Optimal portfolio allocation with volatility and co-jump risk that Markowitz would like*

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April 14, 2018

Abstract

We study a continuous time optimal portfolio allocation problem with volatility and co-jump risk, allowing prices, variances and covariances to jump simultaneously. Differently from the traditional approach, we deviate from affine models by specifying a flexible Wishart jump-diffusion for the co-precision (the inverse of the covariance matrix). The optimal portfolio weights which solve the dynamic programming problem are genuinely dynamic and proportional to the instantaneous co-precision, reconciling optimal dynamic allocation with the static Markowitz-type economic intuition. An application to the optimal allocation problem across hedge fund investment styles illustrates the importance of having jumps in volatility associated with jumps in price.

Keywords: asset allocation, stochastic volatility, co-jumps, co-precision, Wishart process, dynamic programming, HJB equation, hedge funds.

JEL classification: C61, G11

*We thank Claudio Tebaldi, Martino Grasselli, and Alessandro Sbuelz, as well as the participants to the QFW2017 in University of Milano Bicocca, Italy, to the QFW2018 in the University of Roma II, Italy and the 4th Young scholar conference, University of Sussex, for comments and suggestions. Errors and omissions remain our own.

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1 Introduction

Long-term investors in financial markets fear sudden drops in asset prices, especially since empirical research showed that market crashes are accompanied by large spikes in volatilities and covariances. This is what happened, for example, during the well-known *Black Monday* of October 1987, or the market crash of October 2008. In the continuous-time literature, contemporaneous spikes in the level of asset prices (typically, negative) and in their volatility (typically, positive) are modeled using co-jumps, see e.g. Duffie, Pan, and Singleton (2000), Eraker, Johannes, and Polson (2003), Eraker (2004), Chernov, Gallant, Ghysels, and Tauchen (2003), Todorov and Tauchen (2011) and Bandi and Renò (2016), among others. Credible dynamic asset allocation models in continuous time should account for this kind of tail risks, and, at the same time, preserve standard precepts of portfolio theory, the main one being that, all other things being equal, risk-averse investors should invest less in more volatile assets. Unfortunately, this does not happen to be the case in the majority of the models considered in the literature.

This paper presents a mathematical model of dynamic asset allocation in continuous time with volatility and co-jump risks which is mathematically tractable, and such that the optimal allocation is proportional to the expected excess returns and to the inverse covariance matrix, as in the standard Markowitz theory. More precisely, we consider a risk-averse investor who allocates wealth in an economy in which the riskless asset (e.g., a bond) has constant rate of return and the risky assets (e.g., the stocks) are characterized by an expected constant return, a time-varying diffusion term, allowing for stochastic volatility and correlations, and a jump component. The state of this economy is summarized by the value of the covariance matrix among the risky assets. Generalizing the dynamics in Chacko and Viceira (2005) to a multivariate setting, we specify the dynamics of the covariance matrix of the risky assets in terms of its inverse, named *co-precision*, which is a key ingredient toward our proposed solution. The co-precision is assumed to evolve according to a mean-reverting Wishart process, that is a flexible multivariate affine process, with a jump component which can affect both the diagonal and the off-diagonal terms, meant to account for spikes in variances and covariances. We assume that jumps in price and in co-precision occur simultaneously, although they have different random amplitudes and could be correlated. From a mathematical point of view, such an assumption implies that there exists a unique counting process driving all the jumps, as in Das and Uppal (2004).

Our contribution to the existing literature is twofold. The first contribution is to provide an exact solution to the optimal allocation problem in the case without co-jumps when the inverse covariance matrix follows a Wishart process. As in Chacko and Viceira (2005) in the univariate case, the optimal allocation in our non-affine multivariate model is proportional to the inverse covariance matrix, and inversely proportional to the risk-aversion coefficient, with proportionality coefficients which, roughly speaking, depend on expected excess returns (myopic component) and the covariance between returns and volatility (hedging component). The second contribution is to show how to obtain an approximate solution for the optimal allocation strategy in the more realistic case in which co-jumps are present. Also in this case, the optimal allocation is proportional to the inverse covariance matrix and inversely proportional to the risk-aversion coefficient. A third term appears, representing the jump hedging component, which depends on the correlation between the jump amplitudes in prices and in the covariance matrix.

