



SAPIENZA
UNIVERSITÀ DI ROMA

DIPARTIMENTO DI INGEGNERIA ASTRONAUTICA
ELETTRICA ED ENERGETICA

DOTTORATO IN INGEGNERIA AEROSPAZIALE

CICLO XXVI

MHD Reaction Wheel for Spacecraft Attitude Control: Configuration Analysis and Numerical Models

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ANNO ACCADEMICO 2013/2014

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Abstract

A novel actuator for spacecraft attitude control with liquid flywheel is presented. The main characteristic of this new concept of reaction wheel is that a conductive liquid rather than a solid mass is accelerated to change the angular momentum of the equipment and, as a consequence, to provide torque to the spacecraft. The conductive liquid is accelerated using a distributed Magneto-Hydro-Dynamic (MHD) conductive pump. Two different configurations of the device have been studied on the base of the optimization of the dimensionless moment of inertia and the minimization of the viscous shear.

A 2-dimensional Finite Differences Hybrid Model (FDHM) has been developed on the base of the MHD set of equations under the hypothesis of low Magnetic Reynolds. The model solves numerically the time dependent axially symmetric problem of a conductive liquid rotating in a torus with rectangular cross section due to the interaction of a radial magnetic field and an axial electric field. The electric side of the problem has been solved by means of the node method applied to a network of electric resistances and voltage generators representing the back electromotive voltage induced by the spinning liquid through the magnetic field. The fluid-dynamics side of the problem has been solved using a Crank-Nicolson method over a non uniform and collocated grid. The grid generator has been written to be sensitive to the Hartmann number of the problem.

A Lumped Parameter Model (LPM) has been then derived using a stationary and mono-dimensional analytical solution of the MHD set of equations under the hypothesis of low Magnetic Reynolds. The LPM has been compared with the results obtained with the FDHM, showing good agreement in the estimation of the coefficients of the model.

Finally, in order to understand the room for improvement of the performances of the device, a simplified lumped parameter model has been applied to a device with concentrated MHD conductive pumps.



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Introduction

The ability to control spacecraft attitude and the reliability of the attitude control system are key components of space missions. Also, future Earth observation missions as well as astronomical observation missions require satellites which are more agile (moving their direction of pointing) and with high pointing stability (so as to provide better resolution images) over the higher lifetime possible.

A number of techniques have been developed and refined to underpin attitude control and re-pointing but basically since the beginning of the space exploration era all techniques used for high performance attitude control exploit the use of reaction and (in some cases) momentum wheels.

The technology improvement during this time allowed reaction wheels to improve their performance in terms of control resolution and reliability and efficiency of the electronic part but there are some drawbacks related to standard reaction wheel design that were not improved over time. In particular we can mention some of the more important drawbacks like:

- Standard reaction wheels use moving mechanical parts which reduce the overall reliability of the equipment and increase its overall complexity.
- Jitter and other unbalances associated with moving masses lead to unwanted high frequency disturbances on the satellite itself which could be critical for demanding high resolution observation missions.
- The presence of electric motors in standard reaction wheels induce a ripple disturbance on the torque provided.
- The torque provided by standard reaction wheels is typically non linear in the neighbourhood of the zero angular velocity.

In the last years some articles have been published regarding the possibilities to use active fluid loops as actuator for spacecraft attitude control, [1] [2]. This

kind of actuators uses hydraulic pumps to accelerate a low viscosity liquid inside a torus with circular cross section. These devices have the advantage to reduce the mechanical unbalances typical of classical reaction wheels with solid flywheel but they do not increase the reliability of the actuator because of the presence of a spinning electric engine in the hydraulic pump.

A novel actuator for spacecraft attitude control with liquid flywheel has been studied. The main characteristic of this new concept of a reaction wheel is that a conductive liquid rather than a solid mass is accelerated to change the angular momentum of the equipment and, as a consequence, to provide the torque to the spacecraft. The conductive liquid inside an annulus with rectangular cross section is accelerated using the Lorentz force originated from the interaction between an electric current flowing through the fluid and a magnetic field. In this way the electric engine can be removed with the advantage to remove any ripple disturbance and to improve the overall reliability of the equipment. Moreover the absence of an electric motor could lead to an increased linearity of the torque provided in the neighbourhood of zero angular velocity.

In the last four years the interest in this kind actuators increased and some papers and patents has been published, [3] [4] [5] [6]. The lack of accurate mathematical models describing the behaviour of the device brought this study to start from the basis of the magneto-hydro-dynamics problem to obtain a lumped parameter model at the same time simple and able to model the real physics of the problem [7]. The model is then used to have information about performance, limitations of the device and to understand where and how future researches can operate in order to improve the performances of the device.

Configurations: Distributed Magnetic Field

In this chapter, a preliminary analysis of the optimal configuration for an MHD reaction wheel using distributed conductive MHD pumps is presented.

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1.1 Working principle

The main feature of this new concept of reaction wheel is that a conductive fluid rather than a solid mass is accelerated to change the angular momentum and to provide a controlling torque to the spacecraft. The Lorentz force is used to accelerate the conductive liquid. This driving force originates from the interaction between a magnetic field and an electric current flowing through the fluid.

The externally applied magnetic field is provided by permanent magnets because the use of magnetic coils would increase the total power consumption of the actuator and this is not convenient for a device whose use is addressed to space application, where power resources are usually limited. The magnetic field in this section is considered distributed and it affects all the volume where the liquid is placed.

The spin of the conductive liquid can be controlled in intensity and direction by means of the electric current sent to device. The inversion of the electric current direction switch the direction of the Lorentz force acting as body force on the fluid.

The liquid is placed in an annulus with a rectangular cross section and it rotates along the azimuthal direction. The Lorentz force acts in the direction in which the conductive fluid is free to flow.

1.2 Configurations of the device

The maximum Lorentz force that can be obtained from the interaction between an electric current and a magnetic field appears when the two fields are mutually perpendicular. This, together with the advantage to have the conductive liquid placed in a axially-symmetric container lead to two possible arrangements of magnetic field and electric current and two different configurations for the MHD actuator, Figure (1.1):

- *Axial magnetic field configuration*: this configuration takes into account an electric current along the radial direction r and a magnetic field along the axial direction z [5] [3].

- *Radial magnetic field configuration*: this configuration switches the first configuration considering an electric current along the z direction and a magnetic field along the r direction [7] [4].

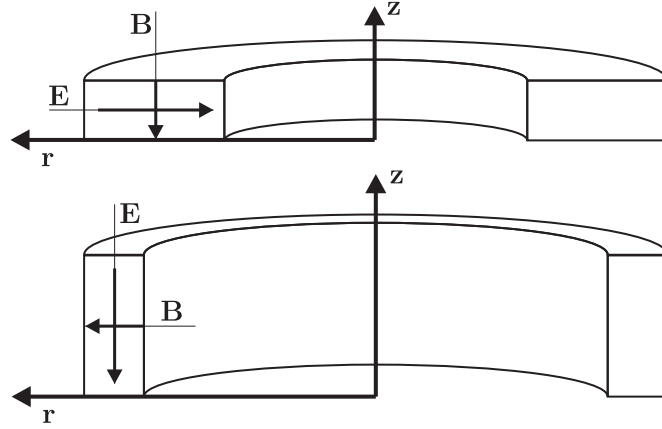


Figure 1.1: First (top) and second configuration (bottom) for electric field E and magnetic field B

Both the configurations provide a force in the requested direction. The advantage to use one of two configurations has been evaluated on the base of the maximization of the dimensionless moment of inertia.

1.3 Dimensionless Moment of Inertia

The dimensionless moment of inertia $\tilde{I}_z$ give a numerical indication of how efficiently a mass is used to generate angular momentum. To obtain the value of $\tilde{I}_z$ for a hollow cylinder we need to divide the dimensional moment of inertia by the mass and the squared external radius of the cylinder. We obtain:

$$I_z = \frac{m(r_e^2 + r_i^2)}{2} \quad \tilde{I}_z = \frac{I_z}{mr_e^2} = \frac{1 + \beta^2}{2} \quad (1.3.1)$$

where r_e is the outer radius of the hollow cylinder, r_i is the inner radius of the hollow cylinder, β is the ratio r_i/r_e and m is the mass of the liquid.

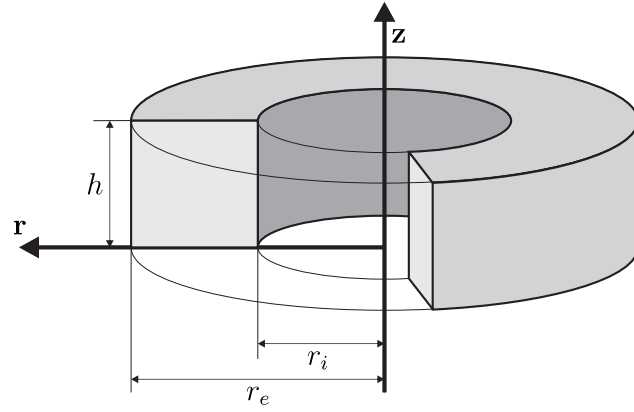


Figure 1.2: *Hollow Cylinder and symbols meaning*

We can see that $\tilde{I}_z$ is independent from the height of the cylinder. It assume its maximum value when the inner radius approaches the outer radius, i.e. for thin cylinders.

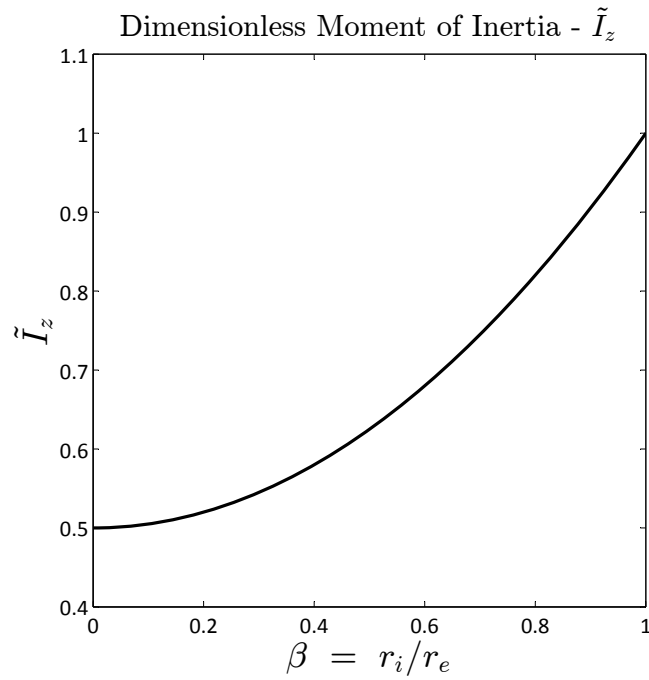


Figure 1.3: *Dimensionless Moment of Inertia*

1.4 Comparison between configurations

We start pointing out that in both the configurations the dimension parallel to the magnetic field is constrained in order to have a homogeneous magnetic field in the air gap of the magnetic circuit. It means that the height h of the device in the first configuration and the radial length of the cross section, $L = r_e - r_i$, in the second one can not be greater than a certain value.

In Figure (1.4) it is evident that where the dimensionless moment of inertia has its maximum we have the minimum of the dimensional moment of inertia. For the axial magnetic field configuration if the device has an almost optimal dimensionless moment of inertia it is not possible to have a great dimensional moment of inertia due to the limitation of the height h of the device.

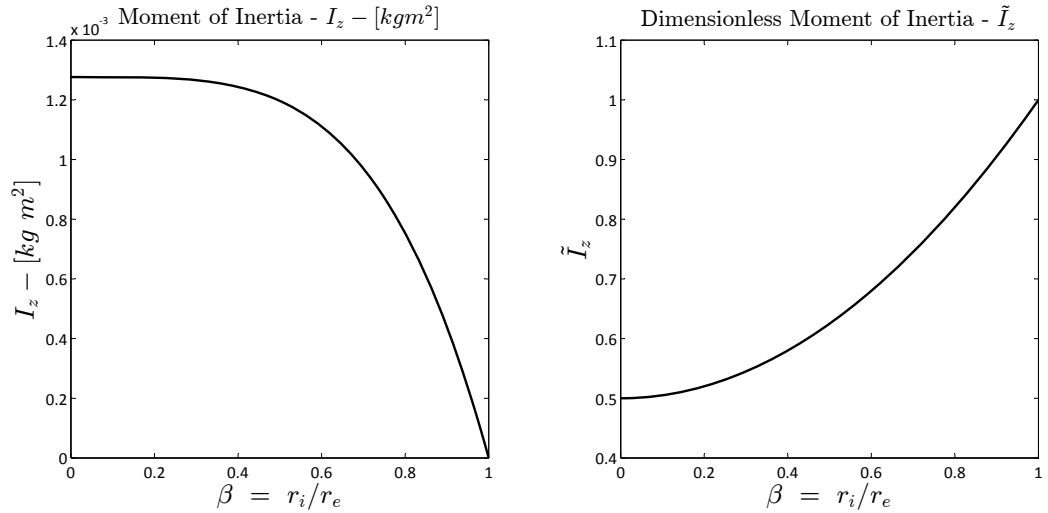


Figure 1.4: Dimensional and Dimensionless Moment of Inertia - 1st Configuration

For the radial magnetic field configuration the device can be characterized by an almost optimal dimensionless moment of inertia and it is possible to obtain the desired dimensional moment of inertia modifying the height of the device, not constrained in this configuration.

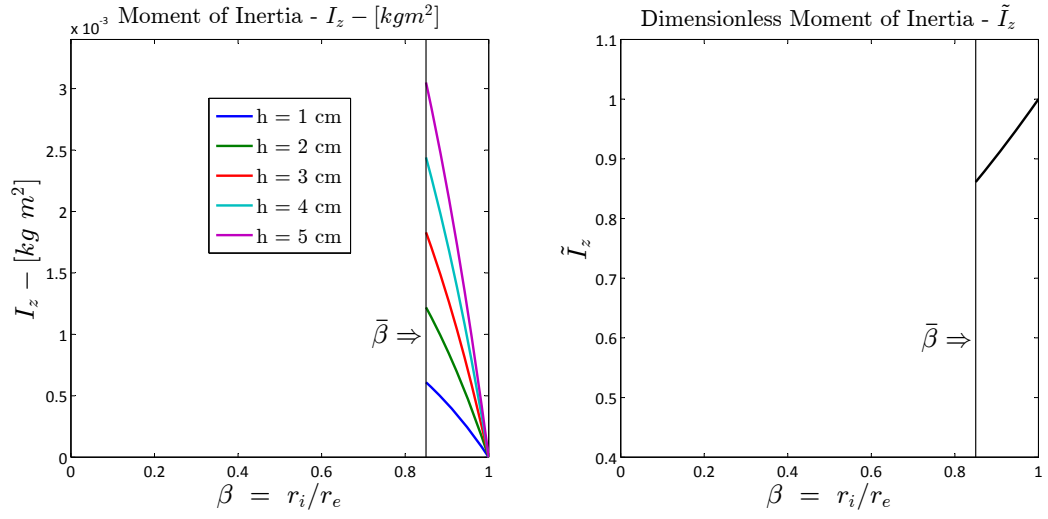


Figure 1.5: *Dimensional and Dimensionless Moment of Inertia - 2st Configuration*

We could think to stack several annuli according to the axial magnetic field configuration in order to obtain the desired moment of inertia while optimizing the dimensionless moment of inertia, Figure (1.6), as already studied in [5]. This approach, anyway, would increase the contact surface between the liquid and cavity, and then the viscous shear, more than how it would increase choosing the second configuration.

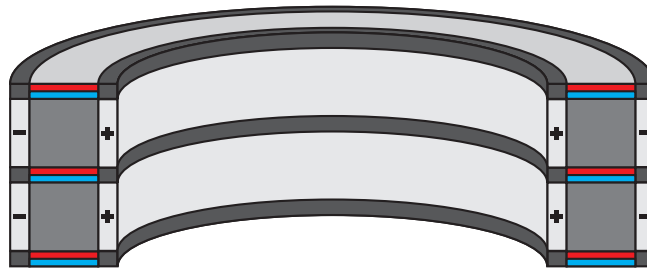


Figure 1.6: *MHD Fluid Loop - 1st Configuration*

We can conclude that the second configuration, characterized by a radial magnetic field and an axial electric field, maximizes the dimensionless moment of inertia while optimizing the contact surface of the conductive liquid. It represents a more easily scalable configuration in which the required moment of

inertia of the liquid flywheel can be met just varying the height of device. The device, according to the selected configuration, has electric plates on the top and on the bottom sides and two concentric cylindrical magnets generating a radial magnetic field. The conductive fluid is placed in the air gap of the hollow cylinder, Figure (1.7). The spin direction of the liquid can be switched by changing the direction of the electric field between the two plates.

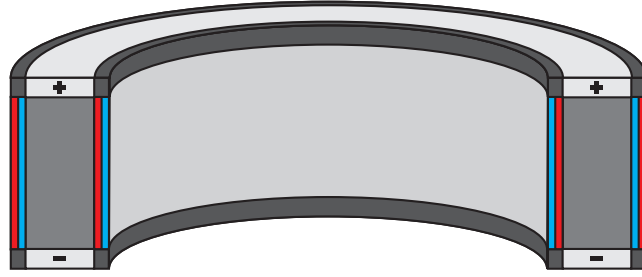


Figure 1.7: *MHD Fluid Loop - 2nd Configuration*

Axially symmetric Finite Differences Model

In this chapter, a 2-dimensional Finite Differences model will be developed on the base of the MHD set equation. The model solves numerically the axially symmetric problem of a conductive liquid rotating in a torus with rectangular cross section due to the interaction of a radial magnetic field and an axial electric field.

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2.1 Complete MHD set of equations

The physics of the Magneto-Hydro-Dynamics (MHD) phenomena is generally described by a set of equations that takes into account the mutual interactions among the velocity, the electric field and the magnetic field.

The complete MHD set of equations includes Maxwell's equations applied to a moving conductive medium, together with the fluid-dynamics equation for a viscous fluid on which acts an electromagnetic body force.

$$\nabla \times \mathbf{E} = -\frac{\partial \mathbf{B}}{\partial t} \quad (2.1.1)$$

$$\nabla \cdot \mathbf{B} = 0 \quad (2.1.2)$$

$$\nabla \times \mathbf{B} = 4\pi\mu\mathbf{J} \quad (2.1.3)$$

$$\nabla \cdot \mathbf{J} = 0 \quad (2.1.4)$$

$$\mathbf{J} = \sigma(\mathbf{E} + \mathbf{U} \times \mathbf{B}) \quad (2.1.5)$$

$$\rho \frac{\partial \mathbf{U}}{\partial t} + \rho(\mathbf{U} \cdot \nabla)\mathbf{U} = -\nabla P_n + \nu\rho\Delta\mathbf{U} + \mathbf{J} \times \mathbf{B} \quad (2.1.6)$$

$$\nabla \cdot \mathbf{U} = 0 \quad (2.1.7)$$

In order to simplify the set three assumptions are considered:

- The flow is considered laminar
- The problem is considered axially-symmetric and it is solved over the $r - z$ domain.
- The hypothesis of Low Magnetic Reynolds, Low- Re_m is applied.

The hypothesis of axially-symmetric problem is a direct consequence of the distribution of magnetic field considered to be radial all over the annulus and the presence of electric armatures that extend all over the upper and

lower surface of the actuator. The hypothesis of laminar flow is consistent considering a limited maximum velocity of the flow and the typical stabilizing effect that Lorentz force induce in MHD flows. The hypothesis of Low- Re_m will be discussed in detail in the next section.

2.1.1 The Low Magnetic Reynolds Approximation

The Magnetic Reynolds number, Re_m , gives information on how the magnetic field is influenced by the velocity field and the electric current. We look at the effect of a prescribed magnetic field on the flow. To ensure that $\mathbf{B}$ remains unaffected by $\mathbf{U}$ we must restrict ourselves to low magnetic Reynolds numbers:

$$R_m = \frac{U_0 L}{\eta} = \mu_m \sigma U L \ll 1 \quad (2.1.8)$$

where U_0 is the mean velocity of the fluid. However, this is not overly restrictive, at least not in the case of liquid-metal MHD. For example for case we are going to analyze $\eta \sim 1 \text{ m}^2/\text{s}$, $L \sim 0.005$ and internal friction keeps U_0 to a level of around $0.01 \text{ m/s} \sim 1 \text{ m/s}$. This gives $R_m \sim 10^{-3} \div 10^{-4}$.

The essence of the low Magnetic Reynolds R_m approximation is that the magnetic field associated with induced currents, $J \sim \sigma U_0 \times B$, is negligible by comparison with the imposed magnetic field.

There are three distinct cases which commonly arise.

- 1 The imposed magnetic field is static, the flow is induced by some external agent, and friction keeps u to a modest level in the sense that $U_0 \ll \eta/l$
- 2 The imposed magnetic field travels or rotates uniformly. This induces a flow which, due to friction in the fluid, is somewhat slower than the speed of the field.
- 3 The imposed magnetic field oscillates extremely rapidly, in the sense that the skin-depth $\delta = (2/\mu_m \sigma f)^{1/2}$ is much less than the characteristic length of the problem, f being the field frequency. The magnetic field is then excluded from the interior of the conductor and inertia or friction in the fluid ensures that $U_0 \ll fL$

Categories (1) – (3) cover the majority of flows in engineering applications. Typical examples are the magnetic damping of jets, vortices or turbulence (1), magnetic stirring using a rotating magnetic field (2) and magnetic levitation (3). In the cases we are going to study the applied magnetic field will be steady and as said, the Magnetic Reynolds is $R_m \ll 1$. We can then assert that all the case we are going to simulate belong to the first case.

We now discuss the simplifications which result in the governing equations when R_m is low and the imposed magnetic field is steady. Let $\mathbf{E}_0$, $\mathbf{J}_0$ and $\mathbf{B}_0$ represent the fields which would exist in a given situation if $\mathbf{U} = 0$, and let $\mathbf{e}$, $\mathbf{j}$ and $\mathbf{b}$ be the infinitesimal perturbations in $\mathbf{E}$, $\mathbf{J}$ and $\mathbf{B}$ which occur due to the presence of a vanishingly small velocity field. These quantities are governed by:

$$\nabla \times \mathbf{E}_0 = 0 \quad (2.1.9)$$

$$\mathbf{J}_0 = \sigma \mathbf{E}_0 \quad (2.1.10)$$

$$\nabla \times \mathbf{e} = -\frac{\partial \mathbf{b}}{\partial t} \quad (2.1.11)$$

$$\mathbf{j} = \sigma (\mathbf{e} + \mathbf{U} \times \mathbf{B}_0) \quad (2.1.12)$$

where we have neglected the second-order term $\mathbf{U} \times \mathbf{b}$ in (2.1.12). Now Faraday's equation gives $\mathbf{e} \sim \mathbf{U} \mathbf{b}$ and so the perturbation in the electric field may also be neglected in (2.1.11). Ohm's law now becomes:

$$\mathbf{J} = \sigma (\mathbf{E}_0 + \mathbf{U} \times \mathbf{B}_0) \quad (2.1.13)$$

However, $\mathbf{E}_0$ is irrotational and so may be written as $-\nabla \phi$, where ϕ is an electrostatic potential. Our final version of Ohm's law is therefore:

$$\mathbf{J} = \sigma (-\nabla \phi + \mathbf{U} \times \mathbf{B}_0) \quad (2.1.14)$$

while the leading-order term in the Lorentz force per unit volume is:

$$\mathbf{F} = \mathbf{J} \times \mathbf{B}_0 \quad (2.1.15)$$

equation (2.1.14) and (2.1.15) are all that we need to evaluate the Lorentz force in low- R_m MHD.

There is no need to calculate $\mathbf{b}$ since it does not appear in the Lorentz force. Moreover, $\mathbf{J}$ is uniquely determined by (2.1.14) since:

$$\nabla \cdot \mathbf{J} = 0 \quad (2.1.16)$$

$$\nabla \times \mathbf{J} = \sigma \nabla \times (\mathbf{U} \times \mathbf{B}_0) \quad (2.1.17)$$

and a vector field is unambiguously determined if its divergence and curl are known.

From now on we shall drop the subscript on $\mathbf{B}_0$ and $\mathbf{E}_0$, on the understanding that $\mathbf{B}$ represents the imposed, steady magnetic field and $\mathbf{E}$ is the externally applied electric field [8].

2.2 Axially symmetric MHD equations

The set of equations on which the numerical model has been built are:

$$\nabla \cdot \mathbf{B} = 0 \quad (2.2.1)$$

$$\nabla \times \mathbf{E} = 0 \quad (2.2.2)$$

$$\nabla \cdot \mathbf{J} = 0 \quad (2.2.3)$$

$$J = \sigma(\mathbf{E} + \mathbf{U} \times \mathbf{B}) \quad (2.2.4)$$

$$\rho \frac{\partial \mathbf{U}}{\partial t} + \rho(\mathbf{U} \cdot \nabla) \mathbf{U} = -\nabla P_n + \nu \rho \Delta \mathbf{U} + \mathbf{J} \times \mathbf{B} \quad (2.2.5)$$

$$\nabla \cdot \mathbf{U} = 0 \quad (2.2.6)$$

In the previous set of equations Eq.(2.2.1) states the conservation of the magnetic field flux density, the Eqs.(2.2.2)-(2.2.4) model the electric side of the problem while the Eqs.(2.2.5)-(2.2.5) the fluid-dynamics side.

The shape of the previous set of equations in the cylindrical coordinates r and z is given in detailed way in the following paragraphs.

2.2.1 Magnetic Field

The Eq.(2.2.1) represents the conservation of the magnetic flux density:

$$\nabla \cdot \mathbf{B} = \frac{1}{r} \frac{\partial (rB_r)}{\partial r} + \frac{\partial B_z}{\partial z} = 0 \quad (2.2.7)$$

The low- Re_m approximation implies that the induced magnetic field generated by the electric current flowing through the liquid is neglectable compared to the applied magnetic field. In this way the magnetic field $\mathbf{B}$ can be considered steady over the time and it can be solved independently from Eqs.(2.2.2)-(2.2.6).

As the device is characterized by a pure radial magnetic field with an assigned value B_0 at the inner radius r_{in} , the (2.2.6) can be reduced to:

$$\nabla \cdot \mathbf{B} = \frac{1}{r} \frac{\partial (rB_r)}{\partial r} = 0 \quad (2.2.8)$$

and then:

$$rB_r = const = r_{in}B_0 \Rightarrow B = \frac{r_{in}B_0}{r} \quad (2.2.9)$$

From now we will not specify anymore the direction of magnetic field.

2.2.2 Navier-Stokes equations and continuity equation for the velocity

In this paragraph we will simplify both the (2.2.5) and (2.2.6) using the hypothesis of laminar flow and axial symmetry.

The (2.2.6) is the continuity equation for the velocity and it can be rewritten as:

$$\nabla \cdot \mathbf{U} = \frac{1}{r} \frac{\partial (rU_r)}{\partial r} + \frac{\partial U_z}{\partial z} = 0 \quad (2.2.10)$$

in which the tangential component has been neglected due to the hypothesis of axial symmetry.

Limiting the analysis to the laminar case, we can consider the vorticity equation along the direction θ :

$$\nabla \times \mathbf{U} = \frac{\partial U_r}{\partial z} - \frac{\partial U_z}{\partial r} = 0 \quad (2.2.11)$$

and introducing the potential of the velocity ϕ we can say that the (2.2.11) is satisfied everywhere in the plane $r - z$ if:

$$\begin{cases} U_r = -\frac{\partial \Phi}{\partial r} \\ U_z = -\frac{\partial \Phi}{\partial z} \end{cases} \quad (2.2.12)$$

Substituting the (2.2.12) into the (2.2.10) we obtain:

$$\nabla \cdot \mathbf{U} = \frac{1}{r} \frac{\partial}{\partial r} \left(r \frac{\partial \phi}{\partial r} \right) + \frac{\partial^2 \Phi}{\partial z^2} = \nabla^2 \Phi \quad (2.2.13)$$

The (2.2.13) is the Laplace equation in the space $r - z$ and it is well-known that if there is no source term and all the boundary conditions are Neumann homogeneous boundaries condition the solution is constant all over the domain. We can then say that the radial and axial components of the velocity are zero under the hypothesis of laminar flow.

The (2.2.5) thanks to the results just found can be reduced to the only azimuthal component:

$$\rho \frac{\partial U_\theta}{\partial t} = \mu \left[\frac{1}{r} \frac{\partial}{\partial r} \left(r \frac{\partial U_\theta}{\partial r} \right) + \frac{\partial^2 U_\theta}{\partial z^2} - \frac{U_\theta}{r^2} \right] + J_z \mathbf{B} \quad (2.2.14)$$

in which the hypothesis of laminar flow has been applied and pressure gradient along the azimuthal coordinate has been neglected.

2.2.3 Electric current density and Back electromagnetic force

The electric side of the problem is classically solved reducing the Eqs.(2.2.2)-(2.2.4) to a single equation in the electric potential from which is possible to deduce the electric current necessary to integrate the Navier-Stokes equation. It is known in the literature that this method leads to error in the conservation of electric current and it appears to be effective in the only cases where the Hartmann number is low. This model will be solved with a classical Finite difference algorithm and for this reason will be called *FDM*, Finite Difference Model.

The (2.2.2), also called Faraday Law, states that a variable magnetic field generates a not null curl of the electric field. In the case in study the magnetic field is constant over the time and the electric field will be considered irrotational. Moreover the low- Re_m approximation implies that the induced magnetic field generated by the electric current flowing through the liquid is neglectable compared to the applied magnetic field. We then have

$$\nabla \times \mathbf{E} = \hat{\theta} \left(\frac{\partial E_r}{\partial z} - \frac{\partial E_z}{\partial r} \right) = 0 \quad (2.2.15)$$

In order to satisfy the (2.2.15) we introduce the potential of the electric field ϕ defined as:

$$\begin{cases} E_r = -\frac{\partial \phi}{\partial r} \\ E_z = -\frac{\partial \phi}{\partial z} \end{cases} \quad (2.2.16)$$

In this way the (2.2.15) is automatically solved.

The (2.2.3) describes the conservation of electric current; as for the (2.2.2) it can be expressed in cylindrical coordinates as:

$$\nabla \cdot \mathbf{J} = \frac{J_r}{r} + \frac{\partial J_r}{\partial r} + \frac{\partial J_z}{\partial z} = 0 \quad (2.2.17)$$

Finally the (2.2.4) also called Ohm's law for moving conductors links the velocity field with the magnetic field, electric current and electric field. We rewrite its components along r and z and using the (2.2.16) we have:

$$\begin{aligned} J_r &= \sigma (E_r + (\mathbf{U} \times \mathbf{B}) \cdot \hat{r}) = \\ &= \sigma E_r = -\sigma \frac{\partial \phi}{\partial r} \end{aligned} \quad (2.2.18)$$

$$\begin{aligned} J_z &= \sigma (E_z + (\mathbf{U} \times \mathbf{B}) \cdot \hat{z}) = \\ &= \sigma (E_z - U_\theta \mathbf{B}) = -\sigma \left(\frac{\partial \phi}{\partial z} + U_\theta \mathbf{B} \right) \end{aligned} \quad (2.2.19)$$

It is convenient to assemble the (2.2.16), (2.2.17), (2.2.18) and (2.2.19) in order to have all the electric quantities function of the only electric potential.

We compute the derivatives with respect of r and z of the homologous components of the electric current density:

$$\frac{\partial J_r}{\partial r} = -\sigma \frac{\partial^2 \phi}{\partial r^2} \quad (2.2.20)$$

$$\frac{\partial J_z}{\partial z} = -\sigma \left(\frac{\partial^2 \phi}{\partial z^2} + \frac{\partial U_\theta}{\partial z} \mathbf{B} \right) \quad (2.2.21)$$

Substituting these results in the (2.2.17) we obtain the equation in the electric potential:

$$\frac{1}{r} \frac{\partial \phi}{\partial r} + \frac{\partial^2 \phi}{\partial r^2} + \frac{\partial^2 \phi}{\partial z^2} = -\frac{\partial U_\theta}{\partial z} \mathbf{B} \quad (2.2.22)$$

An alternative method to investigate the electric current distribution is to apply the node-voltage analysis to an electric network consistent with the mesh-grid used for solving the Navier-Stokes equation. Thanks to low- Re_m , we can say that the electric network will be inductors-free and only made by electric resistances and voltage generators representing the back electromotive voltage induced by the spinning liquid through the magnetic field. This approach will be described in the following during the discussion of the numerical method used to solve the problem, and it will be called *FDHM*, Finite Differences Hybrid Model because the Finite differences approach will be used only for the fluid-dynamics part of the problem while the algebraic node-voltage analysis will be used for the electric part of the problem.

2.2.4 Boundary and Initial Conditions

The complete set of equations for the low- R_m MHD has been reduced to a system of two PDE equations, (2.2.14) and (2.2.22), in the unknowns U_θ , velocity of the fluid, and ϕ , electrostatic potential. A relation more is needed to link the two PDE equations; the relation chosen is the (2.2.19) which provide the conversion of the electrostatic potential to the z -component of the electric current density needed to compute the azimuthal velocity field.

In order to obtain a solution for the problem we have to set the right boundary conditions for all the unknown taken into account.

The liquid is enclosed in a torus with rectangular cross section. Neglecting the effect of the capillarity in the zones next to the corners and the effect that a not wetting liquid could have on the velocity field, we set the no-slip condition on all the boundaries of the cross section. It means that the tangential velocity of the fluid at the boundaries is set to the same velocity of the torus, and considering the torus at rest and bounded to a fixed reference system the velocity of the fluid at the boundaries is set to zero.

$$U_\theta(r_{in}, z, t) = U_\theta(r_{out}, z, t) = U_\theta(r, 0, t) = U_\theta(r, h, t) = 0 \quad (2.2.23)$$

The initial condition for the velocity is:

$$U_{\theta}(r, z, 0) = 0 \quad (2.2.24)$$

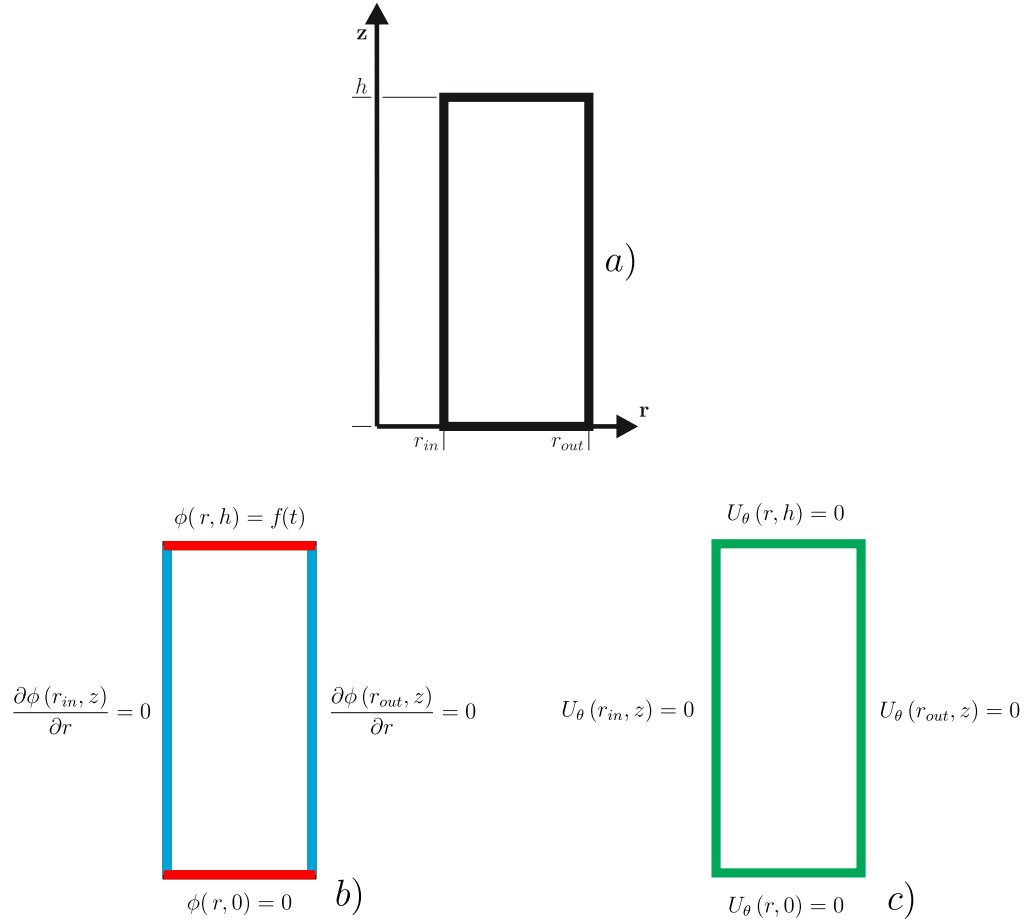


Figure 2.1: Boundary Condition applied to the Finite Difference model.

- a) Reference system and dimension of the cross section*
- b) Electric boundary conditions*
- c) Fluid-dynamics boundary conditions*

On the electric side we shall impose boundary conditions that allow the electric current to flow through the section. We choose an external electric field along z and then we consider a net flux of electric current only on the boundaries parallel to the r axis while insulating the boundaries parallel to the z axis. The external electric field is generated by time varying electrostatic potential

at the upper boundary, $z = h$ and we consider a null, constant over the time, reference electric potential at the lower boundary, $z = 0$.

$$\phi(r, 0) = 0 \quad (2.2.25)$$

$$\phi(r, h) = f(t) \quad (2.2.26)$$

The condition of isolated wall for the boundaries parallel to the z axis can be assigned with the use of (2.2.18), putting the radial component of the velocity equal to zero:

$$J_r = -\sigma \frac{\partial \phi}{\partial r} = 0 \quad (2.2.27)$$

and obtaining:

$$\frac{\partial \phi(r_{in}, z)}{\partial r} = 0 \quad (2.2.28)$$

$$\frac{\partial \phi(r_{out}, z)}{\partial r} = 0 \quad (2.2.29)$$

2.3 FDM: Finite Differences Model

The *FDM* has been solved numerically using a consistent and conservative scheme over a collocated not uniform mesh [9] applied to the following set of equations:

$$\rho \frac{\partial U_\theta}{\partial t} = \mu \left[\frac{1}{r} \frac{\partial}{\partial r} \left(r \frac{\partial U_\theta}{\partial r} \right) + \frac{\partial^2 U_\theta}{\partial z^2} - \frac{U_\theta}{r^2} \right] + J_z \mathbf{B} \quad (2.3.1)$$

$$\frac{1}{r} \frac{\partial \phi}{\partial r} + \frac{\partial^2 \phi}{\partial r^2} + \frac{\partial^2 \phi}{\partial z^2} = -\frac{\partial U_\theta}{\partial z} \mathbf{B} \quad (2.3.2)$$

$$J_r = -\sigma \frac{\partial \phi}{\partial r} \quad (2.3.3)$$

$$J_z = -\sigma \left(\frac{\partial \phi}{\partial z} + U_\theta \mathbf{B} \right) \quad (2.3.4)$$

$$\mathbf{B}(r) = \frac{r_{in} B_0}{r} \quad (2.3.5)$$

In this collocated grid the velocity, U_θ , and electric potential, ϕ , are calculated at the center cell while the radial and axial electric current densities J_r and J_z are evaluated on the cell faces respectively parallel to the z and r axes.

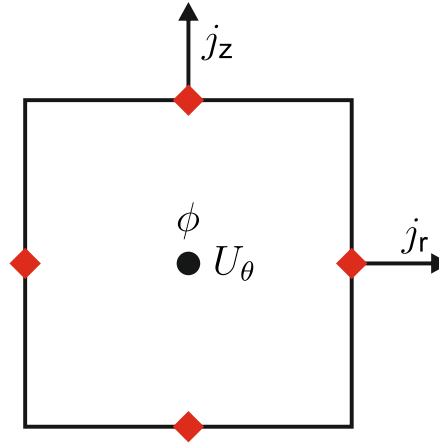


Figure 2.2: *Arrangement of the physical quantities in a collocated grid.*

The algorithm starts from the calculation of the electric potential at the center cell starting from a null initial condition for the velocity field U_θ ; once known the electric potential in all the domain, it is possible to calculate the current densities by means of Eqs.(2.3.3)-(2.3.4). The so calculated electric current densities refer to the cell faces and in order to proceed to the calculation of velocity field they need to be interpolated to the center cell in order to have the Lorentz force at the center cell. The mono-dimensional and time-dependant Navier-Stokes equation is then solved used the Crank-Nicolson method over a not uniform grid over the space $r - z$ and a uniform time sequence. In the following the discrete formulas for Eqs.(2.3.1)-(2.3.4) will be given in details.

2.3.1 Conservative and Consistent Laplace and divergence operator in cylindrical coordinates

The conservation of the electric current in differential form refer in the collocated grid to the cell faces.

$$\nabla \cdot \mathbf{J} = \frac{J_r}{r} + \frac{\partial J_r}{\partial r} + \frac{\partial J_z}{\partial z} = 0 \quad (2.3.6)$$

The terms of the (2.3.6) can be expressed in the discrete form as follows:

$$\frac{J_r}{r} = \frac{1}{r_{i,j}} \frac{(J_r)_{i+1/2,j} + (J_r)_{i-1/2,j}}{2} \quad (2.3.7)$$

$$\frac{\partial J_r}{\partial r} = \frac{(J_r)_{i+1/2,j} - (J_r)_{i-1/2,j}}{r_{i+1/2} - r_{i-1/2}} \quad (2.3.8)$$

$$\frac{\partial J_z}{\partial z} = \frac{(J_z)_{i,j+1/2} - (J_z)_{i,j-1/2}}{z_{i+1/2} - z_{i-1/2}} \quad (2.3.9)$$

It is possible to separate the contribution of the electric potential to the electric current $\mathbf{J}^s$ from the the one generated by the electric induction $\mathbf{J}^u$.

$$\mathbf{J} = -\nabla\phi + \mathbf{U} \times \mathbf{B} = \mathbf{J}^s + \mathbf{J}^u \quad (2.3.10)$$

The components of the conductive part of electric current density at the cell faces can be related with the values of the electric potential at center cell.

$$\begin{cases} (J_r^s)_{i+1/2,j} = -\frac{\phi_{i+1,j} - \phi_{i,j}}{r_{i+1} - r_i} \\ (J_r^s)_{i-1/2,j} = -\frac{\phi_{i,j} - \phi_{i-1,j}}{r_i - r_{i-1}} \end{cases} \quad (2.3.11)$$

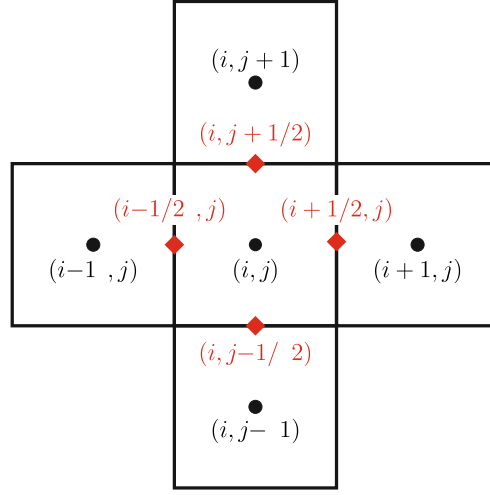


Figure 2.3: *Finite Differences Model stencil.*

$$\begin{cases} (J_z^s)_{i+1/2,j} = -\frac{\phi_{i,j+1} - \phi_{i,j}}{z_{j+1} - z_j} \\ (J_z^s)_{i-1/2,j} = -\frac{\phi_{i,j} - \phi_{i,j-1}}{z_j - z_{j-1}} \end{cases} \quad (2.3.12)$$

Substuting Eqs.(2.3.11)-(2.3.12) into Eqs.(2.3.7)-(2.3.9), for the Eq.(2.3.7) we have:

$$\frac{J_r^s}{r} = a_{11} \phi_{i+1,j} + a_{12} \phi_{i,j} + a_{13} \phi_{i-1,j} \quad (2.3.13)$$

$$\begin{aligned} a_{11} &= \frac{1}{2r_{i,j}} \left(-\frac{1}{r_{i+1} - r_i} \right) \\ a_{12} &= \frac{1}{2r_{i,j}} \left(\frac{1}{r_{i+1} - r_i} - \frac{1}{r_i - r_{i-1}} \right) \\ a_{13} &= \frac{1}{2r_{i,j}} \left(\frac{1}{r_i - r_{i-1}} \right) \end{aligned} \quad (2.3.14)$$

for the Eq.(2.3.8):

$$\frac{\partial J_r^s}{\partial r} = a_{21} \phi_{i+1,j} + a_{22} \phi_{i,j} + a_{23} \phi_{i-1,j} \quad (2.3.15)$$

$$\begin{aligned} a_{21} &= - \left(\frac{1}{r_{i+1/2} - r_{i-1/2}} \right) \left(\frac{1}{r_{i+1} - r_i} \right) \\ a_{22} &= - \left(\frac{1}{r_{i+1/2} - r_{i-1/2}} \right) \left(-\frac{1}{r_{i+1} - r_i} - \frac{1}{r_i - r_{i-1}} \right) \\ a_{23} &= - \left(\frac{1}{r_{i+1/2} - r_{i-1/2}} \right) \left(\frac{1}{r_i - r_{i-1}} \right) \end{aligned} \quad (2.3.16)$$

for the Eq.(2.3.9):

$$\frac{\partial J_z^s}{\partial z} = a_{31} \phi_{i,j+1} + a_{32} \phi_{i,j} + a_{33} \phi_{i,j-1} \quad (2.3.17)$$

$$\begin{aligned} a_{31} &= - \left(\frac{1}{z_{j+1/2} - z_{j-1/2}} \right) \left(\frac{1}{z_{j+1} - z_j} \right) \\ a_{32} &= - \left(\frac{1}{z_{j+1/2} - z_{j-1/2}} \right) \left(-\frac{1}{z_{j+1} - z_j} - \frac{1}{z_j - z_{j-1}} \right) \\ a_{33} &= - \left(\frac{1}{z_{j+1/2} - z_{j-1/2}} \right) \left(\frac{1}{z_j - z_{j-1}} \right) \end{aligned} \quad (2.3.18)$$

Considering that the velocity $\mathbf{U}$ is purely azimuthal while the magnetic field $\mathbf{B}$ is purely radial, the inductive part of the Eq.(2.3.10) can be written as:

$$\mathbf{J}^u = \mathbf{U} \times \mathbf{B} = \begin{vmatrix} \hat{r} & \hat{\theta} & \hat{z} \\ U_r & U_\theta & U_z \\ B_r & B_\theta & B_z \end{vmatrix} = \begin{vmatrix} \hat{r} & \hat{\theta} & \hat{z} \\ 0 & U_\theta & 0 \\ B_r & 0 & 0 \end{vmatrix} = -(B_r U_\theta) \hat{z} \quad (2.3.19)$$

The divergence of $\mathbf{J}^u$ has only the component parallel to the z axis and using the values of B_r and U_θ at the cell faces we have:

$$\begin{aligned}
 \nabla \cdot \mathbf{J}^u &= \frac{\partial J_z^u}{\partial z} = \frac{(J_z^u)_{i,j+1/2} - (J_z^u)_{i,j-1/2}}{z_{i+1/2} - z_{i-1/2}} \\
 &= \frac{-(B_r)_i (U_\theta)_{i,j+1/2} + (B_r)_i (U_\theta)_{i,j-1/2}}{z_{i+1/2} - z_{i-1/2}} = -C
 \end{aligned} \tag{2.3.20}$$

Finally the equation in the electric potential at the cell center can be expressed as:

$$\begin{aligned}
 (a_{11} + a_{21}) \phi_{i+1,j} + (a_{12} + a_{22} + a_{32}) \phi_{i,j} + (a_{13} + a_{23}) \phi_{i-1,j} + \\
 a_{31} \phi_{i,j+1} + a_{33} \phi_{i,j-1} = C
 \end{aligned} \tag{2.3.21}$$

2.3.2 Interpolation Operator for not equispaced grid

The Navier-Stokes formula gives values of U_θ at the center cell but in order to compute the inductive part of the Eq.(2.3.20) values of U_θ at the cell faces are needed. On a collocated grid system, an interpolation operator $\Lambda_{c \rightarrow f}$ can be defined, which represents the interpolation from the cell center to cell faces:

$$\mathbf{U}_f = \Lambda_{c \rightarrow f} \mathbf{U}_c \tag{2.3.22}$$

where $\mathbf{U}$ denotes the velocity vector (U_r, U_θ, U_z) . The detailed formula of the interpolation operator on collocated grid is:

$$\mathbf{U}_{i+1/2,j} = \Lambda_{c \rightarrow fr+} \mathbf{U}_c = \alpha_+ \mathbf{U}_{i,j} + (1 - \alpha_+) \mathbf{U}_{i+1,j} \tag{2.3.23}$$

$$\mathbf{U}_{i,j+1/2} = \Lambda_{c \rightarrow fz+} \mathbf{U}_c = \beta_+ \mathbf{U}_{i,j} + (1 - \beta_+) \mathbf{U}_{i,j+1} \tag{2.3.24}$$

The subscript $c \rightarrow fr+$ and $c \rightarrow fz+$ denotes the interpolation from the cell center normal to the r and z axes; the symbol $+$ denotes that the cell faces are located to the right and top of the cell center.

The coefficients of the interpolation operator $\Lambda_{c \rightarrow f}$ are defined on the base of

the grid arrangement:

$$\alpha_+ = \frac{r_{i+1} - r_{i+1/2}}{r_{i+1} - r_i} \quad \beta_+ = \frac{z_{i+1} - z_{i+1/2}}{z_{i+1} - z_i} \quad (2.3.25)$$

An equivalent operator can be defined for the interpolation from the cell faces to center cell and it can be used to obtain the electric current density at center cell once known it's value at the cell faces. Due to the position of the center cell, considered to be in the middle position between two faces of the cell, we have:

$$\mathbf{J}_c = \Lambda_{f \rightarrow c} \mathbf{J}_f \quad (2.3.26)$$

where $\Lambda_{f \rightarrow c}$ has the meaning of:

$$(J_r)_{i,j} = \left[\Lambda_{fr \rightarrow c} (J_r)_{fr} \right]_{i,j} = \frac{(J_r)_{i+1/2,j} + (J_r)_{i-1/2,j}}{2} \quad (2.3.27)$$

$$(J_z)_{i,j} = \left[\Lambda_{fz \rightarrow c} (J_z)_{fz} \right]_{i,j} = \frac{(J_z)_{i,j+1/2} + (J_z)_{i,j-1/2}}{2} \quad (2.3.28)$$

2.3.3 Grid Generator

The mathematical model is solved on a not equispaced grid in order to study with accuracy the boundary effects due to the coupling between the electric and fluid-dynamic problems. It is possible to define two different characteristic lengths for the boundary layers:

- for the boundaries normal to the magnetic field, i.e. parallel to the z axis, the boundary layers are called Hartmann layers and their lengths are:

$$\delta_{Hrtm} = \frac{1}{B} \sqrt{\frac{\mu}{\sigma}} \quad (2.3.29)$$

- for the boundaries parallel to the magnetic field, i.e. parallel to the r

axis, the boundary layers are called Shercliff layers and their lengths are:

$$\delta_{Sh} = \sqrt{\frac{L}{B}} \sqrt{\frac{\mu}{\sigma}} \quad (2.3.30)$$

where L is the difference between the inner and outer radius of the cavity containing the liquid.

With the purpose to make several simulations for different values of magnetic field $\mathbf{B}$, electric conductivity of the fluid σ , length of the magnetic gap L and height of liquid torus h it has been necessary to understand how let the grid vary in function of the dimension of δ_{Hrtm} and δ_{Sh} .

The grid generator create four different kinds of grid. For the r axis:

- when $\delta_{Hrtm} \leq (r_{out} - r_{in})/6$ the grid on the r axis is characterized by an equispaced mesh in the zone:

$$r_{in} \leq r \leq r_{in} + 2\delta_{Hrtm} \quad r_{out} - 2\delta_{Hrtm} \leq r \leq r_{out} \quad (2.3.31)$$

while in the zone $r_{in} + 2\delta_{Hrtm} < r < r_{out} - 2\delta_{Hrtm}$ the grid is logarithmic

- if $\delta_{Hrtm} > (r_{out} - r_{in})/6$ the grid is logarithmic from r_{in} to r_{out} .

For the z axis:

- when $\delta_{Sh} \leq h/6$ the grid on the z axis is characterized by an equispaced mesh in the zone:

$$0 \leq z \leq 2\delta_{Sh} \quad h - 2\delta_{Sh} \leq z \leq h \quad (2.3.32)$$

while in the zone $2\delta_{Sh} < z < h - 2\delta_{Sh}$ the grid is logarithmic

- if $\delta_{Sh} > h/6$ the grid is logarithmic from 0 to h .

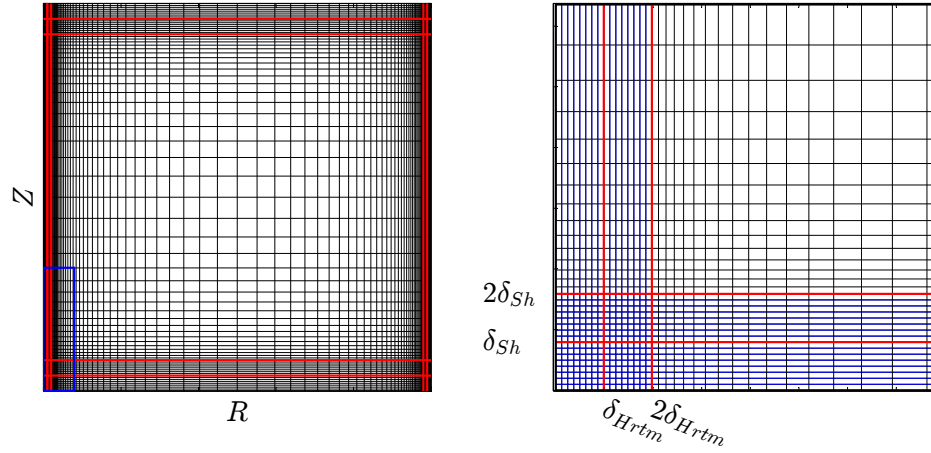


Figure 2.4: Grid for $\delta_{Sh} \leq h/6$ and $\delta_{Hrtm} \leq (r_{out} - r_{in})/6$.

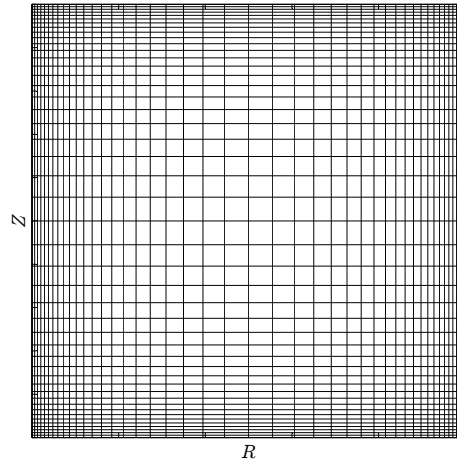


Figure 2.5: Grid for $\delta_{Sh} > h/6$ and $\delta_{Hrtm} > (r_{out} - r_{in})/6$.

2.3.4 Navier-Stokes Equation

The Navier-Stokes equation Eq.(2.3.1):

$$\rho \frac{\partial U_\theta}{\partial t} = \mu \left[\frac{1}{r} \frac{\partial U_\theta}{\partial r} + \frac{\partial^2 U_\theta}{\partial r^2} + \frac{\partial^2 U_\theta}{\partial z^2} - \frac{U_\theta}{r^2} \right] + J_z \mathbf{B} \quad (2.3.33)$$

is solved using a Crank-Nicolson scheme on a five points stencil. The finite difference scheme has been calculated for a not equispaced mesh. The time derivative has been evaluated with a second order finite difference scheme with constant time step. The shape of the discretized version of the derivatives

present in Eq.(2.3.33) is:

$$\frac{1}{r} \frac{\partial U_\theta}{\partial r} \rightarrow b_{1,1} u_{i+1,j} + b_{1,2} u_{i,j} + b_{1,3} u_{i-1,j} \quad (2.3.34)$$

$$\begin{aligned} b_{11} &= \frac{1}{r_{i,j}} \left(\frac{1}{r_{i+1,j} - r_{i,j}} \right) \\ b_{12} &= \frac{1}{r_{i,j}} \left(-\frac{1}{r_{i+1,j} - r_{i,j}} + \frac{1}{r_{i,j} - r_{i-1,j}} \right) \\ b_{13} &= \frac{1}{r_{i,j}} \left(\frac{1}{r_{i,j} - r_{i-1,j}} \right) \end{aligned} \quad (2.3.35)$$

$$\frac{\partial^2 U_\theta}{\partial r^2} \rightarrow b_{2,1} u_{i+1,j} + b_{2,2} u_{i,j} + b_{2,3} u_{i-1,j} \quad (2.3.36)$$

$$\begin{aligned} b_{21} &= \frac{1}{r_{i+1,j} - r_{i,j}} \\ b_{22} &= -\frac{1}{r_{i+1,j} - r_{i,j}} - \frac{1}{r_{i,j} - r_{i-1,j}} \\ b_{23} &= \frac{1}{r_{i,j} - r_{i-1,j}} \end{aligned} \quad (2.3.37)$$

$$\frac{\partial^2 U_\theta}{\partial z^2} \rightarrow b_{3,1} u_{i,j+1} + b_{3,2} u_{i,j} + b_{3,3} u_{i,j-1} \quad (2.3.38)$$

$$\begin{aligned} b_{31} &= \frac{1}{z_{i,j+1} - z_{i,j}} \\ b_{32} &= -\frac{1}{z_{i,j+1} - z_{i,j}} - \frac{1}{z_{i,j} - z_{i,j-1}} \\ b_{33} &= \frac{1}{z_{i,j} - z_{i,j-1}} \end{aligned} \quad (2.3.39)$$

$$\frac{U_\theta}{r^2} \rightarrow \frac{u_{i,j}}{r_{i,j}^2} = -b_{4,2} u_{i,j} \quad (2.3.40)$$

$$\frac{\partial U_\theta}{\partial t} \rightarrow \frac{u_{i,j}^{n+1} - u_{i,j}^n}{\Delta t} \quad (2.3.41)$$

The right side of the discrete version of Eq.(2.3.33) has then the following shape:

$$\begin{aligned}
 F(u, J_z, B, r, z) = & \mu [(b_{1,1} + b_{2,1}) u_{i+1,j} \\
 & + (b_{1,2} + b_{2,2} + b_{3,2} + b_{4,2}) u_{i,j} \\
 & + (b_{1,3} + b_{2,3}) u_{i-1,j} \\
 & + b_{3,1} u_{i,j+1} + b_{3,3} u_{i,j-1}] + (J_z)_{i,j} B_{i,j}
 \end{aligned} \tag{2.3.42}$$

Applying the Crank-Nicolson algorithm we obtain:

$$\frac{u_{i,j}^{n+1} - u_{i,j}^n}{\Delta t} = \frac{1}{2\rho} [F^{n+1}(u, J_z, B, r, z) + F^n(u, J_z, B, r, z)] \tag{2.3.43}$$

The (2.3.43) led to a tridiagonal system that can be easily solved with tridiagonal matrix algorithms.

The computation of the velocity U_θ extends to the all the center cell nodes. The no-slip condition is implemented adding node on the boundaries, where U_θ is known and equal to zero, in $(i - 1/2, j)$ and $(i + 1/2, j)$ for the boundaries parallel to the z axis and nodes in $(i, j - 1/2)$ and $(i, j + 1/2)$ for the boundaries parallel to the r .

2.3.5 Discrete boundary conditions

In order to solve the electodynamic problem (2.2.22) the boundary conditions (2.2.25)(2.2.26) and (2.2.28)-(2.2.29) must be applied.

The Dirichlet conditions on the top and bottom side of the cross section are quite simple to express. They are known values of the electric potential and they appear inside the vector of known terms of the risolutive system.

The Neumann conditions on the right and left side of the cross section needs to be studied. The classical way to implement conditions on the derivative is to use ghost nodes on the grid and to couple this conditions with

the general equation for the electric potential.

The ghost nodes are virtual nodes external to the computational grid for which is not supposed to know the value of any variables.

For the left side boundary we have that the Eq.(2.2.28) can be expressed as:

$$\frac{\partial \phi(r_{in}, z)}{\partial r} = 0 \rightarrow \frac{\phi_{i+1,j} - \phi_{i,j}}{r_{i+1} - r_i} + \frac{\phi_{i,j} - \phi_{i-1,j}}{r_i - r_{i-1}} = 0 \quad (2.3.44)$$

On this side of the cross section the node $(i - 1, j)$ is a ghost node and we can put the electric potential on this node in evidence to obtain:

$$\phi_{i-1,j} = \phi_{i+1,j} \left(\frac{r_i - r_{i-1}}{r_{i+1} - r_i} \right) + \phi_{i,j} \left(1 - \frac{r_i - r_{i-1}}{r_{i+1} - r_i} \right) \quad (2.3.45)$$

If we write the Eq.(2.2.22) and then we introduce the Eq.(2.3.45) we obtain the Neumann condition applied, just using central finite differences scheme and the value of ϕ in the ghost node will disappear.

In the same way for the right boundary of the cross section, we have:

$$\frac{\partial \phi(r_{out}, z)}{\partial r} = 0 \rightarrow \frac{\phi_{i+1,j} - \phi_{i,j}}{r_{i+1} - r_i} + \frac{\phi_{i,j} - \phi_{i-1,j}}{r_i - r_{i-1}} = 0 \quad (2.3.46)$$

which means:

$$\phi_{i+1,j} = \phi_{i,j} \left(1 - \frac{r_{i+1} - r_i}{r_i - r_{i-1}} \right) + \phi_{i-1,j} \left(\frac{r_{i+1} - r_i}{r_i - r_{i-1}} \right) \quad (2.3.47)$$

With the same procedure we obtain the Neumann condition on the right boundary of the cross section.

