (15)$$

In other words $\widehat{K}(t) = K_{\widehat{C}}(t)$.

Remark 4.4. *Note that*

$$K_C(t) = \mu_C(\{(u, v) \in [0, 1]^2 : C(u, v) \leq t\}).$$

where μ_C is the probability measure on $[0, 1]^2$ such that

$$\mu_C((u, u'] \times (v, v']) = C(u, v) + C(u', v') - C(u, v') - C(u', v).$$

The above remark suggests an alternative way to compute $K_C(t)$. This will be done in the following Lemma 4.5, the proof of which is almost immediate and will be omitted.

Let

$$\{u_0 = 0 \leq u_1 \leq \dots \leq u_i \leq u_{i+1} \leq \dots \leq u_n\}$$

be a finite partition of $(0, 1]$, and denote by \mathcal{P} the class of all finite partitions of $(0, 1]$. Set furthermore

$$C_u^{-1}(t) := \sup\{v : C(u, v) \leq t\},$$

the generalized inverse of $v \mapsto C(u, v)$.

Lemma 4.5. Let $C(u, v)$ be a copula, then

$$K_C(t) = \sup_{\mathcal{P}} \left\{ \sum_{i \in I} [C(u_{i+1}, v_i) - C(u_i, v_i)], \text{ with } v_i = C_{u_{i+1}}^{-1}(t) \right\}. \quad (16)$$

We now remind the definition of Positive (Negative) Kendall Dependence (see in particular [14])

Definition 4.6. A copula D is Positive Kendall Dependent (PKD) when

$$K_D(t) \leq K_H(t) = t - t \log(t), \quad t \in (0, 1),$$

while D is Negative Kendall Dependent (NKD) when the reverse inequality holds. Two random variables X, Y are Positive (Negative) Kendall Dependent (PKD/NKD) when their connecting copula C is PKD/NKD.

Two random variables X, Y are Positive (Negative) upper-orthant Kendall Dependent (PuoKD/NuoKD) when their survival copula \hat{C} is PKD/NKD.

Remark 4.7. We notice that the PKD property for the connecting copula C and for the survival copula \hat{C} give rise to different dependence properties for the random variables X, Y .

4.2 Bivariate ageing functions, semi-copulas, and related dependence properties

In the case of exchangeability, a survival function $\bar{F}(x, y)$ takes the form

$$\bar{F}(x, y) = \hat{C}(\bar{G}(x), \bar{G}(y)), \quad (17)$$

where \hat{C} is an exchangeable copula and \bar{G} denotes the common marginal survival function. Restricting our attention to this case we also associate to $\bar{F}(x, y)$ a function that describes the family of its level curves, according to the following definition (see [10]).

Definition 4.8 (Bivariate ageing function [10, 12]). The function

$$B : [0, 1]^2 \rightarrow [0, 1],$$

defined by

$$B(u, v) := \exp\{-\bar{G}^{-1}(\bar{F}(-\log u, -\log v))\}, \quad (18)$$

is called bivariate ageing function of $\bar{F}(x, y)$.

Notice that in the above definition we have used the convention $-\log(0) = +\infty$, $\bar{F}(x, +\infty) = \bar{F}(+\infty, y) = \bar{F}(+\infty, +\infty) = 0$, for all x and y .

From (18) one immediately obtains

$$\bar{F}(x, y) = \bar{G}(-\log(B(e^{-x}, e^{-y}))). \quad (19)$$

Any other survival function $\bar{M}(x, y)$ sharing with $\bar{F}(x, y)$ the same family of level curves, must also share the same ageing function B . Therefore $\bar{M}(x, y)$ must be of the following form

$$\bar{M}(x, y) = \bar{H}(-\log(B(e^{-x}, e^{-y}))),$$

for a marginal survival function \bar{H} . Nevertheless, for arbitrary \bar{H} , it is not guaranteed that $\bar{H}(-\log(B(e^{-x}, e^{-y})))$ is a bona-fide survival function. Conditions on \bar{H} have been given to guarantee that $\bar{H}(-\log(B(e^{-x}, e^{-y})))$ is actually a survival function (see Theorem 2, Remarks 10 and 24 in [41]).

Furthermore, like it happens for any copula, B is a $[0, 1]$ -valued function, defined over $[0, 1]^2$, increasing in each variable, and it is such that

$$B(w, 1) = B(1, w) = w, \quad B(w, 0) = B(0, w) = 0, \quad \forall w \in [0, 1]. \quad (20)$$

However the function B is not always a copula. A bivariate copula, in fact, must have the properties of a bivariate probability distribution function. Whereas, on the contrary, examples of ageing functions B can be given such that

$$B(u', v') - B(u, v') - B(u', v) + B(u, v) < 0 \quad (21)$$

for some values $0 < u < u' < 1$, $0 < v < v' < 1$.

Nevertheless, B turns out to be a copula in some special cases.

Example 4.9. Let us consider the remarkable case of Schur-constant $\bar{F}(x, y)$ survival functions with marginal survival function \bar{G} , i.e.,

$$\bar{F}(x, y) = \bar{G}(x + y),$$

(see [51]). In such a case the function B does coincides with the product copula $\Pi(uv) = uv$. It is immediate to check that the bivariate function $\bar{F}(x, y) = \bar{G}(x + y)$ is a survival function if and only if $\bar{G}(x)$ is convex.

In the limiting case of "comonotone" dependence, (see Example 4.2) one has

$$B(u, v) = \hat{C}(u, v) = M(u, v) = u \wedge v.$$

Thus $B(u, v)$ is a supermigrative copula.

The term *semi-copula* has been proposed to designate functions which, like the ageing function B above, are increasing in each variable, satisfy the margin conditions (20), and are such that the inequality in (21) may hold for some values $0 < u < u' < 1$, $0 < v < v' < 1$. For a more formal definition of semi-copula, basic properties, extensions and technical details about semi-copulas and ageing functions, also in a multivariate context, see in particular, the papers [10–12, 18, 21, 22, 24, 41], and the book [23] with references cited therein. We point out that the class of bivariate ageing functions is strictly contained in the one of semi-copulas (see, e.g., [23]).

Taking into account the expression (17) for the survival function we can also rewrite (18) as

$$B(u, v) = \exp\{-\bar{G}^{-1}(\hat{C}(\bar{G}(-\log u), \bar{G}(-\log v)))\}. \quad (22)$$

By setting

$$\gamma(u) = \exp\{-\bar{G}^{-1}(u)\}, \quad (23)$$

whence

$$\gamma^{-1}(z) = \bar{G}(-\log(z)), \quad (24)$$

we can also write

$$B(u, v) = \gamma(\hat{C}(\gamma^{-1}(u), \gamma^{-1}(v))). \quad (25)$$

The above expression suggests to introduce the following family of operators on the semi-copulas: for any *increasing homeomorphism* of $[0, 1]$, i.e., for any continuous strictly increasing function $\eta : [0, 1] \leftrightarrow [0, 1]$ such that $\eta(0) = 0$ and $\eta(1) = 1$, one may define the operator Φ_η which transforms a semi-copula S into the semi-copula

$$\Phi_\eta(S)(u, v) := \eta(S(\eta^{-1}(u), \eta^{-1}(v))). \quad (26)$$

Then, by using the above notation and recalling that $\bar{G}(0) = 1$ and $x \mapsto \bar{G}(x)$ is continuous strictly decreasing, we can actually write

$$B = \Phi_\gamma(\hat{C}).$$

Viceversa, given the ageing function B and the marginal survival function \bar{G} then the copula \hat{C} is given by

$$\hat{C} = \Phi_{\gamma^{-1}}(B).$$

For our purposes it is useful to extend to semi-copulas definitions of stochastic dependence that have been formulated for copula functions. In particular, it is immediate to define the property of Supermigrativity (Submigrativity) for any semi-copula S : We say that S is supermigrative (submigrative) iff

$$S(us, v) \geq (\leq) S(u, sv), \quad \text{whenever } 0 \leq v \leq u \leq 1, \quad s \in (0, 1). \quad (27)$$

Similarly we point out that all the definitions of dependence properties considered so far can be extended in a direct way, but the PKD/NKD property. The extension to semi-copulas of the latter property requires an appropriate definition of the concept of Kendall distribution for semi-copulas. Such a definition will be given in Subsection 7.1.

4.3 Archimedean copulas and semi-copulas

One important class of copulas is the one of Archimedean copulas:

Let $\phi : (0, 1] \rightarrow [0, +\infty)$ be a continuous, convex, and decreasing function and denote by C_ϕ the bivariate Archimedean copula with additive generator ϕ . Namely, for $0 \leq u, v \leq 1$, set

$$C_\phi(u, v) := \phi^{-1} [\phi(u) + \phi(v)].$$

It is convenient, though not strictly necessary, to assume ϕ strictly decreasing and such that

$$\lim_{u \rightarrow 0^+} \phi(u) = +\infty, \quad \phi(1) = 0,$$

so that the function ϕ^{-1} can be identified with a one-dimensional survival function of the type that we are considering here.

Two basic examples of Archimedean copulas are the independence copula $\Pi(u, v) = uv$, where $\phi(u) = -\log(u)$, and the survival copula of a Schur-constant survival function as in Example 4.9, where $\phi(u) = \bar{G}^{-1}(u)$.

Similarly, we say that a semi-copula S is Archimedean when it has the form

$$S_\varphi(u, v) := \varphi^{-1} [\varphi(u) + \varphi(v)],$$

where $\varphi : (0, 1] \rightarrow [0, +\infty)$ is a continuous decreasing function, not necessarily convex. We maintain the assumptions

$$\lim_{u \rightarrow 0^+} \varphi(u) = +\infty, \quad \varphi(1) = 0.$$

Since C_ϕ is a copula, and S_φ is a semi-copula it makes sense to consider their dependence properties. Since ϕ^{-1} and φ^{-1} are, technically, one-dimensional survival functions, it makes sense to consider their ageing properties. On the other hand, the inverse of any one-dimensional survival function \bar{G} , if it exists, can be seen as the generator of the Archimedean semi-copula

$$S_{\bar{G}^{-1}}(u, v) = \bar{G}(\bar{G}^{-1}(u) + \bar{G}^{-1}(v)). \quad (28)$$

Remark 4.10. When \bar{G} , and then \bar{G}^{-1} , is convex, the above Archimedean semi-copula $S_{\bar{G}^{-1}}$ is a copula. Actually $S_{\bar{G}^{-1}}$ is the survival copula of the Schur-constant model in the Example 4.9.

Remark 4.11. An exchangeable bivariate survival function $\bar{F}(x, y)$ with an Archimedean survival copula $\hat{C} = C_\phi$ and marginal survival function \bar{G} has the form

$$\bar{F}(x, y) = \phi^{-1} [\phi(\bar{G}(x)) + \phi(\bar{G}(y))]. \quad (29)$$

In this case the ageing function $B(u, v)$ is Archimedean as well. More precisely, recalling (22), (26), and (23) one has

$$B(u, v) = \Phi_{\gamma}(\hat{C})(u, v) = \exp\{-\bar{G}^{-1}(\phi^{-1}[\phi(\bar{G}(-\log u)) + \phi(\bar{G}(-\log v))])\}$$

namely

$$B(u, v) = S_{\varphi}(u, v), \quad (30)$$

where

$$\varphi(u) = \phi(\gamma(u)) = \phi(\bar{G}(-\log u)). \quad (31)$$

5 Relating Dependence to Ageing Properties: the IFR/DFR case

As announced at the end of Section 3, we continue to analyze the case of exchangeable variables X_1, \dots, X_n , with two-dimensional marginal survival function $\bar{F}^{(2)}$ given in (5). We denote by \bar{G} and $\hat{C}(u, v)$ the marginal survival function and the survival copula of $\bar{F}^{(2)}$. We are now in a position to discuss the interrelations among dependence properties of PMD/NMD, the univariate positive/negative ageing properties of IFR/DFR, and the bivariate positive/negative ageing conditions of Bayesian biv-IFR/biv-DFR. In this respect we remind the following simple result, which can be obtained as a consequence of the general results given in Section 5 of [12]. We prefer to state this specific result for different reasons. First of all it admits a self-contained proof, different from the proof in [12] (see the Appendix). Furthermore it provides the reader with a paradigmatic scheme of those results, and will help us to explain in the next section the general idea behind them.