Dynamic portfolio allocation in continuous time has already extensively analyzed these issues starting from the seminal contribution of Merton's model (Merton, 1971). In this literature, the trade-off is usually between obtaining tractable (hopefully, closed-form) solutions and, at the same time, include all the sources of risk which are thought to determine the investors's choice. However, as we argue, there is typically a tension between these two requisites. The traditional approach to deal with volatility and jump risk for portfolio allocation is to use affine stochastic volatility models, see Liu (2007); Buraschi, Porchia, and Trojani (2010); Bäuerle and Li (2013) for models without jumps, and Liu, Longstaff, and Pan (2003); Hong and Jin (2016); Branger, Muck, Seifried, and Weisheit (2017) for models with jumps. The assumed price/volatility dynamics in these models, in the spirit of Duffie, Pan, and Singleton (2000), delivers attractive closed-form solutions for the optimal portfolio strategy. However, imposing the affine structure restricts their optimal allocation to be static and, more worryingly, independent on the covariance matrix of the risky assets (in the sense explained in detail in Appendix B of this paper, and discussed, e.g., in Liu, 2007, page 30-31), while what matters is the size of the risk premium, that is the coefficient linking instantaneous expected returns to the instantaneous covariance matrix. Thus, while mathematically neat, these results are impractical since they imply, for example, that the optimal allocation in risky assets does not depend on their variance.[1] Such volatility-independent allocation is unpalatable since it is just against the simple intuition of portfolio theory à la Markowitz.

Outside the affine world, Chacko and Viceira (2005) find an approximate solution to the optimal consumption problem of the investor with stochastic volatility, which has a closed form under some non-trivial parameter restrictions, and an approximate solution in the general case. However, they neither account for jump risk, nor for stochastic correlations. In a similar framework, Kraft, Seiferling, and Seifred (2016) write the solution of a consumption problem with stochastic volatility, without jumps, in terms of the solution of a partial differential equation, to be obtained numerically. Das and Uppal (2004), extended by Sbuelz (2017) for default risk, consider a multivariate system affected by a systemic jump component, and provide a nearly-closed form solution, but the volatilities of the stocks are constant so that there is no volatility risk. Ascheberg, Branger, Kraft, and Seifried (2016) solve the allocation problem with jumps in price only, by approximating them by diffusing components.

Our paper tries to reconcile dynamic asset allocation under both volatility and co-jump risk with Markowitz theory, by proposing a multivariate non-affine dynamics which extends the univariate model of Chacko and Viceira (2005). In the realistic case in which co-jumps are present, we solve for the allocation strategy under a suitable approximation which is similar to that in Ascheberg et al. (2016). Our approximation relies on assuming that the size of the price jump multiplied by the allocation in risksy assets is "small". The approximation appears sensible, since the largest daily drowdown experienced by S&P500 is -23% (the Black Monday) and typical allocation in stocks is at $20 - 80\%$ in most investment styles. Numerical experiments with realistic parameter values indicate indeed that the approximate solution is not far from the actual one in a simple case in which the latter can be found explicitly, as specified in details in Appendix C. The most important feature of our proposed solution is that the optimal strategy in our case is state-dependent and proportional to the co-precision, that is inverse to the covariance matrix. This result implies both a temporary and

[1] As Liu (2007) points out, in affine models high variance is associated with high risk premium and thus the myopic component is independent on stochastic volatility. The fact that the optimal allocation does not depend on the average volatility level of the risky assets can be seen by inspection of formulas (17)-(18) at page 240 of Liu, Longstaff, and Pan (2003), reporting the optimal allocation in stocks, which does not depend neither on α, which determines the variance mean-reversion level, nor V_t, the dynamic variance level. The same consideration applies to Formula (15), page 403 of Buraschi, Porchia, and Trojani (2010); Formula (5.2), page 1034 of Bäuerle and Li (2013) in the multivariate case with stochastic volatility; and Formula (22), page 69 of Branger et al. (2017) in the multivariate case with co-jumps. See Appendix B for a technical explanation.

a permanent impact on optimal asset allocation. After a spike in variances and covariances, optimal allocation in risky assets changes (typically, reduces with respect to the risk-free asset) by a fraction that depends on jump size and model parameters. This effect is transient since, after a jump, both volatility and optimal allocation revert back to their mean-reversion level. Our result also implies a permanent effect, in the sense that, all other things being equal, average optimal allocation in risky assets will depend on the average co-precision level. Thus, contrary to the results in the affine models, our specification implies that a risk-averse investor will invest less in more volatile assets, and will act genuinely dynamically by changing her allocation in risky assets after a market crash.

