Sapienza
Università di Roma

Faculty of Mathematical, Physical and Natural Sciences

Physics Department

P.H.D. Thesis

On the consistency of DSR models for multiparticle systems

Academic Year 2018/2019

Candidate:
Palmisano Michelangelo

Supervisors:
Prof. Giovanni Amelino-Camelia
Prof. Valeria Ferrari

Contents

Chapter 1

Introduction

In the research studies on quantum gravity (QG) there has been a tenacious interest on the fate of Lorentz symmetry at the Planck scale [1–3] i.e. the energy (length) scale $E_P = \sqrt{\hbar c^5/G} \approxeq 10^{28}$ eV ($L_P = \sqrt{\hbar G/c^3} \approxeq 10^{-35}$ m) we believe should be characteristic of the QG regime.
Indeed, over the last decade, some approaches have focused on the possibility that, due to the combined effects of quantum mechanics and general relativity [4], L_P may set an intrinsic minimum allowed length we can have access to and, in particular, suggested that this feature may need to be implemented in a relativistic invariant manner [5–7].
As a consequence, it is believed that this scale may identify a regime beyond which our notion of space-time comes to fall and has to be replaced with a totally new concept of "quantum" space-time.
While it is not fully clear yet what exactly the properties of this space-time should be in a definitive theory of quantum gravity, several approaches suggest that at least in a flat limit a good candidate may be the so-called κ-Minkowski space-time, a 3+1-dimensional structure whose key property is the non-commutativity of its coordinates

$$[x_0, x_i] = i\ell x_i, \tag{1.1}$$

where x_0 is the time, x_i are the three spatial directions and ℓ is some deformation parameter usually identified with the Planck length L_P itself.
This quantum space-time is naturally coupled with its symmetry structure, the κ-Poincaré Hopf algebra which is an highly non linear deformation of the standard Poincaré algebra and contains a new relativistic invariant κ, which again is often identified with the Planck energy E_P.
This is interpreted as the maximum energy of a free particle, which is very analogous to the role played by the speed of light c in Special Relativity.

In this regard the κ-Poincaré algebra provides a very important example of a **DSR** theory, where DSR stands for **Doubly** or **Deformed Special Relativity** [8–13].
Probably the most striking feature of this algebra is given by the intrinsic non-cocommutativity of its coalgebra. As a result, the κ-Poincaré algebra prescribes a totally new set of non-commutative composition laws of momenta, similarly to what happens for the composition of speeds in special relativity, and a new dispersion relation of mass (MDR), both turning into the special relativistic ones in the limit $\kappa \to \infty$ or $\ell \to 0$.
If we consider two elementary particles with momenta p and q, this new composition law reads

$$p \oplus q = \begin{cases} p_0 + q_0, \\ p_i + e^{-\ell p_0} q_i, \end{cases} \tag{1.2}$$

while the dispersion relation of a particle with mass m is given by

$$m^2 = \frac{4}{\ell^2} \sinh^2 \frac{\ell}{2} p_0 - e^{\ell p_0} |p_i|^2. \tag{1.3}$$

As we can easily see the deformation is stronger as the energy increases with respect to κ, thus, paradoxically, the deviations from the commutative case should be more evident for macroscopic systems, which clearly is not true.
This question is known in the literature as the "soccer-ball problem", an issue which has been given some attention so far [14–16] but has never been convincingly resolved.
Here we explore the possibility that the solution of this problem lies in the analysis of the relativistic properties of a composite system.
More specifically, we want to show that the deformation effects should gradually suppress when the systems under consideration are composed of more and more particles.
As firstly pointed out in [16], which provides some of the main motivations of this thesis, this can be naively understood by noting that the center of mass coordinates

$$x^T_\mu = \frac{x^A_\mu + x^B_\mu}{2} \tag{1.4}$$

of a system composed of two identical particles A and B satisfy the commutation rule

$$\left[x^T_i, x^T_0\right] = i\frac{\ell}{2} x^T_i, \tag{1.5}$$

and more in general the center of mass coordinates of a system composed of N identical particles obey the rule

$$\left[x_i^T, x_0^T\right] = i\frac{\ell}{N}x_i^T. \tag{1.6}$$

This is a first, elementary indication of the fact that for systems made up of a very large number N of particles the deformation effects should become practically negligible, so that we recover the standard Poincaré kinematics on a macroscopic scale.
It's interesting to note that in this context one would naturally have to deal with a "non-universal" relativistic theory, where different systems of particles (and single particles alone) would obey to rules of transformation dictated by different symmetry algebras [17, 18].
Another key aspect we will point out during this exploration is the role held by the relative motion of the particles composing the system. In particular we will show that, contrarily to the standard Poincaré case, the relative motion and the center of mass motion are strictly connected, and as a consequence the total and relative generators of the two particle system symmetry algebra will be connected as well in a complex fashion, as manifested by the commutators and coproducts of the algebra itself.
Although this will make in some sense tricky to fully understand the meaning and the properties of the formal structure of the symmetry algebra, a pivotal result we will stress several times in the continuation is that for small relative motion (compared to κ) the symmetry algebra of the system is simply a 2κ-Poincaré algebra.
We believe that this result together with the observation given by equations (1.5) and (1.6) provides an important step forward in the solution of the soccer-ball problem.
At the same time we will try to address another related issue in the DSR scenario, that is the construction of a fully relativistic interaction theory between two or more particles in the so-called galilean limit[1] where $c \to \infty$, which is still basically an open topic in the field[2].
To summarize our work is organized as follows.
In chapter II we briefly discuss the mathematical tools we make use of in this work, namely the Hopf algebras theory. Then we review a well-known

[1]The reason for this is to be found in the famous "No-interaction" theorem of special relativity [21–23], which states the impossibility of building an interaction theory between a finite number of particles covariant under the symmetries of the Lorentz algebra, that we expect to remain true also in the context of DSR theories.

[2]See for instance [19] and [20] for some interesting takes on **deformed quantum mechanics** different from the one we propose in this thesis.

example of deformation of the Poincaré symmetries, i.e. the κ-Poincaré Hopf algebra.
In chapter III we show how interpreting the κ-Poincaré composition of momentum as the total momentum of a two particle system inevitably leads to a κ deformed algebra, so that the soccer-ball problem can't find any solution in that case.
After discussing how the center of mass coordinates naturally involve a 2κ (or $\frac{\ell}{2}$) deformation as in (1.5), in chapter IV we propose a new symmetry algebra for the composite system where the rescaling of the deformation parameter is evident and even turns out to be just a 2κ-Poincaré algebra for systems with small relative motion.
In chapter V we briefly review the so-called pregeometric formalism which allows us to represent the κ-Poincaré algebra in a convenient way for our purposes.
Next we use this representation in chapter VI to discuss another candidate for the role of algebra of symmetry of our two particle system, which by construction has the advantage to be fully compatible with the infinitesimal transformations of the single particles composing the system itself.
We then show how this novel symmetry algebra is in turn consistent with a $\ell \to \frac{\ell}{2}$ rescaling and actually equal to a 2κ-Poincaré algebra in the limit where the relative motion between the particles is negligible.
We stress that these results provide a clear indication of the fact that the soccer-ball problem should gradually dissipate when the number of particles involved grows higher and higher.
Afterwards in chapter VII we briefly examine a more general case where particles with different masses are involved, in order to investigate if our proposed solution of the soccer-ball problem still holds.
In chapter VIII we analyze the galilean limits of our novel algebras in relation to the possibility of building an interaction theory in the non-commutative framework, safe from the soccer-ball problem. In particular we study in depth the case of an harmonic potential between two particles, showing its deviations from the standard quantum mechanics case.
Finally, we report our conclusions in chapter IX.

Chapter 2

Mathematical preliminaries

The purpose of this chapter is to introduce some abstract algebra's notions which we will make intensively use of in the rest of this work.
For the sake of simplicity the definition and discussion of the main properties of these structures will be given as synthetically as possible, and in particular from here on we will often suppose tacitly known some more elementary mathematical facts, limiting ourselves to briefly mentioning them as they arise.
For more details you can consult for example [24–26].

2.1 Algebras

An algebra A over a field $\mathbb{K}$ is a vector space with a bilinear product $\times : A \times A \to A$. In other words an algebra is a quintuple $(A, \mathbb{K}, +, \cdot, \times)$ where A is a set of points, the quadruple $(A, \mathbb{K}, +, \cdot)$ is a vector space over $\mathbb{K}$ and the operations $+, \cdot$ and $\times$ are all compatible between themselves ($\cdot$ is the scalar multiplication by an element of $\mathbb{K}$).
More precisely we require that for every $x, y, z \in A$ and $c, d \in \mathbb{K}$ the following relations are satisfied:

$$\begin{aligned} (x+y) \times z &= x \times z + y \times z && \text{(distributivity to the right)}, \\ z \times (x+y) &= z \times x + z \times y && \text{(distributivity to the left)}, \\ (c \cdot x) \times (d \cdot y) &= (cd) \cdot (x \times y) && \text{(products compatibility)}. \end{aligned} \tag{2.1.1}$$

The product $\times$ may or may not be associative (commutative), so that we may talk about associative (commutative) algebras or non-associative (non-commutative) algebras.
For example, given a positive integer n, the ring of the real square matrices of order n is an associative (non-commutative for $n > 1$) algebra over the field $\mathbb{R}$, where the product $\times$ is trivially given by the standard matrix multiplication.
An example of a non-associative algebra is instead given by the three-dimensional Euclidean space $\mathbb{R}^3$ equipped with the familiar cross product between vectors.
It's worth noting that given an associative and non-commutative algebra A over a field $\mathbb{K}$ it is always possible to define a new non-associative version A' which is identical as a vector space, but is equipped with a new product

$$[x, y] := x \times y - y \times x, \quad x, y \in A, \tag{2.1.2}$$

which is non-associative and satisfies the Jacobi identity

$$[x, [y, z]] + [y, [z, x]] + [z, [x, y]] = 0. \tag{2.1.3}$$

Since $[x, x] = 0$ for every x in A, this new algebra is a **Lie Algebra**.
We will discuss more in depth this pivotal class of algebras in the next section.
An algebra will be called **unital** if it owns a neutral element, a *unit* e with respect to the product $\times$, or in other words if for every $x \in A$ we have

$$x \times e = e \times x = x \qquad \text{(unitality)}. \tag{2.1.4}$$

If this element exists, it is necessarily unique.
The algebra of square matrices of order n is an example of unital algebra where the neutral element is simply the identity matrix.
Given two algebras A and B on a field $\mathbb{K}$, a map $\phi : A \to B$ such that

$$\begin{aligned} &\phi(k \cdot x) = k \cdot \phi(x) \\ &\phi(x + y) = \phi(x) + \phi(y) \\ &\phi(x \times y) = \phi(x) \times \phi(y) \end{aligned} \tag{2.1.5}$$

for every $x, y \in A$ and $k \in \mathbb{K}$ is an algebra homomorphism.
In particular if ϕ is a bijective application, we will call it an isomorphism between the algebras A and B, and we will say that the algebras A and B are isomorphic.

If A and B are unital algebras we will also ask that the map $\phi : A \to B$ sends the neutral element of A into the neutral element of B.
A simple example of (unital) algebra homomorphism is given by the application $\phi : \mathbb{C} \to \mathbb{R}$ that maps every complex number $z = x + iy$ in its real part: $\phi(z) = x$.
A complementary example is given by the morphism $\phi : \mathbb{R} \to \mathbb{C}$ that sends any real number x in the complex number x, which is the inclusion of $\mathbb{R}$ in $\mathbb{C}$ as a subalgebra.
More in general a subalgebra B of an algebra A over a field $\mathbb{K}$ is a linear subspace closed over the product $\times$.
This makes B an algebra on its own, whose product is simply the one induced by the algebra A itself.
A similar construction is that of a left (right) ideal.
A left (right) ideal I of an algebra A over a field $\mathbb{K}$ is a linear subspace such that the left (right) product of an element of A by an element of I is still an element of I.
A two-sided ideal is a left ideal that is also a right ideal, and we will call it just ideal.
It's pretty clear that an ideal I of an algebra A is a subalgebra of A, whereas the vice-versa doesn't generally hold.
This kind of structure is very frequent and important, for example it is possible to prove that the kernel of an homomorphism ϕ between two algebras A and B is an ideal of A, while the image is a subalgebra of B.
Let us consider an ideal I of an algebra A over $\mathbb{K}$, and let us define the quotient A/I as the set of the equivalence classes $[v]$ of vectors $v + i$ with $i \in I$. A/I is still a linear space over $\mathbb{K}$, moreover given two vectors $v + i_1$ and $w + i_2$ with $i_1, i_2 \in I$ we find that

$$(v + i_1) \times (w + i_2) = v \times w + v \times i_2 + i_1 \times w + i_1 \times i_2 = v \times w + i_3 \quad (2.1.6)$$

where $i_3 = v \times i_2 + i_1 \times w + i_1 \times i_2 \in I$.
Therefore the product $\times$ in A induces a product on A/I defined by $[v] \times [w] = [v \times w]$, which can be proven to be compatible with the axioms (2.1.1).
This means that A/I is an algebra over $\mathbb{K}$, called quotient algebra.
One of the most important examples is given by the canonical isomorphism $A/\mathrm{Ker}(\phi) \cong \mathrm{Im}(\phi)$, where $\phi : A \to B$ is a given homomorphism between the algebras A and B.
Before concluding this paragraph, let us discuss a couple more constructions we will make use of later on.

Given a family of algebras $\{A_f\}_{f\in\mathcal{F}}$ over $\mathbb{K}$ we can easily define the direct product $\Pi_{f\in\mathcal{F}}A_f$ as the algebra of the $\mathcal{F}$-uple of elements belonging to the algebras A_f, with sum, scalar multiplication and product defined component by component from the corresponding operations of the starting algebras.
The direct product $\Pi_{f\in\mathcal{F}}A_f$ contains the direct sum $\bigoplus_{f\in\mathcal{F}}A_f$ of the $\mathcal{F}$-uple with only a finite number of components different from zero as a subalgebra.
In a similar fashion given two algebras A and B over a field $\mathbb{K}$, it is possible to equip the tensor product $A\otimes_{\mathbb{K}}B$ with an algebra structure over $\mathbb{K}$ through the definition of a binary operator $\times$ as

$$(a_1\otimes b_1)\times(a_2\otimes b_2)=(a_1\times a_2)\otimes(b_1\times b_2) \tag{2.1.7}$$

and by extending it over all $A\otimes_{\mathbb{K}}B$ through bilinearity.
It's easy to check that this product satisfies the algebra axioms (2.1.1), and that in particular the algebra $A\otimes_{\mathbb{K}}B$ inherits many structural properties of A and B: it is associative, commutative or unital if the starting algebras are, and in the last case if 1_A and 1_B are the units of A and B respectively, the neutral element in $A\otimes_{\mathbb{K}}B$ is just $1_A\otimes 1_B$.
Thanks to the notion of tensor product of algebras we can express the associativity axiom through the language of category theory: an algebra A over a field $\mathbb{K}$ is associative if the following diagram commutes

$$\begin{tikzcd} A\otimes A\otimes A \arrow[r, "\nabla\otimes\mathbb{1}"] \arrow[d, "\mathbb{1}\otimes\nabla"'] & A\otimes A \arrow[d, "\nabla"] \\ A\otimes A \arrow[r, "\nabla"'] & A \end{tikzcd}$$

where we replaced the cartesian product $A\times A$ with the tensor product $A\otimes A$ and the map $\nabla : A\otimes A\to A$ is naturally induced by the product $\times$ itself:

$$\nabla(a\otimes b)=a\times b. \tag{2.1.8}$$

Similarly we will say that an algebra A over $\mathbb{K}$ is commutative if the diagram

$$\begin{tikzcd} A\otimes A \arrow[rr, "\iota"] \arrow[dr, "\nabla"'] & & A\otimes A \arrow[dl, "\nabla"] \\ & A & \end{tikzcd}$$

is, where ι is the inversion map that sends $a \otimes b$ to $b \otimes a$.
Finally we will say that and algebra A over $\mathbb{K}$ is unital if there exists a map $\eta : \mathbb{K} \to A$ such that the following diagram is commutative:

$$\begin{tikzcd}
A \otimes \mathbb{K} \arrow[d, "\mathbb{1}\otimes\eta"'] \arrow[dr, "\cong"] & \\
A \otimes A \arrow[r, "\nabla"] & A \\
\mathbb{K} \otimes A \arrow[u, "\eta\otimes\mathbb{1}"] \arrow[ur, "\cong"'] &
\end{tikzcd}$$

The sideward arrows represent the canonical isomorphisms between $A \otimes \mathbb{K}$ and A or $\mathbb{K} \otimes A$ and A which map $a \otimes 1_{\mathbb{K}}$ or $1_{\mathbb{K}} \otimes a$ to a, where $1_{\mathbb{K}}$ is the identity of $\mathbb{K}$.
In order to understand the equivalence between thiese definitions it suffices to note that the commutativity of the diagram implies that $a \times \eta\left(1_{\mathbb{K}}\right) = \eta\left(1_{\mathbb{K}}\right) \times a = a$ for every $a \in A$, which implies in turn that $\eta\left(1_{\mathbb{K}}\right)$ is the neutral element of A.
This way of expressing the properties of the algebraic structures at stake will prove useful in the following thanks to some duality arguments typical of category theory.

2.2 Lie algebras

A Lie Algebra $\mathfrak{g}$ over a field $\mathbb{K}$ is a vector space equipped with a non-associative product $[\cdot,\cdot] : \mathfrak{g} \times \mathfrak{g} \to \mathfrak{g}$ called “Lie bracket” satisfying the axioms

$$\begin{aligned}
&[ax + by, z] = a\,[x,z] + b\,[y,z] && \text{(left bilinearity)},\\
&[z, ax + by] = a\,[z,x] + b\,[z,y] && \text{(right bilinearity)},\\
&[x,x] = 0 && \text{(alternativity)},\\
&[x,[y,z]] + [y,[z,x]] + [z,[x,y]] = 0 && \text{(Jacobi identity)},
\end{aligned} \tag{2.2.1}$$

where $a, b \in \mathbb{K}$ and $x, y, z \in \mathfrak{g}$.
Since for every $x, y \in \mathfrak{g}$ we have that

$$0 = [x+y, x+y] = [x,x] + [x,y] + [y,x] + [y,y]\,, \tag{2.2.2}$$

the Lie brackets are trivially anti-commutative:

$$[x, y] = -[y, x]. \tag{2.2.3}$$

Clearly all the terminology we developed in the previous paragraph can be directly applied to the Lie Algebras theory, so that we can talk about Lie subalgebras, ideals, quotient spaces and Lie algebra homomorphisms.
As we mentioned in the previous section for every associative algebra A over a field $\mathbb{K}$ it is possibile to build a Lie algebra $\mathfrak{l}(A)$, identical as a vector space and equipped with a Lie bracket

$$[x, y] = xy - yx. \tag{2.2.4}$$

The associativity of A's product implies the validity of the Jacobi identities, thus $\mathfrak{l}(A)$ is effectively a Lie algebra.
The associative algebra A is called **universal enveloping algebra** of $\mathfrak{l}(A)$, and plays a pivotal role in the Lie algebra theory. We will discuss this more in depth in the next paragraph.
We have already seen an example of Lie algebra with the Euclidean space $\mathbb{R}^3$ with the familiar cross product operation.
More complex examples are given by matrix algebras.
One of the best-known cases is the vector space $\mathfrak{gl}_n(\mathbb{K})$ of square matrices of order n over a field $\mathbb{K}$, called general linear algebra. This is a Lie algebra associated with the universal enveloping algebra composed by the same matrices of order n equipped with the ordinary product between matrices.
$\mathfrak{gl}_n(\mathbb{K})$ also contains a Lie subalgebra given by the space of the null trace matrices $\mathfrak{sl}_n(\mathbb{K})$, which is commonly called special linear algebra. It is interesting to note that its universal enveloping algebra is a subalgebra of the algebra of the matrices of order n.
However, not all Lie algebras are directly relatable to a universal enveloping algebra.
For example, consider the vector space $\mathfrak{u}(n)$ of the complex antihermitian matrices of order n. It is easy to see that this is a Lie algebra with Lie brackets given by the usual matrix product commutator, and yet this same product does not provide it with an associative algebra structure.
In fact, $\mathfrak{u}(n)$ is a typical example of a Lie algebra associated with a Lie group, in this case to the group of unitary matrices $U(n)$.
The correspondence between Lie algebras and Lie groups is very narrow: each Lie group gives rise to a well-determined Lie algebra (the tangent space to the identity), while for every finite dimensional Lie algebra there is a connected Lie group that generates it.

This group, however, is not univocally determined, but if two connected Lie groups have the same Lie algebra then they are locally isomorphic.
In particular, given a group of real matrices G, the corresponding Lie algebra $\mathfrak{g}$ consists of all the matrices X such that $\exp(tX) \in G$ for every $t \in \mathbb{R}$, where exp is the exponential map

$$\exp(X) = \sum_{k=0}^{\infty} \frac{X^k}{k!}. \tag{2.2.5}$$

Note that by definition of tangent space a Lie algebra has the same dimension (as a vector space) of the parent group manifold.
One has for example that $\mathfrak{gl}_n(\mathbb{C})$ and $\mathfrak{sl}_n(\mathbb{C})$ are respectively the Lie algebras of the general linear group of the invertible matrices of order n and the special linear group of the invertible matrices of the same order with unitary determinant, denoted with $GL_n(\mathbb{C})$ and $SL_n(\mathbb{C})$.
On the other hand the group $O(n)$ of the ortogonal matrices of order n and its subgroup $SO(n)$ composed by the matrices with unitary determinant share the same Lie algebra $\mathfrak{o}(n)$ composed by the real skew matrices of order n.
The most relevant case for us, however, is that of the group of isometries of the Minkowski space-time, i.e. the Poincaré group.
As a group it contains the four-dimensional translations and the Lorentz transformations, and its Lie algebra is commonly called Poincaré algebra.
We will further study this structure in the last paragraphs of this chapter.

2.3 Universal enveloping algebras

As we have seen every associative algebra A over a field $\mathbb{K}$ becomes a Lie algebra over $\mathbb{K}$ with Lie brackets given by the commutators of its elements, algebra that we denoted by $\mathfrak{l}(A)$.
It is interesting to ask whether it is possible to somehow operate an inverse construction, that is, to pass from a given Lie algebra $\mathfrak{l}$ over a field $\mathbb{K}$ to an associative (and unital) algebra which reflects its essential properties.
More precisely, let $\mathfrak{l}$ be such a Lie algebra. Given a unital associative algebra $U(\mathfrak{l})$ over $\mathbb{K}$ and a homomorphism of Lie algebras $h : \mathfrak{l} \to \mathfrak{l}(U)$ we say that $U(\mathfrak{l})$ is the universal enveloping algebra of $\mathfrak{l}$ if for any unital associative algebra A on $\mathbb{K}$ and for each Lie algebra homomorphism $f : \mathfrak{l} \to \mathfrak{l}(A)$ there exists a unique homomorphism of unital algebras $g : U(\mathfrak{l}) \to A$ such

that $f = g_{\mathfrak{l}} \circ h$, where $g_{\mathfrak{l}} : \mathfrak{l}(U) \to \mathfrak{l}(A)$ is the Lie algebra homomorphism naturally induced by g.
It is not difficult to prove that every Lie algebra admits an unique universal enveloping algebra, aside from algebra isomorphisms.
Let us consider in fact the tensor algebra[1] $\mathcal{T}(\mathfrak{l})$, and the ideal I generated by linearity by the elements of the form

$$a \otimes b - b \otimes a - [a, b] \tag{2.3.3}$$

where $a, b \in \mathfrak{l}$.
We can define the algebra $U(\mathfrak{l}) = \mathcal{T}(\mathfrak{l}) / I$, equipped with a product naturally induced by $\mathcal{T}$.
The inclusion map $\mathfrak{l} \to \mathcal{T}(\mathfrak{l})$ induces a linear application $h : \mathfrak{l} \to U(\mathfrak{l})$, and it's not hard to check that the couple $(U(\mathfrak{l}), h)$ satisfies the universal property discussed above.
Unfortunately the universal enveloping algebras do not always have a simple description and are often much larger (as vector spaces) than the starting Lie algebras.
An interesting example is given by an abelian Lie algebra $\mathfrak{l}$ over a field $\mathbb{K}$, that is an algebra whose Lie brackets are always equal to 0. In this case the universal enveloping algebra $U(\mathfrak{l})$ is commutative and, if n is the dimension of $\mathfrak{l}$, it can be identified with the polynomial algebra in n variables over $\mathbb{K}$.
Another typical case is that in which $\mathfrak{l}$ acts by means of infinitesimal transformations (this is for example the case of Lie algebras associated with symmetry groups such as the Poincaré one); it can be found that its universal enveloping algebra $U(\mathfrak{l})$ acts by means of differential operators of all orders.

---

[1]The tensor algebra of an algebra V is defined as

$$\mathcal{T}(V) := \bigoplus_{k=0}^{\infty} \mathcal{T}^k(V) = \mathbb{K} \oplus V \oplus (V \otimes V) \oplus ... \tag{2.3.1}$$

where

$$\mathcal{T}^k(V) := V^{\otimes k} = \overbrace{V \otimes V \otimes ... V}^{k \text{ times}} \tag{2.3.2}$$

and $\mathcal{T}^0(V) = \mathbb{K}$.

2.4 Coalgebras

In the category theory, each algebraic structure has a dual structure that is obtained simply by exchanging the direction of the arrows in the commutative diagrams that establish its properties.
In our case, reversing the direction of the maps in the two diagrams that define the algebraic features of associative and unital algebras, we find the structure that goes under the name of coalgebra.
Formally a coalgebra over a field $\mathbb{K}$ is a vector space C over $\mathbb{K}$ along with two linear applications $\Delta : C \to C \otimes C$ and $\epsilon : C \to \mathbb{K}$ called respectively coproduct and counit, such that

$$\begin{aligned} (\mathbb{1} \otimes \Delta) \circ \Delta &= (\Delta \otimes \mathbb{1}) \circ \Delta \qquad && \text{(coassociativity)}, \\ (\mathbb{1} \otimes \epsilon) \circ \Delta &= (\epsilon \otimes \mathbb{1}) \circ \Delta \qquad && \text{(counitality)}, \end{aligned} \tag{2.4.1}$$

that is

$$\begin{tikzcd} C \arrow[r, "\Delta"] \arrow[d, "\Delta"'] & C \otimes C \arrow[d, "\mathbb{1}\otimes\Delta"] \\ C \otimes C \arrow[r, "\Delta\otimes\mathbb{1}"'] & C \otimes C \otimes C \end{tikzcd} \qquad \begin{tikzcd} C \otimes \mathbb{K} & \\ C \otimes C \arrow[u, "\mathbb{1}\otimes\epsilon"] \arrow[d, "\epsilon\otimes\mathbb{1}"'] & C \arrow[l, "\Delta"'] \arrow[ul, "\cong"'] \arrow[dl, "\cong"] \\ \mathbb{K} \otimes C & \end{tikzcd}$$

are commutative diagrams.
Finally the coalgebra C is cocommutative if the diagram

$$\begin{tikzcd} & C \arrow[dl, "\Delta"'] \arrow[dr, "\Delta"] & \\ C \otimes C \arrow[rr, "\iota"'] & & C \otimes C \end{tikzcd}$$

commutes, where the application ι is the inversion map that sends $c \otimes d$ in $d \otimes c$, $c, d \in C$.
The examples of coalgebras are generally more complicated than the algebras ones. As a paradigmatic case we can consider the tensor algebra $\mathcal{T}(V)$ of a vector space V over a field $\mathbb{K}$.
We can equip $\mathcal{T}(V)$ of a coalgebra structure in the following way.

Let the coproduct Δ be defined as

$$\Delta(x) := x \otimes 1 + 1 \otimes x, \qquad \Delta(1) = 1 \otimes 1, \tag{2.4.2}$$

$x \in V$, and the counit ϵ defined as

$$\epsilon(v) = 0 \qquad \forall v \in V. \tag{2.4.3}$$

It's easy to prove that by extending this definition through linearity over the entire space $\mathcal{T}(V)$, we actually obtain a coalgebra over $\mathbb{K}$.
Given two coalgebras $(C_1, \Delta_1, \epsilon_1)$ and $(C_2, \Delta_2, \epsilon_2)$ over a field $\mathbb{K}$, a linear application $\phi : C_1 \to C_2$ is a coalgebra morphism if

$$\begin{aligned} &(\phi \otimes \phi) \circ \Delta_1 = \Delta_2 \circ \phi && \text{(compatibility with coproducts),} \\ &\epsilon_2 \circ \phi = \epsilon_1 && \text{(compatibility with counits).} \end{aligned} \tag{2.4.4}$$

A bijective morphism between two coalgebras is a coalgebra isomorphism, and the coalgebras will be called isomorphic.
At this point it is easy to re-adapt the definitions of the substructures given in the context of the algebras to the case under consideration.
For example a vector subspace D of a coalgebra C over $\mathbb{K}$ is a sub-coalgebra if the coproduct restricted to D is a linear application from D to $D \otimes D$. In this way D itself is a coalgebra, whose counit is simply the restriction of ϵ to D.
The notion corresponding to the ideal is more complicated: a vector subspace I in C will be called coideal if it is contained in the kernel of ϵ and the image of the restriction of the coproduct to I is contained in the subspace $I \otimes C + C \otimes I$ of $C \otimes C$.
With this definition the quotient space C/I inherits a coalgebra structure, in fact if we represent a generic equivalence class $[c] \in C/I$ through a vector $c + i \in C$ where i is an element of I, then we have that

$$\Delta(c + i) = \Delta(c) + \Delta(i), \qquad \Delta(i) \in I \otimes C + C \otimes I, \tag{2.4.5}$$

thus the coproduct Δ induces an application $\Delta : C/I \to C/I \otimes C/I$ defined by $\Delta([c]) = [\Delta(c)]$, which is itself a coproduct on C/I.
Moreover we have that

$$\epsilon(c + i) = \epsilon(c) + \epsilon(i) = \epsilon(c), \tag{2.4.6}$$

for every element of the form $c + i \in C$, thus we deduce that the counit of C/I is simply given by the functional $\epsilon([c]) = [\epsilon(c)]$.

It's easy to verify for example that the kernel of a coalgebra morphism $\phi : C_1 \to C_2$ is a coideal of C_1, whereas its image is a subcoalgebra of C_2. In particular $C_1/\mathrm{Ker}\,(\phi) \cong \mathrm{Im}\,(\phi)$.
We want to conclude this paragraph by brielfy discussing the so-called Sweedler's notation [27], which is a very useful tool that allows us to simplify considerably the mathematics behind the coalgebra theory.
Given a coalgebra (C, Δ, ϵ) over a field $\mathbb{K}$ and an element $c \in C$ we can write

$$\Delta\,(c) = \sum_i c^i_{(1)} \otimes c^i_{(2)} \tag{2.4.7}$$

for two specific sequences of elements $c^i_{(1)}$ and $c^i_{(2)}$ belonging to C.
In the Sweedler's notation this is expressed by writing

$$\Delta\,(c) = \sum_c c_{(1)} \otimes c_{(2)}, \tag{2.4.8}$$

or more concisely

$$\Delta\,(c) = c_{(1)} \otimes c_{(2)} \tag{2.4.9}$$

where the sum is implicit.
This formal notation allows us to greatly simplify the axioms (2.4.1): the counit property can be rewritten as

$$c = \epsilon\left(c_{(1)}\right)c_{(2)} = c_{(1)}\epsilon\left(c_{(2)}\right), \tag{2.4.10}$$

whereas the coassociativity becomes

$$c_{(1)} \otimes \left(c_{(2)} \otimes c_{(3)}\right) = \left(c_{(1)} \otimes c_{(2)}\right) \otimes c_{(3)} \tag{2.4.11}$$

and the cocommutativity can be reformulated simply as

$$c_{(1)} \otimes c_{(2)} = c_{(2)} \otimes c_{(1)}. \tag{2.4.12}$$

Finally a linear application $\phi : (C, \Delta_c, \epsilon_c) \to (D, \Delta_d, \epsilon_d)$ between coalgebras is a coalgebra morphism if

$$\phi\left(c_{(1)}\right) \otimes \phi\left(c_{(2)}\right) = \phi\,(c)_{(1)} \otimes \phi\,(c)_{(2)}. \tag{2.4.13}$$

We see then that thanks to the Sweedler's notation the various coalgebra properties can be looked over as formally identical to the corresponding algebra ones.

2.5 Bialgebras

A bialgebra B over a field $\mathbb{K}$ is a vector space over $\mathbb{K}$ which is at the same time an unital associative algebra and a coalgebra, in a compatible way.
If we denote with ∇ ed η the product and the unit and with Δ and ϵ the coproduct and the counit, the properties of a bialgebra $(B, \nabla, \eta, \Delta, \epsilon)$ are summarized by the commutativity of the following diagrams:

$$\begin{tikzcd}
 & B \arrow[dr, "\Delta"] & \\
B \otimes B \arrow[ur, "\nabla"] \arrow[d, "\Delta\otimes\Delta"'] & & B \otimes B \\
B \otimes B \otimes B \otimes B \arrow[rr, "\mathbb{1}\otimes\iota\otimes\mathbb{1}"'] & & B \otimes B \otimes B \otimes B \arrow[u, "\nabla\otimes\nabla"']
\end{tikzcd}
\qquad
\begin{tikzcd}
B \otimes B \arrow[rr, "\nabla"] \arrow[dr, "\epsilon\otimes\epsilon"'] & & B \arrow[dl, "\epsilon"] \\
 & \mathbb{K} \otimes \mathbb{K} \cong \mathbb{K} \arrow[dl, "\eta\otimes\eta"'] \arrow[dr, "\eta"] & \\
B \otimes B & & B \arrow[ll, "\Delta"]
\end{tikzcd}$$

$$\begin{tikzcd}
 & B \arrow[dl, "\epsilon"'] & \\
\mathbb{K} \arrow[rr, "\mathbb{1}"'] & & \mathbb{K} \arrow[ul, "\eta"']
\end{tikzcd}$$

The first one represents the compatibility between product and coproduct, where ι is the inversion application that maps $b_1 \otimes b_2$ in $b_2 \otimes b_1$ for every $b_1, b_2 \in B$.
The second one ensures the compatibility between the product and the counit or the coproduct and the unit.
Finally the last one points out the consistence between the unit and the counit of the bialgebra.
These three diagrams substantially express the fact that in a bialgebra the coproduct and the counit are algebra homomorphisms, or equivalently that the product and the unit are coalgebra morphisms.
In order to clarify this point, let us consider the spaces $(\mathbb{K}, \nabla_{\mathbb{K}}, \eta_{\mathbb{K}})$ and $(B \otimes B, \nabla_{\otimes}, \eta_{\otimes})$.

