$\mathcal{C}^{1,\gamma}$ regularity for singular or degenerate fully nonlinear operators and applications

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Abstract

In this note, we prove $\mathcal{C}^{1,\gamma}$ regularity for solutions of some fully nonlinear degenerate elliptic equations with "superlinear" and "subquadratic " Hamiltonian terms. As an application, we complete the results of [6] concerning the associated ergodic problem, proving, among other facts, the uniqueness, up to constants, of the ergodic function.

1 Introduction

The goal of the present paper is to establish $C^{1,\gamma}$ regularity results and to obtain a priori estimates for viscosity solutions of a class of fully nonlinear elliptic equations which may be singular or degenerate at the points where the gradient of the solution vanishes.

Regularity properties of viscosity solutions of fully nonlinear elliptic equations have been studied since a long time, starting with the seminal paper of Caffarelli [8] in 1989, which contains in particular $\mathcal{C}^{1,\gamma}$ estimates for $F(x, D^2u) = f$ when $f \in L^p$, $p > n$. His results were extended to L^p viscosity solutions of operators $F(x, D^2u, Du)$ at most linear in the gradient by Swiech in [16]. Later, Winter [17] proved $\mathcal{C}^{1,\gamma}(\overline{\Omega})$ estimates in the presence of a regular boundary datum.

In the recent preprint [15], Saller Nornberg proves $\mathcal{C}^{1,\gamma}$ and $W^{2,p}$ results for L^p viscosity solutions, when $F(x, D^2u, Du)$ is fully nonlinear, uniformly elliptic and at most quadratic in the gradient.

In [1], the first two authors of the present paper consider singular or degenerate equations of the form
$$|\nabla u|^\alpha F(D^2u) = f(x, \nabla u)\,,$$
where F is fully nonlinear uniformly elliptic, $\alpha > -1$, and f has growth at most of order $1+\alpha$ in the gradient. Lipschitz regularity results are proved in [1], and $\mathcal{C}^{1,\gamma}$ regularity up to the boundary in the case $\alpha \leq 0$ in [2], for the Dirichlet problem with homogeneous boundary conditions. Later, $\mathcal{C}^{1,\gamma}$ interior regularity was obtained in the case $\alpha > 0$ by Imbert and Silvestre [11], when f does not depend on ∇u. These results have been extended to the case where f depends on the gradient with growth at most $\alpha+1$, and to boundary $\mathcal{C}^{1,\gamma}$ results in the presence of sufficiently regular boundary datum, in [3], [4], [5].

In this note we prove $\mathcal{C}^{1,\gamma}$ interior and boundary regularity results when the equation possesses some Hamiltonian "superlinear" but at most "quadratic" in the gradient. More precisely, we consider equations of the form

$$-\,|\nabla u|^\alpha F(D^2u) + b(x)|\nabla u|^\beta = f(x)\,, \tag{1.1}$$

where the coefficient functions b and f are continuous, and the exponents α and β always satisfy $\alpha > -1$ and $\alpha + 1 < \beta \leq \alpha + 2$. On the second order operator F, we assume it is a continuous function $F : \mathcal{S}_N \to \mathbb{R}$ defined on the set $\mathcal{S}_N$ of $N \times N$ symmetric matrices, positively homogeneous of the degree one, and satisfying further the uniform ellipticity condition

$$a\,\mathrm{tr}(N) \leq F(M+N) - F(M) \leq A\,\mathrm{tr}(N)\,, \tag{1.2}$$

for any $M, N \in \mathcal{S}_N$, with $N \geq 0$, for given positive constants $A \geq a > 0$.
The considered equations include, as a very special case, the semilinear equation

$$-\Delta u + b(x)|Du|^\beta = f(x)$$

with $1 < \beta \leq 2$, and it is for this reason that the growth of the first order terms of equations (1.1) is referred to as "superlinear" and "subquadratic".
The definition of viscosity solution we adopt is the one firstly introduced in [1], which is equivalent to the usual one in the case $\alpha \geq 0$, and, in any cases, allows not to test points where the gradient of the test function is zero, except in the locally constant case.

In the paper [7], we proved local and global Lipschitz regularity results for viscosity solutions of (1.1). In particular, we showed that if u satisfies in the viscosity sense equation (1.1) in a domain $\Omega \subseteq \mathbb{R}^N$, then, for any pair of bounded subdomains $\omega \subset\subset \omega' \subset\subset \Omega$, there exists a positive constant M depending on $a, A, N, \alpha, \beta, \|u\|_{L^\infty(\omega')}, \|f\|_{L^\infty(\omega')}, \|b\|_{W^{1,\infty}(\omega')}$ and on ω, ω' such that

$$|\nabla u(x)| \leq M \qquad \text{a.e. in } \omega\,.$$

Our main result in the present paper reads as follows.

