

Some classes of graphs that are not PCGs [☆]

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Abstract

A graph $G = (V, E)$ is a *pairwise compatibility graph* (PCG) if there exists an edge-weighted tree T and two non-negative real numbers d_{min} and d_{max} , $d_{min} \leq d_{max}$, such that each node $u \in V$ is uniquely associated to a leaf of T and there is an edge $(u, v) \in E$ if and only if $d_{min} \leq d_T(u, v) \leq d_{max}$, where $d_T(u, v)$ is the sum of the weights of the edges on the unique path $P_T(u, v)$ from u to v in T . Understanding which graph classes lie inside and which ones outside the PCG class is an important issue. In this paper we show that some interesting classes of graphs have empty intersection with PCG; they are wheels, strong product of a cycle and P_2 and the square of an n node cycle, with n sufficiently large. As a side effect, we show that the smallest planar graph not to be PCG has not 20 nodes, as previously known, but only 8 (it is C_8^2).

Keywords: Phylogenetic Tree Reconstruction Problem, Pairwise Compatibility Graphs (PCGs), PCG Recognition Problem, Smallest Planar not PCG, Wheel.

1. Introduction

Graphs we deal with in this paper are motivated by a fundamental problem in computational biology, that is the reconstruction of ancestral relationships [13]. It is known that the evolutionary history of a set of organisms is represented by a *phylogenetic tree*, i.e. a tree where leaves represent distinct

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known taxa while the internal nodes possible ancestors that might have led through evolution to this set of taxa. The edges of the tree are weighted in order to represent a kind of evolutionary distance among species. Given a set of taxa, the *phylogenetic tree reconstruction problem* consists in finding the “best” phylogenetic tree that explains the given data. Since it is not completely clear what “best” means, the performance of the reconstruction algorithms is usually evaluated experimentally by comparing the tree produced by the algorithm with those partial subtrees that are unanimously recognized as “sure” by biologists. However, the tree reconstruction problem is proved to be NP-hard under many criteria of optimality, moreover real phylogenetic trees are usually huge, so testing these heuristics on real data is in general very difficult. This is the reason why it is common to exploit *sample techniques*, extracting relatively small subsets of taxa from large phylogenetic trees, according to some biologically-motivated constraints, and to test the reconstruction algorithms only on the smaller subtrees induced by the sample. The underlying idea is that the behavior of the algorithm on the whole tree will be more or less the same as on the sample. It has been observed that using in the sample very close or very distant taxa can create problems for phylogeny reconstruction algorithms [9] so, in selecting a sample from the leaves of the tree, the constraint of keeping the pairwise distance between any two leaves in the sample between two given positive integers d_{min} and d_{max} is used. This motivates the introduction of *pairwise compatibility graphs* (PCGs): given a phylogenetic tree T , and integers d_{min}, d_{max} we can associate a graph G , called the pairwise compatibility graph of T , whose nodes are the leaves of T and for which there is an edge between two nodes if the corresponding leaves in T are at weighted distance within the interval $[d_{min}, d_{max}]$.

From a more theoretical point of view, we highlight that the problem of sampling a set of m leaves from a weighted tree T , such that their pairwise distance is within some interval $[d_{min}, d_{max}]$, reduces to selecting a clique of size m uniformly at random from the associated pairwise compatibility graph. As the sampling problem can be solved in polynomial time on PCGs [11], it follows that the *max clique problem* is solved in polynomial time on this class of graphs, if the edge-weighted tree T and the two values d_{min}, d_{max} are known or can be provided in polynomial time.

The previous reasonings motivate the interest of researchers in the so called *PCG recognition problem*, consisting in understanding whether, given a graph G , it is possible to determine an edge-weighted tree T and two integers d_{min}, d_{max} such that G is the associated pairwise compatibility graph.

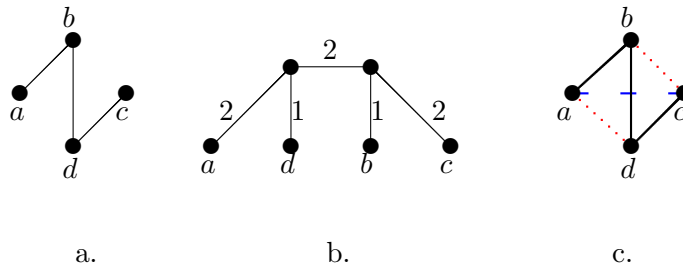


Figure 1: a. A graph G . b. An edge-weighted caterpillar T such that $G = PCG(T, 4, 5)$. c. G where the PCG-coloring induced by triple $T, 4, 5$ is highlighted.

In Figure 1.a a small graph that is $PCG(T, 4, 5)$ is depicted and, in Figure 1.b, T is shown. In general, T is not unique; here T is a *caterpillar*, i.e. a tree consisting of a central path, called spine, and nodes directly connected to that path. Due to their simple structure, caterpillars are the most used witness trees to show that a graph is PCG. However, it has been proven that there are some PCGs for which it is not possible to find a caterpillar as witness tree [4].

Due to the flexibility afforded in the construction of instances (i.e. choice of tree topology and values for d_{min} and d_{max}), when PCGs were introduced, it was also conjectured that all graphs are PCGs [11]. This conjecture has been confuted by proving the existence of some graphs not belonging to PCG. Namely, Yanhaona et al. [15] show a not PCG bipartite graph with 15 nodes (Figure 2.a). Mehnaz and Rahman [12] generalize the technique in [15] to provide a class of bipartite graphs that are not PCGs. More recently, Durochet et al. [8] prove that there exists a not bipartite graph with 8 nodes that is not PCG (Figure 2.b); this is the smallest graph that is not PCG, since all graphs with at most 7 nodes are PCGs [4]. The authors of [8] provide also an example of a planar graph with 20 nodes that is not PCG (Figure 2.c). Finally, it holds that, if a graph H is not PCG, every graph admitting H as induced subgraph is not PCG, too [5].

From the other side, many graph classes have been proved to be in PCG, such as cliques and trees, cycles, single chord cycles, cacti, tree power graphs [16, 15], interval graphs [2], triangle-free outerplanar 3-graphs [14], Dilworth 2 and Dilworth k graphs [6, 7].

However, despite these results, it remains unclear which is the boundary of the PCG class. In this paper, we move a step in the direction of searching new graph classes that are not PCGs, by showing that the three following classes of graphs have empty intersection with PCG:

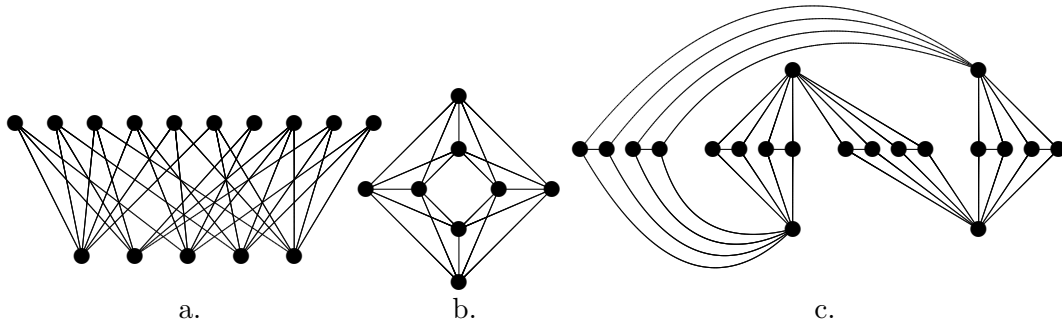


Figure 2: a. The first graph proven not to be a PCG. b. The graph of smallest size proven not to be a PCG. c. A planar graph that is not PCG.

- wheels, for which it was left as an open problem in [3] to understand whether they were PCGs or not;
- graphs obtained as strong product between a cycle and P_2 , that are a generalization of the smallest known not PCG [8];
- graphs constructed as the square of a cycle.

As a side effect we find the smallest planar graph (with 8 nodes) that is not PCG: the square of the 8 node cycle; the smallest planar graph previously known not to be PCG has 20 nodes.

We highlight that in [10] it is stated that a graph is not PCG if its complement has two 'far' induced subgraphs which are either a chordless cycle of at least four nodes or the complement of a cycle of length at least 5; two induced subgraphs are 'far' if not only they are node disjoint but there is no edge connecting them. The hypotheses of this result do not hold for any graph classes we consider.

The rest of this paper is organized as follows. Since all the three graph classes are handled with the same technique, we describe it in Section 2; in Section 3 we list some forbidden configurations, useful in the following; in Sections 4, 5 and 6 the results dealing with the previously named classes are presented. In Section 7, for any graph G in each one of the three classes, we show that by deleting any node from G we get a PCG, so proving that it does not contain any induced subgraph that is not PCG, i.e. we prove that the graphs inside all the three graph classes are *minimal not PCGs*. We conclude the paper with Section 8, where we address some open problems.

2. Proof Technique

In this section, after introducing some definitions, we describe the general proof technique we exploit to prove that all the three considered classes of graphs have empty intersection with the class of PCGs, formally defined as follows.