They both are unital associative algebras, the first in an obvious way and the second through the product

$$\nabla_\otimes := (\nabla \otimes \nabla) \circ (\mathbb{1} \otimes \iota \otimes \mathbb{1}) : (B \otimes B) \otimes (B \otimes B) \to B \otimes B \tag{2.5.1}$$

and the unit

$$\eta_\otimes := \eta \otimes \eta : \mathbb{K} \otimes \mathbb{K} \cong \mathbb{K} \to B \otimes B. \tag{2.5.2}$$

In this way given $x_1, x_2, y_1, y_2 \in B$ the product in $B \otimes B$ is specified by

$$\nabla_\otimes ((x_1 \otimes x_2) \otimes (y_1 \otimes y_2)) = \nabla (x_1 \otimes y_1) \otimes \nabla (x_2 \otimes y_2), \tag{2.5.3}$$

which we rewrite for simplicity as

$$(x_1 \otimes y_1)(x_2 \otimes y_2) = x_1 y_1 \otimes x_2 y_2. \tag{2.5.4}$$

At the same time $(\mathbb{K}, \Delta_\mathbb{K}, \epsilon_\mathbb{K})$ is trivially a coalgebra, and $B \otimes B$ can be made so as well by defining a coproduct

$$\Delta_\otimes := (\mathbb{1} \otimes \iota \otimes \mathbb{1}) \circ (\Delta \otimes \Delta) : B \otimes B \to (B \otimes B) \otimes (B \otimes B) \tag{2.5.5}$$

and a counit

$$\epsilon_\otimes := \epsilon \otimes \epsilon : B \otimes B \to \mathbb{K} \otimes \mathbb{K} \cong \mathbb{K}. \tag{2.5.6}$$

The first diagram and the lower half of the second one express the fact that $\Delta : B \to B \otimes B$ is an homomorphism between the algebras (B, ∇, η) and $(B \otimes B, \nabla_\otimes, \eta_\otimes)$:

$$\Delta \circ \nabla = \nabla_\otimes \circ (\Delta \otimes \Delta) : B \otimes B \to B \otimes B, \tag{2.5.7}$$

or, simply,

$$\Delta (xy) = \Delta (x) \Delta (y). \tag{2.5.8}$$

Moreover we have that

$$\Delta \circ \eta = \eta_\otimes : K \to B \otimes B, \tag{2.5.9}$$

that is

$$\Delta (1_B) = 1_{B \otimes B} \tag{2.5.10}$$

where 1_B is the unit of B.

On the other hand the third diagram and the other half of the second one tell us that $\epsilon : B \to \mathbb{K}$ is an algebra homomorphism between (B, ∇, η) and $(\mathbb{K}, \nabla_{\mathbb{K}}, \eta_{\mathbb{K}})$ since

$$\epsilon \circ \nabla = \nabla_{\mathbb{K}} \circ (\epsilon \otimes \epsilon) : B \otimes B \to K, \tag{2.5.11}$$

that is

$$\epsilon (xy) = \epsilon (x) \epsilon (y) \tag{2.5.12}$$

and

$$\epsilon \circ \eta = \eta_{\mathbb{K}} : K \to K, \tag{2.5.13}$$

which is simply

$$\epsilon (1_B) = 1_{\mathbb{K}}. \tag{2.5.14}$$

Likewise we can state that the first diagram and the second's upper half express the fact that the application $\nabla : B \otimes B \to B$ is a morphism between the coalgebras $(B \otimes B, \Delta_{\otimes}, \epsilon_{\otimes})$ and (B, Δ, ϵ), being

$$\nabla \otimes \nabla \circ \Delta_{\otimes} = \Delta \circ \nabla : B \otimes B \to B \otimes B \tag{2.5.15}$$

and

$$\epsilon \circ \nabla = \nabla_{\mathbb{K}} \circ \epsilon_{\otimes} : B \otimes B \to \mathbb{K}. \tag{2.5.16}$$

Finally the last diagram and the lower half of the second one give us a coalgebra morphism $\eta : \mathbb{K} \to B$ between $(\mathbb{K}, \Delta_{\mathbb{K}}, \epsilon_{\mathbb{K}})$ and (B, Δ, ϵ), as we can deduce from the relations

$$\Delta \circ \eta = \eta_{\otimes} \circ \Delta_{\mathbb{K}} : \mathbb{K} \to (B \otimes B) \tag{2.5.17}$$

and

$$\epsilon \circ \eta = \eta_{\mathbb{K}} \circ \epsilon_{\mathbb{K}} : \mathbb{K} \to \mathbb{K}. \tag{2.5.18}$$

Clearly we will call a linear application ϕ between two bialgebras B and B' a bialgebra morphism if it is separately an algebra homomorphism and a coalgebra morphism.

We will say that a vector space B' of a bialgebra B is a sub-bialgebra if it is a subalgebra and a subcoalgebra at the same time, and similarly a bialgebra ideal I will be an algebra ideal and a coalgebra coideal all together.

From this, given a bialgebra B with an ideal I, we can give the definition of bialgebra quotient B/I as usual.
In particular if $\phi : B \to B'$ is a bialgebra morphism, $\mathrm{Ker}\,(\phi)$ is an ideal of B and the bialgebra isomorphism $B/\mathrm{Ker}\,(\phi) \cong \mathrm{Im}\,(\phi)$ holds.
To conclude this paragraph, we want to discuss a very interesting example of a bialgebra.
Let us consider the set $\mathbb{K}^G$ of functions from a group G to a field $\mathbb{K}$, which can be thought as a vector space with a base $\left\{1_{\mathbb{K}}^{g}\right\}_{g\in G}$ composed by the functions equal to $1_{\mathbb{K}}$ in g and null elsewhere.
$\mathbb{K}^G$ is an associative algebra with a product trivially given by multiplication of functions and unit given by the constant function $1_{\mathbb{K}}$.
We can define a coproduct Δ over $\mathbb{K}^G$ by specifying its action on the base $\left\{1_{\mathbb{K}}^{g}\right\}_{g\in G}$ and by extending it through linearity over the entire space.
For instance we can put

$$\Delta\left(1_{\mathbb{K}}^{g}\right) = 1_{\mathbb{K}}^{g} \otimes 1_{\mathbb{K}}^{g}, \tag{2.5.19}$$

with a trivial counit

$$\epsilon\left(1_{\mathbb{K}}^{g}\right) = 1_{\mathbb{K}}. \tag{2.5.20}$$

It's quite easy to verify that, given these operations, $\mathbb{K}^G$ is a fully fledged bialgebra.

2.6 Hopf algebras

The final and most important structure we want to discuss, is that of an **Hopf algebra**.
An Hopf algebra is a bialgebra H over a field $\mathbb{K}$ together with a linear application

$$S : H \to H \tag{2.6.1}$$

called *antipode* such that the following diagram is commutative:

$$
\begin{tikzcd}
& H \otimes H \arrow[rr, "S \otimes \mathbb{1}"] & & H \otimes H \arrow[dr, "\nabla"] & \\
H \arrow[ur, "\Delta"] \arrow[rr, "\epsilon"] \arrow[dr, "\Delta"'] & & \mathbb{K} \arrow[rr, "\eta"] & & H \\
& H \otimes H \arrow[rr, "\mathbb{1} \otimes S"'] & & H \otimes H \arrow[ur, "\nabla"'] &
\end{tikzcd}
$$

where $\Delta, \nabla, \epsilon, \eta$ are the coproduct, the product, the counit and the unit of H.
In the Sweedler's notation this can be concisely written as

$$S\left(c_{(1)}\right) c_{(2)} = c_{(1)} S\left(c_{(2)}\right) = \epsilon\left(c\right) \tag{2.6.2}$$

for every c in H.
Clearly an Hopf algebra will be called commutative (cocommutative) if it is commutative (cocommutative) as an algebra (coalgebra).
Given two Hopf algebras H and H', an homomorphism between H and H' is a bialgebra morphism $\phi : H \to H'$ which is consistent with the antipodes S and S':

$$\phi\left(S\left(x\right)\right) = S'\left(\phi\left(x\right)\right) \tag{2.6.3}$$

for every $x \in H$.
A bijective Hopf algebra morphism will be called isomorphism, and two Hopf algebra connected by such a morphism will be called isomorphic.
It is possible to prove that S is an anti-homomorphism of H in H, that is for every $x, y \in H$

$$S\left(xy\right) = S\left(y\right) S\left(x\right). \tag{2.6.4}$$

Thus, in particular, S is certainly an homomorphism if H is commutative.

Anyhow, S^2 is always an homomorphism, in fact for example we immediately find that

$$S^2(xy) = S(S(xy)) = S(S(y)S(x)) = S^2(x)S^2(y) \tag{2.6.5}$$

and similarly for the other axioms.
Finally, S is an automorphism, i.e. an isomorphism $S : H \to H$, if it is an invertible application.
A very important case is that when $S^2 = \mathbb{1}$; if this happens the Hopf algebra H is said to be involutive and the algebraic sector of H with antipode S (renamed involution) is called *-algebra. This is the case of the algebra of observables in quantum mechanics, where the involution is given by the linear application that sends any operator in its adjoint.
One of the most interesting aspects of Hopf's algebra theory is that it can be revised as a sort of generalization of the more elementary group theory, from which it inherits many results and properties.
As in the case of algebras and coalgebras we can naturally define the concept of Hopf subalgebra A of an Hopf algebra H: it is a sub-bialgebra of H such that the antipode S maps A in A.
In other words, an Hopf subalgebra A of H is a vector subspace of H which is also an Hopf algebra, with product, co-product, unit, counit and antipode given by the restrictions of the corresponding maps of H to A.
What makes this definition richer than before is the possibility of introducing the notion of a normal Hopf subalgebra, in analogy with the case of normal subgroups of a given group G.
Given an Hopf algebra H, a subalgebra A is right normal if $\mathrm{ad}_r(h)(A) \subseteq A$ for every $h \in H$, where the *right adjoint* ad_r is defined in the Sweedler's notation by

$$\mathrm{ad}_r(h)(a) = S\left(h_{(1)}\right) a h_{(2)} \tag{2.6.6}$$

for every $a \in A$ and $h \in H$.
Note the analogy with the conjugation operation $c_g : h \mapsto g^{-1}hg$ which defines the normality of a subgroup N of a group G (that is N is normal if c_g sends N to N).
In a similar way an Hopf subalgebra A is called left normal if $\mathrm{ad}_l(h)(A) \subseteq A$, where this time the *left adjoint* ad_l acts on A according to

$$\mathrm{ad}_l(h)(a) = h_{(1)} a S\left(h_{(2)}\right). \tag{2.6.7}$$

If A is at the same time a right and left normal Hopf subalgebra, it will be called a normal subalgebra of H.
Note that if S is bijective the two conditions are equivalent, for example if H is commutative or cocommutative.
Let us now define the subset $A^+ = \text{Ker}\left(\epsilon|_A\right)$ of A; we have that the subset HA^+ of H composed by the products of the type ha, $h \in H$ and $a \in A^+$ is an Hopf ideal of H, i.e. a vector subspace that is a bialgebra ideal and is invariant under the action of the antipode S.
Consequently the subspace H/HA^+ is an Hopf algebra, and the structure of H can be reconstructed starting from those of H/HA^+ and A itself.
This is very reminiscent to what happens in group theory, although not identical. But if $A = A^+$ we have that the vector space on H/A is in fact an Hopf algebra through the quotient maps inherited in a canonical way by H itself.
The analogy with groups can be fully understood by noting that the axioms that establish their structure can be expressed by commutative diagrams identical to those characterizing the properties of an Hopf algebra. Specifically, it is sufficient to replace a set G (the group) with a vector space H (the Hopf algebra) and the set containing a single point with the field $\mathbb{K}$.
In this case the counit is defined in an obvious way as the constant map that sends an element $g \in G$ in the single element of the codomain, while the coproduct is given by the diagonal map $g \mapsto (g, g)$.
The product obviously represents the product in the group while the unit represents the neutral element of the group with respect to this product.
Finally, the antipode S is the inverse operator.
A very important example of Hopf algebra can be built from the universal enveloping algebra (2.3).
Recalling equations (2.4.2) and (2.4.3), it's not hard to verify that, given a vector space V over a field $\mathbb{K}$, the tensor algebra $\mathcal{T}(V)$ is in fact a bialgebra. If we now define the map $S : \mathcal{T}(V) \to \mathcal{T}(V)$ specified by

$$S(v) = -v, \quad \forall v \in V \tag{2.6.8}$$

and extended through linearity over the entire space, we recognize that the tensor algebra is indeed an Hopf algebra, with antipode S.
However, we showed that given a Lie algebra $\mathfrak{l}$, the universal enveloping algebra $U(\mathfrak{l})$ is obtained as a quotient from the tensor algebra $\mathcal{T}(\mathfrak{l})$; as a consequence it inherits an Hopf algebra structure from it, with the same (induced) coproduct, counit and antipode.

It's important to note that these two Hopf algebras are both cocommutative, i.e., they both have **primitive** coproducts, while in general are not commutative ($\mathcal{T}(V)$ is commutative if and only if V has dimension 0 or 1 over $\mathbb{K}$, $U(\mathfrak{l})$ is commutative if and only if $\mathfrak{I}$ is abelian).
We already mentioned a concrete example of such an Hopf algebra, which is indeed a very familiar one, even though its Hopf algebra structure is rarely referred to.
We will discuss it thoroughly in the next section.

2.7 The Poincaré algebra

The Poincaré algebra $\mathcal{P}$ is defined as the Lie algebra of the Poincaré group, composed by every coordinate transformation that leaves invariant the distance

$$x^\mu y_\mu = g_{\mu\nu} x^\mu y^\nu \tag{2.7.1}$$

between two points x, y of the Minkowski space-time $\mathcal{M}$, where $g_{\mu\nu}$ is the well-known metric tensor.
This group of symmetries can be understood as a semi-direct product $T^4 \rtimes O(1,3)$ of the quadri-dimensional translation group T^4 and the non-abelian group $O(1,3)$ of the Lorentz transformations, with composition law given by

$$g_1 g_2 = (a_1 + \Lambda_1 a_2, \Lambda_1 \Lambda_2) \tag{2.7.2}$$

with $g_i = (a_i, \Lambda_i)$.
Its Lie algebra is precisely the Poincaré algebra, which is composed of 10 **generators**, or base's elements, satisfying the following commutation rules

$$\begin{aligned}
&\left[P_\mu, P_\nu\right] = 0, && \left[R_i, P_0\right] = 0, && \left[R_i, P_j\right] = \epsilon_{ijk} P_k \\
&\left[R_i, R_j\right] = \epsilon_{ijk} R_k, && \left[R_i, N_j\right] = \epsilon_{ijk} N_k, && \left[N_i, P_0\right] = P_i, \\
&\left[N_i, P_j\right] = \delta_{ij} P_0, && \left[N_i, N_j\right] = -\epsilon_{ijk} R_k,
\end{aligned} \tag{2.7.3}$$

where the operators P_μ are the quadri-dimensional translation generators and R_i, N_i are respectively the generators of rotations and boosts around the direction x_i.

It's worth mentioning the so-called quadratic Casimir of the Poincaré algebra

$$C^2 = P_0^2 - P_1^2 - P_2^2 - P_3^2 \equiv P_0^2 - |\bar{P}|^2 \tag{2.7.4}$$

which, as one can easily verify, commutes with every generator of $\mathcal{P}$.
The physical interpretation of this operator is very familiar: it gives us the dispersion relation of masses of free particles, ensuring its invariance.
In this regard, let us recall that the Poincaré group is the group of symmetries of Einstein's theory of relativity, that is the group of transformations that allow passing from one inertial reference frame to another. The nevralgic point of the theory is that all physical equations are **covarianti** with respect to the Poincaré group, that is they are left unchanged (in form) by the elements of the group.
In particular, there is a physical quantity that assumes the same value in all inertial reference frames: the speed of light in the vacuum c.
Here and in the following chapters we will tacitly assume to work in natural units with $c = 1$, but it is good to remember for future utility (see the galilean limit discussion in chapter (8)) that the Poincaré group (and therefore its Lie algebra $\mathcal{P}$) always depends on the parameter c.
It's now convenient to consider the universal enveloping algebra $U(\mathcal{P})$ of the Poincaré algebra, which as we have seen is an Hopf algebra.
In particular we know that $U(\mathcal{P})$ is generated as an algebra by $\mathcal{P}$ itself, so to make its structure explicit it is sufficient to define coproducts, counit and antipodes on the Poincaré algebra generators.
Thus we get the cocommutative Hopf algebra[2] specified by the coproducts

$$\begin{aligned}
\Delta\left(P_\mu\right) &= P_\mu \otimes 1 + 1 \otimes P_\mu, \\
\Delta\left(R_i\right) &= R_i \otimes 1 + 1 \otimes R_i, \\
\Delta\left(N_i\right) &= N_i \otimes 1 + 1 \otimes N_i,
\end{aligned} \tag{2.7.5}$$

the counit actions

$$\begin{aligned}
\epsilon\left(P_\mu\right) &= 0, \\
\epsilon\left(R_i\right) &= 0, \\
\epsilon\left(N_i\right) &= 0
\end{aligned} \tag{2.7.6}$$

[2]Here and in the following with slight abuse of notation we will always identify the Poincaré algebra $\mathcal{P}$ with its universal enveloping algebra $U(\mathcal{P})$.

and the antipodes

$$\begin{aligned} S\left(P_\mu\right) &= -P_\mu, \\ S\left(R_i\right) &= -R_i, \\ S\left(N_i\right) &= -N_i. \end{aligned} \tag{2.7.7}$$

There are other examples of Hopf algebras that are neither commutative nor cocommutative, often referred to (though improperly) as quantum groups [28,29].
These algebras are extremely important in the realm of non-commutative geometry and quantum gravity, and were originally introduced precisely to study some type of "generalized" or "quantized" spaces (see for instance the so-called κ-Minkowski space-time we will discuss in depth later on) through an appropriate, deformed algebra of functions.
In general there exist various mathematical procedures, for instance the one due to Drinfel'd and Jimbo [30–33], that allow to deform a cocommutative Hopf algebra into a non-cocommutative one.
We will not analyze in full details these formal procedures, instead we will just point out that these Hopf algebras are constructed in such a way that they reduce to the starting universal enveloping algebras when one or more parameters on which they depend on tend to a prefixed limit.
In particular the most famous deformation of the Poincaré algebra is a quantum group called κ-Poincaré algebra $\kappa\mathcal{P}$ [34,35].
In our case the κ-Poincaré algebra depends on an additional parameter (besides the speed of light c) that we will denote by $\ell \equiv \frac{1}{\kappa}$, which has the dimensions of the inverse of an energy and therefore in natural units those of a length.
We will devote the next and final paragraph of this mathematical introduction to present some of the key properties of the κ-Poincaré algebra, as well as its physical interpretation.

2.8 The k-Poincaré algebra

The κ-Poincaré algebra is a particular deformation of the standard Poincaré algebra which enforces the **doubly special relativity** (DSR) principle [8–13].

In other words replacing the Poincaré algebra with this new, deformed one, introduces a new **invariant** energy κ alongside the old invariant speed of light c, usually thought to be equal or close to the Plank energy E_p and interpreted as the maximum energy attainable by a particle.
This is very analogous to what happens in the transition from the Galilei relativity to the Einstein relativity, where the deformation parameter is the speed of light c itself.
In order to achieve this result the old commutators of the Poincaré algebra has to be deformed by κ in a non-linear way.
Even more significantly the κ-Poincaré algebra will feature a new striking property, that is a non primitive coalgebra structure.
In this paragraph we will show and discuss the form of both the algebra and coalgebra sectors of the κ-Poincaré algebra as they will be used in the rest of the thesis.
Since we will mostly use natural units, it's useful to introduce a new deformation length scale $\ell = \frac{1}{\kappa}$ which we can interpret as a quantity equal or close to the Planck length ℓ_P.
Written in the so-called **bi-cross-product basis** [35], the four-dimensional κ- Poincaré algebra reads:

$$\begin{aligned}
&[P_\mu, P_\nu] = 0, && [R_i, P_0] = 0, \\
&[R_i, P_j] = i\epsilon_{ijk}P_k, && [R_i, R_j] = i\epsilon_{ijk}R_k, \\
&[R_i, N_j] = -i\epsilon_{ijk}N_k, && [N_i, P_0] = iP_i, \\
&[N_i, P_j] = i\delta_{ij}\left(\frac{1-e^{-2\ell P_0}}{2\ell} + \frac{\ell}{2}\vec{P}^2\right) - i\ell P_i P_j, && [N_i, N_j] = -i\epsilon_{ijk}R_k,
\end{aligned} \tag{2.8.1}$$

with a deformed mass Casimir

$$m^2 = \frac{2}{\ell}\sinh^2\frac{\ell}{2}P_0 - e^{\ell P_0}\vec{P}^2 \tag{2.8.2}$$

which is left invariant by the modified generators in (2.8.1) (see (2.7.4) in comparison).
The second important difference between the κ-Poincaré algebra and the Poincaré algebra, as aforementioned, lays in its non-trivial coalgebra structure.
As we saw the Poincarè algebra can be equipped with a coalgebra structure specified by the coproducts

$$\begin{aligned}\Delta\left(P_0\right) &= P_0 \otimes 1 + 1 \otimes P_0 \\ \Delta\left(P_i\right) &= P_i \otimes 1 + 1 \otimes P_i \\ \Delta\left(N_i\right) &= N_i \otimes 1 + 1 \otimes N_i \\ \Delta\left(R_i\right) &= R_i \otimes 1 + 1 \otimes R_i.\end{aligned} \tag{2.8.3}$$

The common interpretation of this is that the coproducts dictate the form of the Leibniz rule for the generators, that is how they act on products of functions.
In our case we have

$$X \triangleright (fg) = (X \triangleright f)\, g + f\,(X \triangleright g) \tag{2.8.4}$$

for every generator X of the Poincaré algebra.
This also reflects the fact that in the standard special relativity the conservation law describing the point-interactions of particles is linear with respect to the momenta, i.e.

$$p_0 + k_0 = 0, \qquad p_i + k_i = 0 \tag{2.8.5}$$

in the case of two particles with momenta p and k.
In a similar way the (trivial) antipodes (2.7.7) of the Poincaré algebra simply express what inversion rule is compatible with this linear composition law, or, explicitly, that the inverse of a momenta p is simply $-p$.
On the other hand the coalgebra section and antipodes of the κ-Poincaré algebra turn out to be naturally non-trivial. In the bi-cross-product basis (2.8.1) the coproducts are given by

$$\begin{aligned}\Delta\left(P_0\right) &= P_0 \otimes 1 + 1 \otimes P_0 \\ \Delta\left(P_i\right) &= P_i \otimes 1 + e^{-\ell P_0} \otimes P_i \\ \Delta\left(N_i\right) &= N_i \otimes 1 + e^{-\ell P_0} \otimes N_i + \ell \epsilon_{ijk} P_j \otimes R_k \\ \Delta\left(R_i\right) &= R_i \otimes 1 + 1 \otimes R_i,\end{aligned} \tag{2.8.6}$$

with antipodes

$$\begin{aligned}S\left(P_0\right) &= -P_0, \\ S\left(P_i\right) &= -e^{lP_0} P_i, \\ S\left(R_i\right) &= -R_i, \\ S\left(N_i\right) &= -e^{lP_0} N_i + l \epsilon_{ijk} e^{lP_0} P_j R_k.\end{aligned} \tag{2.8.7}$$

As a consequence the usual conservation law no longer holds, instead it has to be modified as

$$p_0 + k_0 = 0, \qquad p_i + e^{-\ell p_0} k_i = 0. \tag{2.8.8}$$

Looking more closely to (2.8.2) and (2.8.8), one immediately notes that the deformation effects resulting from these equations should become more and more evident when the energy of the particles involved becomes higher and higher.
Thus, if the κ-Poincaré relativistic theory was to be taken as it is on every scale (i.e. if it would be an **universal** relativistic theory), then its deviation from the standard special or galilean relativity should be most prominent for macroscopic bodies, which is absurd.
This is the aforementioned soccer-ball problem.
Several proposals have been raised in the past years to address this contradiction, see in particular [14–16].
However, we believe that the proper route to follow in order to better understand the soccer-ball problem is to study more in depth the relativistic properties of a composite system in the κ-Poincaré framework, a topic that has received little to none attention in the literature.
In particular the goal of the next chapters will be to show that through an appropriate definition of the composite system coordinates it is possible in principle to build a deformed symmetry algebra such that the deformation parameter scales with the number of particles involved.
Before doing that, we find useful to conclude this introduction by briefly discussing the space-time naturally connected with the κ-Poincaré algebra, that is the so-called **κ-Minkowski** space-time.
From now on for the sake of simplicity we will work in 1+1-dimensions, thus our single particle κ-Poincaré algebra will be composed only by the three generators P_0, P_1 and N.
As in standard quantum mechanics the time and spatial translation generators can be represented as derivatives with respect to a time coordinate x_0 and a space coordinate x_1, forming a flat $\mathbb{R}^2$ space-time:

$$P_0 = i\frac{\partial}{\partial x_0}, \qquad P_1 = -i\frac{\partial}{\partial x_1}. \tag{2.8.9}$$

The difference here is that the coordinates x_0 and x_1, in an operator sense, do no longer commute between themselves.
The easiest way to see this is to act on the commutator $[x_1, x_0]$ with P_1 and using its coproduct from (2.8.6):

$$\begin{aligned}
P_1 \triangleright [x_1, x_0] &= P_1 \triangleright x_1 x_0 - P_1 \triangleright x_0 x_1 \\
&= -ix_0 - \left(e^{-\ell P_0} \triangleright x_0\right) P_1 \triangleright x_1 \\
&= -ix_0 + i\left(1 - \ell P_0\right) \triangleright x_0 \\
&= \ell \\
&= i\ell P_1 \triangleright x_1,
\end{aligned} \tag{2.8.10}$$

thus

$$[x_1, x_0] = i\ell x_1. \tag{2.8.11}$$

This is usually interpreted as a way to enforce non-commutativity in the space-time itself, even though in a very preliminary sense.
At the same time it's easy to see that this non-commutativity must have strong consequences on the algebra of functions over the space-time.
In fact, while for standard spaces like the Minkowski space-time this algebra turns out to be a commutative algebra quite naturally, now we deal with a non-commutative one [36], as a direct consequence of equation (2.8.11).
Indeed, given a generic function $f(x_0, x_1)$ over the κ-Minkowski space-time, we recognize immediately that the order whereby we write the coordinates x_0 and x_1 themselves is essential.
In general we will always work with the so-called **time-to-the-right** prescription, that is, as the name suggests, putting the time coordinates always to the right of the space coordinates.
In particular we will often use the set of κ-Minkowski's time-to-the-right plane waves as a base of the algebra of functions, that is the family $\left\{e^{ikx}\right\}_{k\in\mathbb{R}}$ defined as

$$e^{ikx} = e^{ik_1 x_1} e^{-ik_0 x_0}. \tag{2.8.12}$$

The action of the symmetry generators on the algebra of functions carries a lot of informations on the symmetry algebra itself, thus we will often make use of the action of the generators on the plane waves to grasp some of these informations.
As a final remark on this topic, we find useful to stress that the plane waves family is still closed with respect to the product as in the commutative case, i.e. that the product of two plane waves is still a plane wave.
To see this let us explicitly perform the calculation as

$$
\begin{aligned}
e^{ipx}e^{iqx} &= e^{ip_1x_1}e^{-ip_0x_0}e^{iq_1x_1}e^{-iq_0x_0} \\
&= e^{ip_1x_1}\left[\sum_{n=1}^{\infty}\frac{(-ip_0)^n}{n!}x_0^n\right]\left[\sum_{m=1}^{\infty}\frac{(iq_1)^m}{m!}x_1^m\right]e^{-iq_0x_0} \\
&= e^{ip_1x_1}\left[\sum_{n,m=1}^{\infty}\frac{(-ip_0)^n(iq_1)^m}{n!m!}x_0^{n-1}x_0x_1x_1^{m-1}\right]e^{-iq_0x_0} \\
&= e^{ip_1x_1}\left[\sum_{n,m=1}^{\infty}\frac{(-ip_0)^n(iq_1)^m}{n!m!}x_0^{n-1}x_1(x_0-i\ell)x_1^{m-1}\right]e^{-iq_0x_0} \\
&= e^{ip_1x_1}\left[\sum_{n,m=1}^{\infty}\frac{(-ip_0)^n(iq_1)^m}{n!m!}x_0^{n-1}x_1^2(x_0-2i\ell)x_1^{m-2}\right]e^{-iq_0x_0} \\
&= e^{ip_1x_1}\left[\sum_{n,m=1}^{\infty}\frac{(-ip_0)^n(iq_1)^m}{n!m!}x_0^{n-1}x_1^m(x_0-im\ell)\right]e^{-iq_0x_0} \\
&= e^{ip_1x_1}\left[\sum_{n,m=1}^{\infty}\frac{(-ip_0)^n(iq_1)^m}{n!m!}x_1^m(x_0-im\ell)^n\right]e^{-iq_0x_0} \\
&= e^{ip_1x_1}\left[\sum_{m=1}^{\infty}\frac{(iq_1)^m}{m!}x_1^me^{-\ell mp_0}\right]e^{-ip_0x_0}e^{-iq_0x_0} \\
&= e^{i\left(p_1+e^{-\ell p_0}q_1\right)x_1}e^{-i(p_0+q_0)x_0} \\
&= e^{i(p\oplus q)x},
\end{aligned}
\tag{2.8.13}
$$

where throughout the middle passages of (2.8.13) we used the κ-Minkowski commutation relations (2.8.11) in order to invert the position of $e^{-ip_0x_0}$ and $e^{iq_1x_1}$.

Thus composing two plane waves of momenta p and q produces a plane wave with momenta $\left(p_1+e^{-\ell p_0}q_1, p_0+q_0\right)$, which is coherent with the composition law (2.8.8).

Indeed one can immediately see that the translation generators have standard action on the plane waves:

$$
P_\mu \triangleright e^{ikx} = k_\mu e^{ikx}, \tag{2.8.14}
$$

so that we can interpret (2.8.13) as a direct result of the form of the coproduct $\Delta(P_1)$.

The last thing we want to point out in the conclusion of this chapter, is that the Heisenberg algebra our "quantum" coordinates satisfy with their dual translation generators is deformed itself as

$$
\begin{aligned}
&[P_0, x_0] = i, && [P_0, x_1] = 0, \\
&[P_1, x_0] = -i\ell P_1, && [P_1, x_1] = -i.
\end{aligned}
\tag{2.8.15}
$$

To prove this we can easily compute the third commutator, the deformed one, that is:

$$
\begin{aligned}
[P_1, x_0] \triangleright f(x) &= P_1 \triangleright (x_0 f(x)) - x_0 P_1 \triangleright f(x) \\
&= \left(e^{-\ell P_0} \triangleright x_0\right) P_1 \triangleright f(x) - x_0 P_1 \triangleright f(x) \\
&= -i\ell P_1 \triangleright f(x)\,.
\end{aligned}
\tag{2.8.16}
$$

These commutators provide us with a differential representation for the boost generator N:

$$
N = x_1 \left(\frac{1 - e^{-2\ell P_0}}{2\ell} + \frac{\ell}{2} P_1^2 \right) - x_0 P_1. \tag{2.8.17}
$$

It's easy to verify that the relevant commutators in (2.8.1) are satisfied by this operator.
We will often use the Heisenberg algebra (2.8.15) as a guide to refer to during our explorations.
The main purpose of this thesis is then to generalize these constructions to composite systems.
For the sake of simplicity we will mostly consider two particle systems; furthermore we will always require these two particles to be identical, so that they have the same mass[3].
This request makes it easier to define the center of mass coordinates and the symmetry algebra of the systems.
In the next section we will follow a purely κ-Poincaré inspired approach to the question, showing why exactly the soccer-ball problem can't be addressed in that playground.
Then we will discuss our novel proposals, where conversely the soccer-ball problem could actually find a proper solution.

[3] We will briefly discuss the case of more than two eventually different particles in chapter 7.

Chapter 3

Composite system with deformed center of mass coordinates

3.1 The case of two particles of equal mass

Let us consider the standard composition of momenta of two identical particles A and B in the k-Poincaré algebra in 1+1-dimensions:

$$\begin{aligned} P_0^T &= P_0^A + P_0^B, \\ P_1^T &= P_1^A + e^{-\ell P_0^A} P_1^B. \end{aligned} \tag{3.1.1}$$

We want to search for a couple of coordinates operators x_0^T, x_1^T, which could be interpreted as a generalization of the center of mass coordinates, such as P_μ^T and x_ν^T are dual, i.e. the set of operators $\left\{P_\mu^T, x_\nu^T\right\}$ encloses an (eventually deformed) Heisenberg Algebra.
The standard choice

$$\begin{aligned} x_0^T &= \frac{x_0^A + x_0^B}{2}, \\ x_1^T &= \frac{x_1^A + x_1^B}{2} \end{aligned} \tag{3.1.2}$$

is clearly inadequate in this context, since

$$\begin{aligned}\left[P_1^T, x_1^T\right] &= \left[P_1^A + e^{-\ell P_0^A} P_1^B, \frac{x_1^A + x_1^B}{2}\right] \\ &= -\frac{i}{2}\left(1 + e^{-\ell P_0^A}\right).\end{aligned} \tag{3.1.3}$$

Nonetheless this calculation suggests us to consider a slightly deformed version of the coordinate x_1^T above, i.e.

$$x_1^T = \frac{x_1^A + e^{\ell P_0^A} x_1^B}{2} \tag{3.1.4}$$

which correctly gives

$$\left[P_1^T, x_1^T\right] = -i. \tag{3.1.5}$$

On the other hand, while the commutators between P_0^T and x_μ^T are respectively i and 0, the commutator $\left[P_1^T, x_0^T\right]$ isn't closed:

$$\begin{aligned}\left[P_1^T, x_0^T\right] &= \left[P_1^A + e^{-\ell P_0^A} P_1^B, \frac{x_0^A + x_0^B}{2}\right] \\ &= -i\frac{\ell}{2}P_1^A + \frac{1}{2}\left[e^{-\ell P_0^A}, x_0^A\right] P_1^B - i\frac{\ell}{2}e^{-\ell P_0^A} P_1^B \\ &= -i\ell\left(\frac{P_1^A}{2} + e^{-\ell P_0^A} P_1^B\right).\end{aligned} \tag{3.1.6}$$

In order to maintain the nice commutation relations between P_0^T and x_0^T, we could try to modify the latter by adding a function of x_1^A and/or x_1^B, since trivially $\left[P_0^A, x_1^A\right] = \left[P_0^B, x_1^B\right] = 0$.
The final choice is given by the operator

$$x_0^T = \frac{x_0^A + x_0^B + \frac{\ell}{2}\left\{x_1^A, P_1^A\right\}}{2}, \tag{3.1.7}$$

which gives the correct commutator with P_1^T:

$$\begin{aligned}\left[P_1^T, x_0^T\right] &= -i\ell\left(\frac{P_1^A}{2} + e^{-\ell P_0^A} P_1^B\right) - i\frac{\ell}{2}P_1^A \\ &= -i\ell\left(P_1^A + e^{-\ell P_0^A} P_1^B\right) \\ &= -i\ell P_1^T.\end{aligned} \tag{3.1.8}$$

In conclusion these novel center of mass coordinates

$$\begin{aligned} x_0^T &= \frac{x_0^A + x_0^B + \frac{\ell}{2}\left\{x_1^A, P_1^A\right\}}{2}, \\ x_1^T &= \frac{x_1^A + e^{\ell P_0^A} x_1^B}{2} \end{aligned} \tag{3.1.9}$$

satisfy the commutation relation

$$\begin{aligned} \left[x_1^T, x_0^T\right] &= \frac{1}{4}\left(\left[x_1^A, x_0^A + \ell x_1^A P_1^A\right] + \left[e^{\ell P_0^A} x_1^B, x_0^A + x_0^B\right]\right) \\ &= i\frac{\ell}{4}x_1^A + i\frac{\ell}{4}x_1^A + i\frac{\ell}{4}e^{\ell P_0^A} x_1^B + i\frac{\ell}{4}e^{\ell P_0^A} x_1^B \\ &= i\ell\left(\frac{x_1^A + e^{\ell P_0^A} x_1^B}{2}\right) \\ &= i\ell x_1^T \end{aligned} \tag{3.1.10}$$

and coherently lead us to a k-deformed Heisenberg algebra

$$\begin{aligned} \left[P_0^T, x_0^T\right] &= i, & \left[P_0^T, x_1^T\right] &= 0, \\ \left[P_1^T, x_0^T\right] &= -i\ell P_1^T, & \left[P_1^T, x_1^T\right] &= -i. \end{aligned} \tag{3.1.11}$$

The next step should be to define the relative coordinates of the system[1]. From what we learned about the center of mass case, the most educated proposal is

$$\begin{aligned} x_0^R &= x_0^A - x_0^B - \frac{\ell}{2}\left\{x_1^A, P_1^A\right\}, \\ x_1^R &= x_1^A - e^{\ell P_0^A} x_1^B. \end{aligned} \tag{3.1.12}$$

It's quite trivial to see that x_1^T and x_1^R commutes between themselves; on the other hand we find that

$$\begin{aligned} \left[x_1^R, x_0^T\right] &= \frac{1}{2}\left(\left[x_1^A, x_0^A + \ell x_1^A P_1^A\right] - \left[e^{\ell P_0^A} x_1^B, x_0^A + x_0^B\right]\right) \\ &= i\frac{\ell}{2}x_1^A + i\frac{\ell}{2}x_1^A - i\frac{\ell}{2}e^{\ell P_0^A} x_1^B - i\frac{\ell}{2}e^{\ell P_0^A} x_1^B \\ &= i\ell\left(x_1^A - e^{\ell P_0^A} x_1^B\right) \\ &= i\ell x_1^R. \end{aligned} \tag{3.1.13}$$

[1] Note that we start with 3+3 generators between single particles translations and boosts, so the final number of generators must still be equal to 6.

