A CHARACTERIZATION OF MINIMAL VARIETIES OF
$\mathbb{Z}_p$-GRADED PI ALGEBRAS

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Abstract. Let $F$ be a field of characteristic zero and $p$ a prime. In
the present paper it is proved that a variety of $\mathbb{Z}_p$-graded associ-
tive PI $F$-algebras of finite basic rank is minimal of fixed $\mathbb{Z}_p$-exponent
$d$ if, and only if, it is generated by an upper block triangular ma-
trix algebra $UT_{\mathbb{Z}_p}(A_1, \ldots, A_m)$ equipped with a suitable elementary $\mathbb{Z}_p$-
grading, whose diagonal blocks are isomorphic to $\mathbb{Z}_p$-graded simple al-
gebras $A_1, \ldots, A_m$ satisfying $\dim_F(A_1 \oplus \cdots \oplus A_m) = d$.

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gular Matrix Algebras.

1. Introduction

Let $F$ be a field of characteristic zero and $A$ an associative $F$-algebra
satisfying a polynomial identity (in the sequel we shall refer to these algebras
as PI algebras) graded by a finite group $G$. In the last two decades there has
been an extensive investigation of graded polynomial identities satisfied by
$A$. Apart from their own interesting features, the main motivations for this
type of work come from the significant information on quite general questions
that the additional graded structure and related objects may provide. This
is the case, for instance, for the solution of the Specht problem due to Kemer
(see [19]) in which $\mathbb{Z}_2$-gradings play a key role. Along this line, we are going
to focus on the growth of graded codimensions.

More specifically, let $F\langle X_G \rangle$ be the free associative $G$-graded algebra on
a countable set over $F$ and $\text{Id}_G(A)$ the ideal of $G$-graded identities of a $G$-
graded $F$-algebra $A$. Then one considers the relatively free $G$-graded algebra
$F\langle X_G \rangle / \text{Id}_G(A)$ and the dimension of the subspace of its multilinear elements
in $n$ free generators, denoted by $c_n^G(A)$. The sequence $\{c_n^G(A)\}_{n \geq 1}$ is called
the sequence of the $G$-graded codimensions of $A$. It was introduced by
Regev in the ungraded case (which actually corresponds to when $G = \langle 1_G \rangle$)
in the seminal paper [21], where it was proved that in presence of a non-zero
polynomial identity it is exponentially bounded. As a consequence of this

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fact, in [13] it was stated that $c_n^G(A)$ is exponentially bounded if, and only, $A$ is a PI algebra. In such an event, Aljadeff and Giambruno [1] have recently captured the exponential growth of this sequence proving that

$$\exp_G(A) := \lim_{m \to +\infty} m^{\sqrt[n]{c_n^G(A)}}$$

exists and is a non-negative integer, which is called the $G$-graded exponent or simply the $G$-exponent of $A$ (for the sake of completeness, we have to mention that this was established in [12] when $G$ is finite and abelian and, previously, in [2] under the extra assumption that $A$ is finite-dimensional). This extends the deep contribution of Giambruno and Zaicev ([14] and [15]) on the existence of the exponent of a PI algebra.

These results are the most striking culminating points of quantitative investigations of (non-necessarily) graded polynomial identities satisfied by a PI algebra: in fact, the existence of the graded exponent allows us to classify varieties on an integral scale whose steps are the minimal varieties of given $G$-exponent $d$, namely those varieties of $G$-exponent $d$ such that every proper subvariety has $G$-exponent strictly less than $d$.

In this direction, Giambruno and Zaicev [17] proved that in the ungraded case a variety of exponential growth is minimal if, and only if, it is generated by the Grassmann envelope of a so called minimal superalgebra. Previously the same authors in [16] stated that a variety of PI algebras of finite basic rank (that is generated by a finitely generated PI algebra) is minimal if, and only if, it is generated by an upper block triangular matrix algebra $UT(m_1, \ldots, m_n)$. Motivated by all these results, in [10] varieties of $\mathbb{Z}_2$-graded PI algebras or supervarieties of finite basic rank (that is, generated by a finitely generated superalgebra satisfying an ordinary polynomial identity) which are minimal of fixed $\mathbb{Z}_2$-graded exponent have been investigated. In particular, it has been established that any such supervariety is generated by one of the above mentioned minimal superalgebras. But despite some partial results in [8] and [9], until this moment their complete characterization is still unknown.

In the present paper we deal with varieties of $\mathbb{Z}_p$-graded PI algebras of finite basic rank where $p$ is a prime and our goal is the classification of those which are minimal of a given $\mathbb{Z}_p$-exponent (obviously this solves the problem for the above discussed case $p = 2$ as well). In more detail, given a $m$-tuple of $\mathbb{Z}_p$-graded simple algebras $(A_1, \ldots, A_m)$ we construct an upper block triangular matrix algebra $UT_{\mathbb{Z}_p}(A_1, \ldots, A_m)$ equipped with a suitable elementary $\mathbb{Z}_p$-grading, whose $i$-th diagonal block is isomorphic to the graded algebra $A_i$. The main result we prove is that a variety of $\mathbb{Z}_p$-graded PI-algebras of finite basic rank is minimal of $\mathbb{Z}_p$-exponent $d$ if, and only if, it is generated by a $\mathbb{Z}_p$-graded algebra $UT_{\mathbb{Z}_p}(A_1, \ldots, A_m)$ satisfying $\dim_F(A_1 \oplus \cdots \oplus A_m) = d$.

2. Preliminaries and minimal $G$-graded algebras

Throughout the rest of the paper, unless otherwise stated, $G$ is a finite group, $F$ is a field of characteristic zero and all the algebras are assumed
to be associative and to have the same ground field \( F \). For any ring \( R \) and pair of positive integers \( s \) and \( t \) the symbol \( M_{s \times t}(R) \) means the space of all matrices with \( s \) rows and \( t \) columns over \( R \) and set \( M_s(R) := M_{s \times 1}(R) \). In the case in which \( R = F \), we simply write \( M_{s \times t} \) and \( M_s \) instead of \( M_{s \times t}(F) \) and \( M_s(F) \), respectively. Finally, if \( m_1, \ldots, m_n \) is a sequence of positive integers, let \( UT(m_1, \ldots, m_n) \) be the upper block triangular matrix algebra of size \( m_1, \ldots, m_n \) over \( F \).

An algebra \( A \) is \( G \)-graded if it has a vector space decomposition \( A = \bigoplus_{g \in G} A_g \) such that \( A_g A_h \subseteq A_{gh} \) for all \( g, h \in G \). The elements of \( A_g \) are called homogeneous of degree \( g \). An element \( w \) of \( A \) is homogeneous if there exists \( g \in G \) such that \( w \in A_g \) (and in this case denote its degree by \( |w|_A \)), whereas a subalgebra or an ideal \( V \subseteq A \) is homogeneous if \( V = \bigoplus_{g \in G} (V \cap A_g) \).

Let \( X_G := \bigcup X_g \) be the union of disjoint countable sets \( X_g := \{x_1^g, x_2^g, \ldots\} \), where \( g \in G \). The free algebra \( F(X_G) \) freely generated by \( X_G \) is equipped in a natural way with a \( G \)-graded algebra structure if we require that the variables from \( X_g \) have degree \( g \) (and then extend this grading to the monomials on \( X_G \)). This algebra is said to be the free \( G \)-graded algebra over \( F \). An element \( f(x_1^g_1, \ldots, x_n^g_n) \) of \( F(X_G) \) is a \( G \)-graded polynomial identity for the \( G \)-graded algebra \( A = \bigoplus_{g \in G} A_g \) if \( f(a_1, \ldots, a_n) = 0_A \) for every \( a_1 \in A_{g_1}, \ldots, a_n \in A_{g_n} \). We use the symbol \( \text{Id}_G(A) \) to indicate the set of all the \( G \)-graded polynomial identities satisfied by \( A \), which is easily seen to be a \( T_G \)-ideal of \( F(X_G) \), namely a two-sided ideal of the free \( G \)-graded algebra stable under every endomorphism of \( F(X_G) \) preserving the grading.

Given a \( T_G \)-ideal \( I \) of \( F(X_G) \), the variety of \( G \)-graded algebras \( \mathcal{V}^G \) associated to \( I \) is the class of all the \( G \)-graded algebras \( A \) such that \( I \) is contained in \( \text{Id}_G(A) \). The \( T_G \)-ideal \( I \) is denoted by \( \text{Id}_G(\mathcal{V}^G) \). The variety \( \mathcal{V}^G \) is generated by the \( G \)-graded algebra \( A \) if \( \text{Id}_G(\mathcal{V}^G) = \text{Id}_G(A) \), and in this case we write \( \mathcal{V}^G = \text{var}_G(A) \) and set \( e_n^G(\mathcal{V}^G) := e_n^G(A) \), the \( n \)-th \( G \)-graded codimension of the variety \( \mathcal{V}^G \). It is defined as the dimension of \( \frac{P_n^G}{\text{Id}_G(\mathcal{V}^G)} \), where \( P_n^G \) is the space of multilinear polynomials of degree \( n \) of \( F(X_G) \) in the variables \( x_i^g \) for \( 1 \leq i \leq n \) and \( g \in G \). Since \( F \) has characteristic zero, \( \text{Id}_G(\mathcal{V}^G) \) is completely determined by the multilinear polynomials it contains. Thus the behaviour of the sequence of its graded codimensions in some sense measures the rate of growth of the \( G \)-graded polynomial identities of \( \mathcal{V}^G \).