Theorem 1.1. *Suppose that Ω is an open subset in $\mathbb{R}^N$, that $f \in \mathcal{C}(\Omega)$, and $b \in W^{1,\infty}_{loc}(\Omega)$. Let $u \in C(\Omega)$ be a viscosity solution of (1.1) in $\Omega \subseteq \mathbb{R}^N$. Then, there exists $0 < \gamma \leq \frac{1}{1+\alpha^+}$ depending on the data, such that u belongs to $C^{1,\gamma}_{loc}(\Omega)$ and, moreover, for any pair of subsets $\omega \subset\subset \omega' \subset\subset \Omega$ one has*

$$|\nabla u|_{C^{0,\gamma}(\omega)} \leq C_1\left(|u|_{L^\infty(\omega')}, |b|_{W^{1,\infty}(\omega')}, |f|_{L^\infty(\omega')}\right)\,. \tag{1.3}$$

Note that some more explicit bound is the following, depending on the Lipschitz norm of u:

$$|\nabla u|_{C^{0,\gamma}(\omega)} \leq C\left(|u|_{W^{1,\infty}(\omega')} + |b|_\infty^{\frac{1}{1+\alpha}}|u|^{\frac{\beta}{1+\alpha}}_{W^{1,\infty}(\omega')} + |f|^{\frac{1}{1+\alpha}}_{L^\infty(\omega')}\right)\,.$$

Theorem 1.1 covers the case $\beta \leq \alpha + 1$, treated in [3]. We remark that the arguments used in [3] are different and fail in the case $\beta > \alpha + 1$.

Furthermore, Theorem 1.1 include the results of [15] for $\alpha = 0$. We observe that in [15] the author uses essentially the ABP estimate of [12] for fully nonlinear elliptic equations with quadratic growth in the gradient, and Caffarelli's iterative method. This method cannot be employed when $\alpha \neq 0$. The main ingredients in the proof of Theorem 1.1 are the Lipschitz continuity of solutions, a fixed point argument, the existence and uniqueness result for Dirichlet problem proved in [7], and the $\mathcal{C}^{1,\gamma}$ estimates [8] for $\alpha = 0$, of [11] and [4] for $\alpha > 0$, and the one proved in Proposition 2.1 when $\alpha < 0$.

As an application of the $C^{1,\gamma}_{loc}$ regularity of viscosity solutions of equation (1.1), we prove the uniqueness of the ergodic function associated with the considered operators. Let us recall that, as recently proved in [6], if $\Omega \subset \mathbb{R}^N$ is an open, bounded, C^2 domain, if $d(x)$ denotes the distance function from $\partial\Omega$ and if the operator F satisfies the extra regularity assumption

$$F(\nabla d(x) \otimes \nabla d(x)) \text{ is } \mathcal{C}^2 \text{ in a neighborhood of } \partial\Omega\,, \tag{1.4}$$

then, given a locally Lipschitz continuous datum f, there exists a unique constant c_Ω, called the ergodic constant or additive eigenvalue of F, such that the infinite boundary condition problem

$$\begin{cases} -|\nabla u|^\alpha F(D^2u) + |\nabla u|^\beta = f + c_\Omega & \text{in } \Omega \\ u = +\infty & \text{on } \partial\Omega \end{cases} \tag{1.5}$$

has a solution $u \in C(\Omega)$, called ergodic function.

In Section 3 we will prove the following result.

Theorem 1.2. *Let $\Omega \subset \mathbb{R}^N$ be a bounded domain of class C^2 and let F satisfy (1.2) and (1.4). Assume further that $\alpha > -1$, $\alpha + 1 < \beta < \alpha + 2$ and that $f \in C(\Omega)$ is bounded and locally Lipschitz continuous. Then, problem (1.5) has a unique (up to additive constants) solution, provided that, when $\alpha \neq 0$, $\sup_\Omega f < -c_\Omega$ and $\partial\Omega$ is connected.*

2 Proof of Theorem 1.1.

In all this section we set $|v|_{W^{1,\infty}} := |v|_\infty + |\nabla v|_\infty$.

We begin by proving some $\mathcal{C}^{1,\gamma}$ interior estimate for the case $\beta = 0$, which completes the result in [2]. The proof is very similar to the one employed in [2], but we reproduce it here for the sake of completeness. We denote by C_{lip} some constant such that, by [1], for any $u \in W^{1,\infty}(B(0,1))$ and for any g continuous and bounded in $B(0,1)$, any solution w of

$$\begin{cases} -|\nabla w|^\alpha F(D^2 w) = g & \text{in } B(0,1) \\ w = u & \text{on } \partial B(0,1) \end{cases} \tag{2.1}$$

satisfies

$$|w|_{W^{1,\infty}(B(0,1))} \leq C_{lip}(|u|_{W^{1,\infty}(B(0,1))} + |g|^{\frac{1}{1+\alpha}}_{L^\infty(B(0,1))}). \tag{2.2}$$

We then have the following

Proposition 2.1. *Under the above assumptions, any solution w of (2.1) satisfies : for any $r < 1$, there exists C_r such that*

$$|w|_{\mathcal{C}^{1,\gamma}(B(0,r))} \leq C_r(|u|_{W^{1,\infty}(B(0,1))} + |g|^{\frac{1}{1+\alpha}}_{L^\infty(B(0,1))}) \tag{2.3}$$

Proof. This result is well known in the case $\alpha = 0$, [8], while the case $\alpha > 0$ is proved in [11]. It remains to consider the case $\alpha < 0$.

Take $\epsilon = \frac{1}{4C_{lip}}$ and $\delta \leq 1$. Let

$$\mathcal{K}_R = \{v \in \mathcal{C}^1(B(0,1)) \cap W^{1,\infty}(B(0,1)), |v|_{W^{1,\infty}} \leq R\}$$

where R is chosen large enough, fixed such that $R \geq |w|_{W^{1,\infty}}$ and

$$R \geq 2\, C_{lip}(|f|_\infty(1 + R^{-\alpha}) + \frac{R^{1+\alpha}}{2C_{lip}}) + |u|_{W^{1,\infty}})$$

which is possible since both $-\alpha$ and $1+\alpha$ are lesser than 1.