2.3.6 Voltage Drive and Current Drive

The model as it is has a natural predisposition to simulate the reaction of the conductive liquid to a certain kind of law $f(t)$ for the electrostatic potential assigned between the lower and the upper boundaries of the cross section.

It does not seem to be able to simulate an assigned trend $g(t)$ of the electric current; in other words it is not possible to give directly as boundary condition

a certain law for the electric current flowing through the conductive liquid. In order to make it possible we have to consider the (2.1.14). Being the external electric field along z we can replace the divergence with the total derivative of the electrostatic potential with respect to z . We have:

$$\mathbf{J} = -\sigma \frac{\partial \phi}{\partial z} + \sigma \mathbf{U} \times \mathbf{B}_0 \quad (2.3.48)$$

Integrating over the surface in the plane $r - \theta$ we obtain the total current flowing in the conductive liquid:

$$\mathbf{I} = -\sigma \int_S \frac{\partial \phi}{\partial z} dS + \sigma \int_S \mathbf{U} \times \mathbf{B}_0 dS \quad (2.3.49)$$

In order to simplify the derivative of the electric potential with respect of z component we proceed integrating along z too:

$$\int_0^h \mathbf{I} dz = -\sigma \int_0^h \int_S \frac{\partial \phi}{\partial z} dS dz + \sigma \int_0^h \int_S \mathbf{U} \times \mathbf{B}_0 dS dz \quad (2.3.50)$$

Due to the conservation of electric current and being the electric potential just function of the z we have:

$$\mathbf{I} = -\frac{\sigma S}{h} \Delta \phi - \frac{\sigma}{h} \int_0^h \int_S \mathbf{U} \times \mathbf{B}_0 dS dz \quad (2.3.51)$$

What we have found is the relation between the total electric current and the voltage applied to the upper boundary and it takes into account the back electro motive force. We can say that we can numerically impose a certain electric current of known law $I(t)$ trough the following relation:

$$\phi(r, h, t) = -\frac{h}{\sigma S} I(t) - \frac{1}{S} \int_0^h \int_S \mathbf{U} \times \mathbf{B}_0 dS dz \quad (2.3.52)$$

where with $\phi(r, h, t)$ we intend the electric potential at the upper boundary being zero its value at the lower boundary.

2.3.7 Iterative scheme

The solving algorithm use in sequence all the relation presented in previous section. In order to have a clearer vision of the algorithm a flux diagram is presented.

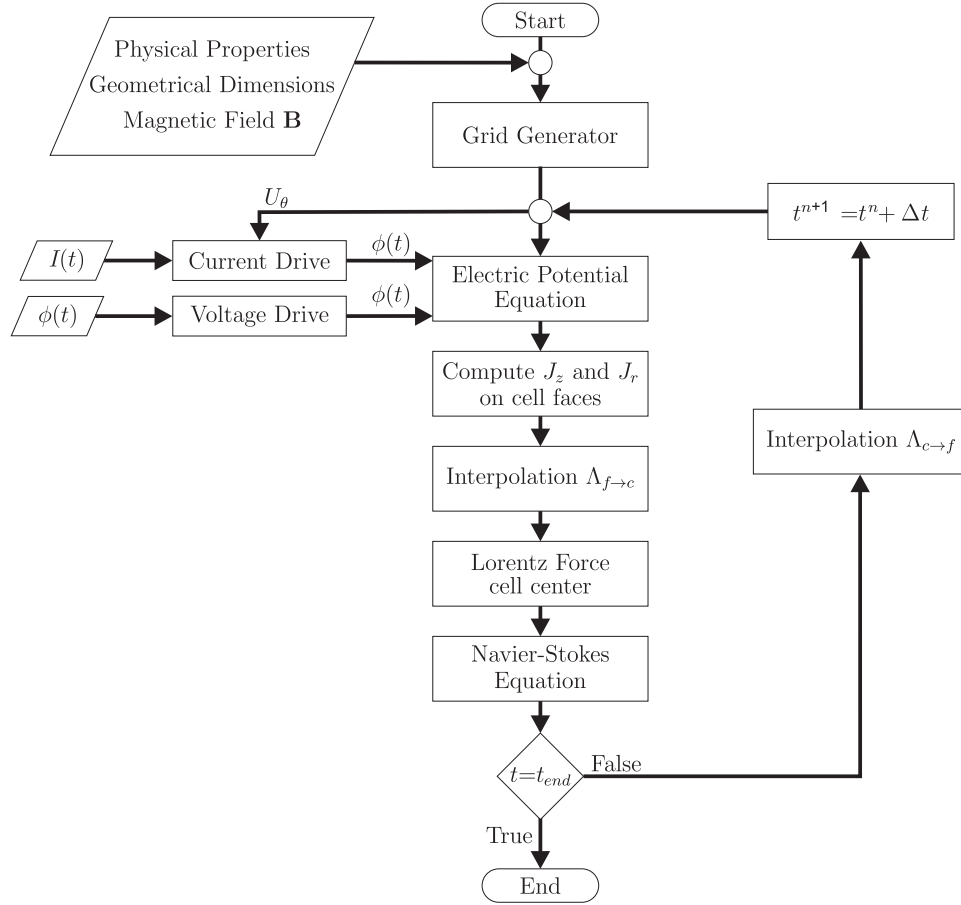


Figure 2.6: *Flux Diagram of the FDM model*

2.3.8 Conservation problems

The FDM shows problem of conservation of electric current. Figure (2.7) shows the total current flowing through the cross section. It is evident that the current should be constant for each value of z because of the conservation of the electric current. The algorithm shows anyway great loss of current near the armatures

of the device. This is not acceptable because this conservation error lead to an underestimation of the Lorentz force acting on the conductive liquid.

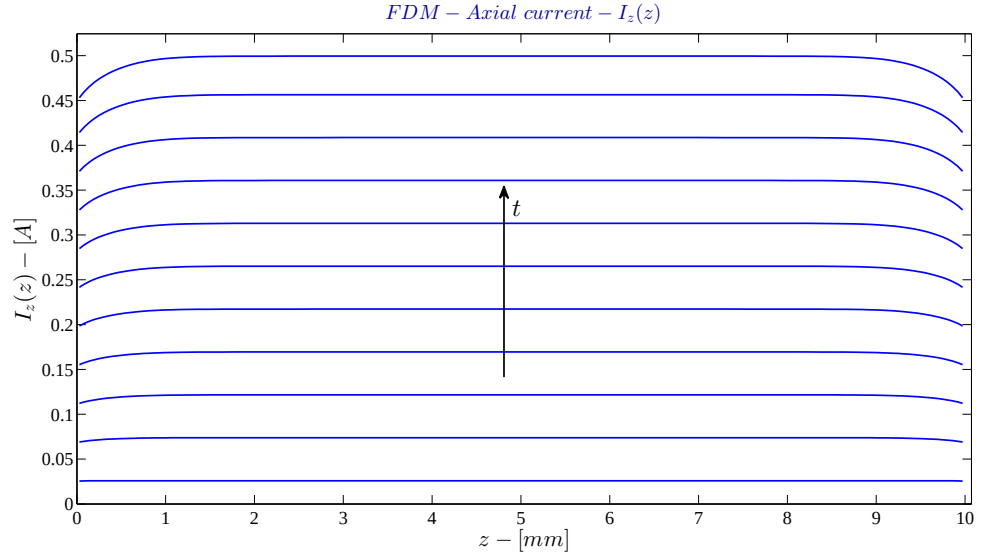


Figure 2.7: FDM - Total axial electric current

The radial component of the electric current, Figure(2.8) has no problem and it shows a perfect antisymmetric trend with null mean.

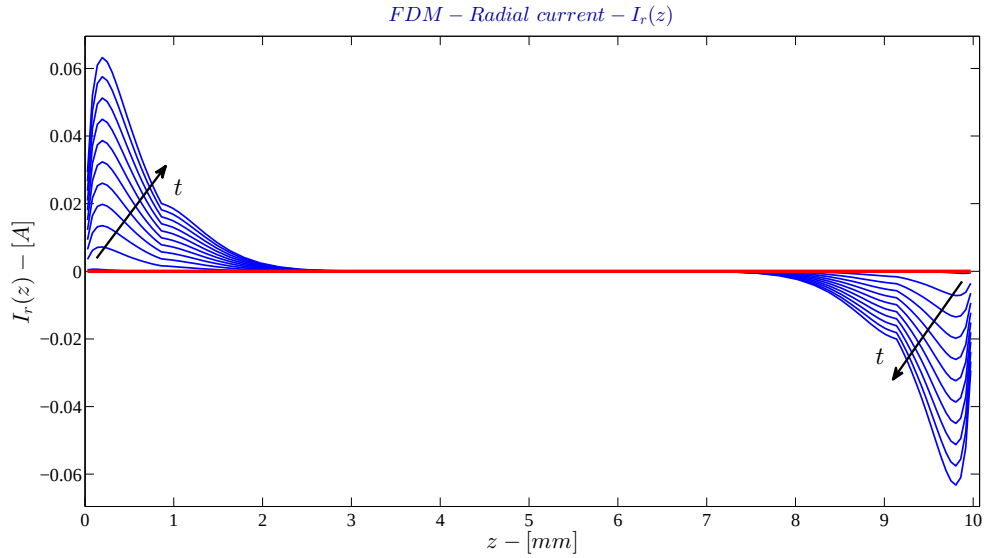


Figure 2.8: FDM - Total radial electric current

2.4 FDHM: Finite Differences Hybrid Model

The Finite Difference Hybrid Model is an alternative algorithm to solve the MHD equations. The difference with the FDM is just on the electric part of the model. Instead of solving the electric potential equation, an electric network is created in order to obtain the shape of the voltage and the electric current distribution inside the cross section of the device. The electric network is then solved applying the nodal analysis.

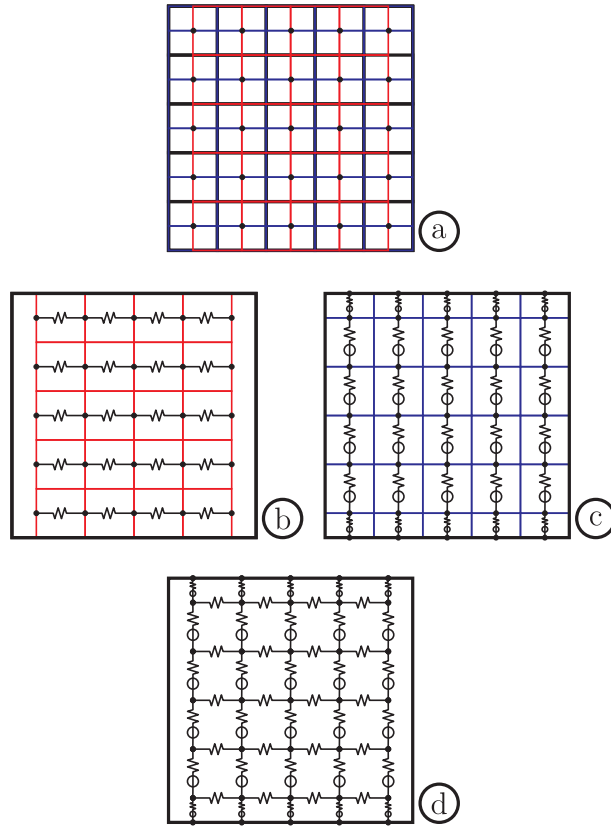


Figure 2.9: *Equivalent electric network.*

- a) *Nodes and cells*
- b) *Electric radial resistances*
- c) *Electric axial resistances and back electromotive voltage*
- d) *Assembled electric network*

The same grid generator of the FDM is used in the FDHM. The cell centers become here the nodes of electric network while the cell faces are the branches.

2.4.1 Equivalent electric network

The equivalent electric network is made of radial and axial resistances and voltage generators.

The radial resistances link the nodes in the radial direction. Applying the conservation of the electric current along the radius on a sequence of cylindrical surface, a law for the density of electric current can be found:

$$\begin{cases} \frac{\partial(r J_r)}{\partial r} = 0 \\ J_r(r_1) = J_0 \end{cases} \Rightarrow J_r(r) = \frac{J_0 r_1}{r} \quad (2.4.1)$$

Introducing first the laws linking the electric current to the electric current density and the voltage to the electric field:

$$I = \iint_S J_r(r) dS = \int_0^l \int_0^{2\pi} J_r r dz d\theta = 2\pi J_0 r l \quad (2.4.2)$$

$$V = - \int_{r_i}^{r_{i+1}} E_r dr = - \frac{1}{\sigma} \int_{r_i}^{r_{i+1}} \frac{J_0 r_1}{r} dr = \frac{J_0 r_1}{\sigma} \ln \left(\frac{r_i}{r_{i+1}} \right) \quad (2.4.3)$$

and then the Ohm law, we obtain:

$$R_r = \frac{V}{I} = \frac{1}{2\pi\sigma} \frac{\ln \left(\frac{r_i}{r_{i+1}} \right)}{l} \quad (2.4.4)$$

The axial resistances link the nodes in direction parallel to z :

$$R_z = \frac{V}{I} = \frac{1}{\sigma} \frac{l}{\pi (r_i^2 - r_{i+1}^2)} \quad (2.4.5)$$

The voltage generators model the back electromotive voltage generated by the

conductive liquid spinning in the magnetic field. Their values can be computed integrated the cross product between velocity and magnetic field over the length parallel to z . Using the Ohm Law for moving conductor we have:

$$V_{ind} = \int_0^{\delta z_j} \mathbf{U} \times \mathbf{B} dz = -U_\theta B(r) \delta z_j \quad (2.4.6)$$

2.4.2 Boundary Conditions

The FDHM does not need specific equations to impose the boundary conditions. The boundary conditions on the side of the cross section parallel to the z axis are imposed not taking into account any radial resistances that could bring electric current in the direction of the boundary itself. So, the nodes adjacent to vertical side of the cross section have just three branches: two axial branches and one radial branch towards the inside of the cross section.

The boundary conditions on the side of the cross section parallel to the r axis are applied imposing the same known value of the electric potential to the nodes on the boundaries.

2.4.3 Voltage and Current drive

In the FDHM the voltage and current drive are realized with two different conditions on the nodes on the upper and lower boundaries.

In order to simulate the voltage drive the voltage on the boundaries are assumed to be known. The lower armature is set to a reference voltage while the nodes on the the upper armature are set to same voltage, in general, function of the time.

In the current drive it is necessary to control the current flowing through the conductive liquid and to have an equipotential surface on each armature. In order to respect these two constrains the nodes on each armature have been collapsed to only one node where to impose a net electric current.

2.4.4 Iterative scheme

The algorithm to solve the FDHM is presented in the following flux diagram.

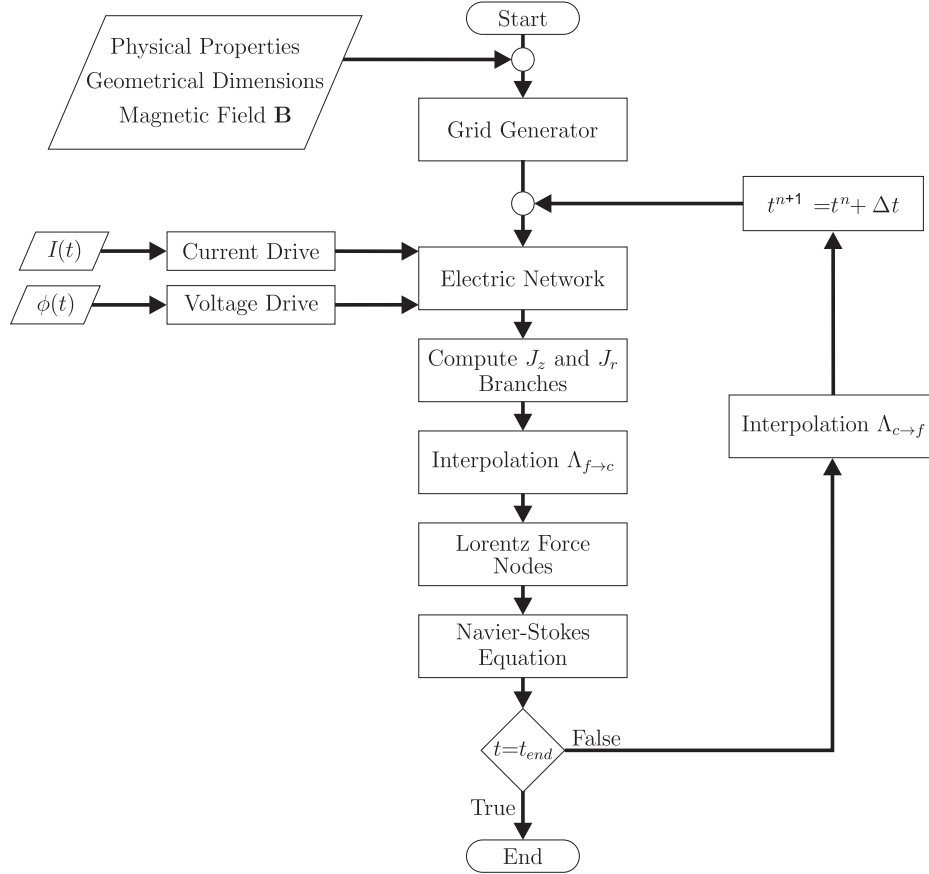


Figure 2.10: *Flux Diagram of the FDHM model*

2.4.5 Results

The FDHM appears to solve the conservation problem of the FDM algorithm, Figures (2.11)(2.12). The total axial electric current is now constant all over the cross section and equal to the supply value.

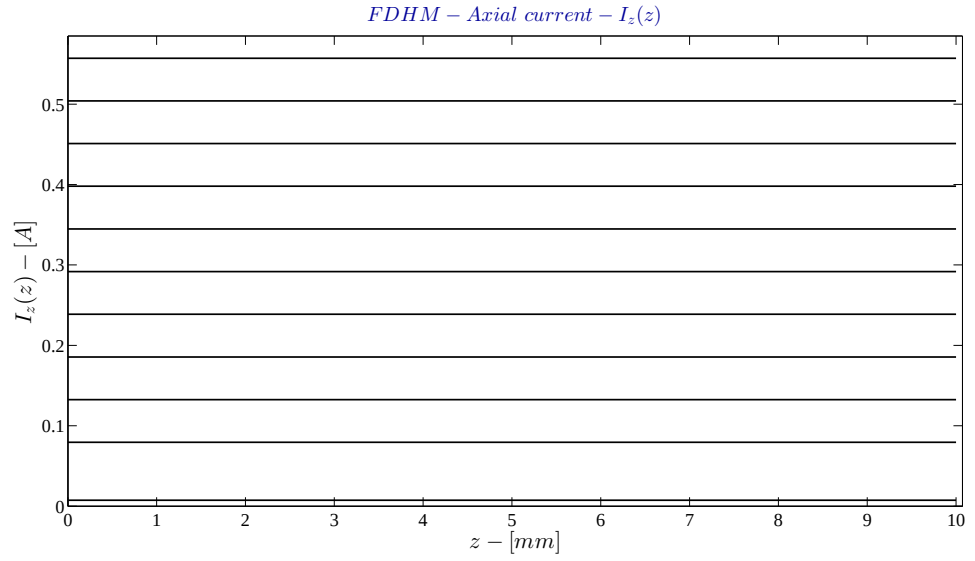


Figure 2.11: *FDHM - Total axial electric current*

The radial component of the electric current shows, as in FDM, an antisymmetric trend with null mean.

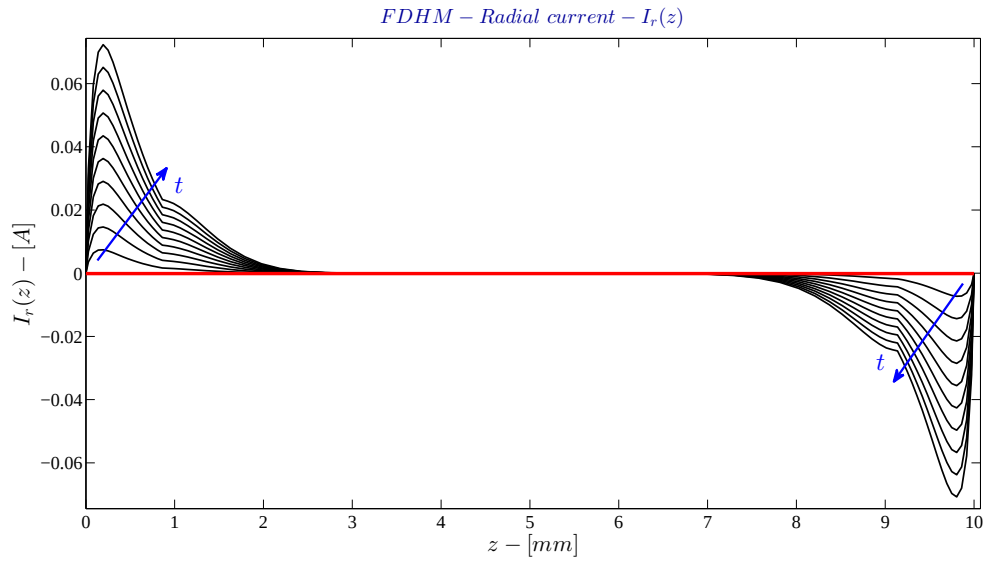


Figure 2.12: *FDHM - Total radial electric current*

The relative errors of the electric and fluid-dynamic part can be computed and they are respectively of the order of 10^{-10} and 10^{-5} .

The error of the electric problem is computed as the standard deviation of the difference between the commanded electric current and computed electric current flowing through the cross section.

$$Err_{el} = \sigma [I_{alim} - I_z(z)] \quad (2.4.7)$$

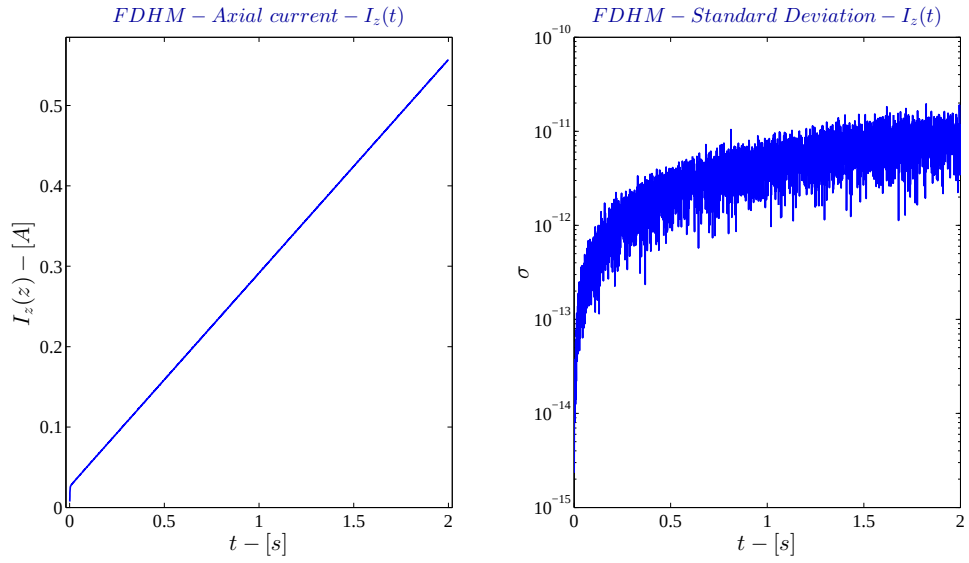


Figure 2.13: *FDHM - Electric problem - Relative error*

The error of the dynamic problem is computed as the difference between the absolute value of the torque computed with the numerical velocity distribution and the difference between the viscous shear stress moment and the moment of Lorentz force:

$$Err_{dyn} = |\dot{\Gamma}| - |M_{Lor} - M_{vis}| \quad (2.4.8)$$

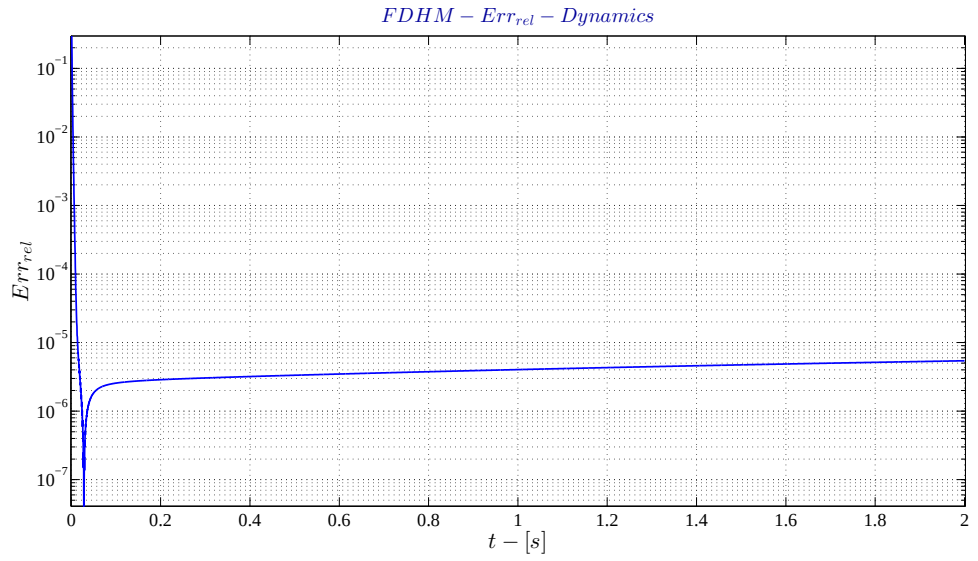


Figure 2.14: *FDHM - Dynamic problem - Relative error*

Finally typical distributions of axial and radial electric current density, linear velocity and angular velocity are shown below.

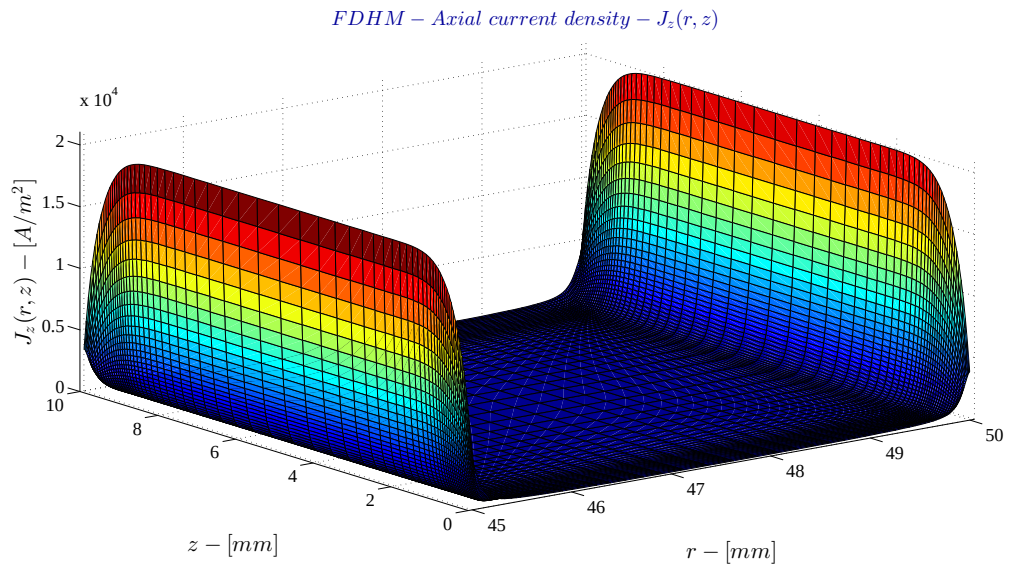


Figure 2.15: *FDHM - Axial electric current density*

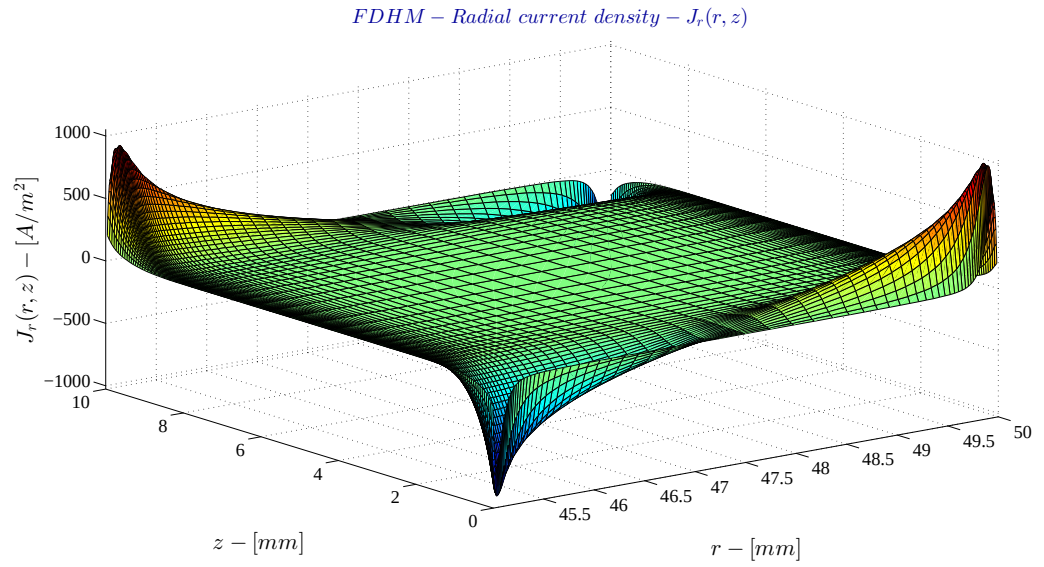


Figure 2.16: *FDHM - Radial electric current density*

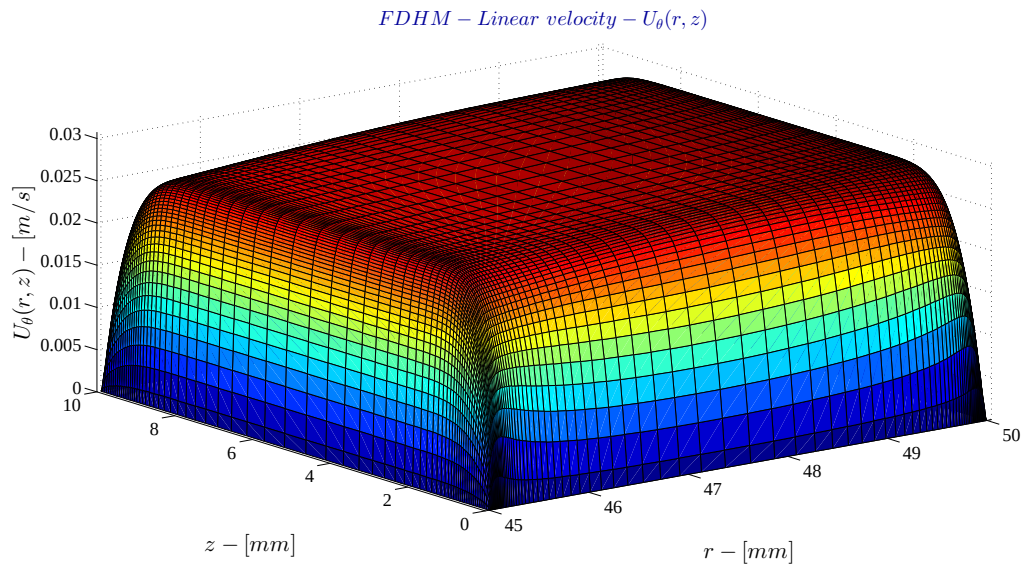


Figure 2.17: *FDHM - Linear velocity*

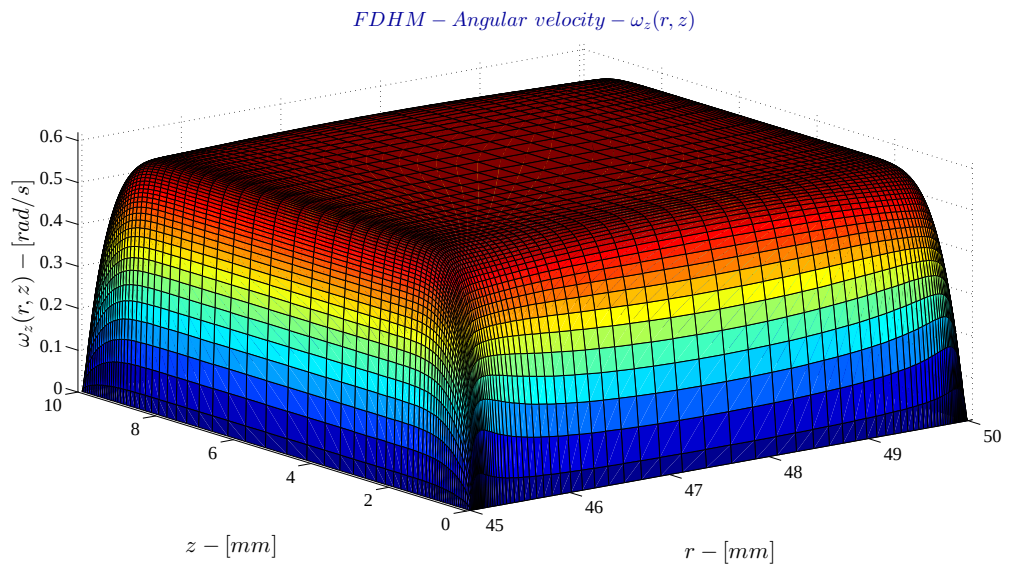


Figure 2.18: *FDHM - Angular velocity*

Lumped Parameter Model

In this chapter, a Lumped Parameter model similar to the model used for classical reaction wheels. The coefficients of the model are computed on the basis of two analytical solutions

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3.1 Lumped Parameter Model

The lumped parameter model of a classical reaction wheel with rigid flywheel is well-known and it take into account the electric and dynamic phenomena happening while the flywheel spins. It has the following shape:

$$\begin{cases} I_z \dot{\omega}_w = K_I I \\ \Delta\phi = K_V \omega_w + R_{eq} I \end{cases} \quad (3.1.1)$$

The purpose of this chapter is to derive a lumped parameter model for the device in study. The model is supposed to be very similar to the classical reaction wheel one. Some contributions need anyway to be added in order to take into account viscous phenomena that are neglectable in the classical reaction wheel.

The general Lumped Parameter Model (LPM) for MHD Reaction Wheel has the following shape:

$$\begin{cases} I_z c_\omega \dot{\omega}_w = -K_{vis} \omega_w + K_I I \\ \Delta\phi = K_V \omega_w + R_{eq} I \end{cases} \quad (3.1.2)$$

In the model the angular velocity of the fluid has the meaning of relative angular velocity between the fluid and the case of the reaction wheel.

The model is similar to the model of a classical reaction wheel with an extra-term taking into account the viscous torque, proportional to the relative angular velocity ω_w . This term leads to a slight different behaviour of the Torque-Current characteristic: while, for the classical reaction wheel, the torque provided is constant when the electric current fed to the device is constant, here, the torque is constant if the electric current varies linearly over the time. This come directly from the nature of the model itself. It models a dumped system and the angular velocity of the liquid can be increased only providing an increasing torque acting on the liquid.

The first equation of (3.1.2) models the dynamic part of the problem stating that the provided torque is the resultant between the moment of the viscous shear and moment of the Lorentz force; the second equation of (3.1.2) model

the electric part of the problem stating that the applied voltage is equal to sum of the resistive voltage losses and the induced voltage generated by the conductive liquid spinning through the magnetic field.

In the following two sections the coefficients of the dynamic and electric parts of the model will be derived [7].

3.2 Lumped Parameter Dynamic Model

In order to derive the dynamic part of the Lumped Parameter model we consider the Navier Stokes equation through an integral balance of moments along the z direction.

$$\begin{aligned} \int_V \rho r \hat{\mathbf{r}} \times \frac{\partial U_\theta}{\partial t} \hat{\boldsymbol{\theta}} dV &= \int_V r \hat{\mathbf{r}} \times [J_z B(r)] \hat{\boldsymbol{\theta}} dV \\ &+ \int_{S \perp r} \mu r \hat{\mathbf{r}} \times \left[r \frac{\partial}{\partial r} \left(\frac{U_\theta}{r} \right) \right] \hat{\boldsymbol{\theta}} dS + \int_{S \perp z} \mu r \hat{\mathbf{r}} \times \left[\frac{\partial U_\theta}{\partial z} \right] \hat{\boldsymbol{\theta}} dS \end{aligned} \quad (3.2.1)$$

A stationary solution needs to be derived and used to obtain quantitative evaluation of the viscous shear at the boundaries together with an equivalent moment of inertia of the spinning liquid and the back electromotive force generated by the spinning liquid inside the superimposed magnetic field.

To compute the moment of the viscous shear, we consider the last term of right side of Eq.(3.2.1) and we obtain for the boundaries of the rectangular cross section:

$$M_{vis} = \int_{S \perp r} \mu r^2 \frac{\partial}{\partial r} \left(\frac{U_\theta}{r} \right) dS + \int_{S \perp z} \mu r \frac{\partial U_\theta}{\partial z} dS \quad (3.2.2)$$

For the terms on the right of the Eq.(3.2.2) a more detailed expression can be given :

$$\begin{aligned} \int_{S \perp r} \mu r^2 \frac{\partial}{\partial r} \left(\frac{U_\theta}{r} \right) dS &= 2\pi \mu r_{in}^3 \int_0^h \frac{\partial \omega_z(r, z)}{\partial r} \Big|_{r=r_{in}} dz \\ &+ 2\pi \mu r_{out}^3 \int_0^h \frac{\partial \omega_z(r, z)}{\partial r} \Big|_{r=r_{out}} dz \end{aligned} \quad (3.2.3)$$

$$\begin{aligned} \int_{S \perp z} \mu r \frac{\partial U_\theta}{\partial z} dS &= 2\pi \mu \int_{r_{in}}^{r_{out}} r^3 \frac{\partial \omega_z(r, z)}{\partial z} \Big|_{z=0} dr \\ &+ 2\pi \mu \int_{r_{in}}^{r_{out}} r^3 \frac{\partial \omega_z(r, z)}{\partial z} \Big|_{z=h} dr \end{aligned} \quad (3.2.4)$$

The moment generated by the viscous shear can then be written as:

$$M_{vis} = K_{vis} \omega_w \quad (3.2.5)$$

where:

$$\begin{aligned} K_{vis} &= \frac{2\pi \mu r_{in}^3}{\omega_0} \int_0^h \frac{\partial \omega_z(r, z)}{\partial r} \Big|_{r=r_{in}} dz + \frac{2\pi \mu r_{out}^3}{\omega_0} \int_0^h \frac{\partial \omega_z(r, z)}{\partial r} \Big|_{r=r_{out}} dz \\ &+ \frac{2\pi \mu}{\omega_0} \int_{r_{in}}^{r_{out}} r^3 \frac{\partial \omega_z(r, z)}{\partial z} \Big|_{z=0} dr + \frac{2\pi \mu}{\omega_0} \int_{r_{in}}^{r_{out}} r^3 \frac{\partial \omega_z(r, z)}{\partial z} \Big|_{z=h} dr \end{aligned} \quad (3.2.6)$$

The torque generated by the Lorentz Force can be derived from the first term of the right side of the (3.2.1). Introducing the shape of the radial magnetic field we have:

$$M_{Lor} = \int_V \mathbf{r} \times \hat{\theta} (J_z B_r) dV = B_0 r_i h I = K_I I \quad (3.2.7)$$

where:

$$K_I = B_0 r_i h \quad (3.2.8)$$

The term of the left side of the (3.2.1) represents the resulting torque acting on the conductive fluid, i.e. the time derivative of the angular momentum. In order to have this term function of the only time derivative of the average velocity of the fluid we start considering the angular momentum in the reference frame of the spacecraft:

$$\Gamma = \int_V \rho r^2 \omega_z(r) \hat{\mathbf{z}} dV = \omega_w c_\omega I_z \hat{\mathbf{z}} \quad (3.2.9)$$

where:

$$c_\omega = \frac{2\pi\rho \int_{r_i}^{r_e} \int_0^h r^3 \omega_z(r, z) dr dz}{\omega_0 I_z} \quad (3.2.10)$$

and I_z is the moment of inertia of a rigid hollow cylinder with the same dimensions of the cavity containing the liquid. In this way the time-derivative of the angular momentum can be considered as:

$$\dot{\Gamma} = \dot{\omega}_w c_\omega I_z \quad (3.2.11)$$

due to invariance with respect of the time of the coefficient c_ω .

3.3 Lumped Parameter Electric Model

A formula for the lumped parameter electric model of the proposed MHD reaction wheel can be obtained by integrating over the volume the microscopic Ohm's Law for moving conductor and introducing the relative velocity between the liquid and the magnetic field considered stiffly linked to the boundaries:

$$\int_V E_z dz dS = \int_V \frac{J_z}{\sigma} dz dS + \int_V r \omega_z(r) B_r dz dS \quad (3.3.1)$$

Considering the electric potential ϕ uniform over the surface and applying the conservation of the electric current we obtain:

$$\Delta\phi = I \frac{h}{\sigma S} + \frac{\int_V r \omega_z(r, z) B_r dz dS}{S} \quad (3.3.2)$$

where the coefficient of the electric current is the equivalent electric resistance of the conductive liquid:

$$R_{eq} = \frac{h}{\sigma S} \quad (3.3.3)$$

The second term on the right side of the (3.3.1) is the induced tension that can be written as:

$$\Delta\phi_{ind} = K_V\omega_w \quad (3.3.4)$$

where:

$$K_V = 2\pi B_0 r_i \frac{\int_{r_i}^{r_e} \int_0^h r \omega_z(r, z) dr dz}{\omega_0 S} \quad (3.3.5)$$

3.4 Analytical solution for the angular and linear velocity

The principal problem to be solved to derive the LPM is to find an analytical solution for the angular and linear velocity of the conducting liquid. This step is necessary to compute the LPM coefficients which appear to be function of this distributions and because the LPM is referred to the average angular of the fluid.

In this paragraph two different solutions will be developed and then compared with the numerical solution provided by the FDHM model.

3.4.1 Monodimensional solution under the hypothesis of Low-Magnetic Reynolds

For convenience we proceed writing the whole set of equation in dimensionless variables. We obtain:

$$\nabla' \cdot \mathbf{b} = 0 \quad (3.4.1)$$

$$\nabla' \cdot \mathbf{j} = 0 \quad (3.4.2)$$

$$\mathbf{j} = \mathbf{e} + \mathbf{u} \times \mathbf{b} \quad (3.4.3)$$

$$\nabla' \times \mathbf{e} = 0 \quad (3.4.4)$$

$$\frac{\partial \mathbf{u}}{\partial \tau} = \frac{1}{Re} \Delta' \mathbf{u} + \frac{Ha^2}{Re} \mathbf{j} \times \mathbf{b} \quad (3.4.5)$$

$$s = \frac{r}{r_e} \quad (3.4.6)$$

$$\nabla' \cdot \mathbf{u} = 0 \quad (3.4.7)$$

in which the following substitutions have been done:

$$\mathbf{u} = \frac{\mathbf{U}}{U_0} \quad (3.4.8)$$

$$\mathbf{b} = \frac{\mathbf{B}}{B_0} \quad (3.4.9)$$

$$\mathbf{e} = \frac{\mathbf{E}}{B_0 U_0} \quad (3.4.10)$$

$$\mathbf{j} = \frac{\mathbf{J}}{\sigma B_0 U_0} \quad (3.4.11)$$

$$\tau = \frac{t U_0}{L} \quad (3.4.12)$$

$$\nabla' = L \nabla \quad (3.4.13)$$

The coefficients in Eq.(3.4.5) are the Hartmann Number, Ha defined as $B_0 L \sqrt{\sigma/\mu}$ and the Reynolds number, Re , defined as $\rho L U_0 / \mu$. In Eqs.(3.4.8)-(3.4.13) U_0 has the meaning of average velocity of the fluid, B_0 is the magnetic field density at the outer radius and r_e is the outer radius of the liquid flywheel.

The main simplifying assumption adopted is to consider infinite the length of the annulus along the axis of symmetry. It implies the neglect of the boundary effects next to the two electric armatures and it implies moreover that all the derivative along z are considered neglectible. The superimposed magnetic field is assumed to be dependent just from the radial coordinate. This assumption together with the uniformity of the imposed electric field $\mathbf{e}$ makes the problem axially symmetric, i.e. all the derivative with respect of the azimuthal direction can be neglected.

We start by analyzing the continuity equations of the magnetic flux density $\mathbf{b}$, current density $\mathbf{j}$ and velocity $\mathbf{u}$.

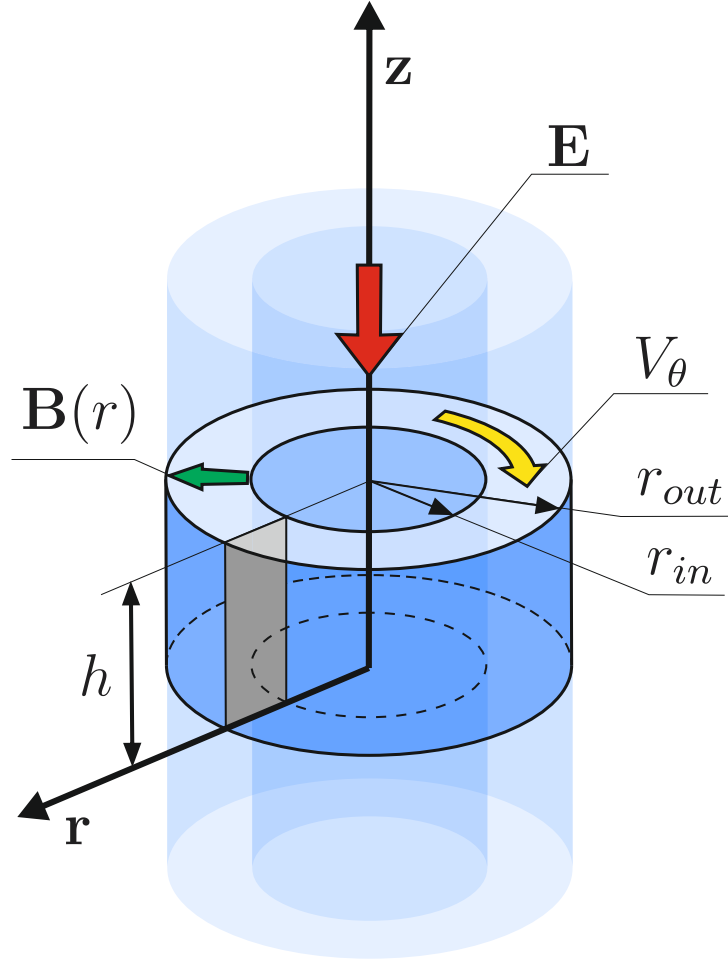


Figure 3.1: Section of an infinite annulus with applied electric field $\mathbf{E}$ and radial magnetic field $\mathbf{B}(r)$.

It is possible to simplify the continuity equations for the dimensionless magnetic flux density $\mathbf{b}$, Eq.(3.4.1), the dimensionless electric current density $\mathbf{j}$, Eq.(3.4.2), and the dimensionless velocity of the fluid $\mathbf{u}$, Eq.(3.4.7). Introducing the expression of the divergence in cylindrical coordinates and simplifying we obtain:

$$\nabla \mathbf{b} = \frac{1}{s} \frac{\partial}{\partial s}(s b_r) + \frac{1}{s} \frac{\partial b_\theta}{\partial \theta} + \frac{\partial b_z}{\partial z} = \frac{\partial}{\partial s}(s b_r) = 0 \quad (3.4.14)$$

$$\nabla \mathbf{j} = \frac{1}{s} \frac{\partial}{\partial s}(s j_r) + \frac{1}{s} \frac{\partial j_\theta}{\partial \theta} + \frac{\partial j_z}{\partial z} = \frac{\partial}{\partial s}(s j_r) = 0 \quad (3.4.15)$$

$$\nabla \mathbf{u} = \frac{1}{s} \frac{\partial}{\partial s}(su_r) + \frac{1}{s} \frac{\partial u_\theta}{\partial \theta} + \frac{\partial u_z}{\partial z} = \frac{\partial}{\partial s}(su_r) = 0 \quad (3.4.16)$$

with the following boundary condition at the outer radius:

$$\begin{aligned} b_r(s_0, \tau) &= 1 \\ s_0 = 1 \quad j_r(s_0, \tau) &= 0 \\ u_r(s_0, \tau) &= 0 \end{aligned} \quad (3.4.17)$$

Solving Eqs.(3.4.14)-(3.4.16) using Eqs.(3.4.17) we obtain:

$$b_r(s, \tau) = \frac{1}{s} \quad (3.4.18)$$

$$j_r(s, \tau) = 0 \quad (3.4.19)$$

$$u_r(s, \tau) = 0 \quad (3.4.20)$$

The axial component of the dimensionless velocity, u_z , can be studied taking into account the Eq.(3.4.5) along z :

$$\frac{\partial u_z}{\partial \tau} = \frac{\lambda^2}{Re} \left[\frac{1}{s} \frac{\partial}{\partial s} \left(s \frac{\partial u_z}{\partial s} \right) \right] + \frac{Ha^2}{Re} j_\theta b_r \quad (3.4.21)$$

Using the hypothesis of axial-symmetry it is possible to say that the component along θ of the current density j_θ is null both because of the radial distribution of the magnetic field and because of the uniformity of the electric field applied. This result leads the Eq. (3.4.21) to be a homogeneous PDE with homogeneous boundary condition due to the no-slip condition applied at the boundaries to the velocity field. Furthermore and without lack of generality we can impose that the fluid is at rest at the initial time $\tau = 0$. This problem gives a null solution over the time for the u_z :

$$u_z(s, \tau) = 0 \quad (3.4.22)$$

Writing the Eq.(3.4.3) along the three directions of cylindrical reference frame and being the dimensionless velocity purely azimuthal and the magnetic flux density purely radial we have:

$$\begin{aligned} e_r &= j_r - u_\theta b_z + u_z b_\theta = 0 \\ e_\theta &= j_\theta - u_z b_r + u_r b_z = 0 \\ e_z &= j_z - u_r b_\theta + u_\theta b_r = j_z + u_\theta b_r \end{aligned} \quad (3.4.23)$$

The Eq.(3.4.4) that describes the irrotationality of the applied electric field is solved taking into account the hypothesis that the component of the electric field along the z direction is uniform. In other words if we consider each armature equipotential we can see that the electric field along the z direction does not vary along the radial and azimuthal directions.

Finally we obtain the complete set of equation for magneto-fluid problems under the hypothesis of Low-Magnetic Reynolds reduced to the the following system of three equations:

$$b_r = \frac{1}{s} \quad (3.4.24)$$

$$j_z = e_z - u_\theta b_r \quad (3.4.25)$$

$$\frac{\partial u_\theta}{\partial \tau} = \frac{\lambda^2}{Re} \frac{\partial}{\partial s} \left[\frac{1}{s} \frac{\partial}{\partial s} (s u_\theta) \right] + \frac{H a^2}{Re} j_z b_r \quad (3.4.26)$$

in which Eq.(3.4.24) is already solved and just Eqs.(3.4.25)-(3.4.26) are coupled. Starting from the hypothesis of Low-Magnetic Reynolds the same equations as in [10] has been found and we can use the same solution proposed in the paper.

Writing the Eq.(3.4.26) with the same symbology as in Reference [10] and

introducing the Eq.(3.4.25) we have:

$$\frac{\partial u_\theta}{\partial \tau} = C_1 \left[\frac{\partial^2 u_\theta}{\partial s^2} + \frac{1}{s} \frac{\partial u_\theta}{\partial s} - \frac{m^2 u_\theta}{s^2} \right] + C_2 \frac{e_z}{s} \quad (3.4.27)$$

where:

$$\begin{aligned} C_1 &= \lambda^2 / Re & C_2 &= Ha^2 / Re \\ \lambda &= L / R & m^2 &= 1 + Ha^2 / \lambda^2 \end{aligned} \quad (3.4.28)$$

In [10] two solutions are given: one steady-state solution and one time-dependent solution. Looking at the purpose to obtain a lumped parameter model we show more interest in the stationary solution in which the influence of the electric current and the shape of velocity profile can be easily uncoupled instead of considering the transient solution that gives a time dependent solution but by which is not possible to derive a lumped parameter model.

The steady-state solutions for the velocity and the angular velocity is:

$$U_\theta(r) = \frac{E_z}{B_0} \left[A_1 r - A_2 r^m - A_3 \frac{1}{r^m} \right] \quad (3.4.29)$$

$$\omega_z(r) = \frac{U_\theta(r)}{r} = \frac{E_z}{B_0} \left[A_1 - A_2 r^{m-1} - A_3 \frac{1}{r^{m+1}} \right] \quad (3.4.30)$$

where:

$$A_1 = \frac{1}{R} \quad A_2 = \frac{1 - \beta^{m+1}}{(1 - \beta^{2m}) R^m} \quad A_3 = \frac{\beta^{m+1} (1 - \beta^{m-1}) R^m}{1 - \beta^{2m}} \quad (3.4.31)$$

and

$$\beta = 1 - \lambda \quad (3.4.32)$$

The shape of the dimensional magnetic field B can be obtained from the

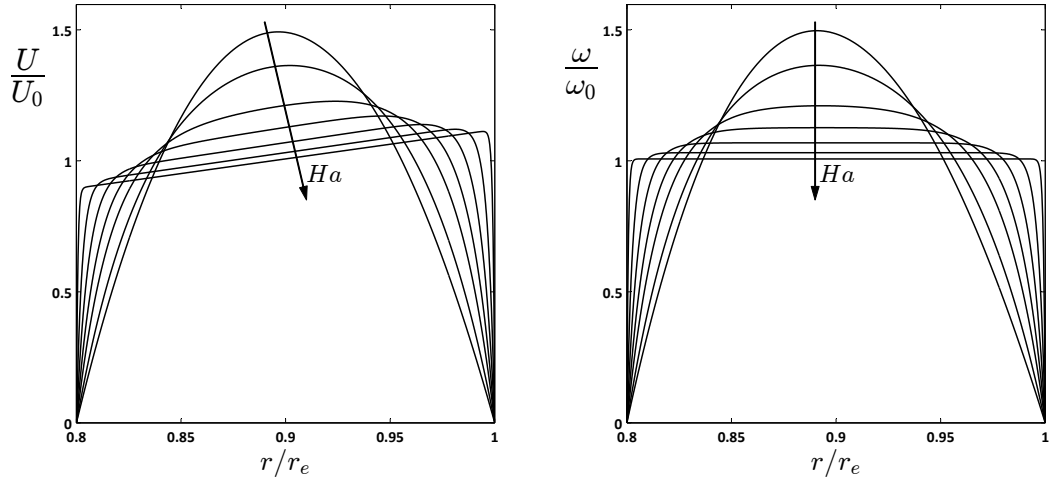


Figure 3.2: *Analytical solutions for dimensionless linear and angular velocities at different Hartmann Number Ha .*

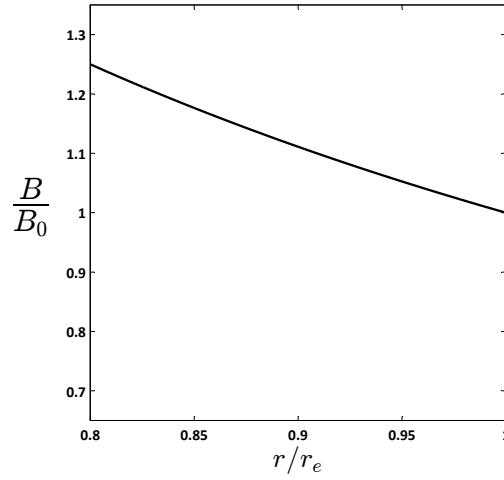


Figure 3.3: *Magnetic Field Density $\mathbf{B}$ normalized with respect of the Magnetic Field Density B_0 at the outer radius.*

Eqs.(3.4.9)-(3.4.24):

$$\mathbf{B}(r) = \frac{B_0 r_e}{r} \quad (3.4.33)$$

where B_0 is the value of the magnetic field at $r = r_e$. The solution is expressed in dimensional quantities. The coefficients m and β in Eq.(3.4.29) are function of the physical property of the conductive fluid and of the dimension along the

direction r of the cavity containing the fluid. The solution can be considered expressed in the reference frame of the spacecraft with the meaning of relative velocity between the liquid and the case. With respect of [10] this solution has been derived explicitly under the hypothesis of Low- Re_m and this can help to understand the range of validity of the solution.

3.4.2 Bi-dimensional solution extended to the cross-section

The first attempt is to find an asymptotic bi-dimensional analytical solution for the complete set of the MHD equations in cylindrical coordinates with radial magnetic field and axial electric current. Hunt and Baylis [12] already succeeded in finding this analytical solution but in the case of axial magnetic field and radial electric current. The same procedures can be followed to try to find a solution for the case in study.

The complete set of equations needs to be used:

$$\nabla \times \mathbf{E} = -\frac{\partial \mathbf{B}}{\partial t} \quad (3.4.34)$$

$$\nabla \cdot \mathbf{B} = 0 \quad (3.4.35)$$

$$\nabla \times \mathbf{H} = 4\pi \mathbf{J} \quad (3.4.36)$$

$$\nabla \cdot \mathbf{J} = 0 \quad (3.4.37)$$

$$J = \sigma(\mathbf{E} + \mathbf{V} \times \mathbf{B}) \quad (3.4.38)$$

$$-\nabla P_n + \nu \rho \Delta \mathbf{V} + \mathbf{J} \times \mathbf{B} = 0 \quad (3.4.39)$$

$$\nabla \cdot \mathbf{V} = 0 \quad (3.4.40)$$

Expanding the Eq.(3.4.38) and considering that:

$$\mathbf{E} = E_r \hat{r} + E_z \hat{z} = -\frac{\partial \phi}{\partial r} \hat{r} - \frac{\partial \phi}{\partial z} \hat{z} \quad (3.4.41)$$

we have:

$$J_r = -\sigma \frac{\partial \phi}{\partial r} \quad (3.4.42)$$

$$J_z = -\sigma \left(\frac{\partial \phi}{\partial z} + U_\theta \hat{\theta} \times \mathbf{B} \right) \quad (3.4.43)$$

Applying to Eq.(3.4.42) and Eq.(3.4.43) respectively the derivative with respect of z and r we obtain:

$$\frac{\partial J_r}{\partial z} = -\sigma \frac{\partial^2 \phi}{\partial r \partial z} \quad (3.4.44)$$

$$\frac{\partial J_z}{\partial r} = \sigma \left[-\frac{\partial^2 \phi}{\partial r \partial z} + \frac{\partial (U_\theta \hat{\theta} \times \mathbf{B})}{\partial r} \right] \quad (3.4.45)$$

Taking now into account the Eq.(3.4.36) we have:

$$J_r = -\frac{1}{4\pi} \frac{\partial H_\theta}{\partial z} \quad (3.4.46)$$

$$J_z = \frac{1}{4\pi} \frac{1}{r} \frac{\partial (r H_\theta)}{\partial r} \quad (3.4.47)$$

Proceeding as before we obtain:

$$\frac{\partial J_r}{\partial z} = -\frac{1}{4\pi} \frac{\partial^2 H_\theta}{\partial z^2} \quad (3.4.48)$$

$$\frac{\partial J_z}{\partial r} = \frac{1}{4\pi} \left[\frac{1}{r} \frac{\partial H_\theta}{\partial r} + \frac{\partial^2 H_\theta}{\partial r^2} - \frac{H_\theta}{r^2} \right] \quad (3.4.49)$$

It is possible then to compare the Eq.(3.4.44) with Eq.(3.4.48):

$$\frac{\partial J_r}{\partial z} = -\sigma \frac{\partial^2 \phi}{\partial r \partial z} = -\frac{1}{4\pi} \frac{\partial^2 H_\theta}{\partial z^2} \quad (3.4.50)$$

and the Eq.(3.4.45) with Eq.(3.4.49):

$$\frac{\partial J_z}{\partial r} = \sigma \left[-\frac{\partial^2 \phi}{\partial r \partial z} + \frac{\partial (U_\theta \hat{\theta} \times \mathbf{B})}{\partial r} \right] = \frac{1}{4\pi} \left[\frac{1}{r} \frac{\partial H_\theta}{\partial r} + \frac{\partial^2 H_\theta}{\partial r^2} - \frac{H_\theta}{r^2} \right] \quad (3.4.51)$$

Subtracting the Eq.(3.4.50) to the Eq.(3.4.51) we have:

$$\frac{1}{r} \frac{\partial H_\theta}{\partial r} + \frac{\partial^2 H_\theta}{\partial r^2} + \frac{\partial^2 H_\theta}{\partial z^2} - \frac{H_\theta}{r^2} - 4\pi\sigma \frac{\partial (U_\theta \hat{\theta} \times \mathbf{B})}{\partial r} = 0 \quad (3.4.52)$$

Considering a perfect radial applied magnetic field with equation:

$$\mathbf{B} = B_0 \frac{r_{int}}{r} \quad (3.4.53)$$

The last term of Eq.(3.4.52) can be written as:

$$4\pi\sigma \frac{\partial (U_\theta \hat{\theta} \times \mathbf{B})}{\partial r} = -4\pi\sigma B_0 r_{int} \left(\frac{1}{r} \frac{\partial U_\theta}{\partial r} - \frac{U_\theta}{r^2} \right) \quad (3.4.54)$$

We finally obtain:

$$\frac{\partial^2 H_\theta}{\partial r^2} + \frac{\partial^2 H_\theta}{\partial z^2} + \frac{1}{r} \frac{\partial H_\theta}{\partial r} - \frac{H_\theta}{r^2} + 4\pi\sigma B_0 r_{int} \left(\frac{1}{r} \frac{\partial U_\theta}{\partial r} - \frac{U_\theta}{r^2} \right) = 0 \quad (3.4.55)$$

The Navier stokes equation Eq.(3.4.39) along the θ direction is:

$$\mu \left[\frac{1}{r} \frac{\partial}{\partial r} \left(r \frac{\partial U_\theta}{\partial r} \right) + \frac{\partial^2 U_\theta}{\partial z^2} - \frac{U_\theta}{r^2} \right] + J_z \mathbf{B} = 0 \quad (3.4.56)$$

With the aid of the Eqs.(3.4.47),(3.4.53) the last term become:

$$J_z \mathbf{B} = B_0 r_{int} \left(\frac{1}{r} \frac{\partial H_\theta}{\partial r} + \frac{1}{r^2} H_\theta \right) \quad (3.4.57)$$

and then:

$$\frac{\partial^2 U_\theta}{\partial r^2} + \frac{\partial^2 U_\theta}{\partial z^2} + \frac{1}{r} \frac{\partial U_\theta}{\partial r} - \frac{U_\theta}{r^2} + \frac{B_0 r_{int}}{\mu} \left(\frac{1}{r} \frac{\partial H_\theta}{\partial r} + \frac{H_\theta}{r^2} \right) = 0 \quad (3.4.58)$$

What can be seen is that Eqs.(3.4.56) and (3.4.58) have similar shape. The first four terms of each equation have the same operators respectively applied to U_θ and H_θ . The last terms are different: they have different coefficient and different signes inside the brackets. The traditional technique is to add the two equations after having applied a non-dimensional substitution and then solve the equation in the variable resulting in the sum between U_θ and H_θ . In this case it is not possible because of the shape of the last term. We then try to find an approximated solution that will be compared with the results obtained from the FDHM.