Proposition 5.1. *With the notations introduced above*

- (i) *If \bar{G} is IFR and \hat{C} is PMD, then $\bar{F}^{(2)}$ is Bayesian biv-IFR.*
- (ii) *If \bar{G} is DFR and \hat{C} is NMD, then $\bar{F}^{(2)}$ is Bayesian biv-DFR.*
- (iii) *If $\bar{F}^{(2)}$ is Bayesian biv-IFR and \bar{G} is DFR, then \hat{C} is PMD.*
- (iv) *If $\bar{F}^{(2)}$ is Bayesian biv-DFR and \bar{G} is IFR, then \hat{C} is NMD.*
- (v) *If \hat{C} is PMD and $\bar{F}^{(2)}$ is Bayesian biv-DFR, then \bar{G} is DFR.*
- (vi) *If \hat{C} is NMD and $\bar{F}^{(2)}$ is Bayesian biv-IFR, then \bar{G} is IFR.*

In Proposition 5.1, for a joint exchangeable survival function \bar{F}_X attention has been fixed on the two-dimensional marginal survival function $\bar{F}^{(2)}$ and on the pair (\hat{C}, \bar{G}) of the two-dimensional survival copula and the marginal survival function, so that $\bar{F}^{(2)}$ is given as in (17).

It is convenient however to rephrase that result in terms of the triple (\hat{C}, \bar{G}, B) , where B is the bivariate ageing function of $\bar{F}^{(2)}$ (see (18)). On this purpose we recall a result connecting the Bayesian biv-IFR/DFR properties (3) and (4), with the PMD/NMD properties for B (see [12]).

Lemma 5.2. *The following three conditions are equivalent:*

- (a) $\bar{F}^{(2)}$ is Bayesian biv-IFR. (b) $\bar{F}^{(2)}$ is Schur-concave. (c) B is PMD.
- Analogously also the following dual conditions are equivalent*
- (a') $\bar{F}^{(2)}$ is Bayesian biv-DFR. (b') $\bar{F}^{(2)}$ is Schur-convex. (c') B is NMD.

For the readers' convenience, in the Appendix we provide an autonomous proof adapted to our language and notation.

It is important to stress that the previous result shows that the Super/Submigrativity property for the ageing function B is a bivariate ageing property.

In view of the previous equivalences, we are now in a position to rephrase the result in Proposition 5.1 in terms of the survival copula \hat{C} , the marginal survival function \bar{G} and the bivariate ageing function B , as follows.

Proposition 5.3.

- (i) If \bar{G} is IFR and \hat{C} is PMD, then B is PMD.
- (ii) If \bar{G} is DFR and \hat{C} is NMD, then B is NMD.
- (iii) If B is PMD and \bar{G} is DFR, then \hat{C} is PMD.
- (iv) If B is NMD and \bar{G} is IFR, then \hat{C} is NMD.
- (v) If \hat{C} is PMD and B is NMD, then \bar{G} is DFR.
- (vi) If \hat{C} is NMD and B is PMD, then \bar{G} is IFR.

Remark 5.4. In [4] it was shown that an Archimedean copula, with a generator ϕ is LTI/LTD if and only if ϕ^{-1} is a IFR/DFR survival function, respectively. As pointed out in the previous Section 4, it makes sense however, to extend dependence properties to semi-copulas. It was observed in [12] that also for an Archimedean semi-copula the LTI/LTD property is equivalent to the IFR/DFR property for the inverse of its generator. In particular \bar{G} is IFR/DFR if and only if the semi-copula $S_{\bar{G}^{-1}}$ (see (28)) is LTI/LTD. Moreover, again for Archimedean semi-copulas, the LTI/LTD property is equivalent to the NMD/PMD property (see Remark 4.3). We can conclude therefore that the above result in Proposition 5.3 can be formulated in terms of the survival copula \hat{C} , and the semi-copulas B and $S_{\bar{G}^{-1}}$. For instance item (i) can be reformulated as

$$(i) \quad \text{If } S_{\bar{G}^{-1}} \text{ is NMD and } \hat{C} \text{ is PMD, then } B \text{ is PMD.} \quad (32)$$

and similarly for items (ii)–(vi).

Let us concentrate again our attention on the three semi-copulas \hat{C} , B , and $S_{\bar{G}^{-1}}$: Summarizing the above arguments, we can claim that PMD/NMD conditions imposed over two semi-copulas imply a condition – of type either PMD or NMD – for the third semi-copula.

6 A path to a more general analysis

As mentioned in the previous sections, the Supermigrativity property for a copula is looked at as a condition of positive dependence, while the IFR property of a marginal survival function is a condition of positive one-dimensional ageing. The terms Submigrativity and DFR respectively refer to the corresponding dual conditions of negative dependence and of negative one-dimensional ageing.

For the survival function $\bar{F}(x, y)$ in (17), one can say that Proposition 5.3 concerns with the properties Supermigrativity/Submigrativity and IFR/DFR for \hat{C} and \bar{G} , respectively. It concerns furthermore with Supermigrativity/Submigrativity of B , which in view of Lemma 5.2 can be seen as a notion of positive/negative bivariate ageing.

Several other concepts of stochastic dependence and of ageing have been considered in the literature. Obviously we can say that “dependence” is a property of the survival copula \hat{C} , “one-dimensional ageing” is a property of the marginal survival function \bar{G} . Furthermore one can consider notions of “bivariate ageing” defined in terms of the ageing function B . See [12] for both technical details and some heuristic argument, and [21–24, 41, 52, 53] for further related discussions and results. As it will be also discussed below, positive/negative bivariate ageing can be defined in terms of positive/negative dependence of the ageing function B , in analogy with the IFR/DFR case.

On such a basis, results of the same form as in Proposition 5.3 can be formulated for some other pairs of «dependence, one-dimensional ageing» properties and notions of “bivariate ageing”, appropriately corresponding to them. Informally, we can describe as follows the general format of such results:

1. Positive dependence and positive one-dimensional ageing imply positive bivariate ageing
2. Positive bivariate ageing and negative one-dimensional ageing imply positive dependence

3. Positive bivariate ageing and negative dependence imply positive one-dimensional ageing. Furthermore, dual statements — where the terms “positive” and “negative” are interchanged — do hold.

In order to setting the above statements in a more precise form, we need to give appropriate answers to the following natural questions:

Q1: What are the appropriate notion of dependence and property of one-dimensional ageing, “corresponding” to each other?

Q2: What might be an appropriate notion of bivariate ageing, “corresponding” to a pair «dependence, one-dimensional ageing»?

Q3: What appropriate meaning should be given to the term “corresponding”?

Finally, assuming that one has given satisfying answers to the preceding questions, one wonders furthermore

Q4: What about a general and synthetic method for proving statements of the same form as in Proposition 5.3?

In what follows, on the one hand, we explain the path that gave rise to the general answers to the above questions, as given in [12]. On the other hand we slightly modify those answers in a way which allows us to add a further result to those obtained in [12] (see Section 7).

The path will be divided into several steps, reaching the answers in the last step g).

a) As also pointed out in Remark 2.3, Proposition 2.1 ensures that the IFR/DFR conditions of one-dimensional ageing for \bar{G} can be translated into inequalities for the bivariate survival function $\bar{F}_\Pi(x, y) := \bar{G}(x)\bar{G}(y)$ of two independent variables distributed according to \bar{G} (see (iv) of Proposition 2.1, see also (iii), which turns out to be an inequality in view of (1)). For instance, we can consider, in place of IFR/DFR conditions, the NBU/NWU conditions for \bar{G} , i.e.,

$$\bar{G}(x+y) \leq (\geq) \bar{G}(x)\bar{G}(y),$$

or equivalently

$$\bar{F}_\Pi(x+y, 0) \leq (\geq) \bar{F}_\Pi(x, y). \quad (33)$$

b) As observed in Remark 2.3 the inequalities in the above step a), concerning the IFR/DFR properties, can be seen as properties of the family of the level curves of the function $\bar{F}_\Pi(x, y) = \bar{G}(x)\bar{G}(y)$. Also the NBU/NWU properties (33) are properties of the family of the level curves for \bar{F}_Π . Any exchangeable bivariate survival function $\bar{F}(x, y)$, which shares with $\bar{F}_\Pi(x, y)$ the family of level curves, also shares with \bar{F}_Π qualitative properties of bivariate ageing, as first discussed in [8, 10, 51]. On the other hand, we recall that two different survival functions share the same family of level curves if and only if they share the same ageing function B . Whence, different bivariate ageing properties defined in terms of the family of the level curves of $\bar{F}(x, y)$ turn out to coincide with properties of B .

c) It is remarkable that the bivariate ageing properties, mentioned in step b) and defined in terms of properties of the ageing function B , actually coincide with different “dependence” properties of B (see [12]). The special case of PMD/NMD is considered here in Lemma 5.2. A heuristic argument for the above claim goes as follows. When B is a copula, then $\bar{F}_B(x, y) := B(e^{-x}, e^{-y})$ is a true bivariate survival function, with standard exponential marginal distributions, as well as $\Pi(e^{-x}, e^{-y})$. The special case when B is the independence copula Π corresponds to a bivariate no-ageing property. Furthermore, in general, since B does clearly coincide with the survival copula of $\bar{F}_B(x, y)$, and simultaneously is also the bivariate ageing of it, the dependence property of such a model coincides with the bivariate ageing property. The latter is also the ageing property of \bar{F} , since \bar{F} and \bar{F}_B share the same family of level curves.

Before continuing our conceptual path, we point out that different papers in the literature had been devoted to the analysis of connections between dependence properties of C_ϕ and ageing properties of ϕ^{-1} . Very precise and detailed results had, in particular, been obtained in [4, 40]. Some notions of positive dependence for C_ϕ have been shown to be equivalent to negative ageing properties of ϕ^{-1} . For the reader's ease, we recall the results in [4] concerning the relations between the dependence properties PQD, PKD, LTD, SI for an Archimedean copula C_ϕ , and the ageing properties of the survival functions ϕ^{-1} . For simplicity sake we assume that the generator ϕ is differentiable in $(0, 1)$ and denote by $f_\phi(x)$ the density of ϕ^{-1} .

$$C_\phi \text{ is PQD/NQD} \Leftrightarrow \phi^{-1}(x) \text{ is NWU/NBU}; \quad (34)$$

$$C_\phi \text{ is PKD/NKD} \Leftrightarrow \phi^{-1}(x) \text{ is DFRA/IFRA}; \quad (35)$$

$$C_\phi \text{ is LTD/LTI} \Leftrightarrow \phi^{-1}(x) \text{ is DFR/IFR}; \quad (36)$$

$$C_\phi \text{ is SI/SD} \Leftrightarrow \log(f_\phi) \text{ is convex/is concave}. \quad (37)$$

The DFRA/IFRA properties are recalled below in Definition 7.1. In the literature, property (37) for f_ϕ is a well known property, also called Decreasing/Increasing Likelihood Ratio (DLR/ILR), see, e.g., [42].

