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ERGODIC PAIRS FOR SINGULAR OR DEGENERATE FULLY NONLINEAR OPERATORS

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Abstract. We study the ergodic problem for fully nonlinear operators which may be singular or degenerate when the gradient of solutions vanishes. We prove the convergence of both explosive solutions and solutions of Dirichlet problems for approximating equations. We further characterize the ergodic constant as the infimum of constants for which there exist bounded sub-solutions. As intermediate results of independent interest, we prove *a priori* Lipschitz estimates depending only on the norm of the zeroth order term, and a comparison principle for equations having no zero order terms.

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1. Introduction

In 1989, in a fundamental paper [21], Lasry and Lions study solutions of

$$-\Delta u+|\nabla u|^q+\lambda u=f(x) \text{ in } \Omega \tag{1.1}$$

that blow up on the boundary of Ω. Here, $q>1$ and Ω is a $\mathcal{C}^2$ bounded domain in $\mathbb{R}^N$. Among other things, they prove that in the subquadratic case $q\le 2$ there exists a unique constant c_Ω, called ergodic constant, and there exists a unique, up to a constant, solution of

$$-\Delta\varphi+|\nabla\varphi|^q-c_\Omega=f(x) \text{ in } \Omega,\ \varphi=+\infty \text{ on } \partial\Omega.$$

The couple (φ,c_Ω) is referred to as an ergodic pair. It is well known that for $q=2$, $-c_\Omega$ is just the principal eigenvalue of $(-\Delta+f)(\cdot)$. The paper [21] generated a huge and interesting literature, also in connection with the stochastic interpretation of the problem. In particular, Porretta in [24] observed a direct connection between the ergodic pairs and the behavior as $\lambda\to 0$ of solutions of equation (1.1) satisfying homogeneous Dirichlet boundary conditions. Moreover, a representation formula for the ergodic constant resembling the one known for the principal eigenvalue is proved in [24] even for $q<2$.

Interestingly, while the concept of principal eigenvalue has been extended to fully nonlinear operators of different types (see *e.g.* [7, 12]), the notion of ergodic constant has not been much investigated in fully nonlinear

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settings. The scope of this paper is to give a thoroughly picture of the ergodic pairs and the related blowing up solutions and solutions with Dirichlet boundary condition for approximating equations.

We now detail the main theorems. In the whole paper Ω denotes a $\mathcal{C}^2$ bounded domain of $\mathbb{R}^N$; $\mathcal{S}$ denotes the space of symmetric matrices in $\mathbb{R}^N$. We consider a uniformly elliptic homogenous operator F, *i.e.* a continuous function $F : \mathcal{S} \to \mathbb{R}$ satisfying:

$$\begin{aligned} &\text{there exist } 0 < a < A \text{ such that for all } M, N \in \mathcal{S}, \text{ with } N \geq 0, \text{ and for all } t > 0, \\ &a\,\mathrm{tr}(N) \leq F(M+N) - F(M) \leq A\,\mathrm{tr}(N), \quad F(tM) = tF(M). \end{aligned} \tag{1.2}$$

We will always consider the differential operator

$$G(\nabla u, D^2 u) = -|\nabla u|^\alpha F(D^2 u) + |\nabla u|^\beta$$

with $\alpha > -1$ and $\alpha + 1 < \beta \leq \alpha + 2$. This reduces to the Lasry Lions case for $\alpha = 0$ and $a = A = 1$.

Theorem 1.1. *Suppose that f is bounded and locally Lipschitz continuous in Ω, and that F satisfies* (1.2)*. Consider the Dirichlet problems*

$$\begin{cases} -|\nabla u|^\alpha F(D^2 u) + |\nabla u|^\beta = f & \text{in } \Omega \\ u = 0 & \text{on } \partial\Omega, \end{cases} \tag{1.3}$$

and, for $\lambda > 0$,

$$\begin{cases} -|\nabla u|^\alpha F(D^2 u) + |\nabla u|^\beta + \lambda |u|^\alpha u = f & \text{in } \Omega \\ u = 0 & \text{on } \partial\Omega. \end{cases} \tag{1.4}$$

The following alternative holds.

1. *Suppose that there exists a bounded sub-solution of* (1.3)*. Then the solution u_λ of* (1.4) *satisfies: (u_λ) is bounded and uniformly converging up to a sequence $\lambda_n \to 0$ to a solution of (1.3).*
2. *Suppose that there is no solution for the Dirichlet problem* (1.3)*. Then, (u_λ) satisfies, up to a sequence $\lambda_n \to 0$ and locally uniformly in Ω,*
 (a) *$u_\lambda \to -\infty$;*
 (b) *there exists a constant $c_\Omega \geq 0$ such that $\lambda |u_\lambda|^\alpha u_\lambda \to -c_\Omega$;*
 (c) *c_Ω is an ergodic constant and $v_\lambda = u_\lambda + |u_\lambda|_\infty$ converges to a solution of the ergodic problem*

$$\begin{cases} -|\nabla v|^\alpha F(D^2 v) + |\nabla v|^\beta = f + c_\Omega & \text{in } \Omega \\ v = +\infty & \text{on } \partial\Omega \end{cases} \tag{1.5}$$

 whose minimum is zero.

The standard notion of viscosity solution fails when the operator is singular, *i.e.*, in this paper, when $\alpha < 0$, hence we will consider viscosity solutions as defined in [7].

When $\alpha = 0$ and F is the Laplacian, Theorem 1.1 has been proved by Porretta in [24]. The case of p-Laplacian operators is considered by Leonori and Porretta in [22].

We observe that Theorem 1.1 yields the existence of an ergodic constant, and a sign property, in the case when the Dirichlet problem (1.3) does not have any solution. In order to show the existence of ergodic pairs when problem (1.3) does admit (sub)solutions, it is enough to prove the existence, for $\lambda > 0$, of solutions blowing up on the boundary. This is achieved in Theorem 4.1, where we basically reproduce the construction given in [21] of explosive sub- and super-solutions.

However, in order to establish further properties of the ergodic constant (in particular, its uniqueness), we need to prove new and refined boundary estimates for explosive solutions. These will be obtained under the additional regularity condition

$$F(\nabla d(x)\otimes\nabla d(x)) \text{ is } \mathcal{C}^2 \text{ in a neighborhood of } \partial\Omega\,, \tag{1.6}$$

where $d(x)$ denotes the distance function from $\partial\Omega$. We observe that (1.6) is certainly satisfied if the domain Ω is of class $\mathcal{C}^3$ and the operator F is $\mathcal{C}^2$, but there can also be cases with nonsmooth F satisfying (1.6). For instance, for all operators $F(M)$ which depend only on the eigenvalues of M, such as Pucci's operators, $F(\nabla d(x)\otimes\nabla d(x))$ is a constant function as long as $|\nabla d(x)|=1$.

Under condition (1.6), we will show that the ergodic constant c_Ω is unique and that it shares some properties with the principal eigenvalue. Let us recall that if v solves (1.5) with $F=\Delta$ and $\beta=\alpha+2$, then $w=e^{-v}$ is a positive solution of

$$\begin{cases} -|\nabla w|^\alpha \Delta w + f|w|^\alpha w = -c_\Omega |w|^\alpha w & \text{in } \Omega\,,\\ w=0 & \text{on } \partial\Omega\,,\end{cases}$$

that is $-c_\Omega$ is the principal eigenvalue of the operator $-|\nabla u|^\alpha\Delta u + f|u|^\alpha u$. This shows a deep relationship between ergodic constants and principal eigenvalues, also confirmed by a Faber–Krahn inequality for the ergodic constant of the Laplacian proved by Ferone *et al.* [19].

Even for $\beta\neq\alpha+2$ and F fully nonlinear, the ergodic constant can be characterized by an inf-formula analogous to the one which defines the principal eigenvalues for fully nonlinear operators.

Following [6, 24], we define

$$\mu^\star = \inf\{\mu\,:\,\exists\,\varphi\in\mathcal{C}(\overline{\Omega}), -|\nabla\varphi|^\alpha F(D^2\varphi) + |\nabla\varphi|^\beta \le f+\mu\}\,.$$

Note that $\mu^\star$ depends on f and Ω, but if there is no ambiguity we will not precise this dependence.

Next, we state the second main result of the paper, which extends to the fully nonlinear case the analogous results proved by Porretta [24] for Laplace operator.

Theorem 1.2. *Suppose that f is bounded and locally Lipschitz continuous in Ω, and that F satisfies* (1.2) *and* (1.6)*. Let c_Ω be an ergodic constant for problem* (1.5)*; then:*

1. *c_Ω is unique;*
2. *$c_\Omega=\mu^\star$;*
3. *the map $\Omega\mapsto c_\Omega$ is nondecreasing with respect to the domain, and continuous;*
4. *if either $\alpha=0$ or $\alpha\neq0$ and $\sup_\Omega f + c_\Omega<0$, then $\mu^\star$ is not achieved. Moreover, if $\Omega'\subset\subset\Omega$, then $c_{\Omega'}<c_\Omega$.*

In order to prove these results many questions need to be addressed. Clearly the first one is the existence of solutions for (1.4) when $\lambda>0$, but even though it is fundamental, this is extraneous to the spirit of this note and it can be found in [10]. The interested reader will see that it is done through a Perron's procedure *i.e.* constructing sub- and super-solutions of (1.4) together with a comparison principle and some Lipschitz estimates depending on the L^∞ norm of the solution.

Theorems 1.1 and 1.2 are obtained by means of several intermediate results, most of which are of independent interest. A first fundamental tool is an interior Lipschitz estimate for solutions of equation (1.4) that does not depend directly on the L^∞ norm of the solution but only on the norm of the zero order term. In the linear case, these kind of estimates were obtained by Capuzzo *et al.* [13], and the proof we use is inspired by theirs. In order to extend the result to the present fully nonlinear singular case, we have to address several nontrivial technical difficulties, see Section 2. After that, we give the proof of Theorem 1.1 in Section 3.

In Section 4, we focus on existence and estimates for explosive solutions of the approximating λ-equation, *i.e.* solutions u of

$$\begin{cases} -|\nabla u|^\alpha F(D^2u) + |\nabla u|^\beta + \lambda |u|^\alpha u = f & \text{in } \Omega \\ u = +\infty & \text{on } \partial\Omega. \end{cases}$$

Here, the function f is assumed to be continuous in Ω, but it is allowed to be unbounded on the boundary, as long as its growth is controlled. This is an important feature that will be needed in the proof of Theorem 1.2. Construction of explosive solutions in the fully nonlinear setting includes the works by Alarcón and Quaas [1], by Esteban *et al.* [18] and by Demengel and Goubet [17], where only suitable zero order terms are considered. Capuzzo Dolcetta *et al.* [14, 15] construct radial explosive solutions in some degenerate cases. The construction we do in order to give existence and estimates of blowing up solutions differs from the standard proof for linear operators (see Rem. 4.3), and we obtain solutions satisfying nonconstant boundary asymptotics. Moreover, our proof can be carried on for other classes of operators, as *e.g.* the p-Laplacian or some generalizations such as

$$F(p, M) = |p|^\alpha (q_1 trM + q_2 \frac{Mp \cdot p}{|p|^2}),$$

with $q_1 > 0$ and $q_1 + q_2 > 0$. Note that the above operator reduces to the p-Laplacian for $\alpha = q_2 = p - 2$ and $q_1 = 1$. When $p > 2$, using the variational form of the p-Laplacian, and its linearity with respect to the Hessian, Leonori and Porretta proved in [22] such estimates and existence results. So, our result for $\alpha < 0$ covers the case $1 < p < 2$ that was not considered there.

In Section 5, we give a comparison theorem for sub- and super-solutions of equation (1.3), in which zero order terms are lacking. The change of equation that allows to prove the comparison principle of Theorem 5.1 is sort of standard, but the computation which follows is original and *ad hoc* for our setting. The work of Leonori *et al.* [23] has been a source of inspiration.

Finally, in Section 6, after proving the existence of ergodic constants and estimating the ergodic functions near the boundary, we give the proof of Theorem 1.2.

Let us finally remark that we left open the question of uniqueness (up to constants) of the ergodic functions. This is a delicate issue, strongly related with the simplicity of the principal eigenvalue for degenerate/singular operators, which is in general an open problem, see [2, 8]. We recall that the usual proof for linear operators, see [21, 24], relies on the strong maximum principle, which does not hold for degenerate operators. On the other hand, for p-Laplacian operators the uniqueness of the ergodic function is obtained in [22] for $p \geq 2$ under the condition $\sup_\Omega f + c < 0$. We believe that the proof given in [22] can be extended to the fully nonlinear singular/degenerate setting, provided that one can prove the $C^{1,\gamma}$ regularity of the ergodic functions. We defer this study to a future work [11].