3.4.3 Cartesian bi-dimensional solution

The axially-symmetric MHD problem extended to a torus with square section has different solution compared with the solution that can be obtained from the same problem in the cartesian form for a straight pipe.

There are anyway some qualitative analogies. As can be seen in the results of the FDHM and in Figure (3.4), the angular velocity of the conductive liquid shows a rigid body-like behaviour of the liquid with constant angular velocity in core zone of the flux. Qualitatively this distribution can be compared with the linear velocity profile solution of the MHD problem in cartesian coordinates for a straight pipe with rectangular cross section.

The basic equations to solve the problem are, in dimensionless form [25]:

$$\nabla^2 b + Ha \frac{\partial u}{\partial y} = 0 \quad (3.4.59)$$

$$\nabla^2 u + Ha \frac{\partial b}{\partial y} = 0 \quad (3.4.60)$$

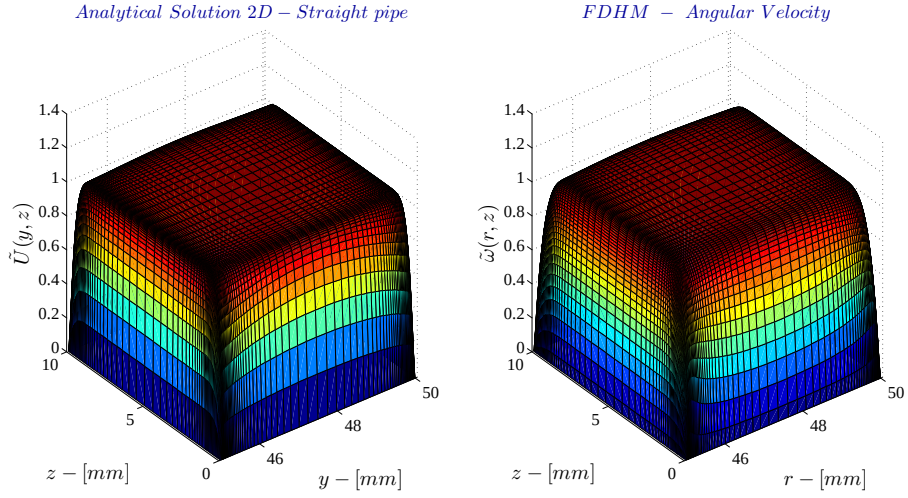


Figure 3.4: Comparison between MHD velocity field in a straight pipe under trasverse magnetic field and MHD angular velocity for an axially-symmetric torus under a radial magnetic field.

The two equations can be solved together with the Elsasser variables:

$$A = u + b, A' = u - b \quad (3.4.61)$$

Adding and subtracting Eq.(3.4.60) from the Eq.(3.4.59) we obtain:

$$\frac{\partial^2 A}{\partial y^2} + \frac{\partial^2 A}{\partial z^2} + Ha \frac{\partial A}{\partial y} = -1 \quad (3.4.62)$$

$$\frac{\partial^2 A'}{\partial y^2} + \frac{\partial^2 A'}{\partial z^2} - Ha \frac{\partial A'}{\partial y} = -1 \quad (3.4.63)$$

The boundary conditions to be applied are no slip condition at all walls, $u = 0$, and considering the boundaries electrically insulating, the induced magnetic field vanishes at the boundaries, $b = 0$. If the boundaries is called Γ , it means that:

$$A = A' = 0 \quad \text{on } \Gamma \quad (3.4.64)$$

Moreover it can be said by symmetry that:

$$A'(y) = A(-y) = 0 \quad (3.4.65)$$

so that it is sufficient to determine A for the solution of the problem. The solution is obtained by expanding the unknown A into a Fourier series as:

$$A(y, z) = A'(-y, z) = \sum_{i=1,3,5}^{\infty} a_i(y) \cos(\lambda_i z) \quad (3.4.66)$$

where we satisfy the boundary conditions at $z = \pm d$ with:

$$\lambda_i = \frac{i\pi}{2d} \quad (3.4.67)$$

Using the Eq.(3.4.61), for the dimensionless linear velocity u and the dimensionless induced magnetic field b , we have:

$$u(y, z) = \sum_{i=1,3,5}^{\infty} u_i(y) \cos(\lambda_i z) \quad (3.4.68)$$

$$b(y, z) = \sum_{i=1,3,5}^{\infty} b_i(y) \cos(\lambda_i z) \quad (3.4.69)$$

Inserting Eq.(3.4.66) into Eq.(3.4.62) and using orthogonality of trigonometric functions yields:

$$\frac{\partial^2 a_i}{\partial y^2} - \lambda_i^2 a_i + Ha \frac{\partial a_i}{\partial y} = -2 \frac{\sin(\lambda_i z)}{\lambda_i d} = -k_i \quad (3.4.70)$$

whose solution is:

$$a_i(y) = \frac{k_i}{\lambda_i^2} + C_{i_1} \cosh \exp -p_{i_1} y + C_{i_2} \exp -p_{i_2} y \quad (3.4.71)$$

where:

$$p_{i1,2} = \frac{1}{2} \left(Ha \mp \sqrt{Ha^2 + 4\lambda_i^2} \right) \quad (3.4.72)$$

In order to obtain the coefficient of the Eqs.(3.4.67),(3.4.68) we can use the symmetry of the problem and Eq.(3.4.61):

$$\begin{aligned} u_i(y) &= \frac{a_i(y) + a_i(-y)}{2} = \\ &= \frac{k_i}{\lambda_i^2} + C_{i1} \cosh(p_{i1}y) - C_{i2} \cosh(-p_{i2}y) \end{aligned} \quad (3.4.73)$$

$$\begin{aligned} b_i(y) &= \frac{a_i(y) - a_i(-y)}{2} = \\ &= -C_{i1} \cosh(p_{i1}y) - C_{i2} \cosh(-p_{i2}y) \end{aligned} \quad (3.4.74)$$

which correspond to the Fourier modes for velocity and magnetic field. Both vanish at the Hartmann walls, $u_i(1) = b_i(1) = 0$, so that the coefficients are determined. We express the Fourier modes for velocity and induced field in Eqs.(3.4.67),(3.4.68) as:

$$u_i(y) = \frac{k_i}{\lambda_i^2} \left[1 - \frac{f_i(y)}{f_i(1)} \right] \quad (3.4.75)$$

$$b_i(y) = \frac{k_i}{\lambda_i^2} \left[\frac{g_i(y)}{f_i(1)} \right] \quad (3.4.76)$$

where:

$$f_i(y) = \alpha_{i2} \cosh(p_{i1}y) - \alpha_{i1} \cosh(p_{i2}y) \quad (3.4.77)$$

$$g_i(y) = \alpha_{i2} \sinh(p_{i1}y) - \alpha_{i1} \sinh(p_{i2}y) \quad (3.4.78)$$

and:

$$\alpha_{i_{1,2}} = \sinh(p_{i_{1,2}}) \quad (3.4.79)$$

The solution $u(y, z)$ found will now be taken as it was the angular velocity profile ω_z instead of the linear velocity profile. To do this, the dimensionless y axis will be renamed r . We then have:

$$\omega_z(r, z) = \sum_{i=1,3,5}^{\infty} \frac{k_i}{\lambda_i^2} \left[1 - \frac{f_i\left(\frac{2r}{r_{out} + r_{in}}\right)}{f_i(1)} \right] \cos\left(\lambda_i \frac{2z}{h}\right) \quad (3.4.80)$$

Comparison between LPM and FDHM

In this chapter, a comparison between the FDHM and the LPM is shown in order to understand the precision of the estimation of the LPM coefficients based on both the analytical solutions.

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4.1 Simulations

In order to test the accuracy of the lumped parameter model several simulation have been done with the FDHM. The aim is to compare the value of the coefficients obtained through the LPM with the ones obtained numerically with the FDHM. The analysis will be done with all the coefficient and extended to two other quantities: the torque provided by the device and the characteristic time to a linear input over the time of the voltage and the electric current provided to the armatures of the device.

The LPM will be tested with the monodimensional analytical solution and the bidimensional cartesian solution obtained in the previous chapter.

The simulations have been computed varying the intensity of the magnetic field B_0 , the electric conductivity of the liquid σ , the inner radius r_{in} and the height h of the the torus containing the liquid. Other parameter have been considered fixed, such as the outer radius r_{out} , the viscosity of the liquid μ and its density ρ . The following table summarizes the grid of values of the simulations.

	U.M	Max Value	Min Value	N.points
ρ	$[kg/m^3]$	1000	1000	1
μ	$[Pa \cdot s]$	10^{-3}	10^{-3}	1
σ	$[S/m]$	10^6	10	6
B_0	$[T]$	0.8	0.2	4
r_{in}	$[mm]$	49	45	5
r_{out}	$[mm]$	50	50	1
h	$[mm]$	10	30	3

Table 4.1: Values grid for the FDHM simulation

4.2 Monodimensional LPM

4.2.1 Coefficient of the viscous shear moment - K_{vis}

The coefficient of the viscous shear moment for the LPM based on the Monodimensional analytical solution can be expressed as:

$$K_{vis} = \frac{2\pi\mu r_i^3 h}{\omega_0} \frac{d\omega_z(r)}{dr} \Big|_{r=r_i} + \frac{2\pi\mu r_e^3 h}{\omega_0} \frac{d\omega_z(r)}{dr} \Big|_{r=r_e} \quad (4.2.1)$$

The derivative present in Eq.(4.2.1) can be computed analytically. The K_{vis} obtained with the Eq.(4.2.1) need to be compared with the numerical K_{vis} obtained with the FDHM. The numerical K_{vis} is obtained evaluating the velocity profile and its derivative with respect of r and z at the boundaries and then multiplying each values for the elementary surface which it is referred to:

$$K_{visFDHM} = \frac{2\mu\pi}{\omega_0} \sum_{j=1}^N (z_{j+1/2} - z_{j-1/2}) \left[r_i \frac{\partial\omega_z(r_i, z_j)}{\partial r} + r_e \frac{\partial\omega_z(r_e, z_j)}{\partial r} \right] + \frac{\mu\pi}{\omega_0} \sum_{i=1}^M (r_{i+1/2}^2 - r_{i-1/2}^2) \left[\frac{\partial\omega_z(r_i, 0)}{\partial r} + \frac{\partial\omega_z(r_i, h)}{\partial r} \right] \quad (4.2.2)$$

where N e M are the number of intervals of the FDHM computational grid along z and r .

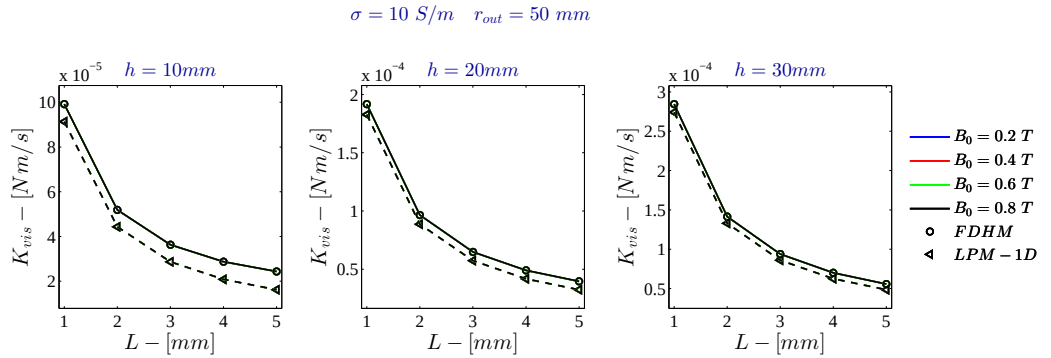


Figure 4.1: K_{vis} for $\sigma = 10 \text{ S/m}$ - Monodimensional LPM

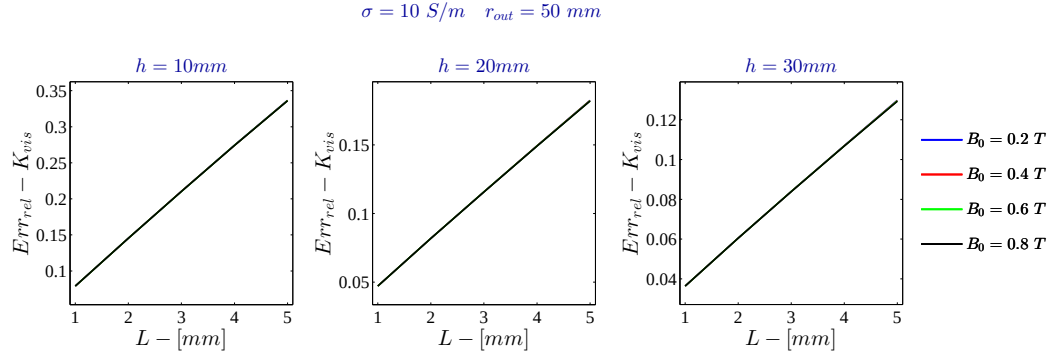


Figure 4.2: Relative Error - K_{vis} for $\sigma = 10 \text{ S/m}$ - Monodimensional LPM

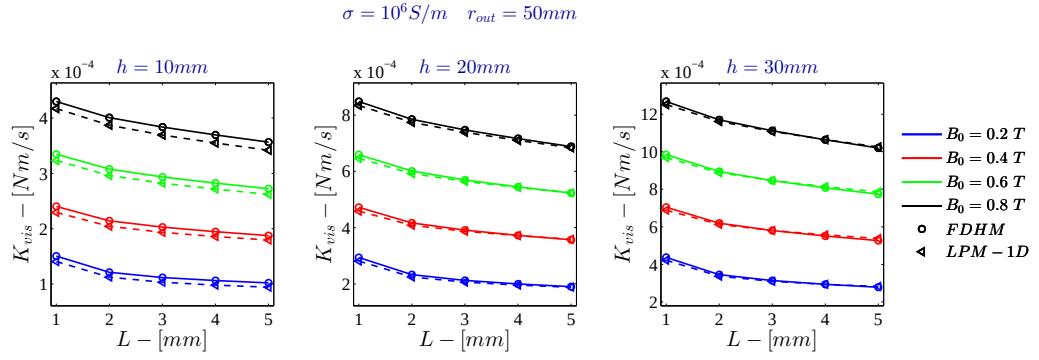


Figure 4.3: K_{vis} for $\sigma = 10^6 \text{ S/m}$ - Monodimensional LPM

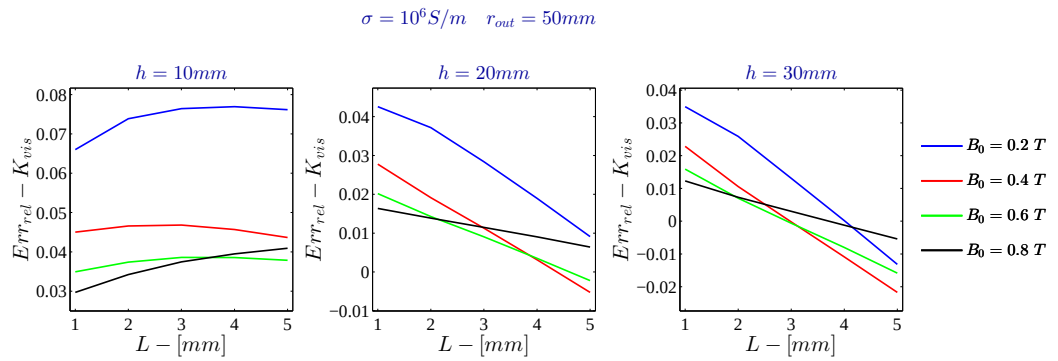


Figure 4.4: Relative Error - K_{vis} for $\sigma = 10^6 \text{ S/m}$ - Monodimensional LPM

4.2.2 Coefficient of the Lorentz Force moment - K_I

The Coefficient of the Lorentz Force moment K_I for the LPM can be ex-

pressed as:

$$K_I = B_0 r_i h \quad (4.2.3)$$

The same coefficient can be computed on the data collected with FDHM. The numerical K_I is calculated as:

$$K_{I_{FDHM}} = \frac{\pi \sum_{j=1}^N \sum_{i=1}^M J_{z_{i,j}} B_{r_{i,j}} (z_{j+1/2} - z_{j-1/2}) (r_{i+1/2}^2 - r_{i-1/2}^2)}{I_{alim}} \quad (4.2.4)$$

For the Current Drive I_{alim} is the electric current fed to the system. In the Voltage Drive the electric current is not know a priori, so it is taken as the average current flowing through the device. This current is computed as:

$$I_{alim} = \frac{1}{N} \sum_{i=1}^N \tilde{I}_{z_j} = \frac{1}{N} \sum_{j=1}^N \sum_{i=1}^M I_{z_{i,j}} \quad (4.2.5)$$

The values of $I_{z_{i,j}}$ come directly from the nodal analysis in the FDHM and they are the average of the electric currents in the branches.

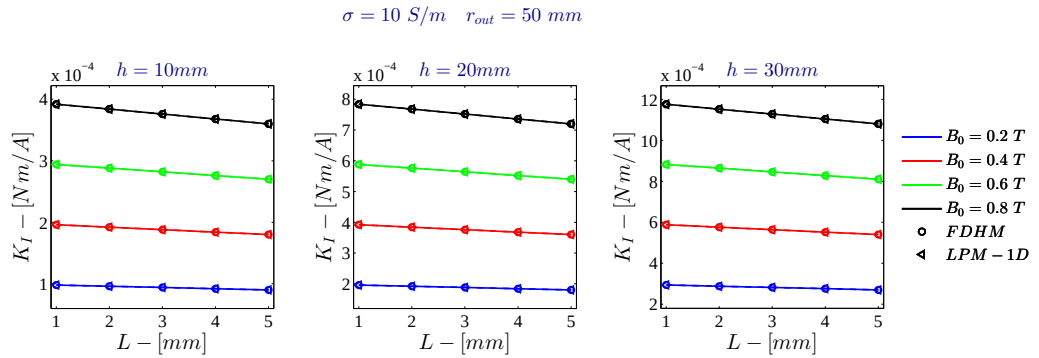


Figure 4.5: K_I for $\sigma = 10 \text{ S/m}$ - Monodimensional LPM

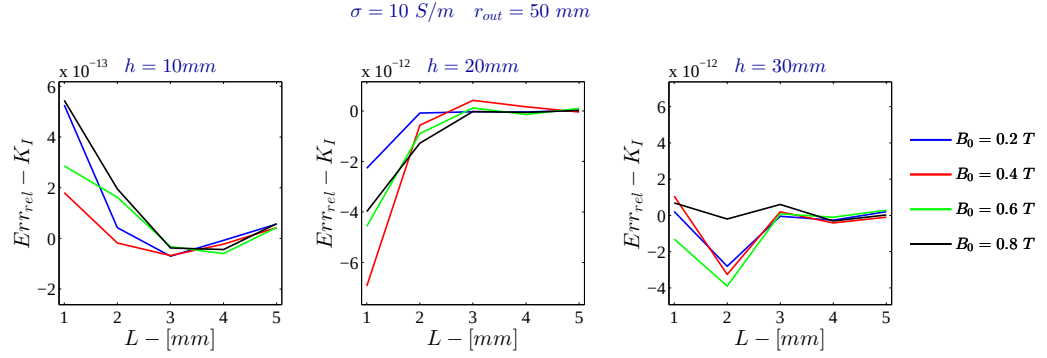


Figure 4.6: Relative Error - K_I for $\sigma = 10 \text{ S/m}$ - Monodimensional LPM

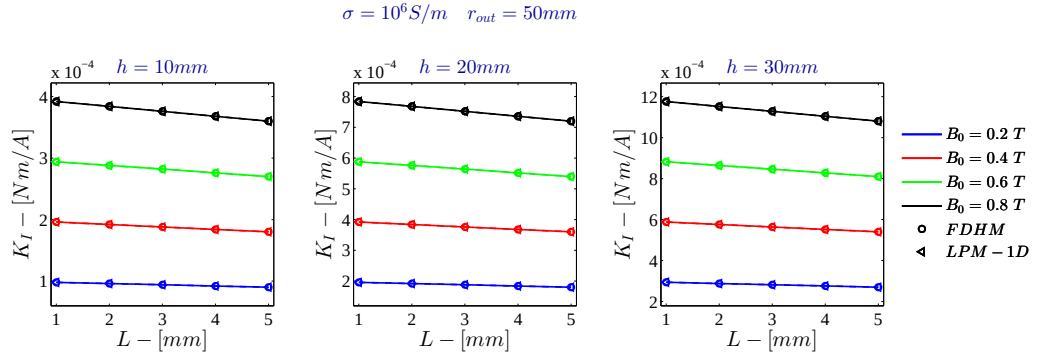


Figure 4.7: K_I for $\sigma = 10^6 \text{ S/m}$ - Monodimensional LPM

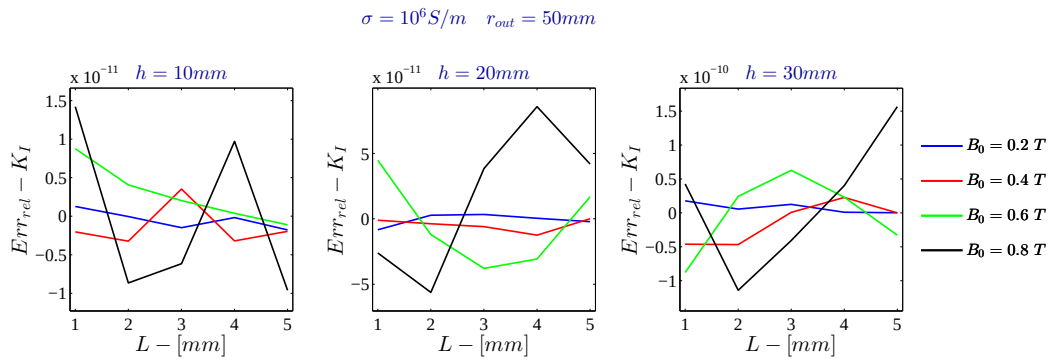


Figure 4.8: Relative Error - K_I for $\sigma = 10^6 \text{ S/m}$ - Monodimensional LPM

4.2.3 Coefficient of the moment of inertia - c_ω

The coefficient c_ω represents the difference between the moment of inertia

of a rigid flywheel with the same dimensions of the torus containing the liquid and the liquid spinning in the torus. In the LPM, with the monodimensional solution, we have:

$$c_\omega = \frac{2\pi\rho h \int_{r_i}^{r_e} r^3 \omega_z(r) dr}{\omega_0 I_z} \quad (4.2.6)$$

In the FDHM the c_ω is computed as:

$$c_\omega = \frac{\pi\rho \sum_{i=1}^M \sum_{j=1}^N \omega_{z_{i,j}} \left(r_{i+1/2}^4 - r_{i-1/2}^4 \right) (z_{j+1/2} - z_{j-1/2})}{2\omega_0 I_z} \quad (4.2.7)$$

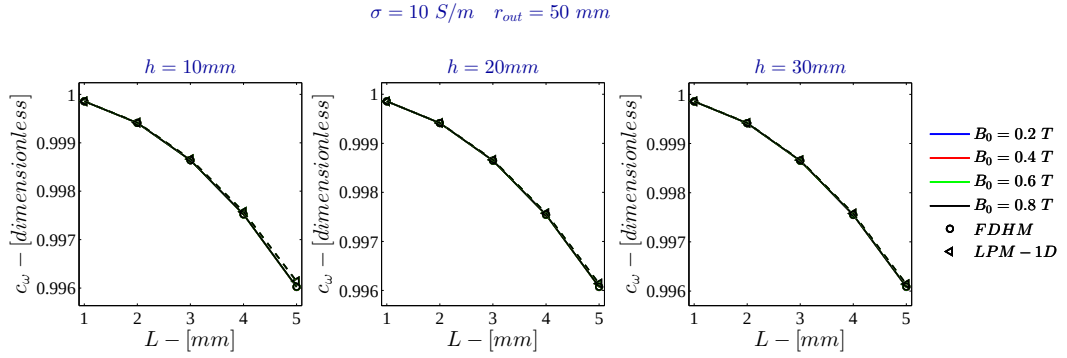


Figure 4.9: c_ω for $\sigma = 10 \text{ S/m}$ - Monodimensional LPM

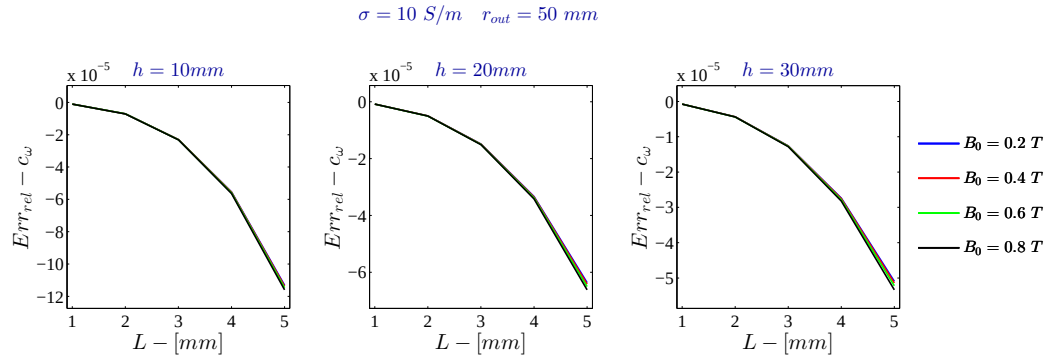


Figure 4.10: Relative Error - c_ω for $\sigma = 10 \text{ S/m}$ - Monodimensional LPM

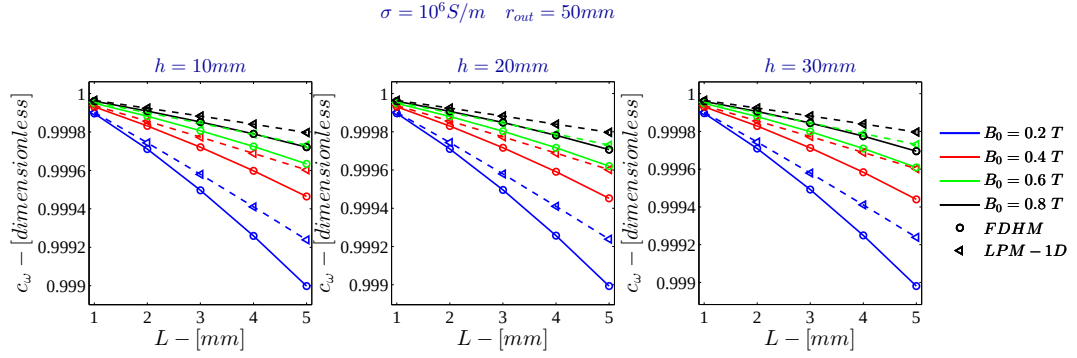


Figure 4.11: c_ω for $\sigma = 10^6 \text{ S/m}$ - Monodimensional LPM

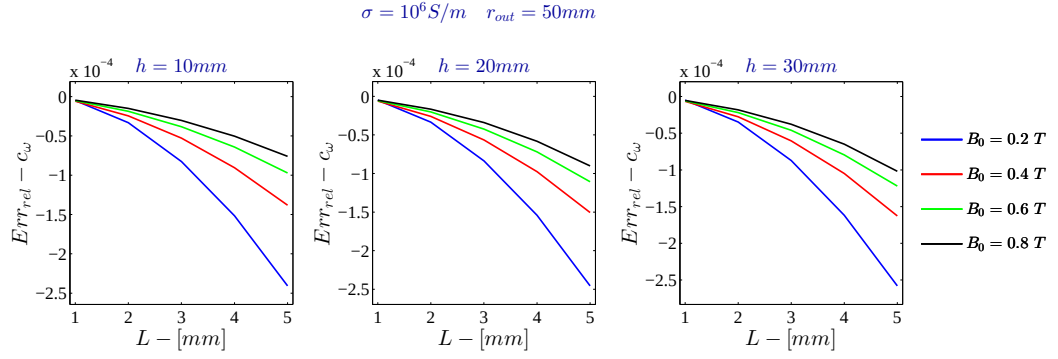


Figure 4.12: Relative Error - c_ω for $\sigma = 10^6 \text{ S/m}$ - Monodimensional LPM

4.2.4 Coefficient of Induced Voltage - K_V

The coefficient of the induced voltage is calculated in monodimensional LPM as:

$$K_V = \frac{2\pi B_0 h \int_{r_i}^{r_e} r \omega_z(r) dr}{\omega_0 S} \quad (4.2.8)$$

where S is the surface perpendicular to z . The coefficient K_V for the FDHM can be computed as:

$$K_{V_{FDHM}} = \frac{B_0 r_i \sum_{j=1}^N \sum_{i=1}^M \omega_{z_{i,j}} (z_{j+1/2} - z_{j-1/2})}{\omega_0} \quad (4.2.9)$$

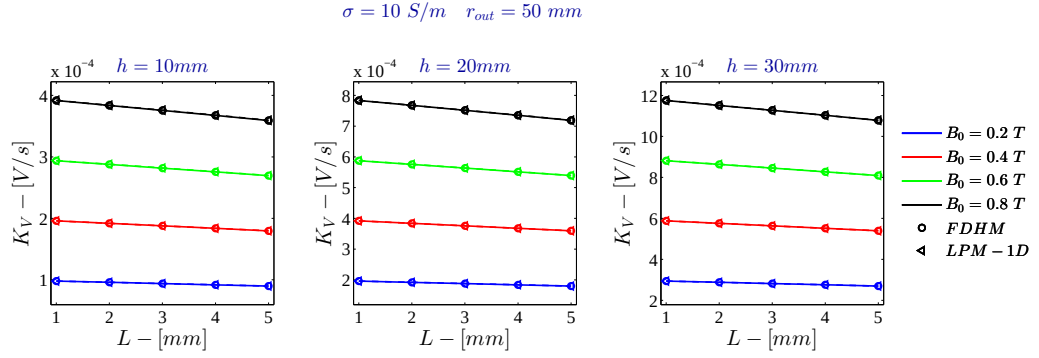


Figure 4.13: K_V for $\sigma = 10 \text{ S/m}$ - Monodimensional LPM

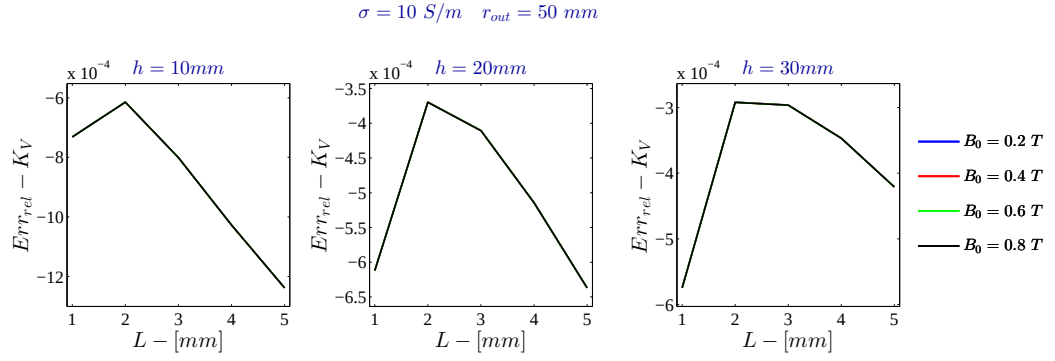


Figure 4.14: Relative Error - K_V for $\sigma = 10 \text{ S/m}$ - Monodimensional LPM

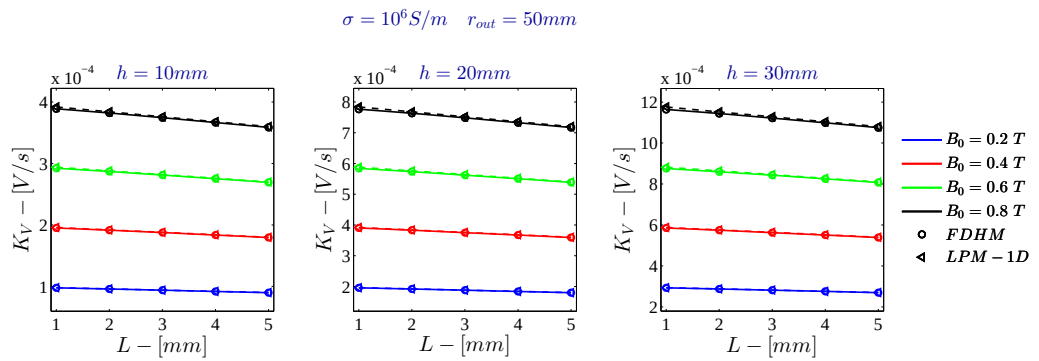


Figure 4.15: K_V for $\sigma = 10^6 \text{ S/m}$ - Monodimensional LPM

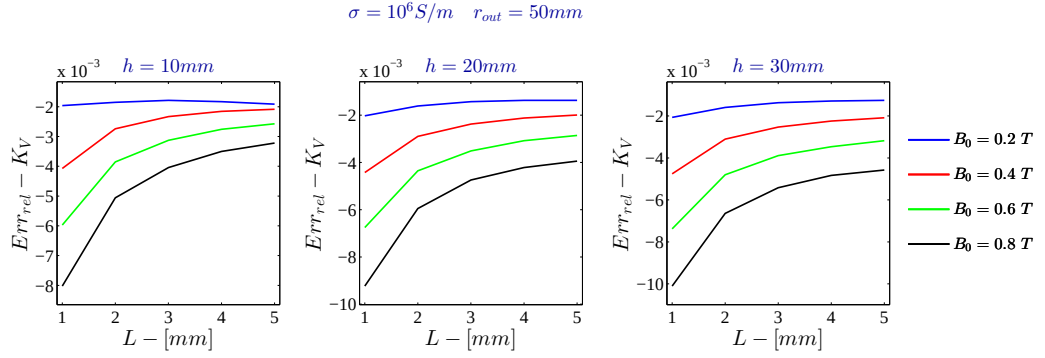


Figure 4.16: Relative Error - K_V for $\sigma = 10^6 \text{ S/m}$ - Monodimensional LPM

4.3 Bidimensional LPM

4.3.1 Coefficient of the viscous shear moment - K_{vis}

In the bidimensional LPM all the boundaries need to be taken into account. The coefficient of viscous shear moment is computed as:

$$\begin{aligned}
 K_{vis} = & \frac{2\pi\mu r_i^3}{\omega_0} \int_0^h \frac{\partial \omega_z(r, z)}{\partial r} \Big|_{r=r_i} dz + \frac{2\pi\mu r_e^3}{\omega_0} \int_0^h \frac{\partial \omega_z(r, z)}{\partial r} \Big|_{r=r_e} dz \\
 & + \frac{2\pi\mu}{\omega_0} \int_{r_i}^{r_e} r^3 \frac{\partial \omega_z(r, z)}{\partial z} \Big|_{z=0} dr + \frac{2\pi\mu}{\omega_0} \int_{r_i}^{r_e} r^3 \frac{\partial \omega_z(r, z)}{\partial z} \Big|_{z=h} dr
 \end{aligned} \quad (4.3.1)$$

The $K_{vis_{FDHM}}$ is computed in same way as showed in the paragraph referring to the monodimensional LPM.

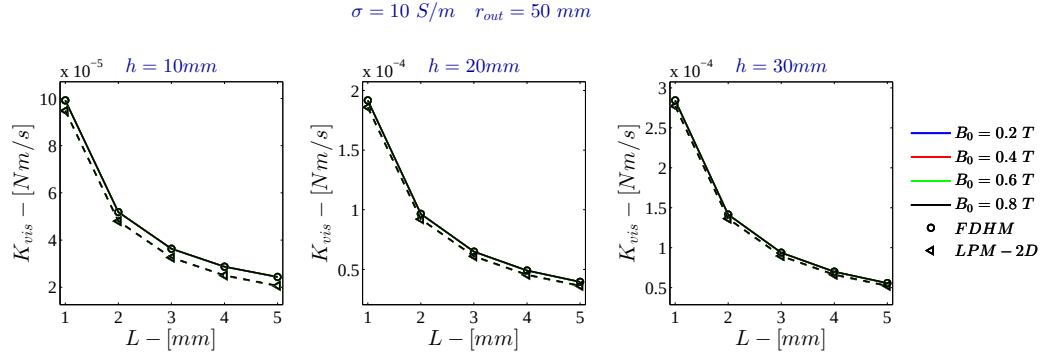


Figure 4.17: K_{vis} for $\sigma = 10 \text{ S/m}$ - Bidimensional LPM

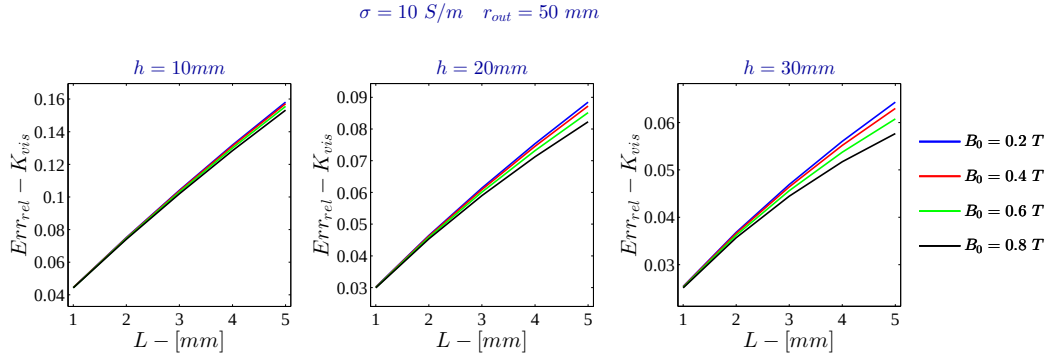


Figure 4.18: Relative Error - K_{vis} for $\sigma = 10 \text{ S/m}$ - Bidimensional LPM

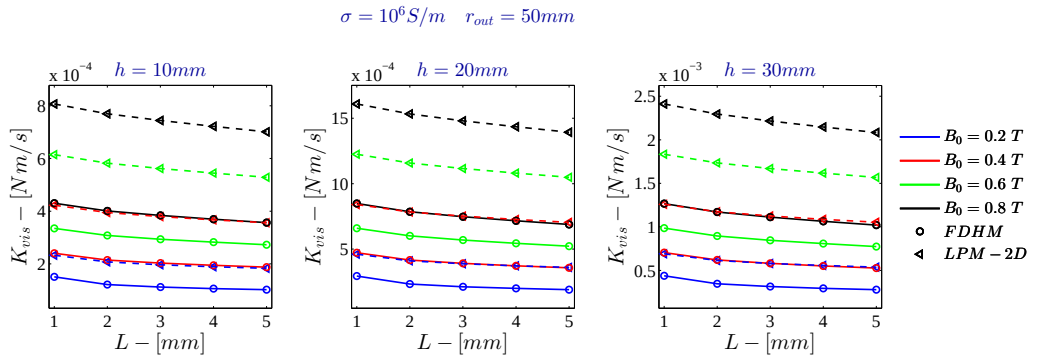


Figure 4.19: K_{vis} for $\sigma = 10^6 \text{ S/m}$ - Bidimensional LPM

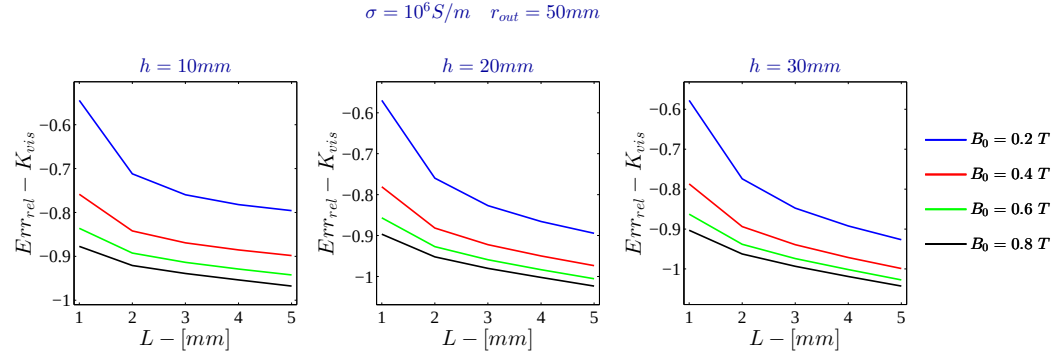


Figure 4.20: Relative Error - K_{vis} for $\sigma = 10^6 \text{ S/m}$ - Bidimensional LPM

4.3.2 Coefficient of the Lorentz Force moment - K_I

The K_I as the $K_{I_{FDHM}}$ is computed as in the paragraph referring to the monodimensional LPM.

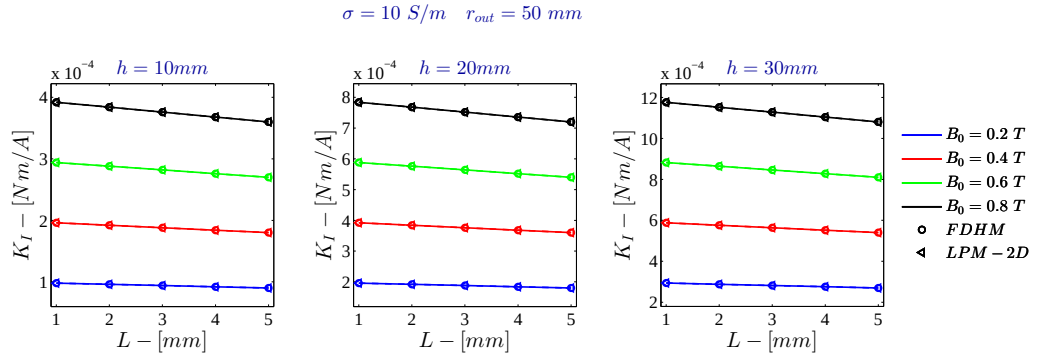


Figure 4.21: K_I for $\sigma = 10 \text{ S/m}$ - Bidimensional LPM

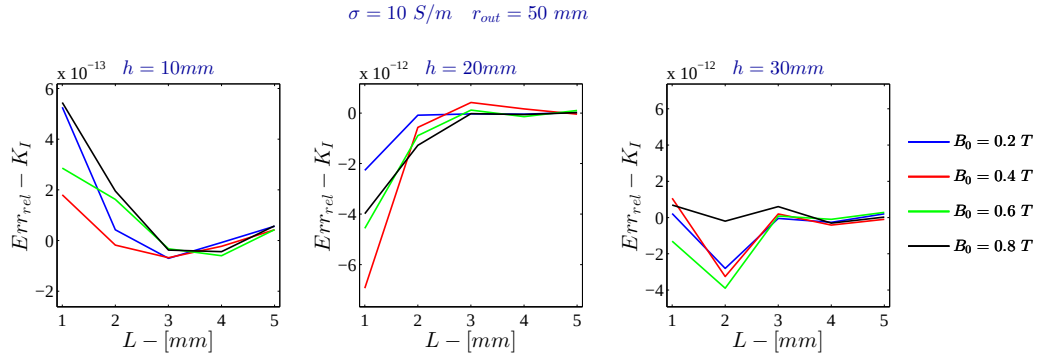


Figure 4.22: Relative Error - K_I for $\sigma = 10 \text{ S/m}$ - Bidimensional LPM

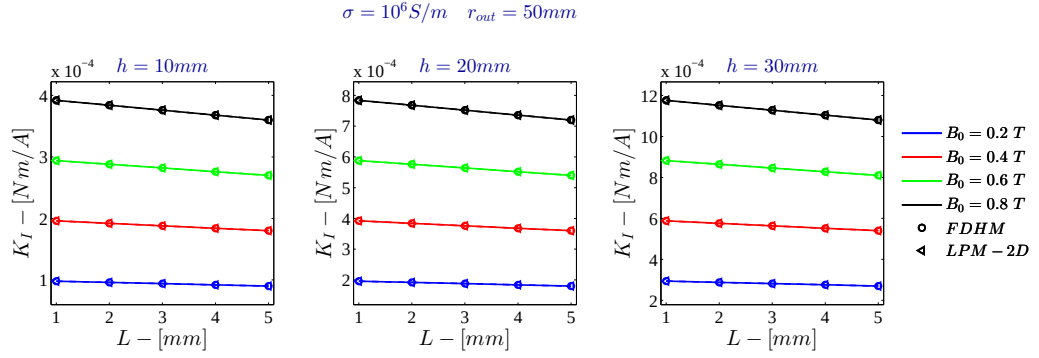


Figure 4.23: K_I for $\sigma = 10^6 \text{ S/m}$ - Bidimensional LPM

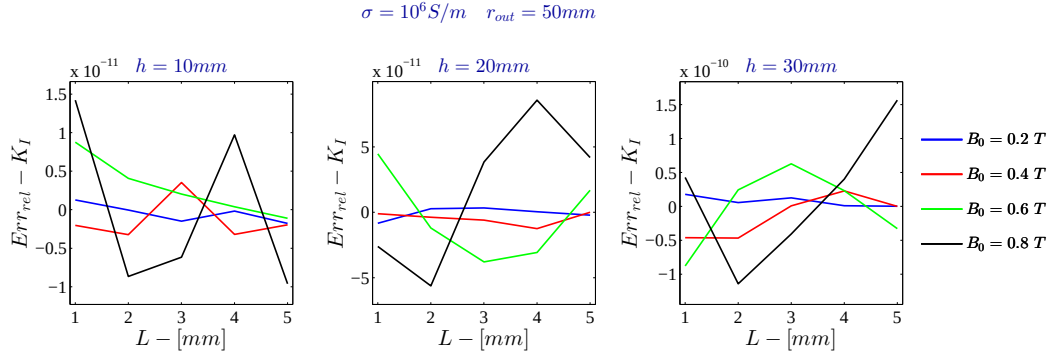


Figure 4.24: Relative Error - K_I for $\sigma = 10^6 \text{ S/m}$ - Bidimensional LPM

4.3.3 Coefficient of the moment of inertia - c_ω

In the bidimensional LPM, the coefficient c_ω is computed as:

$$c_\omega = \frac{2\pi\rho h \int_{r_i}^{r_e} \int_0^h r^3 \omega_z(r, z) dr dz}{\omega_0 I_z} \quad (4.3.2)$$

The coefficient $c_{\omega_{FDHM}}$ is calculated in same way as showed in the paragraph for the monodimensional LPM.

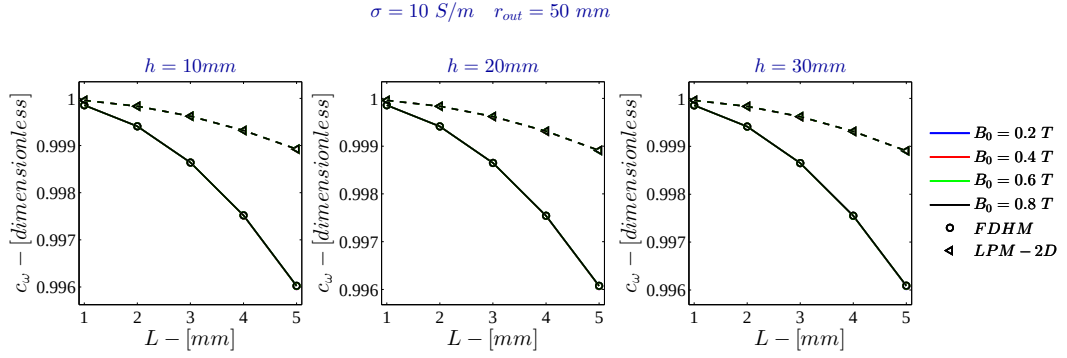


Figure 4.25: c_ω for $\sigma = 10 \text{ S/m}$ - Bidimensional LPM

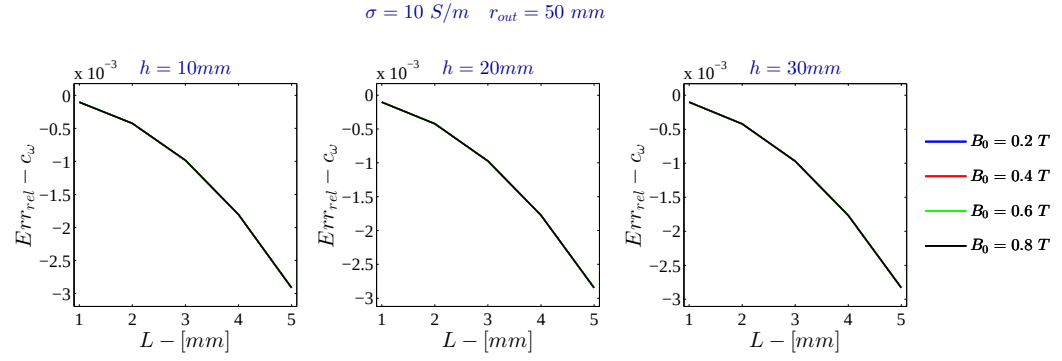


Figure 4.26: Relative Error - c_ω for $\sigma = 10 \text{ S/m}$ - Bidimensional LPM

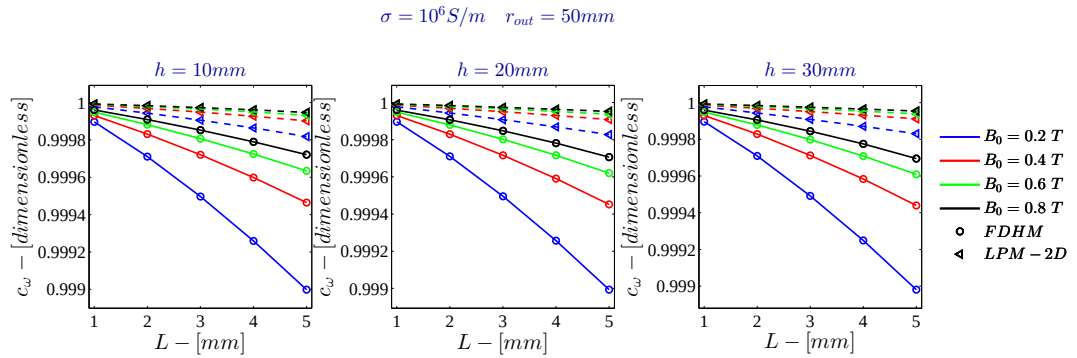


Figure 4.27: c_ω for $\sigma = 10^6 \text{ S/m}$ - Bidimensional LPM

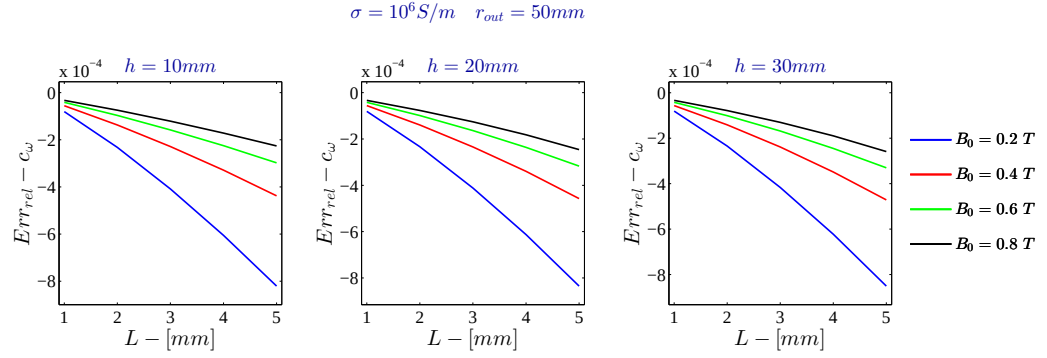


Figure 4.28: Relative Error - c_{ω} for $\sigma = 10^6 \text{ S/m}$ - Bidimensional LPM

4.3.4 Coefficient of Induced Voltage - K_V

In the bidimensional LPM, the coefficient K_V is computed as:

$$K_V = \frac{2\pi B_0 \int_{r_i}^{r_e} \int_0^h r \omega_z(r, z) dr dz}{\omega_0 S} \quad (4.3.3)$$

The coefficient $K_{V_{FDHM}}$ is calculated in same way as showed in the paragraph of the monodimensional LPM.

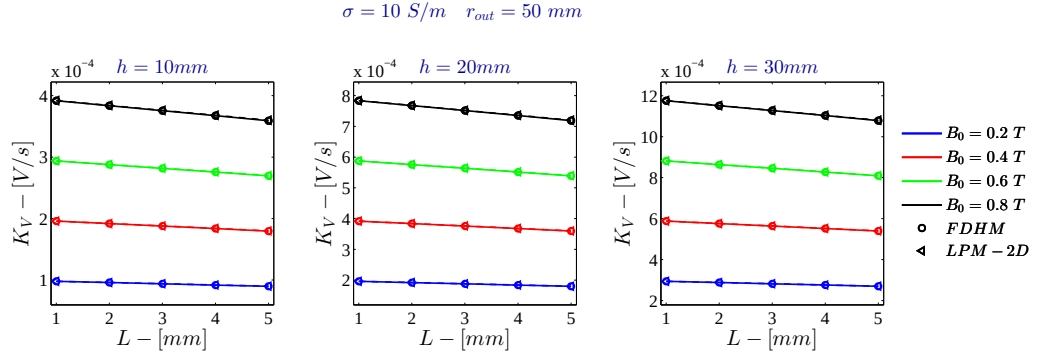


Figure 4.29: K_V for $\sigma = 10 \text{ S/m}$ - Bidimensional LPM

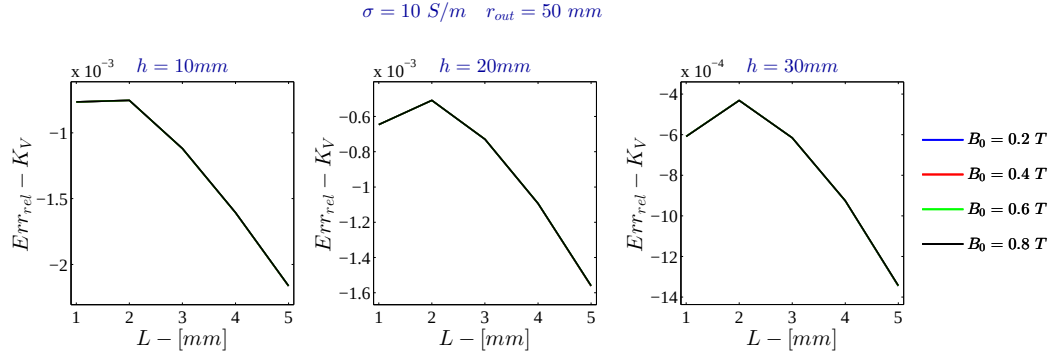


Figure 4.30: Relative Error - K_V for $\sigma = 10 \text{ S/m}$ - Bidimensional LPM

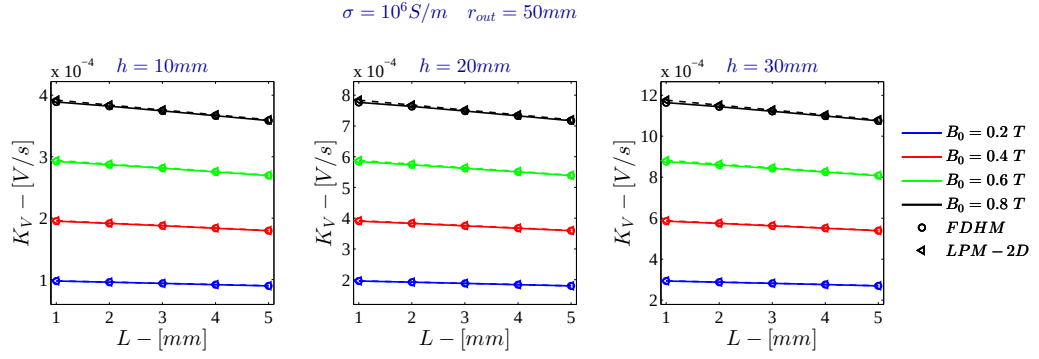


Figure 4.31: K_V for $\sigma = 10^6 \text{ S/m}$ - Bidimensional LPM

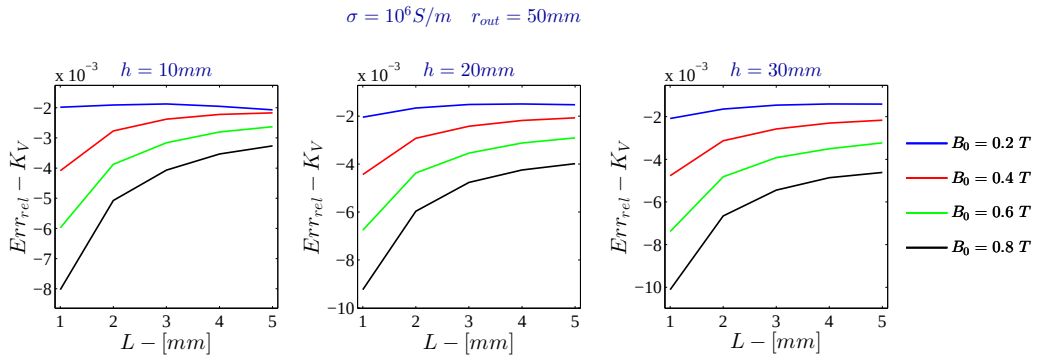


Figure 4.32: Relative Error - K_V for $\sigma = 10^6 \text{ S/m}$ - Bidimensional LPM

4.4 Current Drive

In the case of electric current drive we will solve first only the dynamic part of the LPM for a generic law for the electric current:

$$\dot{\omega} + \frac{K_{vis}}{I'_z} \omega = \frac{K_I}{I'_z} I \quad (4.4.1)$$

with general initial condition:

$$\omega(t_0) = \omega_0 \quad (4.4.2)$$

The solution for ω is:

$$\omega(t) = e^{-\frac{K_{vis}}{I'_z} t} \left[\omega_0 + \int_0^t \frac{K_I}{I'_z} I(t) e^{\frac{K_{vis}}{I'_z} t} dt \right] \quad (4.4.3)$$

If we now apply a linear law to the electric current:

$$I(t) = at \quad (4.4.4)$$

the solution (4.4.3) becomes:

$$\omega(t) = \frac{K_I}{K_{vis}} at - \frac{I'_z K_I a}{K_{vis}^2} \left[1 - e^{-\frac{K_{vis}}{I'_z} t} \right] \quad (4.4.5)$$

and the torque provided is:

$$T = I'_z \dot{\omega} = I'_z \frac{K_I}{K_{vis}} a \left[1 - e^{-\frac{K_{vis}}{I'_z} t} \right] \quad (4.4.6)$$

The (4.4) shows that the reaction of the system has a characteristic time:

$$\tau_I = \frac{I'_z}{K_{vis}} \quad (4.4.7)$$

and the torque provided becomes constant for a linear electric current:

$$T_\infty = \lim_{t \rightarrow \infty} T(t) = I'_z a \frac{K_I}{K_{vis}} \quad (4.4.8)$$

The simulations have been conducted with a current slope $a = 1A/s$.

4.4.1 Monodimensional LPM

4.4.1.1 Torque - $\dot{\Gamma}$

The torque generated in the FDHM is read as the final value obtained during the simulation.