It is worth recalling that if \( G \) has more than one element and a \( G \)-graded algebra satisfies an ordinary polynomial identity, obviously it satisfies a \( G \)-graded polynomial identity, but the converse is in general not true: for instance, it is enough to consider the free algebra \( W \) generated by two indeterminates with the trivial \( G \)-grading. Anyway we shall assume throughout that all the \( G \)-graded algebras \( A \) we deal with are PI algebras because, if this happens, the sequence of the graded codimensions \( \{e_n^G(\text{var}_G(A))\}_{n \geq 1} \) is exponentially bounded and, as reported in the Introduction,

\[
\exp_G(\text{var}_G(A)) := \lim_{m \to +\infty} \sqrt[3]{e_m^G(\text{var}_G(A))}
\]
exists and is a non-negative integer, which is called the \textit{G-graded exponent} or simply the \textit{G-exponent} of the variety \( \text{var}_G(A) \).

In the present paper we actually consider finite-dimensional algebras: in fact, we analyze the behaviour of varieties of \( G \)-graded PI algebras of finite basic rank, i.e. generated by a finitely generated \( G \)-graded PI algebra \( W \). But, in this event, Aljadeff and Kanel-Belov \cite{4} have proved that there exist a field extension \( K \) of \( F \) and a finite-dimensional \( G \)-graded algebra \( A \) over \( K \) such that \( \text{Id}_G(W) = \text{Id}_G(A) \). The fact that \( A \) is a \( K \)-algebra has no effect on our final aims because, since \( F \) has characteristic zero, \( \text{Id}_G(W) = \text{Id}_G(W \otimes_F L) \) (in \( F(\langle X_G \rangle) \)) for any field extension \( L \) of \( F \). Consequently also the \( G \)-graded codimensions of \( W \) do not change upon extension of the base field. Hence for the proofs of our main results we can assume that \( F \) (as well as its extensions) is algebraically closed. This is convenient because over algebraically closed fields it is easier to describe special \( G \)-graded algebras. In particular, a result of Bahturin, Sehgal and Zaicev, which completely determines the structure of finite-dimensional \( G \)-graded simple or \( G \)-simple algebras, will be very useful. We recall that a \( G \)-graded algebra \( A \) is said to be \( G \)-simple if the multiplication is non-trivial and it has no non-trivial homogeneous ideals.

\textbf{Theorem 2.1} ([6]). Let \( F \) be an algebraically closed field and \( A \) be a finite-dimensional \( G \)-simple algebra. Then there exists a subgroup \( H \) of \( G \), a 2-cocycle \( f : H \times H \rightarrow F \setminus \{0_F\} \), where the action of \( H \) on \( F \) is trivial, an integer \( k \) and a \( k \)-tuple \((g_1 := 1_G, g_2, \ldots, g_k) \in G^k \) such that \( A \) is \( G \)-graded isomorphic to \( C := F^f H \otimes M_k \), where \( C_g = \text{span}_F \{ b_h \otimes E_{ij} \mid g = g^{-1}_i h g_j \} \). Here \( b_h \in F^f H \) is a representative of \( h \in H \) and \( E_{ij} \in M_k \) is its \((i,j)\)-matrix unit.

Assume now that \( A \) is a finite-dimensional \( G \)-graded algebra over the algebraically closed field \( F \). Hence, by the generalization of the Wedderburn-Malcev Theorem we can write \( A = A_{ss} + J(A) \), where \( A_{ss} \) is a maximal semisimple subalgebra of \( A \) homogeneous in the \( G \)-grading and \( J(A) \) is its Jacobson radical (which is homogeneous as well). Also \( A_{ss} \) can be written as the direct sum of \( G \)-graded simple algebras. In analogy with the definition given by Giambruno and Zaicev in \cite{17} in the case in which \( G = \mathbb{Z}_2 \), we introduce that of a minimal \( G \)-graded algebra.

\textbf{Definition 2.2.} Let \( F \) be an algebraically closed field. A \( G \)-graded algebra \( A \) is called minimal if it is finite-dimensional and either \( A \) is a \( G \)-simple algebra or \( A = A_{ss} + J(A) \) where

(i) \( A_{ss} = A_1 \oplus \cdots \oplus A_n \), with \( A_1, \ldots, A_n \) \( G \)-simple algebras and \( n \geq 2 \);
(ii) there exist homogeneous elements \( w_{12}, \ldots, w_{n-1,n} \in J(A) \) and minimal homogeneous idempotents \( e_i \in A_1, \ldots, e_n \in A_n \) such that

\[
 e_i w_{i,i+1} = w_{i,i+1} e_{i+1} = w_{i,i+1} \quad 1 \leq i \leq n - 1
\]

and

\[
 w_{12} w_{23} \cdots w_{n-1,n} \neq 0_A;
\]

(iii) \( w_{12}, \ldots, w_{n-1,n} \) generate \( J(A) \) as a two-sided ideal of \( A \).
We notice that the order of the components $A_1, \ldots, A_n$ of the semisimple part $A_{ss}$ of a minimal $G$-graded algebra $A$ is important. For this reason, in the sequel we shall tacitly agree that if $A_{ss} = A_1 \oplus \cdots \oplus A_n$, then $A_1 J(A) A_2 J(A) \cdots J(A) A_n \neq 0_A$. According to the main result of [1], $\exp_G(A) = \dim_F A_{ss}$.

It is easily seen that the minimal $G$-graded algebra $A$ has the following vector space decomposition

$$A = \bigoplus_{1 \leq i \leq j \leq n} A_{ij},$$

with

$$A_{ij} := \begin{cases} A_i & \text{if } i = j, \\ A_i w_{i,i+1} A_{i+1} \cdots A_{j-1} w_{j-1,j} A_j & \text{if } i < j. \end{cases}$$

Moreover $J(A) = \oplus_{i<j} A_{ij}$ and $A_{ij} A_{kl} = \delta_{jk} A_{il}$, where $\delta_{jk}$ is the Kronecker delta. Finally, for every $1 \leq r \leq s \leq n$, set $[r, s] := \{r, r+1, \ldots, s\}$,

(1) $$A_{[r,s]} := \bigoplus_{r \leq i \leq j \leq s} A_{ij}$$

and, for each $1 < k < n$,

(2) $$A^{(k)} := \bigoplus_{1 \leq i < j \leq n; \ i \neq k \neq j} A_{ij}$$

where

$$A'_{ij} := \begin{cases} A_i w_{i,i+1} A_{i+1} \cdots A_{k-1} w_{k-1,k} A_{k+1} \cdots A_{j-1} w_{j-1,j} A_j & \text{if } i < k < j, \\ A_{ij} & \text{otherwise.} \end{cases}$$

We remark that both $A_{[r,s]}$ and $A^{(k)}$ are minimal $G$-graded algebras of $G$-exponent $\dim_F(A_i \oplus \cdots \oplus A_s)$ and $\dim_F(A_1 \oplus \cdots \oplus A_{k-1} \oplus A_{k+1} \oplus \cdots \oplus A_n)$, respectively.

The aim of the present paper is to contribute to the classification of minimal varieties of $G$-graded PI algebras of fixed $G$-exponent. We recall the definition.

**Definition 2.3.** A variety $\mathcal{V}^G$ of $G$-graded PI algebras is said to be minimal of $G$-exponent $d$ if $\exp_G(\mathcal{V}^G) = d$ and $\exp_G(\mathcal{U}^G) < d$ for every proper subvariety $\mathcal{U}^G$ of $\mathcal{V}^G$.

The first technical result along the way is the following

**Lemma 2.4.** Let $A$ be a finite-dimensional $G$-graded algebra over an algebraically closed field. Then there exists a minimal $G$-graded algebra $A' \subseteq A$ such that $\exp_G(A) = \exp_G(A')$.

**Proof.** It is sufficient to use exactly the same arguments of the proof of Lemma 8.1.4 of [18] (which deals with the case $G = \mathbb{Z}_2$). $\square$
The next statement represents the core of this section and establishes an important connection among minimal varieties and minimal $G$-graded algebras.

**Theorem 2.5.** Let $\mathcal{V}^G$ be a variety of $G$-graded PI-algebras of finite basic rank. If $\mathcal{V}^G$ is minimal of $G$-exponent $d$, then there exists a minimal $G$-graded algebra $A$ such that $\mathcal{V}^G = \text{var}_G(A)$.

