# Nonsingular terminal sliding-mode control of nonlinear planar systems with global fixed-time stability guarantees ⋆

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**Abstract**

This paper proposes the use of a novel nonsingular Terminal Sliding surface for the finite-time robust stabilization of second order nonlinear plants with matched uncertainties. Mathematical characteristics of the proposed surface are such that a fixed bound naturally exists for the settling time of the state variable, once the surface has been reached. It will be also presented a simple redesign of the control input able to ensure the feature of fixed-time reaching of the sliding surface, and fixed-time stability will be guaranteed by the proposed Terminal Sliding Mode Control design method. A careful simulation study has been performed using a benchmark system taken from the literature.

*Key words:* Terminal Sliding Mode Control, Fixed-Time Stabilization, Nonsingular Control, Nonlinear Systems

## 1 Introduction

The well known features characterizing Sliding Mode Control (SMC) [20] of robustness against matched perturbations and computational simplicity have made this technique successfully applied in a number of different applicative contexts (i.e. power systems [4], energy devices [5], robotics [6]) for the robust control of linear and nonlinear systems. The design procedure basically consists of two steps: first a surface is designed such that the reduced order system shows the desired properties, next a control law is determined ensuring the attainment of the sliding motion on the designed surface. Standard design methods rely on a linear sliding surface, therefore the associated sliding motion exhibits a linear behaviour with convergence rate arbitrarily fast, tunable by means of the parameters of the siding surface itself. Generally speaking, though the sliding manifold can be reached in an arbitrary finite time, systems states cannot converge to zero in finite time. The so called Terminal Sliding Mode Control (TSMC) [11] has been developed to fill this gap, and to achieve finite time convergence of the system dynamics. Generally speaking, finite-time stability and stabilization problems have been intensively studied for applications requiring severe time response constraints, see [3] [13] [14], and in observation problems when a finite-time convergence of the state estimate to the real values is required [15] [12] [2] [19]. It is worth mentioning that standard TSMC shows singularity problems in some regions of the state space, this hindering its application in real applications. Modified sliding surfaces have been proposed for second-order systems in [22] [9] to avoid the singularity domain. A saturation function has been introduced in [10] for dealing with the singularity problem in the case of chained nonlinear systems with matched perturbations. A finite-time disturbance observer within TSMC design has beed proposed in [21] for dynamical systems with unmatched disturbances.

In addition to the property of finite-time stability, previously discussed, the extra requirement can be considered that a bound exists, for the finite settling time, independent of the initial condition. It this property holds, the system is said to be fixed-time stable [18], since convergence to zero can be guaranteed with a predefined settling time, a-priori known. The phenomenon of fixed-time stability was first discovered in the framework of differentiators design in the paper [7], where a uniform exact differentiator is proposed for the first time and then extended in [1]. Fixed-time stabilization has been studied in [16] with reference to uncertain linear plants, while the control design problem of the robust finite-time and fixed-time stabilization of a chain of integrators by the Implicit Lyapunov Function method has been solved in [17]. Two uniform sliding mode controllers for a second-order uncertain system have been proposed in [8] providing convergence to an arbitrarily small set centered at the origin in finite time, which can be bounded by some constant independent of the initial conditions and uncertainties. In the recent paper by Zuo [23], a previously presented framework addressing finite-time stability for second-order nonlinear plants has been extended to guarantee fixed-time stability in the presence

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of matched perturbations, using a variant of the standard sliding manifold for TSMC together with a continuous sinusoidal function introduced to eliminate the singularity.

In this paper, the problem of designing a fixed-time terminal sliding surface for a second order nonlinear systems, possibly affected by matched perturbations, is approached starting from a different characterization. After the formal definition of the mathematical features of the surface needed for ensuring the attainment of a terminal sliding motion, hence finite-time stability, a proposal is made of a sliding surface drastically novel with respect to previous proposals available in the literature. The second order Single Inverted Pendulum (SIP) system, proposed in [23], has been used as benchmark testbed, and a comparative analysis has been performed with previous available results on this system.