We recall that an Archimedean copula is LTD if and only if is PMD (see also Remark 5.4). We also notice the following implications between notions of positive univariate negative ageing (see, e.g., [49])

$$\log(f_\phi) \text{ convex} \Rightarrow \text{DFR} \Rightarrow \text{DFRA} \Rightarrow \text{NWU}.$$

Whence one obtains the following implications among notions of bivariate dependence for Archimedean copulas

$$\text{SI} \Rightarrow \text{LTD} \Rightarrow \text{PKD} \Rightarrow \text{PQD}.$$

We can now resume our conceptual path by means of the following steps d)–g).

d) We associate to the Archimedean copula C_ϕ the bivariate survival model in $\mathbb{R}^+ \times \mathbb{R}^+$

$$\bar{F}(x, y) = C_\phi(\phi^{-1}(x), \phi^{-1}(y)), \quad (38)$$

i.e., the one with survival copula $\hat{C} = C_\phi$ and marginal survival function $\bar{G} := \phi^{-1}$. In other words this is the Schur-constant model (see Example 4.9 and (29))

$$\bar{F}(x, y) = \phi^{-1}(x + y) = \bar{G}(x + y).$$

The interest of this model is due to the circumstance that the equivalences (34)–(37) become nothing else but equivalences between properties of the survival copula and those of the marginal distribution. We remind that the ageing function B of the above model (38) is the independence copula Π and, in agreement with step c), the condition $B = \Pi$ actually describes a bivariate no-ageing property.

e) We now consider a survival model, with the same survival copula $\hat{C} = C_\phi$ as in step d), but with marginal survival function \bar{G} , different from ϕ^{-1} . In this case the ageing function B is different from Π . As pointed out in Section 4, B is not necessarily a copula, however it turns out to be the Archimedean semi-copula S_φ in (30) with generator φ given in (31), i.e.,

$$\varphi(u) = \phi(\bar{G}(-\log(u))).$$

Actually $B = S_\varphi$ is a copula if and only if φ is convex. Even when this is not the case, one is still allowed to consider “extended” dependence properties of $B = S_\varphi$. We remind that such “extended” dependence properties are looked at as bivariate ageing properties of the survival model. In this perspective, the results obtained in [4, 40] can be extended to Archimedean semi-copulas and reformulated as equivalences between dependence properties of $B = S_\varphi$ and univariate ageing properties of the generator's inverse φ^{-1} .

f) We remind that one purpose of our discussion is the analysis of the interrelations among stochastic dependence of $\bar{F} = \bar{F}^{(2)}$, and ageing properties of the marginal survival function \bar{G} . In the Archimedean case of step e), stochastic dependence of $\bar{F}(x, y) = \bar{F}^{(2)}(x, y) = C_\phi(\bar{G}(x), \bar{G}(y))$ can be characterized by ageing properties of the survival function ϕ^{-1} , in view of the equivalences (34)–(37) above. By the same token, bivariate ageing can be characterized in terms of the univariate ageing property of survival function ϕ^{-1} . By imposing one fixed condition of ageing on each of the three univariate survival functions, ϕ^{-1} , φ^{-1} , \bar{G} , we respectively obtain a property of dependence, of bivariate ageing, and of univariate ageing for the survival model $\bar{F} = \bar{F}^{(2)}$. These survival functions are linked together by the relation (31), which we rewrite in the form

$$\bar{G}(x) = \phi^{-1}(\varphi(e^{-x})), \quad (39)$$

whence, imposing the fixed property of ageing on two of these survival functions implies a condition for the third one. We give an example of this procedure in Lemma 7.21 (see also Remark 7.23).

By taking again into account the equivalence properties of type (34)–(37), we convert such implications (for ϕ^{-1} , φ^{-1} , \bar{G}) into implications (for C_ϕ , S_φ , \bar{G}) of the form appearing in items 1., 2., and 3. at the beginning of this section. Finally, observing that the positive/negative univariate ageing properties of \bar{G} can be characterized by the negative/positive dependence properties of the semi-copula $S_{\bar{G}^{-1}}$, the previous implications can be rephrased as implications on the semi-copulas C_ϕ , S_φ , $S_{\bar{G}^{-1}}$ (as already pointed out in Remark 5.4 for the IFR/DFR property).

The announced conceptual path concludes with the following step.

g) From the arguments in the above steps, and in particular step f), we see that, for the Archimedean case, it is equivalent to state results in terms of stochastic dependence of semi-copulas or in terms of univariate ageing of the inverse of their generators. However the description in term of stochastic dependence allows for a more general analysis. Actually, we can deal with stochastic dependence of semi-copulas, even for non-Archimedean semi-copulas.

In the Introduction of [47] the authors claim that “*many of the modern notions of positive dependence are defined by means of some comparison of the joint distribution of X and Y with their distribution under the theoretical assumption that X and Y are independent*”. We highlight that such an approach does hold for the dependence properties considered in this paper. Indeed they can be characterized in terms of appropriate comparisons between the two models

$$\bar{F}(x, y) = \hat{C}(\bar{G}(x), \bar{G}(y)) \quad \text{and} \quad \bar{F}_\Pi(x, y) = \Pi(\bar{G}(x), \bar{G}(y)),$$

and furthermore these comparisons become suitable comparisons between \hat{C} and Π . For instance the PQD property means that

$$\bar{F}(x, y) \geq \bar{F}_\Pi(x, y) = \bar{G}(x)\bar{G}(y),$$

or equivalently

$$\hat{C}(u, v) \geq \Pi(u, v) = uv.$$

Similarly, properties of bivariate ageing we considered can be characterized in terms of appropriate comparisons with the Schur-constant case, i.e., between the two functions

$$\bar{F}(x, y) \quad \text{and} \quad \bar{G}(x + y).$$

In this respect we observe that the above two functions can be written as

$$\bar{F}(x, y) = \bar{G}(-\log(B(e^{-x}, e^{-y}))) \quad \text{and} \quad \bar{G}(x + y) = \bar{G}(-\log(\Pi(e^{-x}, e^{-y}))),$$

and therefore the mentioned comparisons become now suitable comparisons between B and Π . For instance (see [10]) the bivariate NBU property, which is defined by

$$\bar{F}(x, y) \leq \bar{G}(x + y),$$

can also be written as

$$B(u, v) \geq \Pi(u, v) = uv.$$

The remarks presented so far provide us with a justification of the definition of bivariate ageing properties of $\bar{F}(x, y)$ given in terms of dependence properties of B , as done in [12] and as stated at the beginning of this section. On this basis we are now in a position to reformulate the answers to the questions **Q1–Q3** as given in [12]:

Suppose that we start with a fixed positive dependence property and the related negative dependence property for copulas. Then, as usual, one defines

- **DEP** the model $\bar{F}(x, y)$ is positive/negative dependent iff the survival copula \hat{C} has the positive/negative dependence property.

As discussed in [47] (see Remark 5.6 therein), many positive dependence notions are defined by means of a positive dependence ordering \succeq in terms of the survival copula, rather than through the joint distribution function, namely:

$$\hat{C} \succeq \Pi. \quad (40)$$

Once the positive/negative properties are extended to semi-copulas, then the bivariate and univariate ageing properties “corresponding” to the fixed positive/negative dependence properties are respectively defined as follows

- **biv – AG** the model $\bar{F}(x, y)$ has the bivariate positive/negative ageing property iff the ageing function associated to \bar{F} , i.e.,

$$B = \Phi_\gamma(\hat{C}),$$

has the positive/negative dependence property, where γ is defined in (23) and, for any give increasing homeomorphism η , the operator Φ_η is defined in (26).

- **univ – AG** the model has the univariate positive/negative ageing property iff $S_{\bar{G}}^{-1}$ has the negative/positive dependence property.

The definition of bivariate positive ageing as a positive dependence arises in a natural way as explained in step c), whereas the choice of giving positive univariate ageing the specific form of negative dependence arises from the analysis of the Archimedean case, as justified in step f).

When the considered dependence property is defined by means of a positive dependence ordering \succeq , then the previous definitions **biv – Ag** and **univ – Ag** respectively become

$$B \succeq \Pi, \quad S_{\bar{G}}^{-1} \preceq \Pi. \quad (41)$$

As far as question **Q4** is concerned, again in [12] it is shown that the implications in items 1., 2., and 3. hold, provided suitable technical conditions are verified. Such conditions concern the interrelations between the chosen positive dependence property and the operators Φ_η . In particular we recall the condition that, for any increasing homeomorphism η

$$\Phi_\eta(\Pi) \text{ is positive dependent iff } \Phi_{\eta^{-1}}(\Pi) \text{ is negative dependent.} \quad (42)$$

As a matter of fact condition (42) does hold true for all the positive dependence properties considered in [12] and in this paper.

By taking into account the identity $S_{\bar{G}}^{-1} = \Phi_{\gamma^{-1}}(\Pi)$, when the above condition (42) does hold, then

$$S_{\bar{G}}^{-1} = \Phi_{\gamma^{-1}}(\Pi) \text{ is negative/positive dependent} \quad \text{iff} \quad \Phi_\gamma(\Pi) \text{ is positive/negative dependent,}$$

and therefore, we now define univariate positive/negative ageing property, in a form different, but equivalent to **univ – Ag**:

- **univ – Ag'** the model has the univariate positive/negative ageing property iff the ageing function associated to the model $\bar{F}_\Pi(x, y) = \bar{G}(x)\bar{G}(y)$ has the bivariate positive/negative ageing property, i.e., the Archimedean semi-copula $\Phi_\gamma(\Pi)$ has the positive/negative dependence property.

We also add that when the considered dependence property is defined by means of a positive dependence ordering as in (40), then the above definitions **biv** – **Ag** and **univ** – **Ag'** can be written in terms of the ageing functions of the two models $\bar{F}(x, y)$ and $\bar{F}_\Pi(x, y)$, namely

$$\Phi_\gamma(\hat{C}) \succeq \Pi, \quad \Phi_\gamma(\Pi) \succeq \Pi. \quad (43)$$

As emphasized by (41), the original definitions **biv** – **Ag** and **univ** – **Ag** manifest an asymmetric form, which is eliminated in the alternative formulation **biv** – **Ag** and **univ** – **Ag'**.

Remark 6.1. *The path a)–g), illustrating the definitions of dependence, univariate and bivariate ageing and the interrelations among them, has brought us from the analysis of Archimedean models to the analysis of more general (non-Archimedean) models.*

We notice however that Archimedean semi-copulas play a fundamental role even for the general case. In particular such a role manifests in the definitions of univariate ageing and of Schur-constant models, and also in the condition (42), which actually deals only with Archimedean semi-copulas.

We now conclude this section with a couple of further remarks concerning stochastic dependence.

First of all we point out that in the present setting the dependence properties, to be considered, should not be necessarily defined on the class of all copulas, but only on convenient subclasses. Consequently also the extension of dependence properties to semi-copulas can be restricted to corresponding subclasses. We notice that, also in contexts different from ours, the study of the dependence notions considered in the literature has been often restricted to special subclasses of copulas (see, e.g., [4, 16, 40]). In the frame of our setting, an instance of such a circumstance, emerges in [12] when dealing with the PKD property, which has been analyzed only for the class Archimedean copulas and semi-copulas. In order to treat the PKD property for a larger class of models, we need to introduce the concept of generalized Kendall distribution and the related equivalence classes of semi-copulas (see Definitions 7.3 and 7.8 below). Such an extension will be obtained in the next Section 7.