**Proof.** Since $\mathcal{V}^G$ is of finite basic rank, according to Theorem 1.1 of [4] and the deductions before Theorem 2.1 there exists a finite-dimensional $G$-graded algebra $B$ over an algebraically closed field extension $K$ of $F$ such that $\mathcal{V}^G = \text{var}_G(B)$ and $\exp_G(B) = d$. By Lemma 2.4, $B$ contains a minimal $G$-graded algebra $A$ of $G$-exponent $d$. In particular, the algebra $A$ belongs to $\mathcal{V}^G$ and, by the minimality of $\mathcal{V}^G$, we get that $\mathcal{V}^G = \text{var}_G(A)$. □

3. THE $Z_p$-GRADED ALGEBRA $UT_{Z_p}(A_1, \ldots, A_m)$ RELATED TO A MINIMAL $Z_p$-GRADED ALGEBRA

We are interested to see in more detail the special case in which $p$ is a prime and $G$ is isomorphic to the cyclic group of order $p$, denoted by $Z_p$. To this end, some preliminary considerations are in order. A $G$-grading on the complete matrix algebra $M_m$ is called *elementary* if there exists an $m$-tuple $(g_1, \ldots, g_m) \in G^m$ such that the matrix units $E_{ij}$ of $M_m$ are homogeneous and $E_{ij} \in (M_m)_g$ if, and only if, $g = g_i^{-1}g_j$. In an equivalent manner, we can define a map $\alpha : \{1, \ldots, m\} \to G$ inducing a grading on $M_m$ setting the degree of $E_{ij}$ equal to $\alpha(i)^{-1}\alpha(j)$. Obviously, the algebra of upper block triangular matrices admits an elementary grading: in fact, the embedding of such an algebra into a full matrix algebra with an elementary grading makes it a homogeneous subalgebra. To denote that the grading on $M_m$ (as well as on one of its homogeneous subalgebras) is induced by $\alpha$ we write $(M_m, \alpha)$.

Now, let $\epsilon$ be a primitive $p$-th root of unity in $F$ and assume that $G = \langle \epsilon \rangle \cong Z_p$. Consider $M_p$ endowed with the elementary grading induced by the $p$-tuple $(1, \epsilon, \epsilon^2, \ldots, \epsilon^{p-1})$. Given the permutation $\sigma := (1 \ 2 \ \cdots \ p)$ of the symmetric group $S_p$, let us denote by $D$ the homogeneous subalgebra of $M_p$ generated, as a vector space, by the elements

$$c_{k-1} := \sum_{i=0}^{p-1} E_{\sigma^i(1)\sigma^i(k)} \quad \text{with } k \in [1, p].$$

That is,

$$D = \begin{pmatrix}
a_1 & a_2 & \cdots & a_{p-1} & a_p \\
a_p & a_1 & \ddots & \vdots & \vdots \\
\vdots & \ddots & \ddots & \ddots & \vdots \\
a_3 & \ddots & \ddots & a_2 & a_1 \\
a_2 & a_3 & \cdots & a_p & a_1
\end{pmatrix}, \quad \text{where } a_1, a_2, \ldots, a_p \in F,$$
with its natural grading induced by the $p$-tuple $(1_G, \epsilon, \epsilon^2, \ldots, \epsilon^{p-1})$. Directly from Theorem 2.1 one has the following characterization.

**Proposition 3.1.** Let $F$ be an algebraically closed field and $G = \langle \epsilon \rangle \cong \mathbb{Z}_p$ a group of prime order $p$. If $A$ is a finite-dimensional $G$-simple algebra, then it is isomorphic to one of the following $G$-graded algebras:

(i) $M_n$ with an elementary grading;

(ii) $D \otimes M_r$ with the grading induced by the trivial grading on $M_r$ and the natural one on $D$. In other words, in this case, $A$ is isomorphic to the homogeneous subalgebra $M_r(D)$ of $M_{pr}$ with the grading induced by the $(pr)$-tuple $(1_G, \epsilon, \epsilon^2, \ldots, \epsilon^{p-1}, 1_G, \epsilon, \epsilon^2, \ldots, \epsilon^{p-1})$.

**Proof.** From the deductions of Section 3.1 of [7] it follows that, when $A$ is non-simple graded simple, then it is isomorphic as a graded algebra to $M_r(D)$, where $r$ is such that $r^2p = \dim_F A$.

In the case in which $A$ is simple graded simple the result was established in Theorems 5 and 6 of [5].

We are going to strengthen the conclusion of Theorem 2.5 proving that among the minimal $\mathbb{Z}_p$-graded algebras generating a minimal variety of finite basic rank we can always choose a special upper block triangular matrix algebra. To this end, we need to fix some notation.

For the rest of this section assume that $G = \langle \epsilon \rangle \cong \mathbb{Z}_p$ and $F$ is algebraically closed. Given an $m$-tuple $(A_1, \ldots, A_m)$ of $G$-simple algebras, let

$$
\Gamma_0 := \{ i \mid i \in [1, m], A_i \text{ is simple graded simple} \} \quad \text{and} \quad \Gamma_1 := [1, m] \setminus \Gamma_0.
$$

For every $k \in [1, m]$, let us denote the size of $A_k$ by

$$
s_k := \begin{cases} 
n_k & \text{if } k \in \Gamma_0 \text{ and } A_k \cong M_{n_k}, \\
pm_k & \text{if } k \in \Gamma_1 \text{ and } A_k \cong M_{n_k}(D) \subseteq M_{pm_k}.
\end{cases}
$$

and, set $\eta_0 := 0$, let $\eta_k := \sum_{i=1}^{k} s_i$ and $\text{Bl}_k := [\eta_{k-1} + 1, \eta_k]$. Finally, let us define

- $\mathbf{U} := UT(s_1, \ldots, s_m) := \{(a_{ij})_{i,j \in [1,m]} \mid a_{ij} \in M_{s_i \times s_j} \text{ if } 1 \leq i \leq j \leq m \text{ and } a_{ij} = 0_{M_{s_i \times s_j}} \text{ otherwise}\}$;

- $\mathbf{R} := UT(A_1, \ldots, A_m) := \{(a_{ij}) \in \mathbf{U} \mid a_{kk} \in M_{n_k}(D) \forall k \in \Gamma_1\}$;

and, for all $1 \leq r \leq t \leq m$,

- $\mathbf{U}_{r,t} := \{(a_{ij}) \in \mathbf{U} \mid a_{ij} = 0_{M_{s_i \times s_j}} \forall (i,j) \neq (r,t)\}$;

- $\mathbf{R}_{r,t} := \mathbf{R} \cap \mathbf{U}_{r,t}$.

**A canonical basis for** $UT(A_1, \ldots, A_m)$. For any $i \in [1, s_r]$ and $j \in [1, s_t]$, set

$$
E_{ij}^{(r,t)} := E_{\eta_{r-1}+i, \eta_{t-1}+j}.
$$
where \( E_{\eta r - 1 + i, \eta r - 1 + j} \) is the \((\eta r - 1 + i, \eta r - 1 + j)\)-matrix unit of \( M_{\eta m} \). These elements form a basis of \( U_{r,t} \). Consequently they form also a basis of \( R_{r,t} \) in all the cases except those in which \( r = t \in \Gamma_1 \), which we call canonical.

When \( k \in \Gamma_1 \), if \( a := (a_{ij}) \in R_{k,k} \), then \( a_{kk} \) is a \((n_k \times n_k)\)-matrix with entries in \( D \) and \( a_{ij} = 0_{M_i \times M_j} \) for any pair \((i, j) \neq (k, k)\). Hence we can take as a basis of \( R_{k,k} \) the elements \((c_h \otimes e_{ij})^{(k,k)}\), where \( h \in [0, p - 1] \), \( i, j \in [1, n_k] \), the \( c_h \)'s are defined as in (3) and the \( e_{ij} \)'s are the matrix units of \( M_{n_k} \). We refer to this basis of \( R_{k,k} \) as canonical.

**Definition 3.2.** The basis \( B \) of \( R \) consisting of the union of the canonical basis of \( R_{r,t} \) for every \( 1 \leq r \leq t \leq m \) is said to be the canonical basis of \( R \).

Clearly the element \( b := (c_h \otimes e_{ij})^{(k,k)} \) of \( R_{k,k} \) is a sum of \( p \) distinct matrices in \( U_{k,k} \), write

\[
b = E^{(k,k)}_{i1,j1} + E^{(k,k)}_{i2,j2} + \cdots + E^{(k,k)}_{ip,jp}.
\]

In this case, we say that \( E^{(k,k)}_{i1,j1} \) appears in \( b \) and, when it is convenient, set \( b = \bigoplus E^{(k,k)}_{i1,j1} = \cdots = \bigoplus E^{(k,k)}_{ip,jp} \). Moreover one observes that, for every \( i, j \in [1, s_k] \), there exists a unique canonical basis element \( b \) of \( R_{k,k} \) such that \( E^{(k,k)}_{ij} \) appears in \( b \) (and so \( b = E^{(k,k)}_{ij} \)).

Finally, for each \( k \in \Gamma_1 \), the left (and right) indices of the matrix units \( E^{(k,k)}_{ij} \) appearing in the decomposition (4) of \( b = (c_h \otimes e_{ij})^{(k,k)} \) are pairwise distinct. Therefore, using the fact that \( E^{(r,t)}_{ij} E^{(r',t')}_{ij'} = \delta_{i,i'} \delta_{j,j'} E^{(r,t')}_{ij} \), one has that \( B \) is a multiplicative basis of \( R \), that is, for every \( b_1, b_2 \in B \), if \( b_1 b_2 \neq 0_R \) then \( b_1 b_2 \in B \). As a consequence of this fact one has the following useful result.

**Lemma 3.3.** Let \( b_1, \ldots, b_s \in B \) and assume that \( b := b_1 \cdots b_s \neq 0_R \).

(i) If \( b \in A_k \), then \( b_i \in A_k \) for every \( i \in [1, s] \) and

(a) if \( k \in \Gamma_o \), then there exist \( i, j \in [1, s_k] \) such that \( b = E^{(k,k)}_{ij} \).