Notations

- We use $d(x)$ to denote a $\mathcal{C}^2$ positive function in Ω which coincides with the distance function from the boundary in a neighborhood of $\partial\Omega$,
- For $\delta > 0$, we set $\Omega_\delta = \{x \in \Omega : d(x) > \delta\}$,
- We denote by $\mathcal{M}^+, \mathcal{M}^-$ the Pucci's operators with ellipticity constants a, A, namely, for all $M \in \mathcal{S}$,

$$\mathcal{M}^+(M) = A\,\mathrm{tr}(M^+) - a\,\mathrm{tr}(M^-)$$
$$\mathcal{M}^-(M) = a\,\mathrm{tr}(M^+) - A\,\mathrm{tr}(M^-)$$

and we often use that, as a consequence of (1.2), for all $M, N \in \mathcal{S}$ one has

$$\mathcal{M}^-(N) \leq F(M + N) - F(M) \leq \mathcal{M}^+(N).$$

2. *A priori* Lipschitz-type estimates

In the note [10], we prove the following result

Theorem 2.1. *Assume that f is bounded and continuous in Ω. Then, for any $\lambda > 0$, there exists a unique solution $u_\lambda \in C(\overline{\Omega})$ of (1.4), which is Lipschitz continuous up to the boundary, and satisfies*

$$|u_\lambda|_{W^{1,\infty}(\Omega)} \le C(|u_\lambda|_\infty, |f - \lambda |u_\lambda|^\alpha u_\lambda|_\infty, a, A, \alpha, \beta).$$

This is obtained, through Perron's method, by constructing sub- and super-solutions and using the following general comparison principle.

Theorem 2.2. *Suppose that b is Lipschitz continuous in Ω, $\zeta : \mathbb{R} \to \mathbb{R}$ is a nondecreasing function and f and g are continuous in Ω. Let u be a bounded by above viscosity sub-solution of*

$$-|\nabla u|^\alpha F(D^2 u) + b(x)|\nabla u|^\beta + \zeta(u) \le g$$

and let v be a bounded by below viscosity super solution of

$$-|\nabla v|^\alpha F(D^2 v) + b(x)|\nabla v|^\beta + \zeta(v) \ge f.$$

If either $g \le f$ and ζ is increasing or $g < f$ then, $u \le v$ on $\partial\Omega$ implies that $u \le v$ in Ω.

For the proofs of the above results we refer to [10].

The rest of this section is devoted to prove *a priori* Lipschitz estimates for solutions of the equation

$$-|\nabla u|^\alpha F(D^2 u) + |\nabla u|^\beta + \lambda |u|^\alpha u = f\,, \tag{2.1}$$

that depend on $\lambda |u|_\infty^{\alpha+1}$, but not on $|u|_\infty$. Our estimates will be a consequence of the following result, in which we denote by B the unit ball centered at the origin in $\mathbb{R}^N$.

Proposition 2.3. *Let F satisfy (1.2) and, for $\lambda \ge 0, \alpha > -1$ and $\beta > \alpha + 1$, let u and v be respectively a bounded sub-solution and a bounded from below super solution of equation (2.1) in B, with f Lipschitz continuous in B. Then, for any positive $p \ge \frac{(2+\alpha-\beta)^+}{\beta-\alpha-1}$, there exists a positive constant M, depending only on $p, \alpha, \beta, a, A, N, \|f - \lambda |u|^\alpha u\|_\infty$ and on the Lipschitz constant of f, such that, for all $x, y \in B$ one has*

$$u(x) - v(y) \le \sup_B (u - v)^+ + M \frac{|x-y|}{(1-|y|)^{\frac{\beta+\alpha^-}{\beta-\alpha-1}}} \left[1 + \left(\frac{|x-y|}{(1-|x|)} \right)^p \right].$$

Proof. We argue as in the proof of Theorem 3.1 in [13].

Let us define a "distance" function d which equals $1 - |x|$ near the boundary and it is extended in B as a $\mathcal{C}^2$ function satisfying, for some constant $c_1 > 0$,

$$\begin{cases} d(x) = 1 - |x| & \text{if } |x| > \frac{1}{2} \\ \frac{1-|x|}{2} \le d(x) \le 1 - |x| & \text{for all } x \in \bar{B} \\ |Dd(x)| \le 1\,,\ -c_1 I_N \le D^2 d(x) \le 0 & \text{for all } x \in \bar{B}. \end{cases}$$

For $\xi = \frac{|x-y|}{d(x)}$, we consider the function

$$\phi(x,y) = \frac{k}{d(y)^\tau} |x-y| \left(L + \xi^p \right) + \sup_B (u-v)^+$$

where $p>0$ is a fixed exponent satisfying $p \geq \frac{(2+\alpha-\beta)^+}{\beta-\alpha-1}$, $\tau=\frac{\beta+\alpha^-}{\beta-\alpha-1}$ and L, k are suitably large positive constants to be chosen later.

The statement is proved if we show that for all $(x,y)\in B^2$ one has

$$u(x)-v(y)\leq \phi(x,y)\,.$$

By contradiction, let us assume that $u(x)-v(y)-\phi(x,y)>0$ somewhere. Then, necessarily the supremum is achieved on a pair (x,y) with $x\neq y$ and $d(x)\,,d(y)>0$. Using Ishii's lemma of [20], one gets that on such a point (x,y), for all $\epsilon>0$, there exist symmetric matrices X_ϵ and Y_ϵ such that

$$\begin{gathered}(\nabla_x\phi, X_\epsilon)\in J^{2,+}u(x),\ (-\nabla_y\phi,-Y_\epsilon)\in J^{2,-}v(y)\\ -\left(\frac{1}{\epsilon}+|D^2\phi|\right)I_{2N}\leq \begin{pmatrix} X_\epsilon & O\\ O & Y_\epsilon\end{pmatrix}\leq D^2\phi+\epsilon(D^2\phi)^2.\end{gathered} \tag{2.2}$$

We proceed in the proof by considering separately the cases $\alpha\geq 0$ and $\alpha<0$.

The case $\alpha\geq 0$. Since u is a sub-solution and v a super-solution, by the positive 1-homogeneity of F we have in this case

$$\begin{cases} -F\left(|\nabla_x\phi|^\alpha X_\epsilon\right)+|\nabla_x\phi|^\beta+\lambda|u|^\alpha u(x)\leq f(x)\\ -F\left(-|\nabla_y\phi|^\alpha Y_\epsilon\right)+|\nabla_y\phi|^\beta+\lambda|v|^\alpha v(y)\geq f(y)\end{cases}$$

Subtracting the above inequalities and using also that $u(x)-v(y)>\phi(x,y)\geq 0$, for any $t>0$ we can write

$$\begin{aligned} t\,F\left(|\nabla_x\phi|^\alpha X_\epsilon\right)-\left[F\left((1+t)|\nabla_x\phi|^\alpha X_\epsilon\right)-F\left(-|\nabla_y\phi|^\alpha Y_\epsilon\right)\right]&\\ \leq |\nabla_y\phi|^\beta-|\nabla_x\phi|^\beta+f(x)-f(y)\,,&\end{aligned}$$

and therefore

$$\begin{aligned} t\,|\nabla_x\phi|^\beta\leq\ & F\left((1+t)|\nabla_x\phi|^\alpha X_\epsilon\right)-F\left(-|\nabla_y\phi|^\alpha Y_\epsilon\right)\\ &+|\nabla_y\phi|^\beta-|\nabla_x\phi|^\beta+t\,\left(f-\lambda|u|^\alpha u\right)^+ +f(x)-f(y)\,.\end{aligned}$$

By the uniform ellipticity of F, it then follows that

$$\begin{aligned} t\,|\nabla_x\phi|^\beta\leq\ & \mathcal{M}^+\left((1+t)|\nabla_x\phi|^\alpha X_\epsilon+|\nabla_y\phi|^\alpha Y_\epsilon\right)\\ &+|\nabla_y\phi|^\beta-|\nabla_x\phi|^\beta+t\,\left(f-\lambda|u|^\alpha u\right)^+ +f(x)-f(y)\,.\end{aligned}$$

By multiplying the right inequality of (2.2) on the left and on the right by

$$\begin{pmatrix} \sqrt{1+t}|\nabla_x\phi|^{\alpha/2}I_N & O\\ O & |\nabla_y\phi|^{\alpha/2}I_N\end{pmatrix}$$

and testing the resulting inequality on vectors of the form (v,v) with $v\in\mathbb{R}^N$, we further obtain

$$(1+t)|\nabla_x\phi|^\alpha X_\epsilon+|\nabla_y\phi|^\alpha Y_\epsilon\leq Z_{\alpha,t}+O(\epsilon)\,,$$

with

$$Z_{\alpha,t} = (1+t)|\nabla_x\phi|^\alpha D^2_{xx}\phi + \sqrt{1+t}|\nabla_x\phi|^{\alpha/2}|\nabla_y\phi|^{\alpha/2}\left[D^2_{xy}\phi + \left(D^2_{xy}\phi\right)^t\right] + |\nabla_y\phi|^\alpha D^2_{yy}\phi\,. \tag{2.3}$$

Hence, after letting $\epsilon \to 0$, we get

$$t\,|\nabla_x\phi|^\beta \le \mathcal{M}^+\left(Z_{\alpha,t}\right) + |\nabla_y\phi|^\beta - |\nabla_x\phi|^\beta + t\,\left(f - \lambda|u|^\alpha u\right)^+ + f(x) - f(y)\,. \tag{2.4}$$

We now proceed by evaluating the right hand side terms of (2.4).

An explicit computation shows that, setting $\eta = \frac{|x-y|}{d(y)}$ and $\zeta = \frac{x-y}{|x-y|}$, one has

$$\nabla_x\phi(x,y) = \frac{k}{d(y)^\tau}\left[(L+(1+p)\xi^p)\zeta - p\,\xi^{p+1}\nabla d(x)\right]\,,$$

as well as

$$\nabla_y\phi(x,y) = -\frac{k}{d(y)^\tau}\left[(L+(1+p)\xi^p)\zeta + \tau\,\eta\,(L+\xi^p)\,\nabla d(y)\right]\,.$$

From now on we denote with c possibly different positive constants which depend only on p, N, a, A, α and β. As discussed in [13], for $L > 1$ fixed suitably large depending only on p, one has

$$|\nabla_x\phi| \ge ck\frac{1+\xi^{p+1}}{d(y)^\tau} \tag{2.5}$$

and

$$|\nabla_x\phi|\,,\ |\nabla_y\phi| \le ck\frac{1+\xi^{p+1}}{d(y)^{\tau+1}}\,. \tag{2.6}$$

Moreover, we notice that one has also

$$|\nabla_y\phi| \ge \frac{k}{d(y)^\tau}\left[L+(1+p)\xi^p - \tau\eta(L+\xi^p)|\nabla d(y)|\right] \ge ck\frac{1+\xi^p}{d(y)^\tau} \qquad \text{if}\ \ \tau\,\eta \le \frac{1}{2}\,. \tag{2.7}$$

On the other hand, the second order derivatives of ϕ may be written as follows

$$\begin{aligned} D^2_{xx}\phi = \frac{k}{d(y)^\tau}\Bigg\{&\frac{[L+(1+p)\xi^p]}{|x-y|}B + p(1+p)\frac{\xi^{p-1}}{d(x)}T - p(1+p)\frac{\xi^p}{d(x)}\left(C+C^t\right)\\ &+p(1+p)\frac{\xi^{p+1}}{d(x)}\nabla d(x)\otimes\nabla d(x) - p\,\xi^{p+1}D^2d(x)\Bigg\}\end{aligned}$$

$$\begin{aligned} D^2_{xy}\phi = -\frac{k}{d(y)^\tau}\Bigg\{&\frac{[L+(1+p)\xi^p]}{|x-y|}B + p(1+p)\frac{\xi^{p-1}}{d(x)}T - p(1+p)\frac{\xi^p}{d(x)}C^t\\ &+\frac{\tau\,[L+(1+p)\xi^p]}{d(y)}E - \frac{\tau\,p\,\xi^{p+1}}{d(y)}\nabla d(x)\otimes\nabla d(y)\Bigg\}\end{aligned}$$

$$D^2_{yy}\phi = \frac{k}{d(y)^\tau}\left\{\frac{[L+(1+p)\xi^p]}{|x-y|}B + p(1+p)\frac{\xi^{p-1}}{d(x)}T + \frac{\tau\,[L+(1+p)\xi^p]}{d(y)}\left(E+E^t\right)\right.$$
$$\left. + \frac{\tau\,(\tau+1)\,\eta(L+\xi^p)}{d(y)}\nabla d(y)\otimes\nabla d(y) - \tau\,\eta\,(L+\xi^p)D^2 d(y)\right\}$$

with $B = I_N - \zeta\otimes\zeta\,,\ T = \zeta\otimes\zeta\,,\ C = \zeta\otimes\nabla d(x)$ and $E = \zeta\otimes\nabla d(y)\,.$