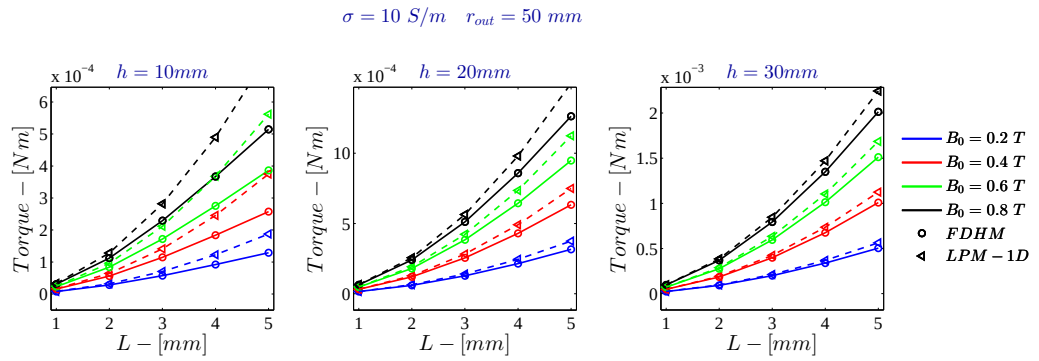


Figure 4.33: $\dot{\Gamma}$ for $\sigma = 10 \text{ S/m}$ - Monodimensional LPM

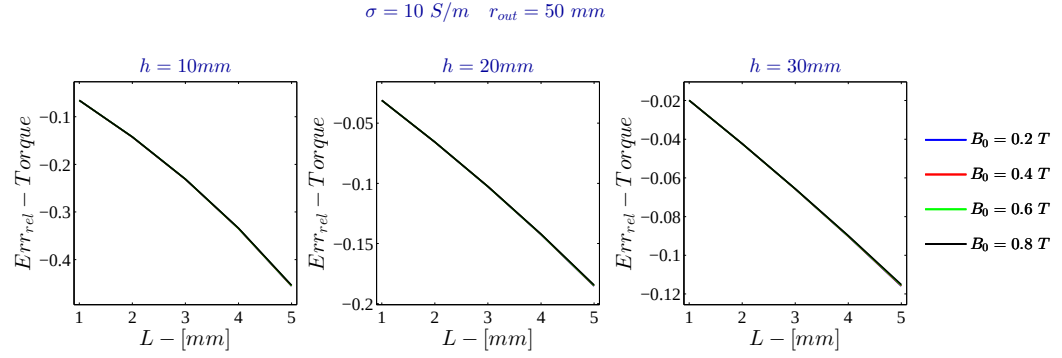


Figure 4.34: Relative Error - $\dot{\Gamma}$ for $\sigma = 10 \text{ S/m}$ - Monodimensional LPM

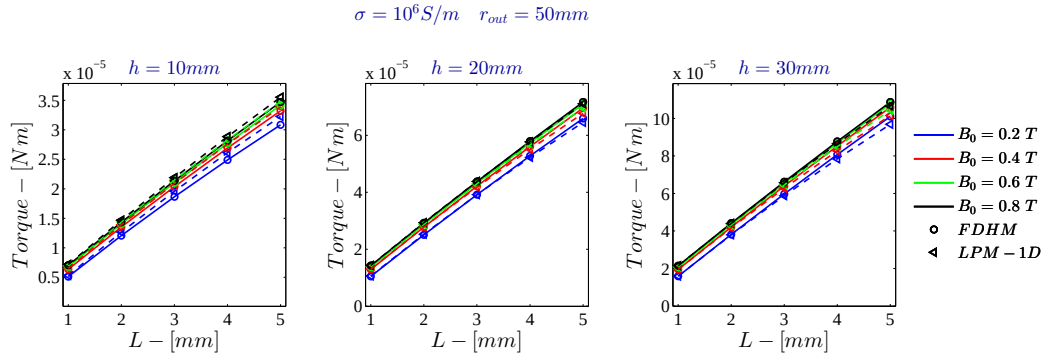


Figure 4.35: $\dot{\Gamma}$ for $\sigma = 10^6 \text{ S/m}$ - Monodimensional LPM

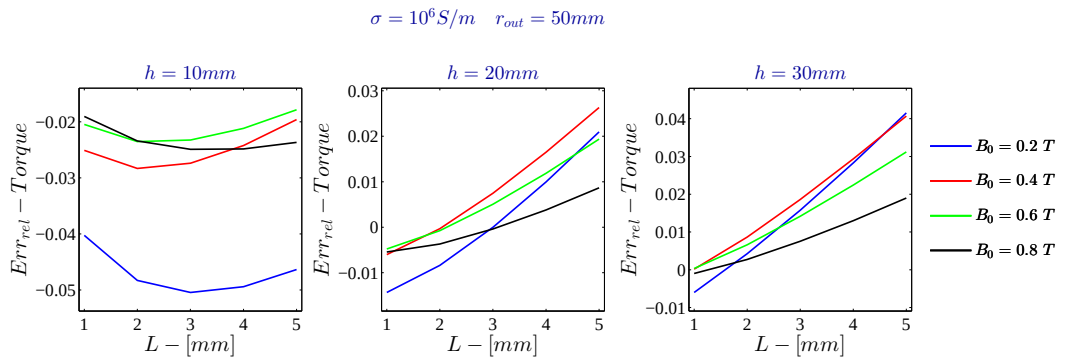


Figure 4.36: Relative Error - $\dot{\Gamma}$ for $\sigma = 10^6 \text{ S/m}$ - Monodimensional LPM

4.4.1.2 Characteristic Time - τ_c

The characteristic time for the FDHM is obtained reading the time when

the Torque reach the value of:

$$t = \tau_{cFDHM} \rightarrow \frac{T(t)}{T_\infty} = 1 - \frac{1}{e} \quad (4.4.9)$$

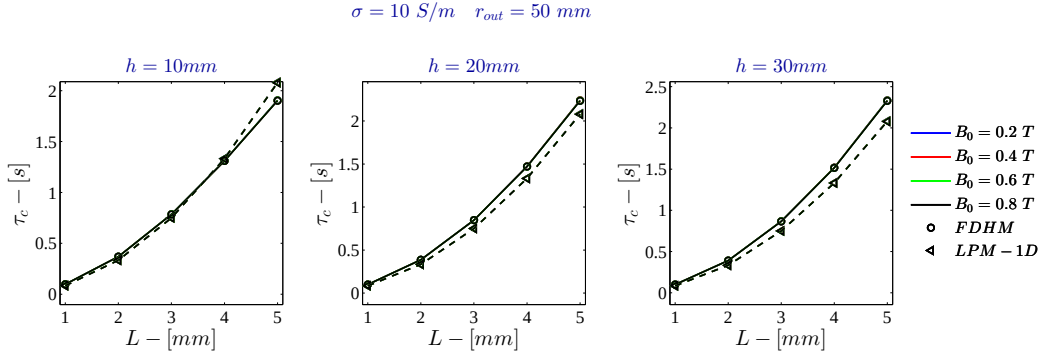


Figure 4.37: τ_c for $\sigma = 10 \text{ S/m}$ - Monodimensional LPM

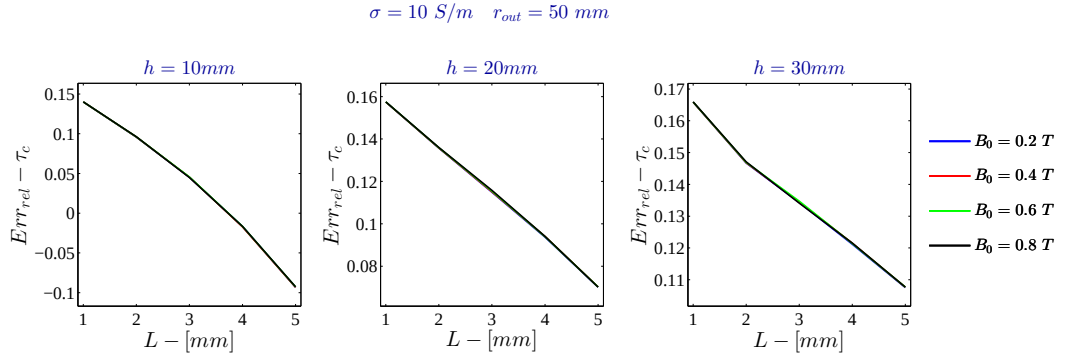


Figure 4.38: Relative Error - τ_c for $\sigma = 10 \text{ S/m}$ - Monodimensional LPM

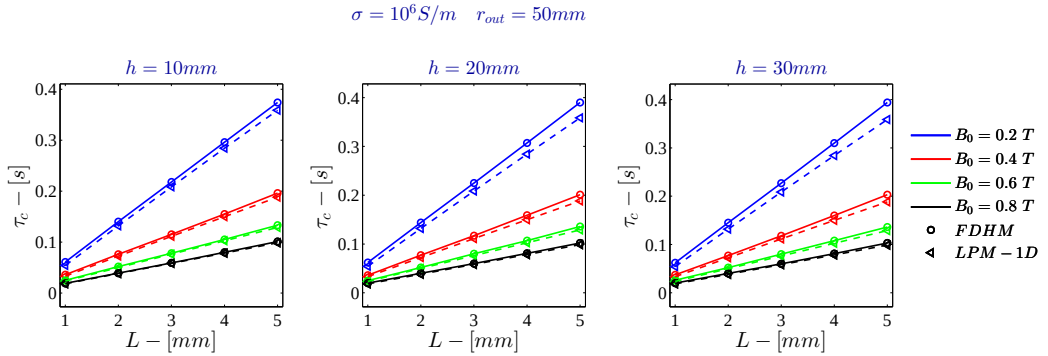


Figure 4.39: τ_c for $\sigma = 10^6 \text{ S/m}$ - Monodimensional LPM

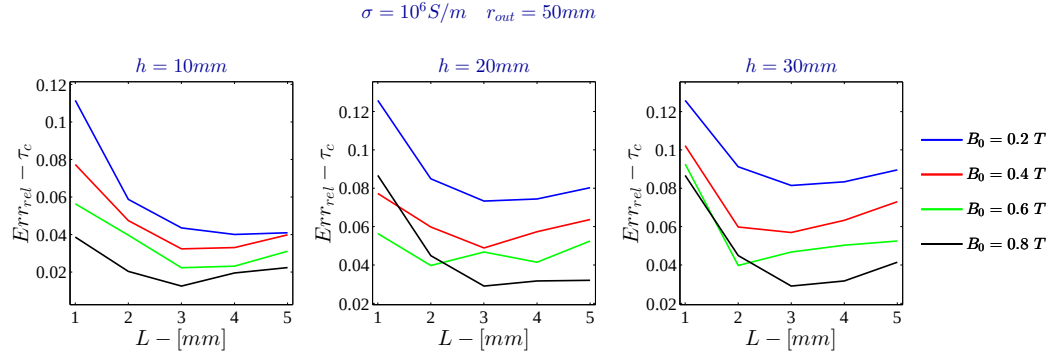


Figure 4.40: Relative Error - τ_c for $\sigma = 10^6 \text{ S/m}$ - Monodimensional LPM

4.4.2 Bidimensional LPM

4.4.2.1 Torque - $\dot{\Gamma}$

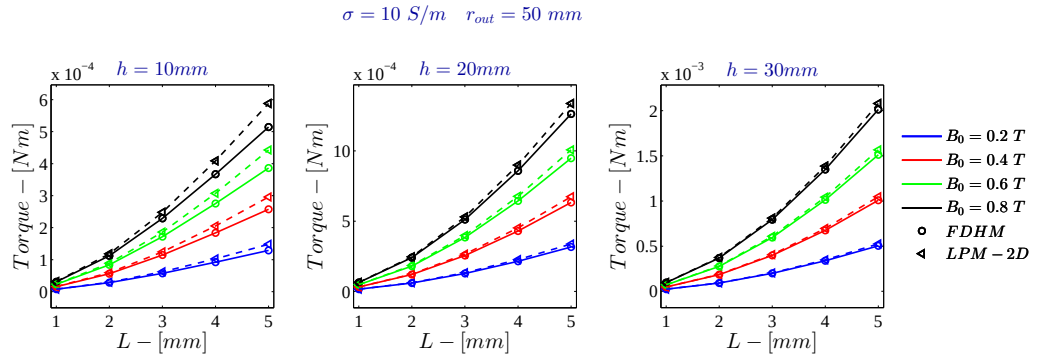


Figure 4.41: $\dot{\Gamma}$ for $\sigma = 10 \text{ S/m}$ - Bidimensional LPM

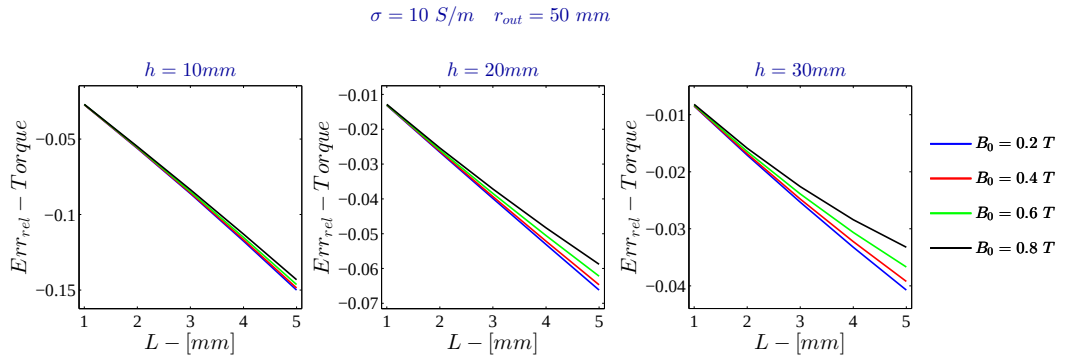
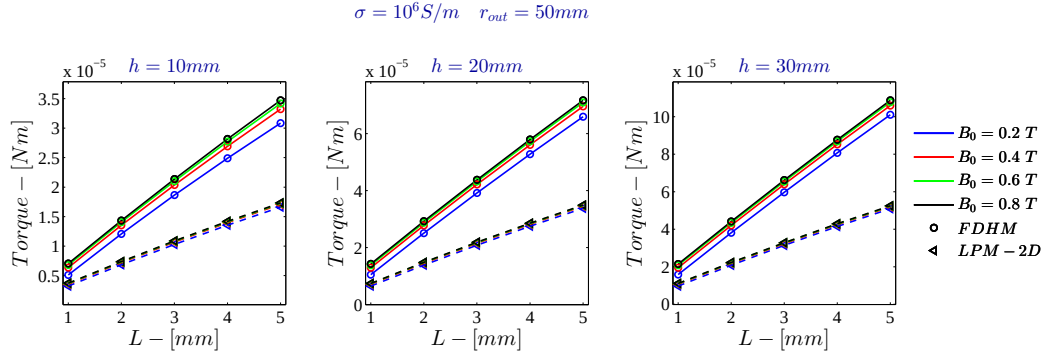
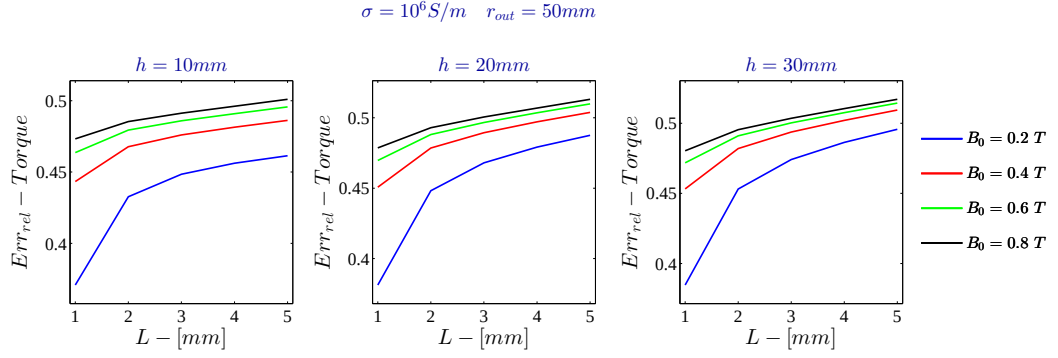
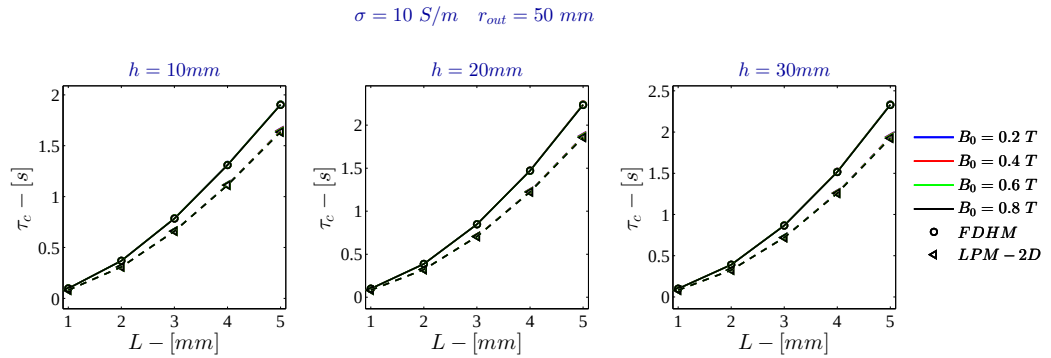


Figure 4.42: Relative Error - $\dot{\Gamma}$ for $\sigma = 10 \text{ S/m}$ - Bidimensional LPM


 Figure 4.43: $\dot{\Gamma}$ for $\sigma = 10^6 \text{ S/m}$ - Bidimensional LPM

 Figure 4.44: Relative Error - $\dot{\Gamma}$ for $\sigma = 10^6 \text{ S/m}$ - Bidimensional LPM

4.4.2.2 Characteristic Time - τ_c


 Figure 4.45: τ_c for $\sigma = 10 \text{ S/m}$ - Bidimensional LPM

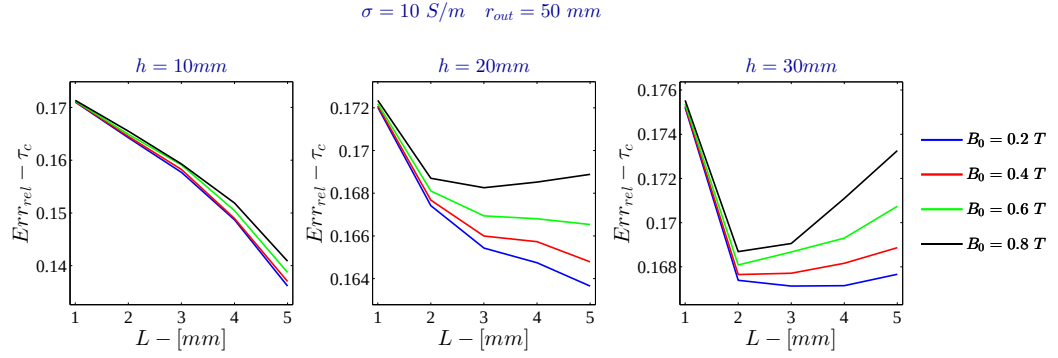


Figure 4.46: Relative Error - τ_c for $\sigma = 10 \text{ S/m}$ - Bidimensional LPM

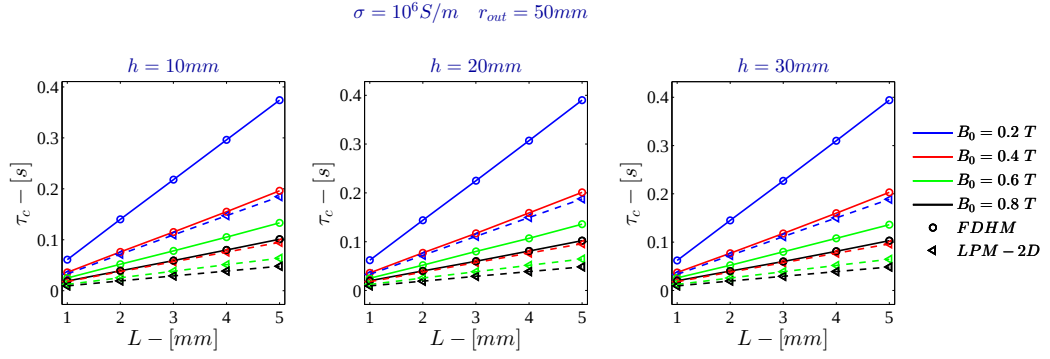


Figure 4.47: τ_c for $\sigma = 10^6 \text{ S/m}$ - Bidimensional LPM

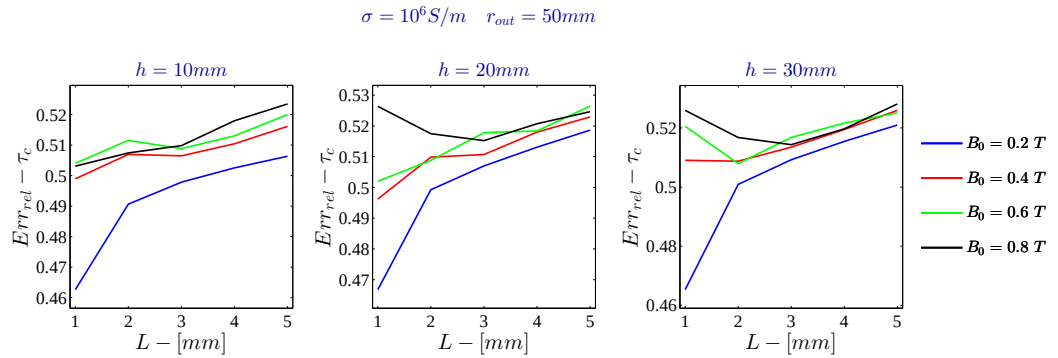


Figure 4.48: Relative Error - τ_c for $\sigma = 10^6 \text{ S/m}$ - Bidimensional LPM

4.5 Voltage Drive

In the case the device is voltage driven we need to consider both the equations of the LPM. Putting them together through the electric current we obtain:

$$\dot{\omega} + \frac{1}{I'_z} \left(K_{vis} + \frac{K_I K_V}{R_{eq}} \right) \omega = \frac{K_I}{R_{eq} I'_z} V \quad (4.5.1)$$

with general initial condition:

$$\omega(t_0) = \omega_0 \quad (4.5.2)$$

The solution for ω is:

$$\omega(t) = e^{-\frac{1}{I'_z} \left(K_{vis} + \frac{K_I K_V}{R_{eq}} \right) t} \left[\omega_0 + \int_0^t \frac{K_I}{R_{eq} I'_z} V(t) e^{-\frac{1}{I'_z} \left(K_{vis} + \frac{K_I K_V}{R_{eq}} \right) t} dt \right] \quad (4.5.3)$$

Considering now a linear law for the voltage:

$$V(t) = at \quad (4.5.4)$$

the solution (4.5.3) becomes:

$$\begin{aligned} \omega(t) = & \frac{K_I}{K_I K_V + K_{vis} R_{eq}} at + \\ & - \frac{I'_z K_I R_{eq} a}{(K_I K_V + K_{vis} R_{eq})^2} \left[1 - e^{-\frac{K_I K_V + K_{vis} R_{eq}}{I'_z R_{eq}} t} \right] \end{aligned} \quad (4.5.5)$$

and the torque provided is:

$$T = I'_z \dot{\omega} = I'_z \frac{K_I}{K_I K_V + K_{vis} R_{eq}} a \left[1 - e^{-\frac{K_I K_V + K_{vis} R_{eq}}{I'_z R_{eq}} t} \right] \quad (4.5.6)$$

The (4.5.6) shows that the reaction of the system has a characteristic time:

$$\tau_V = \frac{I'_z R_{eq}}{K_I K_V + K_{vis} R_{eq}} \quad (4.5.7)$$

and the torque provided becomes constant for a linear voltage:

$$T_\infty = \lim_{t \rightarrow \infty} T(t) = I'_z a \frac{K_I}{K_I K_V + K_{vis} R_{eq}} \quad (4.5.8)$$

The simulations have been conducted with a voltage slope $a = 10^{-3} \text{V/s}$.

4.5.1 Monodimensional LPM

4.5.1.1 Torque - $\dot{\Gamma}$

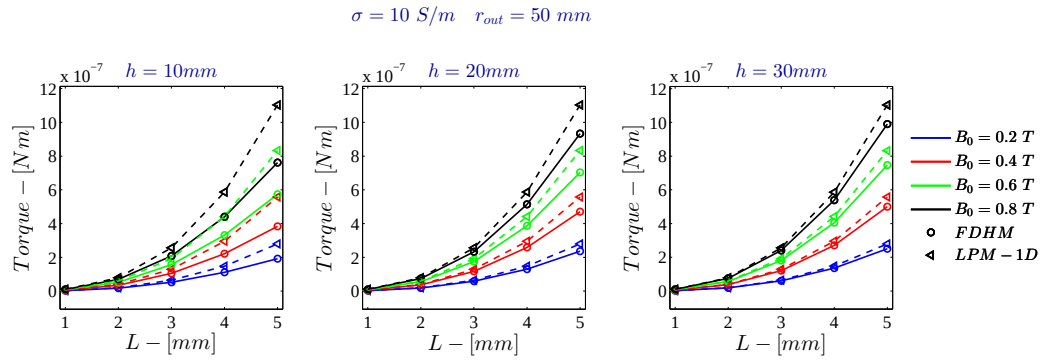


Figure 4.49: $\dot{\Gamma}$ for $\sigma = 10 \text{ S/m}$ - Monodimensional LPM

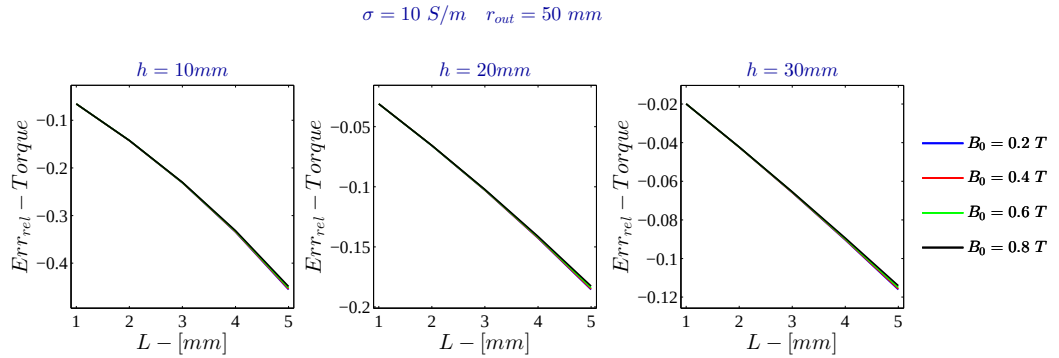
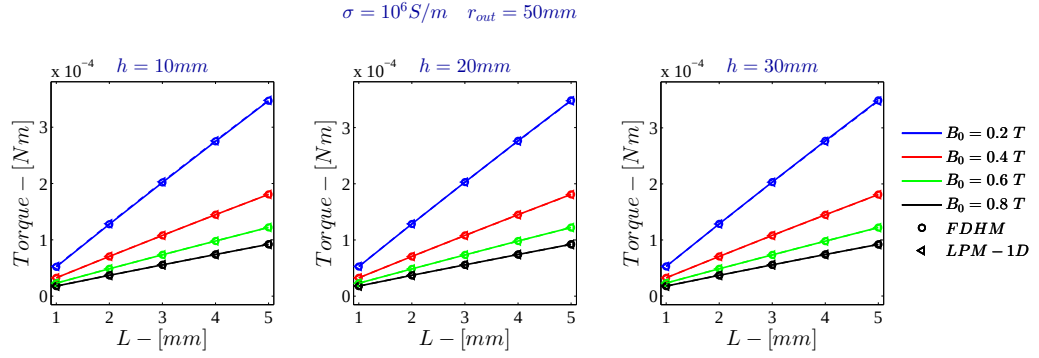
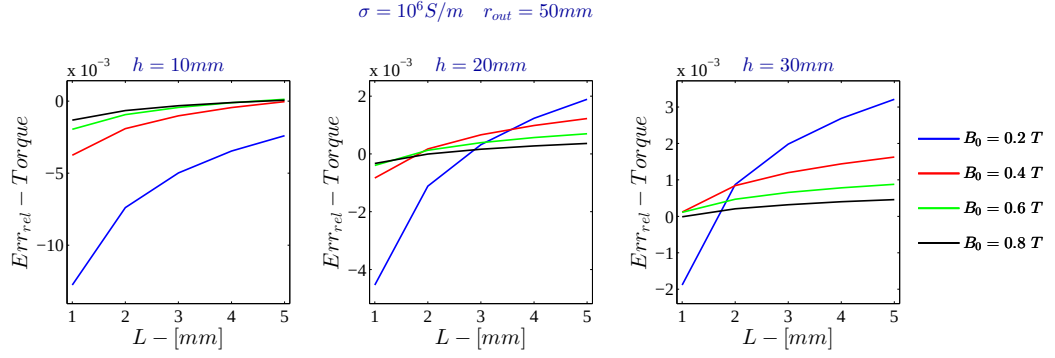
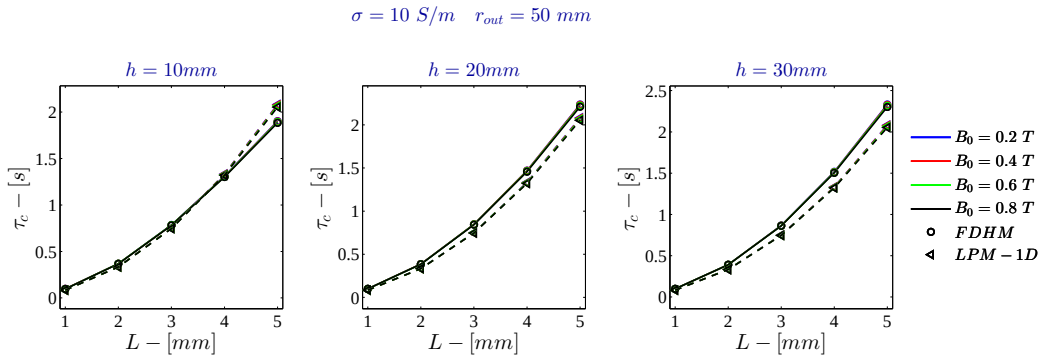


Figure 4.50: Relative Error - $\dot{\Gamma}$ for $\sigma = 10 \text{ S/m}$ - Monodimensional LPM


 Figure 4.51: $\dot{\Gamma}$ for $\sigma = 10^6 \text{ S/m}$ - Monodimensional LPM

 Figure 4.52: Relative Error - $\dot{\Gamma}$ for $\sigma = 10^6 \text{ S/m}$ - Monodimensional LPM

4.5.1.2 Characteristic Time - τ_c


 Figure 4.53: τ_c for $\sigma = 10 \text{ S/m}$ - Monodimensional LPM

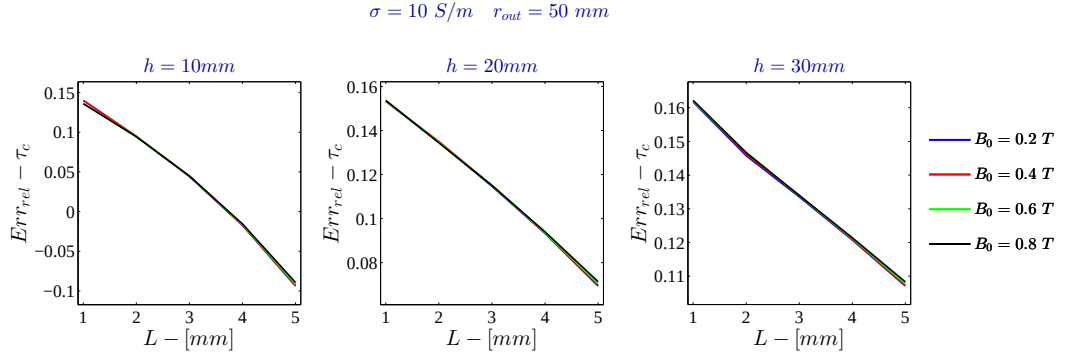


Figure 4.54: Relative Error - τ_c for $\sigma = 10 \text{ S/m}$ - Monodimensional LPM

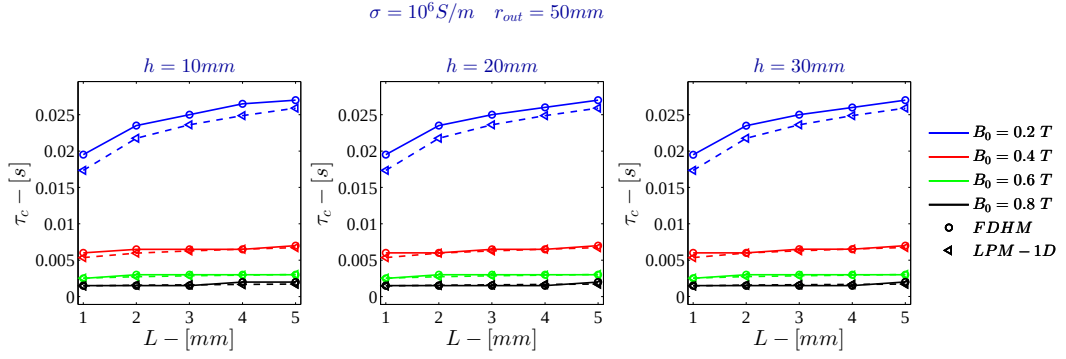


Figure 4.55: τ_c for $\sigma = 10^6 \text{ S/m}$ - Monodimensional LPM

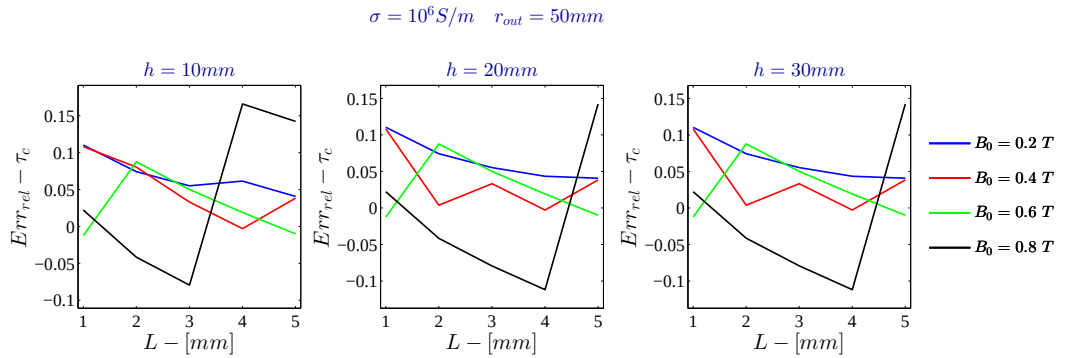
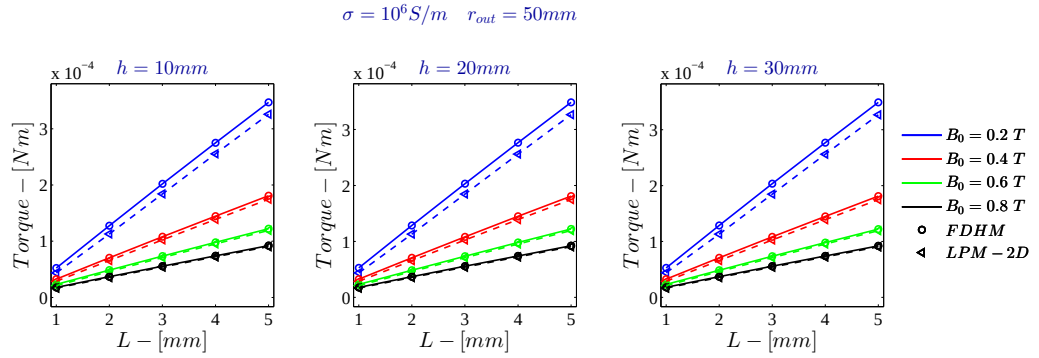
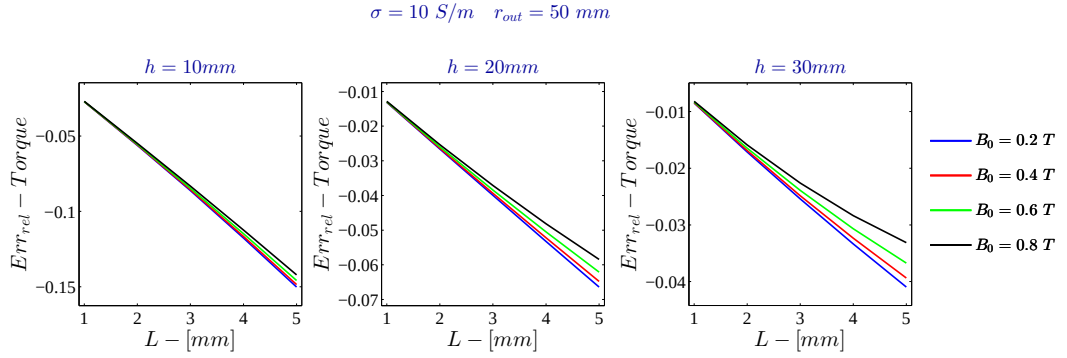
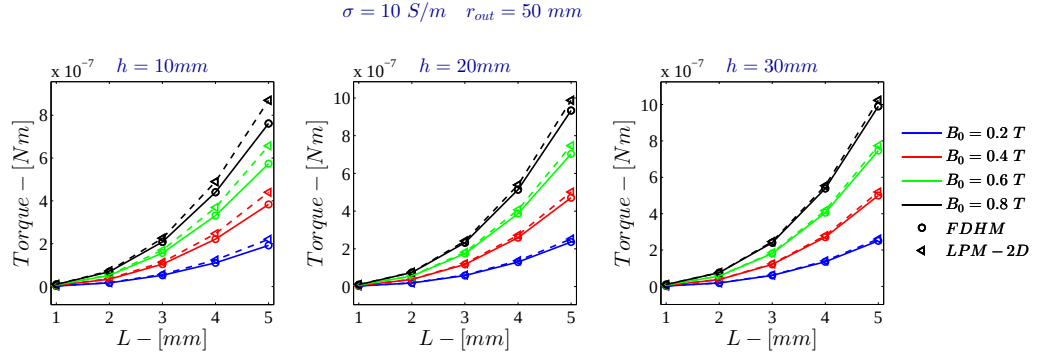


Figure 4.56: Relative Error - τ_c for $\sigma = 10^6 \text{ S/m}$ - Monodimensional LPM

4.5.2 Bidimensional LPM

4.5.2.1 Torque - $\dot{\Gamma}$



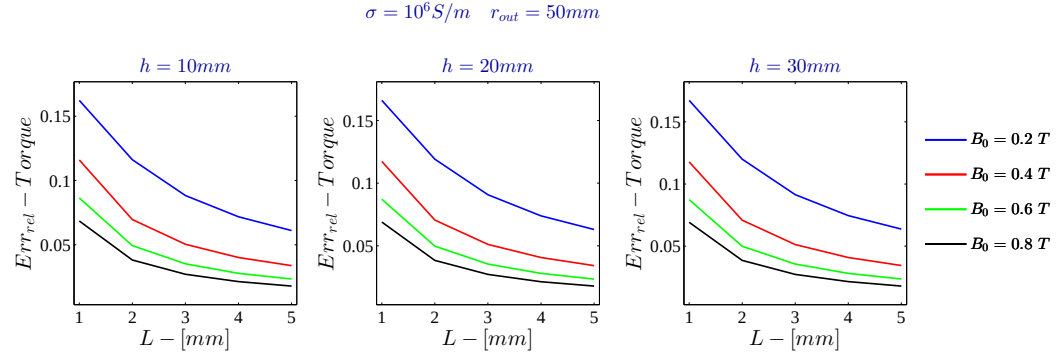


Figure 4.60: Relative Error - $\dot{\Gamma}$ for $\sigma = 10^6 \text{ S/m}$ - Bidimensional LPM

4.5.2.2 Characteristic Time - τ_c

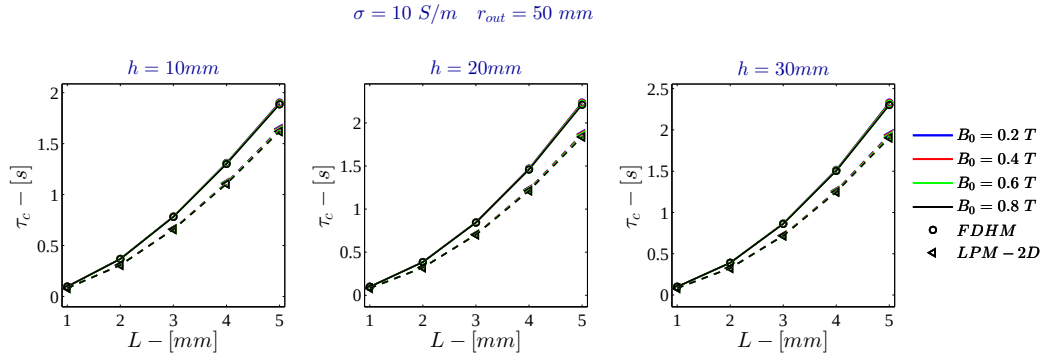


Figure 4.61: τ_c for $\sigma = 10 \text{ S/m}$ - Bidimensional LPM

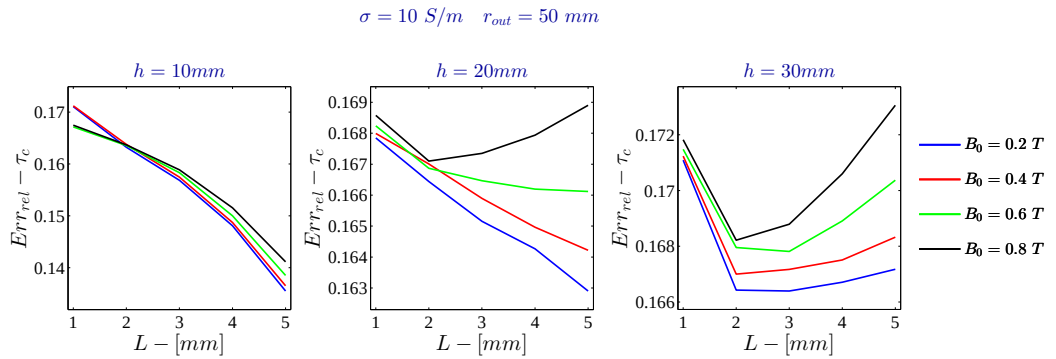


Figure 4.62: Relative Error - τ_c for $\sigma = 10 \text{ S/m}$ - Bidimensional LPM

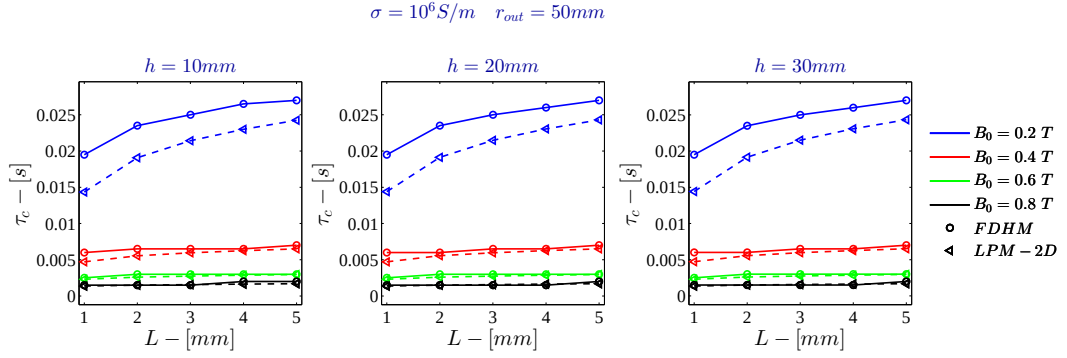


Figure 4.63: τ_c for $\sigma = 10^6 S/m$ - Bidimensional LPM

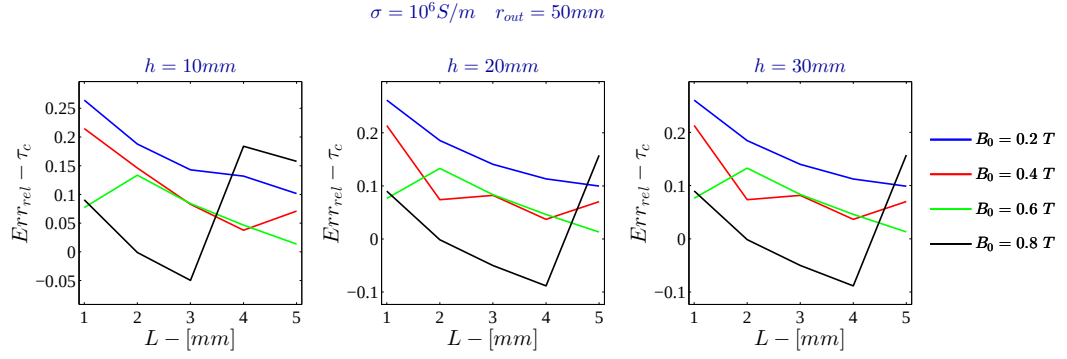


Figure 4.64: Relative Error - τ_c for $\sigma = 10^6 S/m$ - Bidimensional LPM

4.6 Discussion

The comparison between the FDHM and LPM based on the mono and bi-dimensional analytical solutions shows that all the coefficients of the model are well approximated by the LPM apart from the K_{vis} . This coefficient can be estimated better by the LPM based on the mono-dimensional solution for high conductive liquids while for low conductive liquid the LPM based on the bi-dimensional solution is more precise even if the error is of the order of 10^{-1} . This error influences the values of torque computed by the LPM.

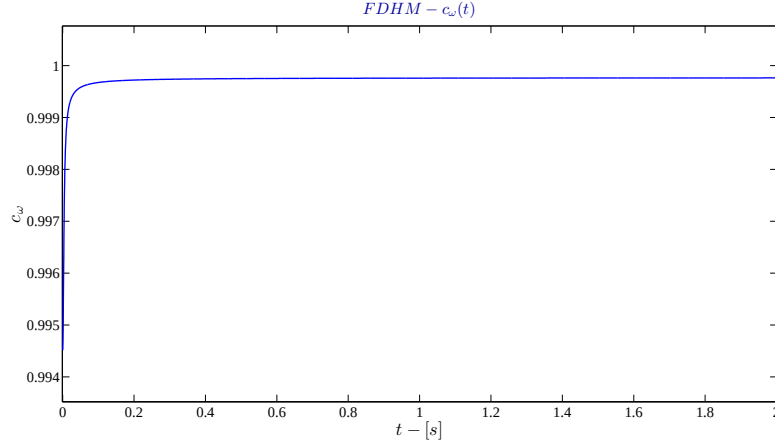


Figure 4.65: Evolution of c_ω

About the characteristic time it can be seen that the error between the characteristic time computed by the FDHM and the LPM is not just the result of the propagation error of the single coefficients used to compute τ_c . A further analysis of the time evolution of the coefficients c_ω and K_{vis} made with the results obtained with the FDHM, shows how these coefficients, considered constant in the LPM, are function of the velocity, Figures(4.65) (4.66) (4.67).

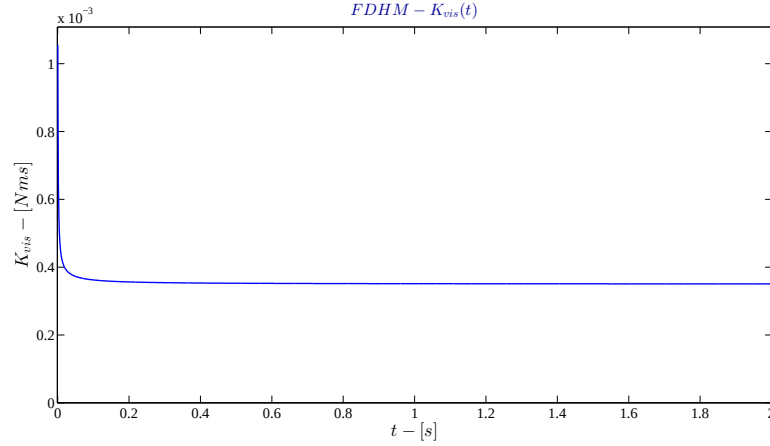


Figure 4.66: Evolution of K_{vis}

This analysis highlights how the device in reality is affected by non-linear effect in the neighbourhood of null angular velocity. This, in some way, resize the hypothetical advantage not to have jump of torque around the zero crossing

[3].

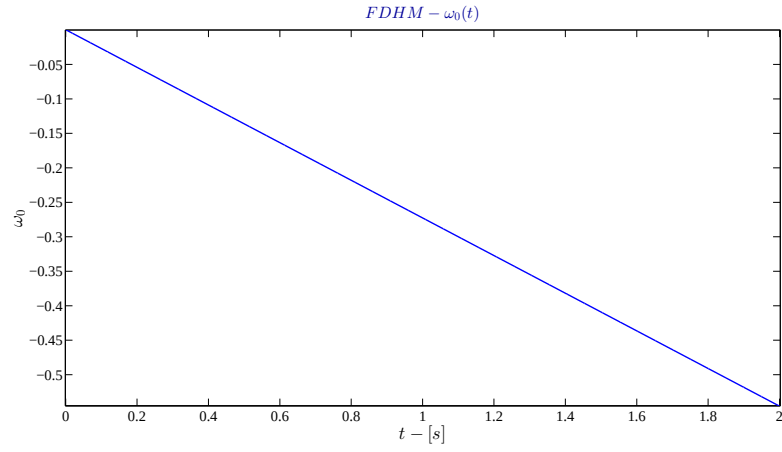


Figure 4.67: *Evolution of ω_0*

Performances analysis.

In this chapter, a simple performance analysis has been conducted by the LPM. The possibility to use liquid metal or a ideal liquid to be designed as been evaluated.

5.1	Liquid Metal	101
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5.1 Liquid Metal

Considering mercury as conductive fluid with the following physical properties:

$$\begin{aligned}\sigma_{Hg} &= 1.04 \cdot 10^6 \text{ S/m} \\ \rho_{Hg} &= 1.36 \cdot 10^4 \text{ Kg/m}^3 \\ \mu_{Hg} &= 1.50 \cdot 10^{-3} \text{ Pa/s}\end{aligned}\tag{5.1.1}$$

the coefficients of the lumped parameter model together with the torque and the power consumption are presented in Figures(5.1)-(5.9). All the values plotted in the following figure, except for c_ω that is dimensionless, are calculated for a torus with rectangular cross section whose height is equal to $h = 1\text{cm}$. The symbols on the y axys of Fig.(5.5), Fig.(5.6) and Fig.(5.9) have the following meaning:

$$\frac{\omega_w}{I} = \frac{\Gamma}{I_z c_\omega I} = \frac{K_I}{K_{vis}}\tag{5.1.2}$$

$$\tilde{T} = \frac{\dot{\Gamma}}{ah} = \frac{1}{h} \left(I_z c_\omega \frac{K_I}{K_{vis}} \right)\tag{5.1.3}$$

$$\tilde{P} = \frac{P}{hI^2} = \frac{1}{h} \left(\frac{K_V K_I}{K_{vis}} + R_{eq} \right)\tag{5.1.4}$$

P is the power consumption of the actuator and a is the time derivative of the electric current. All the following values are obtained by means of the LPM based on the mono-dimensional solution.

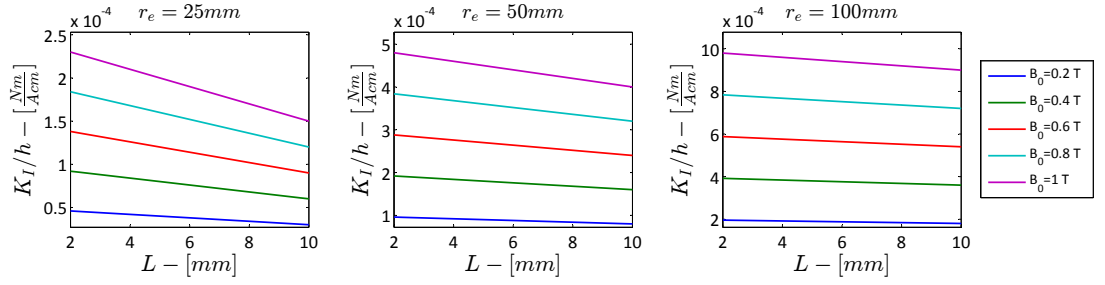


Figure 5.1: K_I function of L for $B_0 = 0.2 \div 1.0$ and 3 different outer radii $r_e = 25, 50, 100\text{mm}$. Liquid: Mercury (Hg).

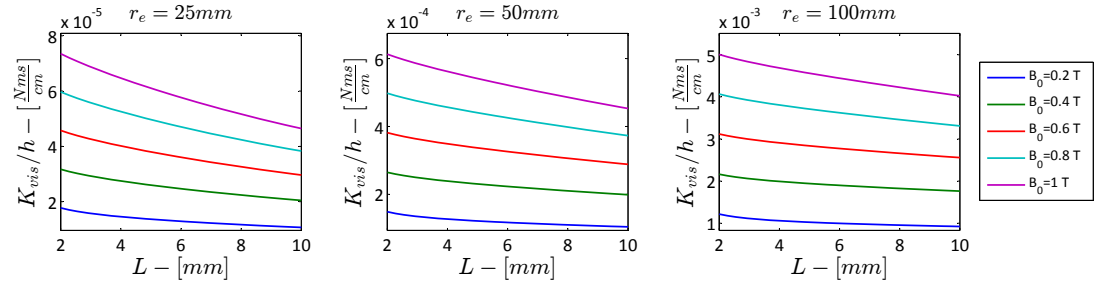


Figure 5.2: K_{vis} function of L for $B_0 = 0.2 \div 1.0$ and 3 different outer radii $r_e = 25, 50, 100\text{mm}$. Liquid: Mercury (Hg).

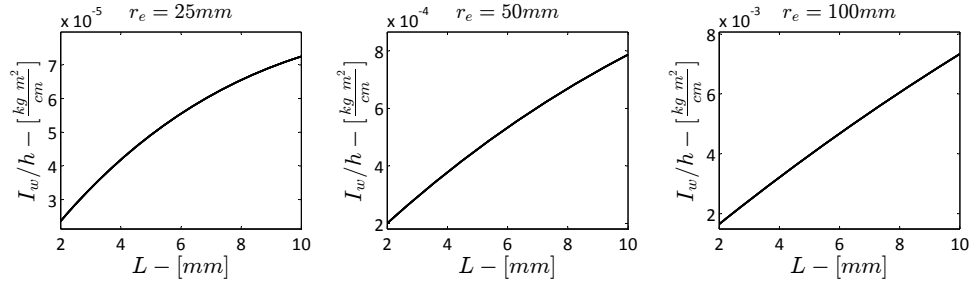


Figure 5.3: I_w function of L for 3 different outer radii $r_e = 25, 50, 100\text{mm}$. Liquid: Mercury (Hg).

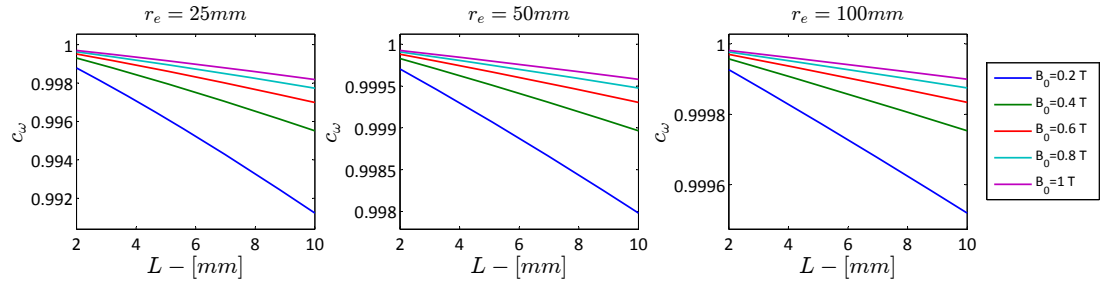


Figure 5.4: c_ω function of L for $B_0 = 0.2 \div 1.0$ and 3 different outer radii $r_e = 25, 50, 100\text{mm}$. Liquid: Mercury (Hg).

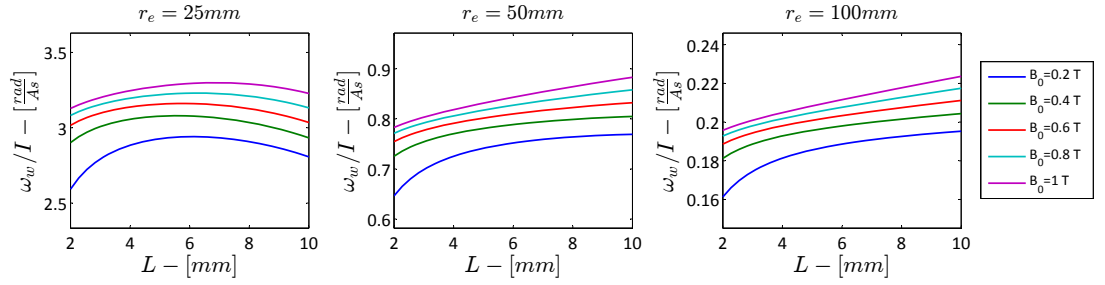


Figure 5.5: Current specific ω_w function of L for $B_0 = 0.2 \div 1.0$ and 3 different outer radii $r_e = 25, 50, 100\text{mm}$. Liquid: Mercury (Hg).

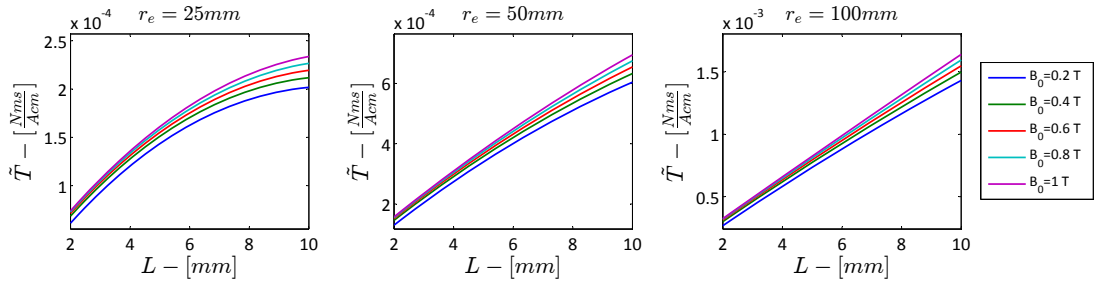


Figure 5.6: Torque function of L for $B_0 = 0.2 \div 1.0$ and 3 different outer radii $r_e = 25, 50, 100\text{mm}$. Liquid: Mercury (Hg).

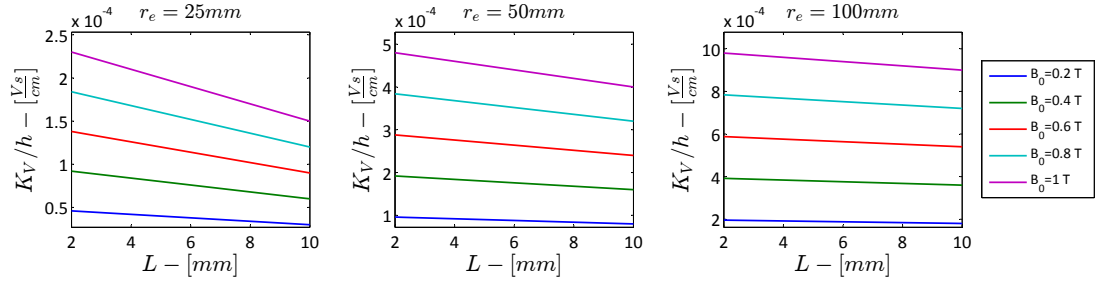


Figure 5.7: K_V function of L for $B_0 = 0.2 \div 1.0$ and 3 different outer radii $r_e = 25, 50, 100\text{mm}$. Liquid: Mercury (Hg).

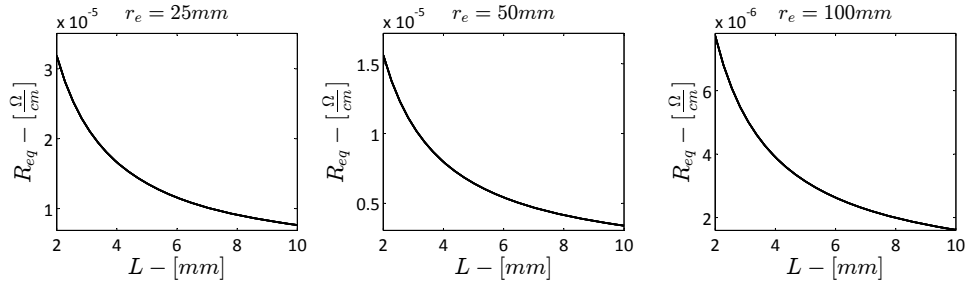


Figure 5.8: R_{eq} function of L for 3 different outer radii $r_e = 25, 50, 100\text{mm}$. Liquid: Mercury (Hg).

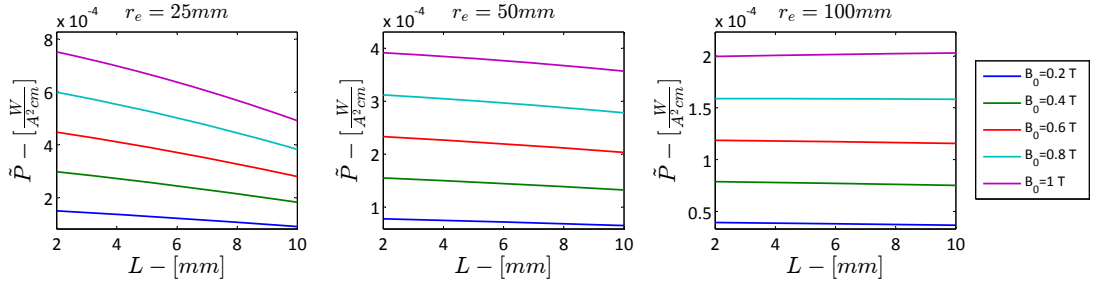


Figure 5.9: Power consumption function of L for $B_0 = 0.2 \div 1.0$ and 3 different outer radii $r_e = 25, 50, 100\text{mm}$. Liquid: Mercury (Hg).

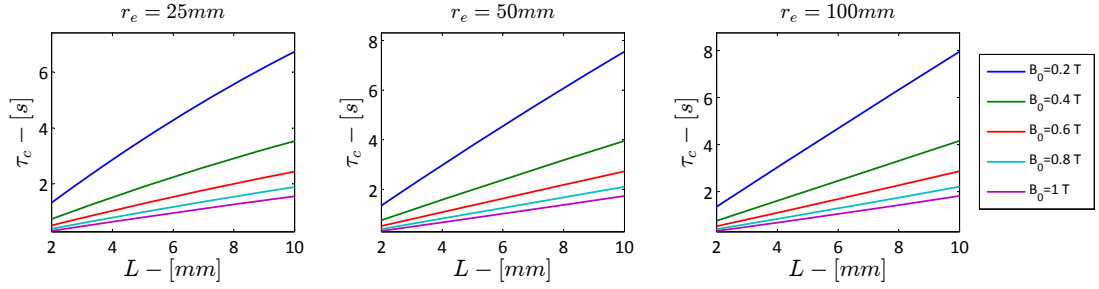


Figure 5.10: Characteristic time function of L for $\sigma = 10 \div 10^6 \text{ S/m}$ and 3 different outer radii $r_e = 25, 50, 100 \text{ mm}$. $B_0 = 1.0 \text{ T}$. $\rho = 1000 \text{ Kg/m}^3$.