Furthermore, for every \( \pi \in S_s \) such that \( b^{\pi} := b_{\pi(1)} \cdots b_{\pi(s)} \neq 0_R \), one has that \( b^{\pi} = E^{(k,k)}_{ij} \) when \( i \neq j \), whereas \( b^{\pi} = E^{(k,k)}_{ij} \) for some \( \ell \in [1, s_k] \) otherwise;

(b) if \( k \in \Gamma_1 \), then \( A_k \cong M_{n_k}(D) \) and \( b = (c_h \otimes e_{ij})^{(k,k)} \) for some \( h \in [0, p - 1] \) and \( i, j \in [1, n_k] \). Moreover, if \( b^{\pi} \neq 0_R \) for some \( \pi \in S_s \), then \( b^{\pi} = (c_h \otimes e_{ij})^{(k,k)} \) when \( i \neq j \), whereas \( b^{\pi} = (c_h \otimes e_{ij})^{(k,k)} \) for some \( \ell \in [1, n_k] \) otherwise.

(ii) If \( b \in J(R) \), then there exist \( 1 \leq r < t \leq m \), \( i \in [1, s_r] \) and \( j \in [1, s_t] \) such that \( b = E^{(r,t)}_{ij} \). Moreover if \( b^{\pi} := b_{\pi(1)} \cdots b_{\pi(s)} \neq 0_R \) for some \( \pi \in S_s \), then \( b^{\pi} = E^{(r,t)}_{ij} \).

**Proof.** Part (i) directly follows from Lemma 1 of [20] taking into account that \( D \) is commutative.

The statement (ii) is a consequence of (i) and the fact that the basis \( B \) is multiplicative. \( \square \)
A grading on $UT(A_1, \ldots, A_m)$. Assume that, in the case in which $k \in \Gamma_0$, $A_k$ is isomorphic to $(M_{n_k}, \alpha_k)$ whereas, when $k \in \Gamma_1$, $A_k$ is isomorphic to $(M_{n_k}(D), \alpha_k)$, where $\alpha_k : [1, s_k] \rightarrow G$ is the map inducing the elementary grading on $A_k$. In particular, if $k \in \Gamma_1$, as seen in Proposition 3.1

$$(\alpha_k(1), \ldots, \alpha_k(s_k)) := (1_G, e, \ldots, e^{p-1}, \ldots, 1_G, e, \ldots, e^{p-1}).$$

Let us define the maps

$$\alpha : [1, \eta_m] \rightarrow G, \quad i \mapsto \alpha_k(i - \eta_{k-1})$$

and, for any $m$-tuple $\tilde{g} := (g_1, \ldots, g_m) \in G^m$,

$$\alpha_{\tilde{g}} : [1, \eta_m] \rightarrow G, \quad i \mapsto g_k \alpha(i),$$

where $k \in [1, m]$ is the (unique) integer such that $i \in \mathrm{Bl}_k$. A very simple, but crucial fact is the following

**Proposition 3.4.** The $G$-graded algebra $(UT(A_1, \ldots, A_m), \alpha_{\tilde{g}})$ is a minimal $G$-graded algebra whose $k$-th graded simple component of its maximal semisimple homogeneous subalgebra is isomorphic to $(M_{n_k}, \alpha_k)$, if $k \in \Gamma_0$, and to $(M_{n_k}(D), \alpha_k)$ otherwise. Furthermore, we can require that in these isomorphisms the minimal homogeneous idempotent $e_k$ of $A_k$ corresponds to $E_{11}^{(k,k)}$ if $k \in \Gamma_0$ and to $(e_{1G} \otimes e_{11})^{(k,k)} = E_{11}^{(k,k)} + E_{22}^{(k,k)} + \cdots + E_{pp}^{(k,k)} = \overline{E}_{11}$ when $k \in \Gamma_1$.

When convenient, any such $G$-graded algebra will be simply denoted by $UT_G(A_1, \ldots, A_m)$.

Now, let $A = A_{ss} + J(A)$ be a minimal $G$-graded algebra such that $A_{ss} = A_1 \oplus \cdots \oplus A_m$. Using the same notation for the homogeneous radical elements defining $A$ which appear in Definition 2.2, set

$$g_1 := 1_G, \quad g_k := |w_{12}w_{23} \cdots w_{k-1,k}|_A \quad \forall k \in [2, m]$$

and $\tilde{g} := (g_1, g_2, \ldots, g_m)$. Let us denote the minimal $G$-graded algebra $(UT(A_1, \ldots, A_m), \alpha_{\tilde{g}})$ by $(UT(A_1, \ldots, A_m), \tilde{\alpha}_A)$ in order to stress that the grading on $UT(A_1, \ldots, A_m)$ depends upon that on $A$.

**Definition 3.5.** The $G$-graded algebra $(UT(A_1, \ldots, A_m), \tilde{\alpha}_A)$ is said to be the upper block triangular matrix algebra related to the minimal $G$-graded algebra $A$.

We are in a position to state the main result of this section.

**Proposition 3.6.** Let $G = \langle e \rangle \cong \mathbb{Z}_p$ be a group of prime order $p$ and $A = A_{ss} + J(A)$ a minimal $G$-graded algebra such that $A_{ss} = A_1 \oplus \cdots \oplus A_m$. Then $(UT(A_1, \ldots, A_m), \tilde{\alpha}_A)$ belongs to $\var_G(A)$. In particular, if $\var_G(A)$ is minimal, then $\var_G(A) = \var_G(UT(A_1, \ldots, A_m), \tilde{\alpha}_A)$.

**Proof.** For convenience of notation, let us denote by $R$ the $G$-graded algebra $(UT(A_1, \ldots, A_m), \tilde{\alpha}_A)$. In order to prove that $R \in \var_G(A)$ we proceed by induction on $m$.

If $m = 1$, then $R = UT(A_1) \cong A_1 = A$ and we are done.
Thus, suppose that \( m \geq 2 \) and let \( f = f(x_1^{h_1}, \ldots, x_n^{h_n}) \in F(X_G) \setminus \text{Id}_G(R) \). We aim to show that \( f \) is not a graded polynomial identity for \( A \). To this end, since the characteristic of the ground field is zero, we can assume that \( f \) is multilinear. Hence there exist \( b_1, \ldots, b_n \) in the canonical basis \( B \) of \( R \), with \( |b_i|_R = h_i \) for each \( i \in [1, n] \), such that \( f(b_1, \ldots, b_n) \neq 0_R \).

Let \( s \) be the number of the \( b_k \)'s which are in \( J(R) \). Since \( J(R) \) is nilpotent of index \( m - 1 \), one has that \( s \leq m - 1 \). Assume first that \( s < m - 1 \). Hence there exists \( i \in [1, m - 1] \) such that, for every \( j \in [i + 1, m] \), none of the elements \( b_1, \ldots, b_n \) is in \( R_{i,j} \). At this stage, we have two cases to distinguish. If there exists \( \ell \in [1, n] \) such that \( b_\ell \in R_{r,\ell} \) for some \( r \in [1, i] \), then all the \( b_k \)'s are in

\[
\bigoplus_{1 \leq r \leq s \leq i} R_{r,s} \cong UT(A_1, \ldots, A_i)
\]

with the induced \( G \)-grading and \( f \notin \text{Id}_G(UT(A_1, \ldots, A_i)) \). Otherwise the basis elements \( b_1, \ldots, b_n \) are in

\[
\bigoplus_{1 \leq r \leq s \leq m; \ r \neq i \neq s} R_{r,s} \cong UT(A_1, \ldots, A_{i-1}, A_{i+1}, \ldots, A_m)
\]

with the induced \( G \)-grading and, hence, \( f \notin \text{Id}_G(UT(A_1, \ldots, A_{i-1}, A_{i+1}, \ldots, A_m)) \). But the \( G \)-graded algebras \( UT(A_1, \ldots, A_i) \) and \( UT(A_1, \ldots, A_{i-1}, A_{i+1}, \ldots, A_m) \) are the upper block triangular matrix algebras related to the homogeneous subalgebras \( A^{[1,i]} \) and \( A^{(i)} \) of \( A \) defined in (1) and (2), respectively. Therefore, by the induction assumption, we conclude that \( f \notin \text{Id}_G(A^{[1,i]}) \) in the former case and \( f \notin \text{Id}_G(A^{(i)}) \) in the latter one. As a consequence, in any event \( f \) does not belong to \( \text{Id}_G(A) \).

Thus suppose that \( s = m - 1 \). Hence there exist \( t_1, \ldots, t_{m-1} \in [1, n] \) such that

\[
b_{t_1} = E_{i_1,1}^{(1,2)}, \ldots, b_{t_{m-1}} = E_{i_{m-1},1}^{(m-1,m)},
\]

where \( i_k \in [1, s_k] \) and \( j_k+1 \in [1, s_{k+1}] \) for every \( k \in [1, m-1] \). Moreover all the remaining elements of the set \( \{b_1, \ldots, b_n\} \) are in the diagonal blocks of \( R \). According to Lemma 3.3,

\[
f(b_1, \ldots, b_n) = \beta E_{i_j,1}^{(1,m)}
\]

for some \( \beta \in F \setminus \{0_F\} \), \( i \in [1, s_1] \) and \( j \in [1, s_m] \). Without loss of generality, we can assume that \( b := b_1 \cdots b_n = E_{i_j,1}^{(1,m)} \). Therefore \( \beta = \sum_\pi \beta_\pi \), where \( \pi \) runs over the elements of the symmetric group \( S_n \) such that \( b_1(1) \cdots b_n(n) \) is not zero in \( R \) and \( \beta_\pi \) is the coefficient of the monomial \( x_1^{h_{\pi(1)}} \cdots x_n^{h_{\pi(n)}} \) in \( f \).