We define the map $\mathcal{T} : v \mapsto w_\delta$ where w_δ satisfies

$$\begin{cases} -F(D^2 w_\delta) = (f - \epsilon|v|^\alpha v + \epsilon|w|^\alpha w)(\delta^2 + |\nabla v|^2)^{\frac{-\alpha}{2}} & \text{in } B(0,1) \\ w_\delta = u & \text{on } \partial B(0,1). \end{cases}$$

The map $\mathcal{T}$ is well defined since the right hand side is continuous and bounded. Furthemore, by (2.2) in the case $\alpha = 0$, one has

$$\begin{aligned} |w_\delta|_{W^{1,\infty}} &\leq C_{lip}\left(\left|(f - \epsilon|v|^\alpha v + \epsilon|w|^\alpha w)(\delta^2 + |\nabla v|^2)^{\frac{-\alpha}{2}}\right|_\infty + |u|_{W^{1,\infty}} \right) \\ &\leq C_{lip}\left((|f|_\infty + 2\frac{R^{1+\alpha}}{4C_{lip}})(\delta^{-\alpha} + R^{-\alpha}) \ + |u|_{W^{1,\infty}} \right) \\ &\leq C_{lip}\left(|f|_\infty(1 + R^{-\alpha}) + \frac{R}{2C_{lip}} + \frac{R^{1+\alpha}}{2C_{lip}} + |u|_{W^{1,\infty}} \right) \\ &\leq R \end{aligned}$$

by the choice of R. hence, $\mathcal{K}_R$ is a closed convex set invariant for $\mathcal{T}$. Moreover, $\mathcal{T}$ is a compact operator (see [1]) so that, by Schauder's theorem, $\mathcal{T}$ possesses a fixed point denoted by $\overline{w_\delta}$. Note that $\overline{w_\delta}$ has a Lipschitz norm independent of δ : indeed, using the convexity inequalities

$$\epsilon\delta^{-\alpha}|\overline{w_\delta}|_\infty^{1+\alpha} \leq (1+\alpha)\epsilon|\overline{w_\delta}|_\infty + (-\alpha)(\delta\epsilon)^{-\alpha}$$

and

$$|f + \epsilon|w|^\alpha w|_\infty |\overline{w_\delta}|^{-\alpha} \leq (-\alpha)\epsilon|\overline{w_\delta}|_\infty + \epsilon^{\frac{\alpha}{1+\alpha}}(|f|_\infty \epsilon|w|_\infty^{1+\alpha})^{\frac{1}{1+\alpha}},$$

one gets that

$$|\overline{w_\delta}|_{W^{1,\infty}} \leq 2C_{lip}\left(\epsilon^{\frac{\alpha}{1+\alpha}}(|f|_\infty \epsilon|w|_\infty^{1+\alpha})^{\frac{1}{1+\alpha}} + (-\alpha)(\delta\epsilon)^{-\alpha} + |u|_{W^{1,\infty}} \right).$$

Furthermore by (2.3) in the case $\alpha = 0$, $\overline{w_\delta}$ satisfies

$$|\overline{w_\delta}|_{\mathcal{C}^{1,\gamma}(B(0,r)} \leq C_r\left((|f|_\infty + \frac{|w|_{W^{1,\infty}}}{4C_{lip}} + \frac{|\overline{w_\delta}|_{W^{1,\infty}}}{4C_{lip}})(1 + |\overline{w_\delta}|_{W^{1,\infty}}^{-\alpha}) + |u|_{W^{1,\infty}} \right) \tag{2.4}$$

Note that $\overline{w_\delta}$ satisfies

$$\begin{cases} -(\delta^2 + |\nabla\overline{w_\delta}|^2)^{\frac{\alpha}{2}} F(D^2\overline{w_\delta}) + \epsilon|\overline{w_\delta}|^\alpha \overline{w_\delta} = f + \epsilon|w|^\alpha w & \text{in } B(0,1) \\ w_\delta = u & \text{on } \partial B(0,1) \end{cases}$$

Using the estimate (2.4) which is independant on δ, up to subsequence, w_δ converges locally uniformly when δ goes to zero, towards a solution $\overline{w}$ of

$$\begin{cases} -|\nabla \overline{w}|^\alpha F(D^2\overline{w}) + \epsilon|\overline{w}|^\alpha \overline{w} = f + \epsilon|w|^\alpha w & \text{in } B(0,1) \\ w = u & \text{on } \partial B(0,1) \end{cases}$$

By uniqueness of solutions of such equation (see [1]), one gets that $\overline{w} = w$ and then by (2.4), w is $\mathcal{C}^{1,\gamma}$ in $B(0,r)$. To get the precise estimate (2.3) let us observe that since w is $\mathcal{C}^1$, then it is a solution of

$$\begin{cases} -F(D^2\varphi) = |\nabla w|^{-\alpha} f & \text{in } B(0,1) \\ \varphi = u & \text{on } \partial B(0,1) \end{cases}$$

In particular one has by (2.3) in the case $\alpha = 0$:

$$\begin{aligned} |w|_{\mathcal{C}^{1,\gamma}(B(0,r))} &\leq C_r(|\nabla w|_\infty^{-\alpha}|f|_\infty + |u|_{W^{1,\infty}}) \\ &\leq C_r(|f|_\infty^{\frac{1}{1+\alpha}} + |w|_{W^{1,\infty}} + |u|_{W^{1,\infty}}) \\ &\leq C_r(1 + C_{lip})(|f|_\infty^{\frac{1}{1+\alpha}} + |u|_{W^{1,\infty}}) \end{aligned}$$

So we get (2.3) with C_r replaced by $C_r(1 + C_{lip})$. In the following we will denote for simplicity C_r this constant.