Finally we point out that the concepts of positive dependence which have been considered in the literature, often require a set of desirable properties, among which the condition that positive dependence be stronger than PQD. Such a condition, also required in [12], does not appear however to be strictly needed. In particular, the positive dependence property PKD, which implies PQD for Archimedean semi-copulas as recalled above, does not imply PQD on the family of all semi-copulas.

7 Relating Dependence to Ageing Properties: the IFRA/DFRA case

In the Reliability literature (see [7, 13, 34, 49]) relevant concepts of positive/negative one-dimensional ageing are those of Increasing/Decreasing Failure Rate in Average (IFRA/DFRA), which, respectively, generalize those of IFR/DFR and are defined as follows.

Definition 7.1. *A one dimensional survival function \bar{G} is IFRA if and only if*

$$x \mapsto -\frac{\log \bar{G}(x)}{x}$$

is an increasing function (see e.g. [7]); \bar{G} is DFRA if and only if it is a non-decreasing function.

This notion of ageing is strictly related with the notion of PKD/NKD. As shown in [4], and recalled in Section 6 (see (35)), an Archimedean copula $\hat{C} = C_\phi$ (with a differentiable generator ϕ) is NKD/PKD if and only if ϕ^{-1} is a IFRA/DFRA survival function, respectively. Namely, as it is well known (see [27]),

$$K_{C_\phi}(t) = t - \phi(t)/\phi'(t), \quad (44)$$

and therefore C_ϕ is PKD if and only if

$$K_{C_\phi}(t) \leq K_\Pi(t) \Leftrightarrow \phi(t)/\phi'(t) \geq t \log(t) \Leftrightarrow \phi^{-1} \text{ is IFRA.}$$

Let us now come to the exchangeable model $\bar{F}(x, y)$ in (29) with marginal survival function \bar{G} and Archimedean survival copula $\hat{C} = C_\phi$ (with a differentiable generator ϕ , as above). As we observed in Remark 4.11, the corresponding ageing function is $B = S_\phi$, where we are using the notation given in (30) and (31), i.e., $\varphi(u) = \phi(\bar{G}(-\log(u)))$. Inspired by the latter circumstance, in [12] the PKD/NKD property was extended to Archimedean ageing functions (not necessarily copulas) by requiring $\varphi(u)/\varphi'(u) \geq u \log(u)$, and it was pointed out that the above equivalence between NKD/PKD and IFRA/DFRA holds even for Archimedean ageing functions.

For such an exchangeable model, a result similar to Proposition 5.3 was given in [12] (see Example 7.4 therein), for what concerns relations between PKD/NKD properties for ageing functions and IFRA/DFRA. Such a result can be rephrased here as follows.

Proposition 7.2. *For the above model the following implications hold.*

- (i) *If \bar{G} is IFRA and $\hat{C} = C_\phi$ is PKD then $B = S_\phi$ is PKD.*
- (ii) *If \bar{G} is DFRA and $\hat{C} = C_\phi$ is NKD then $B = S_\phi$ is NKD.*
- (iii) *If $B = S_\phi$ is PKD and \bar{G} is DFRA then $\hat{C} = C_\phi$ is PKD.*
- (iv) *If $B = S_\phi$ is NKD and \bar{G} is IFRA then $\hat{C} = C_\phi$ is NKD.*
- (v) *If $\hat{C} = C_\phi$ is PKD and $B = S_\phi$ is NKD then \bar{G} is DFRA.*
- (vi) *If $\hat{C} = C_\phi$ is NKD and $B = S_\phi$ is PKD then \bar{G} is IFRA.*

The previous Proposition 7.2 deals then with the concept of PKD for an Archimedean ageing function B . Our aim in this section is to show that the previous result admits a natural extension to a larger class of exchangeable models, where the survival copula \hat{C} is not necessarily Archimedean. On this purpose, in the next subsection we extend the concepts of Kendall distributions and NKD/PDK property to semi-copulas.

7.1 Generalized Kendall distributions and PKD/NKD property for semi-copulas

The next definition of generalized Kendall distributions for semi-copulas is given in analogy with Lemma 4.5.

Definition 7.3. *Let $S : [0, 1]^2 \rightarrow [0, 1]$ be a semi-copula, and set*

$$S_u^{-1}(t) := \sup\{v : S(u, v) \leq t\},$$

the generalized inverse of $v \mapsto S(u, v)$, with the convention that $\sup(\emptyset) = 0$.

The generalized Kendall distribution associated to S is the function $K_S : [0, 1] \rightarrow \mathbb{R}^+$, defined by

$$K_S(t) := \sup_{\mathcal{P}} \left\{ \sum_{i \in I} [S(u_{i+1}, v_i) - S(u_i, v_i)], \text{ with } v_i = S_{u_{i+1}}^{-1}(t) \right\}, \quad (45)$$

where \mathcal{P} is the class of all finite partitions of $[0, 1]$, of the form $\{u_i, i \in I\}$, such that $u_0 = 0$ and $u_i \leq u_{i+1}$.

Example 7.4. *Let $S(u, v) = S_\phi(u, v) = \varphi^{-1}[\varphi(u) + \varphi(v)]$ be an Archimedean semi-copula (see Subsection 4.3) where $\varphi(t)$ is strictly decreasing, differentiable, and such that $\varphi(1) = 0$. Furthermore assume that $\varphi'(t) > 0$ in $(0, 1)$. Then*

$$K_{S_\phi}(t) = t - \frac{\varphi(t)}{\varphi'(t)}.$$

In fact, when $\varphi(t)$ is also convex, then S_ϕ is a copula and the above identity coincides with (44). Moreover, still remaining in this case, by Lemma 4.5, $K_{S_\phi}(t)$ can be computed by (45), i.e.,

$$K_{S_\phi}(t) = t - \frac{\varphi(t)}{\varphi'(t)} = \sup_{\mathcal{P}} \left\{ \sum_{i \in I} [S_\phi(u_{i+1}, v_i) - S_\phi(u_i, v_i)], \text{ with } v_i = (S_\phi)_{u_{i+1}}^{-1}(t) \right\}.$$

The general case can be deduced observing that the latter equality also holds when φ is not convex.

We notice that, with this new definition, the generalized Kendall distribution has the properties

$$K_S(t) \geq t, \quad t \in [0, 1], \quad K_S(0) = 0, \quad K_S(1) = 1, \quad (46)$$

for any semi-copula S as it happens for copulas. Indeed, first of all, w.l.o.g., we may consider only partitions containing t as an element and observe that, when $u_{i+1} \leq t$, then $v_i = S_{u_{i+1}}^{-1}(t) = 1$, so that $S(u_{i+1}, v_i) - S(u_i, v_i) = u_{i+1} - u_i$. Therefore

$$K_S(t) := t + \sup_{\mathcal{P}} \left\{ \sum_{\substack{i \in I \\ u_{i+1} > t}} [S(u_{i+1}, v_i) - S(u_i, v_i)], \text{ with } v_i = S_{u_{i+1}}^{-1}(t) \right\}, \quad (47)$$

and the first property in (46) follows by observing that the increments $S(u_{i+1}, v_i) - S(u_i, v_i)$ are non-negative. The other two properties are obvious.

Furthermore we point out that, when S is a Lipschitz semi-copula, i.e.,

$$|S(u, v) - S(u', v')| \leq L_S(|u - u'| + |v - v'|),$$

then

$$K_S(t) \leq L_S \quad \text{in } [0, 1].$$

Indeed in the latter case one has $0 \leq S(u_{i+1}, v_i) - S(u_i, v_i) \leq u_{i+1} - u_i$. In particular $K_S(t) \leq 1$ when $L_S = 1$, i.e., when S is a quasi-copula (see, e.g., [23]).

Finally we observe that, when S is not a copula, $K_S(t)$ may not be a probability distribution function as shown by the following example.

Example 7.5. Let $S(u, v) = S_\varphi(u, v) := \varphi^{-1}[\varphi(u) + \varphi(v)]$, with $\varphi(t) = \cos(\frac{\pi}{2}t)$, i.e.,

$$S_\varphi(u, v) := \arccos\left(\frac{2}{\pi}[\cos(\frac{\pi}{2}u) + \cos(\frac{\pi}{2}v)]\right).$$

Then, by taking into account Example 7.4,

$$K_S(t) = t - \frac{2}{\pi} \frac{\cos(\frac{\pi}{2}t)}{\sin(\frac{\pi}{2}t)} = t + \frac{2}{\pi} \frac{\cos(\frac{\pi}{2}t)}{\sin(\frac{\pi}{2}t)}, \quad t \in (0, 1),$$

is not a distribution function, since, in particular, $K_S(t)$ converge to ∞ as t goes to 0.

Definition 7.6. A semi-copula S has the PKD property when the generalized Kendall distribution satisfies the following inequality

$$K_S(t) \leq K_\Pi(t) = t - t \log(t), \quad t \in [0, 1], \quad (48)$$

and similarly for the NKD property.

We recall that, for two copulas C_1, C_2 , one says that $C_1 \succeq_{PKD} C_2$ when $K_{C_1}(t) \leq K_{C_2}(t)$. This ordering can be naturally extended to semi-copulas so that the above definition of Positive Kendall Dependence for S is equivalent to $S \succeq_{PKD} \Pi$.

Remark 7.7. We point out that Proposition 7.2 also holds when giving the concept of PKD/NKD property our extended Definition 7.6. Indeed Example 7.4 shows that, for Archimedean semi-copulas, our extension of the PKD property coincides with the one proposed in [12], which concerns exclusively with Archimedean semi-copulas.

Our aim now is to show that Proposition 7.2 admits a natural extension to a larger class of exchangeable models, where the survival copula \widehat{C} is not necessarily Archimedean. On this purpose we start by extending to semi-copulas the definition of the equivalence relation based on the Kendall distribution, introduced in [46] for copulas.

Definition 7.8. Two bivariate semi-copulas $S_1(u, v)$ and $S_2(u, v)$ are Kendall-equivalent, written as $S_1 \equiv_K S_2$, if and only if the Kendall distributions $K_{S_1}(t)$ and $K_{S_2}(t)$ do coincide.

For a fixed semi-copula $S(u, v)$ we denote by \mathcal{K}_S the equivalence class containing S , and by \mathcal{C}_S the subclass of the exchangeable semi-copulas belonging to \mathcal{K}_S .

Before continuing we present three examples related with the above definitions. Example 7.9 shows a case of a non-exchangeable copula belonging to the same Kendall equivalence class of two exchangeable copulas, one of which is Archimedean. Example 7.11 shows that each copula in the Archimax family is Kendall-equivalent to an Archimedean copula or to the maximal copula $M(u, v)$. Starting from these examples, in Example 7.12 we exhibit instances of non-Archimedean exchangeable semi-copulas belonging to the same Kendall equivalence class of Archimedean semi-copulas.

Example 7.9. Consider the following copulas

$$C_0(u, v) = \min\left(u, \max\left(\frac{v}{2}, u + v - 1\right)\right), \quad C_1(u, v) = \max\left(0, u + v - 1, \min\left(u, v - \frac{1}{2}\right), \min\left(v, u - \frac{1}{2}\right)\right),$$

and the Archimedean copula C_ϕ , with generator

$$\phi(t) = \begin{cases} \frac{1}{2t} & \text{if } t \in [0, \frac{1}{2}), \\ 2(1-t) & \text{if } t \in [\frac{1}{2}, 1]. \end{cases}$$

As observed in Example 1 in [46], ϕ is a convex generator and $K_{C_0}(t) = K_{C_1}(t) = K_{C_\phi}(t) = \min(2t, 1)$. Then $C_0, C_1 \in \mathcal{K}_{C_\phi}$, $C_1 \in \mathcal{C}_{C_\phi}$, but $C_0 \notin \mathcal{C}_{C_\phi}$.