Definition 1. [11] *A graph $G = (V, E)$ is a pairwise compatibility graph (PCG) if there exist a tree T , a weight function assigning a positive real value to each edge of G , and two non-negative real numbers d_{min} and d_{max} , $d_{min} \leq d_{max}$, such that each node $u \in V$ is uniquely associated to a leaf of T and there is an edge $(u, v) \in E$ if and only if $d_{min} \leq d_T(u, v) \leq d_{max}$, where $d_T(u, v)$ is the sum of the weights of the edges on the unique path $P_T(u, v)$ from u to v in T . In such a case, we say that G is a PCG of T for d_{min} and d_{max} ; in symbols, $G = PCG(T, d_{min}, d_{max})$.*

In order not to overburden the exposition, in the following, when we speak about a tree, we implicitly mean that it is edge-weighted.

Given a graph $G = (V, E)$, we call *non-edges* of G the edges that do not belong to the graph. A *tri-coloring* of G is an edge labeling of the complete graph $K_{|V|}$ with labels from set { black, red, blue } such that all edges of $K_{|V|}$ that are in G are labeled black, while the other edges of $K_{|V|}$ (i.e. the non-edges of G) are labeled either red or blue. A tri-coloring is called a *partial tri-coloring* if not all the non-edges of G are labeled.

Notice that, if $G = PCG(T, d_{min}, d_{max})$, some of its non-edges do not belong to G because the weights of the corresponding paths on T are strictly larger than d_{max} , while some other edges are not in G because the weights of the corresponding paths on T are strictly smaller than d_{min} . This motivates the following definition.

Definition 2. *Given a graph $G = PCG(T, d_{min}, d_{max})$, we call its PCG-coloring the tri-coloring \mathcal{C} of G such that:*

(u, v) is red in \mathcal{C} if $d_T(u, v) < d_{min}$,

(u, v) is black in \mathcal{C} if $d_{min} \leq d_T(u, v) \leq d_{max}$,

(u, v) is blue in \mathcal{C} if $d_T(u, v) > d_{max}$.

In such a case, we say that triple (T, d_{min}, d_{max}) induces PCG-coloring \mathcal{C} .

In order to read the figures even in gray scale, we draw red edges as red and dotted and blue edges as blue and dashed in all the figures.

In Figure 1.c we highlight the PCG-coloring induced by triple $(T, 4, 5)$ where T is the tree in Figure 1.b.

The following definition formalizes that not all tri-colorings are PCG-colorings.

Definition 3. *A tri-coloring \mathcal{C} (either partial or not) of a graph G is called a forbidden PCG-coloring if no triple (T, d_{min}, d_{max}) inducing \mathcal{C} exists.*

Observe that a graph is PCG if and only if there exists a tri-coloring \mathcal{C} that is a PCG-coloring for G .

Besides, any induced subgraph H of a given $G = PCG(T, d_{min}, d_{max})$ is also PCG, indeed $H = PCG(T', d_{min}, d_{max})$, where T' is the subtree induced by the leaves corresponding to the nodes of H . Moreover, H inherits the PCG-coloring induced by triple (T, d_{min}, d_{max}) from G . Thus, if we were able to prove that H inherits a forbidden PCG-coloring from a tri-coloring \mathcal{C} of G , then we would show that \mathcal{C} cannot be a PCG-coloring for G in any way. This is the core of our proof technique that, given a graph G that we want to prove not to be PCG, consists in:

1. listing some forbidden PCG-colorings of particular graphs that are induced pairwise compatibility subgraphs of G ;
2. showing that each tri-coloring of G induces a forbidden PCG-coloring in at least an induced subgraph;
3. concluding that G is not PCG, since all its tri-colorings are proved to be forbidden.

3. Forbidden Tri-Colorings of Some PCGs

We now highlight some forbidden partial tri-colorings (for short **f-c**). Along the paper, we will use them to show that the three considered classes have empty intersection with PCG.

Given a graph $G = (V, E)$ and a subset $S \subseteq V$, we denote by $G[S]$ the subgraph of G induced by nodes in S .

A *subtree induced by a set of leaves* of T is the minimal subtree of T which contains those leaves. In particular, we denote by T_{uvw} the subtree of a tree induced by three leaves u, v and w .

The following lemma from [15] will be largely used:

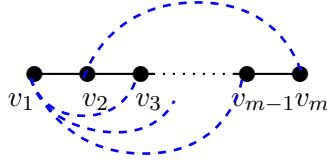


Figure 3: Case of Lemma 2.

Lemma 1. *Let T be a tree, and u, v and w be three leaves of T such that $P_T(u, v)$ is the largest path in T_{uvw} . Let x be a leaf of T other than u, v, w . Then, $d_T(w, x) \leq \max\{d_T(u, x), d_T(v, x)\}$.*

It is immediate to see that the m node path, P_m , is a PCG; the following lemma gives some constraints to the associated PCG-coloring.

Lemma 2. *Let P_m , $m \geq 4$, be a path and let \mathcal{C} be one of its PCG-colorings. If all non-edges (v_1, v_i) , $3 \leq i \leq m-1$, and (v_2, v_m) are colored with blue in \mathcal{C} , then also non-edge (v_1, v_m) is colored with blue in \mathcal{C} .*

PROOF. Refer to Figure 3. Let \mathcal{C} be the PCG-coloring of P_m induced by triple (T, d_{min}, d_{max}) . We apply Lemma 1 iteratively.

First consider nodes v_1, v_2, v_3 and v_4 as u, w, v and x : $P_T(v_1, v_3)$ is easily the largest path in $T_{v_1v_3v_2}$; then $d_T(v_2, v_4) \leq \max\{d_T(v_1, v_4), d_T(v_3, v_4)\} = d_T(v_1, v_4)$ because (v_1, v_4) is a blue non-edge by hypothesis while (v_3, v_4) is an edge.

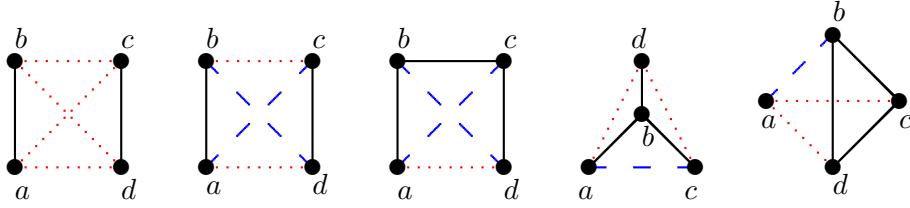
Now repeat the reasoning with nodes v_1, v_2, v_i and v_{i+1} , $4 \leq i < m$, as u, w, v and x , exploiting that at the previous step we have obtained that $d_T(v_2, v_i) \leq d_T(v_1, v_i)$: in $T_{v_1v_iv_2}$, $P_T(v_1, v_i)$ is the largest path and so $d_T(v_2, v_{i+1}) \leq \max\{d_T(v_1, v_{i+1}), d_T(v_i, v_{i+1})\} = d_T(v_1, v_{i+1})$ since (v_1, v_{i+1}) is a blue non-edge while (v_i, v_{i+1}) is an edge.

Posing $i = m-1$, we get that $d_T(v_2, v_m) \leq d_T(v_1, v_m)$; since non-edge (v_2, v_m) is blue by hypothesis, (v_1, v_m) is blue, too. \square

Given a graph, in order to ease the exposition, we call *2-non-edge* a non-edge between nodes that are at distance 2 in the graph.

Lemma 3. *Let P_n , $n \geq 3$, be a path. Any PCG-coloring of P_n that has at least one red non-edge but no red 2-non-edges is forbidden.*

PROOF. If $n = 3$, there is a unique non-edge and it is a 2-non-edge; so, the claim trivially follows.



a. $\mathbf{f-c}(2K_2)a$ b. $\mathbf{f-c}(2K_2)b$ c. $\mathbf{f-c}(P_4)$ d. $\mathbf{f-c}(K_{1,3})$ e. $\mathbf{f-c}(K_3 \cup K_1)$

Figure 4: Some forbidden tri-colorings of small graphs. Acronym $\mathbf{f-c}$ stands for *forbidden coloring*.

So, let it be $n \geq 4$ and consider a triple (T, d_{min}, d_{max}) inducing a PCG-coloring with at least a red non-edge. Among all red non-edges, let (v_i, v_j) be the one such that $j - i$ is minimum. Assume by contradiction, $j - i > 2$. Consider now the subpath P' induced by v_i, \dots, v_j . P' has at least 4 nodes and inherits the PCG-coloring from P_n ; in it, there is only a red non-edge (i.e. the non-edge connecting v_i and v_j). P' satisfies the hypothesis of Lemma 2, hence (v_i, v_j) must be blue, against the hypothesis that it is red. \square

The following lemma is proved in [16] and here translated in our setting:

Lemma 4. *In every PCG-coloring of the n node cycle C_n , $n \geq 4$, there exist at least one red and one blue non-edges.*

Theorem 1. *Let C_n , $n \geq 4$, be a cycle. Then any PCG-coloring of C_n that has no red 2-non-edges is forbidden.*

PROOF. Let $C_n = PCG(T, d_{min}, d_{max})$, $n \geq 4$; from Lemma 4, there exists at least a non-edge (u, v) such that $d_T(u, v) < d_{min}$. In our setting, this means that every PCG-coloring of C_n , $n \geq 4$, has at least a red non-edge. By contradiction, w.l.o.g. assume that this non-edge is (v_1, v_i) , with $4 \leq i < n - 1$. We apply Lemma 3 on the induced P_i and the thesis follows. \square

Theorem 2. *The tri-colorings in Figure 4 are forbidden PCG-colorings.*

PROOF. We prove separately that the tri-colorings in figure are forbidden for PCGs $2K_2$, P_4 , $K_{1,3}$ and $K_3 \cup K_1$.