## 2 Problem statement

Consider the following continuous-time, time invariant, uncertain second order plant described by:

$$\dot{x}_1(t) = x_2(t) \tag{1}$$
$$\dot{x}_2(t) = f(t,\mathbf{x}) + g(t,\mathbf{x})(u(t) + d(t,\mathbf{x})) \tag{2}$$

where: $\mathbf{x}(t) = [x_1(t)\ x_2(t)]^T \in \mathbb{R}^2$ is the state vector (assumed available for measurement), $u(t) \in \mathbb{R}$ is the control input, $f(t,\mathbf{x}), g(t,\mathbf{x}) : \mathbb{R} \times \mathbb{R}^2 \to \mathbb{R}$ are sufficiently smooth vector fields satisfying $f(t,0) = 0$ and $g(t,\mathbf{x}) \neq 0$ over the domain of interest, and the uncertain term $d(t,\mathbf{x})$ summarizes matched parameter variations and/or external disturbances affecting the system.

**Assumption 2.1** *The uncertain term $d(t,\mathbf{x})$ is bounded by a known function $\rho(\mathbf{x})$:*

$$|d(t,\mathbf{x})| \leq \rho(\mathbf{x}) \tag{3}$$

The finite-time stabilization of the plant (1) will be addressed in this note, with the extra requirement that the settling time is upper bounded by an a-priori value not dependent of the initial condition $\mathbf{x}(0)$. A novel terminal sliding surface will be designed to avoid the known singularity problems in the control input, previously discussed, often arising with TSM control as a consequence of the standard form of the terminal sliding surface, traditionally chosen as [23]: $s(t) = x_2(t) + \alpha x_1(t)^{m/n} + \beta x_1(t)^{p/q}$ with proper constraints on the positive integers $m, n, p, q$.

**Problem 2.1** *The problem here considered consists in designing a TSM controller such that:*

- *the origin of (1) is globally finite-time stable;*
- *an upper bound of the finite settling time is available, and is independent of the initial state;*
- *the control input is nonsingular.*

## 3 Motivating example

Consider a second order plant of the form (1) with $n = 2$, and define the following sliding surface

$$s(\mathbf{x}) = x_2(t) + 2\beta\sqrt{|\arctan(x_1(t))|}(1 + x_1^2(t))\sigma(x_1(t)) \tag{4}$$

with $\sigma(x_1(t)) = \text{sign}(x_1(t))$, having redefined the sign function in order to satisfy $\text{sign}(0) = 0$. Once a sliding motion is established on the surface $s(\mathbf{x}) = 0$, the dynamics of the variable $x_1(t)$ are governed by:

$$\dot{x}_1(t) = -2\beta\sqrt{|\arctan(x_1(t))|}(1 + x_1^2(t))\sigma(x_1(t)) \tag{5}$$

In the case $x_1(t) \geq 0 \Leftrightarrow \arctan(x_1(t)) \geq 0$, it holds

$$\frac{dx_1(t)}{\sqrt{\arctan(x_1(t))}(1 + x_1^2(t))} = -2\beta dt \tag{6}$$

and the solution of (6) satifies $\sqrt{\arctan(x_1(t))} - \sqrt{\arctan(x_1(0))} = -\beta t$, this proving that a settling time can be found, independently of the initial condition, given by

$$T_s(x_1(t)) = \frac{1}{\beta}\sqrt{\arctan(x_1(0))} \leq \frac{1}{\beta}\sqrt{\frac{\pi}{2}} \tag{7}$$

such that $x_1(t) = 0$ for $t > \frac{1}{\beta}\sqrt{\frac{\pi}{2}}$ independently of $x_1(0)$. Analogous considerations can be derived in the converse case $x_1(t) < 0$.