For the next example we need to recall the definition of Archimax copulas, introduced in [15].

Definition 7.10. Let $\tilde{\phi}$ be a generator of the Archimedean copula C_ϕ , and let $A : [0, 1] \rightarrow [0, 1/2]$ be a convex function such that $A(t) \in [\max(t, 1-t), 1]$, for all $t \in [0, 1]$. The Archimax copula $C_{\tilde{\phi}, A}(u, v)$ is defined as

$$C_{\tilde{\phi}, A}(u, v) := \tilde{\phi}^{-1}\left([\tilde{\phi}(u) + \tilde{\phi}(v)]A(\tilde{\phi}(u)/[\tilde{\phi}(u) + \tilde{\phi}(v)])\right).$$

As shown in [15], the Kendall- τ coefficient of $C_{\tilde{\phi}, A}$ can be computed as $\tau_{\tilde{\phi}, A} = \tau_A + (1 - \tau_A)\tau_{\tilde{\phi}}$, where

$$\tau_A = \int_0^1 \frac{t(1-t)}{A(t)} dA'(t) \in [0, 1],$$

and $\tau_{\tilde{\phi}}$ is the Kendall- τ of $C_{\tilde{\phi}}$.

Example 7.11. The interest of the family of Archimax copulas in the present context relies on the following observation: Under the condition $\tau_A \in [0, 1)$, the function

$$\phi(t) := (\tilde{\phi}(t))^{1/(1-\tau_A)},$$

is a convex generator and the copula $C_{\tilde{\phi}, A}$ belongs to the equivalence class \mathcal{K}_{C_ϕ} . Indeed, on the one hand, as observed in [28], by Proposition 5.1 in [15],

$$K_{C_{\tilde{\phi}, A}}(t) = t - (1 - \tau_A) \frac{\tilde{\phi}(t)}{\tilde{\phi}'(t)}.$$

On the other hand, we notice that

$$K_{C_\phi}(t) = t - \frac{\phi(t)}{\phi'(t)} = t - \frac{(\tilde{\phi}(t))^{1/(1-\tau_A)}}{(\tilde{\phi}(t)^{1/(1-\tau_A)})'(t)} = t - (1 - \tau_A) \frac{\tilde{\phi}(t)}{\tilde{\phi}'(t)}.$$

When $A(t) = A(1 - t)$, i.e., when $A(t)$ is symmetric with respect to $\bar{t} = 1/2$, then $C_{\tilde{\phi}, A}$ is exchangeable and therefore belongs to \mathcal{C}_{C_ϕ} . As an example we can take, for $\theta \in [0, 1]$, $A(t) = A_\theta(t) := \theta t^2 - \theta t + 1 = 1 - \theta t(1 - t)$, so that

$$C_{\tilde{\phi}, A}(u, v) = \tilde{\phi}^{-1} \left([\tilde{\phi}(u) + \tilde{\phi}(v)] \left(1 - \theta \frac{\tilde{\phi}(u)}{\tilde{\phi}(u) + \tilde{\phi}(v)} \frac{\tilde{\phi}(v)}{\tilde{\phi}(u) + \tilde{\phi}(v)} \right) \right).$$

A different situation is met when $\tau_A = 1$, indeed then $K_{C_{\tilde{\phi}, A}}(t) = t$ and therefore $C_{\tilde{\phi}, A} \in \mathcal{C}_M$. For example one can consider the case $A(t) = \underline{A}(t) := \max(t, 1 - t)$, where

$$C_{\tilde{\phi}, \underline{A}}(u, v) = \tilde{\phi}^{-1} \left([\tilde{\phi}(u) + \tilde{\phi}(v)] \max \left(\frac{\tilde{\phi}(u)}{\tilde{\phi}(u) + \tilde{\phi}(v)}, \frac{\tilde{\phi}(v)}{\tilde{\phi}(u) + \tilde{\phi}(v)} \right) \right) = \max(\tilde{\phi}(u), \tilde{\phi}(v)).$$

Example 7.12. Consider the model with Weibull marginal survival function $\bar{G}(x) = e^{-x^\beta}$, and survival copula $\hat{C} = C_1$, where C_1 is defined in the previous Example 7.9. Then the ageing function

$$B(u, v) = \exp \left\{ -|\log(C_1(e^{-|\log(u)|^\beta}, e^{-|\log(v)|^\beta}))|^{1/\beta} \right\}$$

belongs to $\mathcal{C}_{S_\phi} \subset \mathcal{K}_{S_\phi}$, where

$$\varphi(t) = \begin{cases} \frac{1}{2 \exp\{-|\log(t)|^\beta\}} & \text{if } t \in [0, \frac{1}{2}) \\ 2(1 - \exp\{-|\log(t)|^\beta\}) & \text{if } t \in [\frac{1}{2}, 1] \end{cases}$$

The above claim could be checked directly or it can be seen as a consequence of Proposition 7.18, below.

Note that when $\beta < 1$ then \bar{G} and φ are convex and S_ϕ is copula, while when $\beta > 1$ then \bar{G} and φ are not convex and S_ϕ is only a semi-copula. For $\beta = 1$, i.e., when the marginal distribution is exponential of parameter 1, then $B = \hat{C}(= C_1)$, as already observed in Section 6 (step c)).

Let us now consider the function

$$S_{\tilde{\phi}, A}(u, v) := \tilde{\phi}^{-1} \left([\tilde{\phi}(u) + \tilde{\phi}(v)] A(\tilde{\phi}(u)/[\tilde{\phi}(u) + \tilde{\phi}(v)]) \right), \quad (49)$$

obtained by replacing the convex generator $\tilde{\phi}$ with a non-convex generator $\tilde{\varphi}$ in Definition 7.10 of an Archimax copula. As it is easy to check, $S_{\tilde{\varphi}, A}$ is a semi-copula, and we will call it an Archimax semi-copula. Then, similarly to Example 7.11, one gets that $S_{\tilde{\varphi}, A}$ is Kendall-equivalent to the semi-copula S_ϕ , where $\phi = \tilde{\varphi}^{1/(1-\tau_A)}$, when $\tau_A < 1$. Indeed the generalized Kendall distribution of $S_{\tilde{\varphi}, A}$ can be obtained by following a procedure analogous to the one in Example 7.4 for the Archimedean case.

In view of Definitions 7.6 and 7.8 we can claim that if a semi-copula S is PKD/NKD then all the semi-copulas in the equivalence class \mathcal{C}_S share the same Kendall dependence property. The latter claim suggests us that the appropriate extension of Proposition 7.2 is obtained by replacing the conditions $\hat{C} = C_\phi$ and $B = S_\phi$ with the conditions $\hat{C} \in \mathcal{C}_{C_\phi}$ and $B \in \mathcal{C}_{S_\phi}$, respectively. A precise result will be given in Theorem 7.19 below. To this end we also introduce the following further definition.

Definition 7.13. A semi-copula S is Pseudo-Archimedean whenever there exists an Archimedean semi-copula S_ϕ such that $S \in \mathcal{K}_{S_\phi}$. The generator ϕ of S_ϕ will be referred to as the pseudo-generator of S .

Clearly when a semi-copula S is exchangeable and Pseudo-Archimedean with pseudo-generator ϕ then $S \in \mathcal{C}_{S_\phi}$.

In view of the above definition, Examples 7.11 and 7.12 can be rephrased by saying that, when $\tau_A < 1$, then the Archimax copula $C_{\tilde{\phi}, A}$ and semi-copula $S_{\tilde{\varphi}, A}$ are Pseudo-Archimedean, with pseudo-generators $\phi = (\tilde{\phi})^{1/(1-\tau_A)}$ and $\varphi = (\tilde{\varphi})^{1/(1-\tau_A)}$, respectively.

We end this subsection with an example of a family of PKD copulas.

Example 7.14. Assume that the Archimedean copula $C_{\tilde{\phi}}$ is PKD. Then, any Archimax copula $C_{\tilde{\phi},A}$ in Example 7.11 is PKD, whenever $\tau_A \in [0, 1)$. Indeed by assumption

$$K_{C_{\tilde{\phi}}} = t - \frac{\tilde{\phi}(t)}{\tilde{\phi}'(t)} \leq K_{\Pi}(t) = t - t \log(t)$$

and

$$K_{C_{\tilde{\phi},A}} = t - (1 - \tau_A) \frac{\tilde{\phi}(t)}{\tilde{\phi}'(t)} \leq K_{C_{\tilde{\phi}}}.$$

Obviously the same holds for C_{ϕ} , with $\phi = (\tilde{\phi})^{1/(1-\tau_A)}$, the pseudo-generator of $C_{\tilde{\phi},A}$. In view of Example 7.12 similar considerations hold when considering an Archimax semi-copula. On the contrary, when $C_{\tilde{\phi}}$ is NKD, no similar implication can be obtained.

7.2 Generalization of Proposition 7.2

In this subsection we show the generalization of Proposition 7.2 to the models $\bar{F}(x, y) = \hat{C}(\bar{G}(x), \bar{G}(y))$, with \hat{C} (not necessarily Archimedean) and \bar{G} satisfying the following conditions:

- (P1) for any $u \in (0, 1)$ the function $v \mapsto \hat{C}_u(v) := \hat{C}(u, v)$ is strictly increasing and continuous (and therefore invertible),
- (P2) the function $x \mapsto \bar{G}(x)$ is strictly positive in $[0, \infty)$, strictly decreasing, and continuous (and therefore invertible).

For a copula C satisfying (P1), Genest and Rivest in [28] have proved a transformation result for Kendall distribution (see Proposition 1 therein), which we reformulate in our notation as follows. Fix an increasing homeomorphism $\eta : [0, 1] \rightarrow [0, 1]$, when $C^*(u, v) := \Phi_{\eta}(C)(u, v) = \eta^{-1}(C(\eta(u), \eta(v)))$ is a copula, then the Kendall distribution K_{C^*} can be obtained in terms of K_C as follows:

$$K_{C^*}(t) = t + \eta'(\eta^{-1}(t)) [K_C(\eta^{-1}(t)) - \eta^{-1}(t)].$$

In the perspective of generalizing Proposition 7.2, we will extend the above transformation result to the case when C^* is not necessarily a copula. In this way we will obtain the generalized Kendall distribution for the ageing function B (see Proposition 7.16 below). Indeed, when $\bar{G}(x)$ satisfies (P2), then the functions $\gamma(u)$ and $\gamma^{-1}(z)$ are increasing homeomorphism and then, taking $\eta = \gamma$ and $C = \hat{C}$, the function C^* coincides with the ageing function B , namely

$$B(u, v) = \Phi_{\gamma}(\hat{C})(u, v) = \gamma(\hat{C}(\gamma^{-1}(u), \gamma^{-1}(v))).$$

Remark 7.15. In the proof of Proposition 7.16 we need the following technical property of the ageing function B . The conditions (P1) and (P2), together with the hypotheses that $\gamma(u)$ and $\gamma^{-1}(z)$ are increasing homeomorphisms, imply that for any $u \in (0, 1)$ the function $v \mapsto B_u(v) := B(u, v)$ is strictly increasing and continuous (and therefore invertible). In conclusion the generalized inverse $\hat{C}_u^{-1}(t)$ and $B_u^{-1}(t)$ are true inverse functions.