More interesting are the commutators involving x_0^R. The commutator with x_0^T is obviously zero, but the unexpected result one finds is that x_0^R actually commutes with **all** other coordinates.
This is very different from what we will see in the next chapters, and as a consequence we deduce that x_0^R doesn't play any major role in this setup. Explicitly we have that

$$\begin{aligned}\left[x_1^T, x_0^R\right] &= \frac{1}{4}\left(\left[x_1^A, x_0^A - \ell x_1^A P_1^A\right] + \left[e^{\ell P_0^A} x_1^B, x_0^A - x_0^B\right]\right)\\ &= i\frac{\ell}{4}x_1^A - i\frac{\ell}{4}x_1^A + i\frac{\ell}{4}e^{\ell P_0^A}x_1^B - i\frac{\ell}{4}e^{\ell P_0^A}x_1^B\\ &= 0,\end{aligned} \tag{3.1.14}$$

and

$$\begin{aligned}\left[x_1^R, x_0^R\right] &= \frac{1}{2}\left(\left[x_1^A, x_0^A - \ell x_1^A P_1^A\right] - \left[e^{\ell P_0^A} x_1^B, x_0^A - x_0^B\right]\right)\\ &= i\frac{\ell}{2}x_1^A - i\frac{\ell}{2}x_1^A - i\frac{\ell}{2}e^{\ell P_0^A}x_1^B + i\frac{\ell}{2}e^{\ell P_0^A}x_1^B\\ &= 0.\end{aligned} \tag{3.1.15}$$

It's easy now to define the relative translation generators as

$$\begin{aligned}P_0^R &= \frac{P_0^A - P_0^B}{2},\\ P_1^R &= \frac{P_1^A - e^{-\ell P_0^A}P_1^B}{2},\end{aligned} \tag{3.1.16}$$

so that we can complete the deformed Heisenberg algebra (3.1.11) with the remaining commutators (we left the easy check of these results to the reader)

$$\begin{aligned}\left[P_0^R, x_0^R\right] &= i, & \left[P_0^R, x_1^R\right] &= 0,\\ \left[P_0^T, x_0^R\right] &= 0, & \left[P_0^T, x_1^R\right] &= 0,\\ \left[P_1^R, x_0^R\right] &= 0, & \left[P_1^R, x_1^R\right] &= -i,\\ \left[P_1^T, x_0^R\right] &= 0, & \left[P_1^T, x_1^R\right] &= 0\end{aligned} \tag{3.1.17}$$

and

$$\begin{aligned}\left[P_0^R, x_0^T\right] &= 0, & \left[P_0^R, x_1^T\right] &= 0,\\ \left[P_1^R, x_0^T\right] &= -i\ell P_1^R, & \left[P_1^R, x_1^T\right] &= 0.\end{aligned} \tag{3.1.18}$$

It's important to note that the relative spatial coordinate and translation generator perfectly fit in a κ-Poincaré-Heisenberg translation algebra sector together with the total coordinates and generators, while at the same time the relative time coordinate and translation generator behave as undeformed classical operators.
The last step is to define the boost generators:

$$\begin{aligned} N^T &= N^A + e^{-\ell P_0^A} N^B, \\ N^R &= N^A - e^{-\ell P_0^A} N^B. \end{aligned} \tag{3.1.19}$$

As we will see the commutators they obey won't in general be as nice as the previous ones. The nicest ones are certainly those involving only the total generators:

$$\left[N^T, P_0^T\right] = iP_1^T, \tag{3.1.20}$$

which is very trivial, and

$$\begin{aligned} \left[N^T, P_1^T\right] &= \left[N^A, P_1^A\right] + \left[N^A, e^{-\ell P_0^A}\right] P_1^B + e^{-2\ell P_0^A}\left[N^B, P_1^B\right] \\ &= i\left(\frac{1 - e^{-2\ell P_0^A}}{2\ell} - \frac{\ell}{2} {P_1^A}^2\right) - i\ell e^{-\ell P_0^A} P_1^A P_1^B \\ &\quad + ie^{-2\ell P_0^A}\left(\frac{1 - e^{-2\ell P_0^B}}{2\ell} - \frac{\ell}{2} {P_1^B}^2\right) \\ &= i\left(\frac{1 - e^{-2\ell P_0^T}}{2\ell} - \frac{\ell}{2} {P_1^T}^2\right) \end{aligned} \tag{3.1.21}$$

as expected.
Not surprisingly we have also

$$\begin{aligned} \left[N^T, P_0^R\right] &= iP_1^R, \\ \left[N^R, P_0^T\right] &= iP_1^R, \\ \left[N^R, P_0^R\right] &= iP_1^T. \end{aligned} \tag{3.1.22}$$

On the other hand the remaining commutators cannot be exactly related to the κ-Poincaré algebra ones, even though they show strong similarity and, most importantly, still preserve the deformation parameter ℓ.

Let us then start with

$$\begin{aligned}
\left[N^T, P_1^R\right] &= \left[N^A, P_1^A\right] - \left[N^A, e^{-\ell P_0^A}\right] P_1^B - e^{-2\ell P_0^A}\left[N^B, P_1^B\right] \\
&= \frac{i}{2}\left(\frac{1-e^{-2\ell P_0^A}}{2\ell} - \frac{\ell}{2}P_1^{A\,2}\right) + i\frac{\ell}{2}e^{-\ell P_0^A}P_1^A P_1^B \\
&\quad - \frac{i}{2}e^{-2\ell P_0^A}\left(\frac{1-e^{-2\ell P_0^B}}{2\ell} - \frac{\ell}{2}P_1^{B\,2}\right) \\
&= i\left(\frac{1-2e^{-2\ell P_0^A} + e^{-2\ell P_0^A}e^{-2\ell P_0^B}}{4\ell} - \frac{\ell}{4}P_1^{A\,2}\right. \\
&\qquad \left. + \frac{\ell}{4}e^{-2\ell P_0^A}P_1^{B\,2} + \frac{\ell}{2}e^{-\ell P_0^A}P_1^A P_1^B\right).
\end{aligned} \tag{3.1.23}$$

We can now replace the single particle generators with the total and relative ones by using the equations

$$\begin{aligned}
P_0^A &= \frac{P_0^T}{2} + P_0^R, \qquad & P_0^B &= \frac{P_0^T}{2} - P_0^R, \\
P_1^A &= \frac{P_1^T}{2} + P_1^R, \qquad & e^{-\ell P_0^A}P_1^B &= \frac{P_1^T}{2} - P_1^R
\end{aligned} \tag{3.1.24}$$

which the reader can easily verify.
This leads us to the final form

$$\begin{aligned}
\left[N^T, P_1^R\right] = i&\left(\frac{1-2e^{-\ell P_0^T}e^{-2\ell P_0^R} + e^{-2\ell P_0^T}}{4\ell} - \frac{\ell}{2}P_1^T P_1^R\right. \\
&\left. + \frac{\ell}{8}P_1^{T\,2} - \frac{\ell}{2}P_1^{R\,2}\right).
\end{aligned} \tag{3.1.25}$$

In a similar fashion one finds

$$\begin{aligned}
\left[N^R, P_1^T\right] = i&\left(\frac{1-2e^{-\ell P_0^T}e^{-2\ell P_0^R} + e^{-2\ell P_0^T}}{2\ell} - \ell P_1^T P_1^R\right. \\
&\left. - \frac{\ell}{4}P_1^{T\,2} + \ell P_1^{R\,2}\right)
\end{aligned} \tag{3.1.26}$$

and

$$\left[N^R, P_1^R\right] = i\left(\frac{1-e^{-2\ell P_0^T}}{4\ell} - \ell P_1^{R\,2}\right). \tag{3.1.27}$$

As a last ingredient one can search for the mass Casimir of the algebra, which indeed can easily be defined as

$$C^T = C^A + C^B \tag{3.1.28}$$

where C^A and C^B are the single particle Casimrs, and is quite obviously invariant under both the total and relative boosts.
Explicitly this can be shown to be equal to

$$\begin{aligned} C^T = &\frac{8}{\ell^2}\sinh^2\frac{\ell}{4}P_0^T - \frac{8}{\ell^2}\sinh^2\frac{\ell}{4}P_0^R + \frac{16}{\ell^2}\cosh^2\frac{\ell}{4}P_0^T\sinh^2\frac{\ell}{4}P_0^R \\ &-\frac{1}{2}e^{\ell P_0^T}e^{\ell P_0^R}\cosh\frac{\ell}{2}P_0^T {P_1^T}^2 - 2e^{\ell P_0^T}e^{\ell P_0^R}\cosh\frac{\ell}{2}P_0^T {P_1^R}^2 \\ &+2e^{\ell P_0^T}e^{\ell P_0^R}\sinh\frac{\ell}{2}P_0^T P_1^T P_1^R. \end{aligned} \tag{3.1.29}$$

Now, one could suggest to just ignore the relative coordinates and generators and focus only on the total part.
However, we have not yet taken into account the coalgebra sector of our two particle algebra.
The result we will show is that even though the commutators between the total generators are exactly κ-Poincaré ones, the coproducts still need the relative generators to be fully defined and thus, as an Hopf algebra, the set $\{P_0^T, P_1^T, N^T\}$ is not properly a κ-Poincaré algebra.
As an example let us consider the generator P_1^T.
To start let us recall the coproduct of the operator $e^{-\ell P_0^A}$, that is

$$\Delta\left(e^{-\ell P_0^A}\right) = e^{-\ell P_0^A}\otimes e^{-\ell P_0^A}. \tag{3.1.30}$$

This can be proven by first writing its Taylor expansion

$$e^{-\ell P_0^A} = 1 - \ell P_0^A + \frac{\ell^2}{2}{P_0^A}^2 + ..., \tag{3.1.31}$$

then by using the coproducts of the powers of P_0^A, which obeys the Leibniz rule:

$$\Delta\left({P_0^A}^n\right) = \sum_{i=0}^{n} {P_0^A}^i \otimes {P_0^A}^{n-i}, \tag{3.1.32}$$

from which equation (3.1.31) follows.

Let us now consider the action of P_1^T on the product of two functions f and g:

$$P_1^T \triangleright fg = \left(P_1^A + e^{-\ell P_0^A} P_1^B\right) \triangleright fg. \tag{3.1.33}$$

By using the single particle coproducts this is equal to

$$\begin{aligned} P_1^T \triangleright fg = & \left(P_1^A \triangleright f\right) g + \left(e^{-\ell P_0^A} \triangleright f\right) P_1^A \triangleright g \\ & + \left(e^{-\ell P_0^A} P_1^B \triangleright f\right) e^{-\ell P_0^A} \triangleright g \\ & + \left(e^{-\ell P_0^A - \ell P_0^B} \triangleright f\right) e^{-\ell P_0^A} P_1^B \triangleright g. \end{aligned} \tag{3.1.34}$$

We are not interested in the final form of P_1^T's coproduct in term of the total and relative generators (easily obtainable by using equations (3.1.24)), our goal here is to point out that the action (3.1.34) clearly doesn't allow us to write

$$\Delta\left(P_1^T\right) = P_1^T \otimes 1 + e^{-\ell P_0^T} \otimes P_1^T \tag{3.1.35}$$

due to the inevitable presence of the relative generators P_i^R.
Thus, as aforementioned, although the total commutators are κ-Poincaré ones, the coproducts are not.
Nonetheless it's worth to note that they actually are if the relative motion is small enough compared to κ, so that we can informally put $P_i^R \approx 0$.

3.2 General case

To conclude this chapter we want to briefly address the more general situation with more than two particles with different masses.
Let us procede step by step by first considering two particles with non-zero masses m_A and m_B.
In the following we will focus only on the center of mass coordinates and their dual translation generators.
The latter are of course still the same as the equal mass case, that is

$$\begin{aligned} P_0^T &= P_0^A + P_0^B, \\ P_1^T &= P_1^A + e^{-\ell P_0^A} P_1^B. \end{aligned} \tag{3.2.1}$$

The dual (spatial) coordinate of P_1^T is easily understood to be

$$x_1^T = \frac{m_A x_1^A + m_B e^{\ell P_0^A} x_1^B}{m_A + m_B}, \tag{3.2.2}$$

so that $\left[P_1^T, x_1^T\right] = -i$.
As in the former case the temporal center of mass coordinate is trickier. What we found is that equation (3.1.7) can be generalized as

$$x_0^T = \frac{m_A x_0^A + m_B x_0^B + \frac{\ell}{2} m_B \left\{x_1^A, P_1^A\right\}}{m_A + m_B}, \tag{3.2.3}$$

so that trivially $\left[P_0^T, x_0^T\right] = -i$ and above all

$$\begin{aligned}
\left[P_1^T, x_0^T\right] &= \frac{1}{m_A + m_B}\Bigg(m_A\left[P_1^A, x_0^A\right] + \ell m_B\left[P_1^A, x_1^A\right]P_1^A \\
&\qquad + m_A\left[e^{-\ell P_0^A}, x_0^A\right]P_1^B + m_B e^{-\ell P_0^A}\left[P_1^B, x_0^B\right]\Bigg) \\
&= -\frac{i\ell}{m_A + m_B}\left(m_A P_1^A + m_B P_1^A + m_A P_1^B + m_B P_1^B\right) \\
&= -i\ell P_1^T.
\end{aligned} \tag{3.2.4}$$

Finally we can compute the commutator between the center of mass spatial and temporal coordinates:

$$\begin{aligned}
\left[x_1^T, x_0^T\right] &= \frac{1}{(m_A + m_B)^2}\Bigg(m_A^2\left[x_1^A, x_0^A\right] + \ell m_A m_B\left[x_1^A, P_1^A\right]x_1^A \\
&\qquad + m_A m_B\left[e^{\ell P_0^A}, x_0^A\right]x_1^B \\
&\qquad + m_B^2 e^{\ell P_0^A}\left[x_1^B, x_0^B\right]\Bigg) \\
&= i\ell\frac{m_A(m_A + m_B)x_1^A + m_B(m_A + m_B)e^{\ell P_0^A}x_1^B}{(m_A + m_B)^2} \\
&= i\ell x_1^T,
\end{aligned} \tag{3.2.5}$$

which not surprisingly turns out to be just a κ-Minkowski one, as in the equal mass case.
Thus we found that the κ deformation is preserved even when the particles composing the system have different masses. What about having more than two particles?
Let us consider for simplicity the case of three particles with of equal mass, and let us start by defining the κ-Poincaré inspired total translation generators:

$$\begin{aligned} P_0^T &= P_0^A + P_0^B + P_0^C, \\ P_1^T &= P_1^A + e^{-\ell P_0^A} P_1^B + e^{-\ell P_0^A} e^{-\ell P_0^B} P_1^C. \end{aligned} \tag{3.2.6}$$

We then define the center of mass coordinates of the system as

$$\begin{aligned} x_0^T &= \frac{x_0^A + x_0^B + x_0^C + \ell \left\{ x_1^A, P_1^A \right\} + \frac{\ell}{2} \left\{ x_1^B, P_1^B \right\}}{3}, \\ x_1^T &= \frac{x_1^A + e^{\ell P_0^A} x_1^B + e^{\ell P_0^A} e^{\ell P_0^B} x_1^C}{3}. \end{aligned} \tag{3.2.7}$$

It's not hard to check that the κ-duality relations between generators and coordinates still hold.
For example we have that

$$\begin{aligned} \left[P_1^T, x_0^T \right] = \frac{1}{3} \Bigg(& \left[P_1^A, x_0^A \right] + 2\ell \left[P_1^A, x_1^A \right] P_1^A + \left[e^{-\ell P_0^A}, x_0^A \right] P_1^B \\ & + e^{-\ell P_0^A} \left[P_1^B, x_0^B \right] + \ell \left[P_1^B, x_1^B \right] P_1^B \\ & + \left[e^{-\ell P_0^A}, x_0^A \right] e^{-\ell P_0^B} P_1^C + e^{-\ell P_0^A} \left[e^{-\ell P_0^B}, x_0^B \right] P_1^C \\ & + e^{-\ell P_0^A} e^{-\ell P_0^B} \left[P_1^C, x_0^C \right] \Bigg) \\ = -i\frac{\ell}{3} \Big(& P_1^A + 2P_1^A + e^{-\ell P_0^A} P_1^B + e^{-\ell P_0^A} P_1^B + e^{-\ell P_0^A} P_1^B \\ & + e^{-\ell P_0^A} e^{-\ell P_0^B} P_1^C + e^{-\ell P_0^A} e^{-\ell P_0^B} P_1^C \\ & + e^{-\ell P_0^A} e^{-\ell P_0^B} P_1^C \Big) = -i\ell P_1^T. \end{aligned} \tag{3.2.8}$$

As a consequence the κ-Minkowski commutation rules are still satisfied.
By putting both of these results together we can finally complete our analysis and consider a general system with N particles with $\{m_1, m_2, ..., m_N\}$ masses.
Let M be the total mass of the system, that is

$$M = \sum_{i=1}^{N} m_i \tag{3.2.9}$$

and let $\left\{P_i^j\right\}$ be the single particle generators, $i = 0, 1$, $j = 1, 2, ...N$, such that

$$\begin{aligned} P_0^T &= \sum_{i=1}^{N} P_0^i, \\ P_1^T &= \sum_{i=1}^{N} e^{-\ell \sum\limits_{j=1}^{i-1} P_0^j} P_1^i. \end{aligned} \tag{3.2.10}$$

We can then define the center of mass coordinates of the system as

$$\begin{aligned} x_0^T &= \frac{1}{M} \sum_{i=1}^{N} \left(x_0^i + \frac{\ell}{2} \sum_{j>i}^{N} m_j \left\{ x_1^i, P_1^i \right\} \right), \\ x_1^T &= \frac{1}{M} \sum_{i=1}^{N} m_i e^{\ell \sum\limits_{j=1}^{i-1} P_0^j} x_1^i. \end{aligned} \tag{3.2.11}$$

The duality relations between P_0^T, x_0^T and P_1^T, x_1^T are trivially verified, likewise the commutator between P_1^T and x_0^T, despite some heavy calculus, can be shown to be equal to $-i\ell P_1^T$ without real technical issues.
In particular the center of mass coordinates still live in a κ-Minkowski space-time.
In conclusion we can state in all generality that this recipe leads us from a κ deformation to another κ deformation regardless of the number or the mass of the particles involved.
Therefore, it doesn't provide any solution to the soccer-ball problem.
We have thus shown what deforming the definition of the center of mass leads to, another route would be to modify instead the form of the total and relative generators.

This will take us to a very different scenario where the soccer-ball problem can be properly addressed.

Chapter 4

Symmetry algebra of a system of 2 identical particles - the linear case

4.1 Total and relative generators - basic properties

Let us consider two particles A and B in a two dimensional space-time of equal mass m and coordinates x_μ^A and x_μ^B satisfying

$$\left[x_1^{A,B}, x_0^{A,B}\right] = i\ell x_1^{A,B}, \qquad \left[x_\mu^A, x_\nu^B\right] = 0. \tag{4.1.1}$$

For both of them we can build up two κ-Poincaré algebras with generators $\left\{P_\mu^{A,B}, N^{A,B}\right\}$, where in particular the translation generators $P_\mu^{A,B}$ act on a set of plane waves $e^{ik_1^A x_1^A} e^{-ik_0^A x_0^A}$ and $e^{ik_1^B x_1^B} e^{-ik_0^B x_0^B}$ as

$$P_\mu^{A,B} \triangleright e^{ik_1^{A,B} x_1^{A,B}} e^{-ik_0^{A,B} x_0^{A,B}} = k_\mu^{A,B} e^{ik_1^{A,B} x_1^{A,B}} e^{-ik_0^{A,B} x_0^{A,B}}. \tag{4.1.2}$$

In other words the wave functions

$$e^{ik_1^A x_1^A} e^{-ik_0^A x_0^A}, \qquad e^{ik_1^B x_1^B} e^{-ik_0^B x_0^B} \tag{4.1.3}$$

are a bases of eigenfunctions of the operators P_μ^A and P_μ^B, respectively.
As in the standard relativistic case, we would like to describe our two particle system by separating it in a "total" part and a "relative" part.

To do so let us recall the **undeformed** center of mass coordinates

$$x^T_\mu = \frac{x^A_\mu + x^B_\mu}{2} \tag{4.1.4}$$

and the relative ones[1]

$$x^R_\mu = \frac{x^A_\mu - x^B_\mu}{2}, \tag{4.1.5}$$

satisfying the commutation relations

$$\begin{aligned} \left[x^T_1, x^T_0\right] &= i\frac{\ell}{2}x^T_1, \qquad \left[x^R_1, x^T_0\right] = i\frac{\ell}{2}x^R_1, \\ \left[x^T_1, x^R_0\right] &= i\frac{\ell}{2}x^R_1, \qquad \left[x^R_1, x^R_0\right] = i\frac{\ell}{2}x^T_1. \end{aligned} \tag{4.1.6}$$

The first key observation we can make is that, as we already mentioned in the introduction, the center of mass coordinates satisfy a 2κ-Minkowski commutation relation.
This hints to the possibility that the deformation describing our composite system should somehow account for a halved deformation parameter.
In this way as the number of particles composing the system grows, the deformation effects should become weaker and weaker, thus recovering the standard Poincaré/Galileo physics on a macroscopic scale.
As we will see, however, the possible candidates for the role of symmetry generators will not be uniquely identified; the goal of this thesis is to show two different prescriptions characterized by different sets of properties, strengths and weaknesses.
In order to discuss our first proposal, let us consider the product of two plane waves A and B

$$e^{ik^A_1 x^A_1} e^{-ik^A_0 x^A_0} e^{ik^B_1 x^B_1} e^{-ik^B_0 x^B_0}. \tag{4.1.7}$$

by rewriting the old coordinates $x^{A,B}_\mu$ by means of the new ones $x^{T,R}_\mu$, we easily find that this product is equal to

$$e^{ik^T_1 x^T_1} e^{ik^R_1 x^R_1} e^{-ik^T_0 x^T_0} e^{-ik^R_0 x^R_0} \tag{4.1.8}$$

with

$$k^T_\mu = k^A_\mu + k^B_\mu, \qquad k^R_\mu = k^A_\mu - k^B_\mu. \tag{4.1.9}$$

[1] Here and in the rest of the thesis we define the relative coordinates with an explicit factor $\frac{1}{2}$, contrarily to the standard prescription. This has no real physical consequences on the theory, but simplifies greatly the various results we are going to discuss.

Therefore it is possible to define new translation generators as

$$P^T_\mu = P^A_\mu + P^B_\mu, \qquad P^R_\mu = P^A_\mu - P^B_\mu, \tag{4.1.10}$$

so that

$$P^{T,R}_\mu \triangleright e^{ik^T_1 x^T_1} e^{ik^R_1 x^R_1} e^{-ik^T_0 x^T_0} e^{-ik^R_0 x^R_0} = k^{T,R}_\mu e^{ik^T_1 x^T_1} e^{ik^R_1 x^R_1} e^{-ik^T_0 x^T_0} e^{-ik^R_0 x^R_0}, \tag{4.1.11}$$

which is a consequence of the trivial differential representations of our operators:

$$P^T_1 = -i\frac{\partial}{\partial x^T_1}, \qquad P^R_1 = -i\frac{\partial}{\partial x^R_1}. \tag{4.1.12}$$

These new commutating generators satisfy the following deformed Heisenberg Algebra with the $x^{T,R}_\mu$ coordinates:

$$\begin{aligned}
&\left[P^T_0, x^T_0\right] = i, && \left[P^T_0, x^R_0\right] = 0,\\
&\left[P^T_0, x^T_1\right] = 0, && \left[P^T_0, x^R_1\right] = 0,\\
&\left[P^R_0, x^T_0\right] = 0, && \left[P^R_0, x^R_0\right] = i,\\
&\left[P^R_0, x^T_1\right] = 0, && \left[P^R_0, x^R_1\right] = 0,\\
&\left[P^T_1, x^T_0\right] = -i\frac{\ell}{2}P^T_1, && \left[P^T_1, x^R_0\right] = -i\frac{\ell}{2}P^R_1,\\
&\left[P^T_1, x^T_1\right] = -i, && \left[P^T_1, x^R_1\right] = 0\\
&\left[P^R_1, x^T_0\right] = -i\frac{\ell}{2}P^R_1, && \left[P^R_1, x^R_0\right] = -i\frac{\ell}{2}P^T_1,\\
&\left[P^R_1, x^T_1\right] = 0, && \left[P^R_1, x^R_1\right] = -i
\end{aligned} \tag{4.1.13}$$

and have coproducts

$$\begin{aligned}
\Delta\left(P^{T,R}_0\right) &= P^{T,R}_0 \otimes \mathbb{1} + \mathbb{1} \otimes P^{T,R}_0,\\
\Delta\left(P^T_1\right) &= P^T_1 \otimes \mathbb{1} + e^{-\frac{\ell}{2}P^T_0}\cosh\frac{\ell}{2}P^R_0 \otimes P^T_1 - e^{-\frac{\ell}{2}P^T_0}\sinh\frac{\ell}{2}P^R_0 \otimes P^R_1,\\
\Delta\left(P^R_1\right) &= P^R_1 \otimes \mathbb{1} + e^{-\frac{\ell}{2}P^T_0}\cosh\frac{\ell}{2}P^R_0 \otimes P^R_1 - e^{-\frac{\ell}{2}P^T_0}\sinh\frac{\ell}{2}P^R_0 \otimes P^T_1.
\end{aligned} \tag{4.1.14}$$

It's quite easy to verify these commutators and coproducts from the simple relations (4.1.10) between $P^{T,R}_\mu$ and $P^{A,B}_\mu$.

For instance we have that

$$\begin{aligned}\left[P_1^T, x_0^T\right] &= \frac{1}{2}\left[P_1^A + P_1^B, x_1^A + x_1^B\right] \\ &= \frac{1}{2}\left(-i\ell P_1^A - i\ell P_1^B\right) \\ &= -i\frac{\ell}{2}P_1^T,\end{aligned} \tag{4.1.15}$$

or also

$$\begin{aligned}\Delta\left(P_1^T\right) &= \Delta\left(P_1^A\right) + \Delta\left(P_1^B\right) \\ &= P_1^A \otimes \mathbb{1} + e^{-\ell P_0^A} \otimes P_1^A + P_1^B \otimes \mathbb{1} + e^{-\ell P_0^B} \otimes P_1^B \\ &= P_1^T \otimes \mathbb{1} + \frac{1}{2}e^{-\frac{\ell}{2}P_0^T}\Big[e^{-\frac{\ell}{2}P_0^R} \otimes \left(P_1^T + P_1^R\right) \\ &\qquad\qquad + e^{\frac{\ell}{2}P_0^R} \otimes \left(P_1^T - P_1^R\right)\Big] \\ &= P_1^T \otimes \mathbb{1} + e^{-\frac{\ell}{2}P_0^T}\cosh\frac{\ell}{2}P_0^R \otimes P_1^T - e^{-\frac{\ell}{2}P_0^T}\sinh\frac{\ell}{2}P_0^R \otimes P_1^R.\end{aligned} \tag{4.1.16}$$

Note that the coproducts (4.1.14) express the composition law of total and relative momenta, as well as the composition rule for the products of plane waves.
In order to refine our picture we still need to specify the boost generators of our theory.
The easiest and most natural proposal is to simply emulate the definition of our total and relative translation generators, which is the same definition given in the standard Poincaré case, i.e.

$$\begin{aligned}N^T &= N^A + N^B, \\ N^R &= N^A - N^B.\end{aligned} \tag{4.1.17}$$

Using (2.8.17) the equations in (4.1.17) can be explicitly computed as

$$\begin{aligned}N^T = x_1^T &\left[\frac{1 - e^{-\ell P_0^T}\cosh\ell P_0^R}{\ell} + \frac{\ell}{4}\left({P_1^T}^2 + {P_1^R}^2\right)\right] \\ &+ x_1^R\left(\frac{e^{-\ell P_0^T}\sinh\ell P_0^R}{\ell} + \frac{\ell}{2}P_1^T P_1^R\right) - x_0^T P_1^T - x_0^R P_1^R.\end{aligned} \tag{4.1.18}$$

and

$$
\begin{aligned}
N^R = x_1^T & \left(\frac{e^{-\ell P_0^T} \sinh \ell P_0^R}{\ell} + \frac{\ell}{2} P_1^T P_1^R \right) \\
& + x_1^R \left[\frac{1 - e^{-\ell P_0^T} \cosh \ell P_0^R}{\ell} + \frac{\ell}{4} \left(P_1^{T2} + P_1^{R2} \right) \right] \\
& - x_0^T P_1^R - x_0^R P_1^T .
\end{aligned}
\tag{4.1.19}
$$

Since we can easily go from the set of generators $\left\{ P_\mu^{A,B}, N^{A,B} \right\}$ to the set $\left\{ P_\mu^{T,R}, N^{T,R} \right\}$ and vice versa, the latter clearly forms an enclosed algebra of symmetry.
Firstly we have the (trivial) commutators between the boost and the time translation generators, that are

$$
\begin{aligned}
\left[N^T, P_0^T \right] &= \left[N^A + N^B, P_0^A + P_0^B \right] = i P_1^T , \\
\left[N^T, P_0^R \right] &= \left[N^A + N^B, P_0^A - P_0^B \right] = i P_1^R , \\
\left[N^R, P_0^T \right] &= \left[N^A - N^B, P_0^A + P_0^B \right] = i P_1^R , \\
\left[N^R, P_0^R \right] &= \left[N^A - N^B, P_0^A - P_0^B \right] = i P_1^T .
\end{aligned}
\tag{4.1.20}
$$

Then we can consider for example the commutator $\left[N^T, P_1^T \right]$:

$$
\begin{aligned}
\left[N^T, P_1^T \right] &= \left[N^A, P_1^A \right] + \left[N^B, P_1^B \right] \\
&= \frac{1 - e^{-2\ell P_0^A}}{2\ell} - \frac{\ell}{2} P_1^{A2} + \frac{1 - e^{-2\ell P_0^B}}{2\ell} - \frac{\ell}{2} P_1^{B2} \\
&= \frac{1 - \frac{e^{-2\ell P_0^A} + e^{-2\ell P_0^B}}{2}}{\ell} - \frac{\ell}{2} \left(P_1^{A2} + P_1^{B2} \right) \\
&= \frac{1 - e^{-\ell P_0^T} \cosh \ell P_0^R}{\ell} - \frac{\ell}{4} \left(P_1^{T2} + P_1^{R2} \right) .
\end{aligned}
\tag{4.1.21}
$$

In a similar fashion we find that

$$
\begin{aligned}
\left[N^T, P_1^R\right] &= \left[N^A, P_1^A\right] - \left[N^B, P_1^B\right] \\
&= \frac{1 - e^{-2\ell P_0^A}}{2\ell} - \frac{\ell}{2} P_1^{A2} - \frac{1 - e^{-2\ell P_0^B}}{2\ell} + \frac{\ell}{2} P_1^{B2} \\
&= \frac{-e^{-2\ell P_0^A} + e^{-2\ell P_0^B}}{2\ell} - \frac{\ell}{2}\left(P_1^{A2} - P_1^{B2}\right) \\
&= e^{-\ell P_0^A} e^{-\ell P_0^B} \frac{-e^{-\ell\left(P_0^A - P_0^B\right)} + e^{\ell\left(P_0^A - P_0^B\right)}}{2\ell} \\
&\quad - \frac{\ell}{8}\left[\left(P_1^T + P_1^R\right)^2 - \left(P_1^T - P_1^R\right)^2\right] \\
&= e^{-\ell P_0^T} \frac{\sinh \ell P_0^R}{\ell} - \frac{\ell}{2} P_1^T P_1^R.
\end{aligned}
\tag{4.1.22}
$$

It's easy to realize that the remaining two commutators are actually equal to the previous ones, since they end up connecting the single particle operators A or B with the same signs $+$ or $-$.
Explicitly we have

$$
\left[N^R, P_1^T\right] = \left[N^T, P_1^R\right] = e^{-\ell P_0^T} \frac{\sinh \ell P_0^R}{\ell} - \frac{\ell}{2} P_1^T P_1^R \tag{4.1.23}
$$

and

$$
\left[N^R, P_1^R\right] = \left[N^T, P_1^T\right] = \frac{1 - e^{-\ell P_0^T} \cosh \ell P_0^R}{\ell} - \frac{\ell}{4}\left(P_1^{T2} + P_1^{R2}\right). \tag{4.1.24}
$$

Finally it's important to remark that, in this case, the boost generators satisfy:

$$
\left[N^T, N^R\right] = 0. \tag{4.1.25}
$$

This is not obvious a priori, as we will show in the next chapters.
As for (4.1.14) one can easily compute the coproducts of these two operators, finding

$$
\begin{aligned}
\Delta\left(N^T\right) &= N^T \otimes 1 + e^{-\frac{\ell}{2} P_0^T} \cosh \frac{\ell}{2} P_0^R \otimes N^T \\
&\quad - e^{-\frac{\ell}{2} P_0^T} \sinh \frac{\ell}{2} P_0^R \otimes N^R
\end{aligned}
\tag{4.1.26}
$$

and

$$\Delta\left(N^R\right) = N^R \otimes 1 + e^{-\frac{\ell}{2}P_0^T}\cosh\frac{\ell}{2}P_0^R \otimes N^R \\ - e^{-\frac{\ell}{2}P_0^T}\sinh\frac{\ell}{2}P_0^R \otimes N^T, \tag{4.1.27}$$

as expected.
In conclusion the set of generators $\left\{P_0^T, P_0^R, P_1^T, P_1^R, N^T, N^R\right\}$ is a nicely enclosed Hopf algebra.
Note that, due to the presence of the relative generators P_μ^R, this is not a 2κ-Poincaré algebra as one could have guessed from the 2κ-Minkowski commutator satisfied by the center of mass coordinates.
Still, the deformation parameter ℓ is everywhere rescaled to $\frac{\ell}{2}$.
On top of that if the relative motion is small enough, i.e. $P_\mu^R \ll \kappa$, then the total section of this Hopf algebra is a 2κ-Poincaré one in all regards.
We believe this is a strong indication that the soccer-ball problem disappears on its own when we compose more and more particles together.
Unfortunately, all of this comes at the price of losing one key feature of symmetries in the standard special relativistic framework, i.e. the simple physical principle that **the definition of the total and relative coordinates should be the same in the symmetry-transformed system as in the original one**.
For a generic transformation this can be translated as:

$$U_\epsilon\left(\frac{x_\mu^A + x_\mu^B}{2}\right) = \frac{U_\epsilon\left(x_\mu^A\right) + U_\epsilon\left(x_\mu^B\right)}{2} \tag{4.1.28}$$

or

$$U_\epsilon\left(\frac{x_\mu^A - x_\mu^B}{2}\right) = \frac{U_\epsilon\left(x_\mu^A\right) - U_\epsilon\left(x_\mu^B\right)}{2}, \tag{4.1.29}$$

where ϵ accounts for the amplitude of the transformation itself (see the exponential map (2.2.5) we defined in the Lie Algebra theory review in section 2.2).
We can easily see that our transformations, boosts or translations, no longer obey these conditions.
Let us consider for instance the total (infinitesimal) translation operator

$$T = 1 + \hat{\epsilon}_T P_1^T \tag{4.1.30}$$

for some yet unspecified quantum parameter $\hat{\epsilon}_T$.