Therefore, the matrix $Z_{\alpha,t}$ defined in (2.3) is given by

$$Z_{\alpha,t} = \frac{k}{d(y)^\tau}\left\{\left(\sqrt{1+t}|\nabla_x\phi|^{\alpha/2} - |\nabla_y\phi|^{\alpha/2}\right)^2\left[\frac{(L+(1+p)\xi^p)}{|x-y|}B + p(1+p)\frac{\xi^{p-1}}{d(x)}T\right]\right.$$
$$-\sqrt{1+t}|\nabla_x\phi|^{\alpha/2}\left(\sqrt{1+t}|\nabla_x\phi|^{\alpha/2} - |\nabla_y\phi|^{\alpha/2}\right)\frac{p(1+p)\xi^p}{d(x)}\left(C+C^t\right)$$
$$-|\nabla_y\phi|^{\alpha/2}\left(\sqrt{1+t}|\nabla_x\phi|^{\alpha/2} - |\nabla_y\phi|^{\alpha/2}\right)\frac{\tau(L+(1+p)\xi^p)}{d(y)}\left(E+E^t\right)$$
$$+p(1+t)|\nabla_x\phi|^\alpha\left[\frac{(1+p)\xi^{p+1}}{d(x)}\nabla d(x)\otimes\nabla d(x) - \xi^{p+1}D^2 d(x)\right]$$
$$+\sqrt{1+t}|\nabla_x\phi|^{\alpha/2}|\nabla_y\phi|^{\alpha/2}\frac{\tau p\xi^{(p+1)}}{d(y)}\left[\nabla d(x)\otimes\nabla d(y) + \nabla d(y)\otimes\nabla d(x)\right]$$
$$\left.+\tau\,|\nabla_y\phi|^\alpha\left[\frac{(1+\tau)\eta(L+\xi^p)}{d(y)}\nabla d(y)\otimes\nabla d(y) - \eta(L+\xi^p)D^2 d(y)\right]\right\}\,,$$

and, recalling that $\xi = |x-y|/d(x)$ and that $d<1$ in B, this yields the estimate

$$\mathcal{M}^+(Z_{\alpha,t}) \le \frac{c\,k}{d(y)^\tau}\left\{\left(\sqrt{1+t}|\nabla_x\phi|^{\alpha/2} - |\nabla_y\phi|^{\alpha/2}\right)^2\frac{1+\xi^p}{|x-y|}\right.$$
$$+\sqrt{1+t}|\nabla_x\phi|^{\alpha/2}\left|\sqrt{1+t}|\nabla_x\phi|^{\alpha/2} - |\nabla_y\phi|^{\alpha/2}\right|\frac{\xi^p}{d(x)}$$
$$+|\nabla_y\phi|^{\alpha/2}\left|\sqrt{1+t}|\nabla_x\phi|^{\alpha/2} - |\nabla_y\phi|^{\alpha/2}\right|\frac{1+\xi^p}{d(y)}$$
$$\left.+(1+t)|\nabla_x\phi|^\alpha\frac{\xi^{p+1}}{d(x)} + \sqrt{1+t}|\nabla_x\phi|^{\alpha/2}|\nabla_y\phi|^{\alpha/2}\frac{\xi^{p+1}}{d(y)} + |\nabla_y\phi|^\alpha\eta\frac{1+\xi^p}{d(y)}\right\}.$$

By observing that

$$\left|\sqrt{1+t}|\nabla_x\phi|^{\alpha/2} - |\nabla_y\phi|^{\alpha/2}\right| \le (\sqrt{t+1}-1)|\nabla_x\phi|^{\alpha/2} + \left||\nabla_x\phi|^{\alpha/2} - |\nabla_y\phi|^{\alpha/2}\right|$$

and by applying the trivial inequalities $\sqrt{1+t}-1\le t$, $\sqrt{1+t}\left(\sqrt{t+1}-1\right)\le t$, $\sqrt{1+t}\le 1+t$, after rearranging terms we then deduce

$$\mathcal{M}^+(Z_{\alpha,t}) \le \frac{c\,k}{d(y)^\tau}\left\{t^2|\nabla_x\phi|^\alpha\frac{1+\xi^p}{|x-y|}\right.$$
$$+t\left[|\nabla_x\phi|^\alpha\frac{\xi^p+\xi^{p+1}}{d(x)} + |\nabla_x\phi|^{\alpha/2}|\nabla_y\phi|^{\alpha/2}\left(\frac{1+\xi^p+\xi^{p+1}}{d(y)} + \frac{\xi^p}{d(x)}\right)\right]$$

$$
\begin{aligned}
&+\left(|\nabla_x\phi|^{\alpha/2}-|\nabla_y\phi|^{\alpha/2}\right)^2\frac{1+\xi^p}{|x-y|}\\
&+\left||\nabla_x\phi|^{\alpha/2}-|\nabla_y\phi|^{\alpha/2}\right|\left(|\nabla_x\phi|^{\alpha/2}\frac{\xi^p}{d(x)}+|\nabla_y\phi|^{\alpha/2}\frac{1+\xi^p}{d(y)}\right)\\
&+|\nabla_x\phi|^{\alpha}\frac{\xi^{p+1}}{d(x)}+|\nabla_y\phi|^{\alpha}\frac{(1+\xi^p)\,\eta}{d(y)}+|\nabla_x\phi|^{\alpha/2}|\nabla_y\phi|^{\alpha/2}\frac{\xi^{p+1}}{d(y)}\Bigg\}.
\end{aligned}
\tag{2.8}
$$

We now recall that, as proved in [13], for all $q,\gamma>0$ one has

$$
\frac{\xi^q}{d(x)^\gamma}\leq 2^\gamma\frac{1+\xi^{q+\gamma}}{d(y)^\gamma}\,. \tag{2.9}
$$

Moreover, if $\alpha\geq 2$, the mean value theorem, the bounds (2.6), (2.9) and the explicit expression of $\nabla_x\phi+\nabla_y\phi$ imply that

$$
\begin{aligned}
\left||\nabla_x\phi|^{\alpha/2}-|\nabla_y\phi|^{\alpha/2}\right| &\leq c\,\max\left\{|\nabla_x\phi|^{\alpha/2-1},|\nabla_y\phi|^{\alpha/2-1}\right\}|\nabla_x\phi+\nabla_y\phi|\\
&\leq ck^{\alpha/2}\frac{(1+\xi^{p+1})^{\alpha/2-1}}{d(y)^{\alpha/2(\tau+1)-1}}\left(\frac{\xi^p}{d(x)}+\frac{1+\xi^p}{d(y)}\right)|x-y|\\
&\leq c\left[k\frac{(1+\xi^{p+1})}{d(y)^{\tau+1}}\right]^{\alpha/2}|x-y|.
\end{aligned}
$$

Analogously, if $\alpha<2$ but $\tau\,\eta\leq 1/2$, from (2.5), (2.7) and again (2.9) we deduce

$$
\left||\nabla_x\phi|^{\alpha/2}-|\nabla_y\phi|^{\alpha/2}\right|\leq c\left[k\frac{(1+\xi^p)}{d(y)^\tau}\right]^{\alpha/2-1}k\frac{(1+\xi^{p+1})}{d(y)^{\tau+1}}|x-y|
$$

and therefore, since $\xi\leq\frac{1}{2\tau-1}$ for $\eta\leq\frac{1}{2\tau}$, we obtain in this case

$$
\left||\nabla_x\phi|^{\alpha/2}-|\nabla_y\phi|^{\alpha/2}\right|\leq c\left[k\frac{(1+\xi^{p+1})}{d(y)^{\tau+2/\alpha}}\right]^{\alpha/2}|x-y|\,.
$$

Finally, if $\alpha<2$ and $\tau\,\eta>1/2$, that is $|x-y|>d(y)/2\tau$, we have

$$
\begin{aligned}
&\left||\nabla_x\phi|^{\alpha/2}-|\nabla_y\phi|^{\alpha/2}\right|\\
&\quad\leq|\nabla_x\phi+\nabla_y\phi|^{\alpha/2}\leq c\left[\frac{k(1+\xi^{p+1})|x-y|}{d(y)^{\tau+1}}\right]^{\alpha/2}\\
&\quad\leq c\left[\frac{k(1+\xi^{p+1})}{d(y)^{\tau+1}}\right]^{\alpha/2}\left(\frac{2\tau}{d(y)}\right)^{1-\alpha/2}|x-y|=c\left[k\frac{(1+\xi^{p+1})}{d(y)^{\tau+2/\alpha}}\right]^{\alpha/2}|x-y|\,.
\end{aligned}
$$

Thus, in all cases we obtain

$$
\left||\nabla_x\phi|^{\alpha/2}-|\nabla_y\phi|^{\alpha/2}\right|\leq c\left[k\frac{(1+\xi^{p+1})}{d(y)^{\tau+\max\{1,2/\alpha\}}}\right]^{\alpha/2}|x-y|\leq c\frac{|\nabla_x\phi|^{\alpha/2}}{d(y)^{\max\{\alpha/2,1\}}}|x-y|\,. \tag{2.10}
$$

By using inequalities (2.6), (2.5), (2.9) and (2.10), from estimate (2.8) it then follows

$$\mathcal{M}^+(Z_{\alpha,t}) \leq \frac{ck|\nabla_x\phi|^\alpha}{d(y)^\tau}\left\{t^2\frac{1+\xi^p}{|x-y|}+t\frac{1+\xi^{p+2}}{d(y)^{\alpha/2+1}}+|x-y|\frac{1+\xi^{p+2}}{d(y)^{\alpha+2}}\right\}. \tag{2.11}$$

Moreover, since $p \geq \frac{2+\alpha-\beta}{\beta-\alpha-1}$ and $\tau \geq \frac{\alpha+2}{2(\beta-\alpha-1)}$, by using again (2.5), we further deduce

$$\mathcal{M}^+(Z_{\alpha,t}) \leq ck|\nabla_x\phi|^\alpha\left\{t^2\frac{1+\xi^p}{d(y)^\tau|x-y|}+t\frac{|\nabla_x\phi|^{\beta-\alpha}}{k^{\beta-\alpha}}+\frac{|x-y|(1+\xi^{p+2})}{d(y)^{\tau+\alpha+2}}\right\}.$$

Using the above inequality jointly with (2.4) yields

$$\begin{aligned} t\,|\nabla_x\phi|^{\beta-\alpha} \leq ck&\left\{t^2\frac{1+\xi^p}{d(y)^\tau|x-y|}+t\frac{|\nabla_x\phi|^{\beta-\alpha}}{k^{\beta-\alpha}}+\frac{|x-y|(1+\xi^{p+2})}{d(y)^{\tau+\alpha+2}}\right\}\\ &+|\nabla_x\phi|^{-\alpha}\left(|\nabla_y\phi|^\beta-|\nabla_x\phi|^\beta\right)+t\,|\nabla_x\phi|^{-\alpha}\left(f-\lambda|u|^\alpha u\right)^+\\ &+|\nabla_x\phi|^{-\alpha}\left(f(x)-f(y)\right), \end{aligned}$$

and therefore, being $\beta > \alpha+1$, for k sufficiently large one has

$$\begin{aligned} \frac{t}{2}\,|\nabla_x\phi|^{\beta-\alpha}-t^2\frac{ck(1+\xi^p)}{d(y)^\tau|x-y|} \leq{}& \frac{ck|x-y|(1+\xi^{p+2})}{d(y)^{\tau+\alpha+2}}+|\nabla_x\phi|^{-\alpha}\left(|\nabla_y\phi|^\beta-|\nabla_x\phi|^\beta\right)\\ &+t\,|\nabla_x\phi|^{-\alpha}\left(f-\lambda|u|^\alpha u\right)^++|\nabla_x\phi|^{-\alpha}\left(f(x)-f(y)\right). \end{aligned}$$

We now choose $t>0$ in order to maximize the left hand side, namely

$$t=\frac{|\nabla_x\phi|^{\beta-\alpha}d(y)^\tau|x-y|}{4ck(1+\xi^p)}.$$

We then obtain

$$\begin{aligned} |\nabla_x\phi|^{2(\beta-\alpha)} \leq c&\left\{\frac{k^2(1+\xi^{2(p+1)})}{d(y)^{2\tau+\alpha+2}}+k|\nabla_x\phi|^{-\alpha}\frac{(1+\xi^p)\left(|\nabla_y\phi|^\beta-|\nabla_x\phi|^\beta\right)}{|x-y|d(y)^\tau}\right.\\ &\left.+|\nabla_x\phi|^{\beta-2\alpha}\left(f-\lambda|u|^\alpha u\right)^++k|\nabla_x\phi|^{-\alpha}\frac{(1+\xi^p)\left(f(x)-f(y)\right)}{|x-y|d(y)^\tau}\right\}. \end{aligned}$$

Moreover, arguing as for (2.10) in the case $\alpha \geq 2$, we also have

$$\left||\nabla_y\phi|^\beta-|\nabla_x\phi|^\beta\right| \leq ck^\beta|x-y|\frac{(1+\xi^{p+1})^\beta}{d(y)^{(\tau+1)\beta}},$$

so that

$$\begin{aligned} |\nabla_x\phi|^{2(\beta-\alpha)} \leq C&\left\{\frac{k^2(1+\xi^{(p+1)})^2}{d(y)^{2\tau+\alpha+2}}+|\nabla_x\phi|^{-\alpha}\frac{k^{\beta+1}(1+\xi^{p+1})^{\beta+1}}{d(y)^{\tau(\beta+1)+\beta}}\right.\\ &\left.+|\nabla_x\phi|^{\beta-2\alpha}+|\nabla_x\phi|^{-\alpha}\frac{k(1+\xi^p)}{d(y)^\tau}\right\}, \end{aligned}$$

for some constant $C>0$ depending now also on $\|(f-\lambda|u|^\alpha u)^+\|_\infty$ and on the Lipschitz constant of f.

By inequality (2.5) it then follows

$$
\begin{aligned}
|\nabla_x\phi|^{2(\beta-\alpha)} &\le C\left\{\frac{|\nabla_x\phi|^2}{d(y)^{\alpha+2}}+\frac{|\nabla_x\phi|^{\beta-\alpha+1}}{d(y)^\beta}+|\nabla_x\phi|^{\beta-2\alpha}+|\nabla_x\phi|^{1-\alpha}\right\}\\
&\le C\left\{\frac{|\nabla_x\phi|^{2+\frac{\alpha+2}{\tau}}}{k^{\frac{\alpha+2}{\tau}}}+\frac{|\nabla_x\phi|^{\beta-\alpha+1+\frac{\beta}{\tau}}}{k^{\frac{\beta}{\tau}}}+|\nabla_x\phi|^{\beta-2\alpha}\right\}.
\end{aligned}
$$

Recalling that $\alpha>0$, $\beta>\alpha+1$ and $\tau=\frac{\beta}{\beta-\alpha-1}$, we see that $2(\beta-\alpha)=\beta-\alpha+1+\frac{\beta}{\tau}>2+\frac{\alpha+2}{\tau}$. Hence, from the last inequality and from Young's inequality, we obtain that

$$|\nabla_x\phi|\le C$$

which gives a contradiction to (2.5) for k large enough.