Forbidden tri-coloring f-c($2K_2$)a:

We obtain that the tri-coloring in Figure 4.a is forbidden by rephrasing Lemma 6 of [8] with our nomenclature.

The other proofs are all by contradiction and proceed as follows: for each tri-coloring in Figure 4, we assume that it is a feasible PCG-coloring induced by a triple (T, d_{min}, d_{max}) and show that this assumption contradicts Lemma 1.

Forbidden tri-coloring f-c($2K_2$)b:

From the tri-coloring in Figure 4.b we have that

$$d_T(b, c) < d_{min} \leq d_T(a, b) \leq d_{max} < d_T(a, c).$$

Thus $P_T(a, c)$ is the largest path in $T_{a,b,c}$. By Lemma 1, for leaf d it must be: $d_T(b, d) \leq \max \{d_T(a, d), d_T(c, d)\} = d_T(c, d)$ while from the tri-coloring it holds that $d_T(c, d) \leq d_{max} < d_T(b, d)$, a contradiction.

Forbidden tri-coloring f-c(P_4):

From the tri-coloring in Figure 4.c we have that

$$d_T(a, b), d_T(b, c) \leq d_{max} < d_T(a, c).$$

Thus $P_T(a, c)$ is the largest path in $T_{a,b,c}$. By Lemma 1, for leaf d we have: $d_T(b, d) \leq \max \{d_T(a, d), d_T(c, d)\} = d_T(c, d)$ while from the tri-coloring it holds that $d_T(c, d) \leq d_{max} < d_T(b, d)$, a contradiction.

Forbidden tri-coloring f-c($K_{1,3}$):

From the tri-coloring in Figure 4.d we have that

$$d_T(a, b), d_T(b, c) \leq d_{max} < d_T(a, c).$$

Thus $P_T(a, c)$ is the largest path in $T_{a,b,c}$. By Lemma 1, for leaf d we have: $d_T(b, d) \leq \max \{d_T(a, d), d_T(c, d)\}$ while from the tri-coloring it holds that $d_T(a, d), d_T(c, d) < d_{min} \leq d_T(b, d)$.

Forbidden tri-coloring f-c($K_3 \cup K_1$):

From the tri-coloring in Figure 4.e we have that

$$d_T(a, d), d_T(a, c) < d_{min} \leq d_T(c, d).$$

Thus $P_T(c, d)$ is the largest path in $T_{a,c,d}$. By Lemma 1, for leaf b it must be: $d_T(a, b) \leq \max \{d_T(c, b), d_T(d, b)\}$ while from the tri-coloring it holds that $d_T(c, b), d_T(d, b) \leq d_{max} < d_T(a, b)$, a contradiction. \square

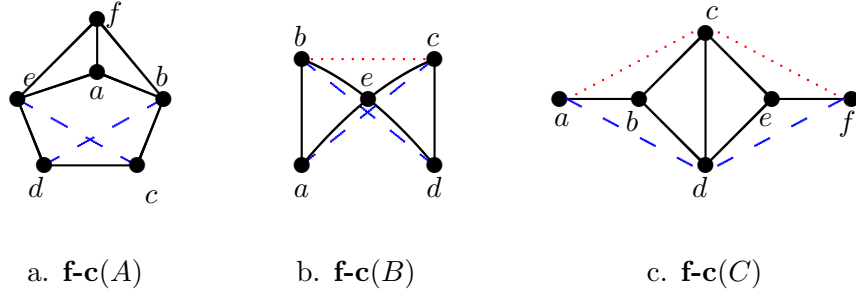


Figure 5: Some forbidden partial tri-colorings of small graphs. Acronym $\mathbf{f-c}$ stands for *forbidden coloring*.

Theorem 3. *The partial tri-colorings in Figure 5 are forbidden PCG-colorings.*

PROOF. Using the results of Theorem 2, we again prove separately that each tri-coloring is forbidden by contradiction.

Forbidden tri-coloring $\mathbf{f-c}(A)$:

Let us assume that the partial tri-coloring in figure 5.a is a PCG-coloring. Consider the PCG-coloring inherited by path $G[b, c, d, e]$. To avoid $\mathbf{f-c}(P_4)$, non-edge (e, b) must be blue. Now consider the PCG-coloring inherited by cycle $G[a, b, c, d, e]$. From Lemma 4, every PCG-coloring of C_n , $n \geq 4$, has at least a red non-edge. Thus at least one of the non-edges between (a, c) and (a, d) is red and w.l.o.g. let assume it is (a, c) . To avoid $\mathbf{f-c}(P_4)$ for path $G[c, d, e, a]$, non-edge (a, d) is red, too. Now, consider the PCG-coloring inherited by the cycle $G[b, c, d, e, f]$; with a similar reasoning, we get that the two non-edges (f, c) and (f, d) are both red. Thus we have four red non-edges, namely (a, c) , (a, d) , (f, c) and (f, d) . This implies $\mathbf{f-c}(2K_2)$ for $G[a, c, d, f]$, a contradiction.

Forbidden tri-coloring $\mathbf{f-c}(B)$:

From the tri-coloring in Figure 5.b we have that

$$d_T(b, c) < d_{min} \leq d_T(b, e), d_T(e, c).$$

Without loss of generality, let assume $d_T(b, e) \leq d_T(e, c)$. Thus $P_T(e, c)$ is the largest path in $T_{b, c, e}$. By Lemma 1, for leaf d we have: $d_T(b, d) \leq \max \{d_T(d, e), d_T(c, d)\}$ while from the tri-coloring it holds that $d_T(d, e), d_T(c, d) \leq d_{max} < d_T(b, d)$, a contradiction.

Forbidden tri-coloring $\mathbf{f-c}(C)$:

From the the tri-coloring in Figure 5.c, extract the inherited *PCG*-colorings for the two subgraphs $G[a, c, d, e]$ and $G[b, c, d, f]$. To avoid $\mathbf{f-c}(K_3 \cup K_1)$, the non-edges (a, e) and (b, f) are both blue. Now we distinguish the two possible cases for the color of non-edge (a, f) :

(a, f) is a red non-edge: consider the *PCG*-coloring for subgraph $G[a, b, e, f]$. To avoid $\mathbf{f-c}(2K_2)b$, non-edge (b, e) has to be blue. This implies that the *PCG*-coloring for path $G[a, b, d, e, f]$ has all the 2-non-edges with color blue while the non-edge (a, f) is red. This is in contradiction with Lemma 3.

(a, f) is a blue non-edge: in this case consider Lemma 1 applied to tree $T_{a,d,f}$. We distinguish the three cases for the largest path among $P_T(a, d)$, $P_T(a, f)$ and $P_T(d, f)$:

$P_T(a, d)$ is the largest path: for leaf b it must be
 $d_T(f, b) \leq \max\{d_T(a, b), d_T(d, b)\}$ while from the tri-coloring
 $d_T(a, b), d_T(d, b) \leq d_{max} < d_T(f, b)$.

$P_T(a, f)$ is the largest path: for leaf c it must be
 $d_T(d, c) \leq \max\{d_T(a, c), d_T(f, c)\}$ while from the tri-coloring
 $d_T(a, c), d_T(f, c) < d_{min} \leq d_T(d, c)$.

$P_T(d, f)$ is the largest path: for leaf e it must be
 $d_T(a, e) \leq \max\{d_T(d, e), d_T(f, e)\}$ while from the tri-coloring
 $d_T(d, e), d_T(f, e) \leq d_{max} < d_T(a, e)$.

In all the three cases, a contradiction arises. □

4. The Wheel

Wheels W_{n+1} are n length cycles C_n whose nodes are all connected with a universal node. They have already been studied from the pairwise compatibility point of view. Indeed, wheel W_{6+1} is PCG and it is the only graph with 7 nodes whose witness tree is not a caterpillar [4] (see Figure 6.a). Moreover, it has been proven in [3] that also the larger wheels up to W_{10+1} do not have a caterpillar as a witness tree but, up to now, no other witness trees are known for these graphs and, in general, it has been left as an open problem whether wheels with at least 8 nodes are PCGs or not. In this section we completely solve this problem.

First we prove that W_{7+1} is PCG.

Theorem 4. *Wheel W_{7+1} is PCG.*

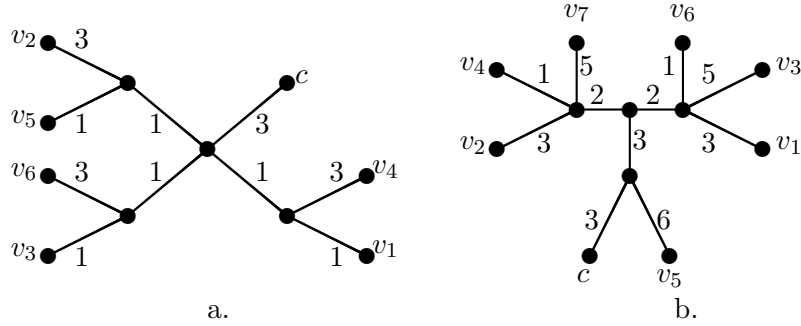


Figure 6: a. Tree T such that $W_{6+1} = PCG(T, 5, 7)$; b. Tree T such that $W_{7+1} = PCG(T, 9, 13)$.