The condition (4.1.28) translates into the request

$$\widehat{\epsilon}_T P_1^T = \widehat{\epsilon}_A P_1^A + \widehat{\epsilon}_B P_1^B \tag{4.1.31}$$

where ϵ_A and ϵ_B are the single particle parameters related to the same transformation.
But then we have that

$$\widehat{\epsilon}_T P_1^A + \widehat{\epsilon}_T P_1^B = \widehat{\epsilon}_A P_1^A + \widehat{\epsilon}_B P_1^B, \tag{4.1.32}$$

which is of course impossible.
One way to solve this issue could be to radically rethink the definition and interpretation of the quantum parameters $\widehat{\epsilon}$ in the non-commutative realm[2].
More studies will be required about the actual infinitesimal or finite transformations generated by our operators in order to solve this puzzle.

4.2 More on the linear total algebra: Heisenberg algebra between boosts and coordinates - mass Casimir

In this paragraph we want to present some extra properties of our algebra that will prove very important in the following chapters, in particular when we'll deal with the galilean limit of our theory.
In order to do so, we find useful in the first place to compute the Heisenberg algebra between the single particle boost generator N and the coordinates x_0, x_1 in the standard κ-Poincaré framework.
Given a generic function $f(x)$ of the κ-Minkowski coordinates, we can use the coproduct

$$\Delta(N) = N \otimes 1 + e^{-\ell P_0} \otimes N \tag{4.2.1}$$

to compute tha action

[2]We will discuss more in depth the role held by these parameters in the κ-Poincaré framework in the next chapter. We remind that in the classical Lie algebra symmetry theory, this parameters are usually simple real numbers.

$$\begin{aligned}[N, x_0] \triangleright f(x) &= (N \triangleright x_0) f(x) + \left(e^{-\ell P_0} \triangleright x_0\right)(N \triangleright f(x)) \\ &\quad - x_0 N \triangleright f(x) \\ &= i x_1 f(x) - i\ell N \triangleright f(x),\end{aligned} \tag{4.2.2}$$

where we used the fact that

$$\begin{aligned}N \triangleright x_0 &= \left[x_1\left(\frac{1-e^{-2\ell P_0}}{2\ell} + \frac{\ell}{2}P_1^2\right) - x_0 P_1\right] \triangleright x_0 \\ &= x_1 \frac{1-1+2\ell P_0 + \ldots}{2\ell} \triangleright x_0 \\ &= i x_1.\end{aligned} \tag{4.2.3}$$

Thus

$$[N, x_0] = i\left(x_1 - \ell N\right). \tag{4.2.4}$$

On the other hand

$$\begin{aligned}[N, x_1] \triangleright f(x) &= (N \triangleright x_1) f(x) + \left(e^{-\ell P_0} \triangleright x_1\right)(N \triangleright f(x)) \\ &\quad - x_1 N \triangleright f(x) \\ &= i x_0 f(x)\end{aligned} \tag{4.2.5}$$

since

$$\begin{aligned}N \triangleright x_1 &= \left[x_1\left(\frac{1-e^{-2\ell P_0}}{2\ell} + \frac{\ell}{2}P_1^2\right) - x_0 P_1\right] \triangleright x_1 \\ &= -x_0 P_1 \triangleright x_0 \\ &= i x_0.\end{aligned} \tag{4.2.6}$$

In conclusion

$$[N, x_1] = i x_0. \tag{4.2.7}$$

Keeping this in mind, it's easy to find the Heisenberg Algebra between boosts and coordinates of the composite system:

$$\begin{aligned}
\left[N^T, x_0^T\right] &= \frac{1}{2}\left[N^A, x_0^A\right] + \frac{1}{2}\left[N^B, x_0^B\right] = i\left(x_1^T - \frac{\ell}{2}N^T\right),\\
\left[N^T, x_0^R\right] &= \frac{1}{2}\left[N^A, x_0^A\right] - \frac{1}{2}\left[N^B, x_0^B\right] = i\left(x_1^R - \frac{\ell}{2}N^R\right),\\
\left[N^T, x_1^T\right] &= \frac{1}{2}\left[N^A, x_1^A\right] + \frac{1}{2}\left[N^B, x_1^B\right] = ix_0^T,\\
\left[N^T, x_1^R\right] &= \frac{1}{2}\left[N^A, x_1^A\right] - \frac{1}{2}\left[N^B, x_1^B\right] = ix_0^R,\\
\left[N^R, x_0^T\right] &= \left[N^T, x_0^R\right] = i\left(x_1^R - \frac{\ell}{2}N^R\right),\\
\left[N^R, x_0^R\right] &= \left[N^T, x_0^T\right] = i\left(x_1^T - \frac{\ell}{2}N^T\right),\\
\left[N^R, x_1^T\right] &= \left[N^T, x_1^R\right] = ix_0^R,\\
\left[N^R, x_1^R\right] &= \left[N^T, x_1^T\right] = ix_0^T.
\end{aligned} \tag{4.2.8}$$

Note in particular that the relative spatial coordinate x_1^R, or rather its associate operator, commutes with the total boost N^T when $x_0^R = 0$, i.e. when the two particles composing our system are considered **at equal time**.
Since as we know $\left[P_\mu^T, x_1^R\right] = 0$, one could think that any function of the form $V\left(x_1^R\right)$ is a good candidate potential for an interacting theory in our non-commutative framework, since it would commute with every symmetry generator at equal times[3].
However, it must be mentioned that the actual symmetry generators in the κ-Poincaré framework are not simply the algebra generators themselves, but instead the products ϵX for a given generator X and a certain quantum parameter ϵ, as we highlighted in the previous section (see (4.1.30)).

[3]One could argue that an interaction theory between two or more particles makes sense only if the particles in consideration live (and interact) at the same physical time.

Indeed some recent works about multi-time wave functions in standard quantum theory [38] point quite strongly towards this direction (note however that the situation is very different if one considers instead a quantum field theory [39, 40]), where the number of particles involved is not fixed due to creation and annihilation mechanisms. This is very similar to what happens for the infamous no-interaction theorem, which we'll discuss more in depth in chapter 8).

These results, though, have been proven only in the special relativity realm, so that they don't directly apply in our picture.

Still, we believe our discoveries imply their validity even in the non-commutative, finite-number of particles framework, although in a somewhat preliminar sense.

Since we don't know yet how to deal with the quantum parameters in the two particle linear algebra case, the interaction potential question can't be properly understood here.
We will explore these facts more in depth when we'll discuss about the galilean limit of our theory in the final chapter of this thesis.
One last ingredient we still miss in our picture is the mass Casimir C^T.
It's quite obvious that the single particle Casimirs C^A and C^B are still invariant in this theory since trivially $\left[O^A, O^B\right] = 0$ for every couple of single particle operators O^A and O^B.
Thus every linear combination of C^A and C^B will still be invariant, for instance their sum

$$C^T = C^A + C^B, \tag{4.2.9}$$

which coherently takes into account the masses m of the two identical starting particles A and B and describes a $2m$ total mass composite system. Explicitely this Casimir can be shown to be equal to

$$\begin{aligned} C^T = &\frac{8}{\ell^2}\sinh^2\frac{\ell}{4}P_0^T - \frac{8}{\ell^2}\sinh^2\frac{\ell}{4}P_0^R \\ &+ \frac{16}{\ell^2}\cosh^2\frac{\ell}{4}P_0^T\sinh^2\frac{\ell}{4}P_0^R - e^{\frac{\ell}{2}P_0^T}\cosh\frac{\ell}{2}P_0^R P_1^{T\,2} \\ &- e^{\frac{\ell}{2}P_0^T}\cosh\frac{\ell}{2}P_0^R P_1^{R\,2} + 2e^{\frac{\ell}{2}P_0^T}\sinh\frac{\ell}{2}P_0^R P_1^T P_1^R, \end{aligned} \tag{4.2.10}$$

which naturally boils down to the 2κ-Poincaré mass Casimir

$$C^T \to \frac{8}{\ell^2}\sinh^2\frac{\ell}{4}P_0^T - e^{\frac{\ell}{2}P_0^T}P_1^{T\,2} \tag{4.2.11}$$

when $P^R \to 0$ and the system can be interpreted as a $2m$ total mass rigid body.

4.3 A final remark about the 2κ-Poincaré restriction

Let us now consider the Hilbert spaces H^A and H^B of the states of the particles A and B respectively, and in particular the bases of eigenfunctions of the translation generators P_μ^A and P_ν^B given by the plane waves

$$e^{ik_1^A x_1^A}e^{-ik_0^A x_0^A}, \qquad e^{ik_1^B x_1^B}e^{-ik_0^B x_0^B}. \tag{4.3.1}$$

In order to describe the composite system $A + B$ we could in the first instance consider in the tensor product $H^A \otimes H^B$ a base of eigenfunctions given by the product of two plane waves of type A and B:

$$e^{ik_1^A x_1^A} e^{-ik_0^A x_0^A} e^{ik_1^B x_1^B} e^{-ik_0^B x_0^B}. \tag{4.3.2}$$

As we saw earlier, with a coordinate change we can rewrite this as

$$e^{ik_1^T x_1^T} e^{ik_1^R x_1^R} e^{-ik_0^T x_0^T} e^{-ik_0^R x_0^R}, \tag{4.3.3}$$

which are explicitily the eigenfunctions of the new generators $P_\mu^{T,R}$.
In particular by restricting ourselves to the subspace of eigenfunctions with $k_\mu^R = 0$, be H^T, the only meaningful generators are now the P_μ^T, which we can fully interpret as translation generators of a 2k-Poincaré algebra, as one can easily see by letting $P_\mu^R \to 0$ in the formulas defining the coproducts and the Heisenberg Algebra.
Therefore if we define the standard boost generator

$$N^T = x_1^T \left(\frac{1 - e^{-\ell P_0^T}}{\ell} + \frac{\ell}{4} {P_1^T}^2 \right) - x_0^T P_1^T \tag{4.3.4}$$

we find that the triad $\left\{ P_\mu^T, N^T \right\}$ actually encloses a 2k-Poincaré algebra on H^T.
We could say then that the scaling $\ell \to \frac{\ell}{2}$, in the case of two identical particles, occurs cleanly at least for rigid bodies.
Let us stress an interesting concurrence: while the deformation scale halves, the mass of the system doubles, as if they were inversely proportional.
We will explore more in depth this intriguing possibility in chapter 7.

Chapter 5

Pregeometry of κ-Minkowski

As we saw in the previous chapters, relativistic symmetry conditions such as (4.1.28) are not naturally carried over in the deformed picture if we adopt the definitions (4.1.10), (4.1.17).
However, we could still ask ourselves if there's a way to enforce these conditions from the start.
In order to answer this question, it's useful to define and discuss briefly the so-called "pregeometric" structure of κ-Minkowski [41], which we will predominantly make use of in the following chapter.
As we know non-commutative coordinates on κ-Minkowski are defined as:

$$[x_1, x_0] = i\ell x_1. \tag{5.0.1}$$

If we call q_0, q_1, π_0, π_1 the standard operators of quantum mechanics obeying

$$\left[\pi_\mu, q_\nu\right] = (-1)^\mu \, i\delta_{\mu\nu}, \tag{5.0.2}$$

where $\mu, \nu = 0, 1$, then it is clear that the following equations:

$$x_0 = q_0, \qquad x_1 = q_1 e^{\ell \pi_0}, \tag{5.0.3}$$

$$x_0 = q_0 + \frac{\ell}{2} \{q_1, \pi_1\}, \qquad x_1 = q_1 \tag{5.0.4}$$

are two possible representations of (5.0.1).
Let us call the first one (5.0.3) the *diagonal* representation since it does not mix the commutative coordinates q_0, q_1, while the latter (5.0.4) the *off-diagonal* in light of the fact that q_0 and q_1 are mixed to give us x_0.

Now we want to describe translation symmetries on this quantum spacetime considering either the diagonal (5.0.3) or the off-diagonal (5.0.4) representation of κ-Minkowski coordinates.

5.1 Diagonal case

Evidently, the usual translation map $x'_\mu = x_\mu + \epsilon_\mu$ does not work.
In fact, if ϵ_μ are trivial translation parameters (i.e. real numbers), we would have that $[x'_1, x'_0] = i\ell x_1 \neq i\ell x'_1$.
This means that (5.0.1) cannot be invariant under classical translations.
Let us define as usual the infinitesimal translation operator T:

$$T = \mathbb{1} + i\widehat{\epsilon}_\mu P_\mu, \tag{5.1.1}$$

where the repeated index μ is summed. Here $\widehat{\epsilon}_\mu$ are "quantum" translation parameters to be specified and P_μ is the generator of infinitesimal translations we will define in a moment.
For the diagonal representation of (5.0.3) it has been introduced the following P_μ [41]:

$$\begin{aligned} P_0 \triangleright f(x_i) &= \left[\pi_0, f\left(x_\mu\right)\right], \\ P_1 \triangleright f(x_i) &= e^{-\ell\pi_0}\left[\pi_1, f\left(x_\mu\right)\right] \end{aligned} \tag{5.1.2}$$

together with the following $\widehat{\epsilon}_\mu$:

$$\widehat{\epsilon}_0 = \epsilon_0, \qquad \widehat{\epsilon}_1 = \epsilon_1 e^{\ell\pi_0} \tag{5.1.3}$$

that have a non-trivial action on κ-Minkowski coordinates:

$$[\widehat{\epsilon}_1, x_1] = [\widehat{\epsilon}_0, x_1] = [\widehat{\epsilon}_0, x_0] = 0, \qquad [\widehat{\epsilon}_1, x_0] = i\ell\widehat{\epsilon}_1. \tag{5.1.4}$$

Given the above equations (5.1.1)-(5.1.2)-(5.1.3)-(5.1.4), it is easy to show that (5.0.1) enjoys translation invariance:

$$x'_0 = x_0 + \widehat{\epsilon}_0, \qquad x'_1 = x_1 + \widehat{\epsilon}_1, \tag{5.1.5}$$

where clearly:

$$[x'_1, x'_0] = i\ell x'_1. \tag{5.1.6}$$

Note that the above defined generators of translations act in the standard way on right ordered plane waves:

$$P_0 \triangleright \left(e^{ik_1x_1}e^{-ik_0x_0}\right) = k_0 e^{ik_1x_1}e^{-ik_0x_0},$$
$$P_1 \triangleright \left(e^{ik_1x_1}e^{-ik_0x_0}\right) = k_1 e^{ik_1x_1}e^{-ik_0x_0} \tag{5.1.7}$$

and they also obey κ-Poincaré coproduct rules i.e.:

$$\Delta\left(P_0\right) = P_0 \otimes \mathbb{1} + \mathbb{1} \otimes P_0, \qquad \Delta\left(P_1\right) = P_1 \otimes \mathbb{1} + e^{-\ell P_0} \otimes P_1. \tag{5.1.8}$$

These facts allow us to identify them with the familiar single particle translation generators introduced in the previous chapters.
Moreover it is also easy to show that the differential operator $\widehat{d} = i\widehat{\epsilon}_\mu P_\mu$ satisfy the Leibniz rule:

$$\widehat{d} \triangleright (fg) = \left(\widehat{d} \triangleright f\right) g + f\left(\widehat{d} \triangleright g\right). \tag{5.1.9}$$

Note that such a property of the differential is crucial e.g. for extending the Noether theorem to the symmetries of quantum space-times [42–44].
Thus, we could say that an acceptable definition of translations must be such that the following requirements are satisfied:

1. P_μ have classical actions on functions of x_μ (i.e. (5.1.7) holds);
2. $\Delta\left(P_\mu\right)$ are those of the bi-cross-product κ-Poincaré Hopf algebra (i.e. (5.1.8) holds);
3. The rule of space-time non-commutativity (5.0.1) must be translation invariant (i.e. (5.1.6) hold);
4. The differential $\widehat{d}$ has a primitive coproduct (i.e. (5.1.9) holds).

At present this picture is not available for the off-diagonal representation. In light of this, even though we will mostly use the diagonal representation in the rest of this work, we find useful to briefly discuss and present it in the next paragraph.

5.2 Off-diagonal case

It is immediate to show that the generators P_μ (5.1.2) defined above for the diagonal representation do not satisfy the first property 1. when they act on functions depending on off-diagonal coordinates (5.0.4). In fact:

$$\begin{aligned} P_1 \triangleright \left(e^{ik_1 x_1} e^{-ik_0 x_0}\right) &= P_1 \triangleright \left(e^{ik_1 q_1} e^{-ik_0\left(q_0 + \frac{\ell}{2}\{q_1, \pi_1\}\right)}\right) \\ &= e^{-\ell\pi_0} k_1 e^{ik_1 x_1} e^{-ik_0 x_0} - \ell k_0 e^{-\ell\pi_0} \pi_1 e^{ik_1 x_1} e^{-ik_0 x_0}. \end{aligned} \tag{5.2.1}$$

Another possibility could be to take trivial generators i.e. $P_0 \equiv [\pi_0, \bullet]$ and $P_1 \equiv [\pi_1, \bullet]$.
However, also this choice does not allow to have the desired standard action (it would just translate in a multiplication by the external factor $e^{-\ell\pi_0}$). Evidently:

$$\begin{aligned} P_1 \triangleright \left(e^{ik_1 x_1} e^{-ik_0 x_0}\right) &= \left[\pi_1, e^{ik_1 q_1} e^{-ik_0\left(q_0 + \frac{\ell}{2}\{q_1, \pi_1\}\right)}\right] \\ &= k_1 e^{ik_1 x_1} e^{-ik_0 x_0} - \ell k_0 \pi_1 e^{ik_1 x_1} e^{-ik_0 x_0}. \end{aligned} \tag{5.2.2}$$

Note that both these choices of momentum generators were suggested by the fact that, as one can easily check, they leave invariant (5.0.1).
At this point we observe that in order to recover property 1. we must find a function $h(\pi_0, \pi_1)$ such that:

$$\left[h(\pi_0, \pi_1), q_0 + \frac{\ell}{2}\{q_1, \pi_1\}\right] = 0. \tag{5.2.3}$$

The above commutator is clearly equivalent to the following partial differential equation:

$$\frac{\partial h}{\partial \pi_0} + \ell\pi_1 \frac{\partial h}{\partial \pi_1} = 0, \tag{5.2.4}$$

which can be easily solved by proposing a factorized form $h(\pi_0, \pi_1) = l(\pi_0)\, m(\pi_1)$:

$$\frac{dl}{d\pi_0} = \ell l, \qquad m = \pi_1 \frac{dm}{d\pi_1}. \tag{5.2.5}$$

Thus the solution is given by:

$$h(\pi_0, \pi_1) = e^{\ell\pi_0} \pi_1. \tag{5.2.6}$$

Now we note that if we identify $P_1 \equiv [h(\pi_0, \pi_1), \bullet]$, then we still do not have recovered property 1.:

$$P_1 \triangleright \left(e^{ik_1x_1}e^{-ik_0x_0}\right) = \left[e^{\ell\pi_0}\pi, e^{ik_1q_1}e^{ik_0\left(q_0+\frac{\ell}{2}\{q_1,\pi_1\}\right)}\right] \\ = e^{\ell\pi_0}k_1e^{ik_1x_1}e^{-ik_0x_0}. \tag{5.2.7}$$

But it is evident that the right choice for the generator of spatial translations is given by:

$$P_1 \triangleright \left(e^{ik_1x_1}e^{-ik_0x_0}\right) = e^{-\ell\pi_0}\left[e^{\ell\pi_0}\pi_1, e^{ik_1x_1}e^{-ik_0x_0}\right] \\ = k_1e^{ik_1x_1}e^{-ik_0x_0}. \tag{5.2.8}$$

It is also easy to prove that the coproducts are given by (5.1.8) and thus 2. is satisfied.
Then we can take the same quantum translation parameters defined for the diagonal case (5.1.3) and check that, given (5.2.7), also 3. holds. In fact:

$$x_1' = x_1 + \widehat{\epsilon}_1, \qquad x_0' = x_0 + \widehat{\epsilon}_1, \tag{5.2.9}$$

which are the same expressions we found for the translated coordinates represented in the diagonal basis (see (5.1.5)).
In light of this, the realization we have found for translation generators (5.2.7) has the advantage of making quantum translations independent from the chosen representation of the κ-Minkowski coordinates.
It does not matter which representation we decide to use, in any case the translated coordinates x'_μ are obtained by adding the same quantum translation parameters $\widehat{\epsilon}_\mu$ to the initial coordinates x_μ.
This is in perfect analogy to standard translations (i.e. translations of commutative coordinates) where the only difference is that we need to make the replacement $\epsilon_\mu \to \widehat{\epsilon}_\mu$.
Finally, using (5.1.3) and (5.2.7) it is easy to prove that also the last requirement 4. is accomplished.
In the following chapters we will always make use of the diagonal prescription, therefore we want to conclude this review by showing the pregeometric form of the κ-Poincaré boost, that is

$$N = e^{-\ell\pi_0}\left[\left(\frac{e^{2\ell\pi_0}-1}{2\ell} + \frac{\ell}{2}\pi_1^2\right)q_1 - \pi_1 q_0, \bullet\right], \tag{5.2.10}$$

which, as one can easily verify, satisfies the correct commutation relations with P_0 and P_1 [41].

Chapter 6

Symmetry Algebra of a system of 2 identical particles - the pregeometric case

6.1 Composite system for low relative energy - preliminar analysis in the pregeometric setting

Before discussing our second proposal for a symmetry algebra of a two particle system, we find useful to re-examine the conclusions obtained in chapter 4 in the light of the pregeometric representation we just defined. That is, we would like to show again that in a composite system in which the two (identical) particles have low relative energy $\ell\pi_0^R = \ell\left(\pi_0^A - \pi_0^B\right) \approx 0$ the symmetry algebra of the system having coordinates

$$x_\mu^T = \frac{x_\mu^A + x_\mu^B}{2} \tag{6.1.1}$$

is actually 2κ-Poincaré, and that this important fact is intrinsically related with the pregeometry of the single particle's κ-Poincaré algebras themselves.

We can show this key property in a constructive way. From the pregeometric representation of the particle's symmetry algebras we can build the operators:

$$\begin{aligned} \pi^T_\mu &= \pi^A_\mu + \pi^B_\mu, & \pi^R_\mu &= \pi^A_\mu - \pi^B_\mu, \\ P^T_0 &= \left[\pi^T_0, \bullet\right], & P^T_1 &= e^{-\frac{\ell\pi^T_0}{2}} \left[\pi^T_1, \bullet\right]. \end{aligned} \tag{6.1.2}$$

It is a trivial exercise to check that the translation generators defined above satisfy the commutation and co-commutation relations of the analogous translation generators of a 2κ-Poincaré algebra, up to corrections of order $\sim \ell\pi^R_0$.
For the boost operator the derivation is only slightly more involved.
The limit $\ell \to 0$ of the pregeometric representation of this operator must be equal to:

$$\eta^T_{\ell\to 0} = \pi^T_0 q^T_1 - \pi^T_1 q^T_0 + \pi^R_0 q^R_1 - \pi^R_1 q^R_0, \tag{6.1.3}$$

where of course

$$q^T_\mu = \frac{q^A_\mu + q^B_\mu}{2}, \qquad q^R_\mu = \frac{q^A_\mu - q^B_\mu}{2}. \tag{6.1.4}$$

The "relative part" of this operator commutes with all the observables relative to the center of mass, so we can neglect it for the purpose of this section.
To obtain the right "total part" of the boost generator it is sufficient to deform π^T_0 in (6.1.3) to the functional form we have for the single particle boost pregeometric representation (with the substitution $\kappa \to 2\kappa$):

$$\pi^T_0 \to \tilde{\pi}^T_0 = \frac{e^{\ell\pi^T_0} - 1}{\ell} + \frac{\ell}{4}{\pi^T_1}^2. \tag{6.1.5}$$

We can thus obtain the right representation for the total boost operator as:

$$N^T = e^{-\frac{\ell}{2}\pi^T_0} \left[\eta^T, \bullet\right] \tag{6.1.6}$$

with $\eta^T = \tilde{\pi}^T_0 q^T_1 - \pi^T_1 q^T_0 + \pi^R_0 q^R_1 - \pi^R_1 q^R_0$.
It is trivial to see that the operators P^T_1, P^T_0 and N^T generate the same algebra and coalgebra structure of the single particle one, except for the substitution $\kappa \to 2\kappa$.
Finally, we want to remark that formulae (6.1.2) and (6.1.6) are no less then the pregeometric representations of the generators (4.1.10) and (4.1.17) when we let P^R_μ go to zero.

6.2 Definition of the pregeometric total and relative generators

We are now ready to show our second proposal for the complete symmetry structure of our 2-particle (A,B) system with standard center of mass

$$x_\mu^T = \frac{x_\mu^A + x_\mu^B}{2} \tag{6.2.1}$$

and relative coordinates

$$x_\mu^R = \frac{x_\mu^A - x_\mu^B}{2}. \tag{6.2.2}$$

It's useful to write down the pregeometric representation of this coordinates, that is

$$\begin{aligned}
x_0^T &= q_0^T, \\
x_0^R &= q_0^R, \\
x_1^T &= q_1^T e^{\frac{\ell}{2}\pi_0^T} \cosh \frac{\ell}{2}\pi_0^R + q_1^R e^{\frac{\ell}{2}\pi_0^T} \sinh \frac{\ell}{2}\pi_0^R, \\
x_1^R &= q_1^T e^{\frac{\ell}{2}\pi_0^T} \sinh \frac{\ell}{2}\pi_0^R + q_1^R e^{\frac{\ell}{2}\pi_0^T} \cosh \frac{\ell}{2}\pi_0^R.
\end{aligned} \tag{6.2.3}$$

Indeed we have that

$$\begin{aligned}
x_1^T = \frac{x_1^A + x_1^B}{2} &= \frac{q_1^A e^{\ell \pi_0^A} + q_1^B e^{\ell \pi_0^B}}{2} \\
&= \frac{1}{2}\left[\left(q_1^T + q_1^R\right) e^{\frac{\ell}{2}\left(\pi_0^T + \pi_0^R\right)} + \left(q_1^T - q_1^R\right) e^{\frac{\ell}{2}\left(\pi_0^T - \pi_0^R\right)}\right] \\
&= \frac{1}{2} q_1^T e^{\frac{\ell}{2}\pi_0^T}\left(e^{\frac{\ell}{2}\pi_0^R} + e^{-\frac{\ell}{2}\pi_0^R}\right) + \frac{1}{2} q_1^R e^{\frac{\ell}{2}\pi_0^T}\left(e^{\frac{\ell}{2}\pi_0^R} - e^{-\frac{\ell}{2}\pi_0^R}\right) \\
&= q_1^T e^{\frac{\ell}{2}\pi_0^T} \cosh \frac{\ell}{2}\pi_0^R + q_1^R e^{\frac{\ell}{2}\pi_0^T} \sinh \frac{\ell}{2}\pi_0^R.
\end{aligned} \tag{6.2.4}$$

and similarly for x_1^R.
As we previously mentioned, we would like to build a new set of symmetry generators $\left\{P_\mu^T, P_\mu^R, N^T, N^R\right\}$ by trying to impose the conditions

(4.1.28) and (4.1.29) from the start, that is

$$U_\epsilon\left(\frac{x_\mu^A + x_\mu^B}{2}\right) = \frac{U_\epsilon\left(x_\mu^A\right) + U_\epsilon\left(x_\mu^B\right)}{2} \tag{6.2.5}$$

and

$$U_\epsilon\left(\frac{x_\mu^A - x_\mu^B}{2}\right) = \frac{U_\epsilon\left(x_\mu^A\right) - U_\epsilon\left(x_\mu^B\right)}{2} \tag{6.2.6}$$

for a generic infinitesimal transformation U_ϵ.
The subscript ϵ means that the transformation is labeled by a set of continuous parameters (common to the two observers and independent of the particle index) which can be made small at will.
Similarly to what we have shown in section 5 when dealing with a single particle, we expect again to find some kind of "quantum" parameters ϵ, with non trivial commutation relations with the coordinates x_μ^T and x_μ^R.
Following [**?**] we have for translations and boosts:

$$\begin{aligned} T &= 1 + \widehat{\epsilon}\left[\pi_\mu, \bullet\right] + \dots \\ B &= 1 + \widehat{\xi}\left[\eta, \bullet\right] + \dots \end{aligned} \tag{6.2.7}$$

and applying (4.1.28) this gives for the respective transformations

$$\begin{aligned} \pi_\mu^T &= \pi_\mu^A + \pi_\mu^B, \\ \pi_\mu^R &= \pi_\mu^A - \pi_\mu^B \end{aligned} \tag{6.2.8}$$

and

$$\begin{aligned} \eta^T &= \eta^A + \eta^B, \\ \eta^R &= \eta^A - \eta^B. \end{aligned} \tag{6.2.9}$$

For the sake of the pregeometric representation we still need to identify the external factors $\widehat{\epsilon}$ and $\widehat{\xi}$.
It's easy to see that the temporal translation generators $P_0^{T,R}$ have trivial representation, i.e.

$$\begin{aligned} P_0^T &= \left[\pi_0^A + \pi_0^B, \bullet\right] = \left[\pi_0^T, \bullet\right], \\ P_0^R &= \left[\pi_0^A - \pi_0^B, \bullet\right] = \left[\pi_0^R, \bullet\right]. \end{aligned} \tag{6.2.10}$$

For the spatial total generator instead using (4.1.28) we find

$$\widehat{\epsilon}_T P_1^T \triangleright x_1^T = \epsilon_T \frac{e^{\ell \pi_0^A} P_1^A \triangleright x_1^A + e^{\ell \pi_0^B} P_1^B \triangleright x_1^B}{2}, \tag{6.2.11}$$

that is

$$\widehat{\epsilon}_T = \epsilon_T \frac{e^{\ell \pi_0^A} + e^{\ell \pi_0^B}}{2}, \tag{6.2.12}$$

where ϵ_T is the real number that quantifies the extent of the translation. Thus in conclusion we have

$$P_1^T = \frac{2}{e^{\ell \pi_0^A} + e^{\ell \pi_0^B}} \left[\pi_1^T, \bullet\right] = \frac{e^{-\frac{\ell}{2}\pi_0^T}}{\cosh \frac{\ell}{2}\pi_0^R} \left[\pi_1^T, \bullet\right]. \tag{6.2.13}$$

It's worth noting that if the relative part is small enough, $\ell \pi_0^R \to 0$, this is just the standard 2κ-Poincaré translation generator as expected.
In a similar fashion using (4.1.29) we can write

$$\widehat{\epsilon}_R P_1^R \triangleright x_1^R = \epsilon_R \frac{e^{\ell \pi_0^A} P_1^A \triangleright x_1^A - e^{\ell \pi_0^B} P_1^B \triangleright \left(-x_1^B\right)}{2} \tag{6.2.14}$$

for the relative counterpart, which again leads us to

$$\widehat{\epsilon}_R = \epsilon_R \frac{e^{\ell \pi_0^A} + e^{\ell \pi_0^B}}{2} \tag{6.2.15}$$

and

$$P_1^R = \frac{e^{-\frac{\ell}{2}\pi_0^T}}{\cosh \frac{\ell}{2}\pi_0^R} \left[\pi_1^R, \bullet\right]. \tag{6.2.16}$$

It's not hard to find that the same result yields even for the total and relative boost generators, which we can now write as

$$N^T = \frac{e^{-\frac{\ell}{2}\pi_0^T}}{\cosh \frac{\ell}{2}\pi_0^R} \left[\eta^T, \bullet\right] \tag{6.2.17}$$

and

$$N^R = \frac{e^{-\frac{\ell}{2}\pi_0^T}}{\cosh \frac{\ell}{2}\pi_0^R} \left[\eta^R, \bullet\right]. \tag{6.2.18}$$

One can easily verify that these definitions are coherent with the structure of our novel two particle space-time, i.e. the translated or boosted coordinates still satisfy the correct commutation relations.

For instance let us consider an infinitesimal translation of the center of mass coordinate x_1^T:

$$x'^T_1 = x_1^T + \widehat{\epsilon}_T = x_1^T + \epsilon_T e^{\frac{\ell}{2}\pi_0^T} \cosh \frac{\ell}{2}\pi_0^R. \tag{6.2.19}$$

We find immediately that

$$\begin{aligned} \left[x'^T_1, x_0^{T'}\right] &= \left[x_1^T, x_0^T\right] + \epsilon_T \left[e^{\frac{\ell}{2}\pi_0^T} \cosh \frac{\ell}{2}\pi_0^R, x_0^T\right] \\ &= i\frac{\ell}{2}x_1^T + i\frac{\ell}{2}\epsilon_T e^{\frac{\ell}{2}\pi_0^T} \cosh \frac{\ell}{2}\pi_0^R \\ &= i\frac{\ell}{2}x'^T_1, \end{aligned} \tag{6.2.20}$$

as claimed.
In the following we will mostly focus on the total generators, in particular on the total boost.
We start by noting that η^T can be rewritten as

$$\eta^T = \tilde{\pi}_0^T q_1^T - \pi_1^T q_0^T + \tilde{\pi}_0^R q_1^R - \pi_1^R q_0^R, \tag{6.2.21}$$

where

$$\begin{aligned} \tilde{\pi}_0^T &= \frac{1}{\ell}\left[e^{\ell\pi_0^T} \cosh \ell\pi_0^R - 1\right] + \frac{\ell}{4}\left[\pi_1^{T2} + \pi_1^{R2}\right], \\ \tilde{\pi}_0^R &= e^{\ell\pi_0^T} \frac{\sinh \ell\pi_0^R}{\ell} + \frac{\ell}{2}\pi_1^T\pi_1^R \end{aligned} \tag{6.2.22}$$

as one can easily verify through some algebraic manipulations.
From the last formulas it is clear that the only departure from a 2κ-Poincaré algebra is given by the relative momentum π_μ^R.
When $\ell\pi_\mu^R \to 0$ the whole algebra reproduces (a pregeometric representation of) the 2κ-Poincaré one.
One can argue that this is the regime we are most interested about, in that both the A and B particles and the composite particle have to be ultrarelativistic - situation which is attainable when the two momenta are both big and parallel.