The case $\boldsymbol{\alpha<0}$. As proved in [9], if $\alpha<0$ a sub-solution u and super-solution v of equation (2.1) satisfy respectively in the viscosity sense

$$
\begin{cases}
-F(D^2u)+|\nabla u|^{\beta-\alpha}+\lambda|\nabla u|^{-\alpha}|u|^\alpha u\le|\nabla u|^{-\alpha}f\\
-F(D^2v)+|\nabla v|^{\beta-\alpha}+\lambda|\nabla v|^{-\alpha}|v|^\alpha v\ge|\nabla v|^{-\alpha}f.
\end{cases}
$$

From (2.2) in this case it then follows that

$$
\begin{cases}
-F(X_\epsilon)+|\nabla_x\phi|^{\beta-\alpha}+\lambda|\nabla_x\phi|^{-\alpha}|u|^\alpha u(x)\le|\nabla_x\phi|^{-\alpha}f(x)\\
-F(-Y_\epsilon)+|\nabla_y\phi|^{\beta-\alpha}+\lambda|\nabla_y\phi|^{-\alpha}|v|^\alpha v(y)\ge|\nabla_y\phi|^{-\alpha}f(y)
\end{cases}
$$

and, arguing as in the previous case, we now obtain for any $t>0$

$$
\begin{aligned}
t\,|\nabla_x\phi|^{\beta-\alpha}\le{}&\mathcal{M}^+(Z_{0,t})+|\nabla_y\phi|^{\beta-\alpha}-|\nabla_x\phi|^{\beta-\alpha}+t\,|\nabla_x\phi|^{-\alpha}(f-\lambda|u|^\alpha u)\\
&-(f-\lambda|u|^\alpha u)\left(|\nabla_y\phi|^{-\alpha}-|\nabla_x\phi|^{-\alpha}\right)+|\nabla_y\phi|^{-\alpha}(f(x)-f(y)),
\end{aligned}
$$

where $Z_{0,t}$ is defined by (2.3) (with $\alpha=0$).

By applying inequalities (2.11) (with $\alpha=0$), (2.5), (2.6), (2.10), in the present case, taking into account that $\beta-\alpha>1$ and $0<-\alpha<1$, we deduce that

$$|\nabla_x\phi|^{2(\beta-\alpha)}\le C\left\{\frac{|\nabla_x\phi|^2}{d(y)^2}+\frac{|\nabla_x\phi|^{\beta-\alpha+1}}{d(y)^{\beta-\alpha}}+|\nabla_x\phi|^{\beta-2\alpha}\right\},$$

for some constant $C>0$ depending on $p,\alpha,\beta,a,A,N,\|f-\lambda|u|^\alpha u\|_\infty$ and on the Lipschitz constant of f. Since now $\tau=\frac{\beta-\alpha}{\beta-\alpha-1}$, we reach a contradiction for k sufficiently large as before. □

As in [13], the above proposition and a scaling argument for solutions of equation (2.1) give the following result.

Theorem 2.4. *Let F satisfy (1.2) and, for $\lambda\ge0,\alpha>-1$ and $\beta>\alpha+1$, let u be a continuous solution of equation (2.1) in $\Omega\subset\mathbb{R}^N$, with f Lipschitz continuous in Ω. Then, u is locally Lipschitz continuous in Ω and* *there exists a positive constant M, depending only on $\alpha, \beta, a, A, N, \|f-\lambda|u|^\alpha u\|_\infty$ and on the Lipschitz constant of f, such that at any differentiability point $x \in \Omega$ one has*

$$|\nabla u(x)| \leq \frac{M}{\mathrm{dist}_{\partial\Omega}(x)^{\frac{1}{\beta-\alpha-1}}}.$$

3. Proof of Theorem 1.1

By using the Lipschitz estimates obtained in the previous section, we can now prove Theorem 1.1.

Proof of Theorem 1.1. Let u_λ be a solution of (1.4). We begin by giving a bound that will be useful in the whole proof. Observe that u_λ^+ is a sub-solution of

$$-|\nabla u_\lambda^+|^\alpha F(D^2 u_\lambda^+) \leq |f|_\infty;$$

from known estimates, see [7], this implies that

$$|u_\lambda^+|_\infty \leq c_1 |f|_\infty^{\frac{1}{1+\alpha}}. \tag{3.1}$$

Let us consider first the case when there exists a sub-solution φ of (1.3). Then, $\varphi - |\varphi|_\infty$ is a sub-solution of equation (1.4), and by the comparison principle we deduce $u_\lambda \geq \varphi - |\varphi|_\infty$. Thus, in this case (u_λ) is uniformly bounded in Ω. The Lipschitz estimates in Theorem 2.1 then yield that u_λ is uniformly converging up to a sequence to a Lipschitz solution of problem (1.3).

We now treat the second case, *i.e.* we suppose that (1.3) has no solutions. In particular $|u_\lambda|_\infty$ diverges, since otherwise we could extract from (u_λ) a subsequence converging to a solution of (1.3).

On the other hand, since $-\left(\frac{|f|_\infty}{\lambda}\right)^{\frac{1}{1+\alpha}}$ is a sub-solution of (1.4), by the comparison principle we obtain $u_\lambda^- \leq \left(\frac{|f|_\infty}{\lambda}\right)^{\frac{1}{1+\alpha}}$, which, jointly with (3.1), yields $\lambda |u_\lambda|_\infty^{1+\alpha} \leq c_1 |f|_\infty$. Hence, there exists $(x_\lambda) \subset \Omega$ such that $u_\lambda(x_\lambda) = -|u_\lambda|_\infty \to -\infty$ and there exists a constant $c_\Omega \geq 0$ such that, up to a subsequence, $\lambda |u_\lambda|_\infty^{1+\alpha} \to c_\Omega$.

We will show, as in [21] (see also [22] and [24]), that $v_\lambda = u_\lambda + |u_\lambda|_\infty = u_\lambda - u_\lambda(x_\lambda)$ converges up to a subsequence to a function v such that the pair (c_Ω, v) solves (1.5).

Clearly, v_λ satisfies in Ω

$$-|\nabla v_\lambda|^\alpha F(D^2 v_\lambda) + |\nabla v_\lambda|^\beta + \lambda (v_\lambda)^{1+\alpha} = f + \lambda (v_\lambda^{\alpha+1} - |u_\lambda|^\alpha u_\lambda) \geq f\,.$$

Next, we set

$$\gamma = \frac{2+\alpha-\beta}{\beta-1-\alpha}\,,$$

and for $s, \delta_0 > 0$ to be chosen sufficiently small, let us consider the function

$$\begin{aligned} \phi(x) &= \frac{\sigma}{(d(x)+s)^\gamma} - \frac{\sigma}{(\delta_0+s)^\gamma} \qquad \text{if } \gamma > 0\,, \\ \phi(x) &= -\sigma\,\log(d(x)+s) + \sigma\,\log(\delta_0+s) \qquad \text{if } \gamma = 0\,, \end{aligned} \tag{3.2}$$

where $\sigma = \left((\gamma+1)\frac{a}{2}\right)^{\frac{1}{\beta-\alpha-1}}\gamma^{-1}$ if $\gamma>0$, $\sigma=\frac{a}{2}$ if $\gamma=0$. A direct computation shows that, for $d(x)\le\delta_0$ with δ_0 small enough, in the case $\gamma>0$ one has

$$-|\nabla\phi|^\alpha\mathcal{M}^-(D^2\phi)+|\nabla\phi|^\beta+\lambda\phi^{1+\alpha}\le-\frac{a(\sigma\gamma)^{\alpha+1}}{2(d+s)^{(\gamma+1)\beta}}+\frac{A(\sigma\gamma)^{\alpha+1}|D^2d|_\infty}{(d+s)^{(\gamma+1)(\alpha+1)}}+\frac{\lambda\sigma^{\alpha+1}}{(d+s)^{\gamma(\alpha+1)}},$$

and, in the case $\gamma=0$,

$$-|\nabla\phi|^\alpha\mathcal{M}^-(D^2\phi)+|\nabla\phi|^\beta+\lambda\phi^{1+\alpha}\le-\frac{\sigma^{\alpha+2}}{(d+s)^{\alpha+2}}+\frac{A\sigma^{\alpha+1}|D^2d|_\infty}{(d+s)^{(\alpha+1)}}+\lambda\left(-\sigma\log(d+s)\right)^{\alpha+1}.$$

In both cases, by the ellipticity of F and for δ_0 and s sufficiently small, we obtain

$$-|\nabla\phi|^\alpha F(D^2\phi)+|\nabla\phi|^\beta+\lambda\phi^{1+\alpha}\le-|f|_\infty\le f(x)\quad\text{ in }\Omega\setminus\overline{\Omega}_{\delta_0}.$$

Moreover, one has $\phi=0\le v_\lambda$ on $\partial\Omega_{\delta_0}$ and $\phi\le|u_\lambda|_\infty=v_\lambda$ on $\partial\Omega$ for λ sufficiently small in dependence of s. The comparison principle then yields

$$v_\lambda\ge\phi>0\quad\text{ in }\Omega\setminus\overline{\Omega}_{\delta_0}. \tag{3.3}$$

Since $v_\lambda(x_\lambda)=0$, from (3.3) we deduce that $(x_\lambda)\subset\overline{\Omega}_{\delta_0}$. The interior Lipschitz estimate of Theorem 2.4 then yields that $v_\lambda=u_\lambda-u_\lambda(x_\lambda)$ is locally uniformly bounded and locally uniformly Lipschitz continuous. This proves both statement 2a of the theorem and that (v_λ) is locally uniformly converging up to a subsequence to a Lipschitz continuous function $v_o\ge0$ in Ω. Moreover, since also (x_λ) converges up to a subsequence to some point $x_o\in\overline{\Omega}_{\delta_0}$, we obtain $v_o(x_o)=0$. We observe further that, locally uniformly in Ω, one has

$$\lim_{\lambda\to0}\lambda|u_\lambda|^\alpha u_\lambda=\lim_{\lambda\to0}\lambda\,|u_\lambda|_\infty^{\alpha+1}\frac{|v_\lambda-|u_\lambda|_\infty|^\alpha(v_\lambda-|u_\lambda|_\infty)}{|u_\lambda|_\infty^{\alpha+1}}=-c_\Omega.$$

This yields statement 2b and, letting $\lambda\to0$ in the equation satisfied by v_λ, also that v_o is a viscosity solution of

$$-|\nabla v_o|^\alpha F(D^2v_o)+|\nabla v_o|^\beta=f+c_\Omega.$$

Finally, letting $\lambda\to0$ in inequality (3.3), we obtain $v_o\ge\phi$ in $\Omega\setminus\overline{\Omega}_{\delta_0}$, which in turn implies, by letting $s\to0$ and $x\to\partial\Omega$, that $v_o(x)\to+\infty$ as $d(x)\to0$. This completely proves statement 2c and concludes the proof of the theorem.

□

4. Explosive solutions

In this section, we prove the existence of solutions of equation (2.1) blowing up at the boundary, which will be used in the proof of existence of ergodic pairs. In what follows we drop the assumption on the boundedness of the right hand side f, and we consider continuous functions in Ω, possibly unbounded as $d(x)\to0$.

Let us introduce the nonnegative exponent

$$\gamma=\frac{2+\alpha-\beta}{\beta-1-\alpha}, \tag{4.1}$$

which plays a crucial role in the next results.

Let us start with a first existence result which follows by the same arguments used in [21] for the linear case, but requires some additional technical care in the construction of explosive sub- and super-solutions, due to the possible singularity of the involved operator.

Theorem 4.1. *Let $\beta \in (\alpha+1, \alpha+2]$, $\lambda > 0$ and let F satisfy (1.2). Let further $f \in \mathcal{C}(\Omega)$ be bounded from below and such that*

$$\lim_{d(x)\to 0} f(x) d(x)^{\frac{\beta}{\beta-1-\alpha}} = 0\,. \tag{4.2}$$

Then, the infinite boundary value problem

$$\begin{cases} -|\nabla u|^\alpha F(D^2 u) + |\nabla u|^\beta + \lambda |u|^\alpha u = f & \text{in } \Omega\,, \\ u = +\infty & \text{on } \partial\Omega\,, \end{cases} \tag{4.3}$$

admits solutions, and any solution u satisfies, for all $x \in \Omega$,

$$\begin{aligned} &\frac{c_0}{d(x)^\gamma} - \frac{D_1}{\lambda^{\frac{1}{\alpha+1}}} \le u(x) \le \frac{C_0}{d(x)^\gamma} + \frac{D_1}{\lambda^{\frac{1}{\alpha+1}}} && \text{if } \gamma > 0\,, \\ &c_0 |\log d(x)| - \frac{D_1}{\lambda^{\frac{1}{\alpha+1}}} \le u(x) \le C_0 |\log d(x)| + \frac{D_1}{\lambda^{\frac{1}{\alpha+1}}} && \text{if } \gamma = 0\,, \end{aligned} \tag{4.4}$$

for positive constants c_0, C_0 and D_1 depending only on $\alpha, \beta, a, A, |d|_{\mathcal{C}^2(\Omega)}$ and on f.