A further feature of the sliding surface (4) is relative to the equivalent control needed for ensuring the achievement of the sliding motion. The condition $\dot{s}_2(\mathbf{x}) = 0$, in the nominal case, provides (the case $x_1(t) > 0$ is considered without loss of generality):

$$f(t,\mathbf{x}) + g(t,\mathbf{x})u(t) + \beta\frac{1 + 4x_1(t)\arctan(x_1(t))}{\sqrt{\arctan(x_1(t))}}x_2(t) = 0 \tag{8}$$

It follows that, after the (arbitrary) reaching time interval $T_r$ needed for the sliding motion to be established by the standard condition $s(\mathbf{x})\dot{s}_2(\mathbf{x}) < -\eta|s(\mathbf{x})|$, for an arbitrary $\eta \in \mathbb{R}^+$. it holds, in view of (5)

$$\begin{aligned} u(t) = &-g(t,\mathbf{x})^{-1}\left(f(t,\mathbf{x}) - 4\beta^2 \cdot \right. \\ &\left.(1 + x_1(t)^2)(1/2 + 2x_1(t)\arctan(x_1(t)))\right) \end{aligned} \tag{9}$$

and $\lim_{x_1 \to 0} u(t) = \gamma, \quad \gamma \in \mathbb{R}, \quad t > T_r$. The reported discussion proves the following result.

**Theorem 3.1** *With reference to the planar uncertain system (1) satisfying Assumption 2.1, the achievement of a sliding motion on the surface $s(\mathbf{x}) = 0$ guarantees the robust finite-time stabilization of the plant. Once the sliding surface is attained ($t > T_r$), the states reach the* *origin within a fixed time:*

$$T_s \leq \frac{1}{\beta}\sqrt{\frac{\pi}{2}} \tag{10}$$

*and the control input shows no singularity as the solution approaches the origin.*

**Remark 3.1** *It may be worth noting that, differently from other implementations of TSM which are ill-posed for null velocities $x_2 = 0$, the chosen control input is well defined for such condition after the reaching phase.*

## 4 A class of finite-time terminal sliding surfaces

To generalize the previous example, consider a real valued function $h(z) : \mathbb{R} \to \mathbb{R}$, sufficiently smooth almost everywhere. Define

$$h'(z) \stackrel{\text{def}}{=} \frac{\partial h(z)}{\partial z} \quad h''(z) \stackrel{\text{def}}{=} \frac{\partial h'(z)}{\partial z}. \tag{11}$$

and consider the following sliding surface

$$s(\mathbf{x}) = x_2 + \beta(h'(x_1))^{-1}; \quad \beta > 0 \tag{12}$$

As exploited in the next definitions and the successive lemmas, a proper choice of the function $h(z)$ might provide some interesting features for sliding motions onto the surface $s(\mathbf{x}) = 0$.

**Definition 4.1** *With reference to the plant (1), the sliding surface (12) is referred to as a finite-time terminal sliding surface if the following conditions hold:*

*i)* $h'(z) \neq 0$ *for all* $z \in \mathbb{R} \setminus \{0\}$;
*ii)* $h(0) = 0$;
*iii)* $\lim_{z\to 0}(h'(z))^{-1} = 0$
*iv) the origin is a globally asymptotically stable equilibrium point of the system*

$$\dot{z}(t) = -\beta(h'(z(t)))^{-1}$$

**Definition 4.2** *Let $h(z)$ define a finite-time terminal sliding surface. If in addition, the asymptotical condition*

$$\lim_{z\to 0} \frac{h''(z)}{h'(z)^3} = \gamma \tag{13}$$

*is fulfilled for a finite $\gamma \in \mathbb{R}$, with $h''(z) \stackrel{\text{def}}{=} \dfrac{\partial h'(z)}{\partial z}$, then the sliding surface is said a finite-time nonsingular terminal sliding surface.*