Set \( j_1 := i \) and \( i_m := j \), let

\[
b_0 := E_{i_j,1}^{(1,1)} \quad \text{and} \quad b_{n+1} := E_{i_{m-1},1}^{(m,m)},
\]

where, in order to use the same notation, \( E_{r,s}^{(k,k)} := E_{r,s}^{(k,k)} \) for every \( r, s \in [1, s_k] \) whenever \( k \in \Gamma_0 \). Then

\[
b_0 f(b_1, \ldots, b_n)b_{n+1} = E_{i_1,1}^{(1,1)} (\beta E_{i_j,1}^{(1,m)}) E_{i_{m-1},1}^{(m,m)} = \beta E_{11}^{(1,m)}.
\]
Now, for each $k \in [1, m - 1]$, let $v_k \in A_k$ and $z_{k+1} \in A_{k+1}$ be the elements corresponding to $E^{(k,k)}_{i_{k+1}}$ and $E^{(k+1,k+1)}_{j_{k+1}}$ in the isomorphisms $A_k \cong R_{k,k}$ and $A_{k+1} \cong A_{1,k+1}$ of $G$-graded algebras of Proposition 3.4, respectively. Denoting by $a_{tk}$ the element $v_kw_{k+1}z_{k+1}$ of $A$ (recall that $w_{k+1}$ is the $k$-th homogeneous radical element defining the minimal $G$-graded algebra $A$), from the equality

$$E^{(k,k+1)}_{i_{k+1}j_{k+1}} = E^{(k,k+1)}_{i_{k+1}} E^{(k+1,k+1)}_{j_{k+1}}$$

it easily follows that

$$|a_{tk}|_A = |v_k|_A |w_{k+1}|_A |z_{k+1}|_A$$

$$= |E^{(k,k)}_{i_{k+1}}|_R |E^{(k,k+1)}_{i_{k+1}}|_R |E^{(k+1,k+1)}_{j_{k+1}}|_R = |E^{(k,k+1)}_{i_{k+1}j_{k+1}}|_R = |b_{tk}|_R.$$  

Furthermore, setting $t_0 := 0$ and $t_m := n + 1$, we notice that, for every $i \in [t_{k-1} + 1, t_k - 1]$, $b_i \in R_{k,k}$. Let $a_i \in A_k$ be the element corresponding to $b_i$, $z_i := a_0$ that corresponding to $b_0$ in $A_1$ and $v_m := a_n + 1$ that corresponding to $b_{n+1}$ in $A_m$ (in the isomorphisms of Proposition 3.4).

We claim that, for every $\pi \in S_n$, one has that $a_0a_{\pi(1)} \cdots a_{\pi(n)}a_{n+1} \neq 0_A$ if, and only if, $b_{\pi(1)} \cdots b_{\pi(n)} \neq 0_R$. In fact, assume first that $a_0a_{\pi(1)} \cdots a_{\pi(n)}a_{n+1}$ is a non-zero element of $A$. Then $\pi(t_k) = t_k$ for all $k \in [1, m - 1]$. Moreover, for every $q \in [1, m]$, the inequality $t_{q-1} < l < t_q$ implies that $t_{q-1} < \pi(l) < t_q$ since $a_{\pi(l)} \in A_q$. Thus

$$0_A \neq z_1a_{\pi(1)} \cdots a_{\pi(n)}v_m = z_1a_{\pi(1)} \cdots a_{\pi(t_1 - 1)}v_1 \cdots a_{\pi(t_1 + 1)} \cdots$$

$$\cdots a_{\pi(t_2 - 1)}v_2w_2z_3 \cdots v_m - w_{m-1}w_{m-1, m}z_m a_{\pi(t_{m-1} + 1)} \cdots a_{\pi(n)}v_m,$$

which yields the following equivalent statements:

$$z_ka_{\pi(t_k + 1)} \cdots a_{\pi(t_k - 1)}v_k \neq 0_A$$

for all $k \in [1, m]$,

$$E^{(1)}_{j_{k-1}j_k}b_{\pi(t_k + 1)} \cdots b_{\pi(t_k - 1)}E^{(1)}_{i_{k+1}j_k} = 0_R$$

for all $k \in [1, m]$,

$$b_{\pi(t_k + 1)} \cdots b_{\pi(t_k - 1)} = 0_R$$

and

$$b_{\pi(1)} \cdots b_{\pi(n)} \neq 0_R.$$  

Conversely, if $b_{\pi(1)} \cdots b_{\pi(n)} \neq 0_R$, according to what was established above one has that $z_ka_{\pi(t_k + 1)} \cdots a_{\pi(t_k - 1)}v_k \neq 0_A$ for all $k \in [1, m]$. Furthermore this product coincides with the minimal homogeneous idempotent $e_k$ of $A_k$ (since $e_k$ corresponds to $E^{(1,k)}_{i_{k+1}}$ and, consequently, $z_1a_{\pi(1)} \cdots a_{\pi(n)}v_m = e_1w_1w_2w_3 \cdots e_{m-1}w_{m-1, m}e_m = w_12 \cdots w_{m-1, m} \neq 0_A$.

The final outcome of these deductions is that

$$a_0f(a_1, \ldots, a_n)a_{n+1} = \beta w_12 \cdots w_{m-1, m} \neq 0_A.$$  

Hence $f$ is not a graded polynomial identity for $A$, and this is enough to conclude that $\langle UT(A_1, \ldots, A_m), \alpha_A \rangle$ is in $\text{var}_G(A)$.

The final part of the statement is an immediate consequence of the fact that $\exp_G(A) = \exp_G(\langle UT(A_1, \ldots, A_m), \alpha_A \rangle)$. \qed
4. Kemer polynomials for $UT_{Z_p}(A_1, \ldots, A_m)$

Throughout this section assume that $G = \langle e \rangle \cong \mathbb{Z}_p$, $F$ is algebraically closed, $(A_1, \ldots, A_m)$ is an $m$-tuple of $G$-simple algebras and $A := UT_G(A_1, \ldots, A_m)$. The aim is to construct a family of $G$-graded polynomials which are not $G$-graded identities for the algebra $A$ but, as we shall see in the sequel, they are in $\text{Id}_G(UT_G(B_1, \ldots, B_n))$ for any sequence of $G$-simple algebras $(B_1, \ldots, B_n)$ such that $\exp_G(A) = \exp_G(UT_G(B_1, \ldots, B_n))$ whenever either $n \neq m$ or $n = m$ and $\dim_F(A_k)_g \neq \dim_F(B_k)_g$ for some $k \in [1, m]$ and $g \in G$. This is a key point in the proof of the main result. More specifically, these graded polynomials are suitable Kemer polynomials for $A$ (for the definition and more details we refer the reader to [4]).

Suppose first that $m = 1$, which means that $A$ is $G$-simple. If $A = M_n$, then one obtains its matrix unit $E_{11}$ as a product of the $n^2$ different canonical basis elements of $A$. Let us fix such a product and refer to it as the standard total product (of basis elements) of $A$.

If $A = M_n(D)$, considering the standard total product of the matrix units $e_{ij}$ of $M_n$, for each $h \in G$ one has a product of all the $n^2$ different basis elements of $A$ of the form $c_h \otimes e_{ij}$ giving $c_h^2 \otimes e_{11}$. Hence, repeating this process for every $h \in G$, we produce the following product of all the $p m^2$ different canonical basis elements of $A$.

$$\Pi_{h \in G}(c_h^2 \otimes e_{11}) = (\Pi_{h \in G} c_h) n^2 \otimes e_{11} = \begin{cases} c_e \otimes e_{11} & \text{if } n \text{ is odd and } p = 2, \\ c_{1G} \otimes e_{11} & \text{otherwise.} \end{cases}$$

We shall still refer to it as the standard total product (of basis elements) of $A$. Notice that, due to the embedding $M_n(D) \subseteq M_{pn}$, we can write $c_{1G} \otimes e_{11} = E_{11}$ and $c_e \otimes e_{11} = E_{12}$.

At this stage, we construct a monomial $m_A \in F\langle X_G \rangle$ (in pairwise different variables) replacing each element appearing in the standard total product of $A$ with a variable of $X_G$ of the same degree. Clearly, $m_A$ has $n^2$ variables if $A = M_n$ and $p m^2$ when $A = M_n(D)$. An evaluation of $m_A$ (by basis elements of $A$) is said to be the standard total if each variable of $m_A$ is evaluated by the corresponding canonical basis element of $A$ appearing in the standard total product of $A$ (used to define $m_A$).