6.3 More on the pregeometric algebra: Heisenberg algebra, coproducts and plane waves

Let us study more deeply the structure of the total and relative generators P^T_μ and P^R_μ, and the Heisenberg algebra they satisfy together with the total and relative coordinates.
First of all let us recall the definition of these generators:

$$\begin{aligned} P^T_0 &= \left[\pi^T_0, \bullet\right], \\ P^R_0 &= \left[\pi^R_0, \bullet\right], \\ P^T_1 &= \frac{e^{-\frac{\ell}{2}\pi^T_0}}{\cosh\frac{\ell}{2}\pi^R_0}\left[\pi^T_1, \bullet\right], \\ P^R_1 &= \frac{e^{-\frac{\ell}{2}\pi^T_0}}{\cosh\frac{\ell}{2}\pi^R_0}\left[\pi^R_1, \bullet\right], \end{aligned} \tag{6.3.1}$$

where of course

$$\pi^T_\mu = \pi^A_\mu + \pi^B_\mu, \qquad \pi^R_\mu = \pi^A_\mu - \pi^B_\mu. \tag{6.3.2}$$

Here and in the following we will mostly ignore the time generators $P^{T,R}_0$, since we already studied their quite trivial behaviour in the previous chapters.
We easily find the actions

$$\begin{aligned} &P^T_1 \triangleright x^T_1 = -i, && P^T_1 \triangleright x^R_1 = -i\tanh\frac{\ell}{2}\pi^R_0, \\ &P^R_1 \triangleright x^T_1 = -i\tanh\frac{\ell}{2}\pi^R_0, && P^R_1 \triangleright x^R_1 = -i, \end{aligned} \tag{6.3.3}$$

and therefore the Heisenberg algebra[1]

[1]The computations are practically identical to the ones we performed for (4.1.13), so we're not going to show them explicitly this time.

$$\begin{aligned}
\left[P_1^T, x_0^T\right] &= -i\frac{\ell}{2}P_1^T, & \left[P_1^T, x_0^R\right] &= -i\frac{\ell}{2}\tanh\frac{\ell}{2}\pi_0^R P_1^T, \\
\left[P_1^T, x_1^T\right] &= -i, & \left[P_1^T, x_1^R\right] &= -i\tanh\frac{\ell}{2}\pi_0^R, \\
\left[P_1^R, x_0^T\right] &= -i\frac{\ell}{2}P_1^R, & \left[P_1^R, x_0^R\right] &= -i\frac{\ell}{2}\tanh\frac{\ell}{2}\pi_0^R P_1^R, \\
\left[P_1^R, x_1^T\right] &= -i\tanh\frac{\ell}{2}\pi_0^R, & \left[P_1^R, x_1^R\right] &= -i.
\end{aligned} \tag{6.3.4}$$

As we can see this algebra actually reduces to the 2κ-Poincaré one when we're dealing with low relative energy $\ell\pi_0^R \approx 0$, as expected.
Let us now talk briefly about the coproducts of these generators, starting for instance from P_1^T.
To do so, we can study the action of P_1^T on the product of two plane waves and then try to "split" it up in two single plane waves actions.
We have

$$\begin{aligned}
P_1^T \triangleright e^{ipx}e^{iqx} &= \frac{e^{-\frac{\ell}{2}\pi_0^T}}{\cosh\frac{\ell}{2}\pi_0^R}\left(\left[\pi_1^T, e^{ipx}\right]e^{iqx} + e^{ipx}\left[\pi_1^T, e^{iqx}\right]\right) \\
&= \left(P_1^T \triangleright e^{ipx}\right)e^{iqx} + \frac{e^{-\frac{\ell}{2}\pi_0^T}}{\cosh\frac{\ell}{2}\pi_0^R}e^{ipx}\left[\pi_1^T, e^{iqx}\right],
\end{aligned} \tag{6.3.5}$$

thus to find the coproduct of our generator we just need to switch the order of the "external factor"

$$\frac{e^{-\frac{\ell}{2}\pi_0^T}}{\cosh\frac{\ell}{2}\pi_0^R} \tag{6.3.6}$$

and the plane wave

$$e^{ipx} = e^{ip_1^T x_1^T}e^{ip_1^R x_1^R}e^{-ip_0^T x_0^T}e^{-ip_0^R x_0^R}. \tag{6.3.7}$$

In order to do that let us rewrite the function $\frac{1}{\cosh\frac{\ell}{2}\pi_0^R}$ as

$$\begin{aligned}
\frac{1}{\cosh\frac{\ell}{2}\pi_0^R} &= 2e^{-\frac{\ell}{2}\pi_0^R}\frac{1}{1+e^{-\ell\pi_0^R}} \\
&= 2\sum_{k=0}^{\infty}(-1)^k e^{-\frac{\ell}{2}(2k+1)\pi_0^R}.
\end{aligned} \tag{6.3.8}$$

Next we use the Baker-Campbell-Hausdorff formula to find that

$$
\begin{aligned}
e^{-\frac{\ell}{2}\pi_0^T} e^{-ip_0^T x_0^T} &= e^{-\frac{\ell}{2}\pi_0^T - ip_0^T x_0^T + \frac{1}{2}\left[-\frac{\ell}{2}\pi_0^T, -ip_0^T x_0^T\right]} \\
&= e^{-\frac{\ell}{2}\pi_0^T - ip_0^T x_0^T - \frac{\ell}{4}p_0^T} \\
&= e^{-\frac{\ell}{2}\pi_0^T - ip_0^T x_0^T + \frac{\ell}{4}p_0^T} e^{-\frac{\ell}{2}p_0^T} \\
&= e^{-ip_0^T x_0^T} e^{-\frac{\ell}{2}p_0^T} e^{-\frac{\ell}{2}\pi_0^T},
\end{aligned}
\tag{6.3.9}
$$

thus

$$
\begin{aligned}
\frac{1}{\cosh\frac{\ell}{2}\pi_0^R} e^{-ip_0^R x_0^R} &= 2\sum_{k=0}^{\infty}(-1)^k e^{-\frac{\ell}{2}(2k+1)\pi_0^R} e^{-ip_0^R x_0^R}, \\
&= e^{-ip_0^R x_0^R} 2\sum_{k=0}^{\infty}(-1)^k e^{-\frac{\ell}{2}(2k+1)\left(p_0^R+\pi_0^R\right)} \\
&= e^{-ip_0^R x_0^R} \frac{1}{\cosh\frac{\ell}{2}\left(p_0^R+\pi_0^R\right)}.
\end{aligned}
\tag{6.3.10}
$$

Using these results we conclude that

$$
\begin{aligned}
P_1^T \triangleright e^{ipx} e^{iqx} &= \left(P_1^T \triangleright e^{ipx}\right) e^{iqx} + e^{ipx} \frac{e^{-\frac{\ell}{2}p_0^T} e^{-\frac{\ell}{2}\pi_0^T}}{\cosh\frac{\ell}{2}\left(p_0^R+\pi_0^R\right)} \left[\pi_1^T, e^{iqx}\right] \\
&= \left(P_1^T \triangleright e^{ipx}\right) e^{iqx} \\
&\quad + e^{ipx} \frac{e^{-\frac{\ell}{2}p_0^T}}{\cosh\frac{\ell}{2}p_0^R + \tanh\frac{\ell}{2}\pi_0^R \sinh\frac{\ell}{2}p_0^R} \left(P_1^T \triangleright e^{iqx}\right),
\end{aligned}
\tag{6.3.11}
$$

that is the coproduct

$$
\Delta\left(P_1^T\right) = P_1^T \otimes 1 + \frac{e^{-\frac{\ell}{2}P_0^T}}{\cosh\frac{\ell}{2}P_0^R + \tanh\frac{\ell}{2}\pi_0^R \sinh\frac{\ell}{2}P_0^R} \otimes P_1^T. \tag{6.3.12}
$$

Of course, since P_1^T and P_1^R share the same external factor (and the same goes for N^T and N^R actually), we have also that

$$
\Delta\left(P_1^R\right) = P_1^R \otimes 1 + \frac{e^{-\frac{\ell}{2}P_0^T}}{\cosh\frac{\ell}{2}P_0^R + \tanh\frac{\ell}{2}\pi_0^R \sinh\frac{\ell}{2}P_0^R} \otimes P_1^R. \tag{6.3.13}
$$

Finally it's useful to spend some words about the actions of these two generators on the plane waves.

We find in particular that

$$\begin{aligned} P_1^T \triangleright e^{ikx} &= \left[k_1^T + \tanh\frac{\ell}{2}\pi_0^R k_1^R\right] e^{ikx}, \\ P_1^R \triangleright e^{ikx} &= \left[\tanh\frac{\ell}{2}\pi_0^R k_1^T + k_1^R\right] e^{ikx}, \end{aligned} \tag{6.3.14}$$

from which we see that, due to the factor $\tanh\frac{\ell}{2}\pi_0^R$, the plane waves e^{ikx} are not eigenfunctions of the translation generators.
This reflects an interesting property of our generators, i.e. the fact that P_1^T and P_1^R **are not** equal to $-i\frac{\partial}{\partial x_1^T}$ and $-i\frac{\partial}{\partial x_1^R}$ respectively as in the standard Poincaré case or in our first proposal (4.1.10).
Rather, from (6.3.14), it is clear that the differential representation[2] of P_1^T and P_1^R is

$$\begin{aligned} P_1^T &= -i\left[\frac{\partial}{\partial x_1^T} + \tanh\frac{\ell}{2}\pi_0^R \frac{\partial}{\partial x_1^R}\right], \\ P_1^R &= -i\left[\tanh\frac{\ell}{2}\pi_0^R \frac{\partial}{\partial x_1^T} + \frac{\partial}{\partial x_1^R}\right], \end{aligned} \tag{6.3.15}$$

which means that P_1^T translates **both**[3] x_1^T and x_1^R, and the same goes for P_1^R.
It is interesting to note that we can still define a new set of plane waves as

$$\begin{aligned} e^{ik'x} =& e^{i\cosh\frac{\ell}{2}\pi_0^R\left(\cosh\frac{\ell}{2}\pi_0^R k_1^T - \sinh\frac{\ell}{2}\pi_0^R k_1^R\right)x_1^T} \\ &\cdot e^{i\cosh\frac{\ell}{2}\pi_0^R\left(-\sinh\frac{\ell}{2}\pi_0^R k_1^T + \cosh\frac{\ell}{2}\pi_0^R k_1^R\right)x_1^R} e^{-ik_0^T x_0^T} e^{-ik_0^R x_0^R} \end{aligned} \tag{6.3.16}$$

which, by using (6.3.14), can be shown to be eigenfunctions of P_1^T and P_1^R:

$$P_1^T \triangleright e^{ik'x} = k_1^T e^{ik'x}, \qquad P_1^R \triangleright e^{ik'x} = k_1^R e^{ik'x}. \tag{6.3.17}$$

In order to highlight the spatial momenta k_1^T and k_1^R of the plane wave, the definition (6.3.16) can be rearranged as

---

[2] Of course P_0^T and P_0^R are still trivially represented by $i\frac{\partial}{\partial x_0^T}$ and $i\frac{\partial}{\partial x_0^R}$ respectively, since they're just equal to the sum or the difference of P_0^A and P_0^B.

[3] Note that this is true even for finite translations $e^{i\hat{\epsilon}_T P_1^T}$ generated by some quantum parameter $\hat{\epsilon}_T$ (see (6.2.7) and below), since in any case $\left[\pi_1^T, x_1^R\right] \neq 0$ due to the pregeometric representation (6.2.4).

$$e^{ikx'} = e^{ik_1^T \cosh\frac{\ell}{2}\pi_0^R\left(\cosh\frac{\ell}{2}\pi_0^R x_1^T - \sinh\frac{\ell}{2}\pi_0^R x_1^R\right)} \cdot e^{ik_1^R \cosh\frac{\ell}{2}\pi_0^R\left(-\sinh\frac{\ell}{2}\pi_0^R x_1^T + \cosh\frac{\ell}{2}\pi_0^R x_1^R\right)} e^{-ik_0^T x_0^T} e^{-ik_0^R x_0^R}, \tag{6.3.18}$$

so that the new set of coordinates

$$\begin{aligned} x'^T_1 &= \cosh\frac{\ell}{2}\pi_0^R\left(\cosh\frac{\ell}{2}\pi_0^R x_1^T - \sinh\frac{\ell}{2}\pi_0^R x_1^R\right), \\ x'^R_1 &= \cosh\frac{\ell}{2}\pi_0^R\left(-\sinh\frac{\ell}{2}\pi_0^R x_1^T + \cosh\frac{\ell}{2}\pi_0^R x_1^R\right) \end{aligned} \tag{6.3.19}$$

allow to express our generators simply as

$$P_1^T = -i\frac{\partial}{\partial x'^T_1}, \qquad P_1^R = -i\frac{\partial}{\partial x'^R_1}. \tag{6.3.20}$$

Using (6.2.4) we can search for the pregeometric representation of this new set of coordinates.
Quite surprisingly they turn out to have a very familiar face, in fact as one can easily check

$$\begin{aligned} x'^T_1 &= q_1^T e^{\frac{\ell}{2}\pi_0^T}\cosh\frac{\ell}{2}\pi_0^R, \\ x'^R_1 &= q_1^R e^{\frac{\ell}{2}\pi_0^T}\cosh\frac{\ell}{2}\pi_0^R. \end{aligned} \tag{6.3.21}$$

This interesting result suggests that any candidate for the role of potential of a given interaction should be a function of x'^R_1 and not x_1^R.
In fact in this way the potential, say $V\left(x'^R_1\right)$, would be invariant under total translations generated by P_1^T, whereas a potential $V\left(x_1^R\right)$ would not.
Of course one would still have to verify its behaviour under the total boost generator.
We will discuss carefully this point in the next paragraphs and especially in chapter 8.

6.4 More on the pregeometric algebra: commutators

Let us consider a possibile definition of the pregeometric generators by means of the single particle ones:

$$\begin{aligned}
P_1^T &= f\left(\pi_0^R\right) P_1^A + f\left(-\pi_0^R\right) P_1^B, \\
P_1^R &= f\left(\pi_0^R\right) P_1^A - f\left(-\pi_0^R\right) P_1^B, \\
N^T &= f\left(\pi_0^R\right) N^A + f\left(-\pi_0^R\right) N^B, \\
N^T &= f\left(\pi_0^R\right) N^A - f\left(-\pi_0^R\right) N^B,
\end{aligned} \tag{6.4.1}$$

where

$$f\left(\pm\pi_0^R\right) = \frac{e^{\pm\frac{\ell}{2}\pi_0^R}}{\cosh\frac{\ell}{2}\pi_0^R}. \tag{6.4.2}$$

This definition comes down quite directly from the pregeometric construction we built up in 6.2.
We can invert relations (6.4.1) to find

$$\begin{aligned}
P_1^A &= \frac{1}{2}e^{-\frac{\ell}{2}\pi_0^R}\cosh\frac{\ell}{2}\pi_0^R\left(P_1^T + P_1^R\right) = \frac{1}{f\left(\pi_0^R\right)}\frac{P_1^T + P_1^R}{2}, \\
P_1^B &= \frac{1}{2}e^{\frac{\ell}{2}\pi_0^R}\cosh\frac{\ell}{2}\pi_0^R\left(P_1^T - P_1^R\right) = \frac{1}{f\left(-\pi_0^R\right)}\frac{P_1^T - P_1^R}{2}, \\
N^A &= \frac{1}{2}e^{-\frac{\ell}{2}\pi_0^R}\cosh\frac{\ell}{2}\pi_0^R\left(N^T + N^R\right) = \frac{1}{f\left(\pi_0^R\right)}\frac{N^T + N^R}{2}, \\
N^B &= \frac{1}{2}e^{\frac{\ell}{2}\pi_0^R}\cosh\frac{\ell}{2}\pi_0^R\left(N^T - N^R\right) = \frac{1}{f\left(-\pi_0^R\right)}\frac{N^T - N^R}{2},
\end{aligned} \tag{6.4.3}$$

which we will make use of in the following.
We wish now to compute the commutators between the generators of our symmetry algebra, and to compare them with the ones we discussed in chapter 4.

To do so, we must consider that the external factors $f\left(\pm\pi_0^R\right)$ don't commute with N^T or N^R, or rather they don't commute with N^A and N^B in (6.4.1).
It's useful to list two algebraic properties satisfied by these factors, that are

$$\frac{f\left(\pi_0^R\right)+f\left(-\pi_0^R\right)}{2}=1, \tag{6.4.4}$$

and

$$\frac{f\left(\pi_0^R\right)-f\left(-\pi_0^R\right)}{2}=\tanh\frac{\ell}{2}\pi_0^R. \tag{6.4.5}$$

Let us then focus for example on

$$N^A=e^{-\ell\pi_0^A}\left[\tilde{\pi}_0^A q_1^A-\pi_1^A q_0^A,\bullet\right], \tag{6.4.6}$$

with

$$\tilde{\pi}_0^A=\frac{e^{2\ell\pi_0^A}-1}{2\ell}+\frac{\ell}{2}{\pi_1^A}^2. \tag{6.4.7}$$

From (2.8.17) we know that N^A can be rewritten as

$$N^A=x_1^A\tilde{P_0^A}-x_0^A P_1^A \tag{6.4.8}$$

with

$$\tilde{P_0^A}=\frac{1-e^{-2\ell P_0^A}}{2\ell}+\frac{\ell}{2}{P_1^A}^2. \tag{6.4.9}$$

Since π_0^R only has non-trivial commutation relations with $x_0^R=\frac{x_0^A-x_0^B}{2}$, we easily find that

$$\left[N^A,\pi_0^R\right]=iP_1^A. \tag{6.4.10}$$

Therefore the commutator between N^A and a generic function $g\left(\pi_0^R\right)$ will simply be

$$\left[N^A,g\left(\pi_0^R\right)\right]=i\frac{dg}{d\pi_0^R}P_1^A. \tag{6.4.11}$$

The same goes obviously for the particle B generators, thus we conclude that in our specific case

$$\begin{aligned}\left[N^A,f\left(\pm\pi_0^R\right)\right]&=\pm i\frac{\ell}{2}\frac{P_1^A}{\cosh^2\frac{\ell}{2}\pi_0^R}\\ \left[N^B,f\left(\pm\pi_0^R\right)\right]&=\mp i\frac{\ell}{2}\frac{P_1^B}{\cosh^2\frac{\ell}{2}\pi_0^R}.\end{aligned} \tag{6.4.12}$$

Using (6.4.3) we can rewrite this as

$$\begin{aligned}\left[N^A, f\left(\pm\pi_0^R\right)\right] &= \pm i\frac{\ell}{4}\frac{e^{-\frac{\ell}{2}\pi_0^R}}{\cosh\frac{\ell}{2}\pi_0^R}\left(P_1^T + P_1^R\right) \\ &= \pm i\frac{\ell}{4} f\left(-\pi_0^R\right)\left(P_1^T + P_1^R\right)\end{aligned} \tag{6.4.13}$$

and

$$\begin{aligned}\left[N^B, f\left(\pm\pi_0^R\right)\right] &= \mp i\frac{\ell}{4}\frac{e^{\frac{\ell}{2}\pi_0^R}}{\cosh\frac{\ell}{2}\pi_0^R}\left(P_1^T - P_1^R\right) \\ &= \mp i\frac{\ell}{4} f\left(\pi_0^R\right)\left(P_1^T - P_1^R\right),\end{aligned} \tag{6.4.14}$$

respectively.
These two results together allow us to compute

$$\begin{aligned}\left[N^T, f\left(\pm\pi_0^R\right)\right] &= \left[f\left(\pi_0^R\right)N^A + f\left(-\pi_0^R\right)N^B, f\left(\pm\pi_0^R\right)\right] \\ &= \pm i\frac{\ell}{2}\frac{P_1^R}{\cosh^2\frac{\ell}{2}\pi_0^R},\end{aligned} \tag{6.4.15}$$

since obviously

$$f\left(\pi_0^R\right)f\left(-\pi_0^R\right) = \frac{1}{\cosh^2\frac{\ell}{2}\pi_0^R}. \tag{6.4.16}$$

In a similar fashion it's easy to see that

$$\begin{aligned}\left[N^R, f\left(\pm\pi_0^R\right)\right] &= \left[f\left(\pi_0^R\right)N^A - f\left(-\pi_0^R\right)N^B, f\left(\pm\pi_0^R\right)\right] \\ &= \pm i\frac{\ell}{2}\frac{P_1^T}{\cosh^2\frac{\ell}{2}\pi_0^R}.\end{aligned} \tag{6.4.17}$$

Equipped with these results, one can easily compute every single commutator of our algebra through the single particle ones.
We perform the explicit derivation in appendix, the results are

$$
\begin{aligned}
\left[N^T, P_1^T\right] &= i\left[\left(1+\tanh^2\frac{\ell}{2}\pi_0^R\right)\frac{1-e^{-\ell P_0^T}\cosh \ell P_0^R}{\ell}\right.\\
&\quad -2\tanh\frac{\ell}{2}\pi_0^R\frac{e^{-\ell P_0^T}\sinh \ell P_0^R}{\ell}-\frac{\ell}{4}P_1^{T\,2}\\
&\quad \left.-\frac{\ell}{2}\tanh\frac{\ell}{2}\pi_0^R P_1^T P_1^R+\frac{\ell}{4}P_1^{R\,2}\right],\\
\left[N^T, P_1^R\right] &= i\left[\left(1+\tanh^2\frac{\ell}{2}\pi_0^R\right)\frac{e^{-\ell P_0^T}\sinh \ell P_0^R}{\ell}\right.\\
&\quad +2\tanh\frac{\ell}{2}\pi_0^R\frac{1-e^{-\ell P_0^T}\cosh \ell P_0^R}{\ell}\\
&\quad \left.-\frac{\ell}{2}\tanh\frac{\ell}{2}\pi_0^R P_1^{R\,2}\right],\\
\left[N^R, P_1^T\right] &= i\left[\left(1+\tanh^2\frac{\ell}{2}\pi_0^R\right)\frac{e^{-\ell P_0^T}\sinh \ell P_0^R}{\ell}\right.\\
&\quad +2\tanh\frac{\ell}{2}\pi_0^R\frac{1-e^{-\ell P_0^T}\cosh \ell P_0^R}{\ell}\\
&\quad \left.-\frac{\ell}{2}\tanh\frac{\ell}{2}\pi_0^R P_1^{T\,2}\right],\\
\left[N^R, P_1^R\right] &= i\left[\left(1+\tanh^2\frac{\ell}{2}\pi_0^R\right)\frac{1-e^{-\ell P_0^T}\cosh \ell P_0^R}{\ell}\right.\\
&\quad +2\tanh\frac{\ell}{2}\pi_0^R\frac{e^{-\ell P_0^T}\sinh \ell P_0^R}{\ell}+\frac{\ell}{2}P_1^{T\,2}\\
&\quad \left.-\frac{\ell}{2}\tanh\frac{\ell}{2}\pi_0^R P_1^T P_1^R-\frac{\ell}{4}P_1^{R\,2}\right].
\end{aligned}
\tag{6.4.18}
$$

As we mentioned the remaining commutators between the boost generators and the energy operators P_0 are all trivial:

$$\begin{aligned}
\left[N^T, P_0^T\right] &= f\left(\pi_0^R\right)\left[N^A, P_0^A\right] + f\left(-\pi_0^R\right)\left[N^B, P_0^B\right] = iP_1^T, \\
\left[N^T, P_0^R\right] &= f\left(\pi_0^R\right)\left[N^A, P_0^A\right] - f\left(-\pi_0^R\right)\left[N^B, P_0^B\right] = iP_1^R, \\
\left[N^R, P_0^T\right] &= f\left(\pi_0^R\right)\left[N^A, P_0^A\right] - f\left(-\pi_0^R\right)\left[N^B, P_0^B\right] = iP_1^R, \\
\left[N^R, P_0^R\right] &= f\left(\pi_0^R\right)\left[N^A, P_0^A\right] + f\left(-\pi_0^R\right)\left[N^B, P_0^B\right] = iP_1^T.
\end{aligned} \tag{6.4.19}$$

Note that, despite the complex and somehow hard to interpret form of the commutators (6.4.18), the desired behaviour for $\pi_\mu^R, P^R \to 0$ is still achieved.
As we note in a final remark in appendix A, in fact, forcing $\ell\pi_0^R \approx 0$ in (6.4.1) actually makes the pregeometric generators equal to the linear ones, so that by working in that regime one would essentially deal with a unique symmetry algebra.
We will return on this point in chapter 8.
As a last interesting observation, the fact that N^T and N^R do not commute with the factors $f\left(\pi_0^R\right)$ and $f\left(-\pi_0^R\right)$ leads us to an important difference between the pregeometric algebra and the linear one: N^T and N^R don't commute between themselves anymore.
Indeed one finds that

$$\begin{aligned}
\left[N^T, N^R\right] &= \left[f\left(\pi_0^R\right)N^A + f\left(-\pi_0^R\right)N^B, f\left(\pi_0^R\right)N^A - f\left(-\pi_0^R\right)N^B\right] \\
&= -f\left(\pi_0^R\right)\left[N^A, f\left(-\pi_0^R\right)\right]N^B - f\left(-\pi_0^R\right)\left[f\left(\pi_0^R\right), N^B\right]N^A \\
&\quad + f\left(-\pi_0^R\right)\left[N^B, f\left(\pi_0^R\right)\right]N^A + f\left(\pi_0^R\right)\left[f\left(-\pi_0^R\right), N^A\right]N^B \\
&= 2f\left(-\pi_0^R\right)\left[N^B, f\left(\pi_0^R\right)\right]N^A - 2f\left(\pi_0^R\right)\left[N^A, f\left(-\pi_0^R\right)\right]N^B \\
&= -2if\left(-\pi_0^R\right)\frac{\ell}{4}f\left(\pi_0^R\right)\left(P_1^T - P_1^R\right)\frac{1}{f\left(\pi_0^R\right)}\frac{N^T + N^R}{2} \\
&\quad + 2if\left(\pi_0^R\right)\frac{\ell}{4}f\left(-\pi_0^R\right)\left(P_1^T + P_1^R\right)\frac{1}{f\left(-\pi_0^R\right)}\frac{N^T - N^R}{2} \\
&= i\frac{\ell}{4}\left[f\left(-\pi_0^R\right)\left(P_1^T + P_1^R\right) - f\left(\pi_0^R\right)\left(P_1^T - P_1^R\right)\right]N^T \\
&\quad - i\frac{\ell}{4}\left[f\left(-\pi_0^R\right)\left(P_1^T + P_1^R\right) + f\left(\pi_0^R\right)\left(P_1^T - P_1^R\right)\right]N^R,
\end{aligned} \tag{6.4.20}$$

thus in conclusion using again properties (6.4.4) and (6.4.5)

$$\left[N^T, N^R\right] = i\frac{\ell}{2}\left(P_1^R + \tanh\frac{\ell}{2}\pi_0^R P_1^T\right)N^T - i\frac{\ell}{2}\left(P_1^T - \tanh\frac{\ell}{2}\pi_0^R P_1^R\right)N^R. \tag{6.4.21}$$

To summarize, as for the coproducts, we are generally left with several functions of π_0^R, which greatly complicate the structure of the algebra itself.
Yet again it's important to stress that if we are dealing with a system with low relative motion the commutators can be approximated with 2κ-Poincaré ones, since both P_μ^R and π_μ^R can be neglected.
In fact the reduced set of generators $\left\{P_\mu^T, N^T\right\}$ is actually a 2κ-Poincaré algebra in that situation, as in the linear case.
To close this chapter we want to spend some final words about the mass Casimir of the pregeometric algebra.
It's quite obvious that the former candidate

$$C^T = C^A + C^B \tag{6.4.22}$$

is still a good one, given its trivial invariance under every pregeometric generator $\left\{P_\mu^T, P_\nu^R, N^T, N^R\right\}$.
However, it clearly has a different representation in this algebra then the previous expression (4.2.10), which we find useful to show explicity.
Without going into much unnecessary computation, by using equations (6.4.3) one finds that

$$\begin{aligned} C^T = & \frac{8}{\ell^2}\sinh^2\frac{\ell}{4}P_0^T - \frac{8}{\ell^2}\sinh^2\frac{\ell}{4}P_0^R + \frac{16}{\ell^2}\cosh^2\frac{\ell}{4}P_0^T\sinh^2\frac{\ell}{4}P_0^R \\ & - \frac{1}{4}e^{\frac{\ell}{2}P_0^T}e^{\frac{\ell}{2}P_0^R}e^{-\ell\pi_0^R}\cosh^2\frac{\ell}{2}\pi_0^R\left(P_1^T + P_1^R\right)^2 \\ & - \frac{1}{4}e^{\frac{\ell}{2}P_0^T}e^{-\frac{\ell}{2}P_0^R}e^{\ell\pi_0^R}\cosh^2\frac{\ell}{2}\pi_0^R\left(P_1^T - P_1^R\right)^2, \end{aligned} \tag{6.4.23}$$

which as usual becomes the standard 2κ-Poincaré Casimir (2.8.2) when the relative momentum of the particles goes to zero.

6.5 Heisenberg algebra between the boost generators and the space-time coordinates

We wish now to study the relations between the pregeometric total and relative boost generators and the center of mass and relative coordinates x_μ^T and x_μ^R.

In other words we want to generalize the single particle Heisenberg algebra structure, which we showed in (4.2.4) and (4.2.7), in our two particle algebra framework, as we did in paragraph 4.2.
Let us start with the commutator between N^T and x_0^T.
Since trivially $\left[\pi_0^R, x_0^T\right] = 0$ and using the representation (6.4.1), we have that

$$\begin{aligned}
\left[N^T, x_0^T\right] &= \frac{1}{2} f\left(\pi_0^R\right)\left[N^A, x_0^A\right] + \frac{1}{2} f\left(-\pi_0^R\right)\left[N^B, x_0^B\right] \\
&= \frac{i}{2} f\left(\pi_0^R\right)\left(x_1^A - \ell N^A\right) + \frac{i}{2} f\left(-\pi_0^R\right)\left(x_1^B - \ell N^B\right) \\
&= \frac{i}{2}\Big[\left(f\left(\pi_0^R\right) + f\left(-\pi_0^R\right)\right) x_1^T \\
&\qquad + \left(f\left(\pi_0^R\right) - f\left(-\pi_0^R\right)\right) x_1^R \\
&\qquad - \ell\left(f\left(\pi_0^R\right) N^A + f\left(-\pi_0^R\right) N^B\right)\Big] \\
&= i\left(x_1^T + \tanh\frac{\ell}{2}\pi_0^R x_1^R - \frac{\ell}{2} N^T\right).
\end{aligned} \tag{6.5.1}$$

On the other hand $\left[\pi_0^R, x_0^R\right] = i$, so

$$\left[f\left(\pm\pi_0^R\right), x_0^R\right] = \pm i \frac{\ell}{2} \frac{1}{\cosh^2 \frac{\ell}{2}\pi_0^R} = \pm i \frac{\ell}{2} f\left(\pi_0^R\right) f\left(-\pi_0^R\right). \tag{6.5.2}$$

Using this result we find that

$$\begin{aligned}
[N^T, x_0^R] &= \frac{1}{2} f\left(\pi_0^R\right)[N^A, x_0^A] + [f\left(\pi_0^R\right), x_0^R] N^A \\
&\quad - \frac{1}{2} f\left(-\pi_0^R\right)[N^B, x_0^B] + [f\left(-\pi_0^R\right)], x_0^R] N^B \\
&= \frac{i}{2} f\left(\pi_0^R\right)\left(x_1^A - \ell N^A\right) - \frac{i}{2} f\left(-\pi_0^R\right)\left(x_1^B - \ell N^B\right) \\
&\quad + i\frac{\ell}{2} f\left(\pi_0^R\right) f\left(-\pi_0^R\right) \frac{1}{2 f\left(\pi_0^R\right)}\left(N^T + N^R\right) \\
&\quad - i\frac{\ell}{2} f\left(\pi_0^R\right) f\left(-\pi_0^R\right) \frac{1}{2 f\left(-\pi_0^R\right)}\left(N^T - N^R\right),
\end{aligned} \tag{6.5.3}$$

that is

$$
\begin{aligned}
[N^T, x_0^R] = \frac{i}{2}\Big[& \left(f\left(\pi_0^R\right) - f\left(-\pi_0^R\right)\right) x_1^T \\
& + \left(f\left(\pi_0^R\right) + f\left(-\pi_0^R\right)\right) x_1^R \\
& - \ell \left(\cancel{f\left(\pi_0^R\right) N^A - f\left(-\pi_0^R\right) N^B}\right)\Big] \\
+ i\frac{\ell}{4}\Big[& - \left(f\left(\pi_0^R\right) - f\left(-\pi_0^R\right)\right) N^T \\
& + \cancel{\left(f\left(\pi_0^R\right) + f\left(-\pi_0^R\right)\right) N^R}\Big] \\
= i & \left(\tanh\frac{\ell}{2}\pi_0^R x_1^T + x_1^R - \frac{\ell}{2}\tanh\frac{\ell}{2}\pi_0^R N^T\right).
\end{aligned}
\tag{6.5.4}
$$

It's easy to evaluate the last two N^T commutators, that are respectively

$$
\begin{aligned}
\left[N^T, x_1^T\right] &= \frac{1}{2} f\left(\pi_0^R\right)\left[N^A, x_1^A\right] + \frac{1}{2} f\left(-\pi_0^R\right)\left[N^B, x_1^B\right] \\
&= \frac{i}{2} f\left(\pi_0^R\right) x_0^A + \frac{i}{2} f\left(-\pi_0^R\right) x_0^B \\
&= \frac{i}{2}\left[\left(f\left(\pi_0^R\right) + f\left(-\pi_0^R\right)\right) x_0^T + \left(f\left(\pi_0^R\right) - f\left(-\pi_0^R\right)\right) x_0^R\right] \\
&= i\left(x_0^T + \tanh\frac{\ell}{2}\pi_0^R x_0^R\right)
\end{aligned}
\tag{6.5.5}
$$

and

$$
\begin{aligned}
\left[N^T, x_1^R\right] &= \frac{1}{2} f\left(\pi_0^R\right)\left[N^A, x_1^A\right] - \frac{1}{2} f\left(-\pi_0^R\right)\left[N^B, x_1^B\right] \\
&= \frac{i}{2} f\left(\pi_0^R\right) x_0^A - \frac{i}{2} f\left(-\pi_0^R\right) x_0^B \\
&= \frac{i}{2}\left[\left(f\left(\pi_0^R\right) - f\left(-\pi_0^R\right)\right) x_0^T + \left(f\left(\pi_0^R\right) + f\left(-\pi_0^R\right)\right) x_0^R\right] \\
&= i\left(\tanh\frac{\ell}{2}\pi_0^R x_0^T + x_0^R\right).
\end{aligned}
\tag{6.5.6}
$$

It's worth noting that the last commutator isn't automatically zero if $x_0^R = 0$, i.e. the two particles are considered at the same time $x_0^A = x_0^B$, contrarily to what we found in the linear case.
On top of that we already know from 6.3.15 that P_1^T also doesn't commute with x_0^R.
This means that, as aforementioned, an hypothetical potential of the form $V\left(x_1^R\right)$ will not be invariant under neither total translations nor total boosts in the pregeometric picture.