**Lemma 4.1** *With reference to the planar uncertain system (1) satisfying Assumption 2.1, the achievement of a sliding motion on a finite-time terminal sliding surface guarantees the robust stabilization of the plant within a finite settling time possibly bounded by a known constant (as in the motivating example).*

**PROOF.** The arguments used for the proof are similar to those developed in [16, Section IV] for the fixed-time control of linear systems.Consider a surface (12) with $h(x_1)$ satisfying Definition 4.1. Whenever a sliding motion is established on the surface $s(\mathbf{x}) = 0$ by means of the standard condition

$$s(\mathbf{x})\dot{s}_2(\mathbf{x}) < -\eta|s(\mathbf{x})|; \quad \eta > 0 \tag{14}$$

the condition $s(\mathbf{x}) = 0$ is attained within a finite time $T_r(s(0)) \stackrel{\text{def}}{=} s(0)/\eta$. As a consequence it holds:

$$\dot{x}_1 = -\beta(h'(x_1))^{-1} \quad t \geq T_1(s(0)) \tag{15}$$

where, by hypothesis, the origin is a globally asymptotically stable equilibrium point of the system (15). Such differential equation provides upon integration

$$h(x_1(t)) - h(x_1(T_r(s(0)))) = -\beta(t - T_r(s(0))), \tag{16}$$

which implies that $h(x_1(t)) = x_1(t) = x_2(t) = 0$ for any $t \geq \bar{T}(\mathbf{x}(0))$, where

$$\bar{T}(\mathbf{x}(0)) \stackrel{\text{def}}{=} \frac{h(x_1(T_r(s(0))))}{\beta} + T_r(s(0)). \tag{17}$$

This concludes the proof. □

The input needed for the establishment of a sliding motion on $s(\mathbf{x}) = 0$ has the form $u(t) = u_{eq}(t) + u_\star(t)$ with

$$u_{eq}(t) = -(g(t,\mathbf{x}))^{-1}\left[f(t,\mathbf{x}) - \beta\frac{h''(x_1)}{h'(x_1)^2}x_2(t)\right] \tag{18}$$

$$u_\star(t) = -(g(t,\mathbf{x}))^{-1}\rho(\mathbf{x})|g(t,\mathbf{x})|\sigma(s(\mathbf{x})) \tag{19}$$

In view of (15), for $t \geq T_r(s(0))$ it holds $x_2(t) = -\beta(h'(x_1(t)))^{-1}$, therefore the control input (19) is nonsingular after the reaching phase under hypothesis (13).

**Remark 4.1** *From the previous discussion, it follows that the control input (19) guarantees that the origin is a globally finite time stable equilibrium point, with settling time $T$ bounded by*

$$T \leq \sup \bar{T}(\mathbf{x}(0)) = \sup \frac{h(x_1(T_r(s(0))))}{\beta} + T_r(s(0)) \tag{20}$$

*In general, the first term of (17) depends on the initial condition $\mathbf{x}(0)$ through the surface reaching time $T_r(s(0))$ as exploited in (17). As shown in the motivating example, such term can be made independent of the initial condition by a proper choice of the function $h(x_1(t))$. In addition, the insertion of extra terms in the control input with suitable growth conditions, can make the overall time $\bar{T}$ independent of the initial datum (i.e. the system can be made fixed-time stable), as it will be shown in the following subsection.*

### 4.1 Fixed-time stability

In this section, the control inputs (18)-(19), previously designed, will be modified to ensure the feature of fixed-time reaching of the sliding surface. Overall, fixed-time stability will be guaranteed by the proposed TSMC design method. To this end, let us recall the following technical lemma on fixed-time vanishing of ODE solutions, which can be inherited from the more general result [16, Lemma 1].