5.2 Ideal Liquid

From Figure(5.2) it is possible to see that the influence of the viscosity is not neglectable and it influences the overall performance of the device. The high value of the K_{vis} is proportional to the electrical conductivity of the fluid due to the coupling between the induced voltage and the velocity profile. The values of K_{vis} have been analyzed for different electrical conductivities of the liquid together with the power consumption $\tilde{P}$ and the generated torque $\tilde{T}$. The values are calculated for a liquid with density $\rho = 1000 \text{ Kg/m}^3$ and viscosity $\mu = 10^{-3} \text{ Pa s}$; the magnetic field is set to $B_0 = 1 \text{ T}$.

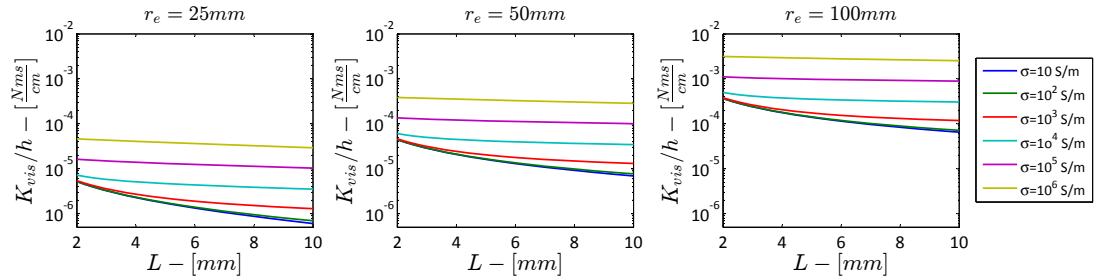


Figure 5.11: K_{vis} function of L for $\sigma = 10 \div 10^6 \text{ S/m}$ and 3 different outer radii $r_e = 25, 50, 100 \text{ mm}$. $B_0 = 1.0 \text{ T}$. $\rho = 1000 \text{ Kg/m}^3$.

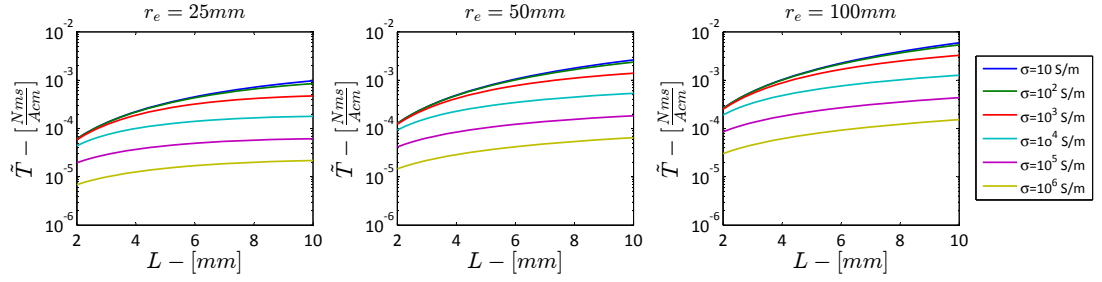


Figure 5.12: Torque function of L for $\sigma = 10 \div 10^6$ S/m and 3 different outer radii $r_e = 25, 50, 100$ mm. $B_0 = 1.0$ T. $\rho = 1000$ Kg/m³.

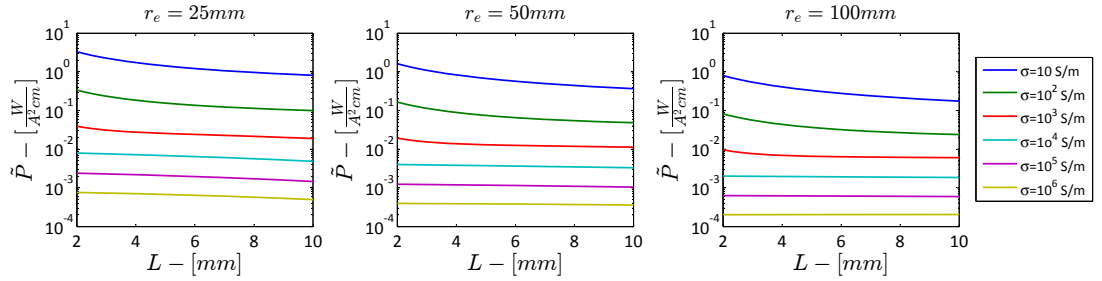


Figure 5.13: Power consumption function of L for $\sigma = 10 \div 10^6$ S/m and 3 different outer radii $r_e = 25, 50, 100$ mm. $B_0 = 1.0$ T. $\rho = 1000$ Kg/m³.

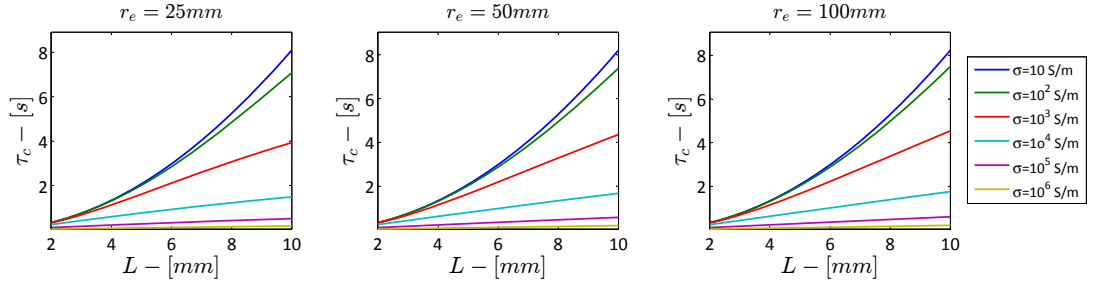


Figure 5.14: Characteristic time function of L for $\sigma = 10 \div 10^6$ S/m and 3 different outer radii $r_e = 25, 50, 100$ mm. $B_0 = 1.0$ T. $\rho = 1000$ Kg/m³.

From Figures(5.11) and (5.12) it is possible to see the improving in the performances for low conducting fluid. Due to the low coupling between the velocity and the magnetic field, the generated torque for a low conductive fluid become bigger of about two orders of magnitude with respect to an high conductive fluid. As expected, the power consumption has an opposite trend but its maximum value is anyway around 1W if an electric current of 1A is applied,

Figure(5.13).

For high conductive fluids the estimation of the parameters is expected to be quite accurate due to the fact that the length of the boundary layers on the upper and bottom walls scale as $\delta_{Sh} \sim 1/\sqrt{Ha}$ while the boundary layers on the lateral surfaces scale as $\delta_{Ha} \sim 1/Ha$. For high Ha these two characteristic length can differ of one order of magnitude, then the error due to the neglection of the Shercliff layers, next to the armatures, is low.

On the other side the prediction of K_{vis} and the available torque could be regarded as not accurate for very low conductive fluid due to the reason that the Shercliff layers on the upper and bottom surface of the torus are in this case of the same order of magnitude as the Hartmann layers on the lateral walls of the torus. Anyway the density of an hypothetical low conductive fluid is expected to be low if compared with the density of a liquid metal. This implies that in order to have the same moment of inertia for low conductive fluid, we need to increase the height of the cross section if the working fluid has low electric conductivity. This implies again that the viscous shear on the top and bottom boundaries could be neglectable with respect of the viscous shear acting on the lateral cylindrical surfaces of the device

LPMHD: Lumped Pump MHD Reaction Wheel

In this chapter, the possibility to use lumped MHD conductive pumps instead of distributed MHD conductive pump is evaluated by means of simplified mathematical model.

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6.1 About the influence of the Magnetic field on K_{vis} and K_I

In the chapter dedicated to the lumped parameter model (LPM) the dependency of the coefficients from the magnetic field has been analytically explained. The attention has been put on the reference value of the magnetic field, B_0 . A further comment should be written about it. The low torque values obtained for a MHD reaction wheel with distributed MHD conductive pump are due to the fact that the magnetic field induces greater viscous shear stresses, through the coupling between the distribution of linear velocity and the distribution of the electric current flowing through the conductive fluid. This effect appears anyway only in the zone where the magnetic field is present. All the previous analysis has been conducted on a device whose magnetic field is distributed all over the torus.

Now we wonder about what could happen if the magnetic field is concentrated in only a portion of the torus. A priori, we expect a reduction of the intensity of the viscous shear stress due to the reduction of the volume where the magnetic field flows. This deduction comes from the fact that the viscous shear stress for MHD flows at high Hartmann numbers is quite higher than a classical Poiseuille flow. We then suppose that, while in the presence of a magnetic field the flow tends to be MHD characteristics, in the portion of the torus where the magnetic field is not present the flow tends to be like a classical Poiseuille flow. On the other hand the coefficient of the moment of the Lorentz force is the same as for distributed MHD conductive pump and so just function of the geometrical dimensions of the device and the reference value of the magnetic field, B_0 . Being the torque $\dot{\Gamma}$ proportional to the K_I and inversely proportional to the K_{vis} we expect to have an increased torque provided by the device [3].

6.2 Lumped Pump MHD Reaction Wheel

In Figure (6.1) a Lumped Pump MHD reaction wheel, LPMHD, is showed. The LPMHD is characterized by an active zone in which a radial magnetic field is present and a passive zone where there is no magnetic field. Inside the active zone two electrodes are placed with the purpose to feed the device with

electric current where the magnetic field is present.

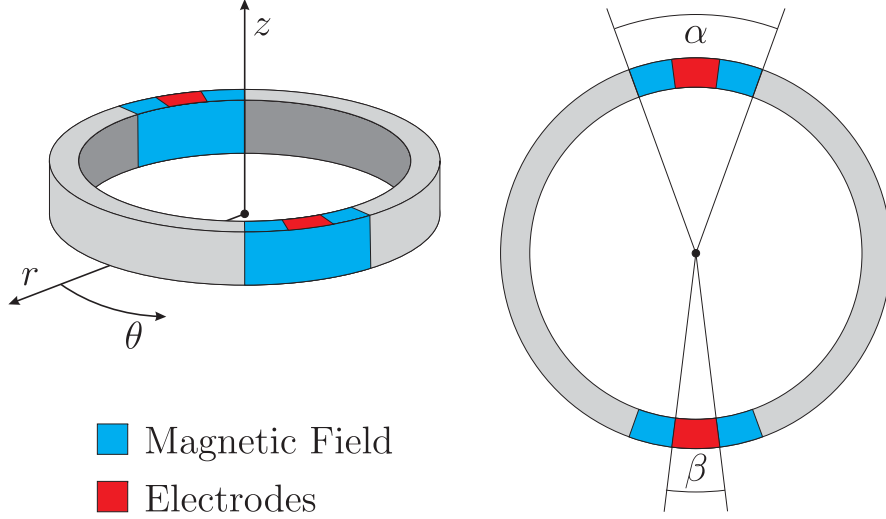


Figure 6.1: LPMHD: Lumped Pump MHD Reaction Wheel

The number of pumps should be even in order to minimize the interaction with earth magnetic field. In effect, in presence of radial magnetic field the single pump can be modelled as magnetic dipole parallel to the radius. If there are two pumps placed in two symmetrical positions we have two dipoles along the radius with opposite directions. Considering the geomagnetic field constant in the neighbourhood of the device, is evident that the two dipoles generate two moments of same intensity with opposite directions. Even if we consider the gradient of the magnetic field in the neighbourhood of the device, for this configuration we have very small disturbing torque.

We define the angular fraction occupied by the magnetic field as:

$$MF = \frac{\alpha N_{pumps}}{2\pi} \quad (6.2.1)$$

while the angular fraction occupied by the electrodes as:

$$EF = \frac{\beta N_{pumps}}{2\pi} \quad (6.2.2)$$

where α and β are the angular width of the magnetic circuit and the electrodes

for each pump.

6.3 Simplified model

The LPMHD is here studied with a simplified model. The aim is to obtain indications about the real convenience to use such a kind of device and to build a model to be used as a simple design tool.

The electric problem is solved in the coordinate $\theta - z$ with the same algorithm used to solve the electric problem in the FDHM [20]. The fluid-dynamic problem is solved considering an MHD flow inside the conductive MHD pumps and a Poiseuille flow in the passive part of the device.

The model is furthermore solved in the domain of the time in order to have information about the torque provided.

6.3.1 Electric model

For symmetry the electric side of the problem will be solved for half of the device because the device considered has two MHD pumps. Furthermore the problem is solved in the $\theta - z$ domain.

The electric network is made up of 3 zones:

- A purely resistive part in the passive zone of the annulus, where no magnetic field is present.
- A part where voltage generators are present in order to model the back electromotive voltage generate by the spinning liquid through the magnetic field. This zone corresponds to the part of the annulus inside the MHD pump where the magnetic field is present.
- A zone where the electric current input is considered. It models the presence of the electrodes.

The horizontal resistances in Figure (6.2) model the electric conduction in azimuthal direction and their values are obtained in the following way:

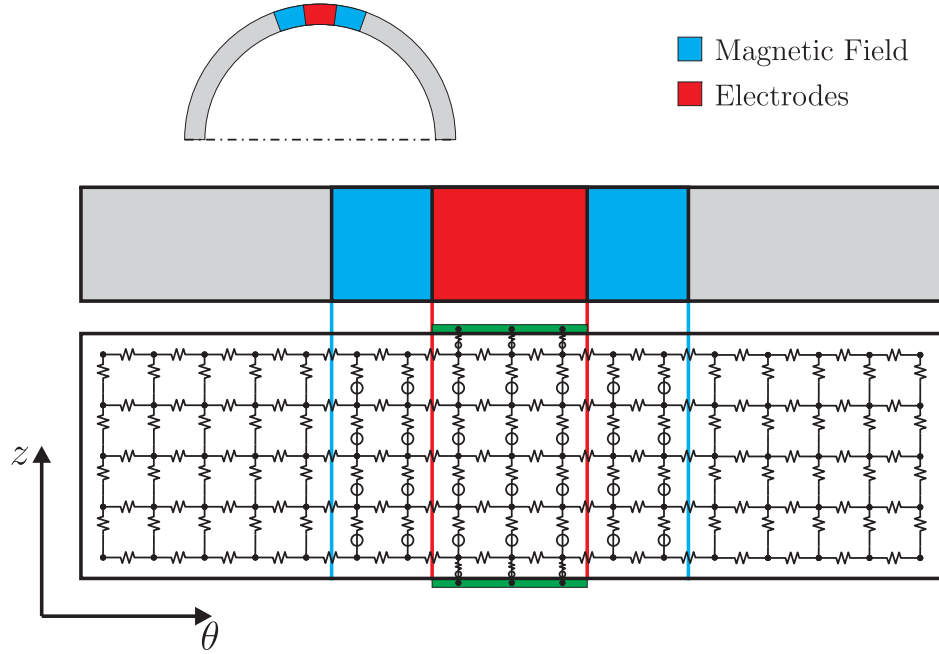


Figure 6.2: LPMHD: Electric network

$$R_{\theta_{i,j}} = \frac{1}{\sigma} \frac{\Delta\theta_i}{\log \frac{R_e}{R_i}} \Delta z_j \quad (6.3.1)$$

While the vertical resistances can be computed as:

$$R_{z_{i,j}} = \frac{1}{\sigma} \frac{\Delta z_j}{\Delta\theta_i (R_e^2 - R_i^2)} \quad (6.3.2)$$

and they represent the conduction along the z direction.

The values of the voltage generator are computed on the base of the 2-dimensional analytical solution commented on the chapter dedicated to the LPM:

$$V_{ind_{i,j}} = K_{V_{i,j}} \omega_w \quad (6.3.3)$$

where:

$$K_V = 2B_0 r_i \frac{\int_{r_i}^{r_e} \int_{z_i}^{z_{i+1}} r \omega_z(r, z) dr dz}{\omega_0 (R_e^2 - R_i^2)} \quad (6.3.4)$$

6.3.2 Dynamic model

The dynamic part of the problem is studied considering a Poiseuille flow in the passive section of the annulus and an MHD flow inside the pump.

The angular velocity of the flow in passive section is computed on the base of the following formula:

$$\omega(r, z) = (ar^2 + br + c)(Az^2 + Bz + C)/r \quad (6.3.5)$$

where:

$$\begin{aligned} a &= -\frac{4}{R_i^2 - 2R_i R_e + R_e^2} & A &= -\frac{4}{h^2} \\ b &= \frac{4(R_i + R_e)}{R_i^2 - 2R_i R_e + R_e^2} & B &= \frac{4}{h} \\ c &= -\frac{4R_i R_e}{R_i^2 - 2R_i R_e + R_e^2} & C &= 0 \end{aligned} \quad (6.3.6)$$

The equation to be solved to have informations about the torque provided by the actuator is:

$$\dot{\Gamma} = I_z \ddot{\omega} = M_{Lor} - \tilde{K}_{vis} \omega_w \quad (6.3.7)$$

The coefficients $\tilde{c}_\omega$ and $\tilde{K}_{vis}$ are computed as the weighted mean of the respective coefficients in the passive section, $K_{vis_{BP}}$ $c_{\omega_{BP}}$, and active section $K_{vis_{MHD}}$ $c_{\omega_{MHD}}$, with respect of MF :

$$\tilde{c}_\omega = c_{\omega_{MHD}} MF + c_{\omega_{BP}} (1 - MF) \quad (6.3.8)$$

$$\tilde{K}_{vis} = K_{vis_{MHD}} MF + K_{vis_{BP}} (1 - MF) \quad (6.3.9)$$

The moment of Lorentz force comes from the numerical integration of the product between the axial electric current along the direction z and the K_I , written as:

$$K_{I_{i,j}} = B_0 r_i \Delta z_j \quad (6.3.10)$$

6.4 Simulations

A parametric study has been conducted varying the values of EF and MF . The liquid considered is Mercury (Hg). In the following table can be found a resume of the input data.

	U.M	Max Value	Min Value	N.points
ρ	$[kg/m^3]$	13534	13534	1
μ	$[Pa \cdot s]$	$1.526 \cdot 10^{-3}$	$1.526 \cdot 10^{-3}$	1
σ	$[S/m]$	$1.041 \cdot 10^6$	$1.041 \cdot 10^6$	1
B_0	$[T]$	0.8	0.8	1
r_{in}	$[mm]$	45	45	1
r_{out}	$[mm]$	50	50	1
I	$[A/s]$	1	1	1
h	$[mm]$	10	30	3
MF	$[\%]$	8	40	5
EF	$[\%]$	2	20	10

Table 6.1: Values grid for the LPMHD simulations

6.4.1 Results

The simulations highlight a dependency between EF and MF and the torque provided. In Figure (6.3) it is possible to see how the torque decrease when the value of EF approaches the value of MF . The meaning of this result is that part of the electric current exiting from the input electrode is pushed outside the active region of the reaction wheel by the back electromotive voltage. This lead to a decreasing total current flowing inside the active part of the pump and a decreasing resulting moment of Lorentz force.

In the same figure it is possible to see a slight increasing of the torque for

increasing MF . The explanation of this effect can be given considering the fact that the $c_{\omega_{BP}}$ is lower than the $c_{\omega_{MHD}}$ and for increasing MF the equivalent moment of inertia $I_z \tilde{c}_\omega$ increases. This lead to lower $\dot{\omega}_w$ and then lower viscous shear stress.

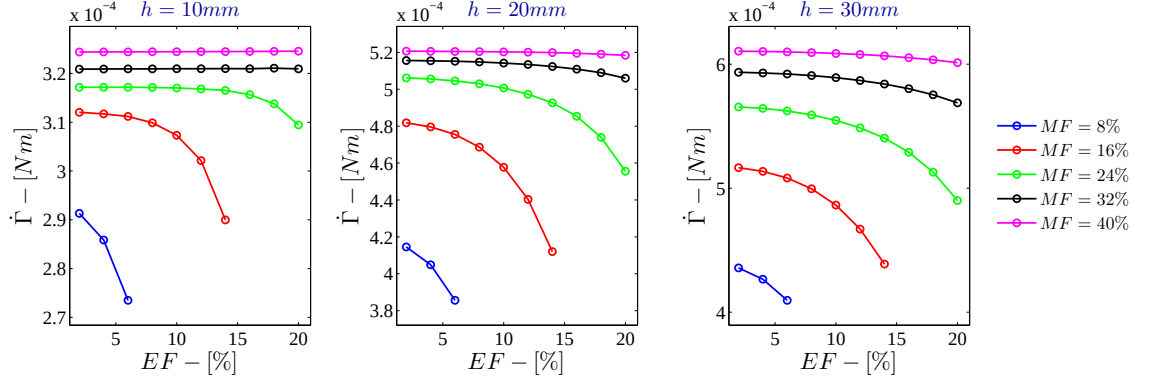


Figure 6.3: LPMHD: Simulations results

If the results of the simulations are compared with the torque obtained with the FDHM we can see that the torque provided by the LPMHD is always lower.

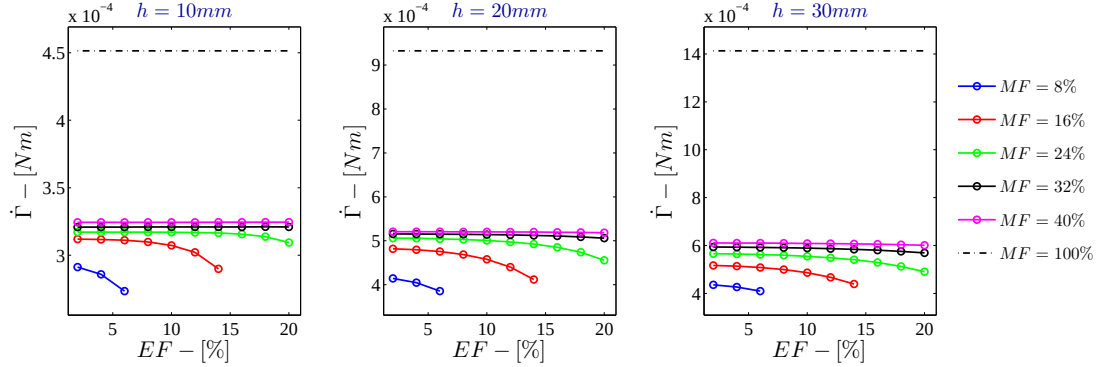


Figure 6.4: LPMHD: Comparison with a distributed MHD conductive pump reaction wheel

The reason resides in some particular electrical loops appearing in the discontinuity zone between the active and passive region of the annulus, Figures (6.5)(6.6)(6.7)(6.8). This loops induce in the active part of the the actuator, electric currents in the opposite direction with respect of the direction of the

electric current flowing through the electrodes. This results in a global loss of moment of Lorentz force and then in a lower torque provided.

Furthermore the loss of torque provided with respect of a distributed MHD conductive pump reaction wheel increase when the height of the torus increase, and not directly proportionally with it.

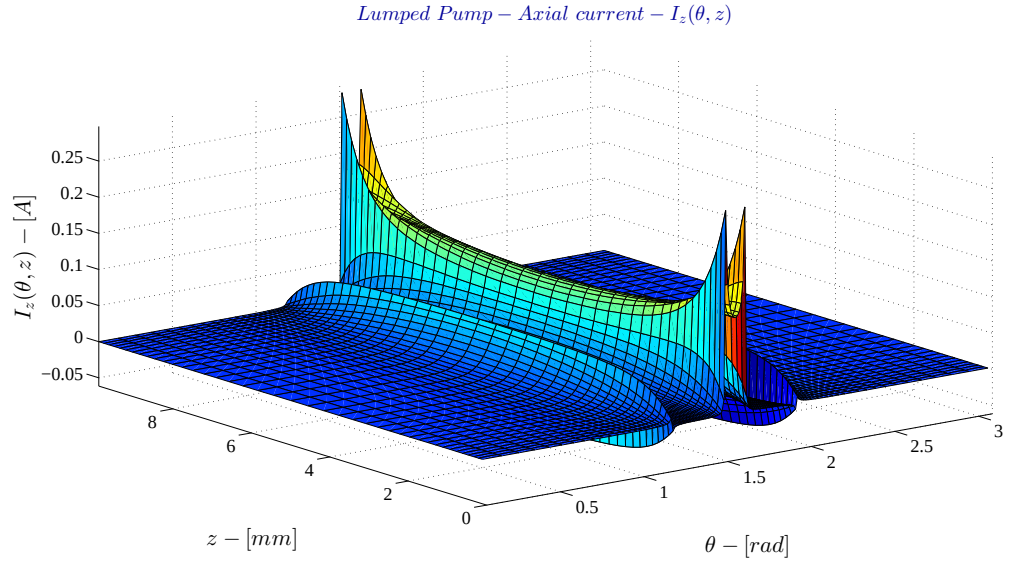


Figure 6.5: LPMHD: Axial electric current

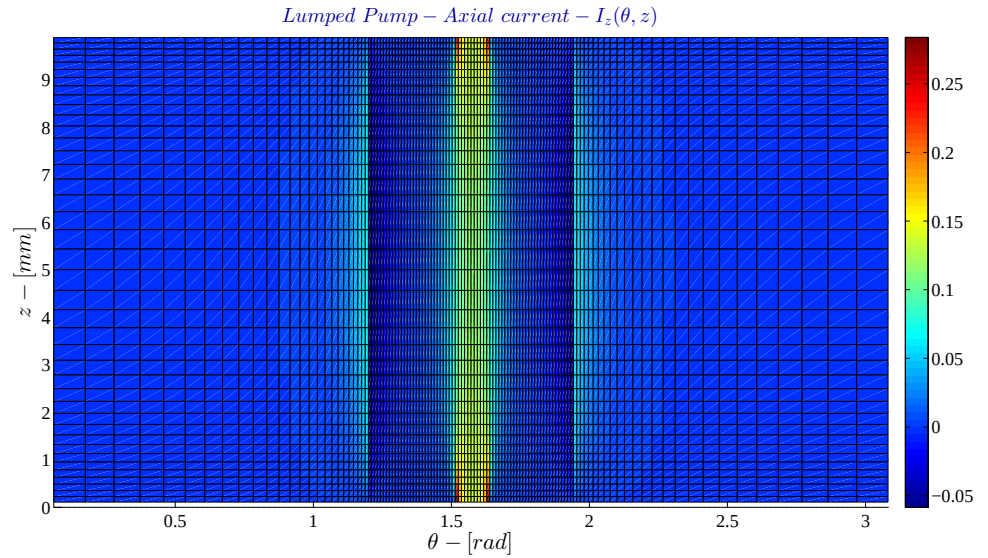


Figure 6.6: LPMHD: Axial electric current - Colorbar

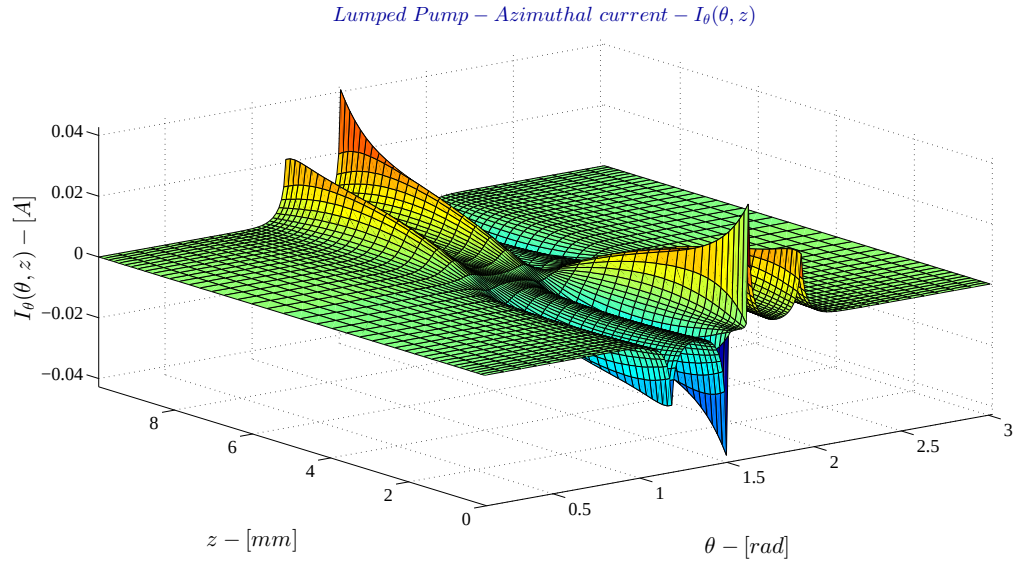


Figure 6.7: LPMHD: Azimuthal electric current

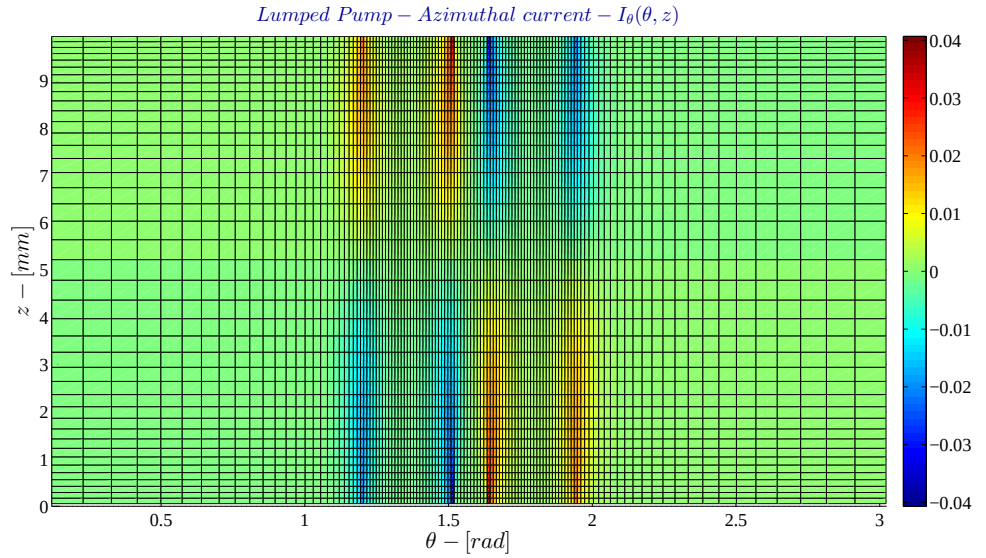


Figure 6.8: LPMHD: Azimuthal electric current - Colorbar

In Figure(6.4) the possibility to use a modular reaction wheel can be evaluated. A LPMHD with an height of $h = 3cm$ appears to be less performing than a composite LPMHD made by three LPMHD whose height is $h = 1cm$. Furthermore this composite LPMHD approaches the value of a MHD reaction wheel with distributed MHD conduction pumps. The advantage of such a kind

of device is technological. It is evident that a device with the magnetic circuit limited to a portion of the annulus is easier to built and it implies an overall mass saving due to the absence of the magnetic circuit extended to all the annulus.

A novel actuator for spacecraft attitude control with liquid flywheel has been studied. The main characteristic of this new concept of reaction wheel is that a conductive liquid rather than a solid mass is accelerated to change the angular momentum of the equipment and, as a consequence, to provide torque to the spacecraft. The conductive liquid is accelerated using a distributed Magneto-Hydro-Dynamic (MHD) conductive pump. Two different configurations of the device have been studied on the base of the optimization of dimensionless moment of inertia and the minimization of the viscous shear. In order to estimate the performances of this actuator, a 2-dimensional Finite Differences Hybrid Model (FDHM) has been developed on the base of the MHD set equation under the hypothesis of low Magnetic Reynolds. The model solves numerically the time dependent axially symmetric problem of a conductive liquid rotating in a torus with rectangular cross section due to the interaction of a radial magnetic field and an axial electric field. The electric side of the problem has been solved by means of the node method applied to a network of electric resistances and voltage generators representing the back electromotive voltage induced by the spinning liquid through the magnetic field. The fluid-dynamics side of the problem has been solved using a Crank-Nicolson method over a non uniform and collocated grid. The grid generator has been written to be sensitive to the Hartmann number of the problem.

A Lumped Parameter Model (LPM) has been then derived using a stationary and mono-dimensional analytical solution of the MHD set of equations under the hypothesis of low Magnetic Reynolds. The LPM has been compared with the results obtained with the FDHM, showing good agreement in the estimation of the coefficients of the model.

The possibility to use an ideal fluid different from liquid metal has been evaluated by means of the LPM. Due to the low coupling between the velocity and the magnetic field, the generated torque for a low conductive fluid become bigger of at about two orders of magnitude with respect of an high conductive fluid. As expected, the power consumption has an opposite trend.

Finally, in order to understand the room for improvement of the performances of the device using liquid metal, a simplified lumped parameter model has been applied to a device with lumped MHD conductive pumps. The results show no improvement in the performances but a particular modular solution could lead to almost the same torque generated by a distributed MHD reaction wheel with the technological advantage to have the magnetic circuit confined in a limited portion of the annulus.

The analysis of the time evolution of the coefficients c_ω and K_{vis} made with the results obtained with the FDHM, shows how these coefficients, considered constant in the LPM, are function of the velocity. This analysis highlights how this kind of device in reality are affected by non-linear effect in the neighbourhood of null angular velocity. This, in some way, resize the hypothetical advantage not to have jump of torque around the zero crossing.

Due to the typical characteristic Torque-Current this kind of device with conductive MHD pumps, appears quite different from a classical reaction wheel and their use could be limited to a role of aid to the traditional Attitude Control Actuators.

Comparison between FDHM and LPM: $\sigma = 10^2 \text{ S/m}$

8.1 Monodimensional LPM

8.1.1 Coefficient of the viscous shear moment - K_{vis}

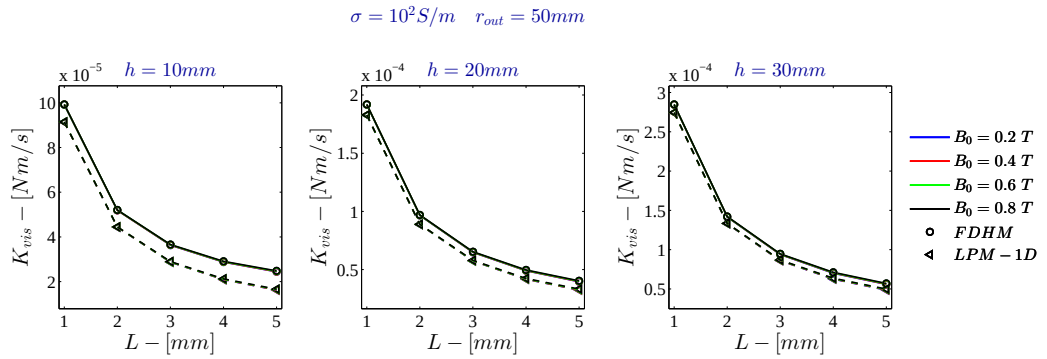


Figure 8.1: K_{vis} for $\sigma = 10^2 \text{ S/m}$ - Monodimensional LPM

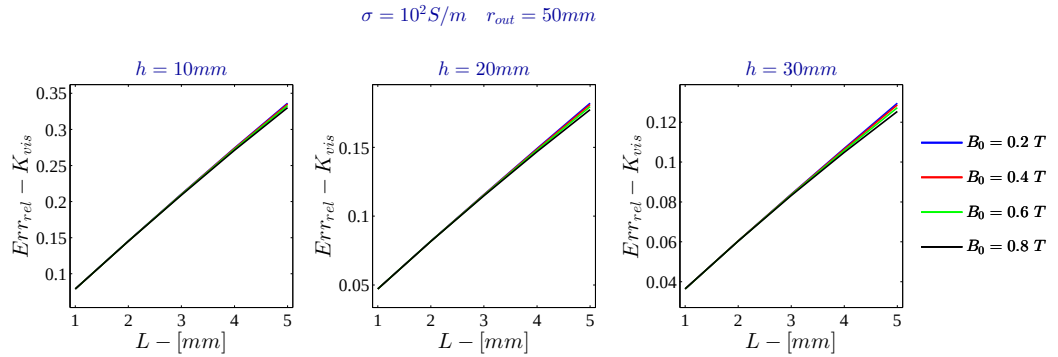


Figure 8.2: Relative Error - K_{vis} for $\sigma = 10^2 \text{ S/m}$ - Monodimensional LPM

8.1.2 Coefficient of the Lorentz Force moment - K_I

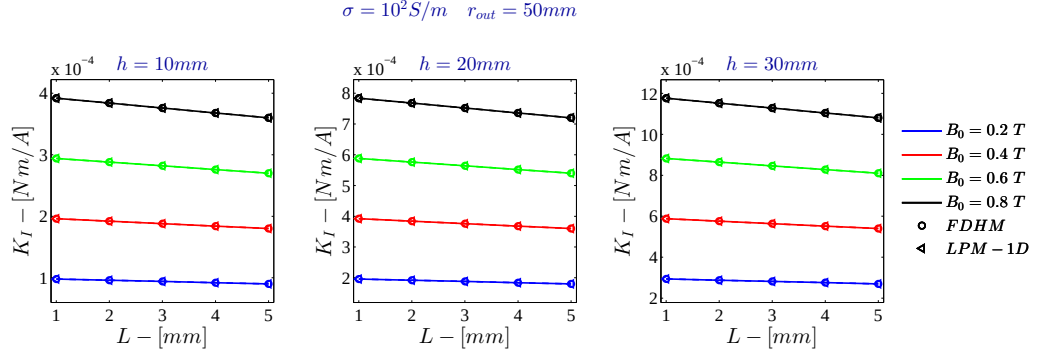


Figure 8.3: K_I for $\sigma = 10^2 \text{ S/m}$ - Monodimensional LPM

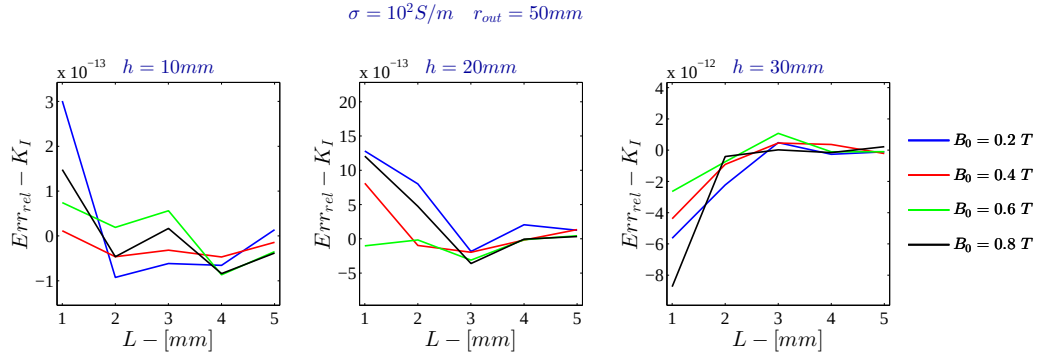
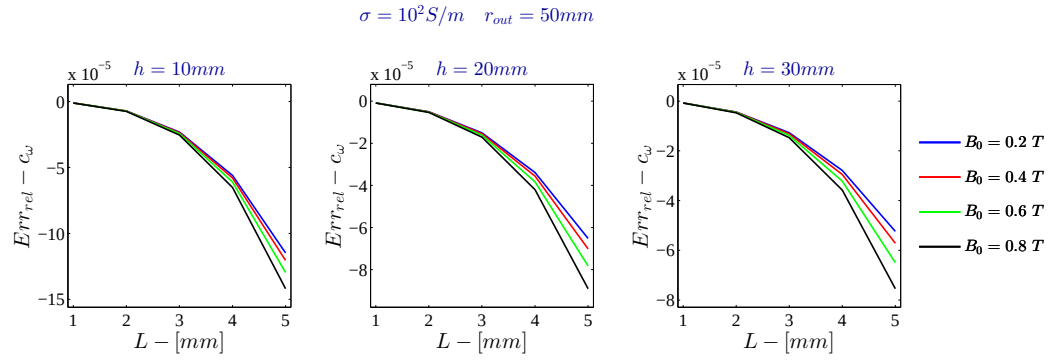
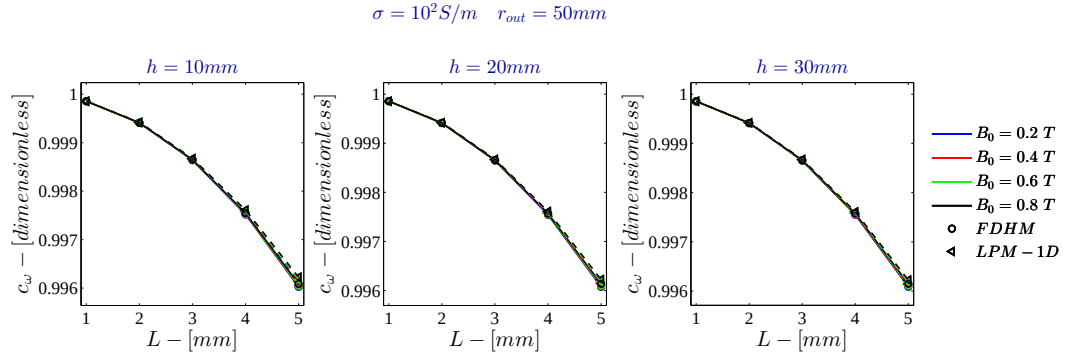


Figure 8.4: Relative Error - K_I for $\sigma = 10^2 \text{ S/m}$ - Monodimensional LPM

8.1.3 Coefficient of the moment of inertia - c_ω



8.1.4 Coefficient of Induced Voltage - K_V

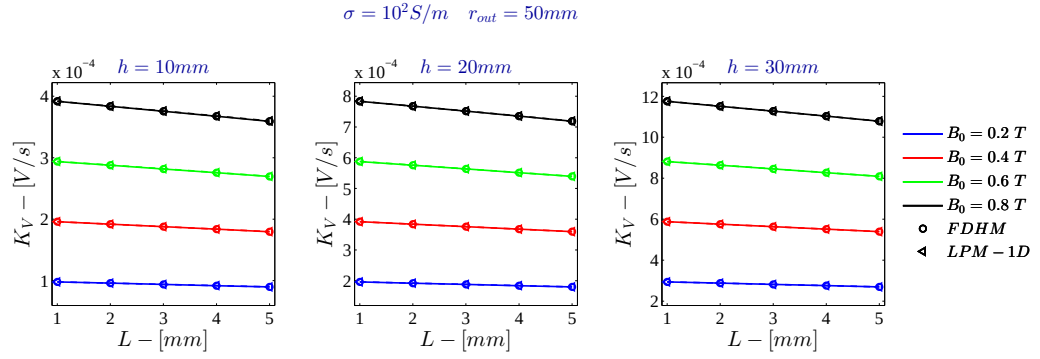


Figure 8.7: K_V for $\sigma = 10^2 \text{ S/m}$ - Monodimensional LPM

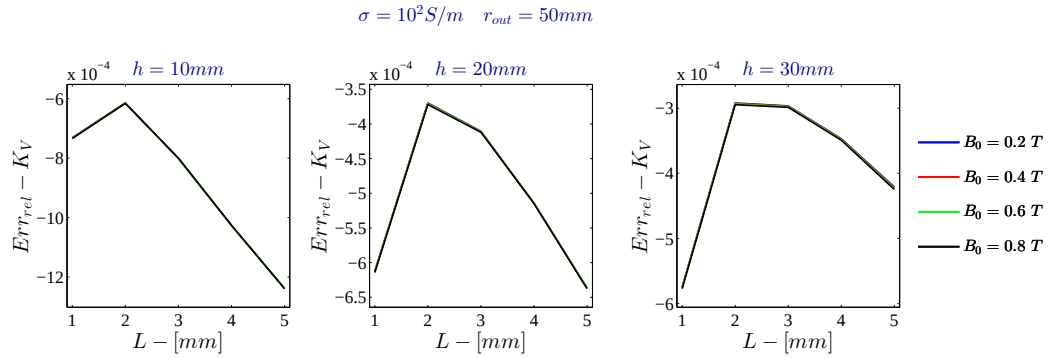


Figure 8.8: Relative Error - K_V for $\sigma = 10^2 \text{ S/m}$ - Monodimensional LPM

8.1.5 Torque - $\dot{\Gamma}$

8.1.5.1 Current Drive

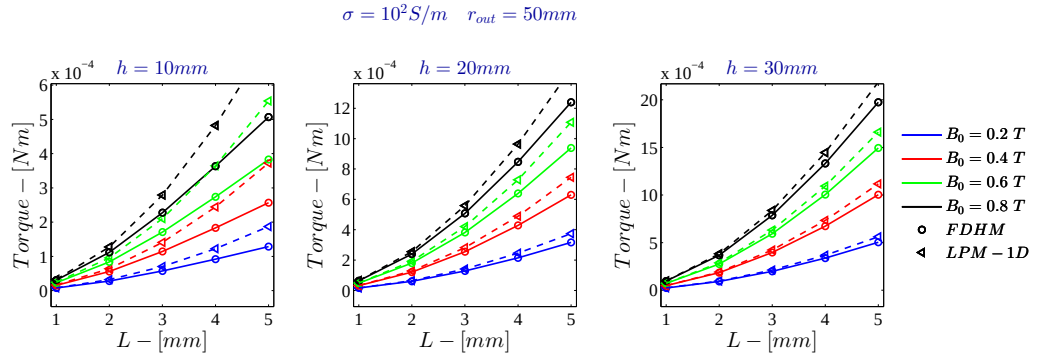


Figure 8.9: $\dot{\Gamma}$ for $\sigma = 10^2 \text{ S/m}$ - Monodimensional LPM

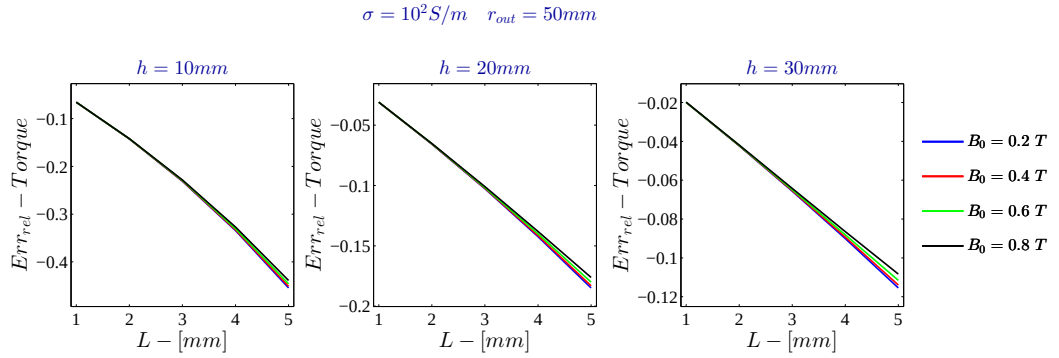


Figure 8.10: Relative Error - $\dot{\Gamma}$ for $\sigma = 10^2 \text{ S/m}$ - Monodimensional LPM

8.1.5.2 Voltage Drive

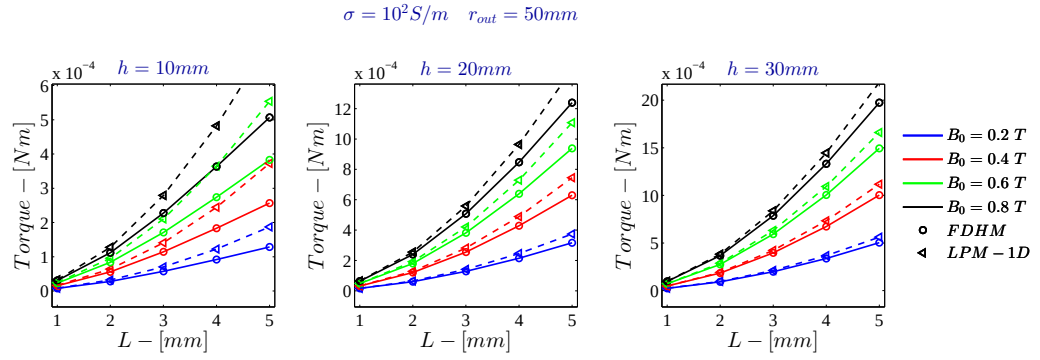


Figure 8.11: $\dot{\Gamma}$ for $\sigma = 10^2 \text{ S/m}$ - Monodimensional LPM

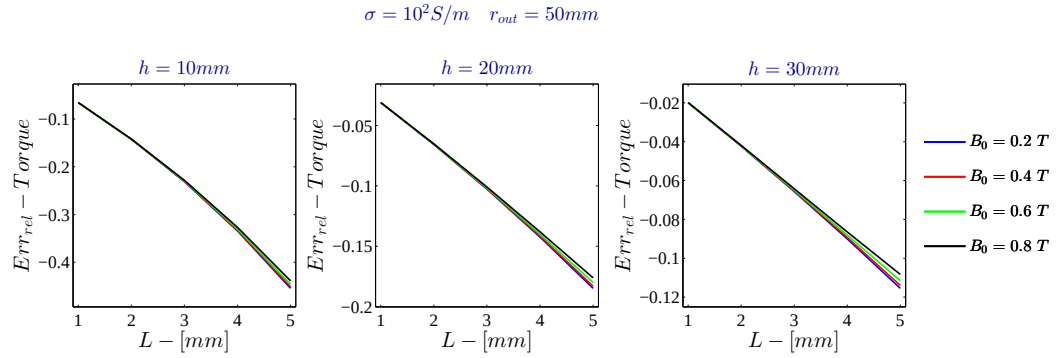


Figure 8.12: Relative Error - $\dot{\Gamma}$ for $\sigma = 10^2 \text{ S/m}$ - Monodimensional LPM

8.1.6 Characteristic Time - τ_c

8.1.6.1 Current Drive

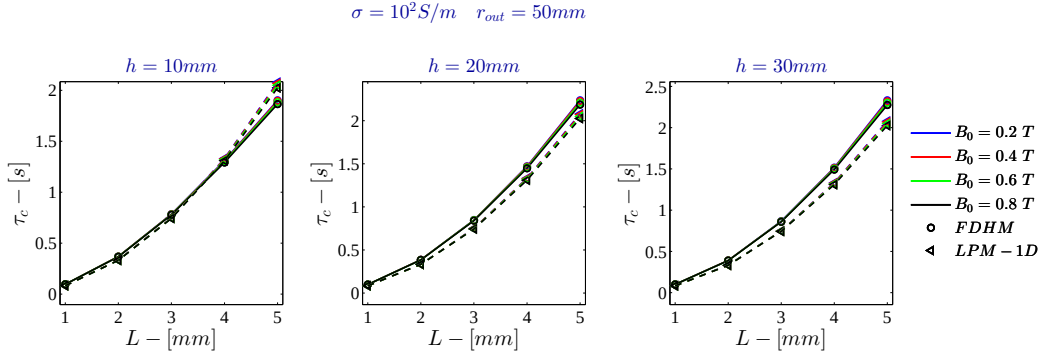


Figure 8.13: τ_c for $\sigma = 10^2 \text{ S/m}$ - Monodimensional LPM

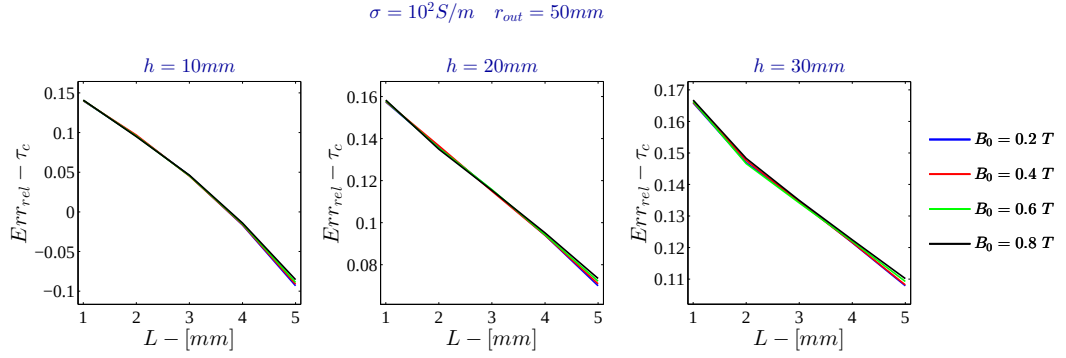


Figure 8.14: Relative Error - τ_c for $\sigma = 10^2 \text{ S/m}$ - Monodimensional LPM

8.1.6.2 Voltage Drive

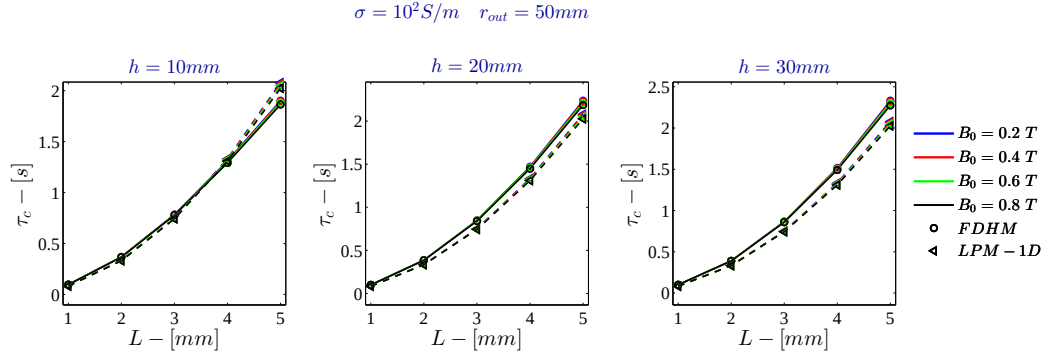


Figure 8.15: τ_c for $\sigma = 10^2 \text{ S/m}$ - Monodimensional LPM

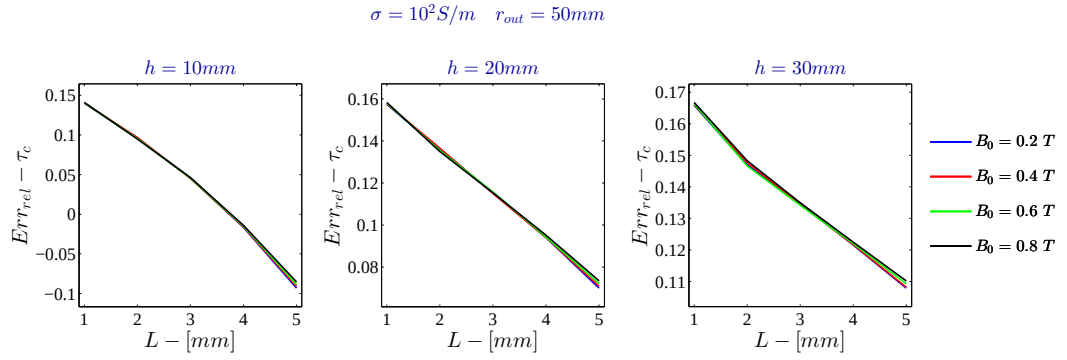


Figure 8.16: Relative Error - τ_c for $\sigma = 10^2 \text{ S/m}$ - Monodimensional LPM

8.2 Bidimensional LPM

8.2.1 Coefficient of the viscous shear moment - K_{vis}

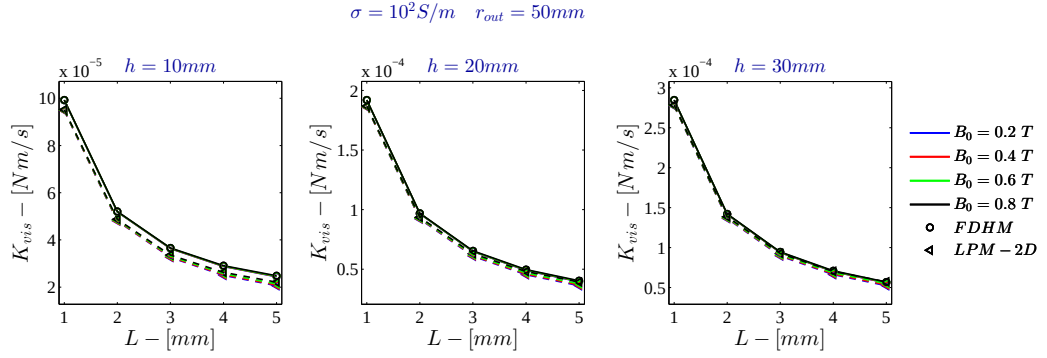


Figure 8.17: K_{vis} for $\sigma = 10^2 \text{ S/m}$ - Bidimensional LPM

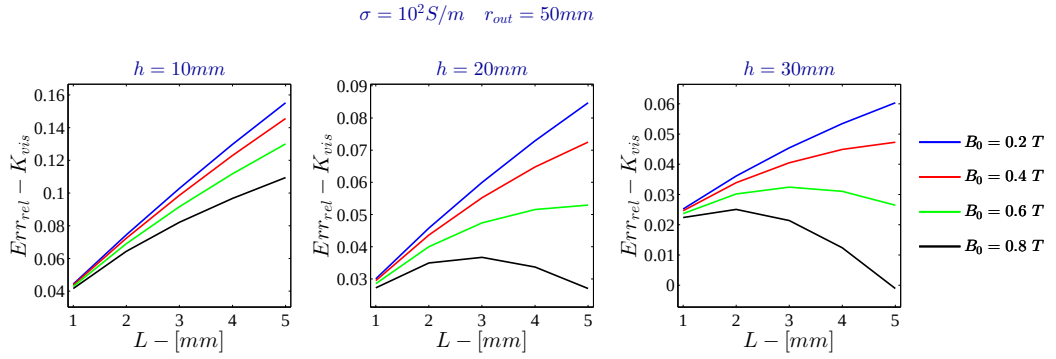
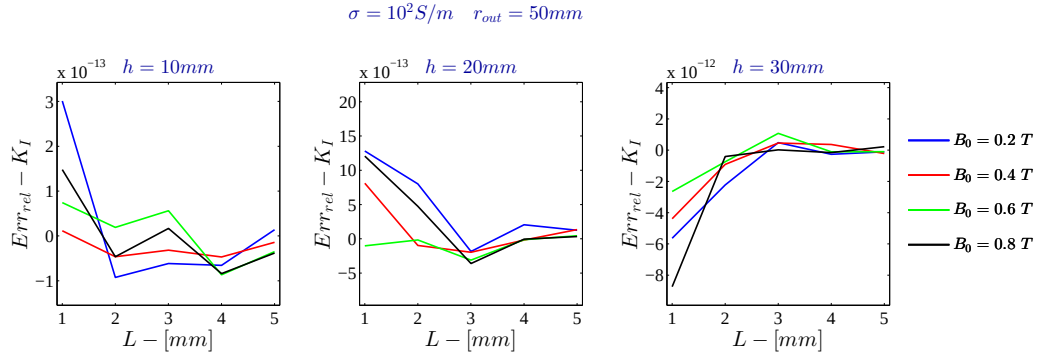
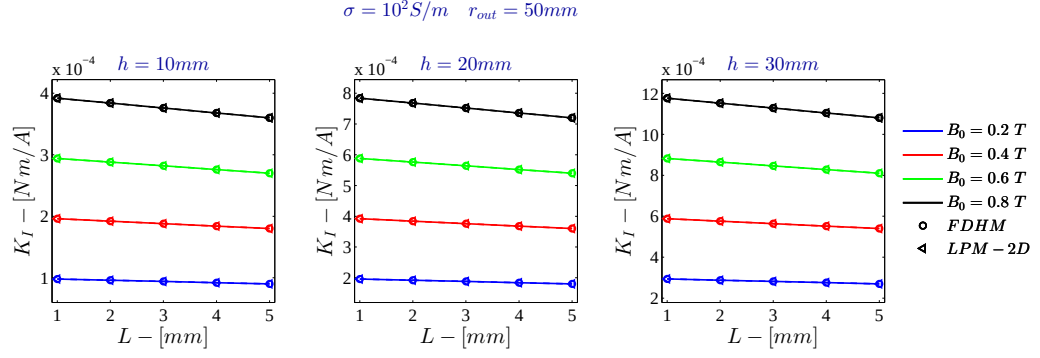
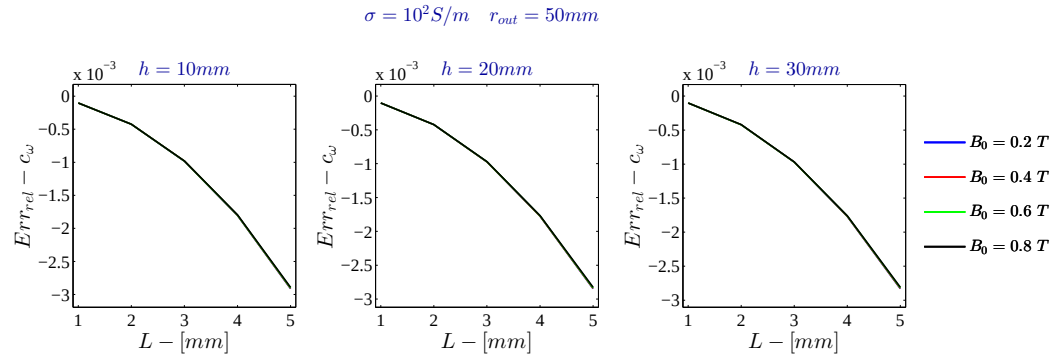
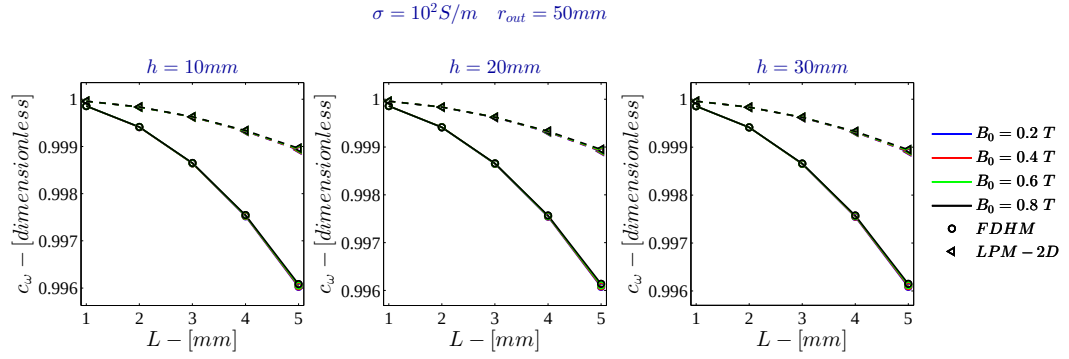