One might then try to find another "relative" coordinate ${x'}_1^R$ which enjoys these properties and consider as valid potentials only the functions of this new coordinate.
We already found a new base $x_1^T, x_1^R \rightarrow {x'}_1^T, {x'}_1^R$ (see (6.3.19)) such that $\left[P_1^T, {x'}_1^R\right] = 0$. But as we'll see in the next paragraph this new coordinate ${x'}_1^R$ still won't commute with N^T even for $x_0^R = 0$.
This issue will be naturally carried over in the Galileian limit of the theory, as we will discuss later on.
To fully complete our picture we just need to evaluate the remaining commutators between the relative boost generator and the coordinates.
Not surprisingly they end up to be just a dual version of the ones we just computed, as can be easily checked following some basically identical steps.
To summarize the resulting deformed Heisenberg Algebra between the boost generators of the pregeometric algebra and the space-time coordinates is

$$\begin{aligned}
\left[N^T, x_0^T\right] &= i\left(x_1^T + \tanh\frac{\ell}{2}\pi_0^R x_1^R - \frac{\ell}{2}N^T\right), \\
\left[N^T, x_0^R\right] &= i\left(\tanh\frac{\ell}{2}\pi_0^R x_1^T + x_1^R - \frac{\ell}{2}\tanh\frac{\ell}{2}\pi_0^R N^T\right), \\
\left[N^T, x_1^T\right] &= i\left(x_0^T + \tanh\frac{\ell}{2}\pi_0^R x_0^R\right), \\
\left[N^T, x_1^R\right] &= i\left(\tanh\frac{\ell}{2}\pi_0^R x_0^T + x_0^R\right), \\
\left[N^R, x_0^T\right] &= i\left(\tanh\frac{\ell}{2}\pi_0^R x_1^T + x_1^R - \frac{\ell}{2}N^R\right), \\
\left[N^R, x_0^R\right] &= i\left(x_1^T + \tanh\frac{\ell}{2}\pi_0^R x_1^R - \frac{\ell}{2}\tanh\frac{\ell}{2}\pi_0^R N^R\right), \\
\left[N^R, x_1^T\right] &= i\left(\tanh\frac{\ell}{2}\pi_0^R x_0^T + x_0^R\right), \\
\left[N^R, x_1^R\right] &= i\left(x_0^T + \tanh\frac{\ell}{2}\pi_0^R x_0^R\right).
\end{aligned} \tag{6.5.7}$$

If we compare this algebra with the previous one (4.2.8), we discover a more intricated relation between the center of mass coordinates and the relative ones, enforced as usual by the familiar term $\tanh\frac{\ell}{2}\pi_0^R$.

6.6 Relation between the total boost generator and ${x'}_1^R$

The goal of this paragraph is to analize more in depth the symmetry properties of the novel relative coordinate ${x'}_1^R$ we first mentioned in (6.3.19), in relation to the possibility of defining a given interaction theory's potential as a function of this coordinate.
Since we already ascertained that ${x'}_1^R$ is well behaved in regard to total translations, i.e. $\left[P_1^T, {x'}_1^R\right] = 0$, we want to establish how it behaves under the total boost generator.
To do so we find useful to recall once again the definition of the new coordinates:

$$\begin{aligned} {x'}_1^T &= \cosh\frac{\ell}{2}\pi_0^R \left(\cosh\frac{\ell}{2}\pi_0^R x_1^T - \sinh\frac{\ell}{2}\pi_0^R x_1^R\right), \\ {x'}_1^R &= \cosh\frac{\ell}{2}\pi_0^R \left(-\sinh\frac{\ell}{2}\pi_0^R x_1^T + \cosh\frac{\ell}{2}\pi_0^R x_1^R\right), \end{aligned} \tag{6.6.1}$$

which can be reverted as

$$\begin{aligned} x_1^T &= {x'}_1^T + \tanh\frac{\ell}{2}\pi_0^R {x'}_1^R, \\ x_1^R &= \tanh\frac{\ell}{2}\pi_0^R {x'}_1^T + {x'}_1^R. \end{aligned} \tag{6.6.2}$$

Since as we can now easily compute

$$\left[N^T, \cosh^2\frac{\ell}{2}\pi_0^R\right] = i\ell\cosh\frac{\ell}{2}\pi_0^R \sinh\frac{\ell}{2}\pi_0^R P_1^R \tag{6.6.3}$$

and

$$\left[N^T, \cosh\frac{\ell}{2}\pi_0^R \sinh\frac{\ell}{2}\pi_0^R\right] = \frac{\ell}{2}\left(\cosh^2\frac{\ell}{2}\pi_0^R + \sinh^2\frac{\ell}{2}\pi_0^R\right) P_1^R, \tag{6.6.4}$$

we have that

$$
\begin{aligned}
\left[N^T, x'^R_1\right] &= \left[N^T, -\cosh\frac{\ell}{2}\pi_0^R \sinh\frac{\ell}{2}\pi_0^R x_1^T + \cosh^2\frac{\ell}{2}\pi_0^R x_1^R\right] \\
&= -\left[N^T, \cosh\frac{\ell}{2}\pi_0^R \sinh\frac{\ell}{2}\pi_0^R\right] x_1^T \\
&\quad - \cosh\frac{\ell}{2}\pi_0^R \sinh\frac{\ell}{2}\pi_0^R \left[N^T, x_1^T\right] \\
&\quad + \left[N^T, \cosh^2\frac{\ell}{2}\pi_0^R\right] x_1^R + \cosh^2\frac{\ell}{2}\pi_0^R \left[N^T, x_1^R\right] \\
&= -i\frac{\ell}{2}\left(\cosh^2\frac{\ell}{2}\pi_0^R + \sinh^2\frac{\ell}{2}\pi_0^R\right) P_1^R x_1^T \\
&\quad - i\cosh\frac{\ell}{2}\pi_0^R \sinh\frac{\ell}{2}\pi_0^R \left(\cancel{x_0^T} + \tanh\frac{\ell}{2}\pi_0^R x_0^R\right) \\
&\quad + i\ell\cosh\frac{\ell}{2}\pi_0^R \sinh\frac{\ell}{2}\pi_0^R P_1^R x_1^R \\
&\quad + i\cosh^2\frac{\ell}{2}\pi_0^R \left(\cancel{\tanh\frac{\ell}{2}\pi_0^R x_0^T} + x_0^R\right),
\end{aligned}
\tag{6.6.5}
$$

where we used the third and fourth equations in (6.5.7).
Now we can use equation (6.6.2) to substitute $\left\{x_1^T, x_1^R\right\}$ with $\left\{x'^T_1, x'^R_1\right\}$ and find

$$
\begin{aligned}
\left[N^T, x'^R_1\right] &= -i\frac{\ell}{2}\left(\cosh^2\frac{\ell}{2}\pi_0^R + \sinh^2\frac{\ell}{2}\pi_0^R\right) P_1^R \left(x'^T_1 + \tanh\frac{\ell}{2}\pi_0^R x'^R_1\right) \\
&\quad + i\ell\cosh\frac{\ell}{2}\pi_0^R \sinh\frac{\ell}{2}\pi_0^R P_1^R \left(\tanh\frac{\ell}{2}\pi_0^R x'^T_1 + x'^R_1\right) \\
&\quad + i\cosh^2\frac{\ell}{2}\pi_0^R \left(1 + \tanh^2\frac{\ell}{2}\pi_0^R\right) x_0^R \\
&= -i\ell\left(\frac{1}{2}\cosh^2\frac{\ell}{2}\pi_0^R + \frac{1}{2}\sinh^2\frac{\ell}{2}\pi_0^R - \sinh^2\frac{\ell}{2}\pi_0^R\right) P_1^R x'^T_1 \\
&\quad + i\ell\tanh\frac{\ell}{2}\pi_0^R (\cosh^2\frac{\ell}{2}\pi_0^R - \frac{1}{2}\cosh^2\frac{\ell}{2}\pi_0^R \\
&\qquad\qquad - \frac{1}{2}\sinh^2\frac{\ell}{2}\pi_0^R) P_1^R x'^R_1 + i x_0^R.
\end{aligned}
\tag{6.6.6}
$$

In conclusion

$$
\left[N^T, x'^R_1\right] = i\left[x_0^R + \frac{\ell}{2}P_1^R\left(-x'^T_1 + \tanh\frac{\ell}{2}\pi_0^R x'^R_1\right)\right]. \tag{6.6.7}
$$

As we anticipated this commutator isn't zero when $x_0^R = 0$.

Not only that: it's easy to recognize that even the commutator

$$\left[\widehat{\epsilon}_T N^T, {x'}_1^R\right] = \widehat{\epsilon}_T \left[N^T, {x'}_1^R\right] \neq 0, \tag{6.6.8}$$

where we reintroduced the quantum parameter $\widehat{\epsilon}_T = \epsilon_T e^{\frac{\ell}{2}\pi_0^T} \cosh \frac{\ell}{2}\pi_0^R$, which clearly commutes with ${x'}_1^R$.
Since ${x'}_1^R$ is the only candidate we have for a generalized relative coordinate given it's good behaviour under total translation, this poses a major obstacle to the possibility of building a proper relativistic interacting theory in the pregeometric framework.
We will analyze more in depth this issue in chapter 8, and discuss in particular how to get around it.

Chapter 7

A more general discussion involving particles with different masses

7.1 The connection between masses and deformation scales

Let us consider three particles with of mass m.
We can define the center of mass coordinates as usual, that is

$$x_\mu^T = \frac{x_\mu^A + x_\mu^B + x_\mu^C}{3}. \tag{7.1.1}$$

If all the three particles belong to a κ-Poincaré algebra with deformation length ℓ, then we find that

$$\begin{aligned} \left[x_1^T, x_0^T\right] &= \frac{1}{9}\left(\left[x_1^A, x_0^A\right] + \left[x_1^B, x_0^B\right] + \left[x_1^C, x_0^C\right]\right) \\ &= i\frac{\ell}{3}x_1^T : \end{aligned} \tag{7.1.2}$$

as expected the deformation length of the composite system scales with the number of particles involved.
We could then proceed as in the previous sections, defining the relative coordinates[1]

[1] This time we don't scale them by any factor.

$$\begin{aligned} x_\mu^{R_1} &= x_\mu^A - x_\mu^B, \\ x_\mu^{R_2} &= x_\mu^A - x_\mu^C, \\ x_\mu^{R_3} &= x_\mu^B - x_\mu^C, \end{aligned} \tag{7.1.3}$$

compute the various commutators between x_μ^T and $x_\mu^{R_i}$, defining the symmetry generators and so on.
Our goal in this chapter however, is to study more in depth the properties of the center of mass coordinates in the most general case when the particles involved possess different masses and/or are associated to different deformation lenghts, i.e. κ-Poincaré algebras.
To start, let us first note that we could consider our three particle system as a sort of two particle system composed by the particle A and the system B, C.
The latter is fully described by the symmetry algebras we discussed in the previous chapters and is identified by its center of mass coordinates

$$x_\mu^{T_{BC}} = \frac{x_\mu^B + x_\mu^C}{2}, \tag{7.1.4}$$

with

$$\left[x_1^{T_{BC}}, x_0^{T_{BC}}\right] = i\frac{\ell}{2} x_1^{T_{BC}}. \tag{7.1.5}$$

Since this system accounts for a $2m$ mass body, it's natural to define the center of mass coordinates of the three particle system as

$$x_\mu^T = \frac{m x_\mu^A + 2m x_\mu^{T_{BC}}}{m + 2m} = \frac{x_\mu^A + 2x_\mu^{T_{BC}}}{3}, \tag{7.1.6}$$

which is actually the same as equation (7.1.1), as it should be.
Using (7.1.5) we find again that

$$\begin{aligned} \left[x_1^T, x_0^T\right] &= \frac{1}{9}\left(\left[x_1^A, x_0^A\right] + 4\left[x_1^{T_{BC}}, x_0^{T_{BC}}\right]\right) \\ &= i\frac{\ell}{3} x_1^T. \end{aligned} \tag{7.1.7}$$

This may look like an overcomputation, but actually suggests how to generalize our construction to systems composed by non identical particles.

Let's say we have two particles A and B with non-zero masses m_A and m_B and deformation lengths[2] ℓ_A and ℓ_B, and let us define the center of mass coordinates as usual:

$$x^T_\mu = \frac{m_A x^A_\mu + m_B x^B_\mu}{m_A + m_B}. \tag{7.1.8}$$

Since

$$\begin{aligned} x^A_\mu &= x^T_\mu + \frac{m_B}{m_A + m_B} x^R_\mu, \\ x^B_\mu &= x^T_\mu - \frac{m_A}{m_A + m_B} x^R_\mu \end{aligned} \tag{7.1.9}$$

where

$$x^R_\mu = x^A_\mu - x^B_\mu \tag{7.1.10}$$

is the relative coordinate, we have that

$$\begin{aligned} \left[x^T_1, x^T_0\right] &= i\frac{\ell_A m_A^2 x^A_1 + \ell_B m_B^2 x^B_1}{(m_A + m_B)^2} \\ &= \frac{\ell_A m_A^2 + \ell_B m_B^2}{(m_A + m_B)^2} x^T_1 + m_A m_B \frac{\ell_A m_A - \ell_B m_B}{(m_A + m_B)^3} x^R_1. \end{aligned} \tag{7.1.11}$$

We see that in general, without further assumptions about the deformation lengths ℓ_A and ℓ_B, equation (7.1.11) mixes up the center of mass coordinates with the relative ones, making it difficult to interpret the commutator $\left[x^T_1, x^T_0\right]$ and its rescaling behaviour.
Nonetheless, it's interesting to note that if for instance the mass of particle A is much larger than the mass of particle B, i.e. $m_A \gg m_B$ or $\frac{m_B}{m_A} \approx 0$, then (7.1.11) can be simplified as

$$\left[x^T_1, x^T_0\right] = i\ell_A x^T_1. \tag{7.1.12}$$

This means that the relativistic properties of the body A, a soccer-ball for instance, are not affected by the interaction with the smaller particle B.
If the body A has a macroscopic mass, however, we still would like it to behave as a special relativistic object.

[2]We already reviewed the possibility of building a **non-universal** relativistic theory for particles belonging to different κ-Poincaré algebras in [17,18].

A possibile solution is to postulate an inverse relation between the deformation length ℓ_A and the mass m_A.
In the working ground we developed in this thesis, this is actually a very natural hypothesis.
In fact we proved that a system composed by many (although identical) particles, say N, would enjoy $\frac{\ell}{N}$-like deformation effects while accounting at the same time for a total mass Nm, ℓ and m being of course the respective single particle parameters.
Howewer in the quantum gravity research field the deformation scale is universally identified with the Planck lenght ℓ_P with some large consesus, whose definition is independent from any given particle one could consider.
More precisely this lenght is generally understood as the scale beyond which the localization of a particle is made impossibile by the combined effects of quantum mechanics (uncertainty principle) and general relativity (black hole creation) [4].
We want to stress that this somehow abstract argument has been developed with a point-particle approximation in mind, thus it could well be missing a key aspect in the spatial extension of the physical bodies, in particular of systems composed by many particles.
Some works [45, 46] actually suggest that the discriminating factor could be the mass-energy density, rather than the mass-energy itself.
As of today, we believe, too little is known about quantum gravity in general and QG low-energy regime in particular to rule out this hypothesis.
Proceeding with our analysis, from equation (7.1.11) we see that if we assume that a particle with non-zero mass m has a deformation parameter

$$\ell = \frac{\widehat{\ell}}{m} \tag{7.1.13}$$

where $\widehat{\ell}$ is some yet unspecified new universal constant[3], then

$$\begin{aligned}\left[x_1^T, x_0^T\right] &= i\frac{\widehat{\ell}}{m_A + m_B} x_1^T \\ &= i\ell_T x_1^T.\end{aligned} \tag{7.1.14}$$

[3] As of now we don't have enough insights on this topic to make assumptions about this quantity, its numerical value and its physical interpretation. Here we will be content to study its formal consequences in the non-commutative multi-particle system theory we developed in this thesis.

This is coherent with the fact that the composite system is a $m_A + m_B$ mass body, thus its deformation length is just

$$\ell_T = \frac{\widehat{\ell}}{m_A + m_B}. \tag{7.1.15}$$

In particular for $m_A = m_B$ we recover the results found in the previous chapters, whereas if $m_A \gg m_B$ then $\ell_T \to \ell_A$.

7.2 Relative coordinates

To conclude this discussion, we would like to show the commutators involving the relative coordinates x^R_μ as well, which will turn out to be quite interesting.
First of all we immediately find that the commutators including one center of mass coordinate x^T_1 or x^T_0 are pretty "standard", in fact

$$\begin{aligned} \left[x^T_1, x^R_0\right] &= \frac{m_A \left[x^A_1, x^A_0\right] - m_B \left[x^B_1, x^B_0\right]}{m_A + m_B} \\ &= i\frac{\widehat{\ell}}{m_A + m_B} x^R_1, \end{aligned} \tag{7.2.1}$$

and the same result yields trivially for $\left[x^R_1, x^T_0\right]$.
The commutator between x^R_1 and x^R_0, however, is quite complex when $m_A \neq m_B$.
By using (7.1.9) we actually find that

$$\begin{aligned} \left[x^R_1, x^R_0\right] &= i\widehat{\ell}\left(\frac{1}{m_A} x^A_1 + \frac{1}{m_B} x^B_1\right) \\ &= i\widehat{\ell}\left[\frac{1}{m_A}\left(x^T_1 + \frac{m_B}{m_A + m_B} x^R_1\right)\right. \\ &\qquad \left. + \frac{1}{m_B}\left(x^T_1 - \frac{m_A}{m_A + m_B} x^R_1\right)\right] \\ &= i\frac{\widehat{\ell}}{\mu} x^T_1 + i\frac{\widehat{\ell}}{\nu} x^R_1, \end{aligned} \tag{7.2.2}$$

where

$$\mu = \frac{1}{\frac{1}{m_A} + \frac{1}{m_B}} \tag{7.2.3}$$

is the well-known reduced **total** mass of the system and

$$\nu = \frac{1}{\frac{1}{m_A} - \frac{1}{m_B}} \tag{7.2.4}$$

is a new characteristic mass which we should call reduced **relative** mass.
Note that for $m_A = m_B$ the reduced relative mass ν diverges, so we can just take

$$\left[x_1^R, x_0^R\right] = i\frac{\widehat{\ell}}{\mu}x_1^T. \tag{7.2.5}$$

Going back to (7.2.2), we recognize that if the mass of particle A is way larger than the mass of particle B, the deformation effects coded in the commutator are governed by the smaller mass m_B rather than the larger one m_A, since $\mu \to m_B$ and $\nu \to -m_B$.
But in the same limit we have that

$$x_\mu^B \approx x_\mu^T - x_\mu^R, \tag{7.2.6}$$

thus

$$\left[x_1^R, x_0^R\right] \approx i\ell_B x_1^B. \tag{7.2.7}$$

We see then that the relative motion will be deformed in any case if one of the particles involved has a very small mass.
One could conjecture that these effects shouldn't be observable on a macroscopic standpoint.
Indeed, when dealing with a macroscopic or rather thermodynamic system, the information we are able to extract from it are usually those concerning its center of mass or total motion, not the relative motion of two of the many particles it is composed of.
On the other hand it can be argued that these effects may add up across all the board, at least in some physical models, resulting in some kind of average ℓ-perturbation of the dynamics of the system itself.
Some sort of "non-commutative" thermodynamics could be needed to actually address this debate.

Here we don't have neither the space nor the tools to accomplish such a demanding task, however we will nonetheless try to lay some basic groundwork in the next and conclusive chapter, by studying the low-velocity limit of our multi-particle non-commutative theory.

Chapter 8

Galilean limit

In this final chapter we want to explore the possibility of building an interaction theory in the non-commutative framework.
To do so, we must first recall a well-known result in special relativity, that is the so-called "no-interaction" theorem [21–23].
This theorem proves that, as a consequence of the existence of an upper limit c to the speed of physical bodies (and therefore to the exhange of signals and informations), there simply can't be a system composed of a finite number of interacting particles.
More precisely it states that if we are given n particles and an Hamiltonian H of the entire system, then the Hamiltonian must be a sum of single particle Hamiltonians.
Thus the interaction is missing from the special relativity realm.
Of course since the theorem holds only for finite number of particles, one could still introduce some kind of interactions by considering an infinite number of them, i.e. the Fock spaces usually enforced in quantum field theories.
Whereas some studies in the last decades have focused on the generalization of field theories to the κ-Poincaré setting [47–51], here we are mostly interested in the finite number of particle case, where we hope to enforce the results we found in the previous chapters.
Since we reasonably believe that the no-interaction theorem should still be valid in the κ-Poincaré realm in some form, even though no generalizations have been fully studied or proposed yet, we find best to work in the so-called **galilean limit** of our multi-particle theories.
The galilean limit in the relativity framework is a specific example of a

more generic mathematical procedure called contraction [52–55].
Given a Lie or Hopf Algebra dependent on some parameter d, the contraction procedure is a mean to obtain a new algebra from it by letting d reach some fixed value, usually zero or infinity.
This procedure not always can be carried out, and sometimes the resulting algebra isn't uniquely determined.
For instance if the starting algebra depends on more than one parameter, say $d_1, d_2, d_3, ...$, and the parameters $d_2, d_3, ...$ depend on the parameter d_1 we want to contract, the contraction limit can sometimes be performed in more than one way.
As we will see the κ-Poincaré algebra is an example of this situation, given the (necessary) dependance of the energy κ on the speed of light c.
Here we are not interested in the technicalities of the contraction procedure in the most general cases, instead we will focus only on the Poincaré and κ-Poincaré ones, which can be understood without much trouble.
For instance, aside from some plain redefinition of the Poincaré algebra generators, the galilean limit of the Poincaré algebra consists simply on sending the speed of light c to infinity, thus obtaining the standard Galilei algebra of classical mechanics.
In the next paragraph we will briefly review this algebra and in particular analyze its connection with the Schrödinger equation of quantum mechanics.
Then we will show how to generalize those considerations to the κ-Poincaré algebra case, in particular when two (identical) particles are involved.
In this way by letting c go to infinity we can easily dodge the no-interaction theorem and its physical consequences, while at the same time still retaining the deformation scale ℓ and, eventually, the non-commutativity of the space-time coordinates.

8.1 The Galilei and the Bargmann algebras

In order to introduce the Galilei algebra, we find useful to start from the Poincaré generators (2.7.3) and unveil their relation with the speed of light c.
Here and in the following we will continue to work in 1+1-dimensions, so that the only meaningful generators are the translation generators P_0, P_1 and the one boost N.

Let us now redefine these generators as

$$\begin{aligned} E &= cP_0, \\ P &= P_1, \\ L &= \frac{N}{c}, \end{aligned} \tag{8.1.1}$$

which by using (2.7.3) are easily found to satisfy the commutation relations

$$\begin{aligned} [E,P] &= [cP_0, P_1] = 0, \\ [L,E] &= \left[\frac{N}{c}, cP_0\right] = iP_1, \\ [L,P] &= \left[\frac{N}{c}, P_1\right] = i\frac{P_0}{c} = i\frac{E}{c^2}. \end{aligned} \tag{8.1.2}$$

In order to enforce duality, we need to redefine the space-time coordinates too as

$$t = \frac{x_0}{c}, \qquad x = x_1, \tag{8.1.3}$$

so that

$$\begin{aligned} [E,t] &= \left[cP_0, \frac{x_0}{c}\right] = i, \qquad & [E,x] &= [cP_0, x_1] = 0, \\ [P,t] &= \left[P_1, \frac{x_0}{c}\right] = 0, \qquad & [P,x] &= [P_1, x_1] = -i. \end{aligned} \tag{8.1.4}$$

Now we just need to let c go to infinity in (8.1.2) to obtain the Galilei algebra

$$\begin{aligned} [E,P] &= 0, \\ [L,E] &= iP, \\ [L,P] &= 0. \end{aligned} \tag{8.1.5}$$

It's interesting to check how this algebra, or rather the boost, acts on the space-time coordinates.
First of all it's convenient to write down the differential representation of the boost generator L.

Using the familiar representations $E = i\hbar\frac{\partial}{\partial t}$ and $P = -i\hbar\frac{\partial}{\partial x}$ of the translation generators[1], it's easy to check that the operator

$$L = i\hbar t\frac{\partial}{\partial x} \tag{8.1.8}$$

satisfies the commutators (8.1.5).
Therefore we get the actions

$$\begin{aligned} L \triangleright t &= 0, \\ L \triangleright x &= i\hbar t. \end{aligned} \tag{8.1.9}$$

The first one tells us that in the galilean relativity time is **absolute**. In fact since a generic boost has the form e^{ivL}, then the transformed time coordinate is just $t' = e^{ivL} \triangleright t = t$.
Therefore if we consider two inertial observers living at the same time, the physics they see will also take place at the same exact time for both, which is a famous assumption of Newtonian mechanics.
On the other hand we have that the transformed spatial coordinate is

$$x' = e^{ivL} \triangleright x = (\mathbb{1} + ivL) \triangleright x = x - vt, \tag{8.1.10}$$

which is just the galilean transformation rule of space for two inertial frames moving with relative velocity v.
A last topic we want to explore is the relation between the Galilei algebra and the Schrödinger equation.
Indeed, it is known that the Schrödinger equation isn't invariant under galilean boosts [56–59].

[1]From now on we will reintroduce the Planck constant $\hbar$. This means that the duality relations between translation generators and coordinates are given by the well-known Heisenberg algebra

$$\begin{aligned} [E, t] &= i\hbar, \qquad [E, x] = 0, \\ [P, t] &= 0, \qquad [P, x] = -i\hbar. \end{aligned} \tag{8.1.6}$$

As a consequence the differential representation of these generators is trivially given by

$$E = i\hbar\frac{\partial}{\partial t}, \qquad P = -i\hbar\frac{\partial}{\partial x}. \tag{8.1.7}$$

To see this let us consider the Schrödinger equation for a free particle with mass m, i.e.

$$i\hbar\frac{\partial}{\partial t}\psi\left(x,t\right)=\frac{P^{2}}{2m}\psi\left(x,t\right). \tag{8.1.11}$$

In the Hilbert formulation of quantum mechanics any symmetry transformation acts on states and observables as

$$\begin{aligned} &\left|\psi\right\rangle\rightarrow\left|\psi'\right\rangle=e^{i\xi X}\left|\psi\right\rangle,\\ &O\rightarrow O'=e^{i\xi X}Oe^{-i\xi X}, \end{aligned} \tag{8.1.12}$$

where X is the generator of the given symmetry and $\xi\in\mathbb{R}$ is the transformation (Lie) parameter.
The Schrödinger equation will be said to be *invariant* if the action of the symmetry transformation results in

$$i\hbar\frac{\partial}{\partial t}\psi'\left(x,t\right)=\frac{P^{2}}{2m}\psi'\left(x,t\right). \tag{8.1.13}$$

This is trivially true for time and space translations, but on the other hand the situation for boost transformations is quite different.
Firstly we immediately recognize that P^2 is actually left invariant by the action of the generic boost e^{ivL}.
In fact it is generally true that given an observable O and a symmetry generator X then

$$O'=e^{i\xi X}Oe^{-i\xi X}=O\iff[X,O]=0; \tag{8.1.14}$$

and P^2 is the quadratic Casimir of the Galilei algebra (8.1.5) as one can trivially check.
However, by using (8.1.8) we find that

$$\begin{aligned} i\hbar e^{ivL}\frac{\partial}{\partial t}\psi\left(x,t\right)&=i\hbar\left(\frac{\partial}{\partial t}e^{ivL}+\left[e^{ivL},\frac{\partial}{\partial t}\right]\right)\psi\left(x,t\right)\\ &=i\hbar\frac{\partial}{\partial t}\psi'\left(x,t\right)+\left[e^{ivL},E\right]\psi\left(x,t\right)\\ &\neq i\hbar\frac{\partial}{\partial t}\psi'\left(x,t\right) \end{aligned} \tag{8.1.15}$$

since

$$\left[e^{ivL},E\right]=-ve^{ivL}P\neq 0. \tag{8.1.16}$$

Thus we conclude that the Schrödinger equation is not invariant under the Galilei algebra.
This can be understood as a consequence of the fact that the Schrödinger operator $E - \frac{P^2}{2m}$ is not invariant under the Galilei algebra, i.e.

$$\left[L, E - \frac{P^2}{2m}\right] \neq 0. \tag{8.1.17}$$

Historically the solution proposed to solve this inconvenience was to substitute the Galilei algebra with a new version having this operator as its quadratic Casimir.
This can be achieved by adding a new generator m to the Galilei algebra which commutes with every other ones (i.e. it's in the *center* of the algebra), but such that the algebra gets modified as

$$\begin{aligned} [E, P] &= 0, \\ [L, E] &= iP, \\ [L, P] &= im. \end{aligned} \tag{8.1.18}$$

This is the so-called **Bargmann** algebra, which, technically speaking, is a *central extension* of the Galilei algebra [60–64].
We will not go anymore deeper in the general theory of central extensions of groups[2], for our purpose it suffices to note that the Schrödinger operator is in fact the quadratic Casimir of the Bargmann algebra:

$$\left[L, E - \frac{P^2}{2m}\right] = iP - 2im\frac{P}{2m} = 0 \tag{8.1.21}$$

and the same goes trivially for the other generators.
Of course the introduction of an external potential in the single particle Schrödinger equation (8.1.13) generally breaks the Bargmann invariance.

[2] In this particular case however, an easy and intuitive way to obtain this algebra is to consider the special relativity dispersion relation for $|P| \ll mc$ (that is the galilean limit in a sense)

$$E = \sqrt{m^2c^4 + P^2c^2} \approx mc^2 + \frac{P^2}{2m}, \tag{8.1.19}$$

which indeed allows us to express the last commutator as

$$[L, P] = i\frac{E}{c^2} = im + i\frac{P^2}{2mc^2} \approx im. \tag{8.1.20}$$

On the other hand when dealing with a system of two interacting particles there's a well-known class of potentials which behave nicely under these transformations, i.e. potentials depending only on the relative space coordinate.
To see this, let us briefly discuss the Bargmann algebra of a two particle system.
We can define as usual the total and relative generators as

$$\begin{aligned} E^T &= E^A + E^B, & E^R &= E^A - E^B, \\ P^T &= P^A + P^B, & P^R &= P^A - P^B, \\ L^T &= L^A - + L^B, & L^R &= L^A - L^B, \end{aligned} \tag{8.1.22}$$

where of course $E^{A,B}$, $P^{A,B}$ and $L^{A,B}$ are the single particle generators.
For the sake of simplicity here and in the following we are assuming that the two particles have the same mass m.
As can be easily checked, the algebra reads as such:

$$\begin{aligned} \left[L^T, E^T\right] &= iP^T, & \left[L^T, E^R\right] &= iP^R, \\ \left[L^T, P^T\right] &= 2im, & \left[L^T, P^R\right] &= 0, \\ \left[L^R, E^T\right] &= iP^R, & \left[L^R, P^R\right] &= iP^T, \\ \left[L^R, P^T\right] &= 0, & \left[L^R, P^R\right] &= 2im, \end{aligned} \tag{8.1.23}$$

where we avoided to explicitly write down the commutators between translation generators since they are all trivially zero.
Note that the commutator between L^T and P^T for example coherently accounts for a system with a total mass $2m$, and that in general $\left[L^T, P^R\right]$ or $\left[L^R, P^T\right]$ ar not zero if the particles have different masses.
The sum of the single particle quadratic Casimirs is itself a Casimir of the algebra (8.1.23), that is

$$\begin{aligned} C^T &= E^A + E^B - \frac{P^{A2}}{2m} - \frac{P^{B2}}{2m} \\ &= E^T - \frac{P^{T2}}{2m} - \frac{P^{R2}}{2m}. \end{aligned} \tag{8.1.24}$$

The Schrödinger equation of a system of two free particles is

$$\left(E^T - \frac{P^{T^2}}{2m} - \frac{P^{R^2}}{2m}\right)\psi = 0, \tag{8.1.25}$$

which is now manifestly invariant.
Of course a system of two free particles isn't particularly interesting, therefore we would like to introduce an *internal* interaction between them while at the same time preserve at least partially the invariance of the Schrödinger equation itself.
Indeed, given a coordinate potential[3] $V\left(x^T, x^R\right)$ we can't expect it to enjoy at the same time both total and relative translation invariance.
What is usually done is to just focus on the total operators E^T, P^T and L^T and ask for invariance under the symmetry transformations they generate.
In order to achieve total translation invariance, we need a potential such that $\left[P^T, V\right] = 0$, which is a condition satisfied only by potentials of the form $V\left(x^R\right)$ since trivially $\left[P^T, x^R\right] = 0$.
If we write down explicitly the total boost generator as

$$L^T = 2mx^T - t^T P^T - t^R P^R \tag{8.1.27}$$

we find that

$$\left[L^T, x^R\right] = i\hbar t^R, \tag{8.1.28}$$

that is zero in our equal-time setup.
Thus we deduce that the Schrödinger equation for two interacting particles with potential $V\left(x^R\right)$ is invariant under the total sector of the Bargmann algebra.
Mentioning this fact is very important to us: our goal in the rest of this chapter is in fact to generalize this same result in the galilean limit of the non-commutative framework we developed in the rest of this thesis.

[3]The total and relative coordinates are defined as usual as

$$\begin{aligned} x^T &= \frac{x^A + x^B}{2}, \qquad t^T = \frac{t^A + t^B}{2}, \\ x^R &= \frac{x^A - x^B}{2}, \qquad t^R = \frac{t^A - t^B}{2}. \end{aligned} \tag{8.1.26}$$

Since we will only deal with *equal-time* interactions (see [38]), t^R is zero and t^T is simply the absolute time t. It is quite obvious that the introduced potential must not depend on the time coordinate.

8.2 Galilean limit of the κ-Poincaré algebra

Before directly exploring the two particle κ-Bargmann theory, we are due to discuss the galilean limit of the κ-Poincaré algebra.
An interesting deviation from the Poincaré case is that the resulting algebra, called κ-Galilei [65–68], depends on how one assumes the deformation scale κ depends on the speed of light c (as we will soon show it must depend on it one way or another).
These algebras are generally presented at every order in ℓ; however, their central extensions have been proven in general quite tough to deal with.
Here we will review a very specific one, such that the non-commutation relations between the coordinates of the resulting κ-**Newton** space-time are preserved in the galilean limit.
Moreover, we will develop our considerations only on a first order expansion in ℓ, which, given the smallness of the deformation scale, will plainly suffice for our purposes.
To start, as we previously did for the Poincaré algebra, we have to explicit any factor of c hidden in the κ-Poincaré algebra's generators and κ-Minkowski coordinates, that is again

$$\begin{aligned} E &= cP_0, \\ P &= P_1, \\ L &= \frac{N}{c} \end{aligned} \tag{8.2.1}$$

and

$$t = \frac{x_0}{c}, \qquad x = x_1. \tag{8.2.2}$$

With this prescription we can rewrite the κ-Poincaré algebra commutators as

$$\begin{aligned} [E, P] &= 0, \\ [L, E] &= iP, \\ [L, P] &= \frac{i}{c}\left(\frac{1 - e^{-2\frac{\ell}{c}E}}{2\ell} - \frac{\ell}{2}P^2\right). \end{aligned} \tag{8.2.3}$$

As aforementioned, we easily recognize that if ℓ is a free parameter in the galilean limit, then the last commutator is zero and we are trivially left with an undeformed Galilei algebra.
At the same time the specific form of this commutator suggests to define a new deformation scale $\tilde{\ell} = \frac{\ell}{c}$ and to assume that $\tilde{\ell}$ remains a non zero finite value for $c \to \infty$.
It's worth noting, however, that in some approaches [67, 68] the deformation scale is redefined as $\tilde{\ell} = \ell c$ and a mass-like generator is directly introduced.
This leads to a very different (centrally extended) algebra, where, for instance, space and time coordinates do commute: $[x, t] = 0$.
Another consequence of this specific galilean limit is that the deformation effects tend to grow bigger with the mass of the particles in consideration, which is a feature we clearly don't wish in our model.
Proceeding with our choice, we can rewrite the last commutator in (8.2.3) as

$$[L, P] = i\left(\frac{1 - e^{-2\tilde{\ell}E}}{2\tilde{\ell}c^2} - \frac{\tilde{\ell}}{2}P^2\right), \tag{8.2.4}$$

which in the limit $c \to \infty$ is just

$$[L, P] = -\frac{\tilde{\ell}}{2}P^2. \tag{8.2.5}$$

Similarly the coproducts can be easily found out to be

$$\begin{aligned}
\Delta(E) &= E \otimes \mathbb{1} + \mathbb{1} \otimes E, \\
\Delta(P) &= P \otimes \mathbb{1} + e^{-\tilde{\ell}E} \otimes P, \\
\Delta(L) &= L \otimes \mathbb{1} + e^{-\tilde{\ell}E} \otimes L,
\end{aligned} \tag{8.2.6}$$

where the first order expansion in $\tilde{\ell}$ is implied.
The fact that the κ-Poincaré coalgebra structure is basically preserved is very important to us: as a direct consequence the κ-Newton cooordinates still coherently satisfy the usual commutation relation

$$[x, t] = i\tilde{\ell}x. \tag{8.2.7}$$

In this way we managed to build up a non-relativistic quantum space-time, which is a very fitting for our goals.