We apply the proposed technology to the problem of optimal allocation in hedge fund indexes, for which event risk is substantial. After calibrating the model, we show that the optimal allocation changes crucially when adding co-jumps, and in particular that neglecting volatility jumps results in non-trivial utility losses, especially for moderate values of the risk-aversion parameter.

The paper is organized as follows. Section 2 describes the model for the risky assets' dynamics. Section 3 states the optimization problem and provides the optimal allocation strategy. Section 4 is devoted to the application to the hedge fund industry. Section 5 concludes. Three appendices contain the mathematical proofs, a discussion of the limitation of affine models, and numerical experiments about the proposed approximation respectively.

2 The model

We assume there exist N risky assets, whose prices are given by $S_t = (S_{t,1}, \dots, S_{t,N})' \in \mathbb{R}^{N\times 1}$, traded in a frictionless market. We further assume there exists a riskless asset M_t. The dynamics of the riskless asset follows

$$dM_t = rM_t \mathrm{d}t,$$

where $r \in \mathbb{R}$ is the instantaneous rate of return of the riskless asset (the risk-free rate), while the dynamics of the risky assets follows

$$\begin{cases} \mathrm{d}S_t = diag(S_t)\left[\alpha \mathrm{d}t + \sqrt{Y_t^{-1}}\mathrm{d}Z_t^{(1)} + J\mathrm{d}N(\lambda)_t\right] \\ \mathrm{d}Y_t = (\Omega\Omega' + KY_t + Y_tK')\mathrm{d}t + \sqrt{Y_t}\mathrm{d}Z_t^{(2)}Q + Q'\left(\mathrm{d}Z_t^{(2)}\right)'\sqrt{Y_t} + \xi(Y_t)\mathrm{d}N(\lambda)_t \end{cases}, \tag{2.1}$$

where $diag(S_t)$ is the square matrix with S_t in the diagonal and 0 on the off-diagonal elements, $\alpha, J \in \mathbb{R}^{N\times 1}$, $Z_t^{(1)} \in \mathbb{R}^{N\times 1}$ and $Z_t^{(2)} \in \mathbb{R}^{N\times N}$ are matrix Wiener processes, $N(\lambda)_t$ is a (non-compensated) Poisson process with intensity $\lambda \in \mathbb{R}$, and $K, Q \in \mathbb{R}^{N\times N}$. Denoting by $GL_N(\mathbb{R})$ the set of invertible matrices in $\mathbb{R}^{N\times N}$, by $\mathbb{S}_N$ the set of symmetric matrices in $\mathbb{R}^{N\times N}$, and by $\mathbb{S}_N^+$ the set of symmetric and positive definite matrices in $\mathbb{R}^{N\times N}$, we further have $\Omega \in GL_N(\mathbb{R})$ with

$$\Omega\Omega' = \psi QQ', \tag{2.2}$$

with $\psi \in \mathbb{R}$ and $\psi > N-1$; $Y_t \in \mathbb{S}_N^+(\mathbb{R})$; $\xi(Y_t) \in \mathbb{S}_N(\mathbb{R})$ such that

$$Y + \xi(Y) \in \mathbb{S}_N^+(\mathbb{R}) \tag{2.3}$$

for all $Y \in \mathbb{S}_N^+(\mathbb{R})$. The Wiener processes determining shocks in the prices S_t and in the variance-covariance matrix $\Sigma_t = Y_t^{-1}$ are correlated according to

$$Z_t^{(1)} = Z_t^{(2)}\rho + \sqrt{1-\rho'\rho}Z_t^{(3)}, \tag{2.4}$$

where $\rho \in \mathbb{R}^{N\times 1}$, $Z_t^{(3)}$ is a Wiener process in $\mathbb{R}^{N\times 1}$ and the elements of $Z_t^{(3)}$ and $Z_t^{(2)}$ are all independent among them.