Proposition 7.16. Assume properties (P1) and (P2). Then, for any $t \in [0, 1]$,

$$K_{\hat{C}}(t) = t + \int_t^1 \frac{\partial \hat{C}}{\partial u}(u, v)|_{v=\hat{C}_u^{-1}(t)} du, \quad (50)$$

Assume furthermore that the density g of G is continuous, then

$$K_B(t) = t + \gamma'(\gamma^{-1}(t)) [K_{\hat{C}}(\gamma^{-1}(t)) - \gamma^{-1}(t)] \quad (51)$$

$$K_B(t) = t + \frac{1}{\frac{d}{dt} \bar{G}(-\log(t))} \left[K_{\hat{C}}(\bar{G}(-\log(t))) - \bar{G}(-\log(t)) \right], \quad (52)$$

$$K_B(t) = t + \int_t^1 \frac{\partial B}{\partial u}(u, v)|_{v=B_u^{-1}(t)} du. \quad (53)$$

Proof. We essentially need to prove (51) for $t \in (0, 1)$. In fact

- Equation (50) is exactly Proposition 1 in [28],
- Equation (52) just amounts to a reformulation of (51), by rewriting $\gamma^{-1}(t)$ and $\gamma'(\gamma^{-1}(t))$ in terms of \bar{G} .
- Equation (53) easily follows by (50) and (51) by taking into account that

$$K_B(t) = t + \gamma'(\gamma^{-1}(t)) \int_{\gamma^{-1}(t)}^1 \frac{\partial \hat{C}}{\partial u}(u, v)|_{v=\hat{C}_u^{-1}(\gamma^{-1}(t))} du, \quad (54)$$

and by the change of variable $u' = \gamma(u)$ within the previous integral. In this respect we point out that $v = \hat{C}_u^{-1}(\gamma^{-1}(t))$ iff $\gamma(v) = B_{\gamma(u)}^{-1}(t)$, $\hat{C}(u, v)|_{v=\hat{C}_u^{-1}(\gamma^{-1}(t))} = t$, and $1 = \gamma^{-1}(1)$.

- Equation (51) is obvious for $t = 0$ and $t = 1$.

In view of (47), the proof of (51) for $t \in (0, 1)$ is obtained by proving that

$$K_B(t) - t = \sup_{\mathcal{P}} \left\{ \sum_{\substack{i \in I \\ u_{i+1} > t}} [B(u_{i+1}, v_i) - B(u_i, v_i)], \text{ with } v_i = B_{u_{i+1}}^{-1}(t) \right\}$$

coincides with $\gamma'(\gamma^{-1}(t)) [K_{\hat{C}}(\gamma^{-1}(t)) - \gamma^{-1}(t)]$. In fact, setting $w_i = \gamma^{-1}(u_i)$, $z_i = \gamma^{-1}(v_i)$ and observing that $z_i = \gamma^{-1}(B_{u_{i+1}}^{-1}(t)) = \hat{C}_{w_{i+1}}^{-1}(\gamma^{-1}(t))$, we can write $K_B(t) - t$ as

$$\sup \left\{ \sum_{\substack{i \in I \\ w_{i+1} > \gamma^{-1}(t)}} [\gamma(\hat{C}(w_{i+1}, z_i)) - \gamma(\hat{C}(w_i, z_i))], z_i = \hat{C}_{w_{i+1}}^{-1}(\gamma^{-1}(t)) \right\}.$$

Setting $\Delta_i = \hat{C}(w_{i+1}, z_i) - \hat{C}(w_i, z_i)$ we immediately get

$$\gamma(\hat{C}(w_{i+1}, z_i)) - \gamma(\hat{C}(w_i, z_i)) = \gamma'(\hat{C}(w_{i+1}, z_i)) \Delta_i + \text{Err}_i,$$

with $\text{Err}_i = [\gamma'(\gamma^{-1}(t) + \theta_i \Delta_i) - \gamma'(\gamma^{-1}(t))] \Delta_i$, for a suitable $\theta_i \in (0, 1)$.

As it is well known, any copula is a Lipschitz function, with Lipschitz constant equal to 1, therefore $\hat{C}(u, v)$ being a copula, we have

$$0 \leq \Delta_i = \hat{C}(w_{i+1}, z_i) - \hat{C}(w_i, z_i) \leq w_{i+1} - w_i \leq \delta,$$

where we have set $\delta := \max(w_{i+1} - w_i; t \leq w_i \leq w_{i+1} \leq 1)$.

Without loss of generality we may assume that $t + \delta < 1$, so that

$$\text{Err}_i \leq \sup_{s \in [t, t+\delta]} |\gamma'(\gamma^{-1}(s)) - \gamma'(\gamma^{-1}(t))| (w_{i+1} - w_i).$$

Finally, observing that $\hat{C}(w_i, z_i) = \gamma^{-1}(t)$, we have that

$$\begin{aligned} & \left| \sum_{\substack{i \in I \\ w_{i+1} > \gamma^{-1}(t)}} [\gamma(\hat{C}(w_{i+1}, z_i)) - \gamma(\hat{C}(w_i, z_i))] - \sum_{\substack{i \in I \\ w_{i+1} > \gamma^{-1}(t)}} \gamma'(\gamma^{-1}(t)) \Delta_i \right| \\ & \leq \sum_{\substack{i \in I \\ w_{i+1} > \gamma^{-1}(t)}} \sup_{s \in [t, t+\delta]} |\gamma'(\gamma^{-1}(s)) - \gamma'(\gamma^{-1}(t))| (w_{i+1} - w_i) \\ & = \sup_{s \in [t, t+\delta]} |\gamma'(\gamma^{-1}(s)) - \gamma'(\gamma^{-1}(t))| \end{aligned}$$

Recalling that $g(x)$ denotes the continuous density of G , we rewrite $\gamma'(\gamma^{-1}(t)) = \frac{t}{g(-\log(t))}$. By assumption the latter function is continuous in $(0, 1)$, and the proof of (51) is accomplished by observing that

$$\sup \left\{ \sum_{\substack{i \in I \\ w_{i+1} > \gamma^{-1}(t)}} \Delta_i, \text{ with } z_i = \hat{C}_{w_{i+1}}^{-1}(\gamma^{-1}(t)) \right\} = K_{\hat{C}}(\gamma^{-1}(t)) - \gamma^{-1}(t).$$

□

Remark 7.17. Apparently, the equation (51) is nothing else but an equivalent form to write the thesis of Proposition 20 in [41]. Actually the difference between the two results relies on the present generalization of Kendall distributions to semi-copulas. In [41] such a generalization was not considered, rather the right hand side of (53) was taken as an operator on ageing functions. Here, on the contrary, $K_B(t)$ is defined directly by (45), or, equivalently by (47), and equation (53) is obtained as a direct consequence.

The present approach allows us to show that B is Pseudo-Archimedean if and only if \widehat{C} is such, and to obtain the pseudo-generators of them. Finally, as a crucial issue, we can thus obtain the relation tying \overline{G} with such pseudo-generators. The precise statements follow:

Proposition 7.18. Assume conditions **(P1)** and **(P2)**. Fix a $t_0 \in (0, 1)$ and define

$$\phi(t) := \exp \left\{ \int_{t_0}^t \frac{1}{s - K_{\widehat{C}}(s)} ds \right\}, \quad (55)$$

and

$$\varphi(t) := \exp \left\{ \int_{\gamma(t_0)}^t \frac{1}{s - K_B(s)} ds \right\}. \quad (56)$$

Then

(i) ϕ is the generator of an Archimedean copula, namely C_ϕ , and φ is the generator of an Archimedean semi-copula, S_φ ,

(ii) \widehat{C} and B are Pseudo-Archimedean and, in particular, $\widehat{C} \in \mathcal{C}_{C_\phi}$ and $B \in \mathcal{C}_{S_\varphi}$.

Assume furthermore that the density g of G is continuous, then the following equivalent relations hold

- (iii)
- (a) $\varphi(t) = \phi(\gamma^{-1}(t)) = \phi(\overline{G}(-\log(t)))$
 - (b) $\overline{G}(x) = \phi^{-1}(\varphi(e^{-x}))$

Proof. By Proposition 1.2 in [27], and as observed in [46], we know that, for any "true" Kendall distribution function $K(t)$ of a bivariate survival function, the function $\phi_K(t) := \exp \left\{ \int_{t_0}^t \frac{1}{s - K(s)} ds \right\}$ is the generator of an Archimedean copula, i.e., $\phi_K(t)$ is increasing and convex, with $\phi_K(0) = 1$, if and only if $K(t^-) > t$ for all $t \in (0, 1)$. Moreover $K(t)$ is the Kendall distribution of the Archimedean copula C_{ϕ_K} .

Under **(P1)** and **(P2)**, by (50) in the previous Proposition 7.16, the Kendall distribution function of \widehat{C} satisfies $K_{\widehat{C}}(t^-) > t$ for all $t \in (0, 1)$, so that $\phi(t)$ is the generator of C_ϕ , and clearly

$$K_{\widehat{C}}(t) = K_{C_\phi}(t) = t - \frac{\phi(t)}{\phi'(t)},$$

and (i)-(ii) follow as far as $\phi(t)$ and \widehat{C} are concerned.

Similar results hold except for the convexity property (see also [41]), as far as $\varphi(t)$ and B are concerned, in particular

$$K_B(t) = K_{S_\varphi}(t) = t - \frac{\varphi(t)}{\varphi'(t)}.$$

Finally, by using the above expressions of $K_{\widehat{C}}(t)$ and $K_B(t)$, together with (51), one immediately gets that $\varphi(t) = \phi(\gamma^{-1}(t))$, i.e., (iii) follows. □

We are now in a position to extend Proposition 7.2 by considering the case when the survival copula \widehat{C} is Pseudo-Archimedean. Actually such a generalization is proved by collecting the results in Proposition 7.18, Remark 7.7, and Proposition 7.2 itself.

Theorem 7.19. Let $F(x, y)$ be a bivariate exchangeable model satisfying the conditions **(P1)** and **(P2)**. Besides the standing assumptions on \bar{G} , assume that the corresponding density $g(x)$ is continuous. Finally assume that the generator $\phi(t)$ defined in (55) is differentiable. Then

- (i) If \bar{G} is IFRA and \hat{C} is PKD then B is PKD.
- (ii) If \bar{G} is DFRA and \hat{C} is NKD then B is NKD.
- (iii) If B is PKD and \bar{G} is DFRA then \hat{C} is PKD.
- (iv) If B is NKD and \bar{G} is IFRA then \hat{C} is NKD.
- (v) If \hat{C} is PKD and B is NKD then \bar{G} is DFRA.
- (vi) If \hat{C} is NKD and B is PKD then \bar{G} is IFRA.

Example 7.20. When $\hat{C} = C_{\tilde{\phi}, A}$ is an Archimax copula, with $A(t) = A(1 - t)$, then the ageing function $B = S_{\tilde{\phi}, A}$, as defined in (49), with $\tilde{\phi} = \phi(\bar{G}(-\log(u)))$. Moreover, when $\tau_A < 1$, by Example 7.14 and Proposition 7.18, then B is Kendall-equivalent to S_ϕ , with $\phi(u) = \phi(\bar{G}(-\log(u)))$, where $\phi = (\tilde{\phi})^{1/(1-\tau_A)}$. In particular, when S_ϕ is PKD and $\bar{G}(x) = e^{-x^\beta}$, with $\beta < 1$, then B is PKD and \bar{G} is DFR and therefore also DFRA. As a consequence of the above Theorem 7.19, the Archimax copula \hat{C} is PKD. Similarly we can obtain that S_ϕ PKD and C_ϕ NKD necessarily imply that \bar{G} is IFRA, whence $\beta \geq 1$.