Figure 8.18: Relative Error - K_{vis} for $\sigma = 10^2 \text{ S/m}$ - Bidimensional LPM

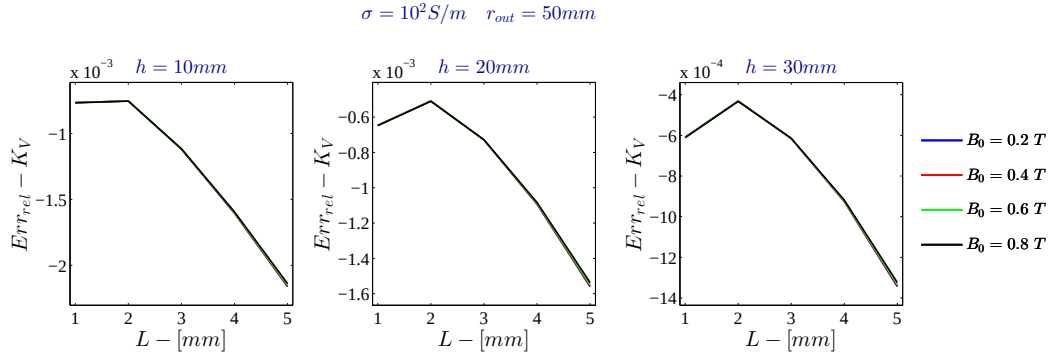
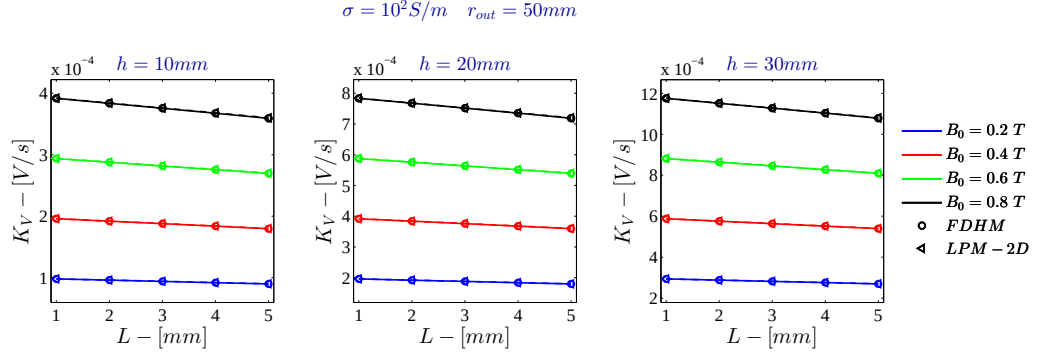
8.2.2 Coefficient of the Lorentz Force moment - K_I



8.2.3 Coefficient of the moment of inertia - c_ω



8.2.4 Coefficient of Induced Voltage - K_V



8.2.5 Torque - $\dot{\Gamma}$

8.2.5.1 Current Drive

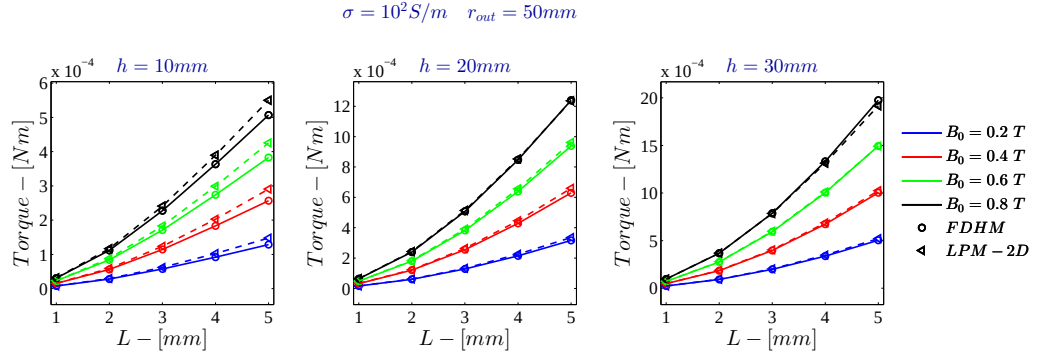


Figure 8.25: $\dot{\Gamma}$ for $\sigma = 10^2 \text{ S/m}$ - Bidimensional LPM

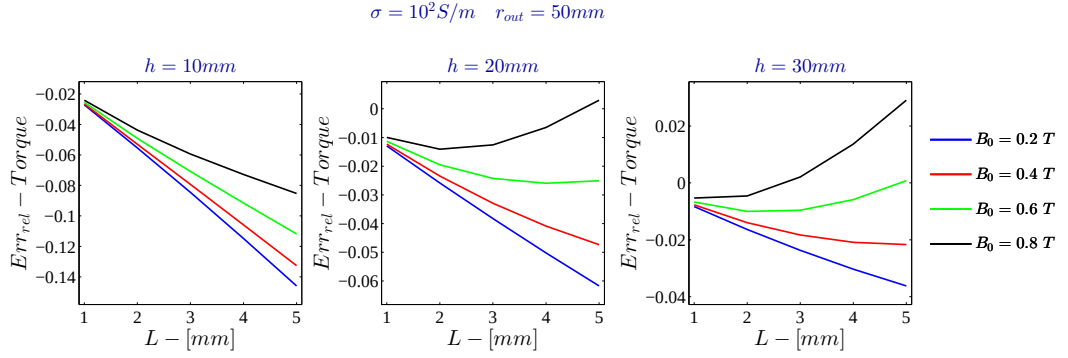


Figure 8.26: Relative Error - $\dot{\Gamma}$ for $\sigma = 10^2 \text{ S/m}$ - Bidimensional LPM

8.2.5.2 Voltage Drive

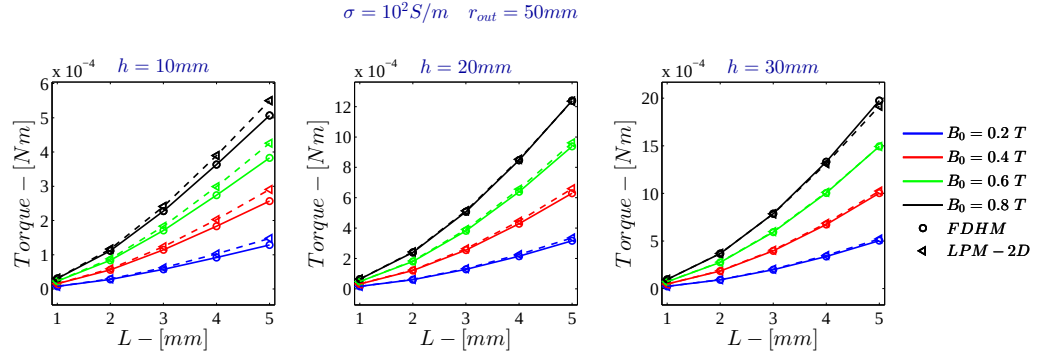


Figure 8.27: $\dot{\Gamma}$ for $\sigma = 10^2 \text{ S/m}$ - Bidimensional LPM

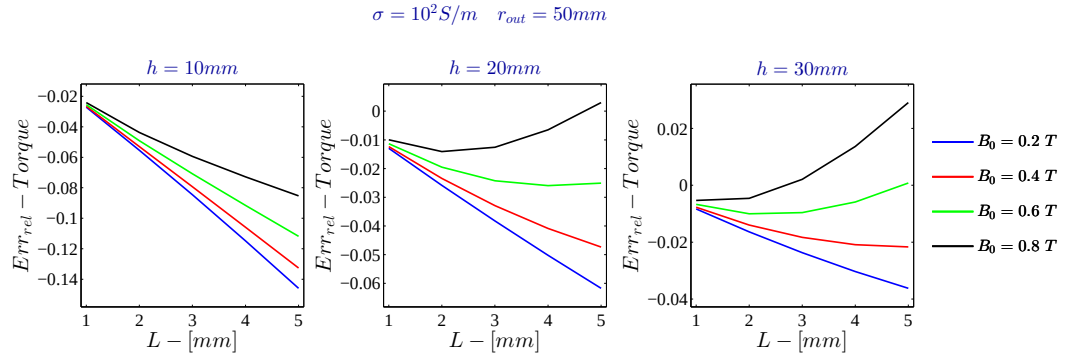


Figure 8.28: Relative Error - $\dot{\Gamma}$ for $\sigma = 10^2 \text{ S/m}$ - Bidimensional LPM

8.2.6 Characteristic Time - τ_c

8.2.6.1 Current Drive

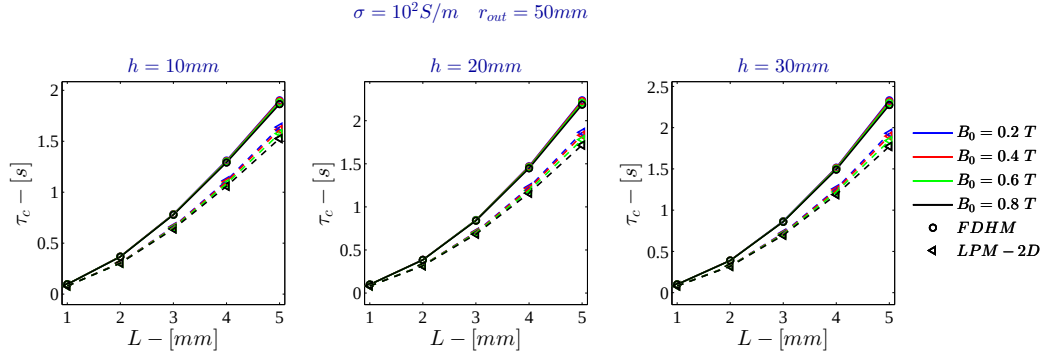


Figure 8.29: τ_c for $\sigma = 10^2 \text{ S/m}$ - Bidimensional LPM

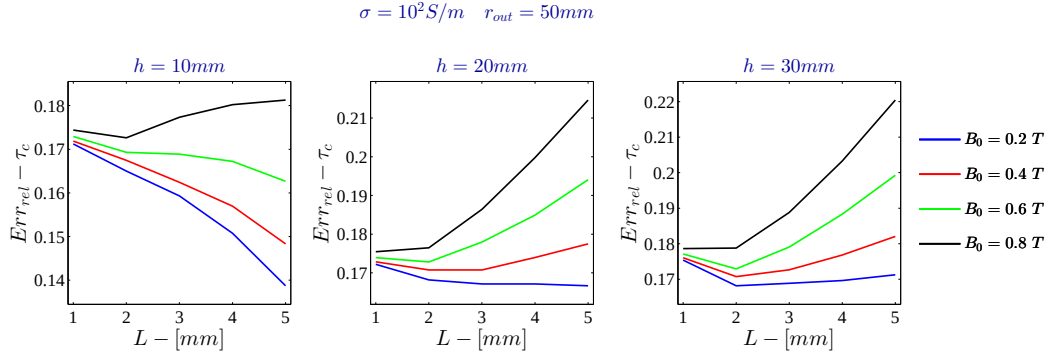


Figure 8.30: Relative Error - τ_c for $\sigma = 10^2 \text{ S/m}$ - Bidimensional LPM

8.2.6.2 Voltage Drive

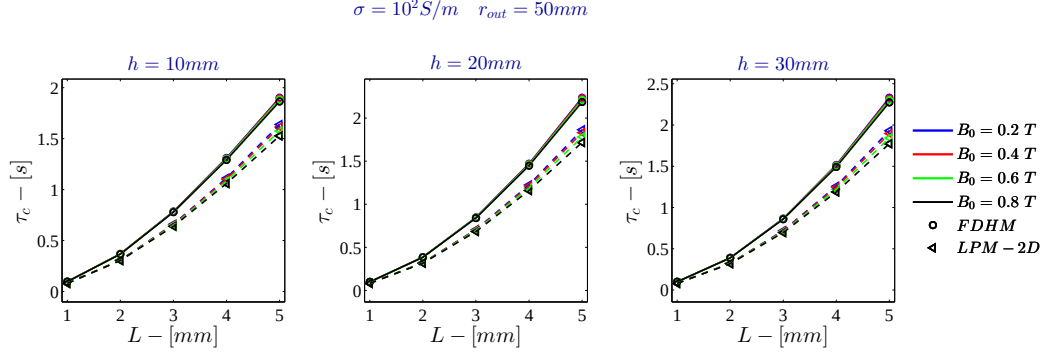


Figure 8.31: τ_c for $\sigma = 10^2 \text{ S/m}$ - Bidimensional LPM

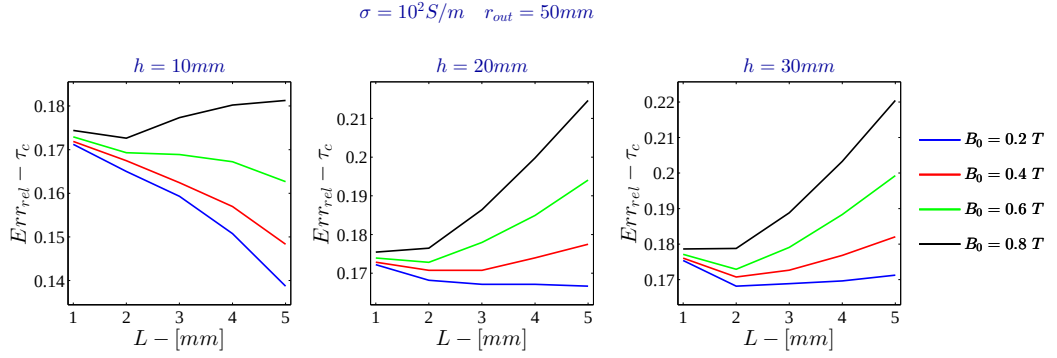


Figure 8.32: Relative Error - τ_c for $\sigma = 10^2 \text{ S/m}$ - Bidimensional LPM

The Crank-Nicolson Method

The Crank-Nicolson method is a well-known finite difference method for the numerical integration of the heat equation and closely related partial differential equations.

It is a second-order method in time, it is implicit in time and can be written as an implicit Runge-Kutta method, and it is numerically stable. The method was developed by John Crank and Phyllis Nicolson in the mid 20th century. Considerind a 1D partial differential equation:

$$\frac{\partial u}{\partial t} = F\left(u, x, \frac{\partial u}{\partial x}, \frac{\partial^2 u}{\partial x^2}\right) \quad (\text{A.0.1})$$

the time derivative on the left hand side can be replaced with a forward finite difference:

$$\left. \frac{\partial u}{\partial t} \right|_{j,n} \approx \frac{u_j^{n+1} - u_j^n}{\Delta t}. \quad (\text{A.0.2})$$

The Crank-Nicolson method appears to be a combination of the forwards and backwards Euler method at two different steps in time, n an $n + 1$.

$$\frac{u_i^{n+1} - u_i^n}{\Delta t} = \frac{1}{2} \left[F_i^{n+1} \left(u, x, \frac{\partial u}{\partial x}, \frac{\partial^2 u}{\partial x^2} \right) + F_i^n \left(u, x, \frac{\partial u}{\partial x}, \frac{\partial^2 u}{\partial x^2} \right) \right] \quad (\text{A.0.3})$$

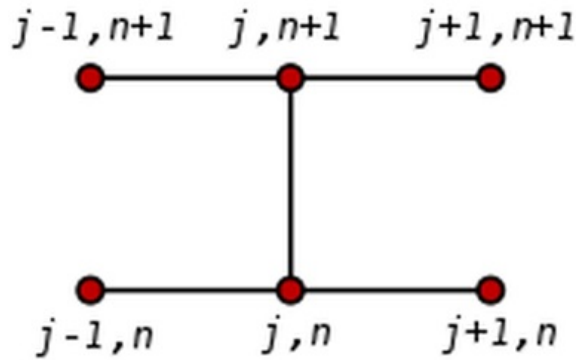


Figure A.1: *Crank-Nicolson stencil for a 1D problem*

As the quantities defined at the time step $n+1$ are unknown, in order to update the solution at the time $n+1$, an algebraic system must be solved:

$$u^{n+1} = Au^n + B \quad (\text{A.0.4})$$

The shape of the matrix A depends on the studied partial differential equation and on the chosen finite difference scheme used to approximate the spatial part of the partial differential equation. The vector B is not null if the source term of the differential equation is not equal to zero.

Comparison between FDHM and LPM: $\sigma = 10^2 \text{ S/m}$

B.1 Monodimensional LPM

B.1.1 Coefficient of the viscous shear moment - K_{vis}

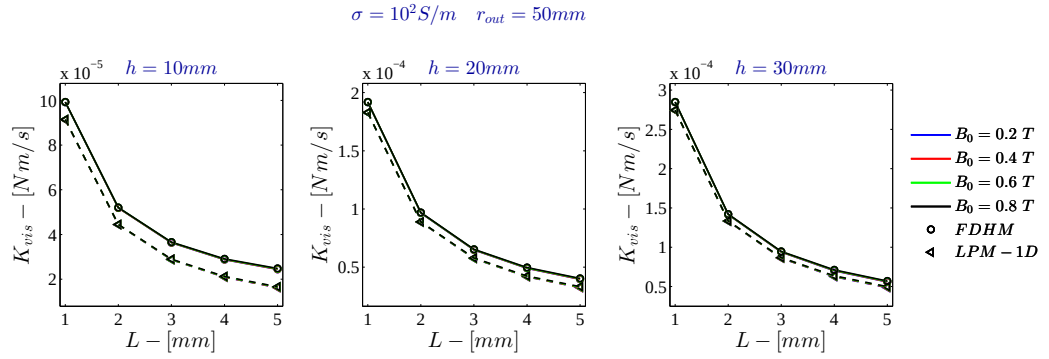


Figure B.1: K_{vis} for $\sigma = 10^2 \text{ S/m}$ - Monodimensional LPM

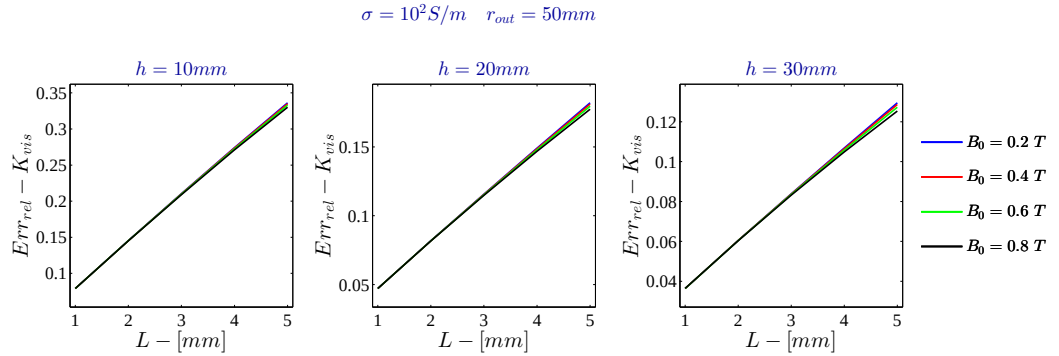


Figure B.2: Relative Error - K_{vis} for $\sigma = 10^2 \text{ S/m}$ - Monodimensional LPM

B.1.2 Coefficient of the Lorentz Force moment - K_I

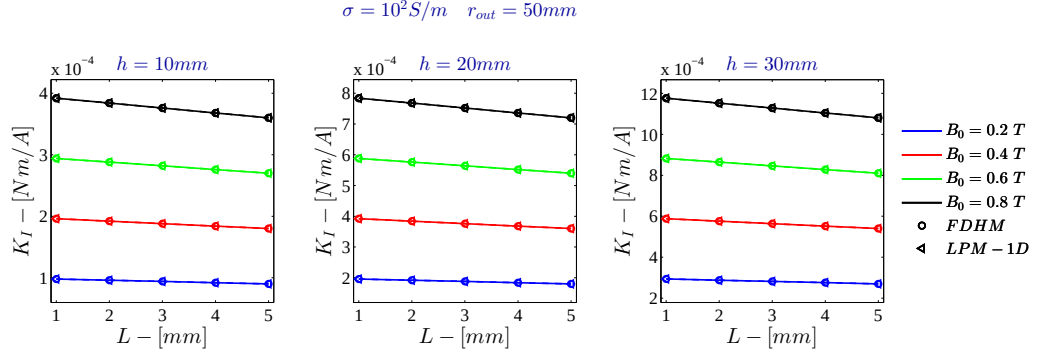


Figure B.3: K_I for $\sigma = 10^2 \text{ S/m}$ - Monodimensional LPM

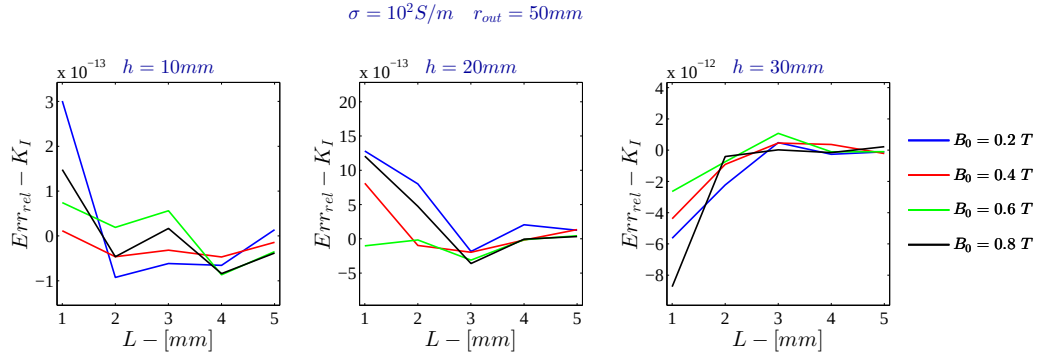
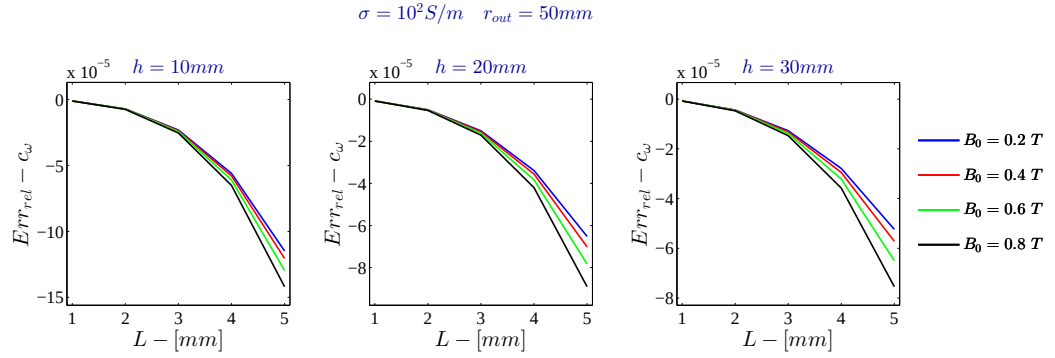
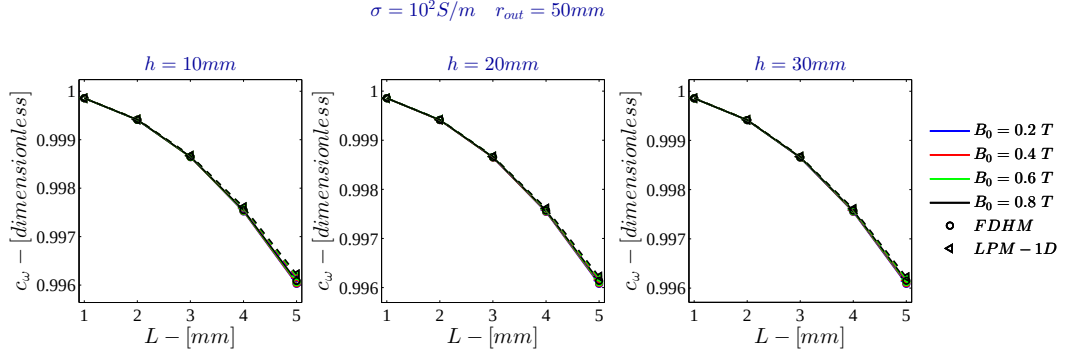


Figure B.4: Relative Error - K_I for $\sigma = 10^2 \text{ S/m}$ - Monodimensional LPM

B.1.3 Coefficient of the moment of inertia - c_ω



B.1.4 Coefficient of Induced Voltage - K_V

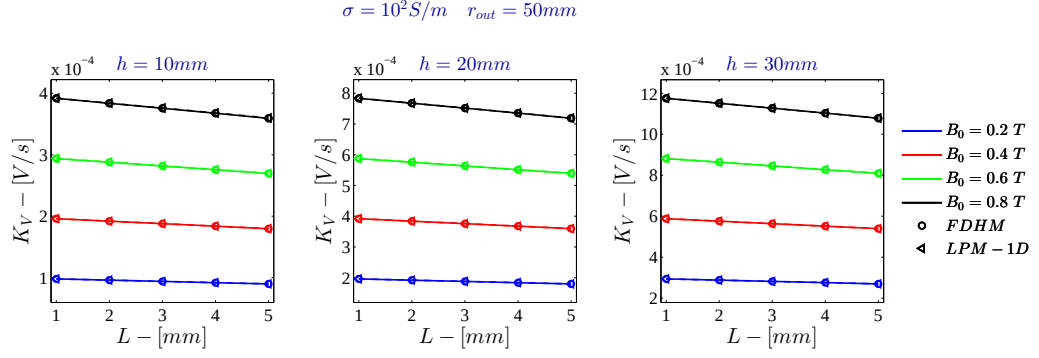


Figure B.7: K_V for $\sigma = 10^2 \text{ S/m}$ - Monodimensional LPM

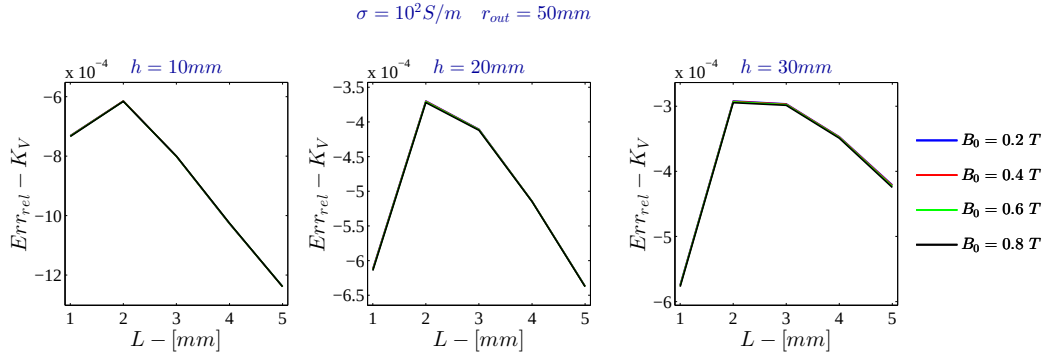


Figure B.8: Relative Error - K_V for $\sigma = 10^2 \text{ S/m}$ - Monodimensional LPM

B.1.5 Torque - $\dot{\Gamma}$

B.1.5.1 Current Drive

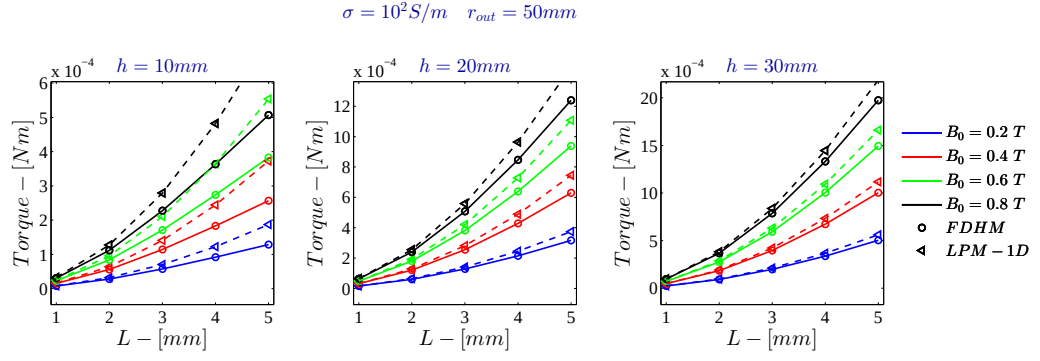


Figure B.9: $\dot{\Gamma}$ for $\sigma = 10^2 \text{ S/m}$ - Monodimensional LPM

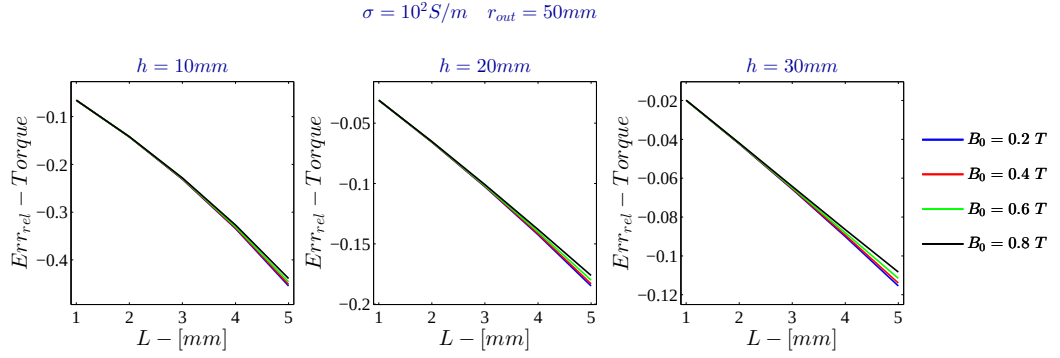


Figure B.10: Relative Error - $\dot{\Gamma}$ for $\sigma = 10^2 \text{ S/m}$ - Monodimensional LPM

B.1.5.2 Voltage Drive

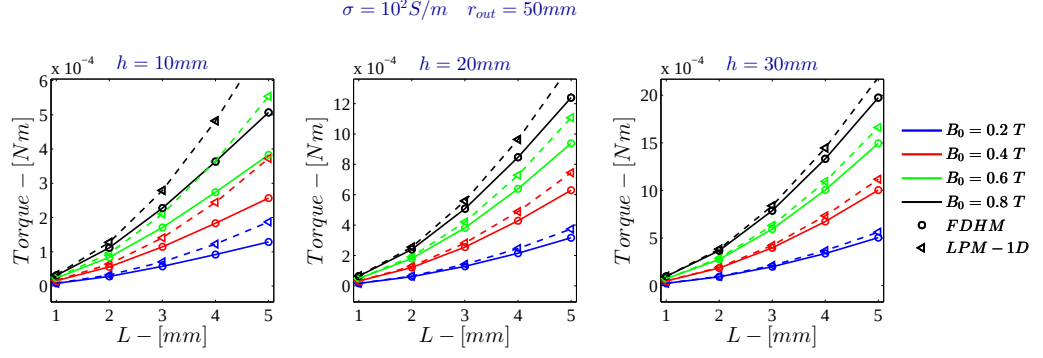


Figure B.11: $\dot{\Gamma}$ for $\sigma = 10^2 \text{ S/m}$ - Monodimensional LPM

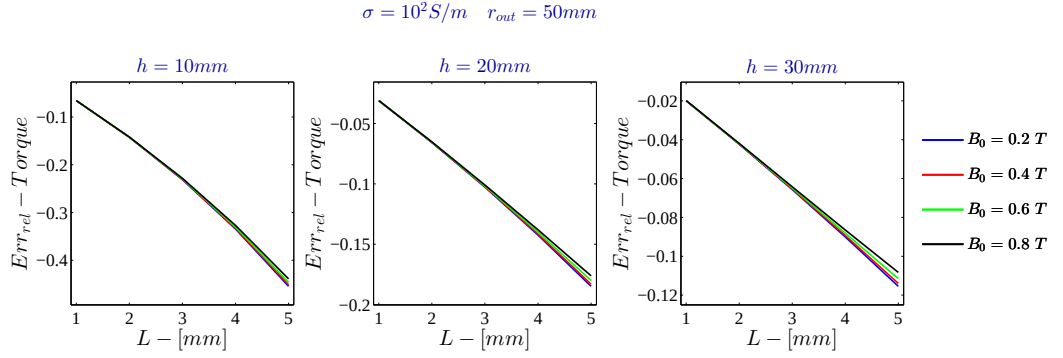


Figure B.12: Relative Error - $\dot{\Gamma}$ for $\sigma = 10^2 \text{ S/m}$ - Monodimensional LPM

B.1.6 Characteristic Time - τ_c

B.1.6.1 Current Drive

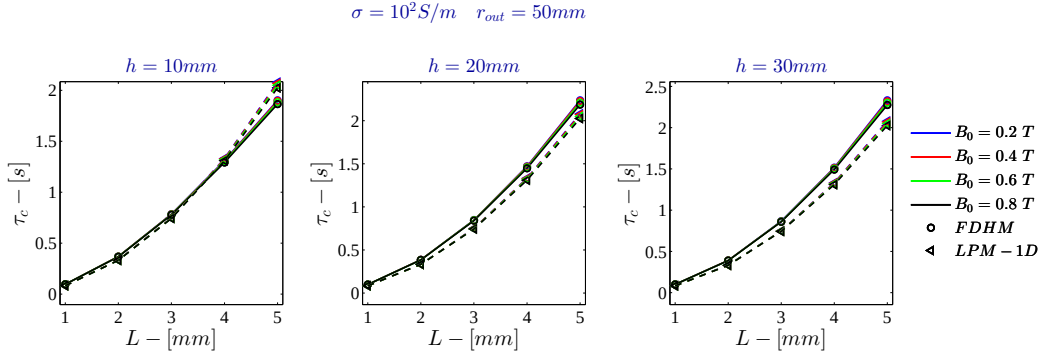


Figure B.13: τ_c for $\sigma = 10^2 \text{ S/m}$ - Monodimensional LPM

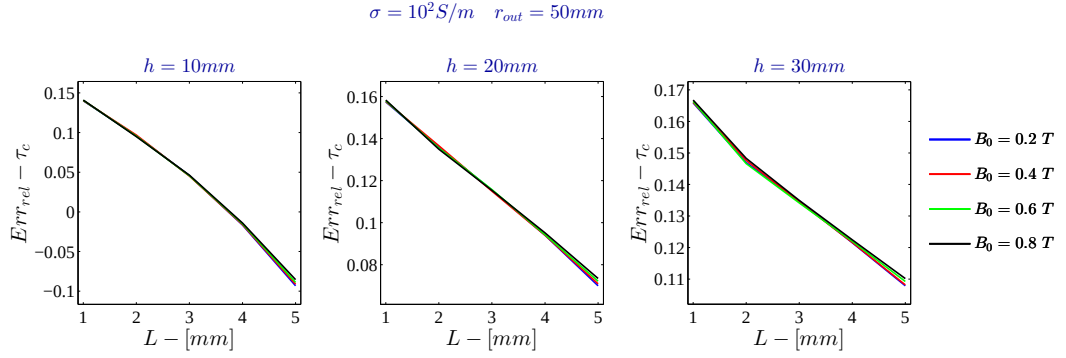


Figure B.14: Relative Error - τ_c for $\sigma = 10^2 \text{ S/m}$ - Monodimensional LPM

B.1.6.2 Voltage Drive

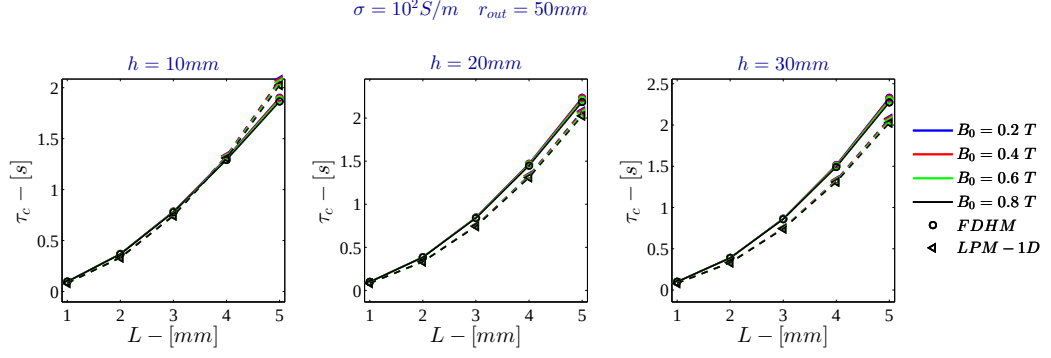


Figure B.15: τ_c for $\sigma = 10^2 \text{ S/m}$ - Monodimensional LPM

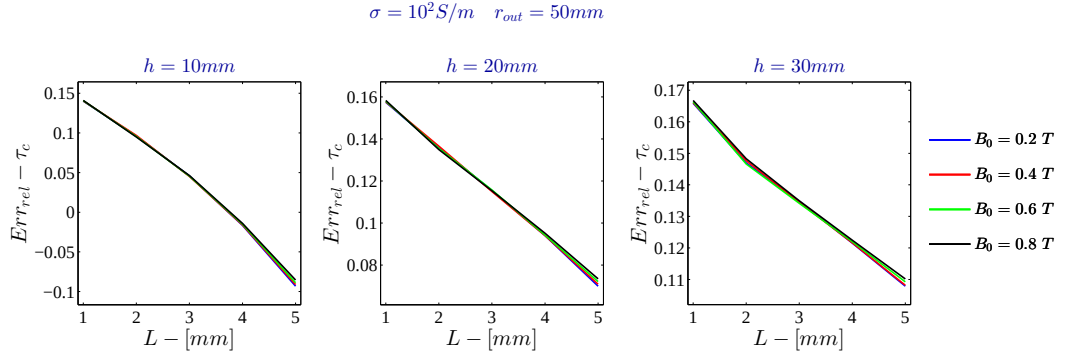


Figure B.16: Relative Error - τ_c for $\sigma = 10^2 \text{ S/m}$ - Monodimensional LPM

B.2 Bidimensional LPM

B.2.1 Coefficient of the viscous shear moment - K_{vis}

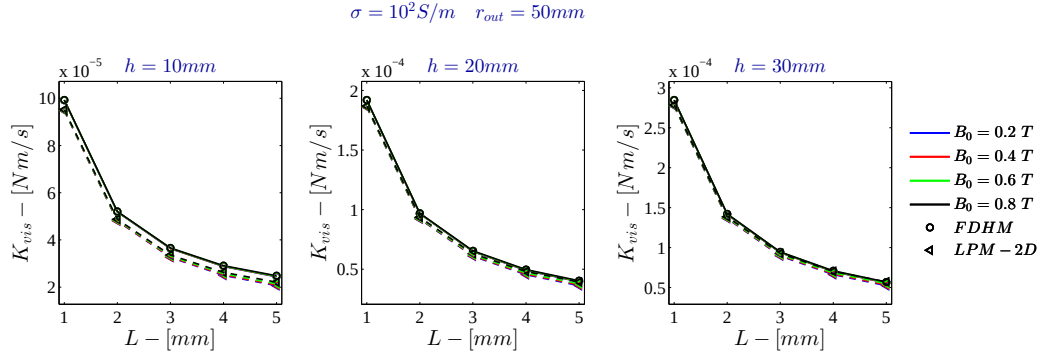


Figure B.17: K_{vis} for $\sigma = 10^2 \text{ S/m}$ - Bidimensional LPM

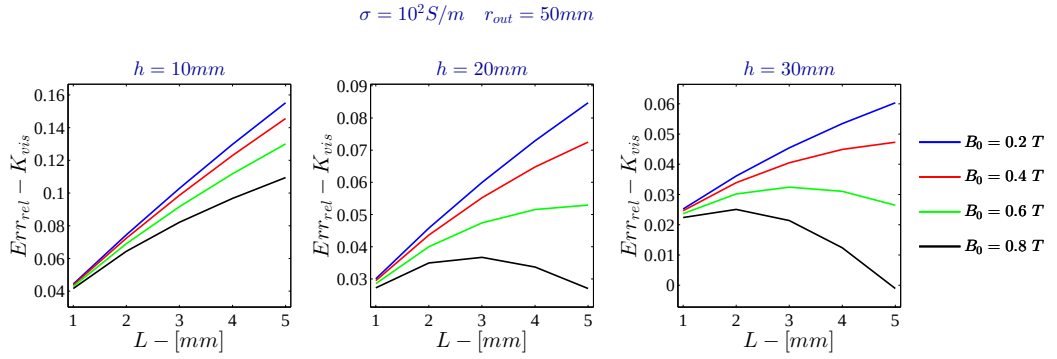
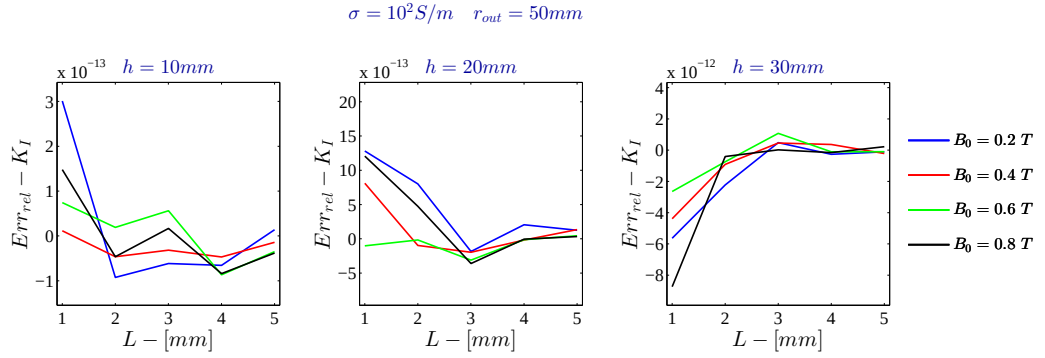
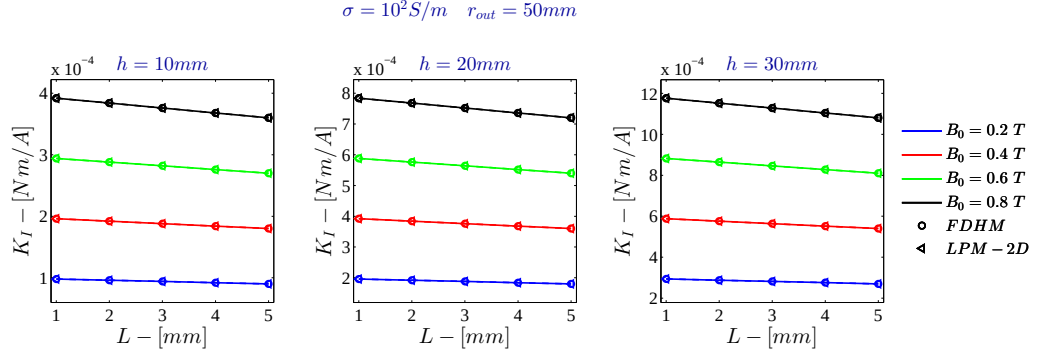
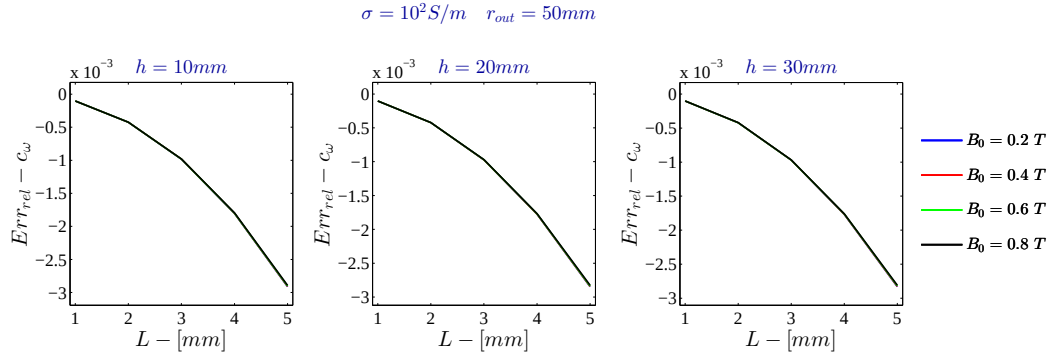
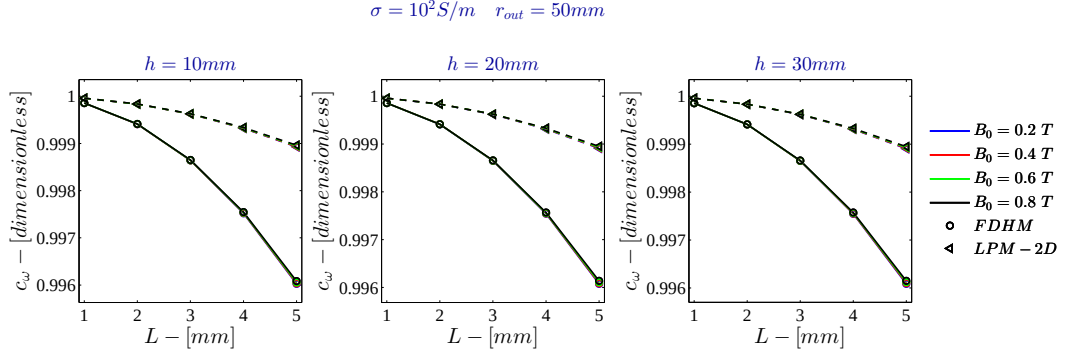


Figure B.18: Relative Error - K_{vis} for $\sigma = 10^2 \text{ S/m}$ - Bidimensional LPM

B.2.2 Coefficient of the Lorentz Force moment - K_I



B.2.3 Coefficient of the moment of inertia - c_ω



B.2.4 Coefficient of Induced Voltage - K_V

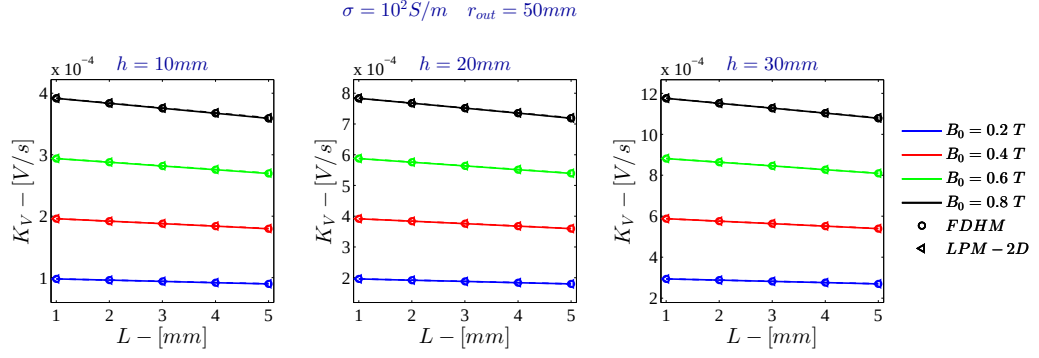


Figure B.23: K_V for $\sigma = 10^2 \text{ S/m}$ - Bidimensional LPM

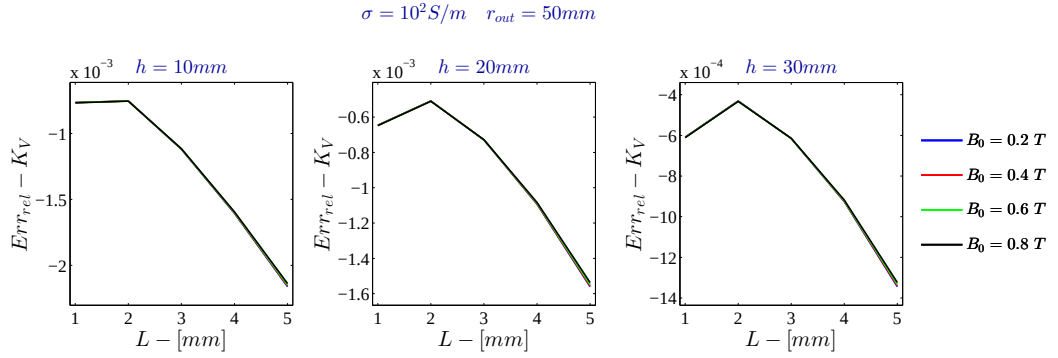


Figure B.24: Relative Error - K_I for $\sigma = 10^2 \text{ S/m}$ - Bidimensional LPM

B.2.5 Torque - $\dot{\Gamma}$

B.2.5.1 Current Drive

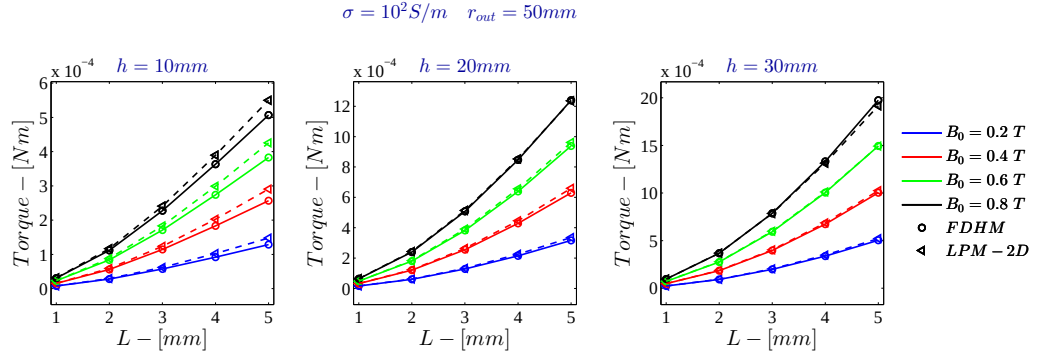


Figure B.25: $\dot{\Gamma}$ for $\sigma = 10^2 \text{ S/m}$ - Bidimensional LPM

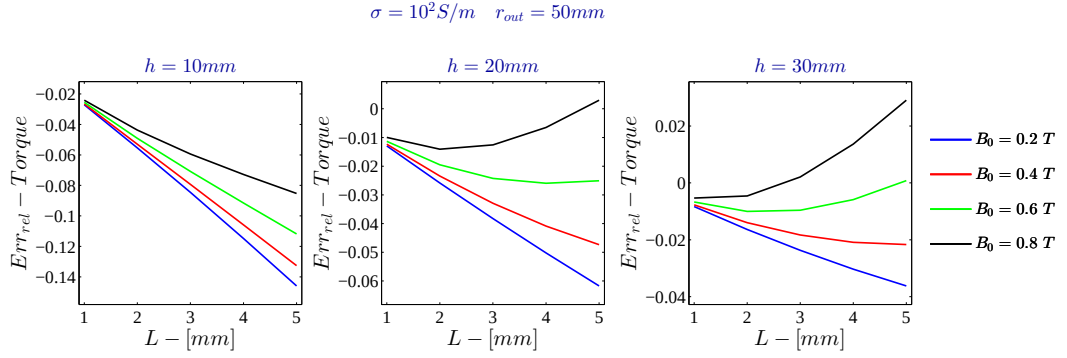


Figure B.26: Relative Error - $\dot{\Gamma}$ for $\sigma = 10^2 \text{ S/m}$ - Bidimensional LPM

B.2.5.2 Voltage Drive

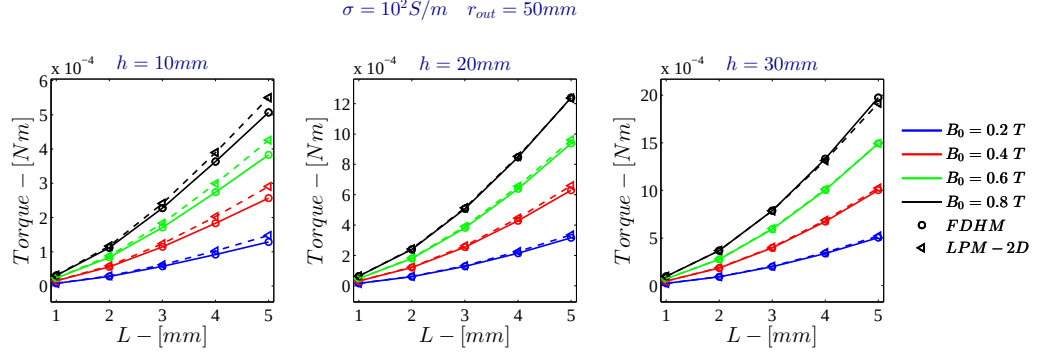


Figure B.27: $\dot{\Gamma}$ for $\sigma = 10^2 \text{ S/m}$ - Bidimensional LPM

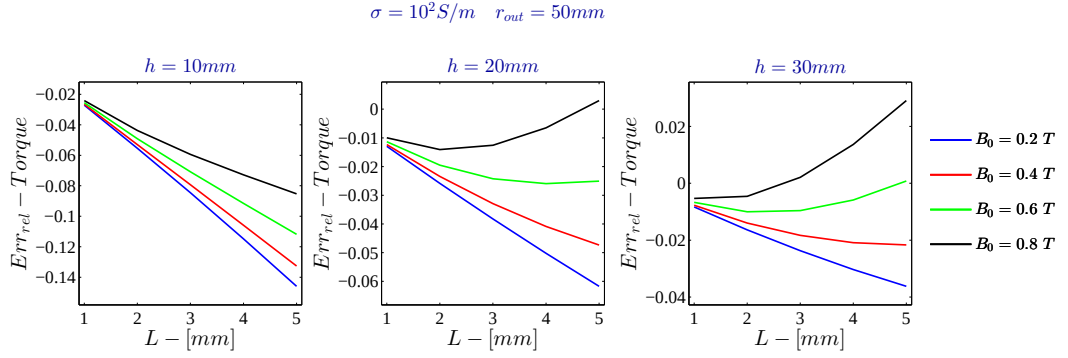


Figure B.28: Relative Error - $\dot{\Gamma}$ for $\sigma = 10^2 \text{ S/m}$ - Bidimensional LPM

B.2.6 Characteristic Time - τ_c

B.2.6.1 Current Drive

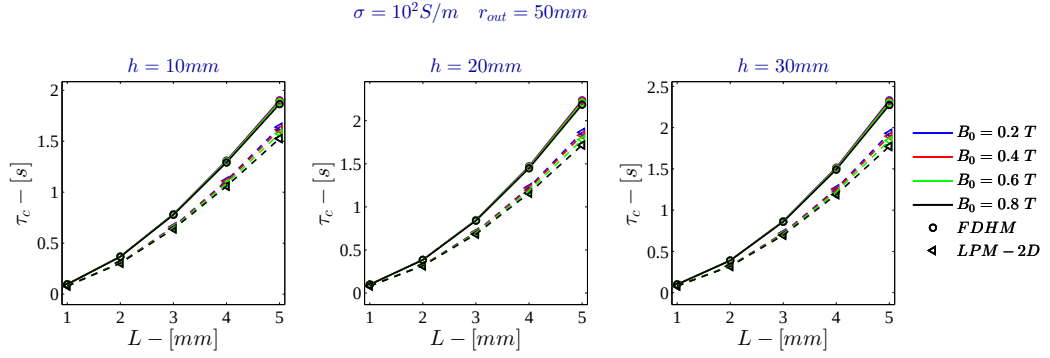


Figure B.29: τ_c for $\sigma = 10^2 \text{ S/m}$ - Bidimensional LPM

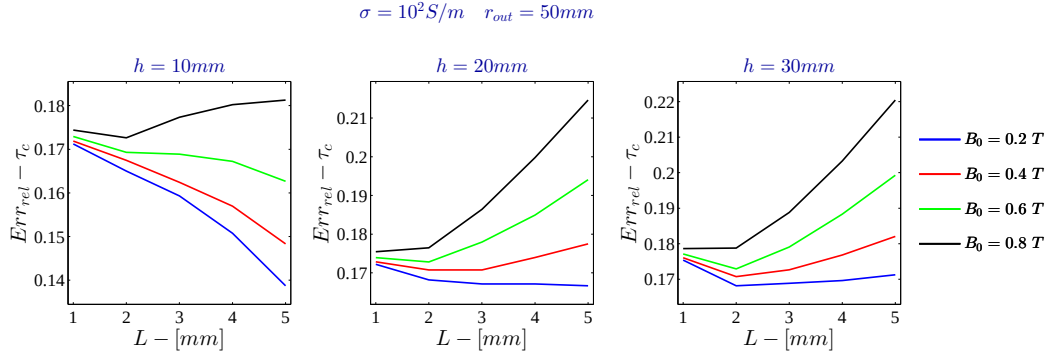


Figure B.30: Relative Error - τ_c for $\sigma = 10^2 \text{ S/m}$ - Bidimensional LPM

B.2.6.2 Voltage Drive

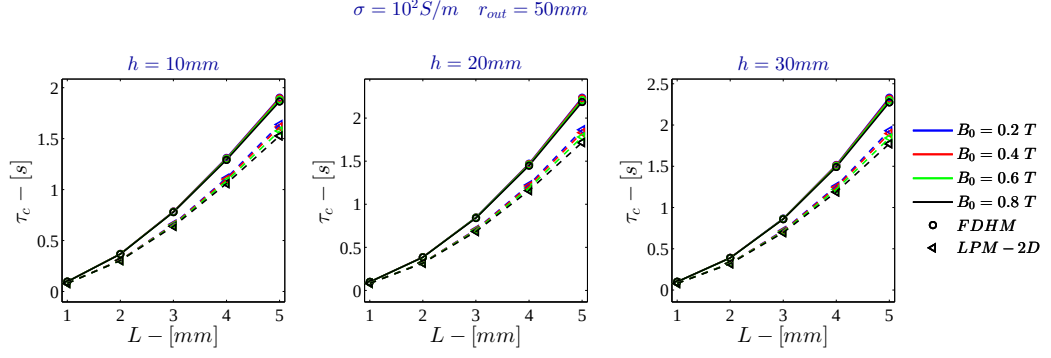


Figure B.31: τ_c for $\sigma = 10^2 \text{ S/m}$ - Bidimensional LPM

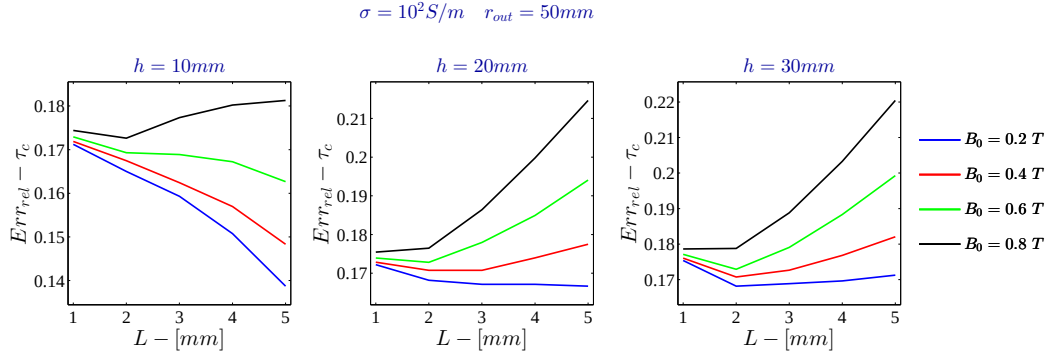


Figure B.32: Relative Error - τ_c for $\sigma = 10^2 \text{ S/m}$ - Bidimensional LPM

Comparison between FDHM and LPM: $\sigma = 10^3 S/m$

C.1 Monodimensional LPM

C.1.1 Coefficient of the viscous shear moment - K_{vis}

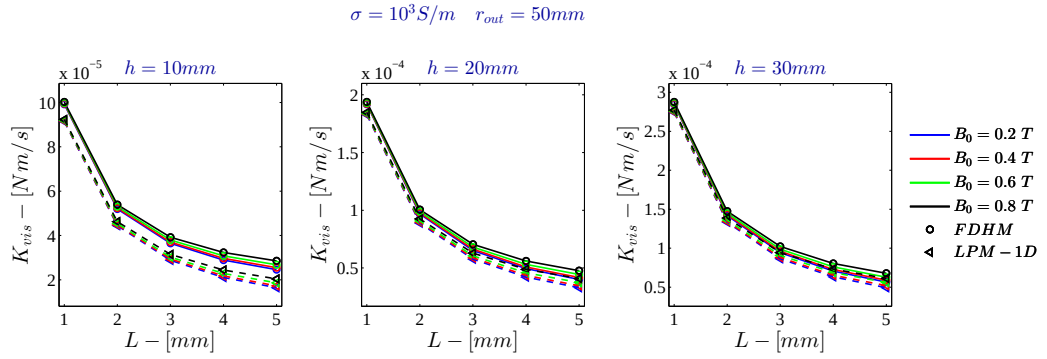


Figure C.1: K_{vis} for $\sigma = 10^3 S/m$ - Monodimensional LPM

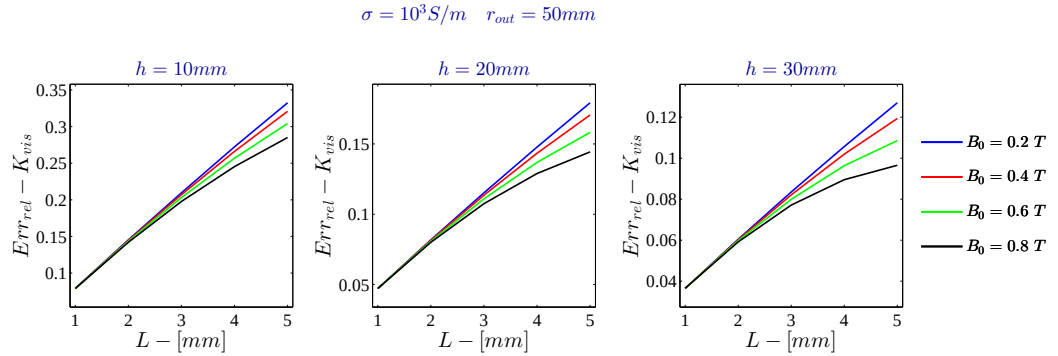


Figure C.2: Relative Error - K_{vis} for $\sigma = 10^3 S/m$ - Monodimensional LPM