The model (2.1) is a multivariate stochastic volatility model in which the inverse of the instantaneous variance-covariance matrix, dubbed *co-precision*, follows a Wishart process. Both prices and co-precision are subject to event risk governed by the unique Poisson process $\mathrm{d}N(\lambda)_t$. When this shock is activated, we have a jump in the prices, with random size J, and a contemporaneous jump in the co-precision, given by the random matrix ξ. Jump sizes J and ξ can be correlated. The model generalizes the existing literature in several respects. With respect to Das and Uppal (2004), who also specify a single Poisson process governing jumps in all stocks, we add stochastic covariance and jumps in the covariance matrix. With respect to Chacko and Viceira (2005), who also specify the dynamics of instantaneous variance in terms of its inverse (the precision), we generalize to a multivariate setting and add jumps in prices and in the covariance matrix. Most typically, the Wishart process is used to model directly Σ_t in stochastic volatility models, see Da Fonseca, Grasselli, and Tebaldi (2008); Buraschi, Porchia, and Trojani (2010); Bäuerle and Li (2013) in the case without jumps, and Branger et al. (2017) in the case with jumps. With respect to these papers the main difference of our model is in the fact that our model is not affine, as for Chacko and Viceira (2005) in the case $N = 1$.

From an economic perspective, modeling a mean-reverting co-precision instead of mean reverting covariance is harmless. However, there are several good reasons for specifying a mean-reverting co-precision instead of a mean-reverting covariance matrix. First, using co-precision can still lead to a tractable dynamic programming problem, as also highlighted by Chacko and Viceira (2005). Second, by modeling co-precision we can write the stochastic optimal control problem in such a way that the optimal weights are (approximately, when co-jumps are present) linear in the co-precision itself, that is inversely related to the spot covariance matrix.

The interpretation of the model parameters is pretty straightforward, it just has to be applied to co-precision instead that to covariance (its matrix inverse). The matrix K represents the speed of mean reversion of the co-precision to its mean-reversion level $\overline{Y}$, which satisfies:

$$\psi QQ' + K\overline{Y} + \overline{Y}K' = 0,$$

and thus depends on K, Q and the parameter ψ. We thus expect the matrix K to be negative semi-definite in practice. The matrix Q is proportional to the volatility of the co-precision matrix (vol-of-inverse-vol). The relation (2.2) with $\psi > N - 1$ is sufficient to ensure the positive definiteness and the mean-reversion of Y_t at all times (Bru, 1991). The relation (2.3) ensures the positive definiteness of the co-precision after a jump. The correlation structure (2.4) allows for correlated shocks to price returns and their variance-covariance matrix. These correlations will be typically negative in practice, as documented, among others, in Branger et al. (2017), that is variances and covariances typically increase while prices decline. This implies that, in financial markets, the shocks to prices and co-precision are expected to be positively correlated.

The model could be potentially extended to allow for idiosyncratic jump in prices and co-precision, as in Bandi and Renò (2016). However, we stick to a single source of jump risk for ease of exposition.

3 Optimal allocation

We assume that the investor has to allocate wealth in the single riskless and N risky assets, and she does so by maximizing the expected utility from terminal wealth with respect to a power utility

$$U(x) = \frac{x^{1-\gamma}}{1-\gamma},\ \gamma > 0,\ \gamma \neq 1\ , \tag{3.1}$$

without considering intermediate consumption, as in Liu, Longstaff, and Pan (2003), Das and Uppal (2004) and Branger et al. (2017), among others. When $\gamma = 1$, the above utility is $U(x) = \log x$. We denote by w_t the $\mathbb{R}^{N\times 1}$ vector of proportional wealth invested in the risky assets at time t. By the budget constraint, the proportion of wealth invested in the riskless asset is given, after denoting by $\mathbf{1}$ the $N \times 1$ vector of ones, by $1 - w_t'\mathbf{1}$. Starting from initial wealth W_0 at time 0, the dynamics of wealth is given by

$$\mathrm{d}W_t = [w_t'(\alpha - r\mathbf{1}) + r]W_t\mathrm{d}t + w_t'W_t\sqrt{Y_t^{-1}}\mathrm{d}Z_t^{(1)} + w_t'JW_t\mathrm{d}N(\lambda)_t\ . \tag{3.2}$$

The investor's optimization problem can be written as follows

$$V(t, Y_t, W_t) := \max_{(w_t)_{0\leq t\leq T}} \mathbb{E}_t\left[\frac{W_T^{1-\gamma}}{1-\gamma}\right]\ , \tag{3.3}$$

where the expectation represents a shorthand for the conditional expectation, namely, $\mathbb{E}_t = \mathbb{E}[\cdot \mid W_t = W, Y_t = Y]$.