□

We now recall the Lipschitz estimate proved in [7].

Theorem 2.2. *Suppose that F is uniformly elliptic, f is continuous in $B(0,1)$, b is locally Lipschitz continuous in $B(0,1)$, $\alpha > -1$ and $\beta \in (0, \alpha+2]$. Let u be a locally bounded viscosity solution of*

$$-|\nabla u|^\alpha F(D^2u) + b(x)|\nabla u|^\beta = f(x) \quad \text{in } B(0,1)$$

Then u is locally Lipschitz continuous in $B(0,1)$, that is, for any $r < r' < 1$, there exists some constant c depending on the ellipticity constants of F, on r, r' and on universal constants, such that

$$|u|_{W^{1,\infty}(B(0,r))} \leq c(|u|_{L^\infty(B(0,r'))}, |f|_{L^\infty(B(0,r'))}, |b|_{W^{1,\infty}(B(0,r'))})$$

Furthermore, if u and f are bounded and b is bounded and Lipschitz continuous in $B(0,1)$, then u is Lipschitz continuous in $B(0,1)$ and there exists c such that

$$|u|_{W^{1,\infty}(B(0,1))} \leq c(|u|_{L^\infty(B(0,1))}, |f|_{L^\infty(B(0,1))}, |b|_{W^{1,\infty}(B(0,1))})$$

Remark 2.3. The assumption that b is Lipschitz continuous is needed only in the case $\beta = \alpha + 2$. For the case $\beta < \alpha + 2$, b bounded is sufficient.

Since we want to prove local estimates inside $B(0, r)$, we can suppose that u is Lipschitz continuous on the whole of $B(0, 1)$ and we will set $M = |u|_{W^{1,\infty}(B(0,1))}$.

Proof of Theorem 1.1. We will use both a truncation method, and a fixed point argument. The previous estimate (2.3) will then enable us to say that a solution of (1.1) is locally $\mathcal{C}^{1,\gamma}$.

In the following, T_M will denote the truncation operator at the level M, more precisely $T_M(s) = \inf\{|s|, M\}\frac{s}{|s|}$.

Let $\epsilon_\alpha = \frac{2^{-1-\frac{(-\alpha)^+}{1+\alpha}}}{C_{lip}}$. Let us observe that using (2.2), when g is continuous and bounded, the unique solution w, (see [1]) of

$$\begin{cases} -|\nabla w|^\alpha F(D^2 w) + \epsilon_\alpha^{1+\alpha}|w|^\alpha w = g & \text{in } B(0,1) \\ w = u & \text{on } \partial B(0,1) \end{cases}$$

satisfies

$$|w|_{W^{1,\infty}(B(0,1)} \leq 2^{1+(\frac{-\alpha)^+}{1+\alpha}} C_{lip}(|g|_\infty^{\frac{1}{1+\alpha}} + |u|_{W^{1,\infty}})$$

Indeed, by (2.2)

$$\begin{aligned} |w|_{W^{1,\infty}(B(0,1)} &\leq C_{lip}(|g - \epsilon_\alpha^{1+\alpha}|w|^\alpha w|_\infty^{\frac{1}{1+\alpha}} + |u|_{W^{1,\infty}})) \\ &\leq 2^{\frac{(-\alpha)^+}{1+\alpha}} C_{lip}(|g|_\infty^{\frac{1}{1+\alpha}} + \epsilon_\alpha |w| + |u|_{W^{1,\infty}})) \\ &\leq \frac{|w|_{W^{1,\infty}(B(0,1)}}{2} + 2^{\frac{(-\alpha)^+}{1+\alpha}} C_{lip}(|g|_\infty^{\frac{1}{1+\alpha}} + |u|_{W^{1,\infty}}). \end{aligned}$$

Let

$$R = 2^{1+\frac{(-\alpha)^+}{1+\alpha}} C_{lip}\left(\left||f|_\infty + \epsilon_\alpha^{1+\alpha}|u|_\infty^{1+\alpha}\right|^{\frac{1}{1+\alpha}} + (|b|_\infty M^\beta)^{\frac{1}{1+\alpha}} + |u|_{W^{1,\infty}(B(0,1)}\right),$$

define

$$\mathcal{K}_R = \{v \in \mathcal{C}^1 \cap W^{1,\infty}(B(0,1)), |v|_\infty + |\nabla v|_\infty \leq R\},$$

and note that $\mathcal{K}_R$ is a closed convex set in $\mathcal{C}^1 \cap W^{1,\infty}(B(0,1))$. We also define the map $\mathcal{T} : v \to w$ where w is the unique solution, (see [1]) of

$$\begin{cases} -|\nabla w|^\alpha F(D^2 w) + \epsilon_\alpha^{1+\alpha}|w|^\alpha w = \left(f + \epsilon_\alpha^{1+\alpha}|u|^\alpha u - b(x)T_M(|\nabla v|)^\beta\right) & \text{in } B(0,1) \\ w = u & \text{on } \partial B(0,1) \end{cases}$$