Assume now that $m$ is an arbitrary positive integer. For each $k \in [1, m]$ and $g \in G$, set

$$d^A_k := \dim_F(A_k), \quad d^A_{k,g} := \dim_F(A_k)_g \quad \text{and} \quad d^A_{ss,g} := \dim_F(A_{ss})_g = \sum_{k \in [1, m]} d^A_{k,g}.$$ 

For any $\nu \geq m$ and $k \in [1, m]$, consider $\nu$ copies of $m_A$ in pairwise disjoint sets of graded variables. For each $i \in [1, \nu]$, let us denote by $m_A^{(i)}$ the $i$-th copy of $m_A$ and by $S(k, i)$ the set of variables of $m_A^{(i)}$, respectively. Clearly we can write $S(k, i) = \bigcup_{g \in G} S(k, i, g)$, where $S(k, i, g)$ is the set of variables of degree $g$ in $S(k, i)$. Hence

$$|S(k, i)| = d^A_k \quad \text{and} \quad |S(k, i, g)| = d^A_{k,g}.$$ 


Furthermore, for every \( i \in [1, \nu] \) and \( g \in G \), let \( T(i, g) := \cup_{k \in [1, m]} S(k, i, g) \). Clearly,
\[
|T(i, g)| = \sum_{k \in [1, m]} d^A_{k,g} = d^A_{ss,g}.
\]

We notice that the standard total evaluations of each monomial \( m_{A_k}^{(i)} \) give a graded evaluation of \( m_{A_k}^{(1)} \cdots m_{A_k}^{(\nu)} \) equal to \( E_{11}^{(k, k)} \) for almost all \( k \in [1, m] \). More precisely, this does not occur only when \( A_k = M_{nk}(D) \), \( n_k \) and \( \nu \) are both odd and \( p = 2 \). In this case the same evaluation coincides with \( E_{11}^{(k, k)} \) and we shall say that \( (k, \nu, p) \) is an exception.

Now, for each \( j \in [1, m - 1] \), consider the homogeneous radical element \( E_{11}^{(j, j+1)} \) of \( A \) if \( (j, \nu, p) \) is not an exception, and \( E_{21}^{(j, j+1)} \) otherwise. Let us call \( g_j \) its degree and choose a variable \( z_j \) of degree \( g_j \) such that all the elements of \((\cup_{i \in [1,\nu],k \in [1,m]} S(k, i)) \cup (\cup_{j \in [1,m-1]} \{z_j\})\) are pairwise different. Setting \( Z_j := T(j, g_j) \cup \{z_j\} \), one has that
\[
|Z_j| = d^A_{ss,g_j} + 1.
\]

Moreover there exists a graded evaluation of the product
\[
\pi_{A,\nu} := m_{A_1}^{(1)} \cdots m_{A_1}^{(\nu)} \cdot m_{A_2}^{(1)} \cdots m_{A_2}^{(\nu)} \cdots \cdot m_{A_m}^{(1)} \cdots m_{A_m}^{(\nu)},
\]
by canonical basis elements of \( A \), equal to \( E_{11}^{(1,m)} \) if \( (m, \nu, p) \) is not an exception, and to \( E_{12}^{(1,m)} \) otherwise.

We extend the monomial \( \pi_{A,\nu} \) by inserting \( \nu(\dim_F A_{ss}) + m \) pairwise different variables of degree 1 of \( A \) and not involved in \( \pi_{A,\nu} \), bordering each variable appearing in \( \pi_{A,\nu} \). We denote the monomial obtained in this way by \( \tilde{\pi}_{A,\nu} \). For each \( k \in [1, m] \) and \( i \in [1, \nu] \), let \( Y(k, i) \) be the set of all the variables inserted on the left of the variables of the set \( S(k, i) \). Still, for each \( k \in [1, m] \), let \( \tilde{y}_k \) be the variable inserted on the right of the monomial \( m_{A_k}^{(\nu)} \) and therefore, for each \( k \in [1, m - 1] \), on the left of the variable \( z_k \) and set \( Y_k := \cup_{i \in [1,\nu]} Y(k, i) \cup \{\tilde{y}_k\} \).

For each \( j \in [1, m - 1] \) we alternate in \( \tilde{\pi}_{A,\nu} \) the variables of the set \( Z_j \) and, for each \( i \in [m, \nu] \) and \( g \in G \), those of \( T(i, g) \), respectively. We denote the graded polynomial obtained in this way (which clearly depends upon \( \nu \)) by \( f_{A,\nu} \). We prove the following

**Lemma 4.1.** For every \( \nu \geq m \) the polynomial \( f_{A,\nu} \) is not a \( G \)-graded polynomial identity for the \( G \)-graded algebra \( A := UT_G(A_1, \ldots, A_m) \).

**Proof.** For every \( k \in [1, m] \) and \( i \in [1, \nu] \), consider the standard total evaluation \( S(k, i) \) of the monomial \( m_{A_k}^{(i)} \) in \( A \). In particular, if \( k \in \Gamma_0 \), for each variable \( v_t^{(i)} \in S(k, i) \) one has that \( \pi_t^{(i)} = E_{rs}^{(k, k)} \) for some \( r, s \in [1, s_k] \). In this case, evaluate the variable \( y_t^{(i)} \in Y(k, i) \) appearing on the left of \( v_t^{(i)} \) by \( E_{rr}^{(k, k)} \). Since \( \tilde{y}_1^{(i)} \pi_t^{(i)} \cdots \tilde{y}_k^{(i)} \pi_t^{(i)} \) \( \tilde{d}^A_{k} \) \( \pi_t^{(i)} \) = \( E_{11}^{(k, k)} \), evaluate the variable \( \tilde{y}_k \) by \( E_{11}^{(k, k)} \).

Similarly, if \( k \in \Gamma_1 \), for each \( v_t^{(i)} \in S(k, i) \) it holds that \( \pi_t^{(i)} = (c_h \otimes \varepsilon_{rs})^{(k, k)} \) for some \( c_h \in D \) and \( r, s \in [1, n_k] \), and then evaluate the variable \( y_t^{(i)} \in Y(k, i) \)
appearing on the left of $v_i^{(i)}$ by $(c_{1G} \otimes e_{rr})^{(k,k)}$. As $\mathcal{Y}_1^{(i)} \mathcal{Y}_2^{(i)} \cdots \mathcal{Y}_d^{(i)} = \mathcal{Y}_d^{(i)}$ is equal either to $(c_{1G} \otimes e_{11})^{(k,k)} = E_{11}^{(k,k)}$ or to $(c_s \otimes e_{11})^{(k,k)} = E_{12}^{(k,k)}$ (when $n_k$ is odd and $p = 2$), evaluate the variable $y_k$ by $(c_{1G} \otimes e_{11})^{(k,k)} = E_{11}^{(k,k)} = E_{12}^{(k,k)}$. Finally, for every $j \in [1, m - 1]$, set $z_j := E_{11}^{(j,j+1)}$ if $(j, \nu, p)$ is not an exception, and $z_j := E_{21}^{(j,j+1)}$ otherwise. In this way we have obtained a graded evaluation of the monomial $\tilde{\pi}_{A,\nu}$ in $A$ equal to $E_{11}^{(1,m)}$ if $(m, \nu, p)$ is not an exception, and to $E_{12}^{(1,m)}$ otherwise. For simplicity, let us denote such an evaluation by $\mathcal{S}_A$.

Now, fixed $i \in [m, \nu]$ and $g \in G$, take a permutation $\sigma$ of the variables of $\tilde{\pi}_{A,\nu}$ which possibly moves only the elements of $T(i, g)$ and assume that the evaluation of the monomial $\sigma(\tilde{\pi}_{A,\nu})$ in $A$ by $\mathcal{S}_A$ is non-zero. We claim that $\sigma$ must be the identity permutation. In fact, since $z_1, \ldots, z_{m-1}$ are not moved by $\sigma$ and $ad' = 0_A$ for any $a \in A_r$ and $a' \in A_s$ when $r \neq s$, then $\sigma$ permutes the variables of $S(k, i, g)$ for each $k \in [1, m]$. Moreover in each monomial of $f_{A,\nu}$, the variables in $Y_k$ appear in the same order. Therefore, by the choice of the elements in $\mathcal{Y}_k$, we conclude that, in order to not get zero, the evaluation of each variable $v_i^{(i)}$ is uniquely determined by those of the bordering variables and by the degree of $v_i^{(i)}$. Since we do not alternate variables of different degrees, the claim is confirmed.

Finally, for each $j \in [1, m - 1]$, applying the same arguments presented above one has that, for any non-trivial permutation $\tau$ of the variables of the set $Z_j$ in $\tilde{\pi}_{A,\nu}$, the evaluation of the monomial $\tau(\tilde{\pi}_{A,\nu})$ by $\mathcal{S}_A$ is zero.

The outcome of all these deductions is that $\tilde{\pi}_{A,\nu}$ is the unique monomial of $f_{A,\nu}$ which does not vanish under the evaluation by $\mathcal{S}_A$, and this concludes the proof.

We shall refer to the graded polynomial $f_{A,\nu}$ as a Kemer polynomial for $A$.

**Remark 4.2.** The polynomial $f_{A,\nu}$ is constructed starting, for each $G$-simple algebra $A_k$, from a choice of a product of all the canonical basis elements whose result is a fixed canonical basis element. In particular, if $A_1$ and $A_m$ are simple graded simple, namely $A_1 = M_{n_1}$ and $A_m = M_{n_m}$, taking $i \in [1, n_1]$ and $j \in [1, n_m]$, we may fix new standard total products (of basis elements) of $A_1$ and $A_m$ equal to $E_{i1}^{(1,1)}$ and $E_{ij}^{(m,m)}$, respectively. Making the necessary changes in the degrees of the variables involved in $\pi_{A,\nu}$ we obtain a new Kemer polynomial, denoted by $f_{A,\nu}^{ij}$, for which there exists a suitable graded evaluation in $A$ equal to $E_{ij}^{(1,m)}$. Clearly $f_{A,\nu}$ is one of the polynomials $f_{A,\nu}^{ij}$.