In the case without co-jumps, we can find an exact (genuinely dynamic) solution for this problem.

Proposition 3.1. *Consider the Hamilton-Jacobi-Bellman (HJB) equation associated with the investment problem* (3.3) *when* $\lambda = 0$ *(no co-jumps). The optimal strategy is given by:*

$$w_t^* = Y_t\left[\frac{(\alpha - r\mathbf{1}) + 2F_tQ'\rho}{\gamma}\right]. \tag{3.4}$$

The value function of the HJB equation is given by:

$$V(t, Y_t, W_t) = \exp\{Tr(F_tY_t) + G_t\}\frac{W_t^{1-\gamma}}{1-\gamma}\ , \tag{3.5}$$

where the matrix function $F_t \in \mathbb{S}_N$ *and the function* $G_t \in \mathbb{R}$ *solve*

$$\begin{cases} \dot{F}_t + (1-\gamma)(\alpha - r\mathbf{1})D_t' + F_tK + K'F_t - \frac{\gamma(1-\gamma)}{2}D_tD_t' \\ +2(1-\gamma)FQ'\rho D_t' + 2F_tQ'QF_t = 0,\ F_T = 0 \\ \dot{G}_t + (1-\gamma)r + Tr(\Omega\Omega'F_t) = 0,\ G_T = 0 \end{cases}\ , \tag{3.6}$$

where

$$D_t := \frac{(\alpha - r\mathbf{1}) + 2F_tQ'\rho}{\gamma}.$$

Proof. See Appendix A. □

The above result can be considered a generalization of the Chacko and Viceira (2005) case to the multivariate case with power utility, with precision (in the univariate problem) replaced by co-precision (in the multivariate problem) and allowed to follow a more flexible Wishart process. The optimal portfolio weight w_t^* is now genuinely dynamic, and it changes proportionally to the matrix Y_t, with dynamic coefficient D_t. However, our main interest is in the more challenging case in which $\lambda > 0$, that is when assets are subject to event risk.

To recover a viable HJB equation in this more realistic case to solve for optimal investment, we follow an approximated approach. Similar approaches have been used, for example, in Chacko and Viceira (2005), regarding an optimal consumption and portfolio-choice problem in incomplete markets with recursive preferences, and in Ascheberg, Branger, Kraft, and Seifried (2016) for a continuous-time portfolio optimization problem with constant relative risk aversion.

The approximation consists of two steps. We first assume that the jump matrix in the co-precision ξ is constant, not depending on Y_t. In this case, co-precision can become negative definite since the condition (2.3) cannot be fulfilled for all levels of Y_t. A negative co-precision would imply a negative definite covariance matrix. The approximation makes practical sense if the intensity of jumps is small, and if the ξ is such that $Y_t + \xi$ does not become negative definite after the jump.

The second approximation consists in the linearization of the jump term appearing in the HJB equation. Using second-order Taylor expansion, we write

$$(1 + w_t'J)^{1-\gamma} = 1 + (1-\gamma)w_t'J + o((w_t'J)^2) \ , \tag{3.7}$$

and the approximation consists in ignoring the $o((w_t'J)^2)$ term in the HJB equation. Our approximation is thus accurate if the product $w_t'J$ is "small", that is if the jump amplitude is small and/or if the optimal investment in the risky assets is small. We test the reliability of the approximation in a realistic economy in Section C. As noted in Ascheberg et al. (2016), the approximation can be interpreted as a "small jumps" approximation in which the approximated HJB equation would be the same of a model in which "the jump-diffusion process for the stock price is replaced by a diffusion process with drift and volatility adjusted in such a way that the expected return and the local variance match those of the original process" (Ascheberg et al., 2016, page 3). For clarity, we use the symbol $\approx$ instead of the equality sign when indicating solutions to the approximated problem, instead of the actual one. We can prove the following.