PROOF. In order to prove the statement, it is enough to show a triple (T, d_{min}, d_{max}) witnessing that W_{7+1} is PCG. Tree T is shown in Figure 6.b, and the values of d_{min} and d_{max} are 9 and 13, respectively. \square

Then, we prove that every larger wheel $W_{n+1}, n \geq 8$, is not a PCG.

Theorem 5. *Let $n \geq 8$. The graph W_{n+1} is not PCG.*

PROOF. As list of useful forbidden PCG-colorings we will use $\mathbf{f-c}(2K_2)$ a, $\mathbf{f-c}(P_4)$, $\mathbf{f-c}(K_{1,3})$, $\mathbf{f-c}(B)$ and the forbidden tri-coloring in Theorem 1.

We now prove that every tri-coloring of W_{n+1} induces a forbidden PCG-coloring for a certain induced pairwise compatibility subgraph.

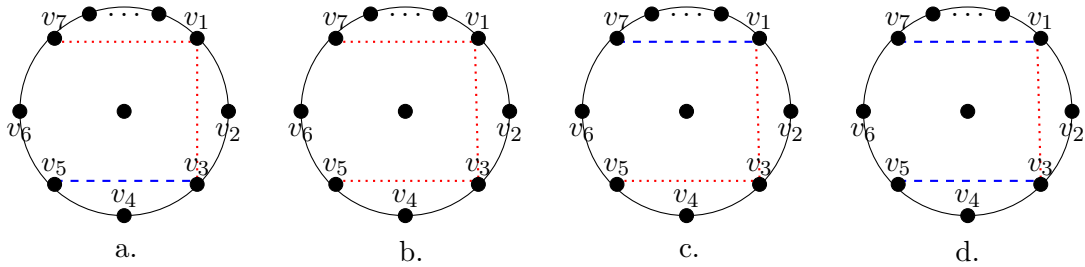


Figure 7: The four cases in the proof of Theorem 5.

Let be given any tri-coloring of W_{n+1} ; in view of Theorem 1, there exists a red 2-non-edge, w.l.o.g. let it be (v_1, v_3) . Let us now consider the three

non-edges (v_7, v_1) , (v_1, v_3) , (v_3, v_5) . There are only 4 possibilities for the colors of these non-edges and we will study them one by one (see Figure 7).

Case in Figure 7.a:

Assume first that (v_4, v_7) is blue; then non-edge (v_3, v_7) is necessarily red in order to avoid $\mathbf{f-c}(K_{1,3})$ on the graph induced by nodes c, v_1, v_3 and v_7 . In the following we summarize this sentence as:

(v_3, v_7) red $\leftarrow \mathbf{f-c}(K_{1,3})$ on $G[c, v_1, v_3, v_7]$.

and a chain of obliged colored non-edges follows, namely:

- (v_3, v_6) red $\leftarrow \mathbf{f-c}(B)$ on $G[c, v_3, v_4, v_6, v_7]$ (indeed, (v_3, v_7) is red and (v_4, v_7) is blue, so (v_3, v_6) cannot be blue)
- (v_1, v_4) blue $\leftarrow \mathbf{f-c}(K_{1,3})$ on $G[c, v_1, v_4, v_7]$
- (v_1, v_6) red $\leftarrow \mathbf{f-c}(K_{1,3})$ on $G[c, v_1, v_3, v_6]$
- (v_4, v_6) blue $\leftarrow \mathbf{f-c}(K_{1,3})$ on $G[c, v_1, v_4, v_6]$

We got a path induced by nodes v_3, v_4, v_5 and v_6 with forbidden coloring $\mathbf{f-c}(P_4)$, a contradiction, meaning that (v_4, v_7) cannot be blue.

So, (v_4, v_7) is red, and we have the following chain of obliged colored non-edges:

- (v_1, v_5) blue $\leftarrow \mathbf{f-c}(K_{1,3})$ on $G[c, v_1, v_3, v_5]$
- (v_1, v_4) red $\leftarrow \mathbf{f-c}(K_{1,3})$ on $G[c, v_1, v_4, v_7]$
- (v_2, v_4) red $\leftarrow \mathbf{f-c}(B)$ on $G[c, v_1, v_2, v_4, v_5]$
- (v_2, v_7) red $\leftarrow \mathbf{f-c}(K_{1,3})$ on $G[c, v_2, v_4, v_7]$
- (v_5, v_7) blue $\leftarrow \mathbf{f-c}(K_{1,3})$ on $G[c, v_1, v_5, v_7]$
- (v_4, v_6) red $\leftarrow \mathbf{f-c}(P_4)$ on $G[v_4, v_5, v_6, v_7]$
- (v_2, v_6) red $\leftarrow \mathbf{f-c}(K_{1,3})$ on $G[c, v_2, v_4, v_6]$
- (v_1, v_6) red $\leftarrow \mathbf{f-c}(K_{1,3})$ on $G[c, v_1, v_4, v_6]$

Graph $G[v_1, v_2, v_6, v_7]$ has forbidden coloring $\mathbf{f-c}(2K_2)$ a, and this is a contradiction, meaning that (v_4, v_7) cannot be red.

Case in Figure 7.b:

Notice that:

- (v_3, v_7) red $\leftarrow \mathbf{f-c}(K_{1,3})$ on $G[c, v_1, v_3, v_7]$
- (v_5, v_7) red $\leftarrow \mathbf{f-c}(K_{1,3})$ on $G[c, v_3, v_5, v_7]$
- (v_1, v_5) red $\leftarrow \mathbf{f-c}(K_{1,3})$ on $G[c, v_1, v_3, v_5]$

Assume now that (v_4, v_7) is blue; then we have the following chain of obliged colored non-edges:

- (v_5, v_8) red $\leftarrow \mathbf{f-c}(B)$ on $G[c, v_4, v_5, v_7, v_8]$
- (v_3, v_8) red $\leftarrow \mathbf{f-c}(K_{1,3})$ on $G[c, v_3, v_5, v_8]$
- (v_1, v_4) blue $\leftarrow \mathbf{f-c}(K_{1,3})$ on $G[c, v_1, v_4, v_7]$
- (v_2, v_5) red $\leftarrow \mathbf{f-c}(B)$ on $G[c, v_1, v_2, v_4, v_5]$

- (v_2, v_8) red \leftarrow $\mathbf{f-c}(K_{1,3})$ on $G[c, v_2, v_5, v_8]$
- (v_2, v_7) red \leftarrow $\mathbf{f-c}(K_{1,3})$ on $G[c, v_2, v_5, v_7]$

so $G[v_2, v_3, v_7, v_8]$ has forbidden coloring $\mathbf{f-c}(2K_2)$ a, a contradiction.

So, (v_4, v_7) must be red, and (v_1, v_4) red \leftarrow $\mathbf{f-c}(K_{1,3})$ on $G[c, v_1, v_4, v_7]$.

Now, we consider the non-edge (v_1, v_6) . If (v_1, v_6) is red:

- (v_4, v_6) red \leftarrow $\mathbf{f-c}(K_{1,3})$ on $G[c, v_1, v_4, v_6]$
- (v_3, v_6) red \leftarrow $\mathbf{f-c}(K_{1,3})$ on $G[c, v_1, v_3, v_6]$

and we have a contradiction arisen from having $\mathbf{f-c}(2K_2)$ a on $G[v_3, v_4, v_6, v_7]$.

If, on the contrary, (v_1, v_6) is blue, then:

- (v_2, v_7) red \leftarrow $\mathbf{f-c}(B)$ on $G[c, v_1, v_2, v_6, v_7]$
- (v_2, v_4) red \leftarrow $\mathbf{f-c}(K_{1,3})$ on $G[c, v_2, v_4, v_7]$
- (v_2, v_5) red \leftarrow $\mathbf{f-c}(K_{1,3})$ on $G[c, v_2, v_5, v_7]$

deducing a contradiction on $G[v_1, v_2, v_4, v_5]$ with forbidden coloring $\mathbf{f-c}(2K_2)$ a.

Case in Figure 7.c:

- (v_3, v_7) blue \leftarrow $\mathbf{f-c}(K_{1,3})$ on $G[c, v_1, v_3, v_7]$
- (v_5, v_7) blue \leftarrow $\mathbf{f-c}(K_{1,3})$ on $G[c, v_3, v_5, v_7]$

Let us now consider in this order the non-edges $(v_5, v_n), (v_5, v_{n-1}), \dots$ and let (v_5, v_i) be the first encountered blue non-edge, surely existing because (v_5, v_7) is blue.

We distinguish two subcases: either $i = n$ or $i < n$.