w is well defined since the right hand is continuous and bounded. By (2.2) w satisfies

$$\begin{array}{rcl}|w|_{W^{1,\infty}(B(0,1)} & \leq & C_{lip}\left((|f|_\infty+\epsilon_\alpha^{1+\alpha}|u|_\infty^{1+\alpha}+|b|_\infty M^\beta)^{\frac{1}{1+\alpha}}+|u|_{W^{1,\infty}(B(0,1))}\right)\\ & \leq & 2^{\frac{(-\alpha)^+}{1+\alpha}}C_{lip}\left((|f|_\infty+\epsilon_\alpha^{1+\alpha}|u|_\infty^{1+\alpha})^{\frac{1}{1+\alpha}}+(|b|_\infty M^\beta)^{\frac{1}{1+\alpha}}+|u|_{W^{1,\infty}}\right)\\ & \leq R & \end{array}$$

by the choice of R, hence $\mathcal{T}$ sends $\mathcal{K}_R$ into itself. Furthermore by classical uniform estimates,(see [1]), $\mathcal{T}$ is a compact operator. Then, by Schauder's fixed point Theorem, it possesses a fixed point $\overline{w}$ which then satisfies

$$\begin{cases} -|\nabla\overline{w}|^\alpha F(D^2\overline{w})+b(x)|T_M(\nabla\overline{w})|^\beta+\epsilon_\alpha^{1+\alpha}|\overline{w}|^\alpha\overline{w}=f+\epsilon_\alpha^{1+\alpha}|u|^\alpha u & \text{in } B(0,1)\\ w=u & \text{on } \partial B(0,1)\end{cases} \tag{2.5}$$

Recall that $|\nabla u|\leq M$, then u satisfies the same equation. By a mere adaptation of the comparison principle in [1], there is uniqueness of the solution to (2.5) , hence $u=\bar{w}$ and by (2.3) one gets that u is $\mathcal{C}^{1,\gamma}$ for the γ allowed by (2.3).

□

Remark 2.4. Let us observe that, in the case $\alpha\leq 0$, and if the operator F is convex or concave in the Hessian argument, then we can repeat the above proof with the $C^{1,\gamma}_{loc}$ estimates replaced by the $W^{2,p}_{loc}$ estimates, for any $1<p<\infty$, see [8, 9]. This yields the local a priori estimate, for any $\omega\subset\subset\omega'\subset\subset\Omega$

$$\|u\|_{W^{2,p}(\omega)}\leq C_p\left(\|u\|_{L^\infty(\omega')},\|f\|_{L^\infty(\omega')},|b|_{W^{1,\infty}(\omega')}\right)$$

for any viscosity solution u of equation (1.1).

Furthermore, suppose that $\Omega\subset\mathbb{R}^N$ is a smooth bounded domain, that $f\in\mathcal{C}(\overline{\Omega})$, $b\in W^{1,\infty}(\Omega)$, that u is a viscosity solution of (1.1) in Ω, which satisfies the boundary condition $u=\psi$ on $\partial\Omega$, with $\psi\in C^{1,\gamma_0}(\partial\Omega)$, by using the up to the boundary estimates of [17], [3], we obtain the global regularity bound

$$\|u\|_{C^{1,\gamma}(\overline{\Omega})}\leq C\left(\|\psi\|_{C^{1,\gamma}(\partial\Omega)},\|f\|_{L^\infty(\Omega)},|b|_{W^{1,\infty}(\Omega)}\right)$$

for some exponent $\gamma\leq\inf(\gamma_0,\frac{1}{1+\alpha^+})$.

Remark 2.5. As in [15], we can extend our results (only for $\alpha\leq 0$) to the case where $F(M)$ is replaced by $F(p,M)$, satisfying the following structural assumptions: there exist positive constants μ, b such that for any $(p,q)\in(\mathbb{R}^N)^2$, $(X,Y)\in(\mathcal{S}_N)^2$ one has

$$\begin{array}{rcl}-b|p-q|-\mu(|p|+|q|)(|p-q|) & + & \mathcal{M}^-(X-Y)\\ & \leq & F(p,X)-F(q,Y)\\ & \leq & \mathcal{M}^+(X-Y)+b|p-q|+\mu(|p|+|q|)(|p-q|)\end{array}$$

where $\mathcal{M}^+$ and $\mathcal{M}^-$ denote the Pucci extremal operators. The fixed point argument and the truncation method can be easily adapted to this case.

3 Gradient estimates and proof of Theorem 1.2.

As an application of the regularity results proved in the previous section, we now focus on the ergodic pairs associated to the class of operators we are considering.

Precisely, given a function $f \in \mathcal{C}(\Omega)$, let $c_\Omega \in \mathbb{R}$ be a constant for which there exist solutions $u \in \mathcal{C}(\Omega)$ of the infinite boundary value problem (1.5).
In [6] we gave sufficient conditions for the existence of ergodic pairs (c_Ω, u) which solve (1.5), and we proved the uniqueness of c_Ω in some cases. Here, we are concerned in particular with the uniqueness of u.