Proposition 3.2. *Consider the HJB equation associated with the investment problem* (3.3) *in which ξ is replaced by a constant matrix and $(1 + w_t'J)^{1-\gamma}$ is replaced by $1 + (1-\gamma)w_t'J$. This approximated HJB equation is solved by:*

$$w_t^* \approx Y_t \left[\frac{(\alpha - r\mathbf{1}) + 2F_tQ'\rho + \lambda \mathbb{E}\left[e^{Tr(F_t\xi)} J \right]}{\gamma} \right] =: Y_t B_t \ , \tag{3.8}$$

where $B_t \in \mathbb{R}^{N\times 1}$. The value function of the approximated HJB equation is given by:

$$V(t, Y_t, W_t) \approx \exp\{Tr(F_tY) + G_t\} \frac{W_t^{1-\gamma}}{1-\gamma} \ , \tag{3.9}$$

where the matrix function $F_t \in \mathbb{S}_N$ and the function $G_t \in \mathbb{R}$ solve

$$\begin{cases} \dot{F}_t + (1-\gamma)(\alpha - r\mathbf{1})B_t' + F_tK + K'F_t - \frac{\gamma(1-\gamma)}{2}B_tB_t' \\ +2(1-\gamma)F_tQ'\rho B_t' + 2F_tQ'QF_t + \lambda(1-\gamma)\mathbb{E}[e^{Tr(F_t\xi)}J]B_t', \ F_T = 0 \\ \dot{G}_t + (1-\gamma)r + Tr(\Omega\Omega'F_t) + \lambda\mathbb{E}[e^{Tr(F_t\xi)} - 1] = 0, \ G_T = 0 \end{cases} . \tag{3.10}$$

Proof. See Appendix A. □

Eq. (3.8) is the main result of this paper. The structure of the (approximate) optimal allocation is unchanged with respect to the (exact) case without co-jumps, that is proportional to Y_t (now with coefficient B_t) and truly dynamic. The set of ordinary differential equations in Eq. (3.10) can be solved by exploiting numerical techniques. The optimal portfolio weights obtained in Eq. (3.8) consist of three terms. The first one represents the generalization of the *myopic demand* component, which is now proportional to inverse volatility (recall that $Y_t = \Sigma_t^{-1}$), and so it takes the typical form of standard mean-variance allocation. The second one is the *intertemporal hedging demand* term, which depends on the correlation coefficients between Wiener processes driving the diffusive components of asset price and co-precision. Its sign depends on the sign of function F_t and the sign of the components of ρ, which are expected to be positive. An extra hedging term appears due to the interaction between jumps in prices and jumps in volatility, which has the same features of an "illiquidity" term, as discussed in Liu, Longstaff, and Pan (2003). Since in practice $J < 0$, we expect this extra term to be negative, that is to reduce the allocation in risky assets with respect to the risk-free asset.

Remark 3.3. *If we assume $N = 1$ (only one risky asset) and $\xi = 0$, Eq. (3.10) corresponds to the result obtained in Das and Uppal (2004, Prop. 2), where the optimal weights are retrieved by solving a nonlinear equation.*

Remark 3.4. *The coefficient B_t in Eq. (3.8) does not depend on the parameter ψ defined in Eq. (2.2), which is crucial in determining the long-run volatility levels of the risky asset. This is due to the fact that ψ is not involved in determining neither the constant part of B_t, nor the function F_t in (3.10), which in turn determines the optimal weights through the hedging components. In the affine case, where the drift is restricted to be linear in the variance to find an analytical solution, this implies that the optimal allocation does not depend on the long-run variance levels, but only on the risk premium coefficient which relates the instantaneous drift to the instantaneous variance. Since the allocation is static, it does not even depend on the instantaneous variance. Thus, variance levels do not determine optimal allocation at all in affine models, which is against standard Markowitz theory, see the discussion in Appendix B. In our case, optimal allocation depends on instantaneous co-precision, that is to the inverse of the instantaneous covariance matrix, reconciling dynamic asset allocation with static Markowitz theory, and correcting optimal allocation for volatility and co-jump risk.*

Remark 3.5. *Our optimal allocation strategy (3.8) implies that, after a jump in precision of size ξ, the investor will change its investment in risky assets from $Y_t B_t$ to $(Y_t + \xi)B_t$. For example, in the case $N = 1$, since ξ is negative to reproduce spikes in volatility, this implies less investment on stocks, on average, with respect to bonds. Thus, differently from affine models which imply static optimal allocation, the investor will rebalance her portfolio after a jump. This result is consistent with the large volumes traded in financial markets immediately during and after a market crash, see e.g. Caporin, Kolokolov, and Renò (2017).*