If $i = n$:

- (v_3, v_n) blue \leftarrow $\mathbf{f-c}(K_{1,3})$ on $G[c, v_3, v_5, v_n]$
- (v_1, v_5) red \leftarrow $\mathbf{f-c}(K_{1,3})$ on $G[c, v_1, v_3, v_5]$
- (v_1, v_6) red \leftarrow $\mathbf{f-c}(B)$ on $G[c, v_n, v_1, v_5, v_6]$
- (v_3, v_6) red \leftarrow $\mathbf{f-c}(K_{1,3})$ on $G[c, v_1, v_3, v_6]$
- (v_6, v_n) blue \leftarrow $\mathbf{f-c}(K_{1,3})$ on $G[c, v_3, v_6, v_n]$

Now, If $n = 8$, then v_7 and v_n are adjacent and $G[v_6, v_7, v_n, v_1]$ has forbidden tri-coloring $\mathbf{f-c}(P_4)$. If, on the contrary, $n > 8$, then we have the forbidden tri-coloring $\mathbf{f-c}(B)$ on $G[c, v_n, v_1, v_6, v_7]$.

If $i < n$, we know that (v_5, v_{i+1}) is red; moreover:

- (v_3, v_{i+1}) red \leftarrow $\mathbf{f-c}(K_{1,3})$ on $G[c, v_3, v_5, v_{i+1}]$
- (v_3, v_i) blue \leftarrow $\mathbf{f-c}(K_{1,3})$ on $G[c, v_3, v_5, v_i]$
- (v_2, v_{i+1}) red \leftarrow $\mathbf{f-c}(B)$ on $G[c, v_2, v_3, v_i, v_{i+1}]$
- (v_2, v_5) red \leftarrow $\mathbf{f-c}(K_{1,3})$ on $G[c, v_2, v_5, v_{i+1}]$
- (v_6, v_{i+1}) red \leftarrow $\mathbf{f-c}(B)$ on $G[c, v_5, v_6, v_i, v_{i+1}]$
- (v_2, v_6) red \leftarrow $\mathbf{f-c}(K_{1,3})$ on $G[c, v_2, v_6, v_{i+1}]$
- (v_3, v_6) red \leftarrow $\mathbf{f-c}(K_{1,3})$ on $G[c, v_3, v_6, v_{i+1}]$

We get subgraph $G[v_2, v_3, v_5, v_6]$ colored with $\mathbf{f-c}(2K_2)$ a.

Case in Figure 7.d:

We distinguish two subcases, according to the color of non-edge (v_1, v_4) .

If (v_1, v_4) is blue:

- (v_3, v_n) red \leftarrow $\mathbf{f-c}(B)$ on $G[c, v_n, v_1, v_3, v_4]$
- (v_5, v_n) blue \leftarrow $\mathbf{f-c}(K_{1,3})$ on $G[c, v_3, v_5, v_n]$
- (v_4, v_n) blue \leftarrow $\mathbf{f-c}(B)$ on $G[c, v_n, v_1, v_4, v_5]$
- (v_3, v_7) blue \leftarrow $\mathbf{f-c}(K_{1,3})$ on $G[c, v_1, v_3, v_7]$

Now we show that (v_3, v_n) red and (v_4, v_n) blue imply (v_3, v_8) red and (v_4, v_8) blue, so obtaining $G[c, v_3, v_4, v_7, v_8]$ with forbidden coloring $\mathbf{f-c}(B)$, a contradiction.

To show the assert it is sufficient to prove that if (v_3, v_i) is red and (v_4, v_i) is blue and $i > 8$, then (v_3, v_{i-1}) is red and (v_4, v_{i-1}) is blue.

- (v_3, v_{i-1}) red \leftarrow $\mathbf{f-c}(B)$ on $G[c, v_3, v_4, v_{i-1}, v_i]$
- (v_1, v_{i-1}) red \leftarrow $\mathbf{f-c}(K_{1,3})$ on $G[c, v_1, v_3, v_{i-1}]$
- (v_4, v_{i-1}) blue \leftarrow $\mathbf{f-c}(K_{1,3})$ on $G[c, v_1, v_4, v_{i-1}]$

and this part of the proof is concluded.

If, instead, (v_1, v_4) is red:

- (v_4, v_7) blue \leftarrow $\mathbf{f-c}(K_{1,3})$ on $G[c, v_1, v_4, v_7]$
- (v_1, v_5) blue \leftarrow $\mathbf{f-c}(K_{1,3})$ on $G[c, v_1, v_3, v_5]$
- (v_4, v_n) red \leftarrow $\mathbf{f-c}(B)$ on $G[c, v_n, v_1, v_4, v_5]$
- (v_3, v_n) blue \leftarrow $\mathbf{f-c}(2K_2)_a$ on $G[v_1, v_n, v_3, v_4]$

Now, if $n = 8$ then the nodes v_7 and v_8 are adjacent and $G[v_3, v_4, v_7, v_n]$ has forbidden tri-coloring $\mathbf{f-c}(B)$. Thus, let us assume $n > 8$.

- (v_4, v_{n-1}) red \leftarrow $\mathbf{f-c}(B)$ on $G[c, v_3, v_4, v_{n-1}, v_n]$
- (v_1, v_{n-1}) red \leftarrow $\mathbf{f-c}(K_{1,3})$ on $G[c, v_1, v_4, v_{n-1}]$
- (v_3, v_{n-1}) red \leftarrow $\mathbf{f-c}(K_{1,3})$ on $G[c, v_1, v_3, v_{n-1}]$
- (v_5, v_{n-1}) blue \leftarrow $\mathbf{f-c}(K_{1,3})$ on $G[c, v_3, v_5, v_{n-1}]$

Similarly to what we did before, now we show that (v_4, v_{n-1}) red and (v_5, v_{n-1}) blue imply (v_4, v_8) red and (v_5, v_8) blue, so obtaining $G[c, v_4, v_5, v_7, v_8]$ with forbidden coloring $\mathbf{f-c}(B)$, a contradiction.

To show the assert it is sufficient to prove that if (v_4, v_i) is red and (v_5, v_i) is blue and $i > 8$, then (v_4, v_{i-1}) is red and (v_5, v_{i-1}) is blue.

- (v_4, v_{i-1}) red \leftarrow $\mathbf{f-c}(B)$ on $G[c, v_4, v_5, v_{i-1}, v_i]$
- (v_1, v_{i-1}) red \leftarrow $\mathbf{f-c}(K_{1,3})$ on $G[c, v_1, v_4, v_{i-1}]$
- (v_5, v_{i-1}) blue \leftarrow $\mathbf{f-c}(K_{1,3})$ on $G[c, v_1, v_5, v_{i-1}]$

We conclude the proof deducing that G is not PCG since all the partial colorings shown in Figure 7 are not feasible. \square

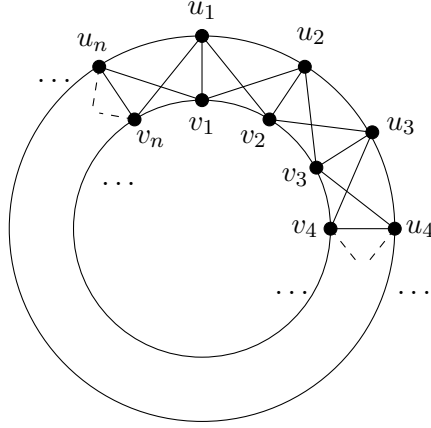


Figure 8: Graph $C_n \square P_2$.

5. The strong product of a cycle and P_2

Given two graphs G and H , their *strong product* $G \square H$ is a graph whose node set is the cartesian product of the node sets of the two graphs, and there is an edge between nodes (u, v) and (u', v') if and only if either $u = u'$ and (v, v') is an edge of H or $v = v'$ and (u, u') is an edge of G .

In the following, we study graph $C_n \square P_2$, a $2n$ node graph in which two cycles are naturally highlighted; we call v_1, \dots, v_n and u_1, u_2, \dots, u_n , respectively, their nodes as shown in Figure 8.

We recall that $C_4 \square P_2$, i.e. the graph depicted in Figure 2.b, has already been proved not to be PCG [8].

We apply our technique to $C_n \square P_2$, by showing that every tri-coloring leads to forbidden tri-coloring $\mathbf{f-c}(C)$. Since this tri-coloring appears only when $n \geq 6$, we need to handle the case $C_5 \square P_2$ separately.

Theorem 6. *Graph $C_5 \square P_2$ is not PCG.*

PROOF. According to the second step of the proof technique, we focus on any tri-coloring of $C_5 \square P_2$ and prove that it is forbidden.

Consider cycle $G[v_1, v_2, v_3, v_4, v_5] = PCG(T, d_{min}, d_{max})$; from Lemma 4, there exists at least a blue non-edge.

Thus, w.l.o.g. assume that non-edge (v_2, v_5) is blue. In order to avoid forbidden coloring $\mathbf{f-c}(A)$ on the induced subgraph $G[v_1, v_2, v_3, u_4, v_4, v_5]$, non-edge (v_1, v_4) must be red. The same reasoning can be used for the following three induced subgraphs: $G[u_1, v_2, v_3, u_3, u_4, v_5]$, $G[u_1, v_2, v_3, v_4, u_4, v_5]$

and $G[u_1, v_2, v_3, v_4, u_4, v_5]$ to prove that non-edges (u_1, v_4) , (v_1, v_3) and (u_1, v_3) must be red, too. We get $\mathbf{f} - \mathbf{c}(2K_2)a$ on the induced subgraph $G[u_1, v_1, v_3, v_4]$, a contradiction.

In view of the last step of the proof technique, $C_5 \square P_2$ is not PCG, so concluding the proof. \square

Theorem 7. *Graph $C_n \square P_2$, $n \geq 6$, is not PCG.*

PROOF. We exploit again the technique described in Section 2.