As proved in [6] and recalled in the introduction, the rate of boundary explosion of any ergodic function can be made precise assuming that the operator F satisfies the "boundary" regularity condition

$$F(\nabla d(x) \otimes \nabla d(x)) \text{ is a } \mathcal{C}^2 \text{ function in a neighborhood of } \partial\Omega \ . \tag{3.1}$$

Here, $d(x)$ denotes the distance function from $\partial\Omega$, and it is of class $\mathcal{C}^2$ in a neighborhood of $\partial\Omega$ by the regularity assumption on the domain. Condition (3.1) is certainly satisfied if the domain Ω is of class $\mathcal{C}^3$ and the operator F is $\mathcal{C}^2$, but there can be also cases with non smooth F satisfying (3.1) in $\mathcal{C}^2$ domains. For instance, when $F(M)$ depends only on the eigenvalues of M, as in the case of Pucci's operators, $F(\nabla d(x) \otimes \nabla d(x))$ is a constant function as long as $|\nabla d(x)| = 1$.

Under assumptions (1.2) and (3.1) on F, and if $f \in \mathcal{C}(\Omega)$ is locally Lipschitz continuous, bounded from below and satisfying

$$\lim_{d(x)\to 0} f(x) d(x)^{\frac{\beta}{\beta-\alpha-1}} = 0\,,$$

then ergodic pairs (c_Ω, u) exist, and any ergodic function u satisfies

$$\lim_{d(x)\to 0} \frac{u(x)\, d(x)^\chi}{C(x)} = 1 \text{ if } \chi > 0, \tag{3.2}$$

and

$$\lim_{d(x)\to 0} \frac{u(x)}{|\log d(x)|\, C(x)} = 1 \text{ if } \chi = 0\,, \tag{3.3}$$

where

$$\chi = \frac{2+\alpha-\beta}{\beta-1-\alpha}\,,$$

and, for x in a neighborhood of $\partial\Omega$,

$$\begin{aligned} C(x) &= ((\chi+1)F(\nabla d(x)\otimes\nabla d(x)))^{\frac{1}{\beta-\alpha-1}}\,\chi^{-1} && \text{if} \quad \chi>0, \\ C(x) &= F(\nabla d(x)\otimes\nabla d(x)) && \text{if} \quad \chi=0. \end{aligned} \tag{3.4}$$

Let us now observe that any ergodic function verifies the assumptions of Theorem 1.1, so that it is a $C^{1,\gamma}_{loc}(\Omega)$ function and satisfies the local a priori bound (1.3). This regularity property, jointly with the asymptotic estimates (3.2), allows us to precise, at least in the case $\chi>0$, the boundary asymptotic behaviour of its gradient. We obtain the analogous of the result proved in [14] for Laplace operator and in [13] for p-Laplace operator.

Theorem 3.1. *Assume that F satisfies* (1.2) *and* (3.1)*, let $f\in\mathcal{C}(\Omega)$ be bounded and suppose that $\chi>0$ (i.e. $\beta<\alpha+2$). Then, for any ergodic function u, one has*

$$\lim_{d(x)\to 0}\frac{d(x)^{\chi+1}\nabla u(x)\cdot\nabla d(x)}{C(x)}=-\chi\,. \tag{3.5}$$

Proof. Let us consider, for $x_0\in\partial\Omega$ fixed and $\delta>0$, the function

$$u_\delta(\zeta)=\delta^\chi u(x_0+\delta\,\zeta)\,,$$

defined for $\zeta\in\frac{1}{\delta}\,(\Omega-x_0)$.

By (3.2), one has

$$\lim_{\delta\to 0}u_\delta(\zeta)=\frac{C(x_0)}{(\nabla d(x_0)\cdot\zeta)^\chi}$$

locally uniformly with respect to ζ in the halfspace $H=\{\zeta\in\mathbb{R}^N\,:\,\zeta\cdot\nabla d(x_0)>0\}$, and uniformly with respect to $x_0\in\partial\Omega$. In particular, u_δ is locally uniformly bounded in H.

Moreover, by direct computation, u_δ satisfies the equation

$$-|\nabla u_\delta|^\alpha F(D^2u_\delta)+|\nabla u_\delta|^\beta=\delta^{\frac{\beta}{\beta-\alpha-1}}\,[f(x_0+\delta\,\zeta)+c] \qquad \text{in } \frac{1}{\delta}\,(\Omega-x_0)\;.$$

Thus, as a consequence of Theorem 1.1, u_δ belongs to $\mathcal{C}^{1,\gamma}_{loc}(H)$ and verifies estimate (1.3). This implies that u_δ is converging in $\mathcal{C}^1_{loc}(H)$, and, therefore,

$$\lim_{\delta\to 0}\nabla u_\delta(\zeta)=-\chi\frac{C(x_0)\,\nabla d(x_0)}{(\nabla d(x_0)\cdot\zeta)^{\chi+1}}$$

locally uniformly with respect to $\zeta\in H$ and, again, uniformly with respect to $x_0\in\partial\Omega$.