5. **Proof of the main result**

Throughout this section assume that $G = \langle \epsilon \rangle \cong \mathbb{Z}_p$. According to Theorem 2.5 and Proposition 3.6 one has that any minimal variety of $G$-graded PI algebras of finite basic rank of fixed $G$-exponent $d$ is generated by a
suitable $G$-graded upper block triangular matrix algebra $UT_G(A_1, \ldots, A_m)$ satisfying $\dim F(A_1 \oplus \ldots \oplus A_m) = d$ (we shall freely use in the sequel that $\exp_G(UT_G(A_1, \ldots, A_m)) = \dim F(A_1 \oplus \ldots \oplus A_m)$). Our final goal is to provide a characterization of these varieties proving that any $G$-graded algebra of the form $UT_G(A_1, \ldots, A_m)$ generates a minimal variety. To this end, the crucial step consists of relating, for any pair of sequences $(A_1, \ldots, A_m)$ and $(B_1, \ldots, B_n)$ of $G$-simple algebras, the $G$-graded polynomial identities of the $G$-graded algebras $A := UT_G(A_1, \ldots, A_m)$ and $B := UT_G(B_1, \ldots, B_n)$ satisfying $\exp_G(A) = \exp_G(B)$ with the existence (or not) of a graded isomorphism between them.

The first result in this direction concerns the algebraic structures of $A$ and $B$ under a condition a bit weaker than we need, namely $\exp_G(A) = d_A^A \geq d_{ss}^B = \exp_G(B)$. A key ingredient is the Kemer polynomial $f_{A,\nu}$ presented in the previous section. The notation we shall use is introduced there.

**Lemma 5.1.** Let $A := UT_G(A_1, \ldots, A_m)$ and $B := UT_G(B_1, \ldots, B_n)$ be $G$-graded algebras satisfying $d_A^A \geq d_{ss}^B$ and $\nu := m + n - 1$. If $f_{A,\nu}$ is not a $G$-graded polynomial identity for $B$, then the following conditions hold:

(i) $d_{ss,g}^B = d_{ss,g}^A$ for every $g \in G$;
(ii) $n = m$;
(iii) $d_{k,g}^B = d_{k,g}^A$ for every $k \in [1, m]$ and $g \in G$.

**Proof.** Since $f_{A,\nu}$ is a multilinear graded polynomial and $f_{A,\nu} \notin \text{Id}_G(B)$, there exists a non-zero graded evaluation $\overline{f}_B$ of $f_{A,\nu}$ by canonical basis elements of $B$. Moreover we may assume, without loss of generality, that $\overline{f}_B$ is a non-zero graded evaluation of the monomial $\overline{f}_{A,\nu}$ of $f_{A,\nu}$. As $J(B)$ is nilpotent of index $n$ and $[m, m + n - 1] = n$, one has that there exists $m \leq \ell \leq m + n - 1$ such that the variables of $\cup_{g \in G} T(\ell, g) = \cup_{k \in [1, m]} S(k, \ell)$ as well as all those of $\cup_{k \in [1, m]} Y(k, \ell)$ must be necessarily evaluated only by semisimple elements in $\overline{f}_B$.

The fact that, for every $g \in G$, $f_{A,\nu}$ alternates in the set $T(\ell, g)$ forces $|T(\ell, g)| \leq d_{ss,g}^B$, and thus (5) yields

$$d_{ss,g}^A = |T(\ell, g)| \leq d_{ss,g}^B \quad \forall g \in G.$$

Combining the above inequality with the original assumption we obtain

$$d_{ss}^A = \sum_{g \in G} d_{ss,g}^A \leq \sum_{g \in G} d_{ss,g}^B = d_{ss}^B \leq d_{ss}^A$$

and, consequently, that $d_{ss,g}^A = d_{ss,g}^B$ for every $g \in G$, which concludes the proof of item (i).

Now, $\cup_{g \in G} T(\ell, g) = \cup_{k \in [1, m]} \overline{S}(k, \ell)$ is a total evaluation of the product $m_1^{(i)} \cdots m_m^{(i)}$ by canonical basis elements of $B_{ss}$, i.e. an evaluation involving all (and only) the canonical basis elements of $B_{ss}$ (and, hence, each one exactly once). Furthermore, as $bb' = 0_B$ for any $b \in B_i$ and $b' \in B_j$ whenever $i \neq j$, for each $k \in [1, m]$ the monomial $m_k^{(i)}$ must be evaluated in a unique block of $B_{ss}$. At this stage assume, if possible, that $m < n$. Then there
exist $h \in [1, m]$ and at least two elements from different $B_i$’s appearing in the same $S(h, \ell)$, which is a contradiction. Hence $n \leq m$.

On the other hand, for every $j \in [1, m-1]$, $f_{A, \nu}$ alternates in the set $Z_j$, which, according to (6), has cardinality

$$|Z_j| = d^A_{ss, s_j} + 1 = d^B_{ss, s_j} + 1.$$ 

Therefore we must have at least $m - 1$ canonical basis elements of $J(B)$ in $\overline{S_B}$. But this implies that $m - 1$ is smaller than the nilpotent index of $J(B)$. The combination with the above disequality gives item (ii).

Finally, as we have seen above, one has that, for every $k \in [1, m]$ the monomial $m^{(\ell)}_{Ak}$ must be evaluated in $B_k$. Since $\overline{S(k, \ell)}$ is a total evaluation of $m^{(\ell)}_{Ak}$ by canonical basis elements of $B_k$, item (iii) follows looking at the number of variables of $m^{(\ell)}_{Ak}$ of degree $g$. \hfill \Box

The next and crucial step is to show that, if $A$ and $B$ have the same $G$-exponent, the condition $\text{Id}_G(B) \subseteq \text{Id}_G(A)$ is sufficient to guarantee that they are isomorphic. It is, in some sense, a generalization of Theorem 3.3 of [11] when $G$ is a group of prime order $p$ (and it was actually proved there for the particular case in which all the algebras $A_1, \ldots, A_m$ are simple graded simple). To this aim we shall use as a main tool the concept of invariance subgroup of a graded simple algebra still introduced in [11]. We recall the definition and fix some notation.

Given the $G$-graded algebra $A := UT_G(A_1, \ldots, A_m)$, assume that $\alpha_A : \bigcup_{k=1}^{m} \text{Bl}_k \to G$ is the map inducing the grading on $A$. For any $1 \leq r \leq s \leq m$ let us denote by $\alpha_A^{[r,s]}$ the restriction of $\alpha_A$ to $\bigcup_{k \in [r,s]} \text{Bl}_k$ and set $\alpha_A^{(r)} := \alpha_A^{[r,r]}$. Thus $A^{[r,s]} \cong (UT(A_r, \ldots, A_s), \alpha_A^{[r,s]})$ and $A^{[r,r]} \cong (A_r, \alpha_A^{(r)})$. In the last case, when convenient, we shall simply write $A_r$. For each $k \in [1, m]$ set

$$w^{(k)}_A(g) := \{ s | s \in \text{Bl}_k, \; \alpha_A^{(k)}(s) = g \} \; \forall \; g \in G.$$ 

The subgroup

$$H^{(k)}_A := \{ h \in G, \; w^{(k)}_A(hg) = w^{(k)}_A(g) \; \text{for all} \; g \in G \}$$

of $G$ is said to be the invariance subgroup of $A_k$. It is immediate that, since $G$ has prime order, either $H^{(k)}_A = (1_G)$ or $H^{(k)}_A = G$.

**Proposition 5.2.** Let $A := UT_G(A_1, \ldots, A_m)$ and $B := UT_G(B_1, \ldots, B_n)$ be $G$-graded algebras satisfying $d^A_{ss} = d^B_{ss}$. If $\text{Id}_G(B) \subseteq \text{Id}_G(A)$, then $A$ and $B$ are isomorphic as $G$-graded algebras.

**Proof.** Since $\text{Id}_G(B) \subseteq \text{Id}_G(A)$, Lemma 4.1 yields that $f_{A, \nu} \notin \text{Id}_G(B)$ for all $\nu \geq m$. In particular, taking $\nu := m + n - 1$, from Lemma 5.1 it follows that $n = m$ and $d^B_{k,g} = d^A_{k,g}$ for every $k \in [1, m]$ and $g \in G$. Consequently,

$$d^A_k = d^B_k \quad \forall \; k \in [1, m].$$

The fact that $\dim_F M_{n_k}(D) = pn_k^2$ whereas $\dim_F M_{n_k} = n_k^2$ allows us to conclude that $A_k$ is non-simple graded simple if, and only if, so is $B_k$. 