C.1.2 Coefficient of the Lorentz Force moment - K_I

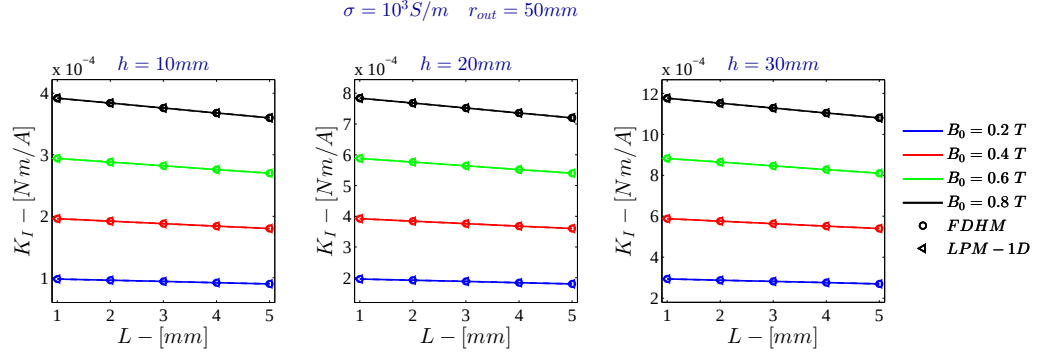


Figure C.3: K_I for $\sigma = 10^3 S/m$ - Monodimensional LPM

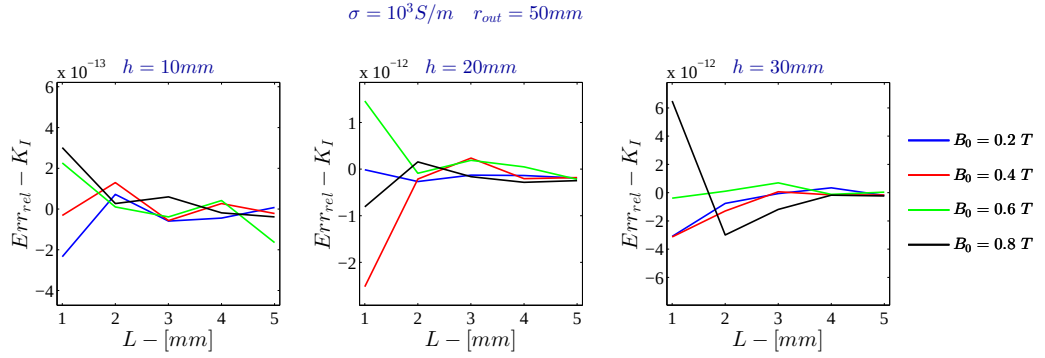
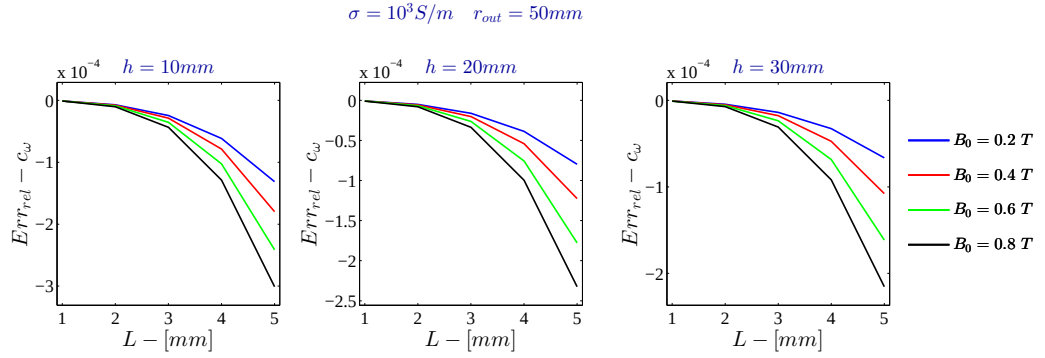
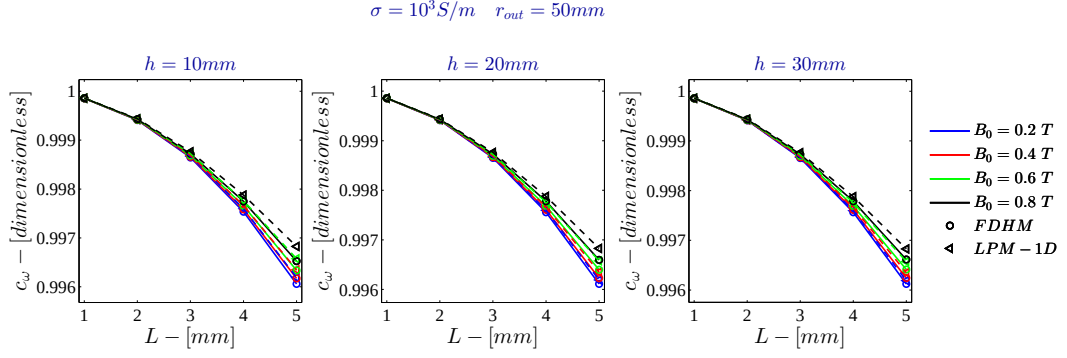


Figure C.4: Relative Error - K_I for $\sigma = 10^3 S/m$ - Monodimensional LPM

C.1.3 Coefficient of the moment of inertia - c_ω



C.1.4 Coefficient of Induced Voltage - K_V

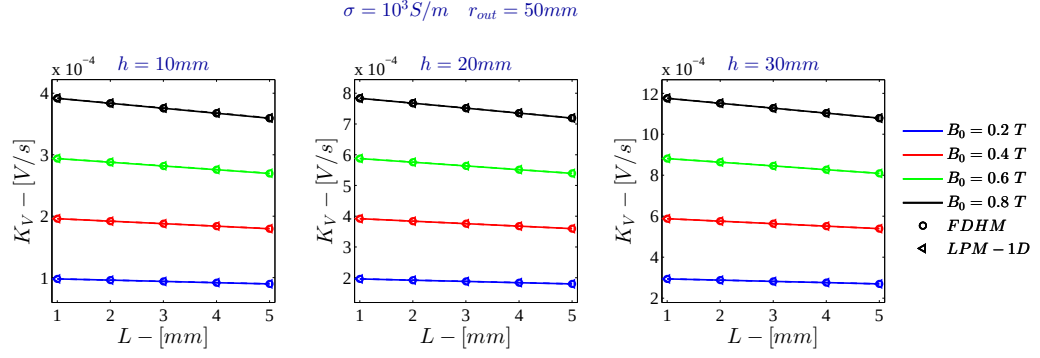


Figure C.7: K_V for $\sigma = 10^3 S/m$ - Monodimensional LPM

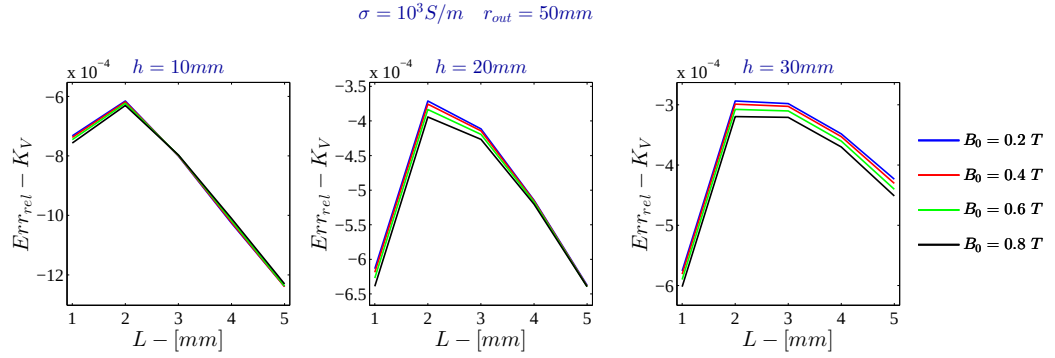


Figure C.8: Relative Error - K_V for $\sigma = 10^3 S/m$ - Monodimensional LPM

C.1.5 Torque - $\dot{\Gamma}$

C.1.5.1 Current Drive

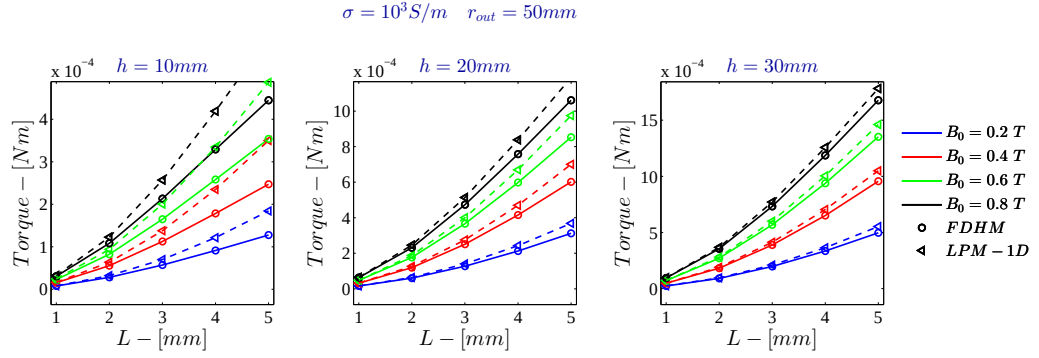


Figure C.9: $\dot{\Gamma}$ for $\sigma = 10^3 S/m$ - Monodimensional LPM

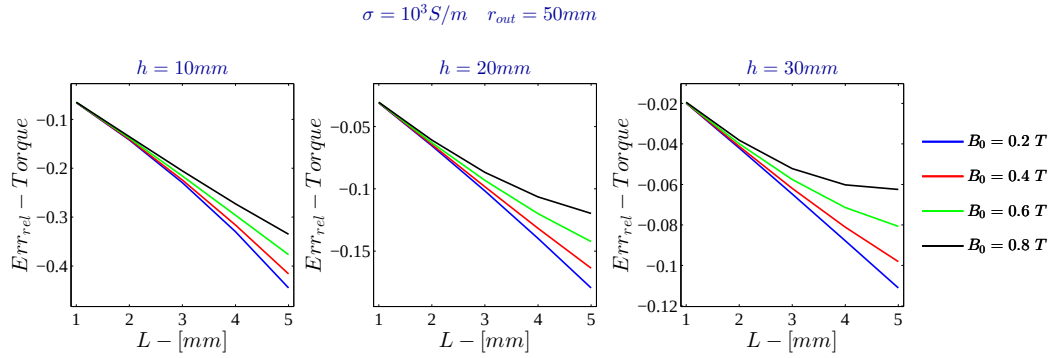


Figure C.10: Relative Error - $\dot{\Gamma}$ for $\sigma = 10^3 S/m$ - Monodimensional LPM

C.1.5.2 Voltage Drive

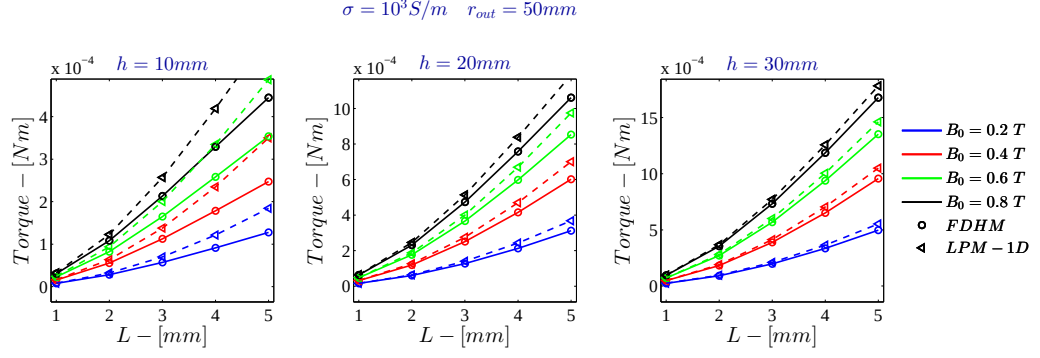


Figure C.11: $\dot{\Gamma}$ for $\sigma = 10^3 S/m$ - Monodimensional LPM

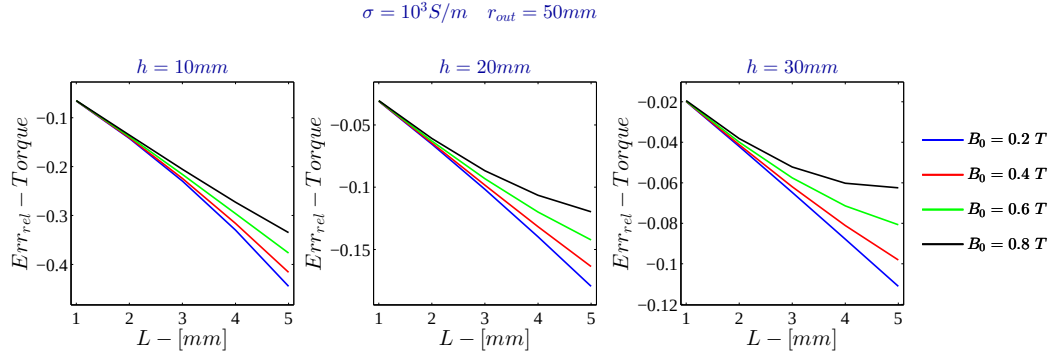


Figure C.12: Relative Error - $\dot{\Gamma}$ for $\sigma = 10^3 S/m$ - Monodimensional LPM

C.1.6 Characteristic Time - τ_c

C.1.6.1 Current Drive

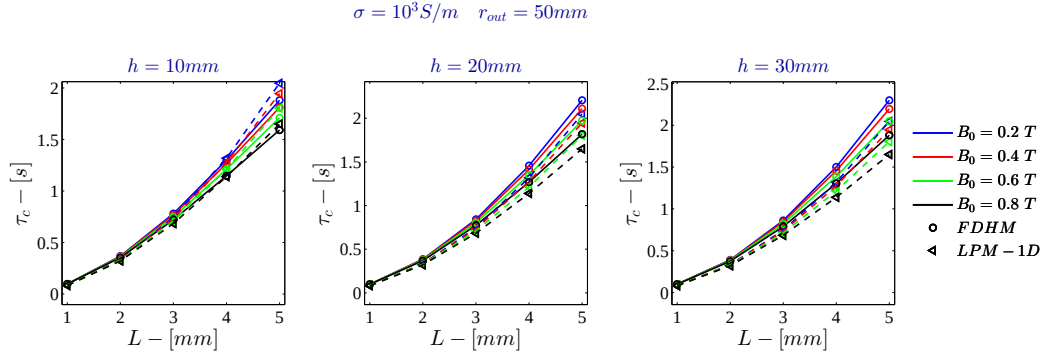


Figure C.13: τ_c for $\sigma = 10^3 S/m$ - Monodimensional LPM

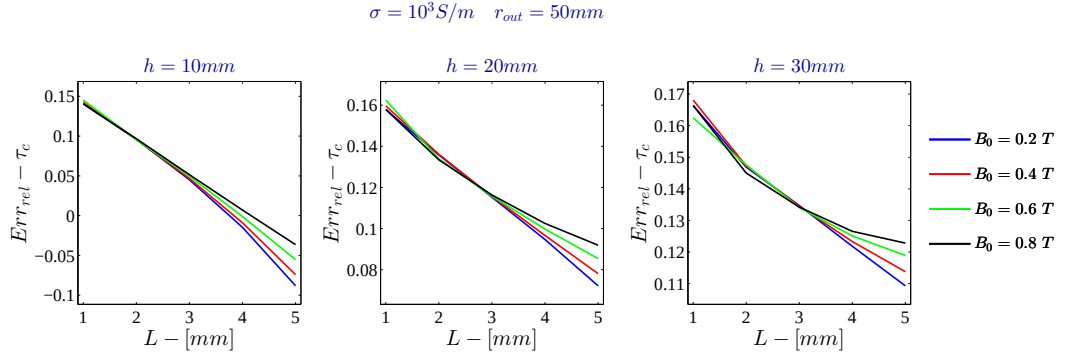


Figure C.14: Relative Error - τ_c for $\sigma = 10^3 S/m$ - Monodimensional LPM

C.1.6.2 Voltage Drive

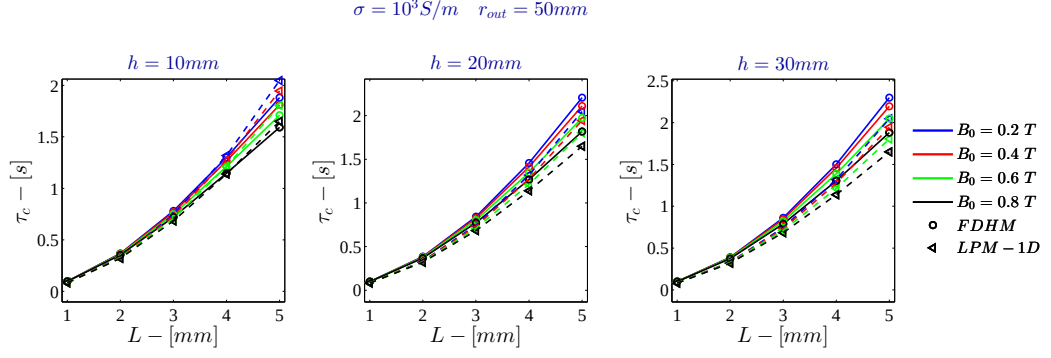


Figure C.15: τ_c for $\sigma = 10^3 S/m$ - Monodimensional LPM

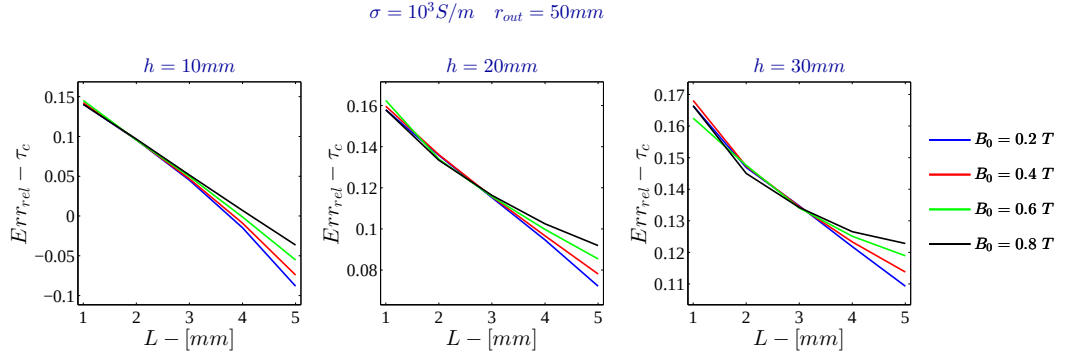


Figure C.16: Relative Error - τ_c for $\sigma = 10^3 S/m$ - Monodimensional LPM

C.2 Bidimensional LPM

C.2.1 Coefficient of the viscous shear moment - K_{vis}

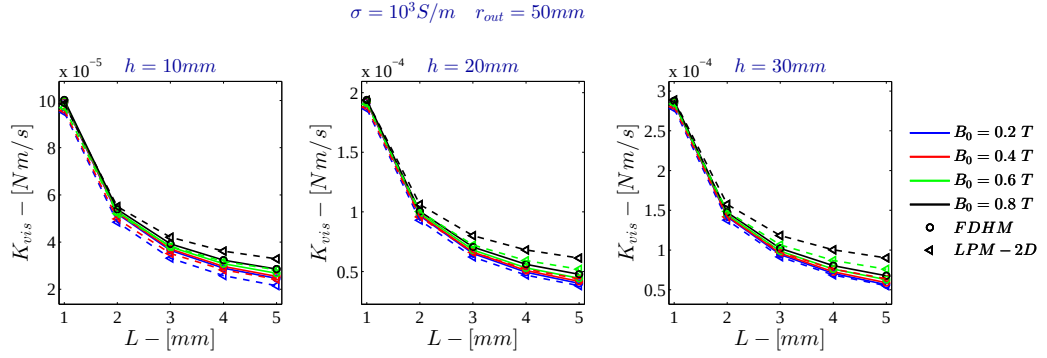


Figure C.17: K_{vis} for $\sigma = 10^3 S/m$ - Bidimensional LPM

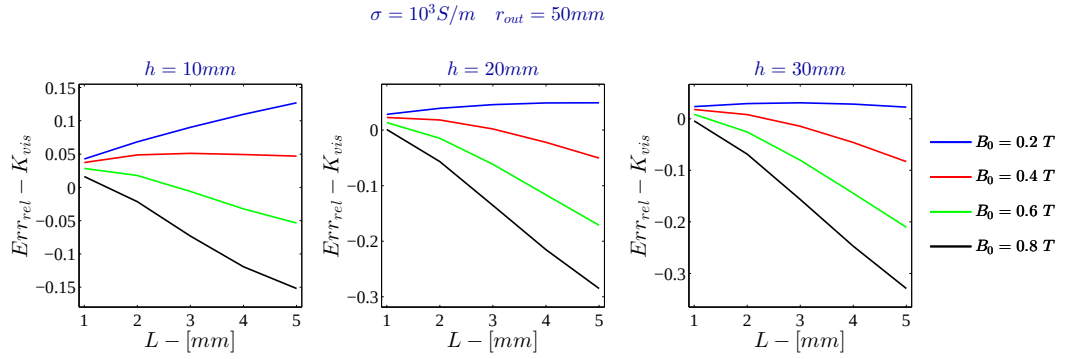


Figure C.18: Relative Error - K_{vis} for $\sigma = 10^3 S/m$ - Bidimensional LPM

C.2.2 Coefficient of the Lorentz Force moment - K_I

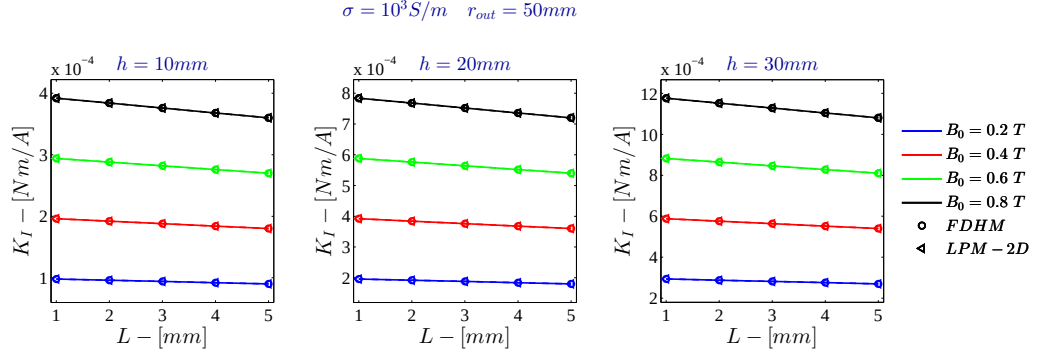


Figure C.19: K_I for $\sigma = 10^3 S/m$ - Bidimensional LPM

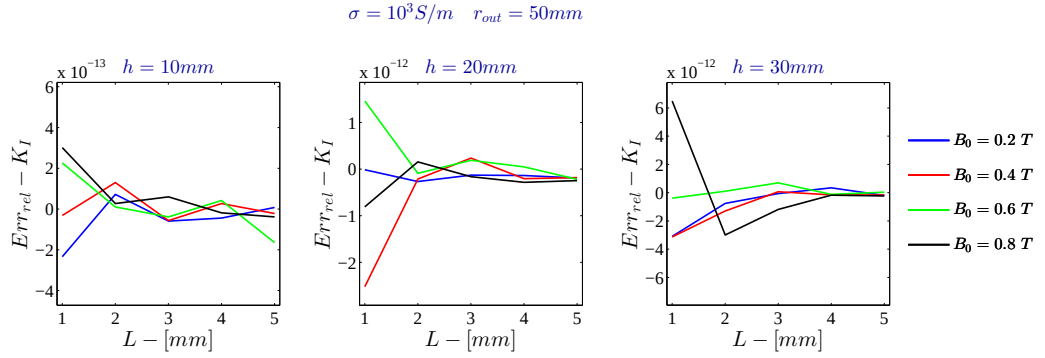
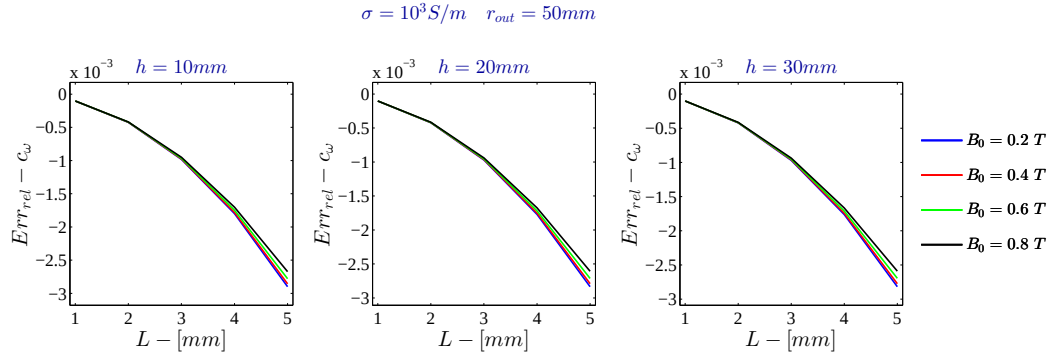
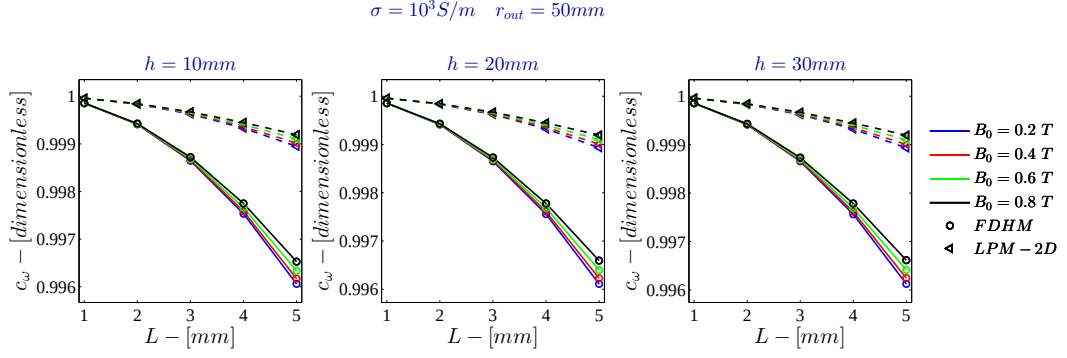


Figure C.20: Relative Error - K_I for $\sigma = 10^3 S/m$ - Bidimensional LPM

C.2.3 Coefficient of the moment of inertia - c_ω



C.2.4 Coefficient of Induced Voltage - K_V

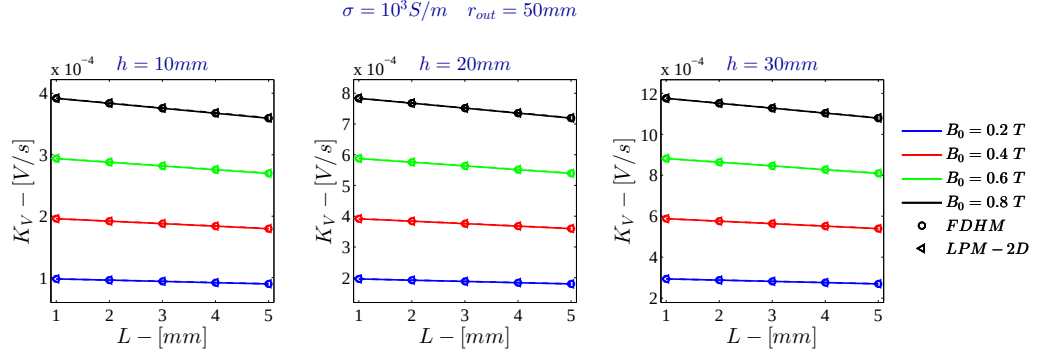


Figure C.23: K_V for $\sigma = 10^3 S/m$ - Bidimensional LPM

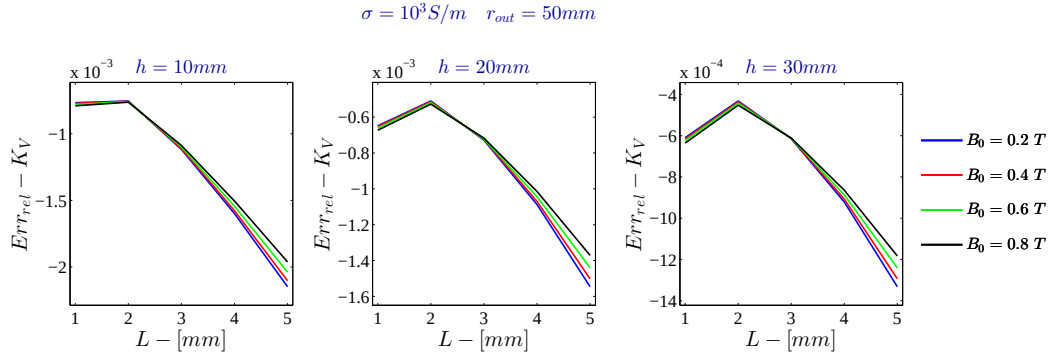


Figure C.24: Relative Error - K_V for $\sigma = 10^3 S/m$ - Bidimensional LPM

C.2.5 Torque - $\dot{\Gamma}$

C.2.5.1 Current Drive

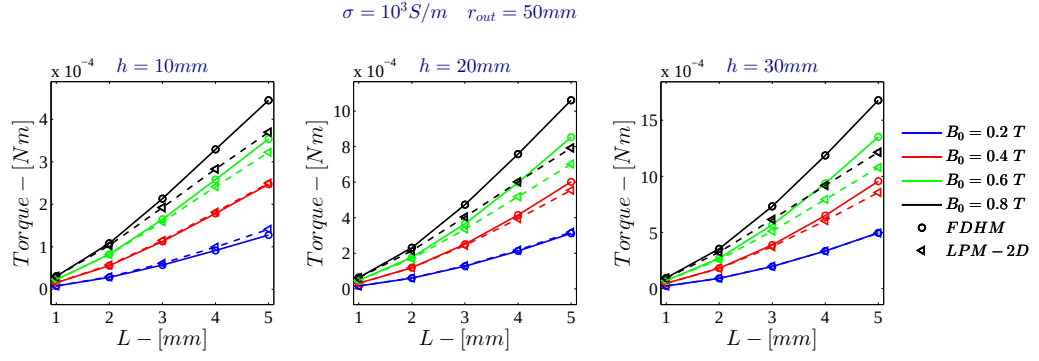


Figure C.25: $\dot{\Gamma}$ for $\sigma = 10^3 S/m$ - Bidimensional LPM

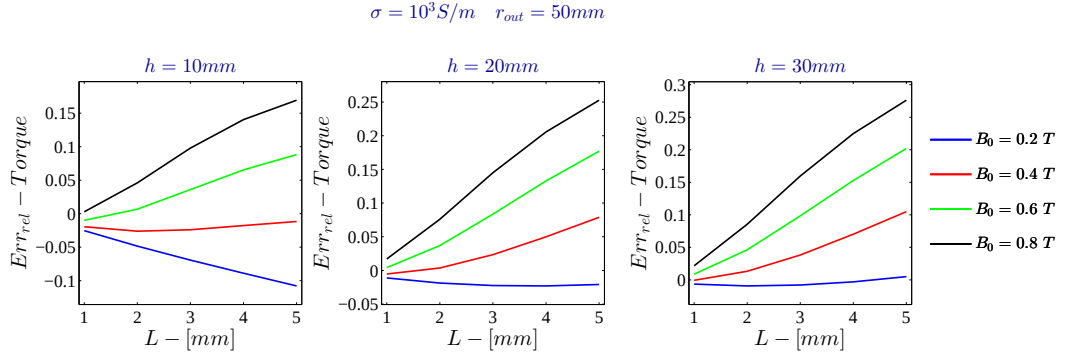


Figure C.26: Relative Error - $\dot{\Gamma}$ for $\sigma = 10^3 S/m$ - Bidimensional LPM

C.2.5.2 Voltage Drive

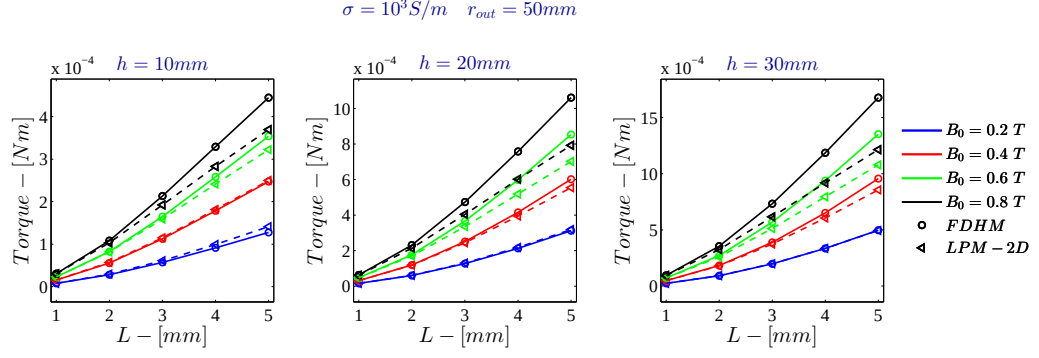


Figure C.27: $\dot{\Gamma}$ for $\sigma = 10^3 S/m$ - Bidimensional LPM

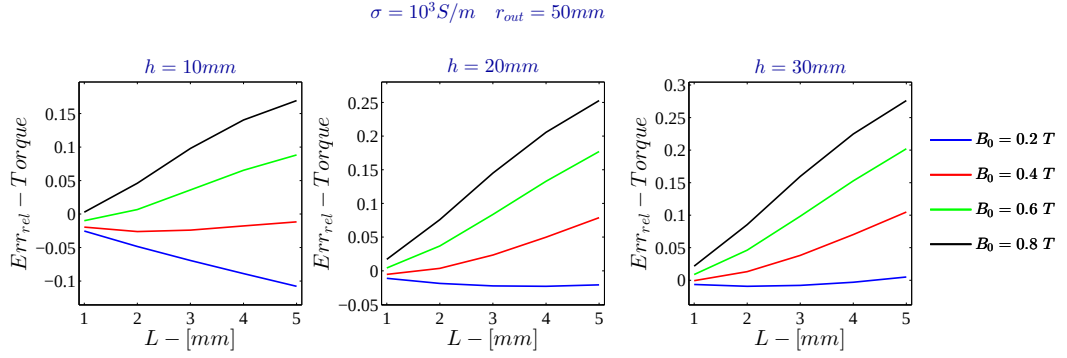


Figure C.28: Relative Error - $\dot{\Gamma}$ for $\sigma = 10^3 S/m$ - Bidimensional LPM

C.2.6 Characteristic Time - τ_c

C.2.6.1 Current Drive

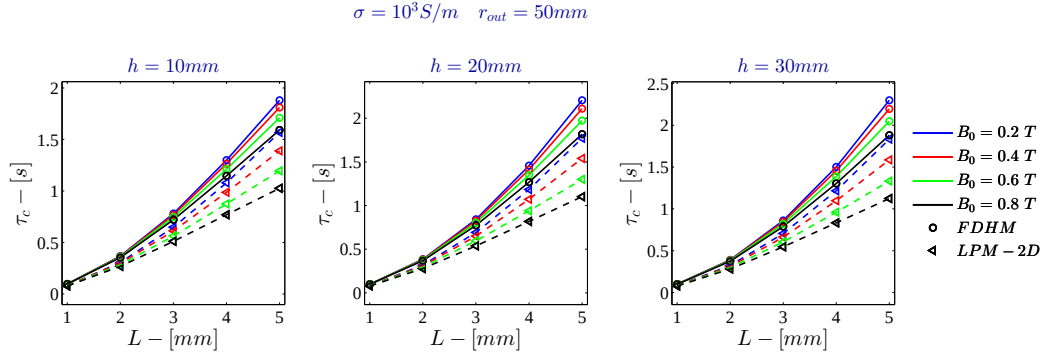


Figure C.29: τ_c for $\sigma = 10^3 S/m$ - Bidimensional LPM

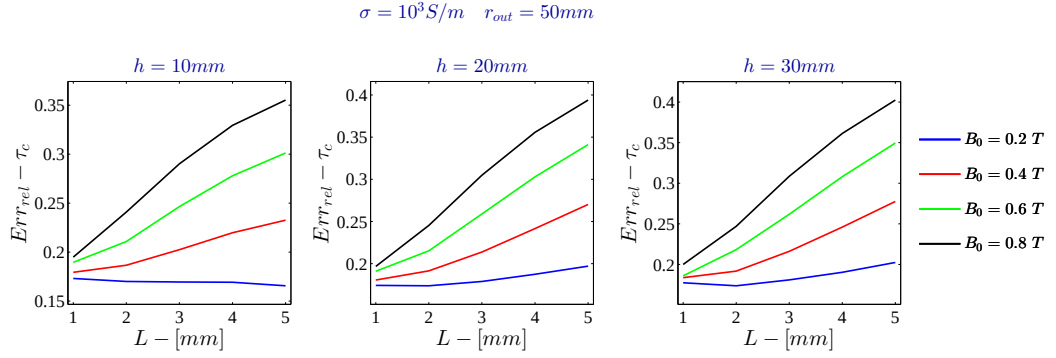


Figure C.30: Relative Error - τ_c for $\sigma = 10^3 S/m$ - Bidimensional LPM

C.2.6.2 Voltage Drive

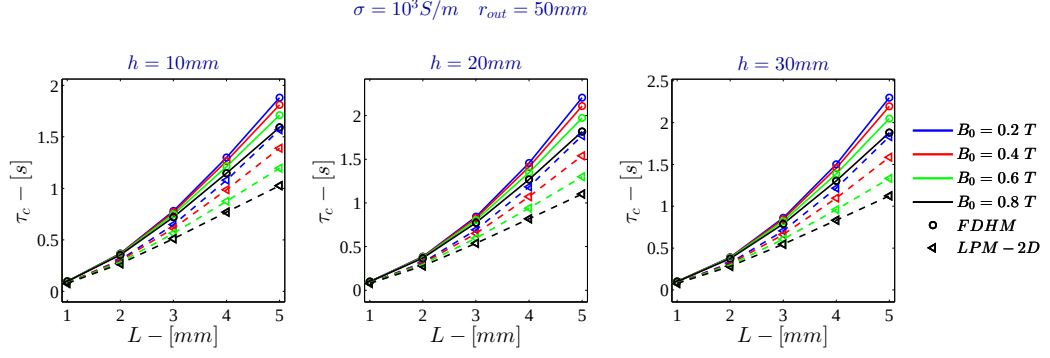


Figure C.31: τ_c for $\sigma = 10^3 S/m$ - Bidimensional LPM

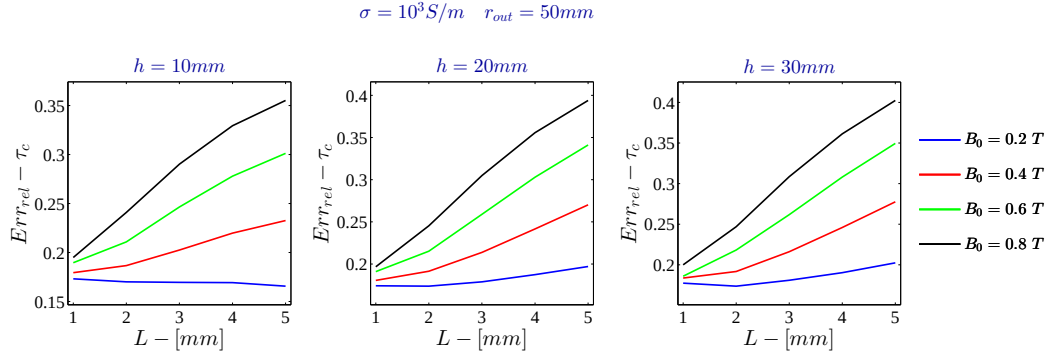


Figure C.32: Relative Error - τ_c for $\sigma = 10^3 S/m$ - Bidimensional LPM

D

Comparison between FDHM and LPM: $\sigma = 10^4 S/m$

D.1 Monodimensional LPM

D.1.1 Coefficient of the viscous shear moment - K_{vis}

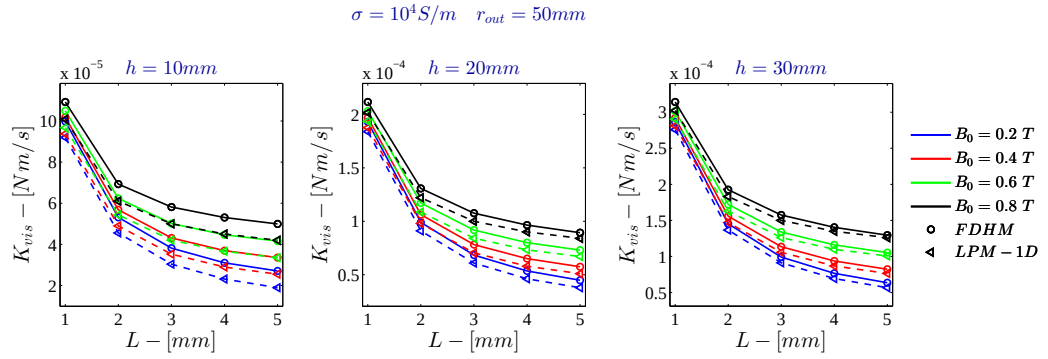


Figure D.1: K_{vis} for $\sigma = 10^4 S/m$ - Monodimensional LPM

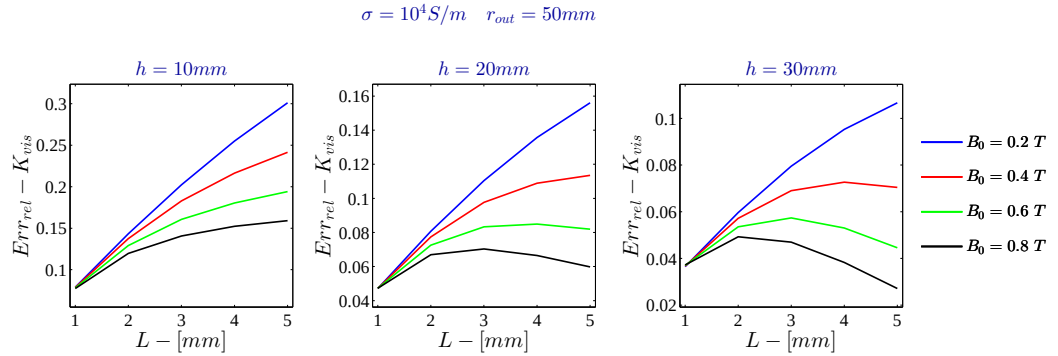
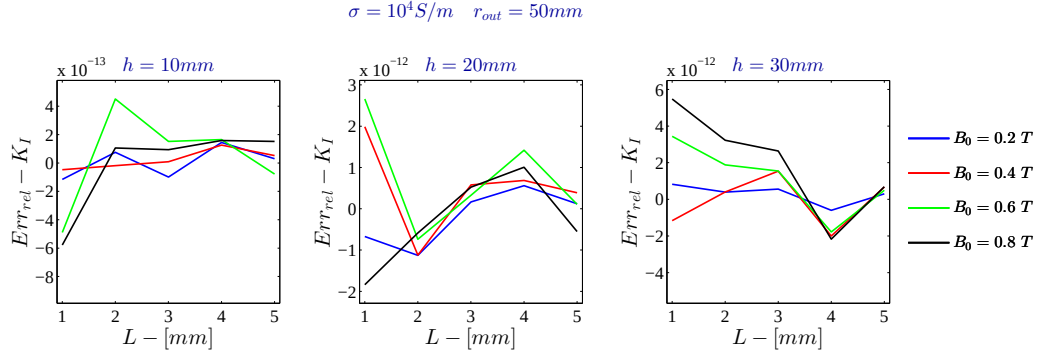
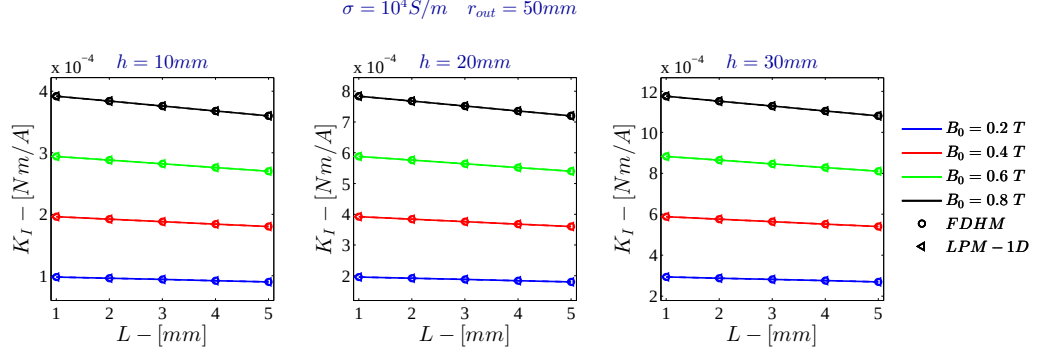
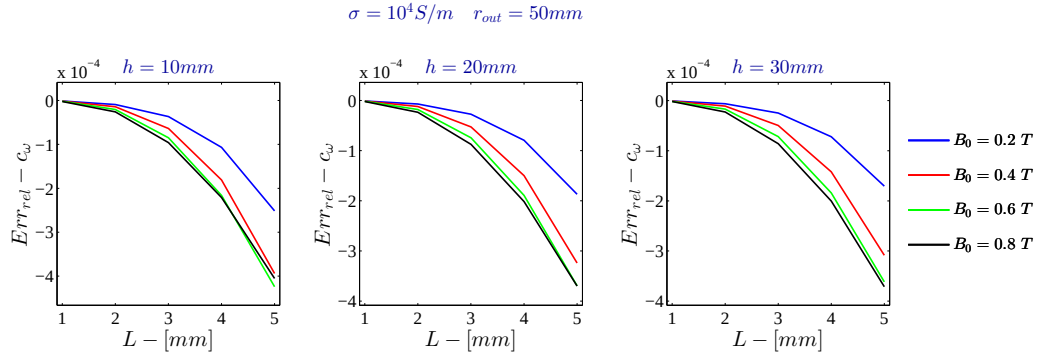
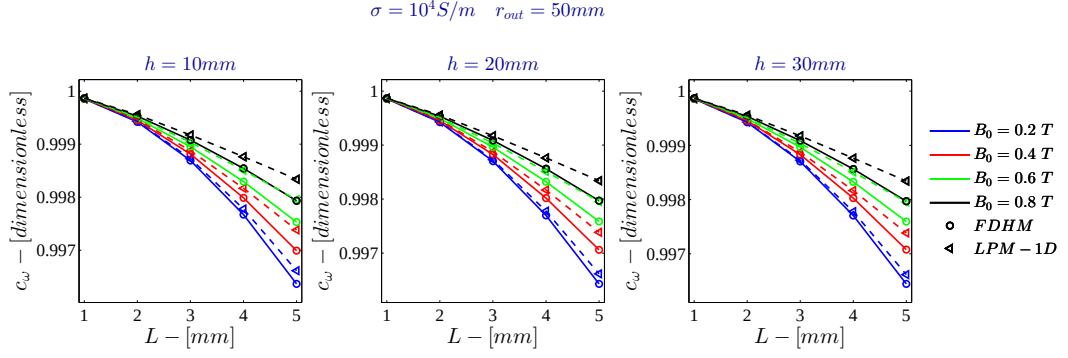


Figure D.2: Relative Error - K_{vis} for $\sigma = 10^4 S/m$ - Monodimensional LPM

D.1.2 Coefficient of the Lorentz Force moment - K_I



D.1.3 Coefficient of the moment of inertia - c_ω



D.1.4 Coefficient of Induced Voltage - K_V

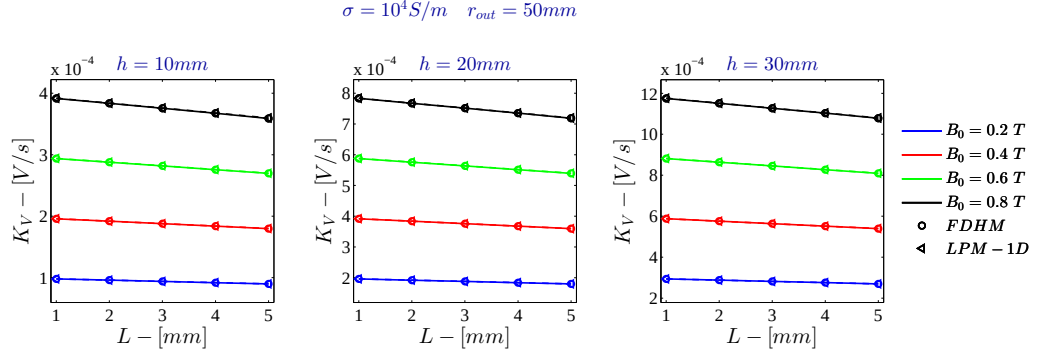


Figure D.7: K_V for $\sigma = 10^4 S/m$ - Monodimensional LPM

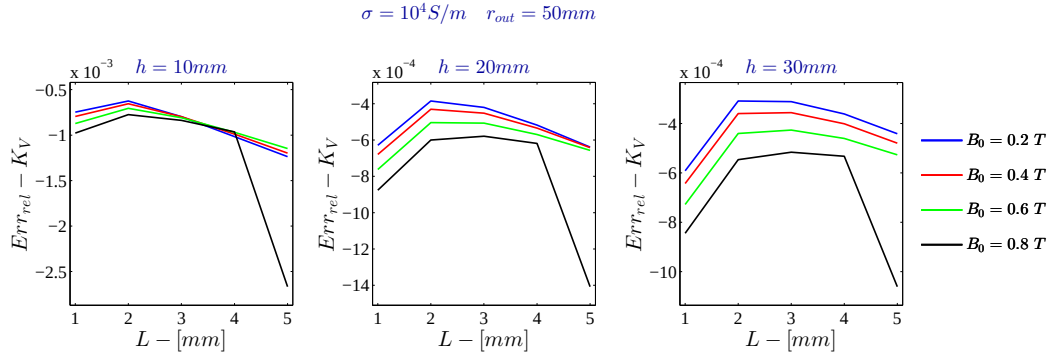


Figure D.8: Relative Error - K_V for $\sigma = 10^4 S/m$ - Monodimensional LPM

D.1.5 Torque - $\dot{\Gamma}$

D.1.5.1 Current Drive

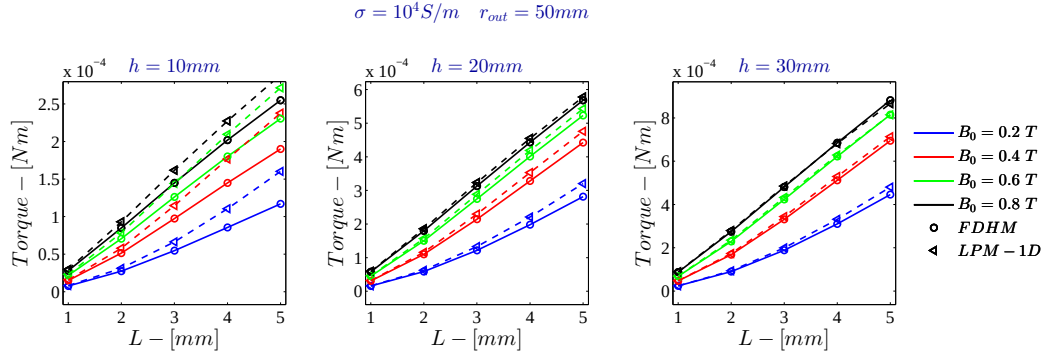


Figure D.9: $\dot{\Gamma}$ for $\sigma = 10^4 S/m$ - Monodimensional LPM

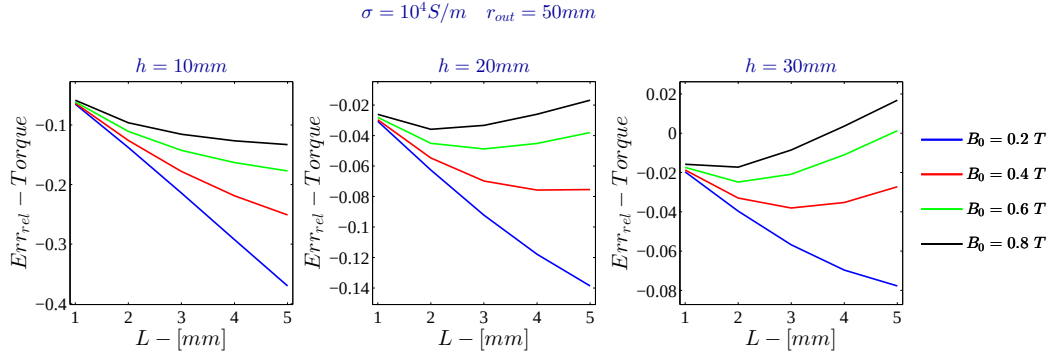


Figure D.10: Relative Error - $\dot{\Gamma}$ for $\sigma = 10^4 S/m$ - Monodimensional LPM

D.1.5.2 Voltage Drive

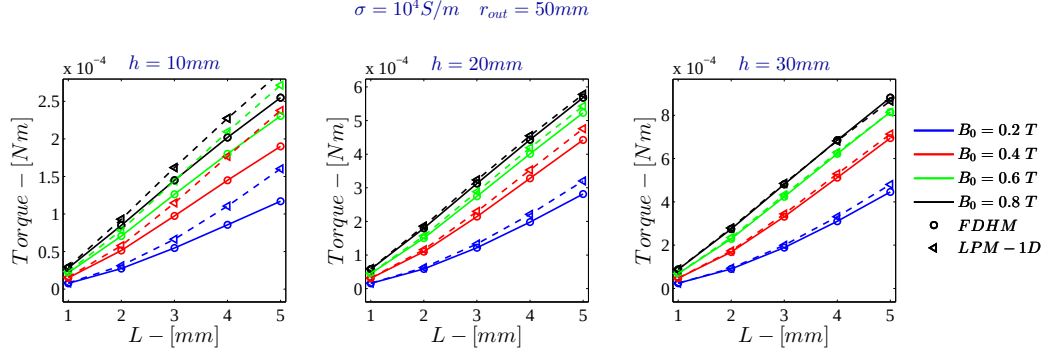


Figure D.11: $\dot{\Gamma}$ for $\sigma = 10^4 S/m$ - Monodimensional LPM

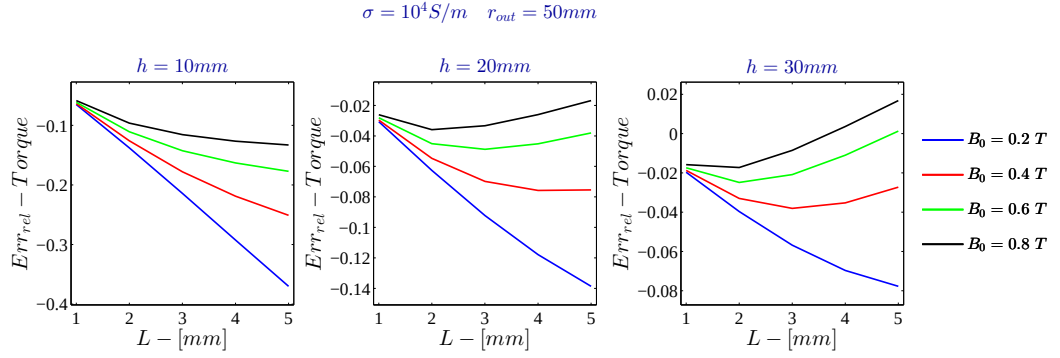


Figure D.12: Relative Error - $\dot{\Gamma}$ for $\sigma = 10^4 S/m$ - Monodimensional LPM

D.1.6 Characteristic Time - τ_c

D.1.6.1 Current Drive

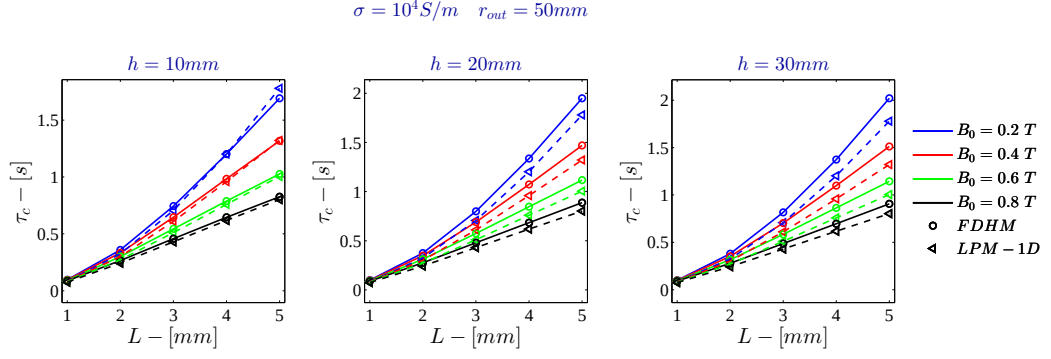


Figure D.13: τ_c for $\sigma = 10^4 S/m$ - Monodimensional LPM

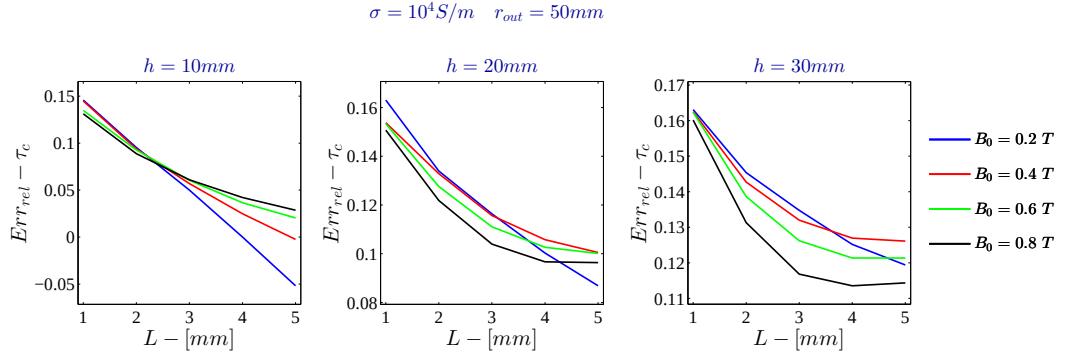


Figure D.14: Relative Error - τ_c for $\sigma = 10^4 S/m$ - Monodimensional LPM

D.1.6.2 Voltage Drive

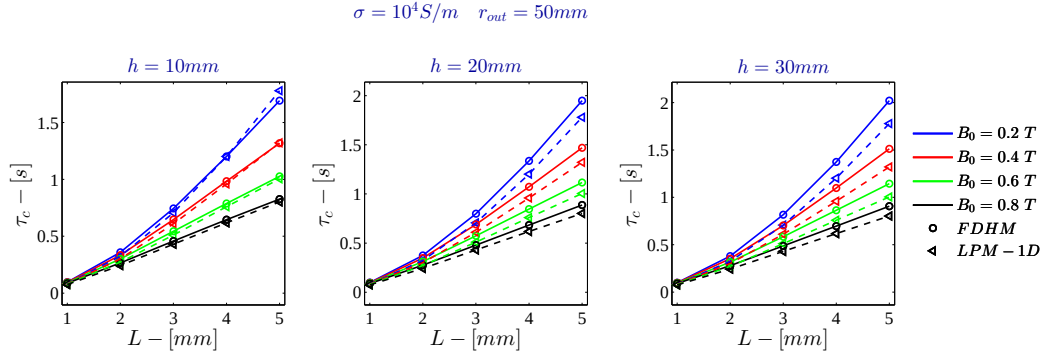


Figure D.15: τ_c for $\sigma = 10^4 S/m$ - Monodimensional LPM

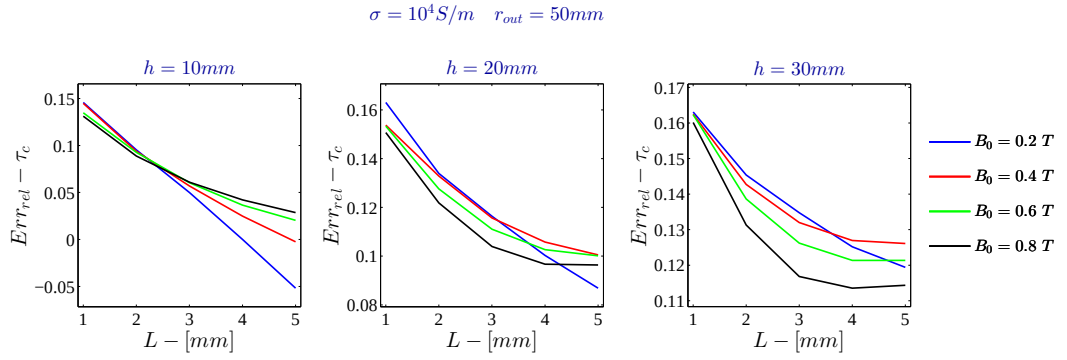


Figure D.16: Relative Error - τ_c for $\sigma = 10^4 S/m$ - Monodimensional LPM

D.2 Bidimensional LPM

D.2.1 Coefficient of the viscous shear moment - K_{vis}

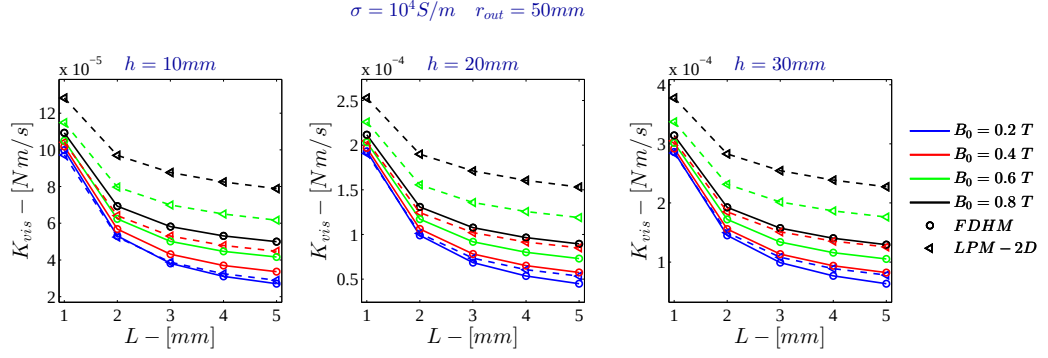


Figure D.17: K_{vis} for $\sigma = 10^4 S/m$ - Bidimensional LPM

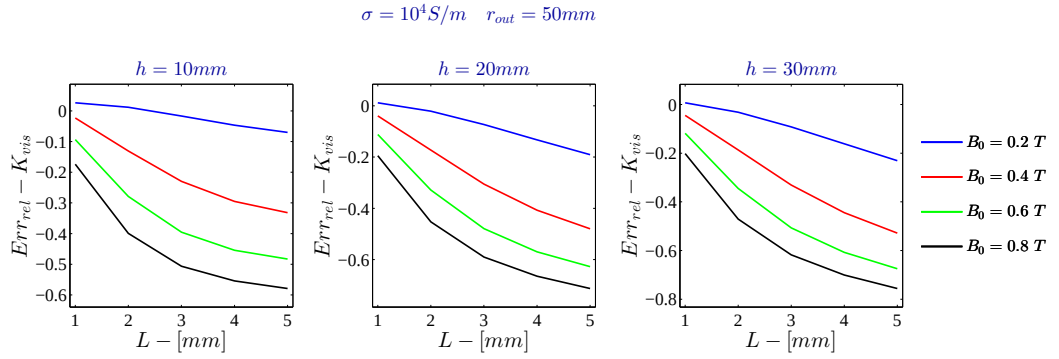


Figure D.18: Relative Error - K_{vis} for $\sigma = 10^4 S/m$ - Bidimensional LPM