We have proved the above Theorem 7.19 by using, among other results, Proposition 7.2, presented in [12]. An alternative proof, avoiding Proposition 7.2, can be given by using the following Lemma 7.21 concerning the IFRA/DFRA property of three survival functions \bar{H}_0 , \bar{H}_1 , and \bar{H}_2 . Indeed, by setting $\bar{H}_0 := \bar{G}$, $\bar{H}_1 := \phi^{-1}$, and $\bar{H}_2 := \varphi^{-1}$ in Lemma 7.21, relation (57) becomes relation (b) in (iii) of Theorem 7.19, and in its turn it coincides with equation (39). Then, in the spirit of step g) in the previous Section 6, under the conditions of Proposition 7.16 Theorem 7.19 follows by taking into account that

- (a) the generalized Kendall distribution of a Pseudo-Archimedean semi-copula coincides with the generalized Kendall distribution of a true Archimedean semi-copula,
- (b) the equivalences (35) relating the NKD/PKD properties of the Archimedean copula C_ϕ and semi-copula S_ϕ to the univariate IFRA/DFRA ageing properties of the survival functions H_1 and H_2 can be extended to Archimedean semi-copulas.

Lemma 7.21. Let $\bar{H}_1(x)$ and $\bar{H}_2(x)$ be two continuous survival functions, strictly decreasing on the common support

$$\sup\{x : \bar{H}_1(x) > 0\} = \sup\{x : \bar{H}_2(x) > 0\}.$$

Let $\bar{G}(x)$ be the survival function defined by

$$\bar{H}_0(x) := \bar{H}_1(\bar{H}_2^{-1}(e^{-x})). \quad (57)$$

Then

- (i) If \bar{H}_0 is IFRA and \bar{H}_1 is DFRA then \bar{H}_2 is DFRA.
- (ii) If \bar{H}_0 is DFRA and \bar{H}_1 is IFRA then \bar{H}_2 is IFRA.
- (iii) If \bar{H}_2 is DFRA and \bar{H}_0 is DFRA then \bar{H}_1 is DFRA.
- (iv) If \bar{H}_2 is IFRA and \bar{H}_0 is IFRA then \bar{H}_1 is IFRA.
- (v) If \bar{H}_1 is DFRA and \bar{H}_2 is IFRA then \bar{H}_0 is DFRA.
- (vi) If \bar{H}_1 is IFRA and \bar{H}_2 is DFRA then \bar{H}_0 is IFRA.

Proof. In order to simplify the proof we deal only with the case when \bar{H}_1 and \bar{H}_2 are strictly positive over $[0, +\infty)$. As recalled in Definition 7.1, the IFRA/DFRA property of a survival functions \bar{H} is a property of its cumulative risk function, i.e., of the function $R_{\bar{H}}(x) = -\log(\bar{H}(x))$, namely that the function $R_{\bar{H}}(x)/x$ is increasing/decreasing. With the above notation, and observing that its inverse function is $R_{\bar{H}}^{-1}(x) = \bar{H}^{-1}(e^{-x})$, we can rewrite equality (57) as

$$R_{\bar{H}_0(x)} := R_{\bar{H}_1}(R_{\bar{H}_2}^{-1}(x)), \quad \text{or equivalently,} \quad R_{\bar{H}_1}(x) = R_{\bar{G}}(R_{\bar{H}_2}(x)).$$

Furthermore, we observe that

$$\frac{R_{\bar{H}_1}(x)}{x} = \frac{R_{\bar{H}_0}(R_{\bar{H}_2}(x))}{R_{\bar{H}_2}(x)} \frac{R_{\bar{H}_2}(x)}{x},$$

and that the function $R_{\bar{H}_0}(R_{\bar{H}_2}(x))/R_{\bar{H}_2}(x)$ is increasing/decreasing if and only if $R_{\bar{H}_0}(x)/x$ is increasing/decreasing, i.e., if and only if \bar{H}_0 is IFRA/DFRA. Then the implications (i)–(vi) follows immediately. For instance, when \bar{H}_0 is IFRA, \bar{H}_1 can be DFRA only when \bar{H}_2 is DFRA. \square

Remark 7.22. It is important to stress that the simplified assumption that the two survival functions are strictly positive on $[0, +\infty)$ is not necessary: what is really important is that the two survival function are strictly positive and invertible on the the same set. Such a property does indeed hold when $\bar{H}_1 = \phi^{-1}$ and $\bar{H}_2 = \varphi^{-1}$, with ϕ and φ defined as in (55) and (56). Moreover when $\bar{H}_0 = \bar{G}$, this property guarantees the desired condition that the function $\bar{G}(x)$ is strictly positive on $[0, +\infty)$.

Remark 7.23. We recall that for two survival functions $\bar{F}_\alpha(x)$ and $\bar{F}_\beta(x)$, $\bar{F}_\alpha \leq_{IFRA} \bar{F}_\beta$ if and only if $\bar{F}_\beta^{-1}(\bar{F}_\alpha(x))$ is star-shaped (see [4] and Definition 5.3 in [7]). Furthermore $\bar{F}_\alpha(x)$ is IFRA if and only if $\bar{F}_\alpha \leq_{IFRA} \bar{F}_0$, where $\bar{F}_0(x) = e^{-x}$, and it is DFRA if and only if $\bar{F}_0 \leq_{IFRA} \bar{F}_\alpha$. By rewriting (57) under the form

$$\bar{H}_1^{-1}(\bar{H}_0(x)) = \bar{H}_2^{-1}(e^{-x})$$

it turns out that H_2 is IFRA, if and only if $\bar{H}_1 \leq_{IFRA} \bar{H}_0(x)$, while \bar{H}_2 is DFRA if and only if $\bar{H}_0(x) \leq_{IFRA} \bar{H}_1$. These observations can be used to give an alternative proof of Lemma 7.21. For example, the implication (i) in the previous lemma follows by observing that the hypotheses on \bar{H}_1 and $\bar{H}_0(x)$ can be written as $\bar{H}_0(x) \leq_{IFRA} \bar{F}_0$ and $\bar{F}_0 \leq_{IFRA} \bar{H}_1$, whence $\bar{H}_0(x) \leq_{IFRA} \bar{H}_1$, which in its turns is equivalent to the thesis, namely that H_2 is DFRA.

7.3 The PKQD dependence property

One of the desirable axioms for a positive dependence notion, as proposed by Kimeldorf and Sampson (see, e.g., [33]), is that it implies PQD. As pointed out at the end of Section 6, however, PKD does not generally imply PQD. This fact is related with the circumstance that \preceq_{PKD} does not satisfy the antisymmetric property, namely it is a preorder, rather than an order. Actually the implication “PKD \Rightarrow PQD” is not needed in the proof of Theorem 7.19, when using Lemma 7.21. Generally when a notion of positive dependence does not satisfy the above mentioned axiom, as it happens for PKD, one may follow two different approaches. A first simple approach amounts to referring to such a notion as a *weak* positive dependence notion. When instead, for some purposes, the PQD property is needed, one may switch to another notion of positive dependence defined by adding the requirement that PQD is also satisfied. More specifically, along the latter approach, for a semi-copula S we now consider the positive dependence condition obtained by requiring S to be simultaneously PKD and PQD. We will refer to this condition with the symbol PKQD. When considering PKQD as positive dependence and IFRA as univariate ageing, under the conditions (P1) and (P2), implications of the kind 1., 2., and 3. in Section 6 still hold, for \hat{C} Pseudo-Archimedean (and exchangeable). The latter claim follows by taking into account the implications of the kind 1., 2., and 3. for the PQD/NQD notions of dependence and the NBU/NWU notions of univariate ageing (see [12]), Theorem 7.19, and the implication “IFRA \Rightarrow NBU”. For example, we obtain that if \bar{G} is IFRA and \hat{C} is PKQD then B is PKQD, as well.

As already mentioned in Section 6, for the Archimedean semi-copulas the implication “PKD \Rightarrow PQD” holds and therefore PKQD coincides with PKD. We now present a class of PKQD semi-copulas, larger than the class of Archimedean ones.

Example 7.24. As in Example 7.14, we consider a generator $\tilde{\phi}$ such that $C_{\tilde{\phi}}$ is PKD, and therefore also PQD. We claim that for any $A(t)$ the Archimax copulas $C_{\tilde{\phi},A}$ are simultaneously PKD and PQD. Indeed the PKD property holds, as already observed in Example 7.14. As far as the PQD property is concerned, we start by observing that $C_{\tilde{\phi},A}$ is PQD if and only if the following condition is verified

$$[\tilde{\phi}(u) + \tilde{\phi}(v)]A\left(\frac{\tilde{\phi}(u)}{\tilde{\phi}(u) + \tilde{\phi}(v)}\right) \leq \tilde{\phi}(uv). \quad (58)$$

Furthermore, when $A(t) = A_0(t) = 1$, then $C_{\tilde{\varphi}, A_0}$ coincides with $C_{\tilde{\varphi}}$ and the PQD property for $C_{\tilde{\varphi}}$ implies the same property for $C_{\tilde{\varphi}, A}$ for any A , since $\max(t, 1 - t) \leq A(t) \leq A_0(t) = 1$.

Same arguments hold for Archimax semi-copulas.

When \bar{G} satisfies **(P2)**, g is continuous, $\hat{C} = C_{\tilde{\varphi}, A}$ satisfies **(P1)**, and $A(t) = A(1 - t)$, the implications of the kind 1., 2., 3. hold for PKQD and IFRA. Note that however we cannot use this result when $A(t) = \max(t, 1 - t)$, since the Archimax copula $C_{\tilde{\varphi}, A}$ is not Pseudo-Archimedean, and therefore it does not satisfies **(P1)**.

Remark 7.25. We observe that Examples 7.14 and 7.24 respectively show that the PKD and PQD conditions for the Archimedean semi-copula $S_{\tilde{\varphi}}$ are inherited by all the Archimax semi-copulas $S_{\tilde{\varphi}, A}$. A further property of this type has been pointed out in [29], where it is shown that the SI dependence condition for the copula $C_{\tilde{\varphi}}$ is inherited by all the Archimax copulas $C_{\tilde{\varphi}, A}$ (see Theorem 3.1, therein). Notice however that for the corresponding negative dependence properties similar implications do not hold.

8 Summary and open problems

For vectors of exchangeable non-negative random variables with the meaning of *lifetimes*, we have reconsidered the relations between dependence and ageing already studied in [12]. Thus, as to stochastic dependence, we have considered properties of described in terms of the associated bivariate survival copula \hat{C} . Concerning the common, one-dimensional, marginal distribution for the single variables, ageing properties are considered, such as IFR/DFR, NBU/NWU, IFRA/DFRA. A basic role in this study is played by the bivariate ageing function B , and bivariate ageing properties defined in term of it. Using the latter properties, in [12] the equivalence relations of [4] have been extended to obtain different types of implications between dependence and ageing. Those equivalence relations can be re-obtained by imposing the condition $B(u, v) = uv$ (i.e., by considering Schur-constant models).

As a main motivation of our work, we made an effort to demonstrate the logic underlying such an approach for exchangeable variables. A further motivation comes from the need to enrich the analysis of the relations between the ageing notions of IFRA/DFRA and the dependence concepts of PKD/NKD. Actually a model is Pseudo-Archimedean (i.e., its survival copula is Kendall equivalent to an Archimedean copula) under suitable regularity conditions and the results obtained in [12] for Archimedean models have been extended here to this subclass of Pseudo-Archimedean models. Even though the latter subclass is rather wide, the problem remains however open whether our extension can be still obtained when dropping the above mentioned regularity conditions and/or the Pseudo-Archimedean property. In this respect one might rely on the result, proved in [46], showing that each Kendall equivalence class of copulas is characterized by a unique associative copula, along with the well-known fact that an associative copula coincides with the ordinal sum of a collection of copulas C_i , where C_i is either Archimedean or equal to M . One step towards the mentioned extension is the characterization of the PKD/NKD property for an associative copula C . On this purpose one can use the formula giving the Kendall distribution of C (see [46]). A further step might amount to limiting attention to models with survival copula Kendall-equivalent to an ordinal sum of Pseudo-Archimedean copulas.