**Lemma 4.2** *Consider the scalar differential equation*

$$\dot{z} = -\lambda\,\sigma(z)|z|^{\alpha} - \mu\,\sigma(z)|z|^{\gamma} \tag{21}$$

*with* $\lambda, \mu > 0$, $\alpha > 1$ *and* $\gamma < 1$. *The equation admits a finite settling time* $T_0$ *uniform with respect to the initial condition* $z(0)$ *and bounded by*

$$T_0 \leq T^{\star}(\alpha, \gamma, \lambda, \mu) := \frac{1}{\lambda(\alpha-1)} + \frac{1}{\mu(1-\gamma)} \tag{22}$$

Bearing the dynamics (21) in mind, the control input (19) can be redesigned as follows

$$u_{\star}(t) = -(g(\mathbf{x}))^{-1}\Big[\rho(\mathbf{x})|g(t,\mathbf{x})|\sigma(s(\mathbf{x})) + \lambda_1\,\sigma(s(\mathbf{x}))|s(\mathbf{x})|^{\alpha_1} + \mu_1\,\sigma(s(\mathbf{x}))|s(\mathbf{x})|^{\gamma_1}\Big] \tag{23}$$

where $\lambda_1, \mu_1, \alpha_1, \gamma_1 > 0$. Thanks to such adjustment, it can be verified that, if the Lyapunov function $V(s) := s^2$ is introduced, it holds $\dot{V}(s_2) \leq -2\lambda_1 V^{\frac{\alpha_1+1}{2}} - 2\mu_1 V^{\frac{\gamma_1+1}{2}}$. This proves that, applying Lemma 4.2 with arbitrary $\alpha_1 > 1$ and $\gamma_1 \in (0,1)$, the variable $s(\mathbf{x})$ has a finite settling-time and uniformly bounded by the bound (22), with $\lambda = 2\lambda_1,\ \mu = 2\mu_1$, and $\alpha = \frac{\alpha_1+1}{2} > 1,\ \gamma = \frac{\gamma_1+1}{2} \in \left(\frac{1}{2}, 1\right)$. The following fixed-time stability result is then granted and allow to compute an uniform estimate for the reaching time of the surface.

**Theorem 4.1** *With reference to the planar uncertain system (1) satisfying Assumption 2.1, the control input* (18), (23) *with* $s(\mathbf{x}) = s(\mathbf{x})$ *guarantees the global fixed-time robust stability of the plant, with a uniformly bounded settling time*

$$T_2(\mathbf{x}(0)) \leq \frac{1}{\beta}\sqrt{\frac{\pi}{2}} + \frac{1}{\lambda(\alpha-1)} + \frac{1}{\mu(1-\gamma)}.$$

## 5 Singularity avoidance

It is clear from the definition that singularity in the control input occurs for $x_1 = 0$ and $x_2 \neq 0$. To overcome this problem, the control input can be modified in a neighborhood $\mathcal{I}_\epsilon \stackrel{\text{def}}{=} \{\mathbf{x} \in \mathbb{R}^2 : x_1 \in (-\epsilon, \epsilon)\}$ of the vertical axis by letting the trajectory "cross the line" without singularity. As a preliminary remark, one can observe that the only critical zones are given by

$$\mathcal{Z}^+ \stackrel{\text{def}}{=} \{\mathbf{x} : s(\mathbf{x}) > 0, x_1 < 0\}$$
$$\mathcal{Z}^- \stackrel{\text{def}}{=} \{\mathbf{x} : s(\mathbf{x}) < 0, x_1 > 0\}$$