D.2.2 Coefficient of the Lorentz Force moment - K_I

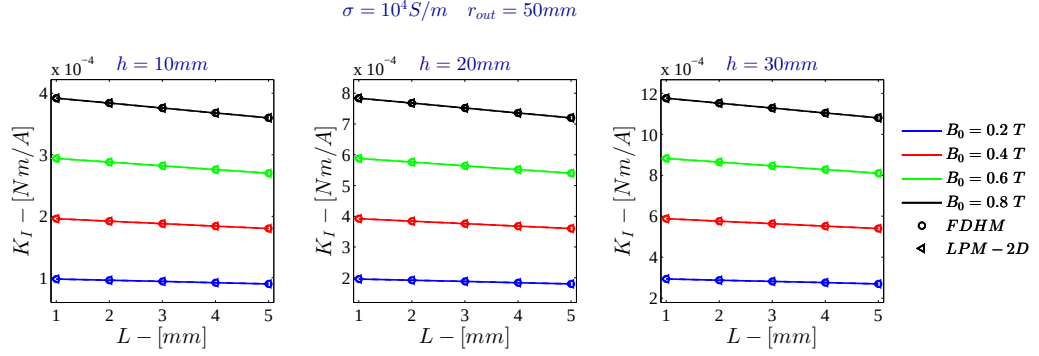


Figure D.19: K_I for $\sigma = 10^4 S/m$ - Bidimensional LPM

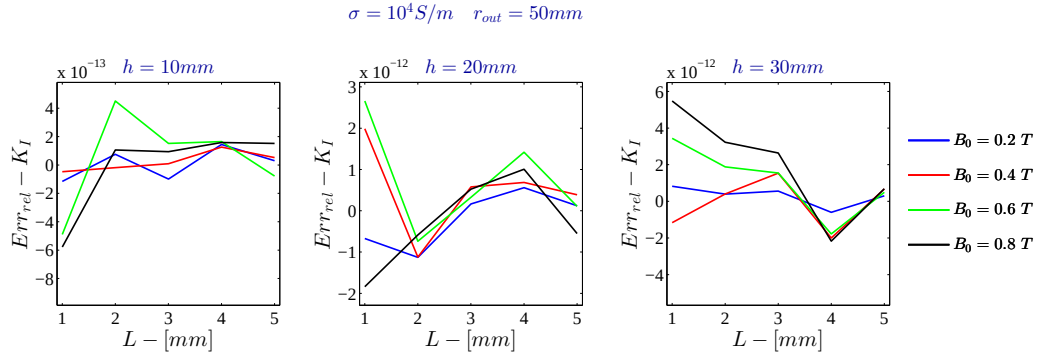
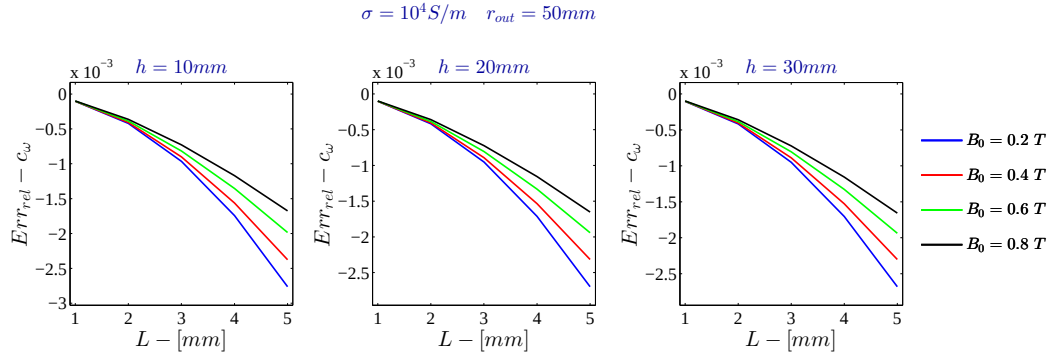
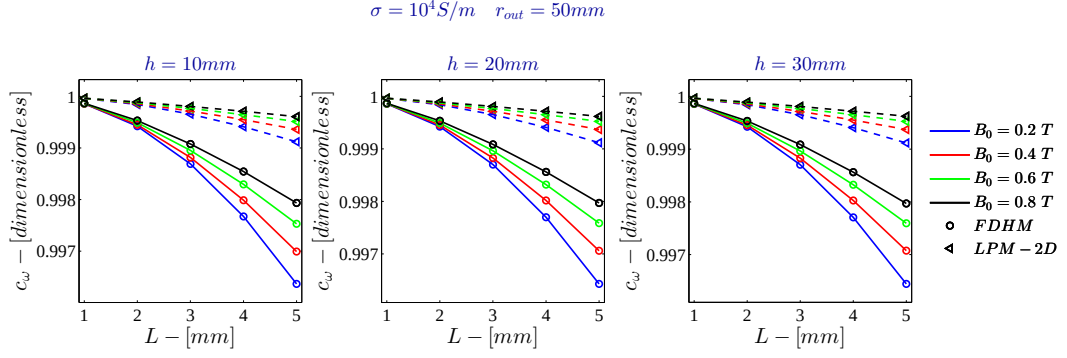


Figure D.20: Relative Error - K_I for $\sigma = 10^4 S/m$ - Bidimensional LPM

D.2.3 Coefficient of the moment of inertia - c_ω



D.2.4 Coefficient of Induced Voltage - K_V

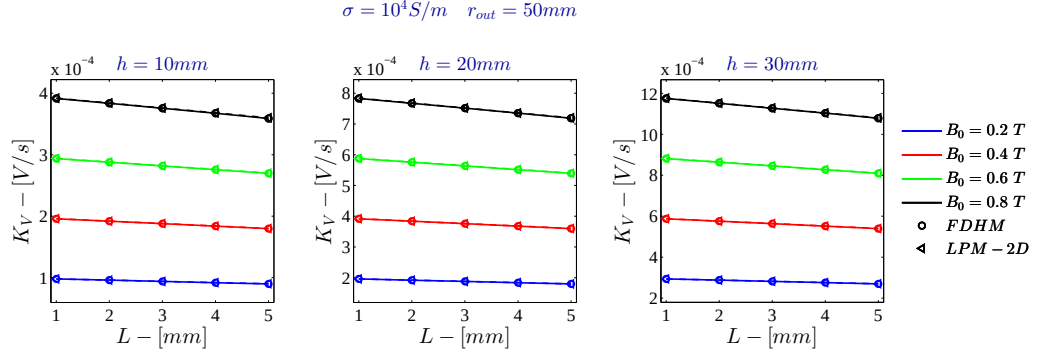


Figure D.23: K_V for $\sigma = 10^4 S/m$ - Bidimensional LPM

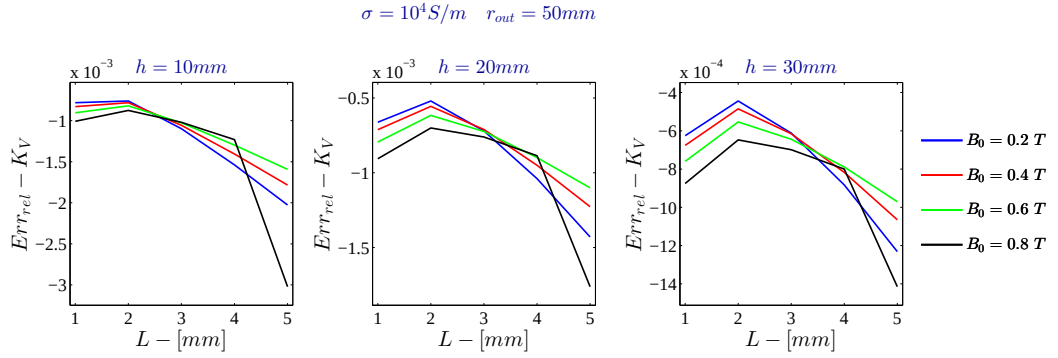


Figure D.24: Relative Error - K_V for $\sigma = 10^4 S/m$ - Bidimensional LPM

D.2.5 Torque - $\dot{\Gamma}$

D.2.5.1 Current Drive

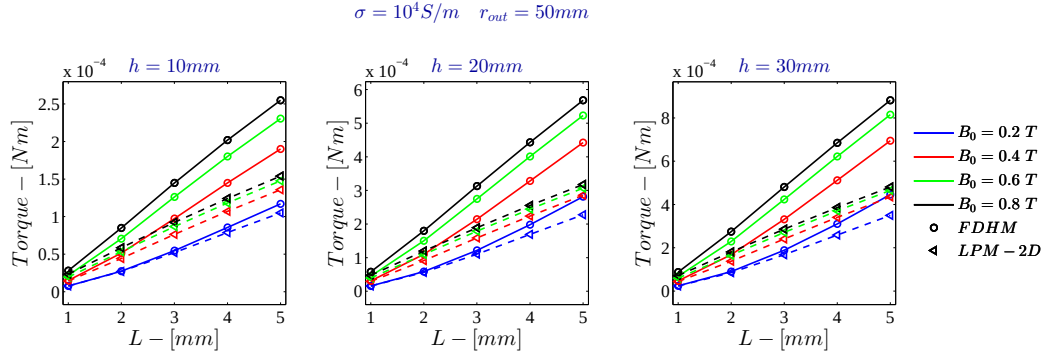


Figure D.25: $\dot{\Gamma}$ for $\sigma = 10^4 S/m$ - Bidimensional LPM

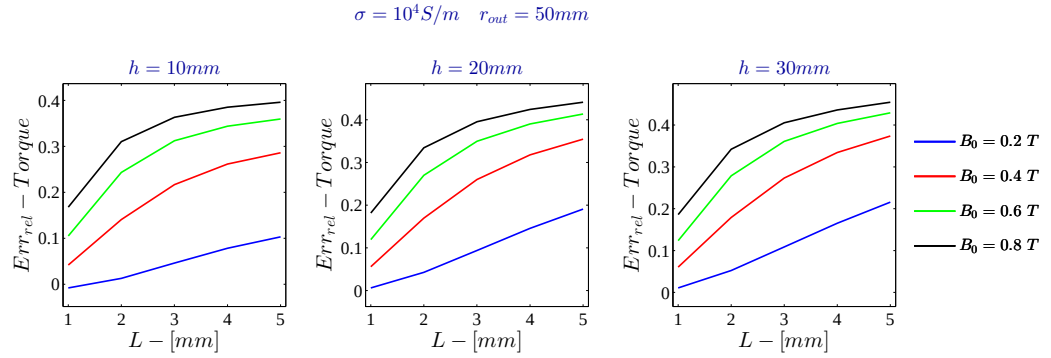


Figure D.26: Relative Error - $\dot{\Gamma}$ for $\sigma = 10^4 S/m$ - Bidimensional LPM

D.2.5.2 Voltage Drive

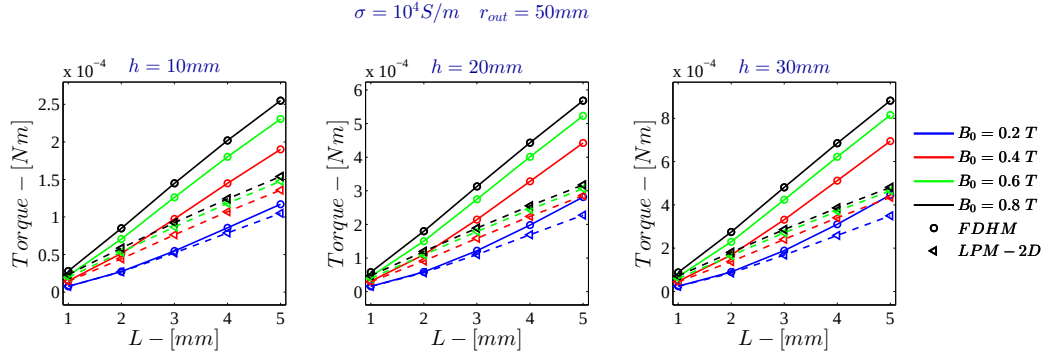


Figure D.27: $\dot{\Gamma}$ for $\sigma = 10^4 S/m$ - Bidimensional LPM

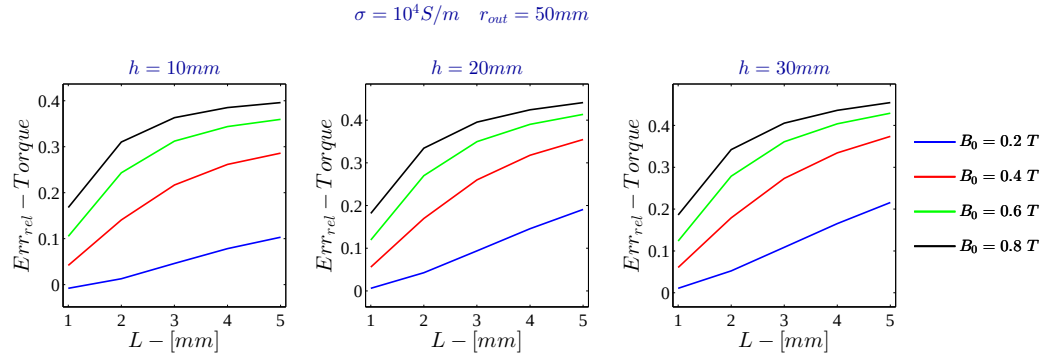
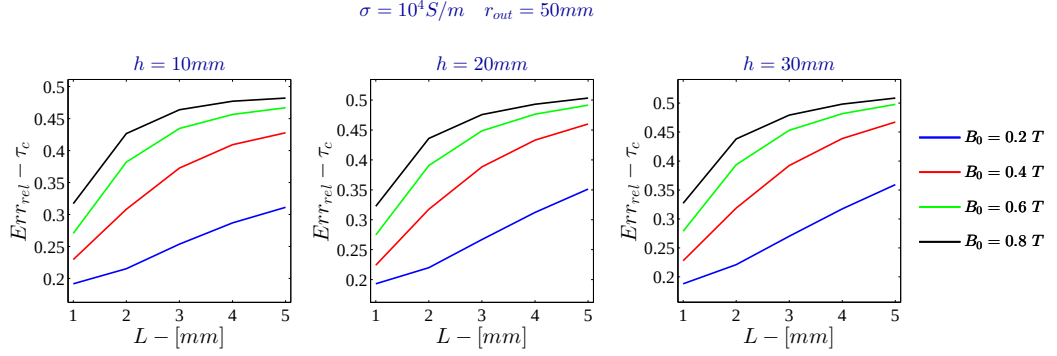
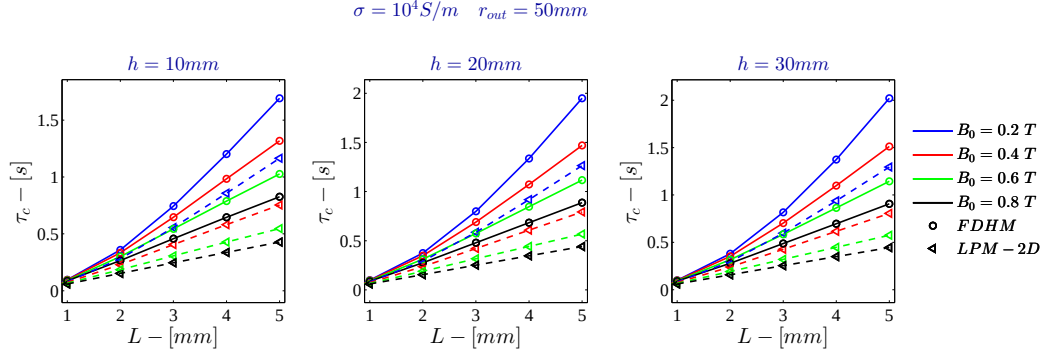


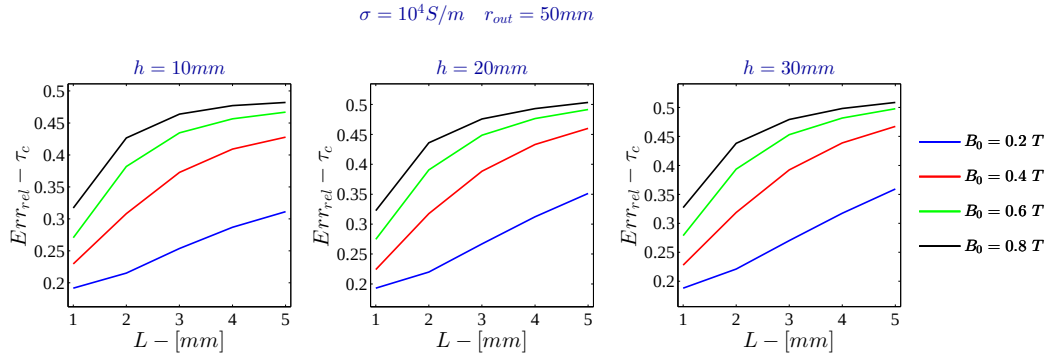
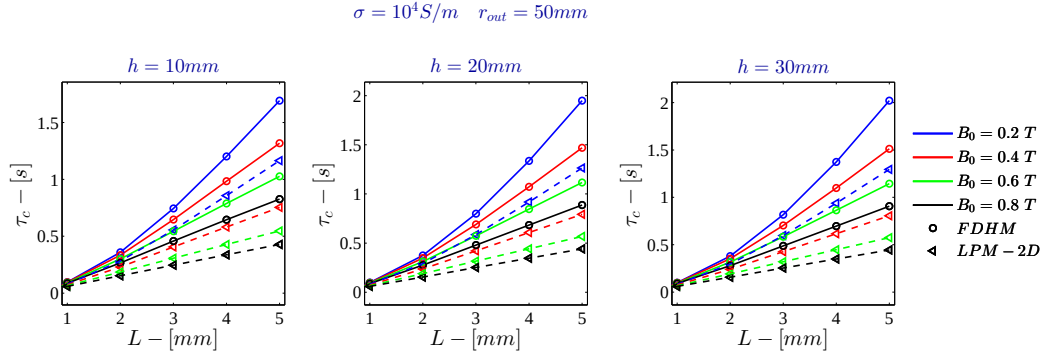
Figure D.28: Relative Error - $\dot{\Gamma}$ for $\sigma = 10^4 S/m$ - Bidimensional LPM

D.2.6 Characteristic Time - τ_c

D.2.6.1 Current Drive



D.2.6.2 Voltage Drive



Comparison between FDHM and LPM: $\sigma = 10^5 S/m$

E.1 Monodimensional LPM

E.1.1 Coefficient of the viscous shear moment - K_{vis}

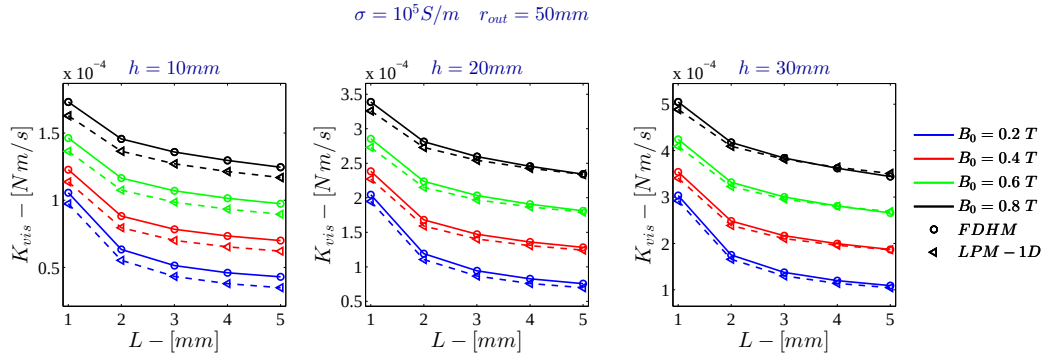


Figure E.1: K_{vis} for $\sigma = 10^5 S/m$ - Monodimensional LPM

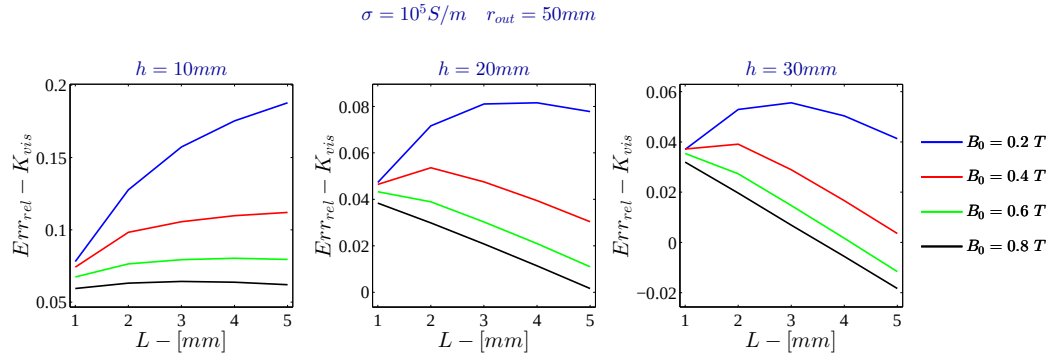


Figure E.2: Relative Error - K_{vis} for $\sigma = 10^5 S/m$ - Monodimensional LPM

E.1.2 Coefficient of the Lorentz Force moment - K_I

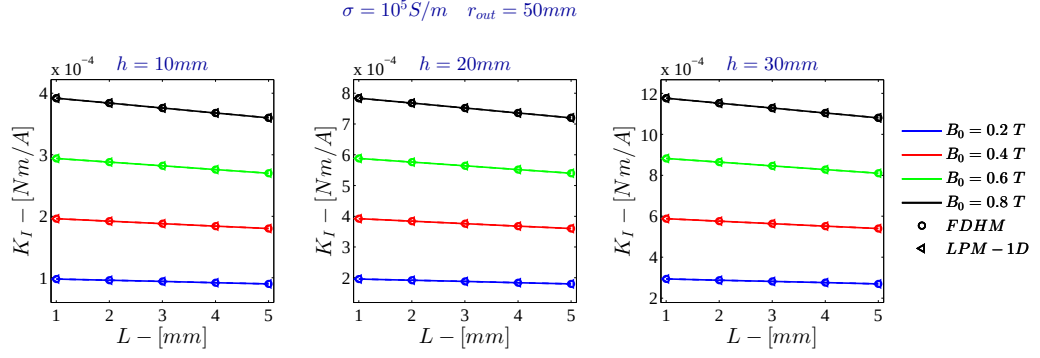


Figure E.3: K_I for $\sigma = 10^5 S/m$ - Monodimensional LPM

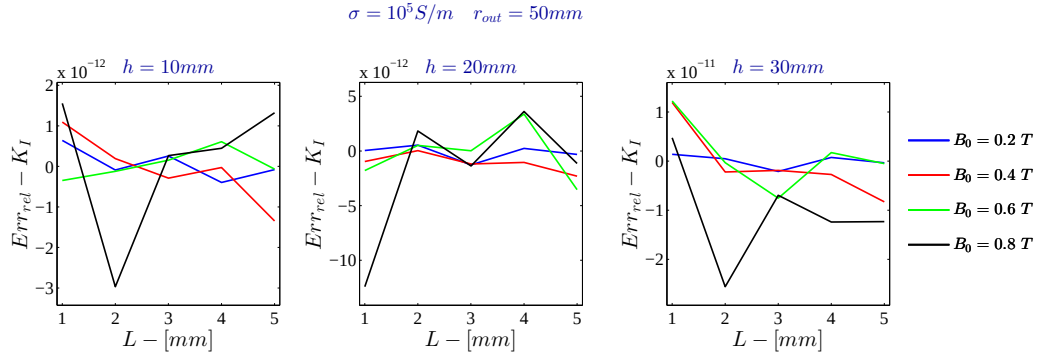
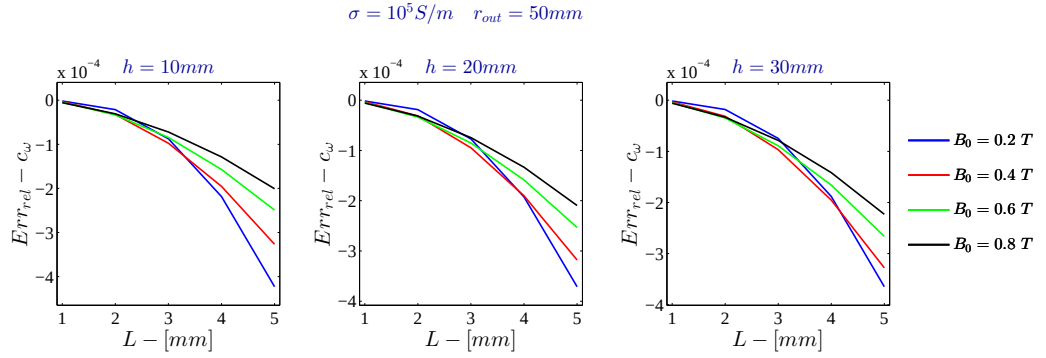
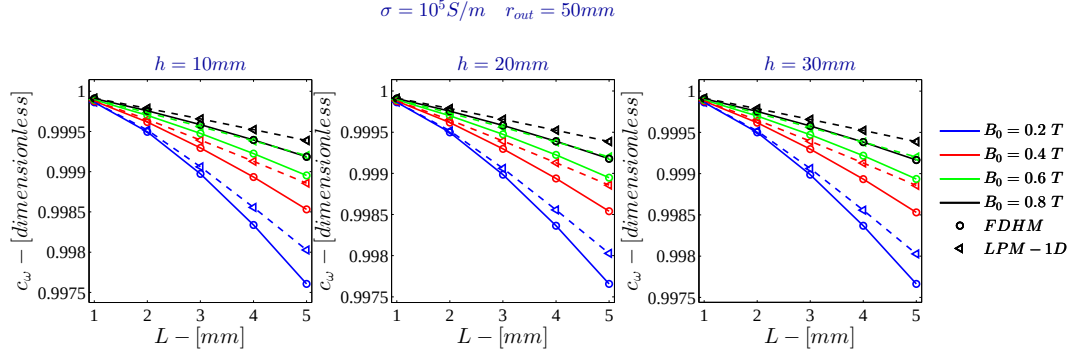


Figure E.4: Relative Error - K_I for $\sigma = 10^5 S/m$ - Monodimensional LPM

E.1.3 Coefficient of the moment of inertia - c_ω



E.1.4 Coefficient of Induced Voltage - K_V

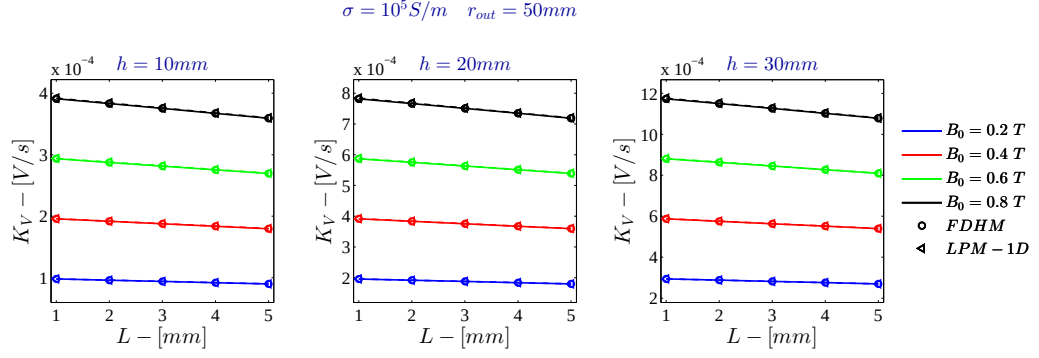


Figure E.7: K_V for $\sigma = 10^5 S/m$ - Monodimensional LPM

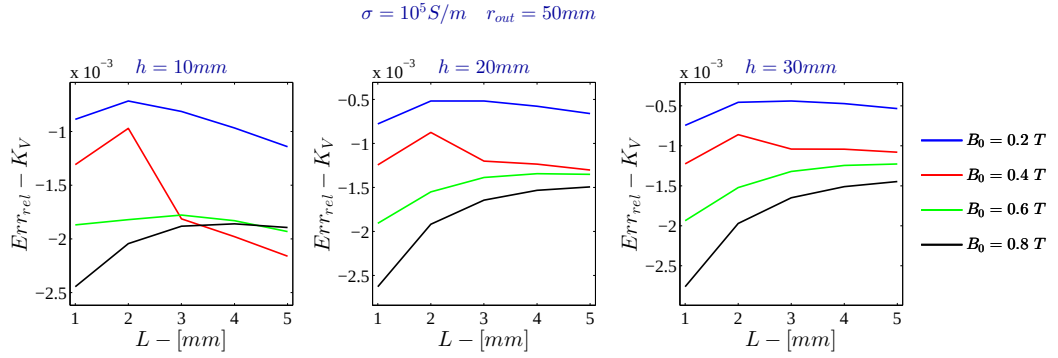


Figure E.8: Relative Error - K_V for $\sigma = 10^5 S/m$ - Monodimensional LPM

E.1.5 Torque - $\dot{\Gamma}$

E.1.5.1 Current Drive

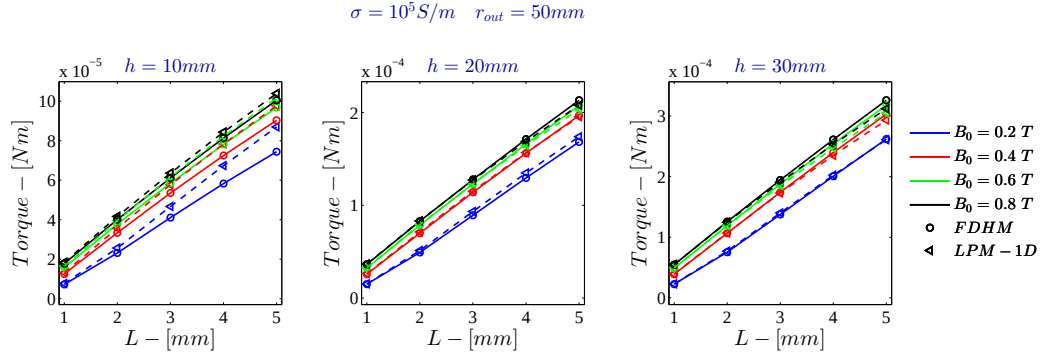


Figure E.9: $\dot{\Gamma}$ for $\sigma = 10^5 S/m$ - Monodimensional LPM

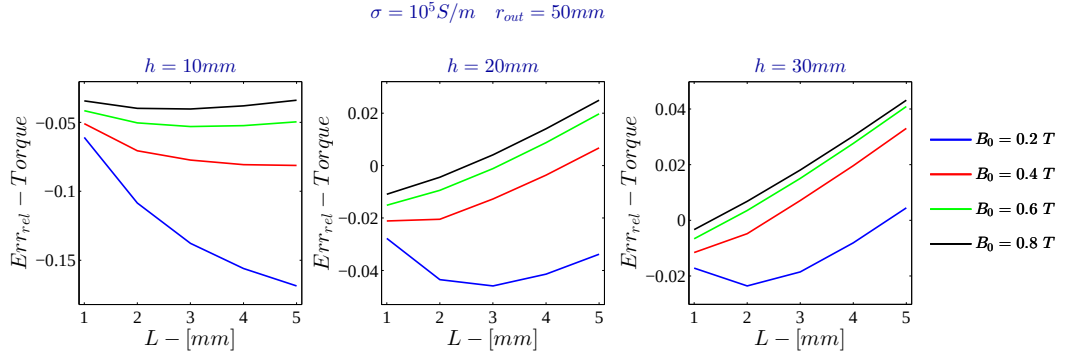


Figure E.10: Relative Error - $\dot{\Gamma}$ for $\sigma = 10^5 S/m$ - Monodimensional LPM

E.1.5.2 Voltage Drive

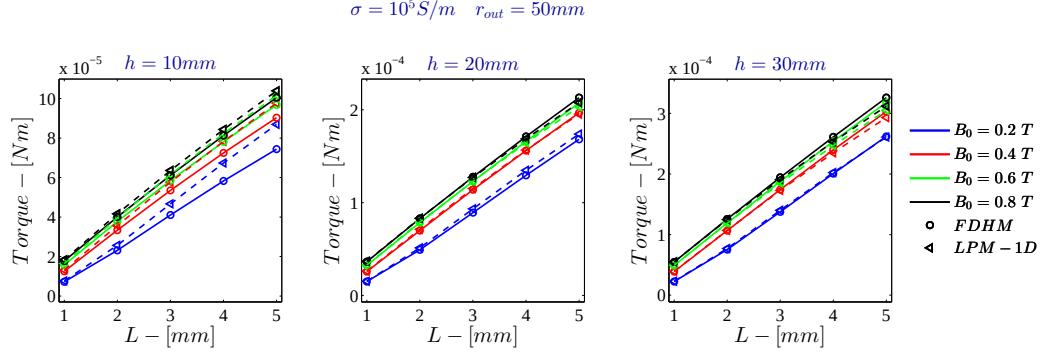


Figure E.11: $\dot{\Gamma}$ for $\sigma = 10^5 S/m$ - Monodimensional LPM

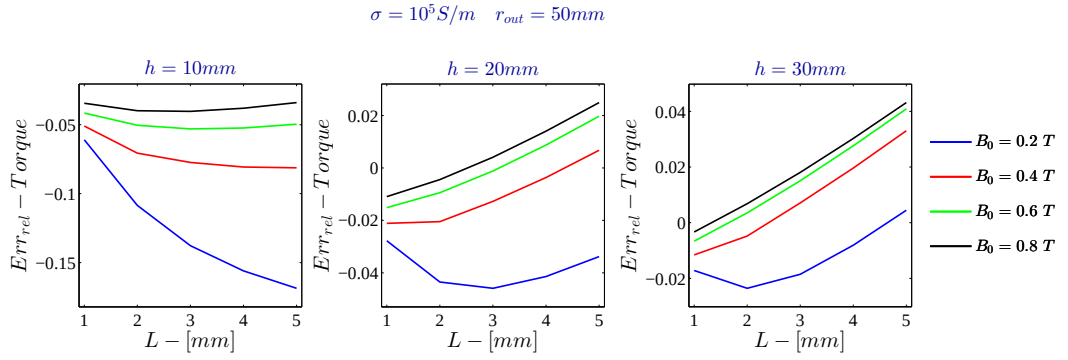


Figure E.12: Relative Error - $\dot{\Gamma}$ for $\sigma = 10^5 S/m$ - Monodimensional LPM

E.1.6 Characteristic Time - τ_c

E.1.6.1 Current Drive

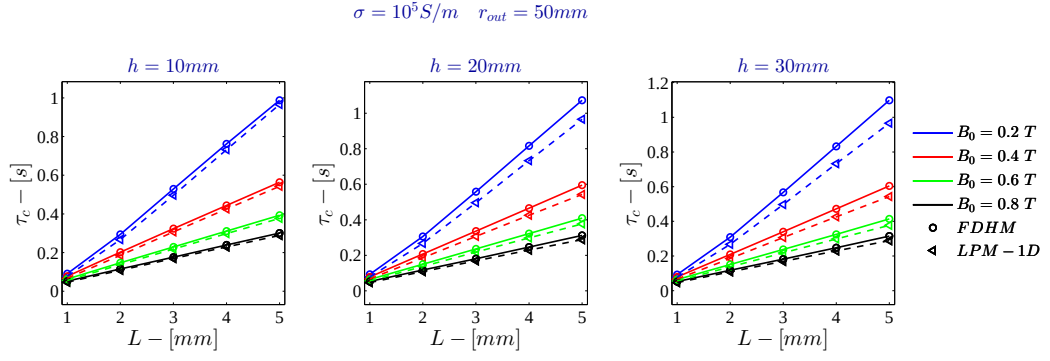


Figure E.13: τ_c for $\sigma = 10^5 S/m$ - Monodimensional LPM

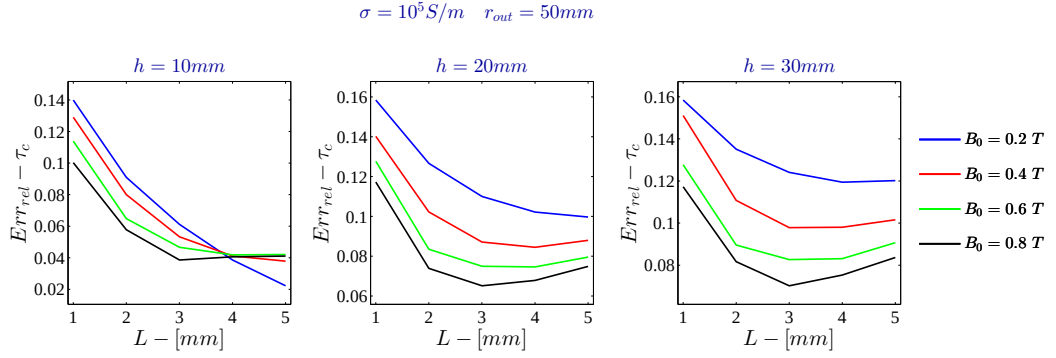


Figure E.14: Relative Error - τ_c for $\sigma = 10^5 S/m$ - Monodimensional LPM

E.1.6.2 Voltage Drive

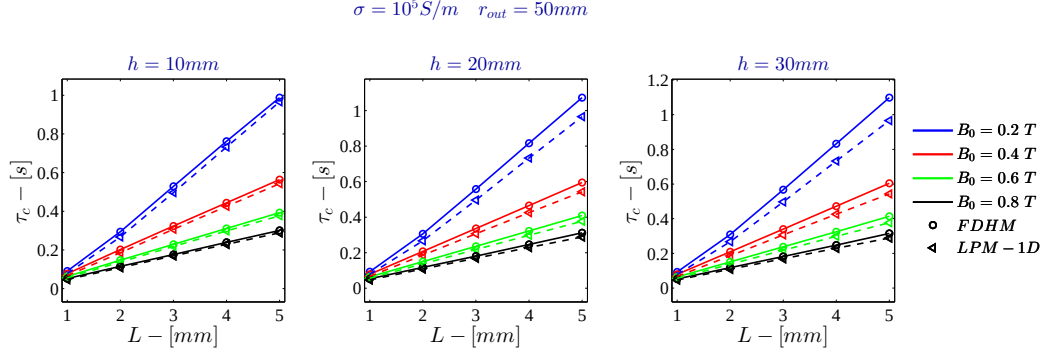


Figure E.15: τ_c for $\sigma = 10^5 S/m$ - Monodimensional LPM

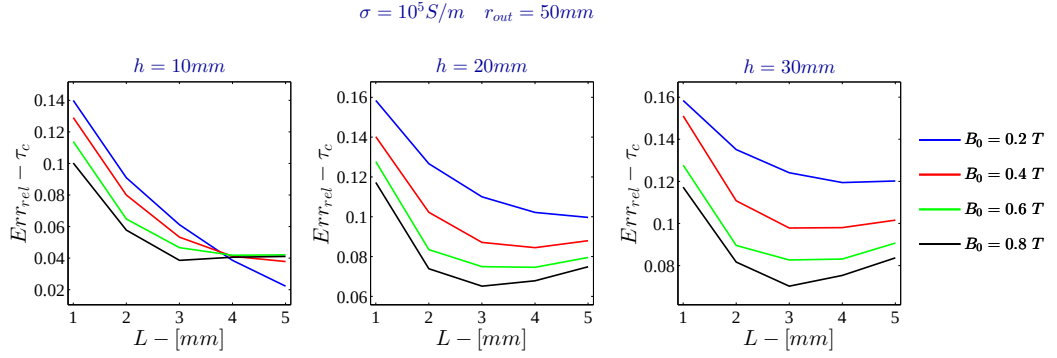
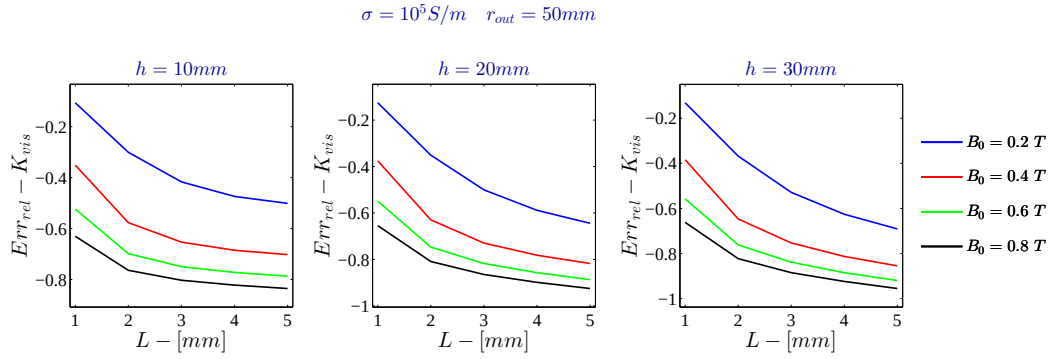
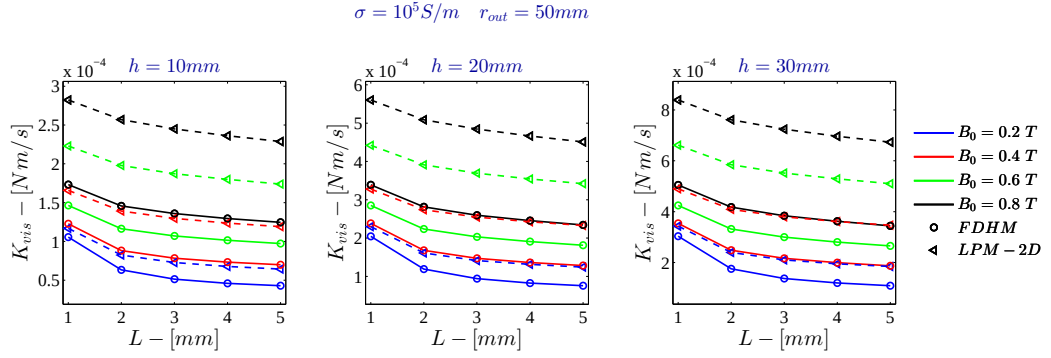


Figure E.16: Relative Error - τ_c for $\sigma = 10^5 S/m$ - Monodimensional LPM

E.2 Bidimensional LPM

E.2.1 Coefficient of the viscous shear moment - K_{vis}



E.2.2 Coefficient of the Lorentz Force moment - K_I

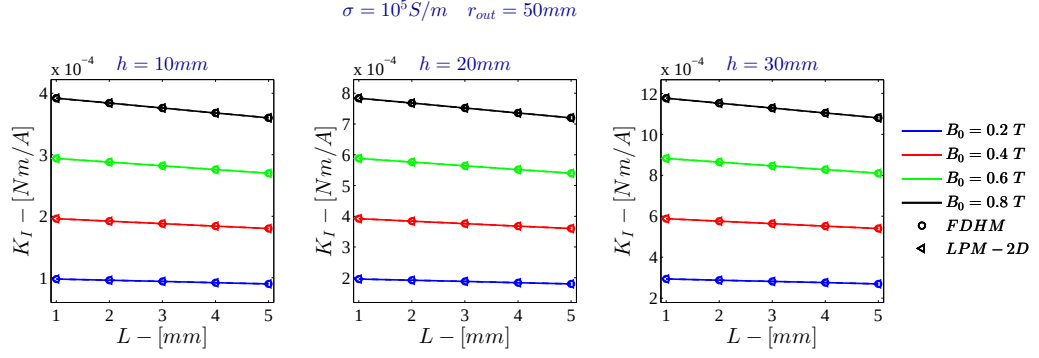


Figure E.19: K_I for $\sigma = 10^5 S/m$ - Bidimensional LPM

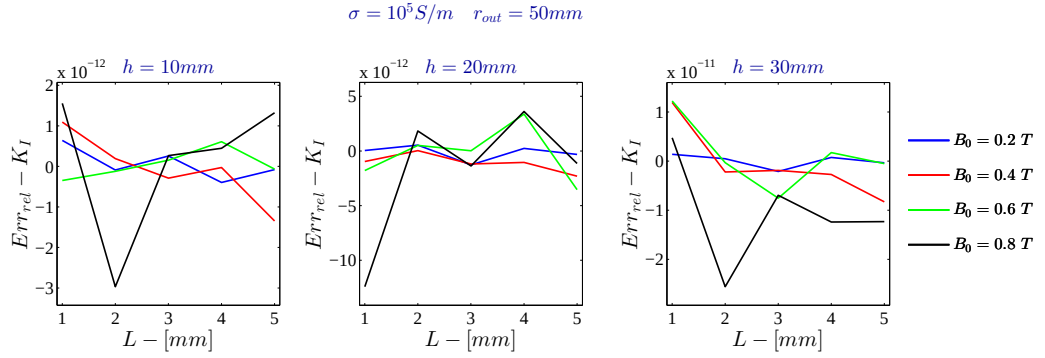
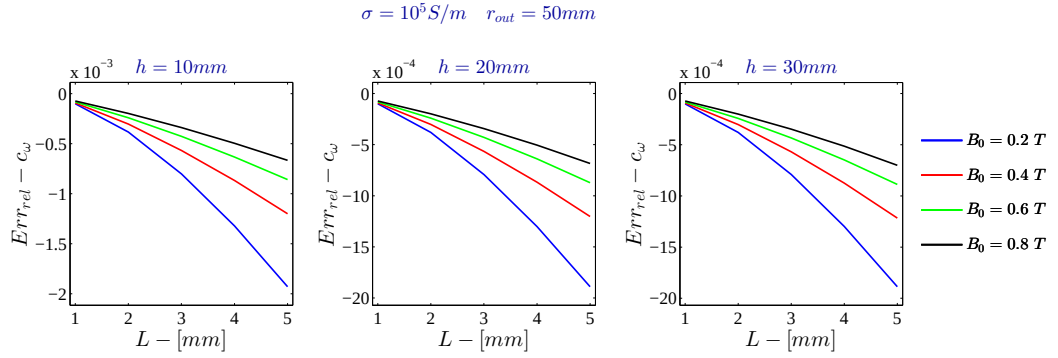
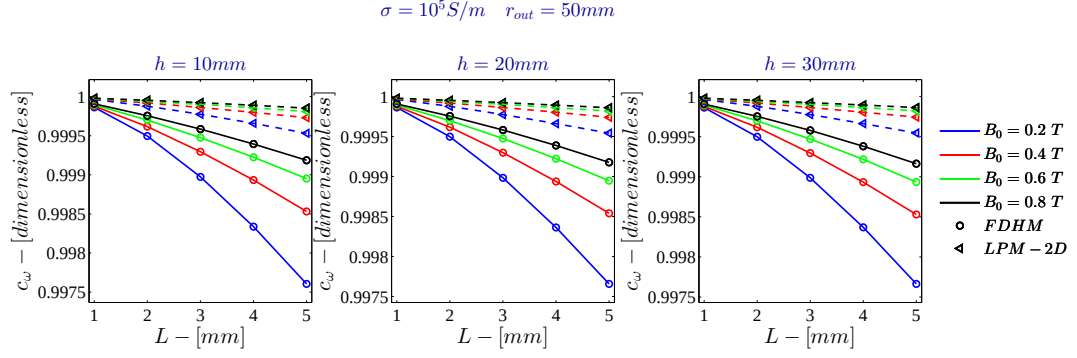
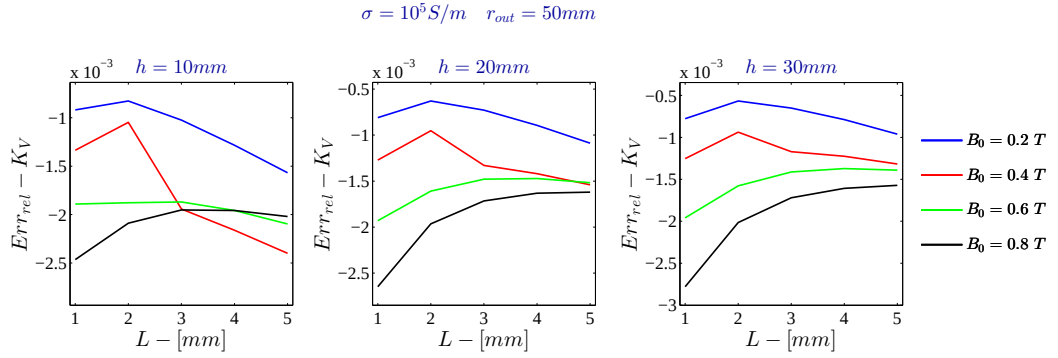
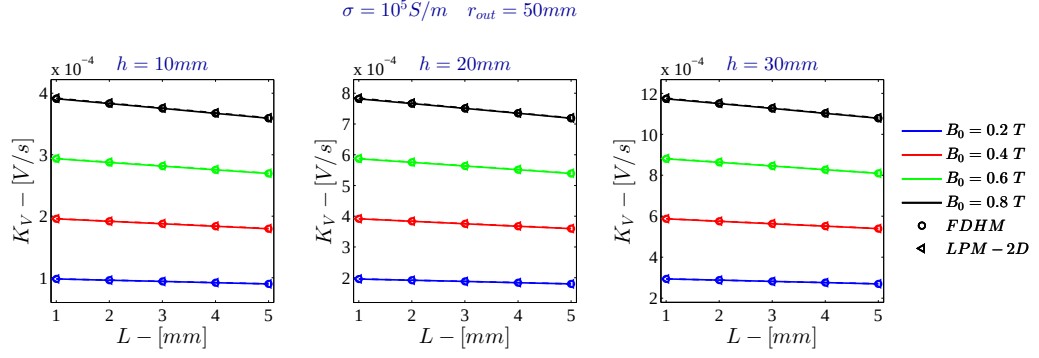


Figure E.20: Relative Error - K_I for $\sigma = 10^5 S/m$ - Bidimensional LPM

E.2.3 Coefficient of the moment of inertia - c_ω



E.2.4 Coefficient of Induced Voltage - K_V



E.2.5 Torque - $\dot{\Gamma}$

E.2.5.1 Current Drive

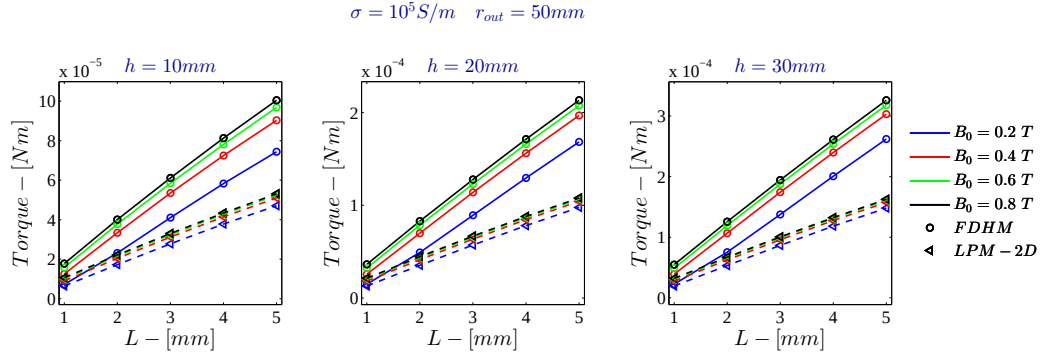


Figure E.25: $\dot{\Gamma}$ for $\sigma = 10^5 S/m$ - Bidimensional LPM

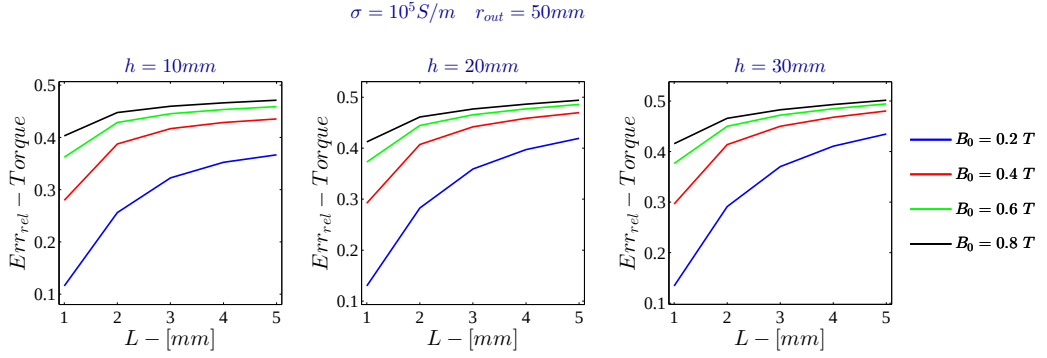


Figure E.26: Relative Error - $\dot{\Gamma}$ for $\sigma = 10^5 S/m$ - Bidimensional LPM

E.2.5.2 Voltage Drive

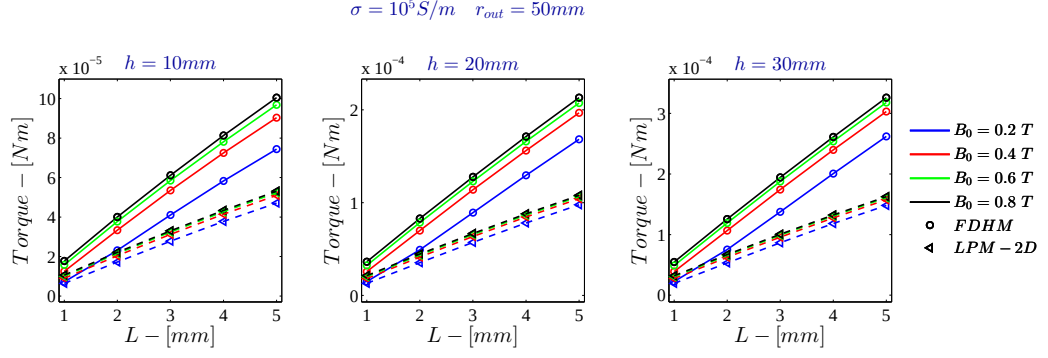


Figure E.27: $\dot{\Gamma}$ for $\sigma = 10^5 S/m$ - Bidimensional LPM

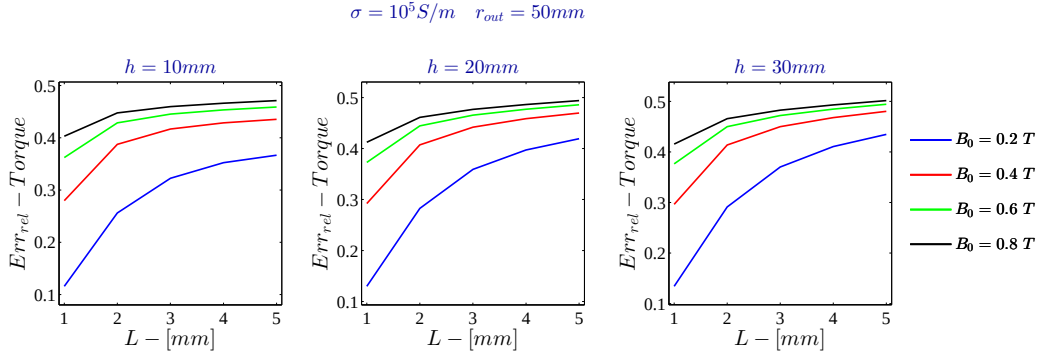


Figure E.28: Relative Error - $\dot{\Gamma}$ for $\sigma = 10^5 S/m$ - Bidimensional LPM

E.2.6 Characteristic Time - τ_c

E.2.6.1 Current Drive

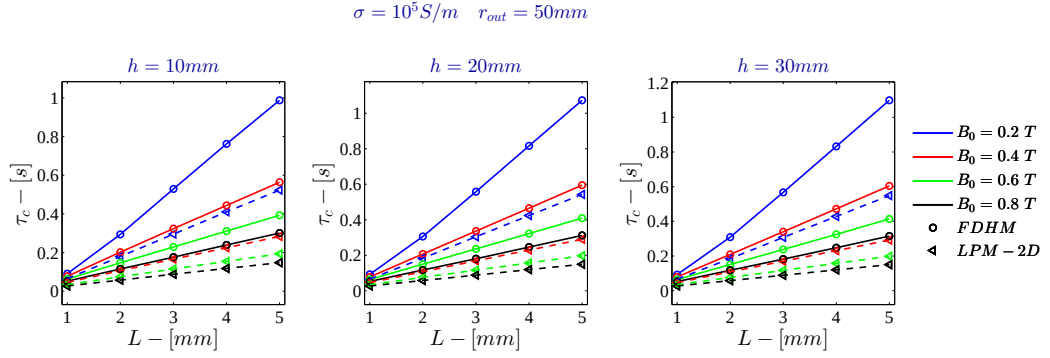


Figure E.29: τ_c for $\sigma = 10^5 S/m$ - Bidimensional LPM

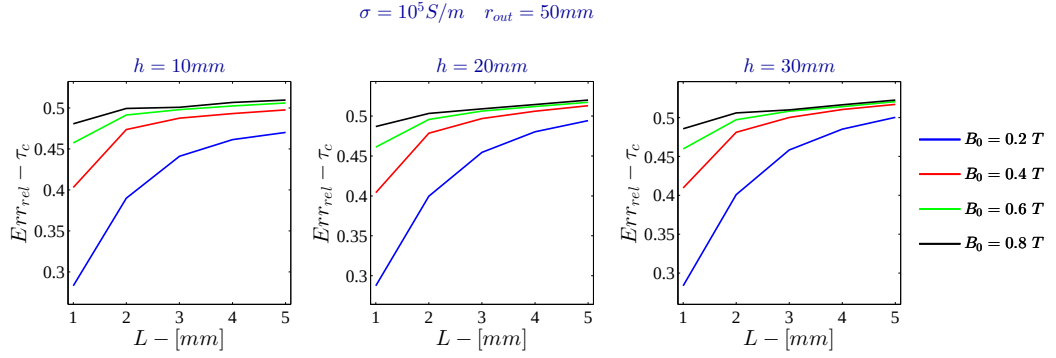


Figure E.30: Relative Error - τ_c for $\sigma = 10^5 S/m$ - Bidimensional LPM

E.2.6.2 Voltage Drive

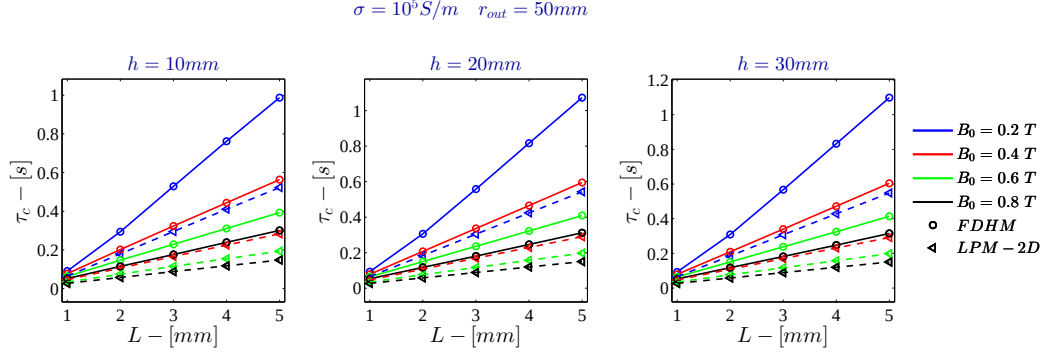


Figure E.31: τ_c for $\sigma = 10^5 S/m$ - Bidimensional LPM

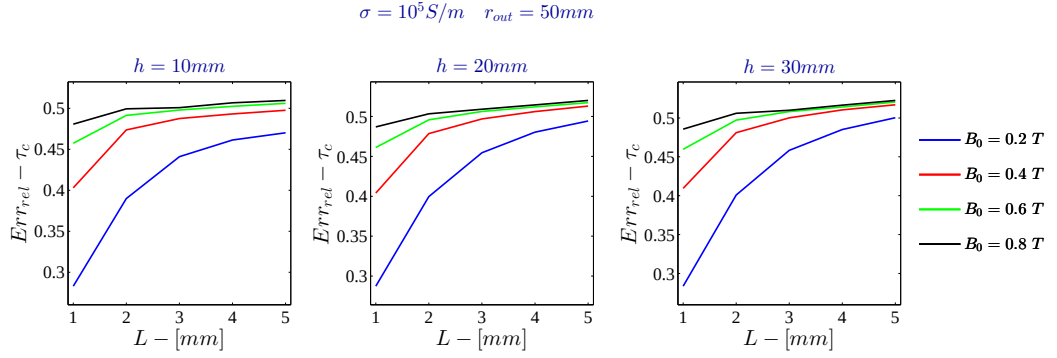


Figure E.32: Relative Error - τ_c for $\sigma = 10^5 S/m$ - Bidimensional LPM

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