As pointed out by the arguments in Subsection 7.3, another issue for further analysis is the characterization of exchangeable copulas such that the implication “PKD \Rightarrow PQD” holds. In any case this implication has not been used to obtain the interrelation results between dependence and ageing in Theorem 7.19. Note that on the one hand PKD does not in general imply PQD, on the other hand it corresponds to a dependence ordering (or preordering) as in (40). This fact may suggest that analogous interrelation results hold for any dependence property defined in terms of a positive ordering, or preordering, even when not implying PQD. Indeed, as mentioned in Section 6 (see in particular (43) at the end of step g)), for this kind of dependence notions, the ageing properties can be written in terms of the ageing functions of the two models $\bar{F}(x, y)$ and $\bar{F}_{\Pi}(x, y)$. This observation can be exploited for a simplification and an extension of the

theoretical method introduced in [12] for proving the interrelations between dependence and ageing.

We highlight that this method has been developed so far only for exchangeable variables and this is the case dealt with in this paper. Potential extensions to non-exchangeable cases is a relevant, still open, problem. According to a hint contained in [12], a solution may hinge on the search of an exchangeable bivariate model $\bar{F}(x, y) = \hat{C}(\bar{G}(x), \bar{G}(y))$ sharing with the given non-exchangeable model the joint distribution of the two order statistics. For such a construction, a basic role is played by the traces on the diagonal of \hat{C} and of the connecting copula C (for this kind of tools, see [20]).

Also worth of further analysis may be the possible connections with problems in Risk Theory. In the papers [54] and [25] the risk-related properties of a single-attribute utility function have been respectively related to the dependence properties of an Archimedean copula and to the one-dimensional ageing property of a survival function. Some conclusion of potential interest may arise from our arguments when dealing with problems related to topics presented therein. More in details, in [54] a risk theory interpretations of Kendall distribution and PKD property are given in terms of the concept of “Happiness of Attaining Wealth 1”. In [25] a problem of optimal burn-in for a single component is related to risk-theory interpretations of univariate ageing properties of its lifetime. Our arguments might be relevant in studying properties of an optimal common burn-in time for the case of two exchangeable components.

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A Appendix

In this appendix we briefly sketch arguments and proofs of some of the results quoted in Sections 2 and 5.

The following arguments demonstrate the reasons why DFR property is maintained under mixture, whereas IFR property is not generally preserved (see Proposition 2.4, Remark 2.5 and Remark 2.6).

We set $\mathcal{E} \equiv \mathbb{R}_+$ and consider, along with T , a non-negative random variable Θ such that the joint distribution of (Θ, T) is absolutely continuous with respect to the product of the two marginal distributions.

In what follows Π_θ denotes the marginal probability distribution of Θ , and, for $\theta \geq 0$, \bar{G}_θ and g_θ , denote the conditional survival function and conditional density function of T given $\Theta = \theta$, respectively, namely

$$\bar{G}_\theta(t) = \int_t^{+\infty} g_\theta(s) ds.$$

With this notation the failure rate function of the conditional distribution of T , given $\Theta = \theta$, is the ratio

$$r_\theta(t) = \frac{g_\theta(t)}{\bar{G}_\theta(t)}.$$

The one-dimensional survival function of T and its probability density function are respectively defined by the mixtures

$$\bar{G}(t) = \int_0^{+\infty} \bar{G}_\theta(t) d\Pi_\theta(\theta); \quad g(t) = \int_0^{+\infty} g_\theta(t) d\Pi_\theta(\theta). \quad (59)$$

Consider now the failure rate function r of T

$$r(t) = \frac{g(t)}{\bar{G}(t)} = \frac{\int_0^{+\infty} g_{\theta}(t) d\Pi_{\theta}(\theta)}{\int_0^{+\infty} \bar{G}_{\theta}(t) d\Pi_{\theta}(\theta)}$$

and the conditional probability distribution of θ given the event $\{T > t\}$, which in view of Bayes' theorem is defined by the equation

$$d\Pi_{\theta}(\theta|T > t) = \frac{\bar{G}_{\theta}(t) d\Pi_{\theta}(\theta)}{\bar{G}(t)} = \frac{\bar{G}_{\theta}(t) d\Pi_{\theta}(\theta)}{\int_0^{+\infty} \bar{G}_{\theta}(t) d\Pi_{\theta}(\theta)}.$$

Thus we can write

$$r(t) = \int_0^{+\infty} r_{\theta}(t) d\Pi_{\theta}(\theta|T > t). \quad (60)$$

We can now compare the mixing measure $\Pi_{\theta}(\theta)$ and the mixing measure $\Pi_{\theta}(\theta|T > t)$, appearing in (59) and (60), respectively. The latter measure does obviously depend on t , while the former does not. This aspect is at basis of the argument aiming to justifying Proposition 2.4, Remark 2.5 and Remark 2.6. In this respect we assume, for simplicity's sake, that Π_{θ} is absolutely continuous, denote its density function by $\pi_{\theta}(\theta)$, and consider, for $0 \leq t_1 < t_2$, the ratio between the conditional densities of θ , given the events $\{T > t_1\}$ and $\{T > t_2\}$. By using the Bayes Formula, we obtain

$$\frac{\pi_{\theta}(\theta|T > t_2)}{\pi_{\theta}(\theta|T > t_1)} = \frac{\pi_{\theta}(\theta) \bar{G}_{\theta}(t_2) \bar{G}(t_1)}{\pi_{\theta}(\theta) \bar{G}_{\theta}(t_1) \bar{G}(t_2)}. \quad (61)$$

As a substantial simplification, furthermore, we consider the case when $r_{\theta}(t)$ is monotone w.r.t. the variable θ , for any $t \geq 0$: for example,

$$r_{\theta'}(t) \leq r_{\theta''}(t), \quad \text{for any } \theta' \leq \theta''. \quad (62)$$

Namely we consider the case when the ratio

$$\frac{\bar{G}_{\theta''}(t)}{\bar{G}_{\theta'}(t)} = \exp \left\{ - \int_0^t [r_{\theta''}(s) - r_{\theta'}(s)] ds \right\}, \quad \text{for } \theta' \leq \theta'',$$

is non-increasing as a function of t . Under the assumption (62), we separately consider now the cases when the IFR property holds for all the conditional distributions of T given $\theta = \theta$ and when the DFR property holds for all the conditional distributions of T given $\theta = \theta$.

Thanks to (62), for $0 \leq t_1 < t_2$, we have that the ratio $\frac{\bar{G}_{\theta}(t_2)}{\bar{G}_{\theta}(t_1)}$, and therefore the ratio in (61), is a non-increasing function of θ . This condition implies (see, e.g., [49]; see also Chapter 3 in [51]) that the conditional distribution $\Pi_{\theta}(\theta|T > t_1)$ is stochastically greater than $\Pi_{\theta}(\theta|T > t_2)$, namely, for any non-decreasing function $\delta(\theta)$, one has

$$\int_0^{+\infty} \delta(\theta) \pi_{\theta}(\theta|T > t_1) d\theta \geq \int_0^{+\infty} \delta(\theta) \pi_{\theta}(\theta|T > t_2) d\theta.$$

Thus, in the conditionally DFR case, when, for any $\theta \geq 0$, the functions $t \mapsto r_{\theta}(t)$ are non-increasing, one obviously have that $t \mapsto r(t)$ is non-increasing as well. Indeed

$$r(t_1) = \int_0^{+\infty} r_{\theta}(t_1) \pi_{\theta}(\theta|T > t_1) d\theta \geq \int_0^{+\infty} r_{\theta}(t_1) \pi_{\theta}(\theta|T > t_2) d\theta$$

and therefore, since $r_{\theta}(t_1) \geq r_{\theta}(t_2)$, one gets

$$r(t_1) \geq \int_0^{+\infty} r_{\theta}(t_2) \pi_{\theta}(\theta|T > t_2) d\theta = r(t_2).$$

On the contrary, in the conditionally IFR case, when, for any $\theta \geq 0$, the functions $t \mapsto r_\theta(t)$ are non-decreasing, one is not allowed to conclude that also $t \mapsto r(t)$ is non-decreasing since, in this case, $r_\theta(t_1) \leq r_\theta(t_2)$.

It is thus clear that, even assuming the IFR property for all the conditional distributions of T given $\Theta = \theta$, we cannot generally conclude that the marginal distribution of T is IFR as well (notice that the latter conclusions would also stand valid under the condition that $r_\theta(t)$ is decreasing w.r.t. the argument θ). In this respect, see also arguments in [5].

A.1 Arguments from Section 5

Proof of Proposition 5.1. It is sufficient to prove the implication in item (i). The other implications can be proven similarly.

Fix $0 \leq x < y$, $t > 0$, so that $\bar{G}(x) > \bar{G}(y)$. Set, furthermore

$$s := \frac{\bar{G}(x+t)}{\bar{G}(x)},$$

so that

$$\frac{\bar{G}(y+t)}{\bar{G}(x)} \leq s \text{ for } \bar{G} \text{ IFR.}$$

By reminding (11) and that \hat{C} is supermigrative we get

$$\begin{aligned} \mathbb{P}(X_i > x+t | X_i > x, X_j > y) &= \frac{\bar{F}^{(2)}(x+t, y)}{\bar{F}^{(2)}(x, y)} \\ &= \frac{\hat{C}(\bar{G}(x+t), \bar{G}(y))}{\bar{F}^{(2)}(x, y)} = \frac{\hat{C}(\bar{G}(x) \cdot s, \bar{G}(y))}{\bar{F}^{(2)}(x, y)} \geq \frac{\hat{C}(\bar{G}(x), \bar{G}(y) \cdot s)}{\bar{F}^{(2)}(x, y)} \end{aligned}$$

By using the assumption of \bar{G} IFR, we have $\bar{G}(y) \cdot s \geq \bar{G}(y+t)$. Furthermore, \hat{C} being a copula and then non-decreasing in each variable, we can conclude

$$\begin{aligned} \mathbb{P}(X_i > x+t | X_i > x, X_j > y) &\geq \frac{\hat{C}(\bar{G}(x), \bar{G}(y) \cdot s)}{\bar{F}^{(2)}(x, y)} \\ &\geq \frac{\hat{C}(\bar{G}(x), \bar{G}(y+t))}{\bar{F}^{(2)}(x, y)} = \mathbb{P}(X_j > y+t | X_i > x, X_j > y). \end{aligned}$$

□

Proof of Lemma 5.2. For $0 \leq x < y$, the inequality (3) just means

$$\bar{F}^{(2)}(x+t, y) \geq \bar{F}^{(2)}(x, y+t), \quad \text{for any } t \geq 0,$$

namely, in view of exchangeability, the condition (1) of Schur-concavity for $\bar{F}^{(2)}$, i.e., condition (a) is equivalent to condition (b). It only remains to prove that condition (b) is equivalent to condition (c). Indeed, taking into account (19), and the fact that the function $\xi \mapsto \bar{G}(-\log(\xi))$ is increasing, the above Schur-concavity condition is equivalent to

$$B(e^{-(x+t)}, e^{-y}) \geq B(e^{-x}, e^{-(y+t)}), \quad \text{for any } t \geq 0, \text{ and } 0 \leq x < y.$$

Setting $u = e^{-x}$, $v = e^{-y}$ and $s = e^{-t}$, the above condition is equivalent to B being PMD, i.e., to the Supermigrativity condition (27) for B .

The proof of the equivalence of the other conditions is similar.

□

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