Proof. We give the proof in the case $\gamma > 0$, the reader can easily see the changes to be made when $\gamma = 0$.

We will get the conclusion by showing that equation (2.1) has a super-solution $\overline{w}$ and a sub-solution $\underline{w}^s$, for any $s > 0$ sufficiently small, satisfying, for $D = D_1/\lambda^{\frac{1}{\alpha+1}}$,

$$\overline{w}(x) \le C_0\, d(x)^{-\gamma} + D\,,$$

$$\underline{w}^s(x) \ge c_0 (d(x) + s)^{-\gamma} - D\,,$$

for all $x \in \Omega$, with equalities holding in a neighborhood of $\partial\Omega$.

Assume for a while that this is proved. Then, for any $R > 0$, we can consider the solution u_R of

$$\begin{cases} -|\nabla u_R|^\alpha F(D^2 u_R) + |\nabla u_R|^\beta + \lambda |u_R|^\alpha u_R = f_R & \text{in } \Omega \\ u_R = R & \text{on } \partial\Omega\,, \end{cases}$$

with $f_R = \min\{f, R\}$. By Theorem 2.2, u_R is monotone increasing with respect to R and satisfies $\underline{w}^s \le u_R \le \bar{w}$, provided that $R > \max_{\partial\Omega} \underline{w}^s(x)$. Moreover, u_R is uniformly locally Lipschitz continuous by Theorem 2.1. Thus, u_R is locally uniformly convergent as $R \to +\infty$ to a solution $\underline{u}$ of (4.3) such that $\underline{w}^0 \le \underline{u} \le \bar{w}$. By definition, $\underline{u}$ is the so-called minimal explosive solution. The maximal explosive solution $\overline{u}$ is then obtained as the limit for $\delta \to 0$ of the minimal explosive solutions in Ω_δ. Thus, it follows that for any solution u of problem (4.3) one has

$$c_0 d(x)^{-\gamma} - D \le \underline{u} \le u \le \overline{u} \le C_0 d(x)^{-\gamma} + D\,.$$

Let us now proceed to the construction of $\overline{w}$ and $\underline{w}^s$.

Let $\delta > 0$ be so small that in the set $\Omega \setminus \Omega_{2\delta} = \{d(x) < 2\delta\}$ the function d satisfies $|\nabla d| = 1$. For $x \in \Omega \setminus \Omega_{2\delta}$, let us consider the function

$$\varphi(x) = C_0\, d(x)^{-\gamma}\,,$$

with $C_0 = \gamma^{-1}\left(2A(\gamma+1)\right)^{\frac{1}{\beta-\alpha-1}}$. By a standard computation and assumption (4.6) on f, it is easy to see that

$$-|\nabla\varphi|^{\alpha}\mathcal{M}^{+}(D^2\varphi)+|\nabla\varphi|^{\beta}\geq f$$

for δ small enough. Hence, for $D>0$ and $\varphi_1(x)=\varphi(x)+D$ we obtain

$$-|\nabla\varphi_1|^{\alpha}F(D^2\varphi_1)+|\nabla\varphi_1|^{\beta}+\lambda\varphi_1^{\alpha+1}\geq f \quad \text{in } \Omega\setminus\Omega_{2\delta}\,.$$

Next, for $x\in\{\delta\leq d(x)\leq 2\delta\}$, we consider the function

$$\varphi_2(x)=\frac{C_0}{\delta^{\gamma}}e^{\frac{1}{d(x)-2\delta}+\frac{1}{\delta}}+D\,,$$

which satisfies

$$|\nabla\varphi_2|^{\alpha}|F(D^2\varphi_2)|+|\nabla\varphi_2|^{\beta}\leq K_1 \quad \text{in } \overline{\Omega}_{\delta}\setminus\Omega_{2\delta}\,,$$

for a positive constant K_1 depending only on N, a, A, α and β. Thus, if D is chosen satisfying $D\geq\left(\frac{|f|_{L^{\infty}(\Omega_\delta)}+K_1}{\lambda}\right)^{\frac{1}{\alpha+1}}$, we obtain

$$-|\nabla\varphi_2|^{\alpha}F(D^2\varphi_2)+|\nabla\varphi_2|^{\beta}+\lambda\varphi_2^{\alpha+1}\geq f(x) \quad \text{in } \overline{\Omega}_{\delta}\setminus\Omega_{2\delta}\,.$$

We then conclude that the function

$$\overline{w}(x)=\begin{cases} C_0\,d(x)^{-\gamma}++D & \text{for } d<\delta\\ \delta^{-\gamma}C_0e^{\frac{1}{d(x)-2\delta}+\frac{1}{\delta}}+D & \text{for } \delta\leq d\leq 2\delta\\ D & \text{for } d>2\delta\end{cases}$$

is the required super solution in Ω. Indeed, in the set Ω_δ, $\overline{w}$ is of class $\mathcal{C}^2$ and it is a super solution by the properties of φ_2 and by the fact that locally constant functions satisfy $|\nabla u|^{\alpha}F(D^2u)=0$. On the other hand, $\overline{w}$ is a super solution in $\Omega\setminus\Omega_{2\delta}$ by the properties of φ_1 and φ_2 and the fact that $\varphi_2(x)<\varphi_1(x)$ for $d(x)>\delta$.

As far as the sub-solution is concerned, for $s>0$, $c_0=\gamma^{-1}\left(\frac{(\gamma+1)a}{2}\right)^{\frac{1}{\beta-\alpha-1}}$ and $x\in\Omega\setminus\Omega_{2\delta}$, let us consider the function

$$\varphi^s(x)=c_0(d(x)+s)^{-\gamma}\,.$$

Symmetric computations as above give that

$$-|\nabla\varphi^s|^{\alpha}F(D^2\varphi^s)+|\nabla\varphi^s|^{\beta}+\lambda|\varphi^s|^{\alpha}\varphi^s\leq-|\nabla\varphi^s|^{\alpha}\mathcal{M}^{-}(D^2\varphi^s)+|\nabla\varphi^s|^{\beta}+\lambda|\varphi^s|^{\alpha}\varphi^s\leq f(x)\,,$$

for δ and s sufficiently small, since f is bounded from below.

Moreover, for $D\geq c_0\delta^{-\gamma}+\left(\frac{|f^-|_\infty}{\lambda}\right)^{\frac{1}{\alpha+1}}$, the constant function $c_0(\delta+s)^{-\gamma}-D$ is also a sub-solution in Ω. Therefore, the function

$$\underline{w}^s(x)=\begin{cases}\varphi^s(x)-D & \text{in } \Omega\setminus\Omega_\delta\\ c_0(\delta+s)^{-\gamma}-D & \text{in } \Omega_\delta\end{cases}$$

is the wanted sub-solution.

□

In order to obtain refined boundary estimates for the explosive solutions, and to show uniqueness of solution in some cases, we need to assume the extra regularity condition (1.6) involving both the operator F and the domain Ω.

Under assumption (1.6), we denote by $C(x)$ a nonnegative function of class $\mathcal{C}^2$ in Ω satisfying in a neighborhood of $\partial\Omega$

$$\begin{aligned} &C(x) = \left((\gamma+1)F(\nabla d(x)\otimes\nabla d(x))\right)^{\frac{1}{\beta-\alpha-1}}\gamma^{-1} && \text{if} \quad \gamma>0,\\ &C(x) = F(\nabla d(x)\otimes\nabla d(x)) && \text{if} \quad \gamma=0, \end{aligned} \tag{4.5}$$

where $\gamma\geq 0$ is defined in (4.1).

Theorem 4.2. *Let $\beta\in(\alpha+1,\alpha+2]$, $\lambda>0$ and let F satisfy (1.2) and (1.6). Let further $f\in\mathcal{C}(\Omega)$ be bounded from below and such that*

$$\lim_{d(x)\to 0} f(x)d(x)^{\frac{\beta}{\beta-1-\alpha}-\gamma_0} = 0\,, \tag{4.6}$$

for some $\gamma_0\geq 0$. Then, any solution u of (4.3) satisfies: for any $\nu>0$ and for any $0\leq\gamma_1\leq\gamma_0$, with $\gamma_1<1$, and $\gamma_1<\gamma$ when $\gamma>0$, there exists $D=\frac{D_1}{\lambda^{1/(\alpha+1)}}$, with $D_1>0$ depending on $\nu,\gamma_1,\alpha,\beta,a,A,|d|_{\mathcal{C}^2(\Omega)},|C|_{\mathcal{C}^2(\Omega)}$ and on f, such that, for all $x\in\Omega$,

$$\begin{gathered} \frac{C(x)}{d(x)^\gamma}-\frac{\nu}{d(x)^{\gamma-\gamma_1}}-D\leq u(x)\leq\frac{C(x)}{d(x)^\gamma}+\frac{\nu}{d(x)^{\gamma-\gamma_1}}+D \quad \text{if } \gamma>0\,,\\ |\log d(x)|\left(C(x)-\nu d(x)^{\gamma_1}\right)-D\leq u(x)\leq|\log d(x)|\left(C(x)+\nu d(x)^{\gamma_1}\right)+D \quad \text{if } \gamma=0\,. \end{gathered} \tag{4.7}$$

Furthermore, the solution u is unique

- *for $\alpha\geq 0$ and any β,*
- *for $\alpha<0$ and any $\beta>\frac{1-\alpha-\alpha^2}{1-\alpha}$, provided that f satisfies (4.6) with $\gamma_0>-\alpha\,\gamma$.*

Proof. As in the previous proof, we detail only the case $\gamma>0$.

1. Refined boundary estimates.

By arguing as in the proof of Theorem 4.1, we get the conclusion by showing that, for every $\nu>0$ and for any $0\leq\gamma_1\leq\gamma_0$, with $\gamma_1<\min\{1,\gamma\}$, there exist $D=\frac{D_1}{\lambda^{1/\alpha+1}}>0$, a super-solution $\overline{w}$ and a sub-solution $\underline{w}^s$, for $s>0$ sufficiently small, satisfying

$$\begin{aligned} \underline{w}^s(x) &\geq C(x)(d+s)^{-\gamma}-\nu(d+s)^{-\gamma+\gamma_1}-D\,,\\ \overline{w}(x) &\leq C(x)d^{-\gamma}+\nu d^{-\gamma+\gamma_1}+D\,, \end{aligned}$$

for any $x\in\Omega$, with equalities holding in a neighborhood of $\partial\Omega$.

Let $\delta>0$ be so small that in the set $\Omega\setminus\Omega_{2\delta}=\{d(x)<2\delta\}$ the function d satisfies $|\nabla d|=1$ and C satisfies (4.5). For $x\in\Omega\setminus\Omega_{2\delta}$, let us consider the function

$$\varphi(x)=C(x)d(x)^{-\gamma}+\nu d(x)^{\gamma_1-\gamma}\,.$$

One has

$$\nabla\varphi(x) = -\gamma\, C(x) d^{-\gamma-1}\left(1+\nu\frac{(\gamma-\gamma_1)}{\gamma\, C(x)}d^{\gamma_1}\right)\nabla d + d^{-\gamma}\nabla C$$

and

$$\begin{aligned} D^2\varphi(x) \quad &= \gamma\,(\gamma+1)C(x)d^{-\gamma-2}\left(1+\nu\tfrac{(\gamma-\gamma_1)(\gamma-\gamma_1+1)}{\gamma\,(\gamma+1)C(x)}d^{\gamma_1}\right)\nabla d\otimes\nabla d\\ &-\gamma\, C(x)d^{-\gamma-1}\left(1+\nu\tfrac{(\gamma-\gamma_1)}{\gamma\, C(x)}d^{\gamma_1}\right)D^2 d\\ &-\gamma\, d^{-\gamma-1}\left(\nabla d\otimes\nabla C+\nabla C\otimes\nabla d\right)+d^{-\gamma}D^2C\,. \end{aligned}$$

By ellipticity of F and by definition of $C(x)$ it then follows

$$\begin{aligned} F(D^2\varphi) \quad &\le \tfrac{\gamma\,(\gamma+1)C(x)}{d^{\gamma+2}}\left(1+\nu\tfrac{(\gamma-\gamma_1)(\gamma-\gamma_1+1)}{\gamma\,(\gamma+1)C(x)}d^{\gamma_1}\right)F\left(\nabla d\otimes\nabla d\right)+\tfrac{K_1}{d^{\gamma+1}}\\ &= \tfrac{(\gamma\, C(x))^{\beta-\alpha}}{d^{\gamma+2}}\left(1+\nu\tfrac{(\gamma-\gamma_1)(\gamma-\gamma_1+1)}{\gamma\,(\gamma+1)C(x)}d^{\gamma_1}\right)+\tfrac{K_1}{d^{\gamma+1}}\,, \end{aligned}$$

for a constant $K_1>0$ depending on $\nu,\alpha,\beta,\gamma_1,a,A,|D^2d|_\infty$ and $|C|_{\mathcal{C}^2(\Omega)}$. In what follows we denote by K_i, $i=1,2\ldots$, different constants depending on these quantities.