In fact, denoting by $Q_1, Q_2, Q_3, Q_4$ the four quadrants of the plane (counted clockwise), one has by definition $x_1\dot{x}_1 > 0$ in $Q_1 \cup Q_3$, and thus the vertical axis cannot be intercepted in such regions. Moreover, in the sectors $Q_2 \setminus \mathcal{Z}^-$ and $Q_4 \setminus \mathcal{Z}^+$, the trajectory of the system necessarily intercepts the sliding surface before reaching the $x_2$-axis, and the singularity is therefore compensated by the sliding mode. Indeed, by a finer reasoning, the critical regions can be further reduced. To a certain extent, the singularity-free regions can be enlarged semiglobally through the choice of the parameter $\beta$: for any $\mathbf{x}_0 \in Q_2$ ($\mathbf{x}_0 \in Q_4$), there always exists a sufficiently large $\beta > 0$ such that $\mathbf{x}_0 \in Q_2 \setminus \mathcal{Z}^-$ ($\mathbf{x}_0 \in Q_4 \setminus \mathcal{Z}^+$). Moreover, as clear from (10) and (20), a larger $\beta$ corresponds to a shorter settling time after the establishment of the sliding motion, and hence to better convergence performances. Let us assume that the quantitative bound $\rho(\mathbf{x}) - |d(t,\mathbf{x})| \geq \xi, \quad \xi > 0$ holds true $\forall \mathbf{x} \in \mathbb{R}^2$. Let us stress that such condition can always be met by considering $\bar{\rho}(\mathbf{x}) = \xi + \rho(\mathbf{x})$. Consider the solutions $\varphi^{\pm}(t)$ of the differential problem

$$\dot{x}_1 = x_2$$
$$\dot{x}_2 = f(t,\mathbf{x}) + g(t,\mathbf{x})u_{eq} \mp \xi$$
$$x_1(0) = 0,\ x_2(0) = 0,$$

with reversed time $t \in (-\infty, 0]$ and define the curves $\Phi^{\pm} := \{\mathbf{x} \in \mathbb{R}^2 : \mathbf{x} = \varphi^{\pm}(t),\ t \in (-\infty, 0]\}$. By construction, the closed regions $\mathcal{S}^{\pm}$ delimited, respectively, by the curves $\Phi^{\pm}$ and by $\mathcal{Z}^{\pm}$ are singularity-free. In fact, the considered reversed-time system corresponds to the worst case $\rho(\mathbf{x}) - |d(t,\mathbf{x})| \equiv \xi$, while the discontinuous input term $u_{\star}(t)$ in the general scenario $\rho(\mathbf{x}) - |d(t,\mathbf{x})| - \xi \geq 0$ guarantees the convergence onto the sliding surface. Such property entails that, whenever the initial condition of the system lies inside $\mathcal{S}^{\pm}$, the sliding surface is reached at some point $\bar{\mathbf{x}}$ with $\bar{x}_1 \neq 0$.
The idea for avoiding singularity is then to modify the control input for $\mathbf{x} \in \mathcal{N}_\epsilon \stackrel{\text{def}}{=} \mathcal{I}_\epsilon \setminus (\mathcal{S}^+ \cup \mathcal{S}^-)$. To this end, let us set

$$u_\sharp := -(g(t,\mathbf{x}))^{-1} f(t,\mathbf{x}) + \sigma(g(t,\mathbf{x})x_2)\rho(\mathbf{x}).$$

and observe that, due to this choice, one has $x_2\dot{x}_2 \geq 0$ and the bound $|\dot{x}_1| \geq |x_2(0)| \geq 2\beta|h'(x_1(0))^{-1}|$ holds. As a consequence, by the comparison principle, one has

$$x_1(t) \leq |x_1(0)| - 2\beta h'(|x_1(0)|)^{-1}t \quad x_1(0) \geq 0$$
$$x_1(t) \geq -|x_1(0)| + 2\beta h'(|x_1(0)|)^{-1}t \;\; x_1(0) \leq 0$$

In both cases, the trajectory exits the critical layer $\mathcal{N}_\epsilon$ within a finite time $T_\epsilon$, which is uniformly bounded by

$$T_\epsilon \leq \max_{\zeta \in [0,\epsilon]} \beta\zeta h'(\zeta). \tag{24}$$

This is the additional time to be included in the evaluation of bounds to the total settling-time, and can be regarded as the cost of singularity avoidance. In conclusion, the non-singular control input is given by

$$u(t) = \begin{cases} u_{eq}(t) + u_\star(t) & \mathbf{x} \in \mathbb{R}^2 \setminus \mathcal{N}_\epsilon \\ u_\sharp(t) & \mathbf{x} \in \mathcal{N}_\epsilon \end{cases}$$