However, we still need a central extension of this κ-Galilei algebra.
As we mentioned constructing a central extension of this algebra is generally a challenging task. Here however we will focus on the easiest possible candidate in a first order approximation, that is the κ-Bargmann algebra

$$\begin{aligned}
&[E,P]=0,\\
&[L,E]=iP,\\
&[L,P]=i\left(m-\frac{\tilde{\ell}}{2}P^2\right).
\end{aligned} \tag{8.2.8}$$

This accounts for the hypothesis that there are no $\sim \tilde{\ell}m$ additional terms to be put in the last commutator.
On top of that we require this algebra to have the same deformed coproducts as in (8.2.6), and thus the same commutation relation (8.2.7) between the space and time coordinates.
Of course we trivially recover the standard Bargmann algebra when $\tilde{\ell}$ goes to zero.
Note that the Heisenberg algebra as well turns out to be deformed as in the κ-Poincaré case; indeed it's easy to check that the duality relations

$$\begin{aligned}
&[E,t]=i\hbar, && [E,x]=0,\\
&[P,t]=-i\hbar\tilde{\ell}P, && [P,x]=-i\hbar.
\end{aligned} \tag{8.2.9}$$

do hold.
The last ingredient we need to complete this picture is the mass Casimir of the κ-Bargmann algebra (8.2.8).
This is given by the operator

$$C^2=\frac{e^{\tilde{\ell}E}-1}{\tilde{\ell}}-e^{\tilde{\ell}E}\frac{P^2}{2m}. \tag{8.2.10}$$

Indeed we have that

$$
\begin{aligned}
\left[L, C^2\right] &= \left[L, \frac{e^{\tilde{\ell}E} - 1}{\tilde{\ell}}\right] - \left[L, e^{\tilde{\ell}E}\frac{P^2}{2m}\right] \\
&= ie^{\tilde{\ell}E}P - \left[L, e^{\tilde{\ell}E}\right]\frac{P^2}{2m} - e^{\tilde{\ell}E}\left[L, \frac{P^2}{2m}\right] \\
&= ie^{\tilde{\ell}E}P - i\tilde{\ell}e^{\tilde{\ell}E}\frac{P^3}{2m} - 2ie^{\tilde{\ell}E}\frac{m - \frac{\tilde{\ell}}{2}P^2}{2m}P \\
&= 0,
\end{aligned}
\tag{8.2.11}
$$

while the other commutators with E and P are as always trivially zero.
It's useful to explicitly write the first order expansion of the mass Casimir (8.2.10), that is

$$
C^2 \approx E - \frac{P^2}{2m} + \frac{\tilde{\ell}}{2}E^2 - \tilde{\ell}E\frac{P^2}{2m}. \tag{8.2.12}
$$

This allows us to generalize the Schrödinger equation for a free particle in the κ-Bargmann picture at the first order in $\tilde{\ell}$, i.e.

$$
\left(E - \frac{P^2}{2m} + \frac{\tilde{\ell}}{2}E^2 - \tilde{\ell}E\frac{P^2}{2m}\right)\psi = 0. \tag{8.2.13}
$$

In the next paragraph we will work out a couple examples of single particle quantum mechanics which will prove useful later on, when we'll discuss a specific model of non-commutative two particle interaction theory.

8.3 Some single particle deformed quantum mechanics

Let us consider a particle in the κ-Newton space-time subject to some potential V.
To determine its possible states we can write the Schrödinger equation[4]

$$
E\psi(x,t) = \left[\frac{P^2}{2m} - \frac{\ell}{2}E^2 + \ell E\frac{P^2}{2m} + V\right]\psi(x,t), \tag{8.3.1}
$$

[4]From now on with slight abuse of notation we will call $\tilde{\ell}$ simply ℓ.

and if the potential V is time independent we can search for solutions of the form $\psi(x,t) = \psi(x)\phi(t)$.
In particular if we denote with

$$H = \left[\frac{P^2}{2m} - \frac{\ell}{2}E^2 + \ell E\frac{P^2}{2m} + V\right] \tag{8.3.2}$$

the time-independent[5] Hamiltonian of the system, we can look for a base $\psi_{E_\alpha}(x)$ of autofunctions

$$H\psi_{E_\alpha}(x) = E_\alpha \psi_{E_\alpha}(x) \tag{8.3.3}$$

so that for each of these we can easily write down the solution of the Schrödinger equation (8.3.1) as $\psi_{E_\alpha}(x,t) = \psi_{E_\alpha}(x)\, e^{-\frac{i}{\hbar}E_\alpha t}$.
We see then that standard quantum mechanics unveils quite normally despite the non-commutative regime we have put ourselves into. Nonetheless some important differences must be highlighted.
First of all the Hamiltonian in (8.3.1) is deformed by ℓ both in presence and absence of an external potential V. Therefore some kind of ℓ-corrections of the standard theory must arise even for free particles.
Moreover, the non-primitive coproducts (8.2.6) force us to use the time to the right order for the wave functions, so that the spatial translation generator doesn't have a weird action on the states in the Schrödinger equation (8.3.1).
This would be in general unavoidable for products of wave functions.
Similarly the non-commutativity of the space-time coordinates makes quite tricky the study of the Schrödinger equation for time-dependent potentials V, and as a consequence for the time-dependent perturbation theory.
Finally there's a more profound interpretational difference from the standard quantum mechanics. As a result of non-commutativity, in fact, here arises a new uncertainty relation between impulse and time in (8.2.9).
Here we won't explore any deeper this last observations, instead our goal is to show the best way to solve the generalized Schrödinger equation for a couple of well-known, useful situations.
In particular we will study two different cases: the case of a free particle in a box and that of an external harmonic potential.
In both cases, and also in the following paragraphs, the best way to address the problem is to make use of the time-independent perturbation theory.

[5] Note that the time-derivative operator, the energy E, can be replaced at the first order in ℓ by a function of only P and V, as we will show explicitly in the continuation.

In fact, for assumption, the deformation scale ℓ is a very small quantity and thus can be used as a perturbation parameter in (8.3.1).
Therefore we will rewrite the Schrödinger equation as

$$E\psi\left(x,t\right)=\left(H_0+\ell\tilde{V}\right)\psi\left(x,t\right), \tag{8.3.4}$$

where $H_0=\frac{P^2}{2m}+V$ is the standard Hamiltonian and $\tilde{V}$ is the perturbation; in our case

$$\tilde{V}=\frac{1}{2}H_0\left(\frac{P^2}{m}-H_0\right)=\frac{1}{2}H_0\left(H_0-2V\right). \tag{8.3.5}$$

If we are dealing with a non-degenerate discrete spectrum labelled by an index n, then the eigenvalues and the eigenstates of the system are modifed as

$$\begin{aligned}E_n&=E_n^{(0)}+\ell\tilde{V}_{nn},\\ \left|n\right\rangle&=\left|n^{(0)}\right\rangle+\ell\sum_{k\neq n}\frac{\tilde{V}_{kn}}{E_n^{(0)}-E_k^{(0)}}\left|k^{(0)}\right\rangle,\end{aligned} \tag{8.3.6}$$

where $\tilde{V}_{kn}=\left\langle k^{(0)}\right|\tilde{V}\left|n^{(0)}\right\rangle$ are the matrix elements of the perturbation $\tilde{V}$.
Note that we coherently kept ourselves at the first order in ℓ.
Let us now consider the case of a particle in a box of length L. The zero order eigenstates and eigenvalues are well-known:

$$\begin{aligned}E_n^{(0)}&=\frac{\pi^2\hbar^2}{2mL^2}n^2,\\ \left|n^{(0)}\right\rangle&=\sqrt{\frac{2}{L}}\sin\frac{n\pi}{L}x.\end{aligned} \tag{8.3.7}$$

Due to the abscence of an external potential we can write $\tilde{V}$ simply as[6] $\frac{E^2}{2}$,thus we easily recognize that the perturbation is actually diagonal: $\tilde{V}_{kn}=0$ when $k\neq n$.
As a consequence the eigenstates of the system remains unmodified by the perturbation: $\left|n\right\rangle=\left|n^{(0)}\right\rangle$.

[6]Here and in the following, even though a bit improperly, we will often identify E with H_0. Wheter it must be interpreted as the Hamiltonian or an eigenvalue should howewer be clear from the context.

However, coherently with the new deformed dispersion relation of our theory, the eigenstates gets modified as

$$\begin{aligned} E_n &= E_n^{(0)} + \frac{\ell}{2} \langle n | E^2 | n \rangle \\ &= E_n^{(0)} \left(1 + \frac{\ell}{2} E_n^{(0)} \right) \\ &= \frac{\pi^2 \hbar^2}{2mL^2} n^2 \left(1 + \ell \frac{\pi^2 \hbar^2}{4mL^2} n^2 \right). \end{aligned} \tag{8.3.8}$$

It's worth to note that the same result would also yield for a free particle: unmodified eigenstates (plane waves, that is) with new characteristic energies $E = E_0 + \frac{\ell}{2} E_0^2$.
The last case we want to study, as mentioned, is that of an external harmonic potential.
The standard quantum mechanics Hamiltonian of the system is given by

$$H = \frac{P^2}{2m} + \frac{1}{2} m \omega^2 x^2, \tag{8.3.9}$$

with eigenstates and eigenvalues

$$\begin{aligned} E_n^{(0)} &= \hbar \omega \left(n + \frac{1}{2} \right), \\ |n^{(0)}\rangle &= \frac{1}{\sqrt{2^n n!}} \left(\frac{m\omega}{\pi \hbar} \right)^{\frac{1}{4}} e^{-\frac{m\omega}{2\hbar} x^2} H_n \left(\sqrt{\frac{m\omega}{\hbar}} x \right). \end{aligned} \tag{8.3.10}$$

Here $H_n(x)$ are the well-known Hermite polynomials:

$$H_n(x) = (-1)^n e^{x^2} \frac{d^n}{d^n x} e^{-x^2}. \tag{8.3.11}$$

It's useful now to list a property of the autofunctions (8.3.10) which we will use later on.
Firstly to lighten the notation let us rewrite them as

$$u_n(x) = \sqrt{\frac{a}{\pi^{\frac{1}{2}} 2^n n!}} H_n(ax) e^{-\frac{a^2}{2} x^2}, \qquad a = \sqrt{\frac{m\omega}{\hbar}}, \tag{8.3.12}$$

so that the harmonic potential also can be rewritten as $V = \frac{1}{2} \hbar \omega a^2 x^2$.

Given the orthonormality of the autofunctions (8.3.12) and by using the recurrence relations [69]

$$2xH_n(x) = H_{n+1}(x) + 2nH_{n-1}(x) \tag{8.3.13}$$

and

$$4x^2H_n(x) = H_{n+2}(x) + 2(2n+1)H_n(x) + 2n(2n-1)H_{n-2}(x)\,, \tag{8.3.14}$$

it can be proven with no difficulty that

$$\int_{-\infty}^{\infty} u_k(x)\, x^2 u_n(x) = \begin{cases} \dfrac{\sqrt{n(n-1)}}{2a^2} & k = n-2, \\ \dfrac{2n+1}{2a^2} & k = n, \\ \dfrac{\sqrt{(n+1)(n+2)}}{2a^2} & k = n+2, \\ 0 & \text{otherwise.} \end{cases} \tag{8.3.15}$$

This tells us how the harmonic potential V connects different eigenstates of our system, that is the value of $\langle n^{(0)}|\, V\, |k^{(0)}\rangle$ for every n and k.
Since from (8.3.9) at the zero order we have that $\frac{p^2}{2m} = E - V$, the perturbation $\tilde{V}$ can be nicely expressed as

$$\tilde{V} = \frac{1}{2}E\left[(2E-2V) - E\right] = \frac{1}{2}E\,(E-2V)\,. \tag{8.3.16}$$

Thus we can now write the matrix elements of the perturbation as

$$\begin{aligned} \langle k^{(0)}|\, \tilde{V}\, |n^{(0)}\rangle &= \frac{1}{2}\left(\langle k^{(0)}|\, E^2\, |n^{(0)}\rangle - \langle k^{(0)}|\, EV\, |n^{(0)}\rangle\right) \\ &= \frac{\hbar^2\omega^2}{2}\left(n+\frac{1}{2}\right)^2 \delta_{n,k} - \hbar\omega\left(k+\frac{1}{2}\right)\langle k^{(0)}|\, V\, |n^{(0)}\rangle\,. \end{aligned} \tag{8.3.17}$$

From (8.3.15) we know that, fixed n, the matrix elements of the harmonic potential V are non-zero only for $k = n \pm 2$ or $k = n$.

As a consequence there turn out to be only three possibilities:

$$\tilde{V}_{kn} = \begin{cases} -\dfrac{\hbar^2\omega^2}{4}\left(k+\dfrac{1}{2}\right)\sqrt{n\left(n-1\right)} & k = n-2, \\ \dfrac{\hbar^2\omega^2}{2}\left[\left(n+\dfrac{1}{2}\right)^2 - \dfrac{1}{2}\left(k+\dfrac{1}{2}\right)\left(2n+1\right)\right] & k = n, \\ -\dfrac{\hbar^2\omega^2}{4}\left(k+\dfrac{1}{2}\right)\sqrt{\left(n+1\right)\left(n+2\right)} & k = n+2, \end{cases} \tag{8.3.18}$$

where we used the fact that $\frac{m\omega^2}{a^2} = \hbar\omega$.
We recognize immediately that the middle one is actually zero, thus with a final algebra manipulation we can rewrite this as

$$\tilde{V}_{kn} = \begin{cases} -\dfrac{\hbar^2\omega^2}{4}\left(n-\dfrac{3}{2}\right)\sqrt{n\left(n-1\right)} & k = n-2, \\ -\dfrac{\hbar^2\omega^2}{4}\left(n+\dfrac{5}{2}\right)\sqrt{\left(n+1\right)\left(n+2\right)} & k = n+2. \end{cases} \tag{8.3.19}$$

We deduce that the energy levels of the harmonic oscillator, contrarily to the previous case, remains unchecked: $E_n = \hbar\omega\left(n+\frac{1}{2}\right)$.
On the other hand, using (8.3.8) we find that the eigenstates get deformed into

$$|n\rangle = |n^{(0)}\rangle - \ell d_{n-2}\,|(n-2)^{(0)}\rangle + \ell d_{n+2}\,|(n+2)^{(0)}\rangle\,, \tag{8.3.20}$$

where we defined the deformation factors

$$\begin{aligned} d_{n-2} &= \frac{\hbar\omega}{8}\left(n-\frac{3}{2}\right)\sqrt{n\left(n-1\right)}, \\ d_{n+2} &= \frac{\hbar\omega}{8}\left(n+\frac{5}{2}\right)\sqrt{\left(n+1\right)\left(n+2\right)}. \end{aligned} \tag{8.3.21}$$

In conclusion we have shown two specific examples of first-order deformed quantum theory. As we will see the results we obtained here will prove very useful in the continuation, when we will address the generalization of all this to an interacting system of two identical particles.

8.4 Two-particle deformed quantum mechanics

Let us consider two particles A and B of equal mass m. We assume both of them are related to two κ-Bargmann algebras (8.2.8) with generators E^A, P^A, L^A and E^B, P^B, L^B.
We know from the previous paragraphs that the space-time coordinates of this particles satisfy a κ-deformed commutation relation:

$$\left[x^A, t^A\right] = i\ell x^A, \qquad \left[x^B, t^B\right] = i\ell x^B, \tag{8.4.1}$$

where again we redefined $\ell = \frac{\ell}{c}$.
As we did in chapters 4 or (6), we can then define the composite system's total and relative coordinates as

$$\begin{aligned} x^T &= \frac{x^A + x^B}{2}, \qquad & t^T &= \frac{t^A + t^B}{2}, \\ x^R &= \frac{x^A - x^B}{2}, \qquad & t^R &= \frac{t^A - t^B}{2}, \end{aligned} \tag{8.4.2}$$

which not surprisingly satisfy the 2κ commutation relations we found in those chapters, i.e.

$$\begin{aligned} \left[x^T, t^T\right] &= i\frac{\ell}{2}x^T, \qquad & \left[x^T, t^R\right] &= i\frac{\ell}{2}x^R, \\ \left[x^R, t^T\right] &= i\frac{\ell}{2}x^R, \qquad & \left[x^R, t^R\right] &= i\frac{\ell}{2}x^T. \end{aligned} \tag{8.4.3}$$

Thus, we have again two possible definitions of the total and relative generators:

$$\begin{aligned} E^T &= E^A + E^B, \qquad & E^R &= E^A - E^B, \\ P^T &= P^A + P^B, \qquad & P^R &= P^A - P^B, \\ L^T &= L^A + L^B, \qquad & L^R &= L^A - L^B, \end{aligned} \tag{8.4.4}$$

or[7]

[7]Note that the pregeometric representation (5.0.3) remains basically untouched by the

$$\begin{aligned}
E^T &= E^A + E^B, & E^R &= E^A - E^B, \\
P^T &= f\left(\epsilon^R\right) P^A + f\left(-\epsilon^R\right) P^B, & P^R &= f\left(\epsilon^R\right) P^A - f\left(-\epsilon^R\right) P^B, \\
L^T &= f\left(\epsilon^R\right) L^A + f\left(-\epsilon^R\right) L^B, & L^R &= f\left(\epsilon^R\right) L^A - f\left(-\epsilon^R\right) L^B,
\end{aligned} \tag{8.4.7}$$

where

$$f\left(\pm\epsilon^R\right) = \frac{e^{\pm\frac{\ell}{2}\epsilon^R}}{\cosh\frac{\ell}{2}\epsilon^R} \approx 1 \pm \frac{\ell}{2}\epsilon^R \tag{8.4.8}$$

are the usual external factors we already know from (6.4.1).
In each case the mass Casimir of the system is simply

$$C^T = C^A + C^B \tag{8.4.9}$$

where $C^{A,B}$ are the single particle Casimirs (8.2.10); however, the actual expression of C^T as a function of E^T, E^R, P^T and P^R depends of course on the specific definition chosen for these generators.
As we highlighted in the previous paragraphs, this Casimir gives us the generalized Schrödinger equation for a system of two **free** identical particles.
In order to describe an actual interaction between the particles involved, though, we need to add some internal potential V in the Schrödinger equation itself.
In particular we require this potential to be compatible with the total symmetries of the two particle algebra, that is $\left[\widehat{\xi}_T O^T, V\right]$ must be zero for each of the total generators O^T multiplied by the appropriate quantum parameter $\widehat{\xi}_T$ in (8.4.4) or (8.4.7) (see the analogous discussion in paragraph 8.1 for the standard quantum mechanics case).
However, both the proposals (8.4.4) and (8.4.7) present some issues.

---

galilean limit. Indeed if we redefine

$$\epsilon = c\pi_0, \qquad \pi = \pi_1 \tag{8.4.5}$$

then the pregeometric factor $\ell\pi_0$ becomes trivially $\tilde{\ell}\epsilon$ (or $\ell\epsilon$ in our simplified notation).
Thus we can easily write the total and relative pregeometric factors as

$$\begin{aligned}
\epsilon^T &= \epsilon^A + \epsilon^B, & \epsilon^R &= \epsilon^A - \epsilon^B, \\
\pi^T &= \pi^A + \pi^B, & \pi^R &= \pi^A - \pi^B.
\end{aligned} \tag{8.4.6}$$

First of all we showed in chapter 4 that the first algebra (8.4.4) is characterized by some inconsistencies, or rather unclear relations between the symmetry generators and the actual finite transformations, as in (4.1.30) and below.
In other words, we don't know how to associate the algebra generators with proper quantum parameters.
On the other hand we showed in the same chapter in section (4.2) that the total boost generator pairs well with any potential of the form $V\left(x^R_1\right)$, that is $\left[N^T, V\right] = 0$ if the particles live at equal times.
It's easy to recognize that this property is inherited by the galilean limit[8], and since $\left[P^T, V\right]$ is obviously zero too, one would be tempted to use $V\left(x^R\right)$ as a good interaction potential.
The situation is inverted for the pregeometric algebra (8.4.7).
In fact the transition from symmetry generators to finite transformations runs smoothly by definition (see (6.2.5) and the following considerations). Unfortunately though, we found that the pregeometric total spatial translation generator acts non-trivially on the relative coordinate x^R_1, and accordingly $\left[P^T, V\left(x^R_1\right)\right] \neq 0$.
Thus we were forced to introduce a new set of coordinates x'^T_1 and x'^R_1 in (6.3.19) such that

$$P^T_1 = -i\frac{\partial}{\partial x'^T_1}, \qquad P^R_1 = -i\frac{\partial}{\partial x'^R_1} \tag{8.4.10}$$

and as a consequence one would recover $\left[P^T_1, V\right] = 0$ for any potential of the form $V\left(x'^R_1\right)$.
Yet in section (6.6) we found that the same potentials can't be invariant under total boosts, that is $\left[N^T, V\left(x'^R_1\right)\right] \neq 0$.
Again, all this properties are naturally inherited in the galilean limit.
Here however, we know that exists a specific regime where both of these issues fade away.
In fact, in the low relative energy approximation $\ell P^R_0 \approx 0$ and $\ell \pi^R_0 \approx 0$ the pregeometric and the linear algebra generators are actually all equal, and enclose a 2κ-Poincaré-like algebra, or a 2κ-Bargmann-like in the galilean limit.

[8]To be fair one should still consider the introduction of the mass term in the κ-Bargmann algebra (8.2.8).
However, it's quite trivial to see that this term doesn't partecipate in the commutator between L^T and x^R since it necessarily appears in the form $\sim mx$, which commutes with x^R.

To quickly see how this unties all the knots, let us consider the total spatial translation generator of the linear algebra (8.4.4).
It's immediate to recognize that it can be written in the pregeometric representation as

$$P^T = e^{-\frac{\ell}{2}\epsilon^T}\left[\pi^T, \bullet\right], \tag{8.4.11}$$

which we can now easily pair with a quantum parameter $\widehat{\xi}_T = \xi_T e^{\frac{\ell}{2}\epsilon^T}$.
The same goes for the total boost generator $L^T = L^A + L^B$.
On the other hand if $\ell\epsilon^R \approx 0$ the pregeometric boost generator in (8.4.7) is actually just $L^T = L^A + L^B$, which we already know satisfies $\left[\widehat{\xi}_T L^T, x^R\right] = 0$ for $t^R = 0$ (note from (6.3.19) that $x'^{R,T} \approx x^{R,T}$ under these conditions).
In conclusion any potential of the form $V\left(x^R\right)$ would be invariant under the total symmetries of our (unique) algebra in this specific case.
There's actually a third reason for which we find desirable to work in this regime.
If we take a closer look at the two particle mass Casimir (8.4.9):

$$\left[E^A - \frac{{P^A}^2}{2m} + \frac{\ell}{2}{E^A}^2 - \ell E^A \frac{{P^A}^2}{2m}\right] + \left[E^B - \frac{{P^B}^2}{2m} + \frac{\ell}{2}{E^B}^2 - \ell E^B \frac{{P^B}^2}{2m}\right] \tag{8.4.12}$$

we immediately recognize that the operator E^R inevitably makes its appareance at the first order in ℓ, regardless of the choice of generators we made between (8.4.4) or (8.4.7)
In general, as we did for the single particle case, we would like to substitute E^T and E^R with their zero order expression wherever they are multiplied by the deformation scale ℓ in the Schrödinger equation.
However, in presence of an internal potential V the relative energy E^R is not a well defined operator on the standard quantum mechanics Hilbert space of solutions of the Schrödinger equation.
In other words, we simply don't know how to treat its action on the wave functions.
Of course all of this gets completely bypassed if, again, we take $\ell E^R \approx 0$.
Some more words are due about this approximation.
In principle, assuming $\ell E^R \approx 0$ is actually pretty reasonable. In fact working in the galilean means that the particles involved travel with speeds way smaller than the speed of light and as a consequence their energy are in general small compared to the deformation energy κ.

The real subtlety here is to justify how at the same time ℓE^R is negligible whereas ℓE^T is not.
Another different route would be instead to consider only weak interactions, namely $V \sim \ell$ potentials. In this way the system would be free at the zero order, and thus at the same order $E^R = \frac{P^{A2}}{2m} - \frac{P^{B2}}{2m}$ would be a well-defined operator.
At any rate, we believe that working with this approximation will suffice at least for this first preliminary exploration, and leave further investigations to other studies.
Let us then consider again the mass Casimir (8.4.9).
In order to find the generalized Schrödinger equation for the composite system, we can substitute the single particle generators with the linear algebra ones in (8.4.4), for instance, and then neglect ℓE^R in the resulting expression.
Using the relations

$$\begin{aligned} E^{A2} + E^{B2} &= \frac{E^{T2} + E^{R2}}{2}, \\ P^{A2} + P^{B2} &= \frac{P^{T2} + P^{R2}}{2} \end{aligned} \tag{8.4.13}$$

we have that

$$\begin{aligned} C^T &= E^A + E^B - \frac{P^{A2}}{2m} - \frac{P^{B2}}{2m} + \frac{\ell}{2}E^{A2} + \frac{\ell}{2}E^{B2} - \ell E^A \frac{P^{A2}}{2m} - \ell E^B \frac{P^{B2}}{2m} \\ &= E^T - \frac{P^{T2}}{4m} - \frac{P^{R2}}{4m} + \frac{\ell}{4}E^{T2} + \frac{\ell}{4}E^{R2} - \ell\left(E^T + E^R\right)\frac{\left(P^T + P^R\right)^2}{16m} \\ &\quad - \ell\left(E^T - E^R\right)\frac{\left(P^T - P^R\right)^2}{16m} \\ &= E^T - \frac{P^{T2}}{4m} - \frac{P^{R2}}{4m} + \frac{\ell}{4}E^{T2} + \frac{\ell}{4}E^{R2} - \ell E^T \frac{P^{T2}}{8m} - \ell E^T \frac{P^{R2}}{8m} \\ &\quad - \ell E^R \frac{P^T P^R}{4m}. \end{aligned} \tag{8.4.14}$$

Thus in conclusion our Casimir is just

$$C^T \approx E^T - \frac{P^{T2}}{4m} - \frac{P^{R2}}{4m} + \frac{\ell}{4}E^{T2} - \ell E^T \frac{P^{T2}}{8m} - \ell E^T \frac{P^{R2}}{8m}, \tag{8.4.15}$$

leading to the generalized Schrödinger equation

$$\left(E^T - \frac{P^{T2}}{4m} - \frac{P^{R2}}{4m} + \frac{\ell}{4}E^{T2} - \ell E^T\frac{P^{T2}}{8m} - \ell E^T\frac{P^{R2}}{8m} - V\right)\psi = 0. \qquad (8.4.16)$$

Note that at the zero order we correctly have

$$H_0\psi = \left(\frac{P^{T2}}{4m} + \frac{P^{R2}}{4m} + V\right)\psi, \qquad (8.4.17)$$

where we identified the total energy E^T with the undeformed Hamiltonian of the system H_0.
As in the single particle deformed quantum theory in section 8.3, the best way to deal with this equation is through perturbation theory, where the perturbation itself is now given by

$$\tilde{V} = \frac{E^{T2}}{4} - E^T\frac{P^{T2}}{8m} - E^T\frac{P^{R2}}{8m} = \frac{E^T}{4}\left(E^T - \frac{P^{T2}}{2m} - \frac{P^{R2}}{2m}\right). \qquad (8.4.18)$$

Using (8.4.17) we can rewrite it as

$$\tilde{V} = \frac{E^T}{4}\left(E^T - 2V\right), \qquad (8.4.19)$$

which actually shows a profound analogy with the single particle case (8.3.5).
Note in particular that following the $\ell \to \frac{\ell}{2}$ rescaling of our theory the perturbation itself gets tuned down by a factor 2.

8.5 Non-commutative quantum harmonic oscillator

To conclude this thesis, we want to apply all the considerations and results obtained in this chapter to one of most important example of multi-particle system in physics, that is the harmonic oscillator.
Let us consider then two particles of equal mass m interacting through an harmonic potential[9] $V = m\omega^2 x^{R2}$.

[9]The abscence of the typical $\frac{1}{2}$ factor should not worry: it essentially matches our definitions of P^R and x^R. In general one can always imagine this external factors being absorbed by ω.

We assume of course that both particles symmetries are that of the κ-Bargmann algebra defined in (8.2.8), and that, for the reasons explained in the previous section, we can neglect the relative energy between the two particles at the first order in ℓ, that is $\ell E^R \approx 0$.
In order to solve the Schrödinger equation (8.4.16) by means of the perturbation theory (8.3.6), we first need to recall the standard quantum mechanics solution of this problem.
For the sake of simplicity we suppose that both particles live in a box of length L.
The classic Schrödinger equation

$$H_0\psi = \left(\frac{P^{T^2}}{4m} + \frac{P^{R^2}}{4m} + m\omega^2 x^{R^2}\right)\psi \tag{8.5.1}$$

can be separeted between its total and relative part $H_0 = H_0^T + H_0^R$.
As a consequence, the zero order eigenstates of the system are of the form $\psi^{(0)}_{m,n}\left(x^T, x^R\right) = \psi^{(0)}_m\left(x^T\right)\psi^{(0)}_n\left(x^R\right)$, where $\psi^{(0)}_m\left(x^T\right)$ are the eigenstates of the free-particle in a box problem and $\psi^{(0)}_n\left(x^R\right)$ are the harmonic oscillator's ones. Explicitly

$$|m^{(0)}\rangle\,|n^{(0)}\rangle = \frac{1}{\sqrt{2^{n-1}n!L}}\left(\frac{2m\omega}{\pi\hbar}\right)^{\frac{1}{4}}\sin\frac{m\pi}{L}x^T e^{-\frac{2m\omega}{2\hbar}x^{R^2}} H_n\left(\sqrt{\frac{2m\omega}{\hbar}}x^R\right) \tag{8.5.2}$$

with eigenvalues

$$E^{(0)}_{mn} = E^{(0)}_m + E^{(0)}_n = \frac{\pi^2\hbar^2}{2mL^2}m^2 + \hbar\omega\left(n + \frac{1}{2}\right), \tag{8.5.3}$$

where $H_n\left(x\right)$ are the Hermite polynomials (8.3.11).
The perturbation is given by

$$\tilde{V} = \frac{H_0}{4}\left(H_0 - 2V\right), \tag{8.5.4}$$

therefore we can express the deformed eigenvalues of our non-commutative problem as

$$\begin{aligned} E_{mn} &= E^{(0)}_{mn} + \frac{\ell}{4}E^{(0)}_{mn}\,\langle (m,n)^{(0)}|\, H_0 - 2V\,|(m,n)^{(0)}\rangle \\ &= E^{(0)}_{mn} + \frac{\ell}{4}E^{(0)}_{mn}\,\langle (m,n)^{(0)}|\, H^T_0 + H^R_0 - 2V\,|(m,n)^{(0)}\rangle\,. \end{aligned} \tag{8.5.5}$$

We know from the single particle case (8.3.18) that

$$\langle (m,n)^{(0)}| \, H_0^R - 2V \, |(m,n)^{(0)}\rangle = \langle n^{(0)}| \, H_0^R - 2V \, |n^{(0)}\rangle = 0 \tag{8.5.6}$$

whereas

$$\langle (m,n)^{(0)}| \, H_0^T \, |(m,n)^{(0)}\rangle = \langle m^{(0)}| \, H_0^T \, |m^{(0)}\rangle = E_m^{(0)}, \tag{8.5.7}$$

thus in conclusion the eigenvalues are

$$\begin{aligned} E_{mn} &= E_{mn}^{(0)} \left(1 + \frac{\ell}{4} E_m^{(0)}\right) \\ &= \left[\frac{\pi^2\hbar^2}{2mL^2} m^2 + \hbar\omega \left(n + \frac{1}{2}\right)\right] \left(1 + \ell \frac{\pi^2\hbar^2}{8mL^2} m^2\right). \end{aligned} \tag{8.5.8}$$

Note that, contrarily to the single particle harmonic oscillator case, the internal energy $E_n^{(0)}$ gets deformed on his own, namely

$$E_n = E_n^{(0)} \left(1 + \frac{\ell}{4} E_m^{(0)}\right). \tag{8.5.9}$$

What's interesting about this is that the deformation is enforced by the center of mass motion, in a sense.
This reflects the common property of two particle systems we highlighted throughout this thesis, that is the intricated relation between the center of mass motion and the relative motion seemingly brought up by the κ-non-commutativity of space-time.
In turn, in standard quantum or classical mechanics the relative motion and the total motion are generally completely independent.
To conclude this analysis, we need only to find the perturbated solution of the Schrödinger equation.
By using the single particle results in section 8.3 this is actually quite easy. Indeed we just need to compute the matrix elements of the perturbation $\tilde{V}_{lm,kn}$, but it's immediate to realize that $\tilde{V}$ can't connect states with different $l \neq m$ quantum numbers.
Therefore we are left with

$$\tilde{V}_{kn} = \frac{1}{4}\left[E_m^{(0)} + \hbar\omega\left(k + \frac{1}{2}\right)\right]\left(E_m^{(0)} - 2\,\langle k^{(0)}| \, V \, |n^{(0)}\rangle\right). \tag{8.5.10}$$

We have already found the last matrix elements in (8.3.18); the resulting deformed eigenstates are

$$|m,n\rangle = |m^{(0)}\rangle \left(|n^{(0)}\rangle + \frac{\ell}{2} d_{m,n-2} \,|(n-2)^{(0)}\rangle - \frac{\ell}{2} d_{m,n+2} \,|(n+2)^{(0)}\rangle \right) \tag{8.5.11}$$

with deformation factors

$$\begin{aligned} d_{m,n-2} &= \frac{\hbar\omega}{8} \left[\frac{E_m^{(0)}}{\hbar\omega} + \left(n - \frac{3}{2} \right) \right] \left(\frac{2E_m^{(0)}}{\hbar\omega} - \sqrt{n\,(n-1)} \right), \\ d_{m,n+2} &= \frac{\hbar\omega}{8} \left[\frac{E_m^{(0)}}{\hbar\omega} + \left(n + \frac{5}{2} \right) \right] \left(\frac{2E_m^{(0)}}{\hbar\omega} - \sqrt{(n+1)\,(n+2)} \right). \end{aligned} \tag{8.5.12}$$

Note that for $m = 0$ (i.e. the minimal energy level for the center of mass motion), this factors are equal to the single particle ones in (8.3.21), that is $d_{0,n\pm 2} = d_{n\pm 2}$.
Thus, as expected, the deformation shows a typical $\frac{\ell}{2}$ rescaling compared to the single particle case.
We can then imagine that for a given number N of interacting particles we would have an $\frac{\ell}{N}$ deformed quantum mechanics.
This is undoubtly a nice result in a purely soccer-ball problem perspective: it shows how this inconsistency naturally fades away on a macroscopic scale on a very practical and physical level, not only on the abstract algebra realm we analyzed in the rest of this thesis.
However, the fact that the deformation effects predicted by our generalized quantum theory are smaller and smaller when the systems involved are larger and larger makes it very difficult for us to observe them, even without taking into account the smallness of the deformation scale ℓ itself.
As of today, the most promising hopes of actually observe the deformed physics forecasted by the DSR theories (or, more in general, Lorentz violations [70–74]) come from the analysis of astrophysical events[10], for instance Gamma Ray Bursts [77–79], where exceptionally high energetic particles are emitted.
As many authors believe, these phenomena can produce particles (photons, neutrinos etc) with energy so extremely high that, during the very

[10] See [75, 76] instead for some nuclear physics inspired experimental proposals and reviews.

long travel time from the source to the earth, the deformation effects may add up and actually become observable[11].