Hence, we deduce that

$$\lim_{\delta\to 0} d(x_0+\delta\,\zeta)^{\chi+1}\nabla u(x_0+\delta\,\zeta) = -\chi\, C(x_0)\,\nabla d(x_0)$$

locally uniformly with respect to $\zeta\in H$ and uniformly with respect to $x_0\in\partial\Omega$. This immediately yields (3.5). □

Remark 3.2. In the case $\chi=0$, one can try to use an analogous argument as above, and to consider the function

$$u_\delta(\zeta) = u(x_0+\delta\zeta) + C(x_0)\log(\delta)\,.$$

By Theorem 4.2 and Theorem 6.3 of [6], it follows that u_δ is uniformly bounded. Moreover, arguing as in the above proof, we obtain that u_δ actually converges in $\mathcal{C}^1_{loc}(H)$ to a solution of

$$\begin{cases} -|\nabla v|^\alpha F(D^2v) + |\nabla v|^{2+\alpha} = 0 & \text{in } H \\ v = +\infty & \text{on } \partial H \end{cases}$$

Using the same argument as in Section 4 of [11], one gets that v satisfies

$$\begin{cases} -F(D^2v) + |\nabla v|^2 = 0 & \text{in } H \\ v = +\infty & \text{on } \partial H \end{cases}$$

Now, consider first the case when F is a *linear operator*, that is $F(M) = a\,\mathrm{tr}(M)$. Then, defining $\varphi = e^{-v/a}$, one sees that φ is positive and harmonic in H, and it satisfies zero boundary conditions. Hence, $\varphi(\zeta) = c\,\nabla d(x_0)\cdot\zeta$ for some constant $c>0$. Coming back to v, one gets that $v(\zeta) = -a\,\log\nabla d(x_0)\cdot\zeta - a\,\log c$, and, by the local $\mathcal{C}^1$ convergence of u_δ to v, we conclude that

$$\lim_{\delta\to 0}\delta\,\nabla u(x_0+\delta\,\zeta) = -\frac{a\,\nabla d(x_0)}{\nabla d(x_0)\cdot\zeta}\,.$$

Observing that in this case $a = C(x_0)$, we deduce for linear operators the asymptotic gradient behaviour

$$\lim_{d(x)\to 0}\frac{d(x)\nabla u(x)\cdot\nabla d(x)}{C(x)} = -1\,,$$

which is the analogous of (3.5) for $\chi=0$.

For general F, an analogous result could be obtained as a consequence of the following Liouville type result: if u is a solution in the half space $H=\{x_N>0\}$ of

$$\begin{cases} -F(D^2u) + |\nabla u|^2 = 0 & \text{in } \{x_N>0\} \\ u(x',0) = +\infty, \end{cases}$$

then there exists some constant c such that

$$u(x) = F(e_N \otimes e_N)|\log x_N| + c\,.$$

By the time being, this is an open question.

We are finally in the position to prove the uniqueness, up to additive constants, of the ergodic function.

Proof of Theorem 1.2. By Theorem 3.1, any ergodic function u satisfies the asymptotic gradient boundary behaviour (3.5). Hence, there exists a positive constant C such that $|\nabla u| \geq Cd^{-\chi-1}$ for $d < \delta$. We can suppose that δ is so small that $(\beta-1-\alpha)Cd^{-(\chi+1)\beta} > 2|f|_\infty(1+\alpha)$.

Suppose now that u and v are two ergodic functions related to the same ergodic constant c_Ω. Recall that $\Omega_\delta = \{x \in \Omega, d(x) < \delta\}$ and consider $u_\epsilon = (1-\epsilon)u$. Let for further computations c_α and c_β some positive constants so that for $\epsilon < \frac{1}{2}$,

$$|(1-\epsilon)^{1+\alpha} - 1 + (1+\alpha)\epsilon| \leq c_\alpha \epsilon^2$$

$$|(1-\epsilon)^{\beta} - 1 + \beta\epsilon| \leq c_\beta \epsilon^2$$

and take $\epsilon < \frac{\beta-\alpha-1}{4} \inf_{\Omega_\delta} \frac{|\nabla u|^\beta}{(c_\alpha+c_\beta)(|\nabla u|^\beta + |f+c|_\infty)}$. Then, u_ϵ is a strict sub-solution in Ω_δ. Indeed

$$\begin{aligned}
-|\nabla u_\epsilon|^\alpha F(D^2 u_\epsilon) &+ |\nabla u_\epsilon|^\beta - (f + c_\Omega)\\
&= (1-\epsilon)^{1+\alpha}\Big(-|\nabla u|^\alpha F(D^2 u) + |\nabla u|^\beta - (f+c_\Omega)\Big)\\
&+ ((1-\epsilon)^\beta - (1-\epsilon)^{1+\alpha})|\nabla u|^\beta + (f+c_\Omega)((1-\epsilon)^{1+\alpha} - 1)\\
&\leq -\epsilon(\beta-1-\alpha)Cd^{-(\chi+1)\beta} + |f+c_\Omega|_\infty \epsilon(1+\alpha)\\
&+ \epsilon^2(c_\alpha + c_\beta)\Big(|\nabla u|^\beta + |f+c|_\infty\Big)\\
&< 0
\end{aligned}$$

By the asymptotic behavior both of u and v, let V_ϵ a neighborhood of the boundary on which $u_\epsilon < v$. Applying the comparison principle in $\Omega_\delta \setminus \overline{V_\epsilon}$ (see [1]), when one of the sub- (super-) solution is strict, one gets that $u_\epsilon \leq v + \sup_{d=\delta}(u_\epsilon - v)$ in $\Omega_\delta \setminus \overline{V_\epsilon}$, hence finally in the whole of Ω_δ, Passing to the limit one gets that $u \leq v + \sup_{d=\delta}(u-v)$ in Ω_δ. On the other hand, using the comparison principle without zero order terms when $\sup(f + c_\Omega) < 0$, proved in [6], one gets that $u \leq v + \sup_{\partial\Omega_\delta}(u-v)$ in $\Omega \setminus \overline{\Omega}_\delta$. We need to prove that u indeed coincides with $v + \sup_{d=\delta}(u-v) := v + m$. The step before says that the supremum of $(u-v)$ in Ω is achieved on $d = \delta$, hence inside Ω. When $\alpha = 0$, the strong maximum principle implies that $u = v + m$.