We will use $\mathbf{f} - \mathbf{c}(2K_2)a$, $\mathbf{f} - \mathbf{c}(K_3 \cup K_1)$, $\mathbf{f} - \mathbf{c}(B)$, $\mathbf{f} - \mathbf{c}(C)$ and the forbidden tri-coloring in Theorem 1.

According to Step 2, we prove that for each tri-coloring of $C_n \square P_2$, with $n \geq 6$, there exists an induced pairwise compatibility subgraph of $C_n \square P_2$ that inherits a forbidden PCG-coloring.

Let fix any tri-coloring of $C_n \square P_2$. Consider the cycle $G[v_1, v_2, \dots, v_n]$; in view of Theorem 1, there exists a red 2-non-edge in the cycle, w.l.o.g. let it be (v_2, v_4) . Consider now the induced subgraph $G[v_2, u_2, v_3, v_4, u_4]$. In order to avoid $\mathbf{f} - \mathbf{c}(B)$, at least one between the non-edges (u_2, v_4) and (v_2, u_4) must be red. Thus, either (v_2, v_4) and (u_2, v_4) are red or (v_2, v_4) and (v_2, u_4) are red. Due to the symmetry of $C_n \square P_2$, it is not restrictive to assume that non-edges (v_2, v_4) and (v_2, u_4) are red. From this, we can prove that all the non-edges incident on v_2 are red. To do that, it is sufficient to show that if non-edges (v_2, v_i) and (v_2, u_i) , $4 \leq i < n$, are red, then non-edges (v_2, v_{i+1}) and (v_2, u_{i+1}) are red, too. To this aim consider the induced subgraph $G[v_2, v_i, u_i, v_{i+1}]$; in order to avoid $\mathbf{f} - \mathbf{c}(K_3 \cup K_1)$, on the three non-edges (v_2, v_i) , (v_2, u_i) and (v_2, v_{i+1}) the red color can not appear exactly twice. Since (v_2, v_i) and (v_2, u_i) are both red, it follows that (v_2, v_{i+1}) must also be red. Analogously, considering the induced subgraph $G[v_2, v_i, u_i, u_{i+1}]$, to avoid $\mathbf{f} - \mathbf{c}(K_3 \cup K_1)$ we get that (v_2, u_{i+1}) is red.

In particular, when $i = n - 1$, we have that (v_2, v_n) and (v_2, u_n) are both red. Consider now the induced subgraph $G[v_2, u_2, v_n, u_n]$; to avoid $\mathbf{f} - \mathbf{c}(2K_2)a$, we have that (u_2, x) , with $x \in \{u_n, v_n\}$, must be a blue non-edge. Analogously, to avoid $\mathbf{f} - \mathbf{c}(2K_2)a$ on the induced graph $G[v_2, u_2, v_4, u_4]$, (u_2, y) , with $y \in \{u_4, v_4\}$ must be a blue non edge. Finally, we get the $\mathbf{f} - \mathbf{c}(C)$ on the induced graph $G[x, v_1, v_2, u_2, v_3, y]$, a contradiction.

Step 3 of the proof technique concludes the proof. \square

6. The square of a Cycle

We recall that all graphs with at most 7 nodes are PCG [4] and that cycles are PCGs [15], so we focus on $n \geq 8$.

For easing the proofs, the nodes of C_n^2 will be indexed with values in the finite group \mathbb{Z}_n of the integers modulo n , i.e. $V(C_n^2) = \{v_0, v_1, \dots, v_{n-1}\}$. As a consequence, for each pair v_i, v_j , the edge (v_i, v_j) belongs to C_n^2 if and only if $j - i \in \{1, 2, n - 1, n - 2\}$.

Before proving that C_n^2 is not PCG, we need some *ad-hoc* forbidden PCG-colorings for C_n^2 .

Given a PCG-coloring of C_n^2 , we call *red-node* a node v of C_n^2 if all the non-edges incident on v are of red color.

Lemma 5. *Let C_n^2 , $n \geq 8$, be a square cycle. Then:*

1. *Any PCG-coloring of C_n^2 where all the 2-non-edges are blue is forbidden.*
2. *Any PCG-coloring of C_n^2 having two red non-edges from a common non red-node to two adjacent nodes is forbidden.*
3. *Any PCG-coloring of C_n^2 having two adjacent red-nodes is forbidden.*

PROOF. We prove separately the three claims.

1. The proof is by contradiction. Let assume a PCG-coloring of C_n^2 without red 2-non-edges in C_n^2 . We distinguish two cases:

n is even Consider the PCG-coloring inherited by cycle $C_{\frac{n}{2}}$ induced by all the vertices of C_n^2 with even index (i.e. $C_{\frac{n}{2}} = G[v_0, v_2, \dots, v_{n-2}]$). The length of this cycle is at least 4 and all its 2-non-edges are blue. This contradicts Theorem 1.

n is odd Consider the PCG-coloring inherited by cycle $C_{\lceil \frac{n}{2} \rceil} = G[v_0, v_2, \dots, v_{n-3}, v_{n-1}]$ induced by all the nodes with even index. All the 2-non-edges in $C_{\lceil \frac{n}{2} \rceil}$ are in the set $\{(v_{2i}, v_{2i+2}) \mid 0 \leq i \leq \frac{n-3}{2}\} \cup \{(v_{n-3}, v_0), (v_{n-1}, v_2)\}$. The set $\{(v_{2i}, v_{2i+2}) \mid 0 \leq i \leq \frac{n-3}{2}\}$ corresponds to 2-non-edges in C_n^2 and, by our assumption, all these non-edges are blue in the PCG-coloring inherited by the cycle. However, the length of $C_{\lceil \frac{n}{2} \rceil}$ is at least 6 thus, by Theorem 1, at least one of the two non-edges $\{(v_{n-3}, v_0), (v_{n-1}, v_2)\}$ is red in the PCG-coloring of $C_{\lceil \frac{n}{2} \rceil}$ inherited from the PCG-coloring of C_n^2 . The red non-edge (v_{n-3}, v_0) implies a PCG-coloring for path $P = G[v_0, v_2, \dots, v_{n-3}]$ of at least 4 nodes where all the 2-non-edges are blue while non-edge (v_0, v_{n-3}) is red. The red non-edge

(v_{n-1}, v_2) implies a *PCG*-coloring for path $P = G[v_2, \dots, v_{n-1}]$ of at least 4 nodes where all the 2-non-edges are blue while the non-edge (v_2, v_{n-1}) is red. In both cases we have a contradiction with Lemma 3.

2. Let v_i and v_{i+1} be the two adjacent nodes and let v_j be the non red-node. The proof is by contradiction:
 - If $j+2 \neq i-1$, consider the induced subgraph $G[v_j, v_{i-1}, v_i, v_{i+1}]$; since $n \geq 8$, due to $\mathbf{f} - \mathbf{c}(K_3 \cup K_1)$ the non-edge (v_j, v_{i-1}) is red. Iterating this reasoning we have that all the non-edges (v_j, v_k) , with $j+2 < k < i$, are red.
 - If $j-2 \neq i+2$, consider the induced subgraph $G[v_j, v_i, v_{i+1}, v_{i+2}]$; since $n \geq 8$, due to $\mathbf{f} - \mathbf{c}(K_3 \cup K_1)$ the non-edge (v_j, v_{i+2}) is red. Iterating this reasoning we have that all the non-edges (v_j, v_k) , with $i+1 < k < j-2$, are red.

Thus we have the contradiction that v_j is a red-node.

3. Let v_i and v_{i+1} be the two adjacent red-nodes and consider the induced subgraph $G[v_i, v_{i+1}, v_{i+4}, v_{i+5}]$; since $n \geq 8$, the subgraph is in fact a K_2 . Thus we obtain the forbidden tri-coloring $\mathbf{f} - \mathbf{c}(2K_2)a$. □

Now we show other two *ad-hoc* forbidden PCG-colorings that hold only for $n \geq 10$ because in the proof we exploit $\mathbf{f} - \mathbf{c}(C)$. Hence, at the end of this section, we will prove separately that C_8^2 and C_9^2 are not PCGs.

Lemma 6. *Let C_n^2 , $n \geq 10$, be a square cycle. Then:*

1. *Any PCG-coloring of C_n^2 with a triple of nodes (v_i, v_{i+4}, v_{i+8}) , $0 \leq i < n$, such that v_{i+8} is the only non red-node is forbidden.*
2. *Any PCG-coloring of C_n^2 with a triple of nodes (v_{i-6}, v_{i-3}, v_i) , $0 \leq i < n$, such that v_{i-6} is the only non red-node is forbidden.*

PROOF. We prove separately the two claims.

1. We consider two cases:

non-edge (v_{i+1}, v_{i+5}) is red Since v_i is a red-node, edge (v_i, v_{i+5}) is a red non-edge, too. Due to lemma 5.2, this implies that node v_{i+5} is a red-node. Hence we have two adjacent red-nodes (i.e. v_{i+4} and v_{i+5}) and, by lemma 5.3, we have a forbidden PCG-coloring.

non-edge (v_{i+1}, v_{i+5}) is blue Consider the induced subgraph $G[v_{i+1}, v_{i+3}, v_{i+4}, v_{i+5}, v_{i+6}, v_{i+8}]$ having the two red non-edges (v_{i+1}, v_{i+4}) and (v_{i+4}, v_{i+8}) . Since $n \geq 10$, due to the forbidden tri-coloring $\mathbf{f} - \mathbf{c}(C)$, non-edge (v_{i+5}, v_{i+8}) is red. Thus there are two red non-edges (v_{i+4}, v_{i+8}) and (v_{i+5}, v_{i+8}) and, in view of lemma 5.2, node v_{i+8} is red.