Hence, we obtain

$$\begin{aligned} &-|\nabla\varphi|^\alpha F(D^2\varphi)+|\nabla\varphi|^\beta\\ &\ge |\nabla\varphi|^\alpha\left[-\tfrac{(\gamma\, C(x))^{\beta-\alpha}}{d^{\gamma+2}}\left(1+\nu\tfrac{(\gamma-\gamma_1)(\gamma-\gamma_1+1)}{\gamma\,(\gamma+1)C(x)}d^{\gamma_1}\right)-\tfrac{K_1}{d^{\gamma+1}}+|\nabla\varphi|^{\beta-\alpha}\right]\\ &\ge |\nabla\varphi|^\alpha\left[-\tfrac{(\gamma\, C(x))^{\beta-\alpha}}{d^{\gamma+2}}\left(1+\nu\tfrac{(\gamma-\gamma_1)(\gamma-\gamma_1+1)}{\gamma\,(\gamma+1)C(x)}d^{\gamma_1}\right)-\tfrac{K_1}{d^{\gamma+1}}\right.\\ &\qquad\left.+\tfrac{(\gamma\, C(x))^{\beta-\alpha}}{d^{(\gamma+1)(\beta-\alpha)}}\left(1+\nu\tfrac{(\gamma-\gamma_1)}{\gamma\, C(x)}d^{\gamma_1}\right)^{\beta-\alpha}-\tfrac{K_2}{d^{(\gamma+1)(\beta-\alpha)-1}}\right]\\ &\ge |\nabla\varphi|^\alpha\left[-\tfrac{(\gamma\, C(x))^{\beta-\alpha}}{d^{\gamma+2}}\left(1+\nu\tfrac{(\gamma-\gamma_1)(\gamma-\gamma_1+1)}{\gamma\,(\gamma+1)C(x)}d^{\gamma_1}\right)-\tfrac{K_1}{d^{\gamma+1}}\right.\\ &\qquad\left.+\tfrac{(\gamma\, C(x))^{\beta-\alpha}}{d^{(\gamma+1)(\beta-\alpha)}}\left(1+\nu\tfrac{(\beta-\alpha)(\gamma-\gamma_1)}{\gamma\, C(x)}d^{\gamma_1}\right)-\tfrac{K_2}{d^{(\gamma+1)(\beta-\alpha)-1}}\right]. \end{aligned}$$

Recalling that $\gamma=\frac{\alpha+2-\beta}{\beta-\alpha-1}$, we finally deduce

$$\begin{aligned} &-|\nabla\varphi|^\alpha F(D^2\varphi)+|\nabla\varphi|^\beta\\ &\ge |\nabla\varphi|^\alpha\left[\nu\tfrac{(\gamma-\gamma_1)(1+\gamma_1)}{(\gamma+1)}\tfrac{(\gamma\, C(x))^{\beta-\alpha-1}}{d^{\gamma+2-\gamma_1}}-\tfrac{K_3}{d^{\gamma+1}}\right]\\ &= \tfrac{|(\gamma\, C(x)+\nu(\gamma-\gamma_1)d^{\gamma_1})\nabla d-d\,\nabla C|^\alpha}{d^{\frac{\beta}{\beta-\alpha-1}-\gamma_1}}\left[\nu\tfrac{(\gamma-\gamma_1)(1+\gamma_1)}{(\gamma+1)}(\gamma\, C(x))^{\beta-\alpha-1}-K_3d^{1-\gamma_1}\right]. \end{aligned}$$

Since $\gamma_1\le\gamma_0$, by assumption (4.6) on f the last inequality implies that, for δ sufficiently small,

$$-|\nabla\varphi|^\alpha F(D^2\varphi)+|\nabla\varphi|^\beta\ge f(x)\qquad\text{in }\Omega\setminus\Omega_{2\delta}\,,$$

and therefore also that

$$-|\nabla\varphi|^\alpha F(D^2\varphi) + |\nabla\varphi|^\beta + \lambda\varphi^{\alpha+1} \ge f(x) \quad \text{in } \Omega\setminus\Omega_{2\delta}\,.$$

Clearly, the same inequality also holds for $\varphi_1(x)=\varphi(x)+D$, for any $D>0$. The function φ_1 can then be extended to the whole of Ω as in the proof of Theorem 4.1, and this construction yields the super-solution $\overline{w}$.

As far as the sub-solution is concerned, for $s>0$, $\nu>0$ and $x\in\Omega\setminus\Omega_{2\delta}$, let us consider the function

$$\varphi^s(x) = C(x)(d(x)+s)^{-\gamma} - \nu(d(x)+s)^{-\gamma+\gamma_1}\,.$$

Analogous computations as above give that

$$\begin{aligned}&-|\nabla\varphi^s|^\alpha F(D^2\varphi^s) + |\nabla\varphi^s|^\beta + \lambda|\varphi^s|^\alpha\varphi^s\\ &\le |\nabla\varphi^s|^\alpha\left[-\tfrac{(\gamma\,C(x))^{\beta-\alpha-1}}{(d+s)^{\gamma+2-\gamma_1}}\nu\tfrac{(\gamma-\gamma_1)(1+\gamma_1)}{\gamma+1} + \tfrac{K_5}{(d+s)^{\gamma+1}}\right] + \lambda\tfrac{A^{\alpha+1}}{(d+s)^{\gamma\,(\alpha+1)}} \le f(x)\,,\end{aligned}$$

for δ and s sufficiently small, since f is bounded from below.

The function $\varphi^s(x)-D$, for suitable large $D>0$, can then be constantly extended in Ω in order to give the wanted sub-solution $\underline{w}^s$.

2. Uniqueness.

We prove that $\underline{u}=\overline{u}$.

Remark that, by estimates (4.7), for any $\theta<1$ and for any $c\in\mathbb{R}$, there exists δ such that $\theta\overline{u}(x)-c\le\underline{u}(x)$ for $d(x)\le\delta$.

The case $\alpha\ge 0$. Observe that, for all $t\in\mathbb{R}$ and $c>0$, one has

$$|t-c|^\alpha(t-c) - |t|^\alpha t \le -2^{-\alpha}c^{\alpha+1}\,.$$

From this, we deduce that

$$-|\nabla(\theta\overline{u}-c)|^\alpha F\left(D^2(\theta\overline{u}-c)\right) + |\nabla(\theta\overline{u}-c)|^\beta + \lambda|\theta\overline{u}-c|^\alpha(\theta\overline{u}-c) \le \theta^{\alpha+1}f(x) - \lambda 2^{-\alpha}c^{\alpha+1}\,,$$

and the choice

$$c = c_\theta = \left(\frac{2^\alpha(1-\theta^{\alpha+1})|f^-|_\infty}{\lambda}\right)^{\frac{1}{\alpha+1}}$$

then yields

$$\begin{aligned}-|\nabla(\theta\overline{u}-c)|^\alpha F\left(D^2(\theta\overline{u}-c)\right) + |\nabla(\theta\overline{u}-c)|^\beta + \lambda|\theta\overline{u}-c|^\alpha(\theta\overline{u}-c)&\\ \le \theta^{\alpha+1}f(x) - (1-\theta^{\alpha+1})\|f^-\|_\infty \le f(x)\,.&\end{aligned}$$

By applying Theorem 2.2, it then follows that $\theta\overline{u}-c_\theta\le\underline{u}$ in Ω, and letting $\theta\to 1$ we obtain the uniqueness of the explosive solution in the case $\alpha\ge 0$.

The case $\alpha<0$. In this case we use the inequality

$$|t-c|^\alpha(t-c) - |t|^\alpha t \le -2^\alpha(\alpha+1)K^\alpha c\,,$$

which holds true for all $0<c<K$ and $t\in\mathbb{R}$ such that $|t|\le K$.

Let $C_1 = \sup_\Omega |\overline{u}(x)|d(x)^\gamma$, which is finite by (4.7). Then, for any $\delta > 0$, one has $|\overline{u}| < \frac{C_1}{\delta^\gamma}$ in Ω_δ. Therefore, for any $0 < \theta < 1$ and $0 < c < \frac{C_1}{\delta^\gamma}$, and for $x \in \Omega_\delta$, we have

$$-|\nabla(\theta\overline{u} - c)|^\alpha F\left(D^2(\theta\overline{u} - c)\right) + |\nabla(\theta\overline{u} - c)|^\beta + \lambda|\theta\overline{u} - c|^\alpha(\theta\overline{u} - c)$$
$$\leq \theta^{\alpha+1} f(x) - \lambda(2C_1)^\alpha(\alpha + 1)\delta^{-\alpha\gamma} c\,.$$

We choose, as before,

$$c = c_{\theta,\delta} = \frac{|f^-|_\infty(1 - \theta^{\alpha+1})}{\lambda(2C_1)^\alpha(\alpha + 1)\delta^{-\alpha\gamma}}\,,$$

which is admissible for δ sufficiently small, since $\alpha > -1$. This yields

$$-|\nabla(\theta\overline{u} - c_{\theta,\delta})|^\alpha F\left(D^2(\theta\overline{u} - c_{\theta,\delta})\right) + |\nabla(\theta\overline{u} - c_{\theta,\delta})|^\beta + \lambda|\theta\overline{u} - c_{\theta,\delta}|^\alpha(\theta\overline{u} - c_{\theta,\delta}) \leq f \quad \text{ in } \Omega_\delta.$$

On the other hand, by estimates (4.7) with $\nu = 1$, we have

$$\theta\overline{u} - c_{\theta,\delta} \leq \underline{u} \quad \text{ on } \partial\Omega_\delta$$

provided that

$$\delta = \delta_\theta = \left(\frac{a(1 - \theta)}{2(1 + D)}\right)^{\frac{1}{\gamma_1}}\,.$$

With this choice of δ, we then deduce from Theorem 2.2 that $\theta\overline{u} - c_{\theta,\delta_\theta} \leq \underline{u}$ in Ω_{δ_θ}. Finally, we let $\theta \to 1$. We observe that, by the restrictions assumed on β and f in the case $\alpha < 0$, we can choose γ_1 satisfying $\gamma_1 > -\alpha\gamma$. Therefore, $c_{\theta,\delta_\theta} \to 0$ as $\theta \to 1$, and we conclude that $\overline{u} \leq \underline{u}$ in Ω.

□

Remark 4.3. Let us put in evidence that estimates (4.7) imply that any solution u of (4.3) satisfies

$$\lim_{d(x)\to 0} \frac{u(x)\, d(x)^\gamma}{C(x)} = 1 \text{ if } \gamma > 0, \qquad \lim_{d(x)\to 0} \frac{u(x)}{|\log d(x)|\, C(x)} = 1 \text{ if } \gamma = 0\,.$$

Moreover, if f satisfies (4.6) with $\gamma_0 = 0$, then necessarily $\gamma_1 = 0$ and (4.7) reduce to

$$(C(x) - \nu)d(x)^{-\gamma} - D \leq u(x) \leq (C(x) + \nu)d(x)^{-\gamma} + D \text{ if } \gamma > 0\,,$$
$$(C(x) - \nu)\,|\log d(x)| - D \leq u(x) \leq (C(x) + \nu)\,|\log d(x)| + D \;\text{ if } \gamma = 0\,,$$

for any $\nu > 0$, with $D > 0$ depending in particular on ν and λ. The above estimates are the classical ones for explosive solutions firstly obtained in the semilinear case in [21], where $C(x)$ is a constant function. Also the case $\gamma_0 = 1$ has been considered in [21], and in this case more refined estimates have been obtained. In the nonlinear case, analogous estimates for $\gamma_0 \geq 1$ would require further regularity assumptions on the nonconstant function $C(x)$. Estimates (4.7) are interesting in the intermediate cases $0 \leq \gamma_0 < 1$, in which they are also new for linear operators and yield a uniqueness result in the nonlinear singular case $\alpha < 0$.

5. A comparison principle for nonlinear degenerate/singular equations without zero order terms

This section is devoted to some comparison principle for fully nonlinear equations without zero order terms. For analogous results concerning nonsingular operators, see [3, 4].

Theorem 5.1. *Let b be a continuous and bounded function in Ω and let f be a bounded continuous function satisfying , when $\alpha \neq 0$, $\inf_\Omega |f| > 0$. Let u and v be respectively sub- and super-solution of*

$$- |\nabla u|^\alpha F(D^2 u) + b(x)|\nabla u|^\beta = f \quad \textit{in } \Omega. \tag{5.1}$$

If u and v are bounded and at least one of the two is Lipschitz continuous then the comparison principle holds i.e.

$$u \leq v \quad on \quad \partial\Omega \Rightarrow u \leq v \quad in \quad \Omega.$$

Proof. Without loss of generality, we will suppose that u is Lipschitz continuous. Moreover, by replacing u and v with $-u$ and $-v$ if necessary, we assume that $f \leq -m < 0$ in Ω.

The case $\alpha = 0$ is quite standard, it is enough to construct strict sub-solutions that converge uniformly to u. We choose $\epsilon > 0$ such that $\frac{a}{\epsilon} > 2B\beta(|\nabla u|_\infty + 1)^{\beta-1}$, where $B = |b|_\infty$. Let $u_\epsilon = u + \epsilon e^{-\frac{x_1}{\epsilon}} - \epsilon$, with *e.g.* $\Omega \subset \{x_1 > 0\}$. Then, u_ϵ is a strict sub-solution of (5.1), being

$$\begin{aligned} F(D^2 u_\epsilon) \quad &\geq F(D^2 u) + \tfrac{a}{\epsilon} e^{-\frac{x_1}{\epsilon}} \\ &\geq b(x)|\nabla u_\epsilon|^\beta - f + b(x)\left(|\nabla u|^\beta - |\nabla u_\epsilon|^\beta\right) + \tfrac{a}{\epsilon} e^{-\frac{x_1}{\epsilon}} \\ &\geq b(x)|\nabla u_\epsilon|^\beta - f + \tfrac{a}{2\epsilon} e^{-\frac{x_1}{\epsilon}}. \end{aligned}$$

Furthermore $u_\epsilon \leq u \leq v$ on $\partial\Omega$. By Theorem 2.2, we obtain that $u_\epsilon \leq v$ in Ω. To conclude, let $\epsilon \to 0$. This computation has been done for a classical solution u, but, with obvious changes, it can be made rigorous for viscosity solutions.