When $\alpha \neq 0$, suppose that $\partial\Omega$ has only one connected component, then Ω_δ is connected. We want to prove that in the whole of Ω_δ, $u = v + m$. Indeed note that Ω_δ has been chosen so that $|\nabla u| \geq Cd^{-\chi-1}$ inside it, hence in particular $\nabla u \neq 0$ in Ω_δ. Then there is an $\bar{x} \in \Omega$, such that $u(\bar{x}) = v(\bar{x}) + m$, $u \leq v + m$, and $\nabla u(\bar{x}) = \nabla v(\bar{x}) \neq 0$. Using the strong comparison principle in [4] one gets that there exists a neighborhood $V_{\bar{x}}$ of $\bar{x}$ where $u \equiv v + m$. Denote $\mathcal{O}_\delta = \{x \in \Omega_\delta, u(x) = v(x) + m\}$. By the previous argument there is one ball $B(\bar{x}, r)$ so that $B(\bar{x}, r) \cap \Omega_\delta \subset \mathcal{O}_\delta$. In particular $\mathcal{O}_\delta$ is non empty, and the same argument proves that $\mathcal{O}_\delta$ is open. By definition it is closed, so $\mathcal{O}_\delta = \Omega_\delta$. Then applying the comparison Theorem without zero order terms in $\Omega \setminus \overline{\Omega}_\delta$, since we have on its boundary $u = v + m$ one gets both that $u \leq v + m$ and $u \geq v + m$. Finally $u = v + m$.

□

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References

[1] I. Birindelli, F. Demengel *First eigenvalue and Maximum principle for fully nonlinear singular operators* , Advances in Differential equations, Vol (2006) **11** (1), 91-119.

[2] I. Birindelli, F. Demengel , *Regularity and uniqueness of the first eigenfunction for singular fully nonlinear operators*. J. Differential Equations 249 (2010), no. 5, 1089-1110.

[3] I. Birindelli, F. Demengel, $\mathcal{C}^{1,\beta}$ *regularity for Dirichlet problems associated to fully nonlinear degenerate elliptic equations*, ESAIM COCV, (2014), **20** (40), 1009-1024.

[4] I. Birindelli, F. Demengel, *Fully nonlinear operators with Hamiltonian: Hölder regularity of the gradient* . NoDEA Nonlinear Differential Equations Appl. 23 (2016), no. 4.

[5] I. Birindelli, F. Demengel, *Hölder regularity of the gradient for solutions of fully nonlinear equations with sub linear first order term* . Geometric methods in PDE's, 257-268, Springer INdAM Ser., 13.

[6] I. Birindelli, F. Demengel, F. Leoni, Ergodic pairs for singular or degenerates fully nonlinear operators To appear in COCV ESAIM, 2019.

[7] I. Birindelli, F. Demengel, F. Leoni, *Dirichlet problems for fully nonlinear equations with "subquadratic" Hamiltonians*, Contemporary Research in elliptic PDEs and related topics, Springer INdAM Ser., 2019.

[8] L. Caffarelli, *Interior a priori estimates for solutions of fully nonlinear equations*, Ann. Math. 130 (1989), 189-213.

[9] L. Caffarelli, M. G. Crandall, M. Kocan, A. Swiech, *on Viscosity solutions of Fully Non linear equations with measurable coefficients* Comm. on Pure and applied Mathematics, XLIX, 365-397 (1996)

[10] I. Capuzzo Dolcetta, F. Leoni, A. Porretta, *Hölder's estimates for degenerate elliptic equations with coercive Hamiltonian*, Transactions of the American Mathematical Society, Vol 362, n.9, (2010), 4511-4536.

[11] C. Imbert, L. Silvestre, $C^{1,\alpha}$ *regularity of solutions of degenerate fully non-linear elliptic equations*, Adv. Math. **233**, (2013), 196-206.

[12] Koike, S.; Swiech, A. *Maximum principle for fully nonlinear equations via the iterated comparison function method.*, Math. Ann. 339, no. 2 (2007), 461-484.

[13] T. Leonori, A. Porretta, *Large solutions and gradient bounds for quasilinear elliptic equations*, Communications in Partial Differential Equations, 41:6 (2016), 952-998.

[14] A. Porretta, *The ergodic limit for a viscous Hamilton- Jacobi equation with Dirichlet conditions,*Rend. Lincei Mat. Appl. 21 (2010), 59-78.

[15] G. Saller Nornberg, $C^{1,\alpha}$ *regularity for fully nonlinear elliptic equations with superlinear growth in the gradient*, Arxiv 1802 01643v4.

[16] Swiech, A. $W^{2,p}$ *-interior estimates for solutions of fully nonlinear, uniformly elliptic equations* . Adv. Diff. Eqs. 2 (6) (1997), 1005-1027.

[17] N. Winter, $W^{2,p}$ *and* $W^{1,p}$ *Estimates at the Boundary for Solutions of Fully Nonlinear, Uniformly Elliptic Equations*, Zeitsch. fur Anal. und ihre Anwen., 28 (2009), n.2, 129-164.