16
Assume first that $A_k \cong M_{n_k}(D) \subseteq M_{p n_k}$. From Proposition 3.1 and its proof it follows that $A_k$ must be isomorphic to $B_k$ (and hence we are done when $m = 1$ and $A_1 = A$ is non-simple graded simple). Furthermore, as seen in Section 3,

$$(\alpha_{A}^{(k)}(\eta_{k-1}+1), \ldots, \alpha_{A}^{(k)}(\eta_{k-1}+pn_k)) = (h_k, h_k \epsilon, \ldots, h_k e_{p-1}, \ldots, h_k, h_k \epsilon, \ldots, h_k e_{p-1})$$

for some $h_k \in G$ (and the same is true for $\alpha_B^{(k)}$ for a suitable $h_k' \in G$). This implies that

$$w_{A}^{(k)}(g) = w_{A}^{(k)}(gh) = n_k = w_{B}^{(k)}(g) = w_{B}^{(k)}(gh) \quad \forall g, h \in G,$$

and hence $H_{A}^{(k)} = H_{B}^{(k)} = G$.

If $A_k$ is simple graded simple, from the previously presented arguments we get that there exists an isomorphism of (ordinary) algebras between $A_k$ and $B_k$, which is not necessarily a $G$-graded isomorphism. More precisely, we may conclude that $A_k$ and $B_k$ are isomorphic to the same matrix algebra $M_{n_k}$, and hence $A$ is isomorphic to $B_1 = B$ as a graded algebra, and the desired conclusion holds.

Therefore assume that $m \geq 2$ and let us consider the $G$-graded algebras $A^{[1,m-1]}$ and $B^{[1,m-1]}$. We claim that $\text{Id}_G(B^{[1,m-1]}) \subseteq \text{Id}_G(A^{[1,m-1]})$. In fact, suppose, if possible, that the inclusion does not hold. Hence there exists a polynomial $f_1 \in \text{Id}_G(B^{[1,m-1]}) \setminus \text{Id}_G(A^{[1,m-1]})$. By virtue of Lemma 4.1, the Kemer polynomial $f_{A^{[m-1,m]}}$ (whose variables can be assumed to be pairwise different from those involved in $f_1$) is not a graded identity for $A^{[m-1,m]}$. But, from (7), one has that

$$d_{ss}^{[m-1,m]} = d_{m-1}^A + d_m^A > d_m^B = d_{ss}^B$$

and, applying Lemma 5.1, obtains that $f_{A^{[m-1,m]}} \in \text{Id}_G(B_m)$. Let $u := \sum_{\ell \in G} x_{\ell}$, where the $x_{\ell}$’s are pairwise different variables of degree $l$ of $F\langle X_G \rangle$ involved neither in $f_1$ nor in $f_{A^{[m-1,m]}}$. Then $f_1 uf_{A^{[m-1,m]}}$ is not a $G$-graded polynomial identity for the algebra $A$. But $f_1 uf_{A^{[m-1,m]}} \in \text{Id}_G(B^{[1,m-1]})$, $\text{Id}_G(B_m) \subseteq \text{Id}_G(B)$, which contradicts the original assumption that $\text{Id}_G(B) \subseteq \text{Id}_G(A)$.

Arguing in the same manner we obtain that $\text{Id}_G(B^{[2,m]}) \subseteq \text{Id}_G(A^{[2,m]})$ and, continuing in a similar fashion, that

$$(8) \quad \text{Id}_G(B^{[k,k']}) \subseteq \text{Id}_G(A^{[k,k']}) \quad \forall 1 \leq k \leq k' \leq m$$

and hence

$$\text{Id}_G(B_k) \subseteq \text{Id}_G(A_k) \quad \forall k \in [1,m].$$

This inclusion allows us to directly apply Corollary 2.3 of [11].

In particular, the combination of the last mentioned result with what we proved when $A_k$ and $B_k$ are non-simple graded simple guarantees for each $k \in [1,m]$ the existence of an element $g_k \in G$ such that

$$w^{(k)}_A(g_k x) = w^{(k)}_B(x) \quad \forall x \in G,$$
that $A_k$ is isomorphic to $B_k$ as a $G$-graded algebra and $H_A^{(k)} = H_B^{(k)}$. For this reason, in the rest of the proof let us denote $H_A^{(k)}$ simply by $H^{(k)}$.

In order to prove that $A$ and $B$ are isomorphic as $G$-graded algebras, it is enough to show that there exists $g \in G$ such that

$$w_A^{(k)}(gx) = w_B^{(k)}(x) \quad \text{for all } k \in [1, m] \text{ and } x \in G.$$  

In fact, if the equality occurs, one can find a permutation $\sigma \in S_{hm}$, satisfying $\sigma(\Gamma_k) = \Gamma_k$ for each $k \in [1, m]$ (and preserving the structure of $A_k$ when $k \in \Gamma_1$) such that the restriction of the endomorphism $\tau$ of $M_{hm}$ induced by $\sigma$ (namely, that defined on the matrix units $E_{ij}$ of $M_{hm}$ by $\tau(E_{ij}) := E_{\sigma(i),\sigma(j)}$) to $A$ is an isomorphism of graded algebras from $A$ into $B$.

To this end we observe that if the integer $k \in [1, m]$ is such that $H^{(k)} = G$, then, for every $g, x \in G$,

$$w_A^{(k)}(gx) = w_A^{(k)}(x) = w_A^{(k)}(gx) = w_B^{(k)}(x).$$

Consequently, if $H^{(k)} = G$ for every $k \in [1, m]$, the equality (9) is satisfied for any choice of $g \in G$. On the other hand, if there exists a unique integer $\ell \in [1, m]$ such that $H^{(\ell)} = \langle 1_G \rangle$, it is sufficient to set $g := g_{\ell}$ to obtain (9).

Therefore we are left with the case in which there exist at least two different integers $k_1, k_2 \in [1, m]$ such that $H^{(k_1)} = H^{(k_2)} = \langle 1_G \rangle$. As $H^{(k)} = G$ for every $k \in \Gamma_1$, one has that $|\Gamma_0| \geq 2$. Now, for any pair of elements $r < s$ in $\Gamma_0$, we replace the polynomial $\Psi_{ij}^{A}$ appearing in the proof of Theorem 3.3 of [11] with the Kemer polynomial $f_{ij}^{A_{r,s}}$ introduced in Remark 4.2. Applying the same arguments used there (replacing the inclusion (1) appearing in that proof with (8)), we conclude that $g^{-1}_{r}g_{s} \in H^{(r)}H^{(s)}$. Invoking again the fact that $H^{(k)} = G$ when $A_k$ is non-simple graded simple, we get that

$$g_r^{-1}g_s \in H^{(r)}H^{(s)} \quad \forall 1 \leq r < s \leq m.$$  

Since there exists $d \in [1, m]$ such that $H^{(d)} = \langle 1_G \rangle$, following again the same reasonings of the final part of the proof of Theorem 3.3 in [11], equation (9) occurs setting $g := g_d$.  

An immediate consequence of the previous result is the following statement, which is in the same spirit of what was proved in [3] for $G$-simple algebras.

**Corollary 5.3.** Let $A := UT_G(A_1, \ldots, A_m)$ and $B := UT_G(B_1, \ldots, B_n)$ be $G$-graded algebras satisfying $d_s^A = d_s^B$. Then $A$ and $B$ are isomorphic as $G$-graded algebras if, and only if, $\text{Id}_G(A) = \text{Id}_G(B)$.

We are now in a position to state the main result of the paper, namely a characterization of minimal varieties of $G$-graded PI algebras of finite basic rank.

**Theorem 5.4.** Let $F$ be a field of characteristic zero and $G$ a group of prime order $p$. A variety $\mathcal{V}^G$ of $G$-graded PI-algebras of finite basic rank is minimal of $G$-exponent $d$ if, and only if, it is generated by a $G$-graded algebra $UT_G(A_1,\ldots,A_m)$ satisfying $\dim_F(A_1 \oplus \cdots \oplus A_m) = d$.  

18
Proof. If $V^G$ is minimal of $G$-exponent $d$, from Theorem 2.5 and Proposition 3.6 it follows that $V^G$ is generated by a suitable $G$-graded algebra $UT_G(A_1, \ldots, A_m)$ satisfying $\dim_F(A_1 \oplus \cdots \oplus A_m) = d$.

Conversely, set $A := UT_G(A_1, \ldots, A_m)$ and consider a subvariety $U^G \subseteq \text{var}_G(A)$ such that $\exp_G(U^G) = \exp_G(\text{var}_G(A))$. Since $\text{var}_G(A)$ satisfies some Capelli identities, according to what was established in Section 7.1 of [4] we conclude that $U^G$ has finite basic rank. Hence, by virtue of Theorem 1.1 of [4], $U^G$ is generated by a finite-dimensional $G$-graded algebra $\hat{B}$ (we recall that, as remarked in Section 2, we can extend the basis field and assume that it is algebraically closed and this has no effect on the codimensions). Lemma 2.4 yields that there exists a minimal $G$-graded algebra $B'$ such that $\text{Id}_G(\hat{B}) \subseteq \text{Id}_G(B')$ and $\exp_G(\hat{B}) = \exp_G(B')$. In particular, Proposition 3.6 guarantees the existence of a $G$-graded algebra $B := UT_G(B_1, \ldots, B_n)$ such that $\text{Id}_G(B') \subseteq \text{Id}_G(B)$ and $\exp_G(B') = \exp_G(B)$. Therefore $\text{Id}_G(A) \subseteq \text{Id}_G(B)$ and $\exp_G(A) = \exp_G(B)$. At this stage, we can directly apply Proposition 5.2 and conclude that $A$ is isomorphic to $B$ as a $G$-graded algebra. Consequently $\text{Id}_G(A) = \text{Id}_G(B)$, and this completes the proof. \]

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References


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