2. The proof is similar to the previous one: now we consider the induced subgraph $G[v_{i-6}, v_{i-4}, v_{i-3}, v_{i-2}, v_{i-1}, v_{i+1}]$ having the two red non-edges (v_{i-6}, v_{i-3}) and (v_{i-3}, v_{i+1}) . Non-edge (v_{i-2}, v_{i+1}) must be blue (otherwise node v_{i-2} would be a red-node like v_{i-3} and, by lemma 5.3, we would have a forbidden PCG-coloring). Due to $\mathbf{f} - \mathbf{c}(C)$, non-edge (v_{i-2}, v_{i-6}) is red. Thus we have two red non-edges (v_{i-6}, v_{i-3}) and (v_{i-6}, v_{i-3}) and, by lemma 5.2, the node v_{i-6} is red. \square

We are now ready to prove that C_n^2 is not PCG.

Theorem 8. *Graph C_n^2 , $n \geq 10$, is not PCG.*

PROOF. The proof is by contradiction. Let (v_i, v_{i+4}) be a red 2-non-edge in C_n^2 (such a non-edge must exist by Lemma 5.1). Consider now the induced path $G[v_i, v_{i+1}, v_{i+3}, v_{i+4}]$. In this path we have the red non-edge (v_i, v_{i+4}) thus, due to $\mathbf{f} - \mathbf{c}(P_4)$, one of the non-edges (v_i, v_{i+3}) and (v_{i+1}, v_{i+4}) is red, too and at least one of the nodes v_i and v_{i+4} is the end-point of two red non-edges toward adjacent nodes. Hence one of these nodes is a red-node (see lemma 5.2). Reindexing the nodes of C_n^2 , this red-node is node v_0 . Consider now the induced subgraph $G[v_{n-3}, v_{n-1}, v_0, v_1, v_2, v_4]$. In this subgraph the non-edges (v_{n-3}, v_0) and (v_0, v_4) are red and, due to $\mathbf{f} - \mathbf{c}(C)$, at least one of the non-edges (v_{n-3}, v_1) and (v_1, v_4) is red. We consider two cases:

non-edge (v_1, v_4) is red The two non-edges (v_0, v_4) and (v_1, v_4) are red so, by Lemma 5.2, node v_4 is a red-node. Considering the triple of nodes (v_0, v_4, v_8) , by Lemma 6.1, node v_8 is a red-node, too. We can iterate this reasoning on the triple (v_4, v_8, v_{12}) and so on finally obtaining that $V^* = \{v_i \mid i \equiv 0 \pmod{4}\}$ is a set of red-nodes in the PCG-coloring. Moreover each node v_i , with $i \not\equiv 0 \pmod{4}$, is adjacent to some node in V^* thus, by lemma 5.3, n is a multiple of 4 (and $n \geq 12$) and set V^* contains all the red-nodes of the PCG-coloring. Consider now the cycle induced by all the nodes having an odd index, i.e. $G[v_1, v_3, v_5, \dots, v_{n-1}]$. This cycle is $\frac{n}{2} \geq 6$ long thus, by Theorem

1, it contains at least a red non-edge. Let (v_i, v_j) be one of these red non-edges. Node v_j is necessarily adjacent to a node in V^* , hence there are two red non-edges from adjacent nodes incident toward v_i in C_n^2 implying that v_i is a red-node (by lemma 5.2). This contradicts the fact that $v_i \notin V^*$.

non-edge (v_{n-3}, v_1) is red The proof is analogous to the previous one: due to the two red non-edges (v_{n-3}, v_0) and (v_{n-3}, v_1) , by lemma 5.2, node v_{n-3} is a red-node. Considering the triple of nodes (v_{n-6}, v_{n-3}, v_0) in lemma 6.2, node v_{n-6} is a red-node, too. We can iterate this reasoning on the triple $(v_{n-9}, v_{n-6}, v_{n-3})$ and so on finally obtaining that $V^* = \{v_i \mid i \equiv 0 \pmod{3}\}$ is a set of red-nodes in the PCG-coloring. Moreover, each node v_i , with $i \not\equiv 0 \pmod{3}$, is adjacent to some node in V^* so, due lemma 5.3, n is a multiple of 3 (and $n \geq 12$) and set V^* contains all the red-nodes of the PCG-coloring. Consider now the cycle induced by all the nodes that are not in V^* , i.e. $G[1, 2, 4, \dots, n-2, n-1]$. This cycle has length at least 8 and, by Theorem 1, there is at least a red non-edge connecting two nodes of the cycle. Let (v_i, v_j) be one of these red non-edges. Node v_j is adjacent to a node in V^* , so v_i is the end-point of two red non-edges toward adjacent nodes in C_n^2 as a consequence v_i is a red-node (by Lemma 5.2). This contradicts the fact that $v_i \notin V^*$.

□

Theorem 9. *Graph C_8^2 is not PCGs.*

PROOF. We exploit again the technique described in Section 2.

We will use **f-c**($2K_2$)a, **f-c**($2K_2$)b, **f-c**($K_3 \cup K_1$), **f-c**(P_4) and the forbidden tri-coloring in Theorem 1.

We then prove that for each tri-coloring of C_8^2 , there exists an induced subgraph of C_8^2 that inherits a forbidden PCG-coloring.

Let fix any tri-coloring of C_8^2 . Consider the induced cycle $G[v_0, v_2, v_4, v_6]$, in view of Theorem 1, there exists a red 2-non-edge in the cycle (i.e. at least one of the non-edges (v_0, v_4) and (v_2, v_6) is red). Consider also the induced cycle $G[v_1, v_3, v_5, v_7]$, again, in view of Theorem 1, there exists a red 2-non-edge in the cycle (i.e. at least one of the non-edges (v_1, v_5) and (v_3, v_7) is red). W.l.o.g. let us assume that the two non edges (v_0, v_4) and (v_1, v_5) are red. Consider now the induced subgraph $G[v_0, v_1, v_4, v_5]$. In order to avoid **f-c**($2K_2$)a and **f-c**($2K_2$)b, exactly one of the two non-edges (v_0, v_5) and (v_1, v_4) is red. Again it is not restrictive to assume (v_0, v_5) red and

(v_1, v_4) blue. Now note that to avoid $\mathbf{f-c}(K_3 \cup K_1)$ on the two subgraphs $G[v_0, v_1, v_4, v_7]$ and $G[v_1, v_4, v_5, v_6]$ both the non-edges (v_4, v_7) and (v_1, v_6) are blue.

We distinguish two subcases, according to the color of the non-edge (v_2, v_6) , and show that in both the cases we get a contradiction:

(v_2, v_6) is blue. Consider the induced cycle $G[v_1, v_2, v_4, v_6, v_7]$. To avoid a forbidden configuration Theorem 1 implies that the non-edge (v_2, v_7) is red. Thus we obtain the subgraph $G[v_2, v_4, v_6, v_7]$ with forbidden coloring $\mathbf{f-c}(P_4)$, a contradiction.

(v_2, v_6) is red. In this case:

- induced subgraph $G[v_1, v_2, v_3, v_6]$ cannot have $\mathbf{f-c}(K_3 \cup K_1)$ so implying (v_3, v_6) blue;
- induced subgraph $G[v_0, v_3, v_4, v_5]$ cannot have $\mathbf{f-c}(K_3 \cup K_1)$ hence (v_0, v_3) is red;
- in order to avoid $\mathbf{f-c}(K_3 \cup K_1)$ on $G[v_0, v_3, v_6, v_7]$, edge (v_3, v_7) must be blue.

It turns out that the subgraph $G[v_1, v_3, v_4, v_6, v_7]$ is a cycle without red non-edges. This is in contrast with Theorem 1. \square

Corollary 1. *Graph C_8^2 is the smallest planar graph that is not PCG.*

Theorem 10. *Graph C_9^2 is not a PCG.*

PROOF. We will use $\mathbf{f-c}(2K_2)$ a, $\mathbf{f-c}(P_4)$ and the forbidden tri-colorings in Theorem 1 and in Lemma 5.

We prove that for each tri-coloring of C_9^2 , there exists an induced subgraph of C_9^2 that inherits a forbidden PCG-coloring. Fix any tri-coloring of C_9^2 and let $V^* \subseteq V(C_9^2)$ be the set of red-nodes of the tri-coloring. We know by Lemma 5.1 and 5.3 that set V^* is not empty and does not contain adjacent nodes. Thus, up to isomorphisms, we can consider only the following four cases:

$$\begin{aligned} V^* &= \{v_0\}, & V^* &= \{v_0, v_3\}, \\ V^* &= \{v_0, v_4\} & \text{and } V^* &= \{v_0, v_3, v_6\}. \end{aligned}$$

Now we show that each of these cases get a contradiction.