**Remark 5.1** *It is worth noticing that, for the choice* $h(z) = \sqrt{\arctan z}$ *proposed in Section 3, the global estimate* $\zeta h'(\zeta) < 0.61$ *holds* $\forall \zeta \in [0, \infty)$*, and hence a uniform estimate on* $T_\epsilon$ *in (24) can be found.*

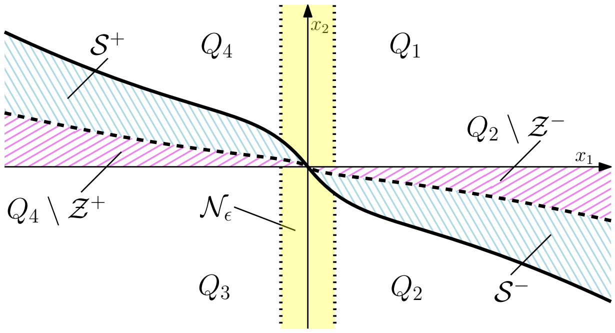


Fig. 1. Singularity-free regions and critical layer

## 6 Simulation results

In order to validate previous theoretical results, the proposed control approach has been applied by simulation on the Single Inverted Pendulum (SIP) used by [23] as a benchmark. The system is described by (1) where:

$$f(t,\mathbf{x}) = \frac{g\sin(x_1(t)) - \dfrac{m\ell x_2(t)^2\cos(x_1(t))\sin(x_1(t))}{m_c + m}}{\ell\left(4/3 - \dfrac{m\cos^2(x_1(t))}{(m_c+m)}\right)} \tag{25}$$

$$g(t,\mathbf{x}) = \frac{\cos(x_1(t))}{\ell\left(4/3(m_c+m) - m\cos^2(x_1(t))\right)} \tag{26}$$

where $g$ is the gravity acceleration, $m_c = 1\ kg$ is the cart mass, $m = 0.1\ kg$ is the mass of the pendulum and $\ell = 0.5\ m$ is its length. The control objective is the tracking of a reference trajectory of the form $r(t) = \sin(0.5\pi t)$, therefore the previous development has been applied to the error variables, expressed by $e_1(t) = x_1(t) - r(t)$, $e_2(t) = x_2(t) - \dot{r}(t)$.

A bounded perturbation of the form $\mathrm{d} = \sin(10x_1) + cos(x_2)$ has been supposed to affect the system in the presented simulations, as in [23]. To better evaluate the performance and features of the proposed approach, the fixed-time controller described in [23] has been implemented and compared with.

The control input (23), guaranteeing fixed time stability, has been tested by simulation, with initial conditions $x_1(0) = 1$, $x_2(0) = \pi/2$. With the design choices $\lambda_1 = 1$, $\mu_1 = 0.5$, $\alpha_1 = 1.5$, $\gamma_1 = 0.001$, the bound on the settling time is 4.125. Figure 3(a), 3(b) display the results obtained by the controller (19), compared with the reference variables. Moreover, the behavior of the sliding variables (4) and (12) in [23] are shown in Fig. 3(c). The convergence times for increasing norm of the initial condition are reported in Fig. 2(a). It should be emphasized that the reported times refer to the time when the boundary layer (of width 0.01) is first intercepted. The simulations show that the real convergence times are by far lower than the estimated bound.

A further comparison has been performed using the academic example reported in [8], under the assumption that the whole state vector is available for measurement. The obtained results are reported in Figs. 4(a)-4(b). The behavior of the system seems insensitive to the change of operation mode, due to the fact that the system dynamics are cancelled in (18). Moreover, the value of the control signal during the transient response for both controllers is highly increased, as one could expect. The price paid is the occurrence of chattering in the control variable.