For the case $\alpha \neq 0$ and $f < 0$, we use the change of function $u = \varphi(z)$, $v = \varphi(w)$ with

$$\varphi(s) = -\gamma(\alpha + 1) \log\left(\delta + e^{-\frac{s}{\alpha+1}}\right).$$

This function is used in [3–5, 22, 23].

We choose δ small enough in order that the range of φ covers the ranges of u and v. The constant γ will be chosen small enough depending only on a, α, β, $\inf_\Omega(-f)$ and $|b|_\infty$; in this proof, any constant of this type will be called universal. Observe that $\varphi' > 0$ while $\varphi'' < 0$.

In the viscosity sense, z and w are respectively sub- and super-solution of

$$- |\nabla z|^\alpha F(D^2 z + \frac{\varphi''(z)}{\varphi'(z)} \nabla z \otimes \nabla z) + b(x)\varphi'(z)^{\beta-\alpha-1}|\nabla z|^\beta + \frac{-f}{(\varphi'(z))^{\alpha+1}} = 0. \tag{5.2}$$

We define

$$H(x, s, p) = \frac{-a\varphi''(s)}{\varphi'(s)}|p|^{2+\alpha} + b(x)\varphi'(s)^{\beta-\alpha-1}|p|^\beta + \frac{-f(x)}{\varphi'(s)^{\alpha+1}}.$$

The point is to prove that at $\bar{x}$, a maximum point of $z-w$, $\frac{\partial H(\bar{x},s,p)}{\partial s}>0$ for all p. This will be sufficient to get a contradiction. A simple computation gives

$$\varphi'=\frac{\gamma e^{-\frac{s}{\alpha+1}}}{\delta+e^{-\frac{s}{\alpha+1}}},\ \varphi''=\frac{-\gamma\delta e^{-\frac{s}{\alpha+1}}}{(\alpha+1)(\delta+e^{-\frac{s}{\alpha+1}})^2}.$$

Hence

$$\left(\frac{-\varphi''}{\varphi'}\right)'=\frac{\delta}{(\alpha+1)^2}\frac{e^{-\frac{s}{\alpha+1}}}{(\delta+e^{-\frac{s}{\alpha+1}})^2}\ \ i.e.\ \left(\frac{-\varphi''}{\varphi'}\right)'=-\frac{\varphi''}{(\alpha+1)\gamma}>0.$$

Differentiating H with respect to s gives:

$$\partial_s H=a|p|^{\alpha+2}\frac{-\varphi''}{(\alpha+1)\gamma}+(-f)\frac{-(\alpha+1)\varphi''}{(\varphi')^{\alpha+2}}+b(x)|p|^\beta(\beta-\alpha-1)(\varphi')^{\beta-\alpha-2}\varphi''.$$

Since $-\varphi''$ is positive, we need to prove that

$$K:=\frac{a|p|^{\alpha+2}}{(\alpha+1)\gamma}+(-f)\frac{\alpha+1}{(\varphi')^{\alpha+2}}-|b|_\infty|p|^\beta(\beta-\alpha-1)(\varphi')^{\beta-\alpha-2}>0.$$

We start by treating the case $\beta<\alpha+2$.

Observe first that the boundedness of u and v implies that there exists universal positive constants c_o and c_1 such that

$$c_o\gamma\le\varphi'\le c_1\gamma.$$

Hence, it is easy to see that there exist three positive universal constants C_i, $i=1,2,3$ such that

$$K>\frac{C_1|p|^{\alpha+2}}{\gamma}+\frac{C_2}{\gamma^{\alpha+2}}-\frac{C_3|p|^\beta}{\gamma^{\alpha+2-\beta}}.$$

We choose $\gamma=\min\left\{1,(\frac{C_3}{C_2})^\beta,(\frac{C_3}{C_1})^{\frac{1}{\alpha+1-\beta}}\right\}$. With this choice of γ, for $|p|\le 1$,

$$\frac{C_1|p|^{\alpha+2}}{\gamma}+\frac{C_2}{\gamma^{\alpha+2}}-\frac{C_3|p|^\beta}{\gamma^{\alpha+2-\beta}}\ge\frac{C_2}{\gamma^{\alpha+2}}-\frac{C_3}{\gamma^{\alpha+2-\beta}}>0;$$

while for $|p|\ge 1$,

$$\frac{C_1|p|^{\alpha+2}}{\gamma}+\frac{C_2}{\gamma^{\alpha+2}}-\frac{C_3|p|^\beta}{\gamma^{\alpha+2-\beta}}\ge\frac{(C_1)|p|^{\alpha+2}}{\gamma}-\frac{C_3|p|^\beta}{\gamma^{\alpha+2-\beta}}>0.$$

If $\beta=\alpha+2$, just take $\gamma<\frac{a}{(\alpha+1)|b|_\infty}$.

This gives that for γ small enough depending only on $\min(-f)$, α, $|b|_\infty$ and β one has, for some universal constant C,

$$\partial_s H(x,s,p)\ge C>0. \tag{5.3}$$

We now conclude the proof of the comparison principle.

We will distinguish the case $\alpha > 0$ and $\alpha < 0$. In the first case we introduce $\psi_j(x,y) = z(x) - w(y) - \frac{j}{2}|x-y|^2$ while in the second case we use $\psi_j(x,y) = z(x) - w(y) - \frac{j}{q}|x-y|^q$ where $q > \frac{\alpha+2}{\alpha+1}$. We detail the case $\alpha > 0$.

Suppose by contradiction that $u > v$ somewhere in Ω, then $z > w$ somewhere, since φ is increasing, while on the boundary $z \le w$. Then the supremum of $z - w$ is positive and it is achieved inside Ω. Hence, ψ_j reaches a positive maximum in $(x_j, y_j) \in \Omega \times \Omega$.

By Ishii's lemma [20], there exists $(X_j, Y_j) \in S \times S$ such that

$$(p_j, X_j) \in \overline{J}^{2,+} z(x_j),\ (p_j, -Y_j) \in \overline{J}^{2,-} w(y_j), \quad \text{with } p_j = j(x_j - y_j)$$

and

$$\begin{pmatrix} X_j & 0 \\ 0 & Y_j \end{pmatrix} \le j \begin{pmatrix} I & -I \\ -I & I \end{pmatrix}.$$

On (x_j, y_j), by a continuity argument, for j large enough one has

$$z(x_j) > w(y_j) + \frac{\sup(z-w)}{2}.$$

Note for later purposes that since z or w are Lipschitz, $p_j = j(x_j - y_j)$ is bounded. Observe that, the monotonicity of $\frac{\varphi''}{\varphi'}$ implies that

$$N = p_j \otimes p_j \left(\frac{\varphi''(z(x_j))}{\varphi'(z(x_j))} - \frac{\varphi''(w(y_j))}{\varphi'(w(y_j))} \right) \le 0.$$

Using the fact that z and w are respectively sub- and super-solutions of the equation (5.2), the estimate (5.3) and that H is decreasing in the second variable, one obtains:

$$\begin{aligned}
0 &\ge \frac{-f(x_j)}{(\varphi')^{\alpha+1}(z(x_j))} - |p_j|^\alpha F(X_j + \frac{\varphi''(z(x_j))}{\varphi'(z(x_j))} p_j \otimes p_j) + b(x_j)|p_j|^\beta \varphi'(z(x_j))^{\beta-\alpha-1} \\
&\ge \frac{-f(x_j)}{(\varphi')^{\alpha+1}(z(x_j))} - |p_j|^\alpha F(-Y_j + \frac{\varphi''(w(y_j))}{\varphi'(w(y_j))} p_j \otimes p_j) \\
&\quad + a|p_j|^{2+\alpha} \left(\frac{\varphi''(w(y_j))}{\varphi'(w(y_j))} - \frac{\varphi''(z(x_j))}{\varphi'(z(x_j))} \right) + |p_j|^\beta b(x_j)(\varphi'(z(x_j))^{\beta-\alpha-1} \\
&\ge \frac{f(y_j) - f(x_j)}{(\varphi'(w(y_j))^{\alpha+1}} + (b(x_j) - b(y_j))|p_j|^\beta (\varphi'(w(y_j)))^{\beta-\alpha-1} \\
&\quad + H(x_j, z(x_j), p_j) - H(x_j, w(y_j), p_j) \\
&\ge C(z(x_j) - w(y_j)) + \frac{o(1)}{\gamma^{\alpha+1}}.
\end{aligned}$$

Here, we have used the continuity of f and b, the boundedness of p_j and that

$$\psi(x_j, y_j) \ge \sup(\psi(x_j, x_j), \psi(y_j, y_j)).$$

Passing to the limit one gets a contradiction, since (x_j, y_j) converges to $(\bar{x}, \bar{x})$ such that $z(\bar{x}) > w(\bar{z})$. This ends the case $\alpha \ge 0$.

In the case $\alpha < 0$, the proof is similar but we need to make sure that one can choose $x_j \neq y_j$. This can be done proceeding as in [10]. □

6. Ergodic pairs

In this section we consider, for $c \in \mathbb{R}$, the equation

$$- |\nabla u|^\alpha F(D^2u) + |\nabla u|^\beta = f + c \text{ in } \Omega. \tag{6.1}$$

Definition 6.1. Suppose that c is some constant (depending on f, Ω, β, α and F) such that there exists $\varphi \in \mathcal{C}(\Omega)$, solution of (6.1), such that $\varphi \to +\infty$ at $\partial\Omega$. We will say that c is an ergodic constant, φ is an ergodic function and (c, φ) is an ergodic pair.

We suppose, as usual, that $\alpha > -1$, $\beta \in (\alpha+1, \alpha+2]$ and recall that $\gamma = \frac{2+\alpha-\beta}{\beta-\alpha-1}$ and $C(x)$ satisfies (4.5). In the following subsections, we prove the existence and show several properties of ergodic pairs.

6.1. Existence of ergodic constants and boundary behavior of ergodic functions

Theorem 1.1 provides the existence of a nonnegative ergodic constant under the assumption that problem (1.3) does not have a solution. In the next result, we obtain the existence of ergodic constants using approximating explosive solutions.

Theorem 6.2. *Let F and f be as in Theorem 4.1, and assume further that f is locally Lipschitz continuous in Ω. Then, there exists an ergodic constant $c \in \mathbb{R}$.*

Proof. By Theorem 4.1, for $\lambda > 0$ there exists a solution U_λ of problem (4.3), which satisfies estimates (4.4). It then follows that $\lambda|U_\lambda|^\alpha U_\lambda$ is locally bounded in Ω, uniformly with respect to $0 < \lambda < 1$. Let us fix an arbitrary point $x_0 \in \Omega$. Then, there exists $c \in \mathbb{R}$ such that, up to a sequence $\lambda_n \to 0$,

$$\lambda|U_\lambda(x_0)|^\alpha U_\lambda(x_0) \to -c\,.$$

On the other hand, Theorem 2.4 yields that U_λ is locally uniformly Lipschitz continuous. Therefore, for x in a compact subset of Ω, one has

$$\lambda\,||U_\lambda(x)|^\alpha U_\lambda(x) - |U_\lambda(x_0)|^\alpha U_\lambda(x_0)| \leq \lambda|U_\lambda(x) - U_\lambda(x_0)|^{\alpha+1} \to 0 \quad \text{ if } \alpha \leq 0\,,$$

as well as, using again estimates (4.4),

$$\lambda\,||U_\lambda(x)|^\alpha U_\lambda(x) - |U_\lambda(x_0)|^\alpha U_\lambda(x_0)| \leq \lambda\frac{K}{\lambda^{\frac{\alpha}{\alpha+1}}}|U_\lambda(x) - U_\lambda(x_0)| \to 0 \quad \text{ if } \alpha > 0\,.$$

It then follows that c does not depend on the choice of x_0 and, up to a sequence and locally uniformly in Ω, one has

$$\lambda|U_\lambda|^\alpha U_\lambda \to -c\,.$$

Moreover, the function $V_\lambda(x) = U_\lambda(x) - U_\lambda(x_0)$ is locally uniformly bounded, locally uniformly Lipschitz continuous and satisfies

$$-|\nabla V_\lambda|^\alpha F(D^2V_\lambda) + |\nabla V_\lambda|^\beta = f - \lambda|U_\lambda|^\alpha U_\lambda \;\text{ in } \Omega\,.$$

If V denotes the local uniform limit of V_λ for a sequence $\lambda_n \to 0$, then one has

$$-|\nabla V|^\alpha F(D^2 V) + |\nabla V|^\beta = f + c \ \text{ in } \Omega\,.$$

Finally, arguing as in the proof of Theorem 1.1 and using Theorem 2.2, we have that, for some $\delta_0 > 0$ sufficiently small,

$$V_\lambda \geq \phi + \min_{d(x)=\delta_0} V_\lambda \quad \text{ in } \Omega \setminus \Omega_{\delta_0}\,,$$

with ϕ defined in (3.2) for arbitrary $s > 0$. Letting $\lambda, s \to 0$ we deduce that $V(x) \to +\infty$ as $d(x) \to 0$. This shows that (c, V) is an ergodic pair and concludes the proof.