$V^* = \{v_0\}$ In this case v_3, v_4 and v_6 are not red-nodes; as a consequence, all the non-edges $(v_1, v_4), (v_4, v_8), (v_3, v_8)$ and (v_1, v_6) are blue (see Lemma 5.3). Thus, to avoid a forbidden coloring in the induced cycle $G[v_1, v_3, v_4, v_6, v_8]$, non-edge (v_3, v_6) must be red (see Theorem 1). So, we have the induced path $G[v_1, v_3, v_6, v_8]$ with forbidden tri-coloring $\mathbf{f-c}(P_4)$, a contradiction.

$V^* = \{v_0, v_3\}$ In this case all the non-edges in the induced cycle $C_5 = G[v_1, v_2, v_4, v_6, v_8]$ are blue (see Lemma 5.3). This is in contradiction with Theorem 1.

$V^* = \{v_0, v_5\}$ In this case v_1 and v_4 are not red-nodes. As a consequence, all the non-edges $(v_4, v_8), (v_1, v_4), (v_1, v_6)$ and (v_2, v_6) are blue (see Lemma 5.3). Moreover, to avoid the forbidden tri-coloring $\mathbf{f-c}(2K_2)$ in the induced subgraph $G[v_2, v_3, v_6, v_8]$, there must be at least one blue non-edge; w.l.o.g. assume it is (v_2, v_8) . Hence, the induced cycle $C_5 = G[v_1, v_2, v_4, v_6, v_8]$ has all the non-edges of color blue. This is in contradiction with Theorem 1.

$V^* = \{v_0, v_3, v_6\}$. In this case all the non-edges not inciding on red-nodes are blue (see Lemma 5.3). In particular, all the non-edges in the induced cycle $C_6 = G[v_1, v_2, v_4, v_5, v_7, v_8]$ are blue. This is in contradiction with Theorem 1. \square

7. Minimality

If a graph contains as induced subgraph a not PCG, then it is not PCG, too. We call *minimal non PCG* a graph that is not PCG and it does not contain any proper induced subgraph that is not PCG.

In this section we prove that every graph G inside each one of the three considered classes we have just proved not to be PCGs is a minimal not PCG. The proof is constructive and it provides an edge-weighted tree T and two values d_{min} and d_{max} such that $PCG(T, d_{min}, d_{max}) = G \setminus \{x\}$ for any node x of G .

The following theorem states that wheels are minimal not PCGs.

Theorem 11. *Let $n \geq 8$. The graph obtained by removing any node from W_{n+1} is PCG. In other words, W_{n+1} is a minimal not PCG.*

PROOF. Notice that, if we remove from W_{n+1} the central node, the resulting graph is a cycle; if we remove any other node, the resulting graph is an interval graph. In both cases, we get a PCG [16, 1]. \square

Now we prove that $C_n \square P_2$ is a minimal not PCG. The proof is constructive and it provides an edge-weighted tree T and two values d_{min} and d_{max} such that $PCG(T, d_{min}, d_{max}) = C_n \square P_2 \setminus \{x\}$ for any node x of $C_n \square P_2$.

Theorem 12. *The graph obtained by removing any node from $C_n \square P_2$, $n \geq 4$, is PCG. In other words, $C_n \square P_2$ is a minimal not PCG.*

PROOF. To prove the statement, we remove from the graph a node x and prove that the new graph G' is PCG. In view of the symmetry of the graph, it is not restrictive to assume that $x = u_n$. We construct a tree T such that $G' = PCG(T, 2n - 2, 2n + 2)$.

We distinguish the following two cases depending on whether n is an even or an odd number:

n is an even number: (refer to Figure 9.a) tree T is a caterpillar with $n - 1$ internal nodes that we denote as x_1, x_2, \dots, x_{n-1} . The internal nodes induce a path from x_1 to x_{n-1} and edges on this path (x_i, x_{i+1}) , $1 \leq i < n - 1$, have all weight 2. Leaves v_i and u_i , $1 \leq i < n$, are connected to x_i with edges of weight n . Finally leaf v_n is connected to the node $x_{\frac{n}{2}}$ with an edge of weight 1.

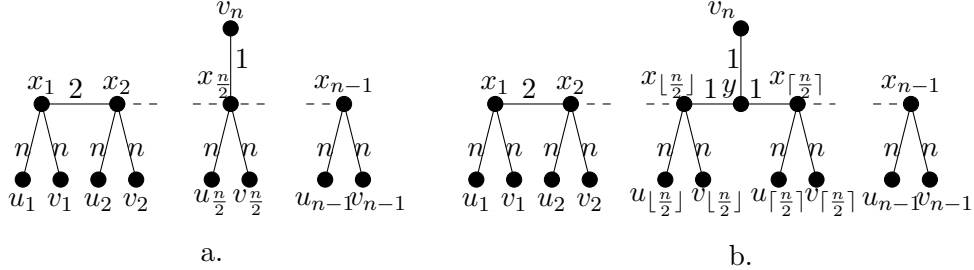


Figure 9: Caterpillars for the proof of Theorem 12.

n is an odd number: (refer to Figure 9.b) tree T is a caterpillar with n internal nodes that we denote as x_1, x_2, \dots, x_{n-1} and y . The internal nodes $x_1, \dots, x_{\lfloor \frac{n}{2} \rfloor}$ induce a path from x_1 to $x_{\lfloor \frac{n}{2} \rfloor}$ and edges (x_i, x_{i+1}) , $1 \leq i < \lfloor \frac{n}{2} \rfloor$, have weight 2. The internal nodes $x_{\lceil \frac{n}{2} \rceil}, \dots, x_{n-1}$ induce a path from $x_{\lceil \frac{n}{2} \rceil}$ to x_{n-1} and edges (x_i, x_{i+1}) , $\lceil \frac{n}{2} \rceil \leq i < n - 1$, have weight 2. Leaves v_i and u_i , $1 \leq i < n$, are connected to x_i with edges of weight n . Finally the internal node y is connected to $x_{\lfloor \frac{n}{2} \rfloor}$, $x_{\lceil \frac{n}{2} \rceil}$ and v_n with edges of weight 1.

In both cases, $G' = PCG(T, 2n - 2, 2n + 2)$. \square

Finally, we prove that also C_n^2 is a minimal not PCG.

Theorem 13. *The graph obtained by removing any node from C_n^2 , $n \geq 8$, is PCG. In other words, C_n^2 is a minimal not PCG.*

PROOF. Consider the graph C_n^2 , $n \geq 8$. To prove the theorem we remove from the graph a node x and prove that the new graph G' is PCG. Without loss of generality assume that $x = v_n$. We construct a tree T such that $G' = \text{PCG}(T, 2n - 2, 2n + 4)$. We consider the following two cases depending on whether n is an even or an odd number:

- *n is an odd number.* Tree T is a caterpillar with $n - 1$ internal nodes we denote as $x_1, x_2, \dots, x_{\frac{n-1}{2}-1}, y, x_{\frac{n-1}{2}}, \dots, x_{n-2}$. The internal nodes induce a path from x_1 to x_{n-2} and edges (x_i, x_{i+1}) , $1 \leq i < (n-1)/2-1$ and $(n-1)/2 \leq i < n-2$, have weight 2. Edges $(x_{\frac{n-1}{2}-1}, y)$ and $(y, x_{\frac{n-1}{2}})$ have weight 1. Leaves v_i , $1 \leq i \leq n-2$, are connected to x_i with edges of weight n . Finally, leaf v_{n-1} is connected to the node y with an edge of weight 3. See Figure 10.a.
- *n is an even number.* Tree T is a caterpillar with $n - 1$ internal nodes we denote as x_1, x_2, \dots, x_{n-1} . The internal nodes x_1, \dots, x_{n-1} induce a path and edges (x_i, x_{i+1}) , $1 \leq i < n-1$, have weight 2.

Leaves v_i , $1 \leq i < n$, are connected to x_i with edges of weight n . Finally v_{n-1} is connected to $x_{\frac{n-2}{2}}$ with an edge of weight 3. See Figure 10.b. \square

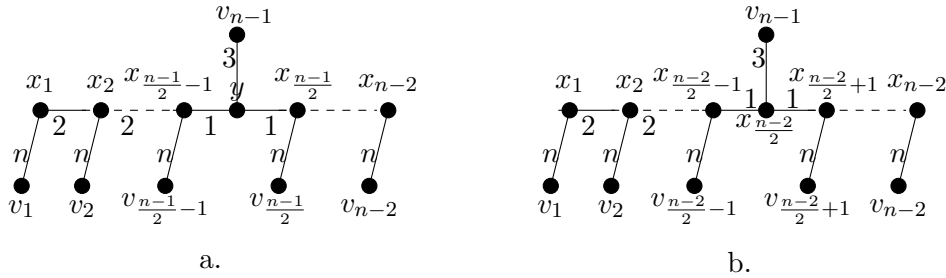


Figure 10: Caterpillars for the proof of Theorem 13: a. n odd; b. n even.

8. Conclusions

In this paper we proved that wheels, $C_n \square P_2$ and the square of cycles are not PCGs. As a side effect, we got that the smallest planar graph not to be PCG has not 20 nodes, as previously known, but only 8.

Even if all the considered classes are obtained by operating on cycles, we think that the same technique we used can be potentially exploited to position outside PCG many other graph classes not related to cycles. This would represent an important step toward the solution of the very general open problem consisting in demarcating the boundary of the PCG class.

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