## 7 Conclusions

This note has investigated the problems of finite-time and fixed-time stabilization of planar nonlinear systems, in the presence of matched perturbations. A novel TS surface is proposed such that a fixed bound naturally exists for the settling time of the state variables, once the surface has been reached. A redesign of the control input has been proposed able to ensure the feature of fixed-time reaching of the sliding surface. A simulation study has been presented using a magnetic levitation system.

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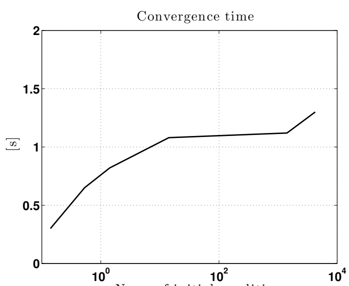


(a)

Fig. 2. Fixed-time controller: converge timed for increasing norm of the initial condition

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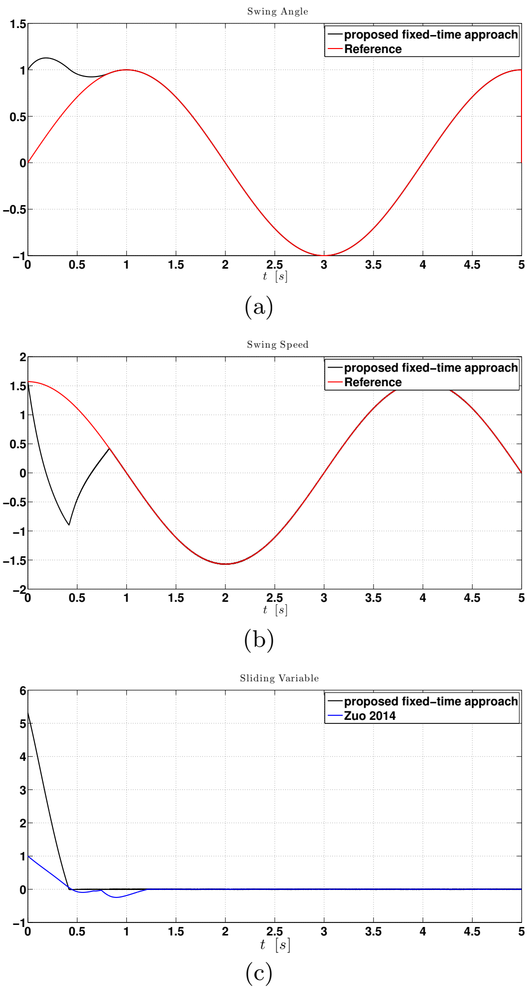


Fig. 3. Fixed-time controller: (a) Swing Angle ; (b) Swing Speed, in the case of $x_2(0) = \pi/2$; (c) Sliding variable (4) compared with (12) in [23]

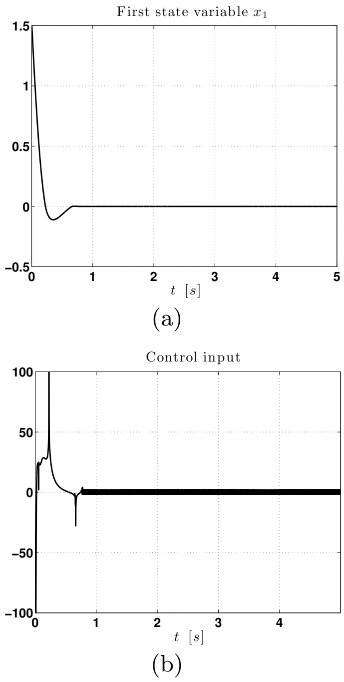


Fig. 4. The academic example reported in [8]: (a) First control variable ; (b) Control input.

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