□

We now prove that, under assumption (1.6), ergodic functions satisfy on the boundary the same asymptotic identities as the explosive solutions of (4.3).

Theorem 6.3. *Let F and f be as in Theorem 4.2. Then, any ergodic function u satisfies*

$$\lim_{d(x)\to 0} \frac{u(x)\, d(x)^\gamma}{C(x)} = 1 \ \textit{if}\ \gamma > 0, \qquad \lim_{d(x)\to 0} \frac{u(x)}{|\log d(x)|\, C(x)} = 1 \ \textit{if}\ \gamma = 0\,. \tag{6.2}$$

Proof. As usual, we consider only the case $\gamma > 0$.

The computations made in the proof of Theorem 4.2 (for $\gamma_1 = 0$) show that, for all $\nu > 0$ and for $\delta_0 > 0$ sufficiently small, the function $\overline{w}_{\nu,\delta}(x) := \frac{C(x)+\nu}{(d(x)-\delta)^\gamma}$ satisfies for $\delta < d(x) < \delta_0$

$$-|\nabla \overline{w}_{\nu,\delta}|^\alpha F(D^2 \overline{w}_{\nu,\delta}) + |\nabla \overline{w}_{\nu,\delta}|^\beta \geq c_1\, \nu (d(x) - \delta)^{-\frac{\beta}{\beta-\alpha-1}}$$

where $c_1 > 0$ is a constant depending on $\alpha, \beta, a, A, |D^2 d|_\infty$ and $|C|_{\mathcal{C}^2(\Omega)}$.

By assumption (4.6) on f, this implies that

$$-|\nabla \overline{w}_{\nu,\delta}|^\alpha F(D^2 \overline{w}_{\nu,\delta}) + |\nabla \overline{w}_{\nu,\delta}|^\beta > f(x) + c = -|\nabla u|^\alpha F(D^2 u) + |\nabla u|^\beta \ \text{ in } \Omega_\delta \setminus \Omega_{\delta_0}$$

for $\delta_0 = \delta_0(\nu)$ small enough. Hence, we are in the hypothesis of Theorem 2.2 and we deduce that

$$u \leq M_\nu + \overline{w}_{\nu,\delta} \ \text{ in } \Omega_\delta \setminus \Omega_{\delta_0},$$

with $M_\nu = \sup_{d(x)=\delta_0} u(x)$. Letting $\delta \to 0$ we obtain that

$$u \leq M_\nu + (C(x) + \nu) d(x)^{-\gamma} \text{ in } \ \Omega \setminus \Omega_{\delta_0}.$$

This in turn implies that

$$\lim_{d(x)\to 0} u(x)^{\alpha+1} d(x)^{\frac{\beta}{\beta-\alpha-1} - \gamma_0} = 0$$

for all γ_0 such that

$$\gamma_0 < \frac{\beta}{\beta - \alpha - 1} - (\alpha + 1)\gamma = \alpha + 2.$$

Since $\alpha+2>1$, we obtain in particular that the function $|u|^\alpha u$ satisfies condition (4.6) with $\gamma_0=1$. Note also that $|u|^\alpha u$ is bounded from below in Ω since it is continuous and blows up on the boundary. Finally, we observe that u satisfies

$$-|\nabla u|^\alpha F(D^2u)+|\nabla u|^\beta+|u|^\alpha u=f+c+|u|^\alpha u\,,$$

where the right hand side $f+c+|u|^\alpha u$ satisfies condition (4.6) with an exponent $\gamma_0=\min\{\gamma_0(f),1\}$, $\gamma_0(f)$ being the exponent appearing in the condition (4.6) satisfied by f. Hence, by applying Theorem 4.2, we obtain that u satisfies the boundary estimates (4.7) with $\lambda=1$ and the constant D also depending on u itself. Estimates (4.7) in turn imply relations (6.2). □

6.2. Uniqueness and further properties of the ergodic constant: proof of Theorem 1.2.

Throughout this section we assume that f is bounded and locally Lipschitz continuous.

In the introduction we have defined $\mu^\star\in\mathbb{R}\cup\{-\infty\}$ as

$$\mu^\star=\inf\{\mu\,:\,\exists\,\varphi\in\mathcal{C}(\overline{\Omega}),-|\nabla\varphi|^\alpha F(D^2\varphi)+|\nabla\varphi|^\beta\le f+\mu\}.$$

It is easy to see that $\mu^\star\le-\inf_\Omega f$. A better upper bound on $\mu^\star$ depending on the domain Ω is given by the following result.

Proposition 6.4. *If $\Omega\subset[0,R]\times\mathbb{R}^{N-1}$, then*

$$\mu^\star\le-\inf_\Omega f-\frac{K_1}{R^{\frac{\beta}{\beta-\alpha-1}}}$$

for a positive constant $K_1=K_1(a,\alpha,\beta)$.

Proof. The function $\varphi(x)=Cx_1^{\frac{\alpha+2}{\alpha+1}}$ with $C=\left[\frac{a}{2(\alpha+1)}\right]^{\frac{1}{\beta-\alpha-1}}\left(\frac{\alpha+1}{\alpha+2}\right)R^{-\frac{\beta}{(\alpha+1)(\beta-\alpha-1)}}$ satisfies, for some constant $K_1=K_1(a,\alpha,\beta)$:

$$-|\nabla\varphi|^\alpha F(D^2\varphi)+|\nabla\varphi|^\beta\le-\frac{a(\alpha+2)^{1+\alpha}}{(\alpha+1)^{2+\alpha}}C^{1+\alpha}+\frac{(\alpha+2)^\beta}{(\alpha+1)^\beta}C^\beta x_1^{\frac{\beta}{\alpha+1}}\le-K_1R^{-\frac{\beta}{\beta-\alpha-1}}.$$

Hence, by its definition, $\mu^\star\le-\inf_\Omega f-K_1R^{-\frac{\beta}{\beta-\alpha-1}}$. □

We are now ready to give the proof of Theorem 1.2.

Proof of Theorem 1.2. Here we set $c_\Omega=c$. Note that the existence of c is given by Theorem 6.2.

Proof of 1. Suppose that c and c' are two ergodic constants, and let φ and φ' be respectively corresponding ergodic functions. By Theorem 6.3 the ratio of φ and φ' goes to 1 as $d(x)\to0$; hence, for any $\theta<1$, the supremum of $\theta\varphi-\varphi'$ is achieved in the interior of Ω since $\theta\varphi-\varphi'$ blows down to $-\infty$ as $d(x)\to0$.

We observe that

$$-|\nabla(\theta\varphi)|^\alpha F(D^2(\theta\varphi))+|\theta\nabla\varphi|^\beta\le\theta^{1+\alpha}(f+c)$$
$$-|\nabla\varphi'|^\alpha F(D^2\varphi')+|\nabla\varphi'|^\beta=f+c.$$

From standard comparison arguments in viscosity solutions theory, see [16], it follows that at a maximum point $\bar{x}$ of $\theta\varphi-\varphi'$ one has $f(\bar{x})+c'\le\theta^{1+\alpha}(f(\bar{x})+c)$. Letting $\theta\to1$, we get $c'\le c$. Exchanging the roles of c and c' we conclude that $c=c'$.

Proof of 2. Let $\mu < c$ and suppose by contradiction that there exists $\varphi \in \mathcal{C}(\overline{\Omega})$ satisfying

$$-|\nabla\varphi|^\alpha F(D^2\varphi) + |\nabla\varphi|^\beta \le f + \mu.$$

Let u be an ergodic function corresponding to c. Clearly, $\sup_\Omega(\varphi - u)$ is attained in Ω. Again by standard viscosity arguments, we obtain that at a maximum point $\bar{x}$ of $\varphi - u$ one has

$$f(\bar{x}) + \mu \ge f(\bar{x}) + c\,,$$

which is a contradiction. Hence, we deduce

$$\{\mu \,:\, \exists\,\varphi \in \mathcal{C}(\overline{\Omega}), -|\nabla\varphi|^\alpha F(D^2\varphi) + |\nabla\varphi|^\beta \le f + \mu\} \subset [c, +\infty)\,,$$

which implies that $\mu^\star$ is finite and $\mu^\star \ge c$.

On the other hand, by definition of $\mu^\star$, for any $\mu < \mu^\star$ the problem

$$\begin{cases} -|\nabla u|^\alpha F(D^2u) + |\nabla u|^\beta = f + \mu & \text{in } \Omega \\ u = 0 & \text{on } \partial\Omega \end{cases}$$

does not have solution. Theorem 1.1 then implies that there exists an ergodic constant $c_{f+\mu} \ge 0$ for the right hand side $f + \mu$. On the other hand, by the uniqueness proved in 1 above, one has $c = \mu + c_{f+\mu}$. Hence, $c \ge \mu$ and, therefore, $c \ge \mu^\star$.

Proof of 3. The nondecreasing monotonicity of c with respect to the domain Ω is an immediate consequence of point 2 above and the definition of $\mu^\star$.

Let us now prove the "continuity" of $\Omega \mapsto c_\Omega$, in the following weak sense. For $\delta > 0$ small, let c_δ denote the ergodic constant in Ω_δ. Then, c_δ is nondecreasing as δ decreases to zero, and $c_\delta \le c = c_\Omega$.

Let u_δ be an ergodic function in Ω_δ, $x_0 \in \Omega$ be a fixed point and let us set $v_\delta = u_\delta - u_\delta(x_0)$.

By Theorem 2.4, v_δ is locally uniformly bounded and locally uniformly Lipschitz continuous in Ω_δ. Thus, up to a sequence $\delta_n \to 0$, v_δ converges locally uniformly in Ω to a solution v of the equation with right hand side $f + \lim_{\delta\to 0} c_\delta$. Moreover, arguing as in the proof of Theorem 6.2, we have that $v_\delta(x) \ge C_0\,(d(x) - \delta)^{-\gamma}$ if $\gamma > 0$, and $v_\delta(x) \ge -C_0 \log\left(d(x) - \delta\right)$ if $\gamma = 0$, for some constant $C_0 > 0$ and for $\delta < d(x) \le \delta_0$. Letting $\delta \to 0$, we get that $v(x) \to +\infty$ as $d(x) \to 0$. Hence, v is an ergodic function in Ω and, by point 1, $\lim_{\delta\to 0} c_\delta$ is the ergodic constant c.

Proof of 4. We prove that the constant $\mu^\star$ is not achieved. Suppose by contradiction that there exists $\varphi \in \mathcal{C}(\overline{\Omega})$ such that

$$-|\nabla\varphi|^\alpha F(D^2\varphi) + |\nabla\varphi|^\beta \le f + \mu^\star = f + c\,.$$

On the other hand, let u be an ergodic function in Ω.

We observe that for all constants M, $\varphi + M$ is still a bounded sub-solution, whereas u is a solution satisfying $u = +\infty$ on $\partial\Omega$. Theorem 5.1 applied in a smaller domain Ω_δ then yields $u \ge \varphi + M$ for arbitrarily large M, which clearly is a contradiction.

A similar argument proves the strict increasing behavior of the ergodic constant. Let $\Omega' \subset\subset \Omega$ and suppose by contradiction that $c_{\Omega'} = c_\Omega$. Let $u_{\Omega'}$ and u_Ω be ergodic functions respectively in Ω' and Ω. For every constant M, both $u_\Omega + M$ and $u_{\Omega'}$ satisfy (6.1) in Ω', with $u_\Omega + M$ bounded and $u_{\Omega'} = +\infty$ on $\partial\Omega'$. Hence, Theorem 5.1 yields the contradiction $u_{\Omega'} \ge u_\Omega + M$ for every M. □

Remark 6.5. We remark that, thanks to Proposition 6.4, the condition $\sup_\Omega f + c < 0$ appearing in Theorem 1.2–4 is satisfied in one of the following cases:

- f is constant in Ω;
- the oscillation $\sup_\Omega f - \inf_\Omega f$ of f is suitably small, in dependence of the length of the projections of Ω on the coordinated axes;
- in at least one direction Ω is suitably narrow, in dependence of the oscillation of f in Ω.

As a direct consequence of Theorems 1.1 and 1.2, we can finally establish a connection between the existence of solutions of the Dirichlet problem (1.3) and the sign of the ergodic constant $c = c_{\Omega,f}$.

Corollary 6.6. *Let F, f be as in Theorem 1.2 and let c denote the ergodic constant in Ω for f. Then:*

(i) if $c < 0$, then problem (1.3) does admit solutions;
(ii) if $c > 0$, then problem (1.3) does not admit any solution.

Proof. From Theorem 1.1 and from the uniqueness of the ergodic constant c proved in Theorem 1.2–1, we deduce that if there is no solution of problem (1.3) then $c \geq 0$, that is statement (i).

On the other hand, if $c > 0$, then, by Theorem 1.2–2, it follows that there does not exist any function $\varphi \in \mathcal{C}(\overline{\Omega})$ satisfying

$$-|\nabla\varphi|^\alpha F(D^2\varphi) + |\nabla\varphi|^\beta \leq f \qquad \text{in } \Omega\,.$$

In particular, there cannot exist any solution of problem (1.3), that is statement (ii).

□

Remark 6.7. If $\sup_\Omega f + c < 0$, by using Theorem 1.2–4 and arguing as above, we deduce that there exist solutions of (1.3) if and only if $c < 0$.

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