In a sense, since these particles can be mostly considered as free, this would be merely related to a κ-Poincaré single particle physics.

Here instead, we can advance the idea of searching for these effects in the study of many-particle systems, even though, as aforementioned, this can prove to be even more difficult.

[11] To be more precise, the κ-Poincaré-based DSR theories predict a κ-dependence between the speed of particles and their energy. In particular two particles produced at the same time from a very distant astrophysical event should arrive on earth with some small delay, if their starting energies are different to begin with. Most importantly, this effect may occour even if the particles are both massless, in contrast with the special relativity prescriptions [80, 81]. The likelihood for this time delay to be measurable depends of course from the distance of the source and the actual magnitude of the deformations.

Chapter 9

Summary and outlook

The main goal of this work was to present a solution to the soccer-ball problem in 1+1-dimensions through the definition and the analysis of the formal properties of a two identical particle system.
As a first attempt we tried to follow a purely κ-Poincaré inspired route, defining the generators of the total symmetry algebra as prescribed by the coproducts of the single particle ones in (3.1.1).
We showed that the center of mass coordinates of the composite system have to be accordingly deformed in order for them to be actually consistent with the translation generators, and as expected we found that these novel coordinates satisfy a κ-Minkowski commutator.
Since the deformation parameter resulting from this approach does not rescale with the number of particles involved, we concluded that the κ-Poincaré composition of momenta can't be adopted as the total momentum of a composite system.
Then we noted that the standard non-deformed center of mass coordinates satisfy a 2κ-Minkowski commutator, so that the deformation length ℓ rescales to $\frac{\ell}{2}$ for a two particle system (and in general to $\frac{\ell}{N}$ for an N-particle system).
We then proposed two different sets of translation and boost generators in chapters 4 and 6, divided in a total subset and a relative subset, both compatible with this fundamental observation.
While emphasizing the weaknesses and the strengths of these two proposal, in each case considered we stressed the nice rescaling behaviour they both provide in relation to the soccer-ball problem.
Moreover, we highlighted the novel role played by the relative energy in the non-commutative realm, finding a deep and unprecedented connec-

tion between the center of mass motion and the relative motion of the systems under consideration.
Finally in the last chapter we presented a first preliminary exploration on the possibility of building interaction theories, i.e. deformed quantum mechanics, in the non-commutative framework.
To do so, we briefly discussed the so-called contraction procedure, emphasizing some of the criticity one has to deal with when applying it to the κ-Poincaré algebra.
In particular by sending the speed of light $c \to \infty$ (that is the same as considering particles with small velocities) we found the galilean limit of the κ-Poincaré algebra, i.e. the κ-Galilei algebra.
In order to write down a generalized Schrödinger equation we then introduced a possible, easiest, central extension of this κ-Galilei algebra at the first order in ℓ, that is the κ-Bargmann algebra (8.2.8).
This allowed us to study the non-commutative harmonic oscillator problem by treating the additional ℓ-deformation terms as a small perturbation in section 8.5.
We decided there to work in a low relative energy $\ell E^R \approx 0$ approximation in order to write down both the two particle algebra and the actual Schrödinger equation in total safety. Additional terms should of course appear if one was able to operate outside this approximation, as they could come out from a more general definition of the κ-Bargmann algebra. However, they would merely result in additional corrections in the perturbation theory, thus we can at least claim that the ones we presented are a correct round of first deformation of the standard quantum mechanics results.
Aside from a purely accademical interest, it's important to stress that this last exploration could also help the experimental research around the space-time non-commutativity.
In fact, for instance, the ℓ-corrections predicted by our harmonic oscillator toy model are in principle observable, even though they are incredibly small. More sophisticated models may have more luck in probing this new testing ground.
To conclude this thesis, we find useful to make a final consideration about an evident but at the same time non trivial consequence of our proposals, that is the non-universal relativistic flavor they naturally embody.
To put it simple, our work leads us to believe that different systems, or even different particles, may be governed by different deformations of relativistic kinematics [17,18].
This is extremely different from the special or galilean relativity case we're

used to, where the Poincaré algebra or the Galilei algebra respectively **universally** control the relativistic symmetries of all particles in the same way.
At first glance, although in a very preliminary sense, the deformation scales dictating different relativistic properties for different systems seem to be inversely proportional to the masses of the systems themselves (see section 7).
At the same time a possible dependence on other quantum numbers, which commonly are not encoded in the symmetry algebra, such as the particles spin can't be excluded.
In order to shed more light on these questions the next step may then be to fully generalize the results of this thesis to the case of an arbitrary number of non-identical particles, as well as better understand the role played by the relative energy E^R in the κ-Poincaré non-commutative realm.

Appendix A

Explicit derivation of the pregeometric algebra commutators

In this appendix we want to explicity compute the commutators between boost and translation generators of the pregeometric algebra in chapter 6. We will use intensively the formal properties of the external parameters (6.4.2), in particular the commutation relations they obey together with the boost generators N^T and N^R, i.e. Eqs. (6.4.15) and (6.4.17).
Let us then start with the commutator between N^T and P_1^T:

$$\left[N^T, P_1^T\right] = \left[N^T, f\left(\pi_0^R\right)P_1^A + f\left(-\pi_0^R\right)P_1^B\right]. \tag{A.1}$$

We can make use of the results mentioned above to write

$$\begin{aligned}\left[N^T, P_1^T\right] = & \, i\frac{\ell}{4}f\left(\pi_0^R\right)f\left(-\pi_0^R\right)P_1^R\frac{1}{f\left(\pi_0^R\right)}\left(P_1^T + P_1^R\right) \\ & + if^2\left(\pi_0^R\right)\left(\frac{1-e^{-2\ell P_0^A}}{2\ell} - \frac{\ell}{2}{P_1^A}^2\right) \\ & - i\frac{\ell}{4}f\left(\pi_0^R\right)f\left(-\pi_0^R\right)P_1^R\frac{1}{f\left(-\pi_0^R\right)}\left(P_1^T - P_1^R\right) \\ & + if^2\left(-\pi_0^R\right)\left(\frac{1-e^{-2\ell P_0^B}}{2\ell} - \frac{\ell}{2}{P_1^B}^2\right),\end{aligned} \tag{A.2}$$

that is

$$\begin{aligned}\left[N^T, P_1^T\right] = i\frac{\ell}{4}P_1^R &\left[f\left(-\pi_0^R\right)\left(P_1^T + P_1^R\right) - f\left(\pi_0^R\right)\left(P_1^T - P_1^R\right)\right] \\ &+ \frac{i}{2l}\left[f^2\left(\pi_0^R\right)\left(1 - e^{-\ell P_0^T}e^{-\ell P_0^R}\right)\right. \\ &\qquad \left. + f^2\left(-\pi_0^R\right)\left(1 - e^{-\ell P_0^T}e^{\ell P_0^R}\right)\right] \\ &- i\frac{\ell}{8}\left[\left(P_1^T + P_1^R\right)^2 + \left(P_1^T - P_1^R\right)^2\right].\end{aligned} \tag{A.3}$$

Using the properties (6.4.4) and (6.4.5) of the external factors we can simplify this expression as

$$\begin{aligned}\left[N^T, P_1^T\right] = i\frac{\ell}{2}P_1^R &\left(P_1^R - \tanh\frac{\ell}{2}\pi_0^R P_1^T\right) - i\frac{\ell}{4}P_1^{T\,2} - i\frac{\ell}{4}P_1^{R\,2} \\ &+ \frac{i}{2l}\left[f^2\left(\pi_0^R\right)\left(1 - e^{-\ell P_0^T}e^{-\ell P_0^R}\right)\right. \\ &\qquad \left. + f^2\left(-\pi_0^R\right)\left(1 - e^{-\ell P_0^T}e^{\ell P_0^R}\right)\right].\end{aligned} \tag{A.4}$$

The last term can be written as

$$\frac{1}{\cosh^2\frac{\ell}{2}\pi_0^R}\left[e^{\ell\pi_0^R} + e^{-\ell\pi_0^R} - e^{-\ell P_0^T}\left(e^{\ell\left(P_0^R - \pi_0^R\right)} + e^{-\ell\left(P_0^R - \pi_0^R\right)}\right)\right], \tag{A.5}$$

that is

$$\frac{1}{\cosh^2\frac{\ell}{2}\pi_0^R}\left[2\cosh\ell\pi_0^R - 2e^{-\ell P_0^T}\cosh\ell\left(P_0^R - \pi_0^R\right)\right]. \tag{A.6}$$

Since

$$\cosh\ell\left(P_0^R - \pi_0^R\right) = \cosh\ell P_0^R\cosh\ell\pi_0^R - \sinh\ell P_0^R\sinh\ell\pi_0^R, \tag{A.7}$$

equation (A.6) can be revaluated as

$$\frac{2}{\cosh^2\frac{\ell}{2}\pi_0^R}\left[\left(1 - e^{-\ell P_0^T}\cosh\ell P_0^R\right) - \sinh\ell\pi_0^R e^{-\ell P_0^T}\sinh\ell P_0^R\right], \tag{A.8}$$

i.e.

$$2\left(1 + \tanh^2\frac{\ell}{2}\pi_0^R\right)\left(1 - e^{-\ell P_0^T}\cosh\ell P_0^R\right) - 4\tanh\frac{\ell}{2}\pi_0^R e^{-\ell P_0^T}\sinh\ell P_0^R. \tag{A.9}$$

Putting all of this together we conclude that our commutator is

$$\begin{aligned}\left[N^T, P_1^T\right] = i \Bigg[& \left(1 + \tanh^2 \frac{\ell}{2}\pi_0^R\right) \frac{1 - e^{-\ell P_0^T} \cosh \ell P_0^R}{\ell} \\ & - 2 \tanh \frac{\ell}{2}\pi_0^R \frac{e^{-\ell P_0^T} \sinh \ell P_0^R}{\ell} - \frac{\ell}{4} P_1^{T2} \\ & - \frac{\ell}{2} \tanh \frac{\ell}{2}\pi_0^R P_1^T P_1^R + \frac{\ell}{4} P_1^{R2} \Bigg].\end{aligned} \tag{A.10}$$

Let us then move on with the commutator between N^R and P_1^T, that is

$$\begin{aligned}\left[N^T, P_1^R\right] &= \left[N^T, f\left(\pi_0^R\right) P_1^A - f\left(-\pi_0^R\right) P_1^B\right] \\ &= \left[N^T, f\left(\pi_0^R\right)\right] P_1^A + f^2\left(\pi_0^R\right)\left[N^A, P_1^A\right] \\ &\quad - \left[N^T, f\left(-\pi_0^R\right)\right] P_1^B - f^2\left(-\pi_0^R\right)\left[N^B, P_1^B\right].\end{aligned} \tag{A.11}$$

Proceeding exaclty as before, we write

$$\begin{aligned}\left[N^T, P_1^R\right] &= i\frac{\ell}{4} f\left(\pi_0^R\right) f\left(-\pi_0^R\right) P_1^R \frac{1}{f\left(\pi_0^R\right)} \left(P_1^T + P_1^R\right) \\ &\quad + i f^2\left(\pi_0^R\right) \left(\frac{1 - e^{-2\ell P_0^A}}{2\ell} - \frac{\ell}{2} P_1^{A2}\right) \\ &\quad + i\frac{\ell}{4} f\left(\pi_0^R\right) f\left(-\pi_0^R\right) P_1^R \frac{1}{f\left(-\pi_0^R\right)} \left(P_1^T - P_1^R\right) \\ &\quad - i f^2\left(-\pi_0^R\right) \left(\frac{1 - e^{-2\ell P_0^B}}{2\ell} - \frac{\ell}{2} P_1^{B2}\right) \\ &= i\frac{\ell}{4} P_1^R \left[f\left(-\pi_0^R\right)\left(P_1^T + P_1^R\right) + f\left(\pi_0^R\right)\left(P_1^T - P_1^R\right)\right] \\ &\quad + \frac{i}{2l}\Big[f^2\left(\pi_0^R\right)\left(1 - e^{-\ell P_0^T} e^{-\ell P_0^R}\right) \\ &\qquad - f^2\left(-\pi_0^R\right)\left(1 - e^{-\ell P_0^T} e^{\ell P_0^R}\right)\Big] \\ &\quad - i\frac{\ell}{8}\left[\left(P_1^T + P_1^R\right)^2 - \left(P_1^T - P_1^R\right)^2\right].\end{aligned} \tag{A.12}$$

We can now make use of properties (6.4.4), (6.4.5) again to find

$$\begin{aligned}\left[N^T, P_1^R\right] = i\frac{\ell}{2}P_1^R\left(P_1^T - \tanh\frac{\ell}{2}\pi_0^R P_1^R\right) - i\frac{\ell}{2}P_1^T P_1^R \\ + \frac{i}{2l}\left[f^2\left(\pi_0^R\right)\left(1 - e^{-\ell P_0^T}e^{-\ell P_0^R}\right)\right. \\ \left. - f^2\left(-\pi_0^R\right)\left(1 - e^{-\ell P_0^T}e^{\ell P_0^R}\right)\right].\end{aligned} \tag{A.13}$$

The term in parentheses can be explicitly written as

$$\frac{1}{\cosh^2\frac{\ell}{2}\pi_0^R}\left[e^{\ell\pi_0^R} - e^{-\ell\pi_0^R} + e^{-\ell P_0^T}\left(e^{\ell\left(P_0^R - \pi_0^R\right)} - e^{-\ell\left(P_0^R - \pi_0^R\right)}\right)\right], \tag{A.14}$$

that is

$$\frac{1}{\cosh^2\frac{\ell}{2}\pi_0^R}\left[2\sinh\ell\pi_0^R + 2e^{-\ell P_0^T}\sinh\ell\left(P_0^R - \pi_0^R\right)\right]. \tag{A.15}$$

Since

$$\sinh\ell\left(P_0^R - \pi_0^R\right) = \sinh\ell P_0^R\cosh\ell\pi_0^R - \cosh\ell P_0^R\sinh\ell\pi_0^R, \tag{A.16}$$

equation (A.15) can be revaluated as

$$\frac{2}{\cosh^2\frac{\ell}{2}\pi_0^R}\left[\sinh\ell\pi_0^R\left(1 - e^{-\ell P_0^T}\cosh\ell P_0^R\right) + \cosh\ell\pi_0^R e^{-\ell P_0^T}\sinh\ell P_0^R\right], \tag{A.17}$$

and finally simplified as

$$4\tanh\frac{\ell}{2}\pi_0^R\left(1 - e^{-\ell P_0^T}\cosh\ell P_0^R\right) + 2\left(1 + \tanh^2\frac{\ell}{2}\pi_0^R\right)e^{-\ell P_0^T}\sinh\ell P_0^R. \tag{A.18}$$

Thus in conclusion

$$\begin{aligned}\left[N^T, P_1^R\right] = i\left[\left(1 + \tanh^2\frac{\ell}{2}\pi_0^R\right)\frac{e^{-\ell P_0^T}\sinh\ell P_0^R}{\ell}\right. \\ + 2\tanh\frac{\ell}{2}\pi_0^R\frac{1 - e^{-\ell P_0^T}\cosh\ell P_0^R}{\ell} \\ \left. - \frac{\ell}{2}\tanh\frac{\ell}{2}\pi_0^R {P_1^R}^2\right].\end{aligned} \tag{A.19}$$

The next commutator we want to compute is the one between N^R and P_1^T, which we expect to be actually quite similar to the one we just evaluated:

$$\begin{aligned}\left[N^R, P_1^T\right] &= \left[N^R, f\left(\pi_0^R\right) P_1^A + f\left(-\pi_0^R\right) P_1^B\right] \\ &= \left[N^R, f\left(\pi_0^R\right)\right] P_1^A + f^2\left(\pi_0^R\right)\left[N^A, P_1^A\right] \\ &\quad + \left[N^R, f\left(-\pi_0^R\right)\right] P_1^B - f^2\left(-\pi_0^R\right)\left[N^B, P_1^B\right].\end{aligned} \tag{A.20}$$

Let us proceed as usual by calculating

$$\begin{aligned}\left[N^R, P_1^T\right] &= i\frac{\ell}{4} f\left(\pi_0^R\right) f\left(-\pi_0^R\right) P_1^T \frac{1}{f\left(\pi_0^R\right)}\left(P_1^T + P_1^R\right) \\ &\quad + i f^2\left(\pi_0^R\right)\left(\frac{1-e^{-2\ell P_0^A}}{2\ell} - \frac{\ell}{2} {P_1^A}^2\right) \\ &\quad - i\frac{\ell}{4} f\left(\pi_0^R\right) f\left(-\pi_0^R\right) P_1^T \frac{1}{f\left(-\pi_0^R\right)}\left(P_1^T - P_1^R\right) \\ &\quad - i f^2\left(-\pi_0^R\right)\left(\frac{1-e^{-2\ell P_0^B}}{2\ell} - \frac{\ell}{2} {P_1^B}^2\right),\end{aligned} \tag{A.21}$$

that is

$$\begin{aligned}\left[N^R, P_1^T\right] &= i\frac{\ell}{2} P_1^T\left(P_1^R - \tanh\frac{\ell}{2}\pi_0^R P_1^T\right) - i\frac{\ell}{2} P_1^T P_1^R \\ &\quad + \frac{i}{2l}\Big[f^2\left(\pi_0^R\right)\left(1 - e^{-\ell P_0^T} e^{-\ell P_0^R}\right) \\ &\qquad - f^2\left(-\pi_0^R\right)\left(1 - e^{-\ell P_0^T} e^{\ell P_0^R}\right)\Big].\end{aligned} \tag{A.22}$$

Since the terms in parentheses are the same ones as in equation (A.13), we conclude that

$$\left[N^R, P_1^T\right] = i\left[\left(1+\tanh^2\frac{\ell}{2}\pi_0^R\right)\frac{e^{-\ell P_0^T}\sinh \ell P_0^R}{\ell} + 2\tanh\frac{\ell}{2}\pi_0^R\frac{1-e^{-\ell P_0^T}\cosh \ell P_0^R}{\ell} - \frac{\ell}{2}\tanh\frac{\ell}{2}\pi_0^R {P_1^T}^2\right]. \tag{A.23}$$

The last commutator of this kind is $\left[N^R, P_1^R\right]$, i.e.

$$\begin{aligned}\left[N^R, P_1^R\right] &= \left[N^R, f\left(\pi_0^R\right)P_1^A - f\left(-\pi_0^R\right)P_1^B\right] \\ &= \left[N^R, f\left(\pi_0^R\right)\right]P_1^A + f^2\left(\pi_0^R\right)\left[N^A, P_1^A\right] \\ &\quad - \left[N^R, f\left(-\pi_0^R\right)\right]P_1^B + f^2\left(-\pi_0^R\right)\left[N^B, P_1^B\right].\end{aligned} \tag{A.24}$$

We then write

$$\begin{aligned}\left[N^R, P_1^R\right] &= i\frac{\ell}{4}f\left(\pi_0^R\right)f\left(-\pi_0^R\right)P_1^T\frac{1}{f\left(\pi_0^R\right)}\left(P_1^T+P_1^R\right) \\ &\quad + if^2\left(\pi_0^R\right)\left(\frac{1-e^{-2\ell P_0^A}}{2\ell} - \frac{\ell}{2}{P_1^A}^2\right) \\ &\quad + i\frac{\ell}{4}f\left(\pi_0^R\right)f\left(-\pi_0^R\right)P_1^T\frac{1}{f\left(-\pi_0^R\right)}\left(P_1^T-P_1^R\right) \\ &\quad + if^2\left(-\pi_0^R\right)\left(\frac{1-e^{-2\ell P_0^B}}{2\ell} - \frac{\ell}{2}{P_1^B}^2\right)\end{aligned} \tag{A.25}$$

,

and finally

$$\begin{aligned}\left[N^R, P_1^R\right] &= i\frac{\ell}{2}P_1^T\left(P_1^T - \tanh\frac{\ell}{2}\pi_0^R P_1^R\right) - i\frac{\ell}{4}{P_1^T}^2 - i\frac{\ell}{4}{P_1^R}^2 \\ &\quad + \frac{i}{2l}\Big[f^2\left(\pi_0^R\right)\left(1-e^{-\ell P_0^T}e^{-\ell P_0^R}\right) \\ &\qquad + f^2\left(-\pi_0^R\right)\left(1-e^{-\ell P_0^T}e^{\ell P_0^R}\right)\Big].\end{aligned} \tag{A.26}$$

We already know the last term from the commutator between N^T and P_1^T, therefore

$$\left[N^R, P_1^R\right] = i\left[\left(1+\tanh^2\frac{\ell}{2}\pi_0^R\right)\frac{1-e^{-\ell P_0^T}\cosh \ell P_0^R}{\ell}\right.$$
$$+2\tanh\frac{\ell}{2}\pi_0^R\frac{e^{-\ell P_0^T}\sinh \ell P_0^R}{\ell}+\frac{\ell}{2}{P_1^T}^2 \tag{A.27}$$
$$\left.-\frac{\ell}{2}\tanh\frac{\ell}{2}\pi_0^R P_1^T P_1^R-\frac{\ell}{4}{P_1^R}^2\right].$$

This concludes the derivation of the commutators between boost and translation generators of the pregeometric algebra.
It's important to note that the limit $\ell\pi_0^R \to 0$ can't be taken as it is in these formulas.
In fact we made intensively use of the commutators between N^T, N^R and the pregeometric factors $f\left(\pm\pi_0^R\right)$ in order to perform these calculations, but these commutators are trivially zero in that regime.
Howewer it's immediate to recognize that in the same limit the boost generators N^T and N^R (and the translation generators as well) can be simply written as the linear sum or difference of the single particle ones, thus we recover the algebra and coalgebra we already found out in chapter 4.

Bibliography

[1] G. Amelino-Camelia, *Quantum spacetime phenomenology*, Living Rev. Rel. 16 5 (2013)

[2] J. Maguejo, L. Smolin, *Generalized Lorentz invariance with an invariant energy scale*, Phys. Rev. D 67, 044017 (2003)

[3] G. Amelino-Camelia, *Fate of Lorentz symmetry in relative-locality momentum spaces*, Phys. Rev. D 85, 084034 (2012)

[4] L. J. Garay, *Quantum gravity and minimum length*, Int.J.Mod.Phys. A10 145-166 gr-qc (1995)

[5] G. Amelino-Camelia, *Testable scenario for Relativity with minimum-length*, Phys.Lett. B510 255-263 (2001)

[6] G. Amelino-Camelia, L. Freidel, J. Kowalski-Glikman, L. Smolin, *The principle of relative locality*, Phys. Rev. D 84, 084010 (2011)

[7] S. Hossenfelder, *Minimal length scale scenarios for quantum gravity*, Living Rev. Relativity 16, 2 (2013)

[8] G. Amelino-Camelia, *Doubly special relativity*, Nature 418 34-35 (2002)

[9] J. Kowalski-Glikman, *Introduction to doubly special relativity*, Lect.Notes Phys. 669 131-159 (2005)

[10] J. Kowalski-Glikman, S. Nowak, *Noncommutative space-time of doubly special relativity theories*, Int.J.Mod.Phys. D12 299-316 (2003)

[11] G. Amelino-Camelia, J. Kowalski-Glikman, G. Mandanici and A. Procaccini, *Phenomenology of Doubly Special Relativity*, Int. J. Mod. Phys. A 20 (2005)

[12] F. Girelli, E. R. Livine, D. Oriti, *Deformed Special Relativity as an effective flat limit of quantum gravity*, Nucl.Phys. B708 411-433 (2005)

[13] G. Amelino-Camelia, *Doubly-Special Relativity: First Results and Key Open Problems*, Int. J. Mod. Phys., D11 (2002)

[14] G. Amelino-Camelia, L. Freidel, J. Kowalski-Glikman, L. Smolin, *Relative locality and the soccer ball problem*, Phys.Rev. D84 087702 (2011)

[15] S. Hossenfelder, *The Soccer-Ball Problem*, SIGMA 10 (2014)

[16] G. Amelino-Camelia, *Planck-Scale Soccer-Ball Problem: A Case of Mistaken Identity*, Entropy 19 no.8, 400 (2017)

[17] G. Amelino-Camelia, *Particle-Dependent Deformations of Lorentz Symmetry*, Symmetry, 4(3), 344-378 (2012)

[18] G. Amelino-Camelia, M. Palmisano, M. Ronco, G. D'Amico, *Mixing coproducts for theories with particle-dependent relativistic properties*, arXiv:1910.05997 [gr-qc] (2019)

[19] J. Lukierski, P. C. Stichel, *Quantum deformations of space-time symmetries and interactions*, Czech.J.Phys. 47 55 (1997)

[20] P. Kosiński, P. Maślanka, *Deformed Galilei Symmetry*, Modern Physics Letters A, Volume 14, Issue 31, pp. 2139-2149 (1999)

[21] H. Leutwyler, *A no-interaction theorem in classical relativistic Hamiltonian particle mechanics*, Il Nuovo Cimento, vol. 37, issue 2, pp. 556-567 (1965)

[22] D.G. Currie, T.F. Jordan, E.C.G. Sudarshan, *Relativistic invariance and Hamiltonian theories of interacting particles*, Rev. Mod. Phys. 35 350-375 (1963)

[23] S. Orenstein, K. Rafanelli, *The origin of the no-interaction theorem*, Lett. Nuovo Cim; v. 23(3); p. 93-96 (1978)

[24] R. G. Underwood, *An introduction to Hopf algebras*, Berlin: Springer-Verlag (2011)

[25] R. G. Underwood, *Foundamentals of Hopf algebras*, Berlin: Springer-Verlag (2015)

[26] J.-P. Serre, *Lie algebras and Lie groups*, Lectures given at Harvard University, Benjamin, New York-Amsterdam, 1965; Springer, New York (1992)

[27] M. E. Sweedler, *Hopf algebras*, Mathematics Lecture Note Series, W. A. Benjamin, Inc., New York (1969)

[28] S. Majid, *Foundations of Quantum Group Theory*, Cambridge University Press, Cambridge, United Kingdom, paperback edn. (2000)

[29] S. Majid, *A Quantum Groups primer*, Cambridge University Press, Cambridge, United Kingdom (2002)

[30] V. G. Drinfel'd, *Quasi hopf algebras*, Leningrad Math. J., 1 (1990)

[31] V.G. Drinfel'd, *Hopf algebras and the quantum Yang–Baxter equation*, Sov. Math. Dokl. 32, 254 (1985)

[32] V.G. Drinfel'd, *Quantum Groups*, (Proceedings of the ICM, Rhode Island, 1987)

[33] M. Jimbo, *Introduction to the Young-Baxter equation*, International Journal of Modern Physics A, Vol 4, No. 15 3759-3777 (1989)

[34] S. Majid, H. Ruegg, *Bicrossproduct structure of κ-Poincaré group and noncommutative geometry*, Phys. Lett., B334 (1994)

[35] S. Majid, *Bicrossproduct hopf algebras and noncommutative spacetime*, In Encyclopedia of Mathematical Physics (edited by J.-P. Françoise, G. L. Naber, and T. S. Tsun), pp. 265–274. Academic Press, Oxford (2006)

[36] A. Connes, *Noncommutative geometry*, Academic Press, San Diego, CA (1994)

[37] F. Girelli, E. R. Livine, *Special relativity as a non commutative geometry: Lessons for deformed special relativity*, Phys.Rev. D81 085041 (2010)

[38] S. Petrat, R. Tumulka, *Multi-Time Schrödinger Equations Cannot Contain Interaction Potentials*, J.Math.Phys. 55 032302 (2014)

[39] S. Petrat, R. Tumulka, *Multi-Time Wave Functions for Quantum Field Theory*, Annals of Physics 345 (2013)

[40] M. Lienert, S. Petrat, R. Tumilka, *Multi-Time Wave Functions*, Journal of Physics: Conference Series 880: 012006 (2017)

[41] G. Amelino-Camelia, V. Astuti, G. Rosati, *Relative locality in a quantum spacetime and the pregeometry of κ-Minkowski*, Eur.Phys.J. C73 no.8 2521 (2013)

[42] A. Agostini, G. Amelino-Camelia, M. Arzano, A. Marciano, R. A. Tacchi, *Generalizing the Noether theorem for Hopf-algebra spacetime symmetries*, Modern Physics Letters A Vol. 22, No. 24, pp. 1779-1786 (2007)

[43] M. Arzano, A. Marciano, *Symplectic geometry and Noether charges for Hopf algebra space-time symmetries*, Phys. Rev. D 75 081701 (2007)

[44] A. Agostini, *Covariant Formulation of Noether's Theorem for κ-Minkowski Translations*, International Journal of Modern Physics A, Vol. 24, p. 1333-1358 (2009)

[45] Hossenfelder S., *Multiparticle states in deformed special relativity*, Phys. Rev. D 75 105005 (2007)

[46] G. J. Olmo, *Palatini actions and quantum gravity phenomenology*, J. Cosmol. Astropart. Phys. 2011 no. 10 018 (2011)

[47] P. Kosiński, P. Maślanka, J. Lukierski, *Local field theory on κ-Minkowski space, star products and noncommutative translations*, Czechoslovak Journal of Physics 50(11):1283-1290 (2000)

[48] P. Kosiński, P. Maślanka, J. Lukierski, A. Sitarz, *Generalized κ-Deformations and Deformed Relativistic Scalar Fields on Noncommutative Minkowski Space*, Topics in Mathematical Physics, General Relativity and Cosmology in Honor of Jerzy Plebanski, pp. 255-277 (2006)

[49] G. Amelino-Camelia, M. Arzano, L. Doplicher, *Field Theories on Canonical and Lie-Algebra Noncommutative Spacetimes*, 2001: A Relativistic Spacetime Odyssey, pp. 497-512 (2003)

[50] G. Amelino-Camelia, M. Arzano, *Coproduct and star product in field theories on Lie-algebra non-commutative space-times*, Phys. Rev. D 65, 084044 (2002)

[51] J. Kowalski-Glikman, G. Rosati, *Multi-particle systems in κ-Poincaré inspired by 2+1-D gravity*, Phys.Rev. D91 no.8, 084061 (2015)

[52] E. Inönü, E. P. Wigner, *On the Contraction of Groups and Their Representations*, Proc Natl Acad Sci U S A. 39(6): 510–524 (1953)

[53] E. J. Saletan, *Contraction of Lie Groups*, Journal of Mathematical Physics 2, 1 (1961)

[54] E. Celeghini, M. Tarlini, *Contractions of group representations I-II-III*, Il Nuovo Cimento, Vol 61, 65 and 68 (1981-1982)

[55] N. A. Gromov, *From Wigner–Inönü Group Contraction to Contractions of Algebraic Structures*, Acta Physica Hungarica 19(3): 209-212 (2004)

[56] M. Hamermesh, *Galilean invariance and the Schrödinger equation*, Annals of Physics Vol. 9(4): 518-521 (1960)

[57] J.-M. Levy-Leblond, *Galilei Group and Nonrelativistic Quantum Mechanics*, Journal of Mathematical Physics 4, 776 (1963)

[58] H. Brown, P. Holland *The Galilean covariance of quantum mechanics in the case of external fields*, American Journal of Physics, 67: 204-214 (1999)

[59] V. Colussi, S. Wickramasekara, *Galilean and $U(1)$-gauge symmetry of the Schrödinger field*, Annals of Physics, 323: 3020-3036 (2008)

[60] G. Hochschild, *Group Extensions of Lie Groups*, Annals of Math. Vol. 54(1) 96–109 (1951)

[61] G. M. Tuynman, W. A. J. J. Wiegerinck, *Central extensions and physics*, J. Geometry and Physics, 4 (2): 207–258 (1987)

[62] S. K. Bose, *The Galilean group in 2+1 space-times and its central extension*, Communications in Mathematical Physics, 169: 385-395 (1995)

[63] D. J. S. Robinson *The Theory of Group Extensions* in: *A Course in the Theory of Groups, Graduate Texts in Mathematics*, vol 80. Springer, New York, NY (1996)

[64] B. Oblak, *Quantum Mechanics and Central Extensions* in *BMS Particles in Three Dimensions*, Springer Theses (Recognizing Outstanding Ph.D. Research). Springer, Cham (2017)

[65] A. Opanowicz, *Two-dimensional centrally extended quantum Galilei groups and their algebras*, Journal of Physics A: Mathematical and General, Volume 33, Number 9 (1999)

[66] J. A. de Azcarraga, J. C. Pérez Bueno, *Deformed and extended Galilei group Hopf algebras*, Journal of Physics A General Physics 29(19) (1996)

[67] J. A. de Azcarraga, J. C. Pérez Bueno, *Relativistic and Newtonian κ-space-times*, Journal of Mathematical Physics, Volume 36, Issue 12, pp.6879-6896 (1995)

[68] S. Giller, C. Gonera, P. Kosiński, P. Maślanka, *The Quantum Galilei Group* Modern Physics Letters A 10(36) (1995)

[69] G. Szegö, *Orthogonal Polynomials*, 4th ed. Providence, RI: Amer. Math. Soc. (1975)

[70] E. E. Antonov, L. G. Dedenko, A. A. Kirillov, T. M. Roganova, G F. Fedorova, E. Yu Fedunin, *Test of Lorentz invariance through observation of the longitudinal development of ultrahigh-energy extensive air showers*, JETP Lett. 73, 446 (2001)

[71] G. Domokos, S. Kovesi-Domokos, *Astrophysical limit on the deformation of the Poincaré group*, Journal of Physics G: Nuclear and Particle Physics, Volume 20, Number 12 (1994)

[72] O. Bertolami, C. S. Carvalho, *Proposed astrophysical test of Lorentz invariance*, Phys. Rev. D 61, 103002 (2000)

[73] T. Jacobson, S. Liberati, D. Mattingly, *Lorentz violation at high energy: Concepts, phenomena, and astrophysical constraints*, Annals of Physics 321(1):150-196 (2006)

[74] L. Gonzalez-Mestres, *Deformed Lorentz Symmetry and Ultra-High Energy Cosmic Rays*, contributed paper HE.1.3.16 to the 26th ICRC, Utah August 17-25 (1999)

[75] L. Gonzalez-Mestres, *High-Energy Nuclear Physics with Lorentz Symmetry Violation*, contributed Paper 435 to the EPS-HEP97 Conference, Jerusalem August 19 - 26, (1997)

[76] J. P. Noordmans, H. W. Wilschut, R. G. E. Timmermans, *Lorentz violation in neutron decay and allowed nuclear β decay*, Phys. Rev. C 87, 055502 (2013)

[77] L. Freidel, L. Smolin, *Gamma ray burst delay times probe the geometry of momentum space*, arXiv:1103.5626v1 [hep-th] (2011)

[78] G. Amelino-Camelia, J. Ellis, N. E. Mavromatos, D. V. Nanopoulos, S. Sarkar, *Tests of quantum gravity from observations of big gamma-ray bursts*, Nature (London) 395, 525 (1998)

[79] G. Amelino-Camelia, S. Majid, *Waves on noncommutative spacetime and gamma-ray bursts*, S Int. J. Mod. Phys. A 15, 4301 (2000)

[80] A. A. Abdo et al., *A limit on the variation of the speed of light arising from quantum gravity effects*, Nature 462, 331 (2009)

[81] G. Rosati, N. Loret, G. Amelino-Camelia, *κ-Poincaré phase space: speed of massless particles and relativity of spacetime locality*, J. Phys.: Conf. Ser. 343 012105 (2012)