4 Application: investing in a fund of hedge funds

We apply the proposed methodology to a challenging financial problem: allocation of wealth in a portfolio composed of hedge fund indexes (the typical problem of a fund of funds). Hedge funds are indeed well known to be subject to substantial event risk (see e.g. Getmansky, Lee, and Lo,

Table 1: Descriptive statistics for the returns (expressed in percentages and in annualized units) of the twelve investment funds composing the portfolio.

	CA	EM	EMN	ED	EDD	EDMS	EDRA	FI	GM	L/S	MF	MS
Mean (%)	6.89	8.24	4.99	8.07	9.15	7.59	5.79	5.40	10.45	9.29	5.35	7.56
Std (%)	6.25	13.01	9.30	5.97	6.04	6.51	3.90	5.08	8.63	8.96	11.50	4.86
Skewness	−2.73	−0.89	−12.47	−2.09	−2.14	−1.63	−0.92	−4.88	0.20	−0.00	0.05	−1.73
Kurtosis	21.58	10.29	189.92	13.07	14.90	9.95	7.68	41.67	8.24	7.20	2.83	10.05

2015), so that ignoring the presence of co-jumps in the model is particularly harmful in their case, as we show below. We consider $N = 12$ constituents of the Credit Suisse Hedge Fund Index. These constituents are hedge fund indexes, that is portfolios of hedge funds categorized by trading style: Convertible Arbitrage (CA), Emerging Markets (EM), Equity Market Neutral (EMN), Event Driven (ED), Event Driven Distressed (EDD), Event Driven Multi-Strategy (EDMS), Event Driven Risk Arbitrage (EDRA), Fixed Income (FI), Global Macro (GM), Long/Short (L/S), Managed Futures (MF), and Multi-Strategy (MS) (we exclude from our analysis the dedicated short-bias index). We use monthly returns from May, 1994 to January, 2018 for a total of $T = 285$ months. The descriptive statistics of monthly returns are reported in Table 1. All the hedge fund indexes display substantial positive mean and low standard deviation, so that they have attractive Sharpe ratios for an investor. However, hedge funds are still extremely risky investment vehicles, as witnessed by their negative skewness and high kurtosis of their realized returns, which are typical signatures of event risk. In particular, Equity Market Neutral (EMN) appears to be particularly subject to event risk given the LTCM episode in its record.

We calibrate the model (2.1) where, for simplicity, we assume that the matrices K, Q, and ξ are diagonal. We also calibrate restricted models with $\xi = 0$ (no jumps in variance) and $\xi = J = 0$ (no jumps at all). Calibration is based on a two-step procedure. In the first step, we determine the monthly dynamics of co-precision and jumps using a kernel estimator and a data-driven threshold respectively, as in Corsi, Pirino, and Renò (2010). We then use a simulated method of moments to estimate the parameters of the model.

In the first step, we proceed as follows. Denote by $r_t^{(i)}$ the return at month t, with $t = 1, \dots, T$, of the i-th hedge fund index, with $i = 1, \dots, N$. We first compute a threshold $\theta_t^{(i)}$ which represents the separation between "continuous" price movements and jumps (see Mancini, 2009 for a theoretical motivation for this technique). The threshold at month t for the i-th index is defined as

$$\theta_t^{(i)} = 3 \cdot \sqrt{\frac{\dfrac{\pi}{2} \displaystyle\sum_{t'=1, t' \neq t}^{T-1} K\left(\frac{t-t'}{L}\right) |r_{t'}^{(i)}||r_{t'+1}^{(i)}|}{\displaystyle\sum_{t'=1, t' \neq t}^{M-1} K\left(\frac{t-t'}{L}\right)}} \qquad t = 1, \dots, T, \tag{4.1}$$

with $L = 12$, $K(x) = \sqrt{\frac{1}{2\pi}} e^{-x^2/2} \mathbf{1}_{\{|x| \leq L\}}$, where $\mathbf{1}_{\{A\}}$ is the indicator function of the event A. Equation

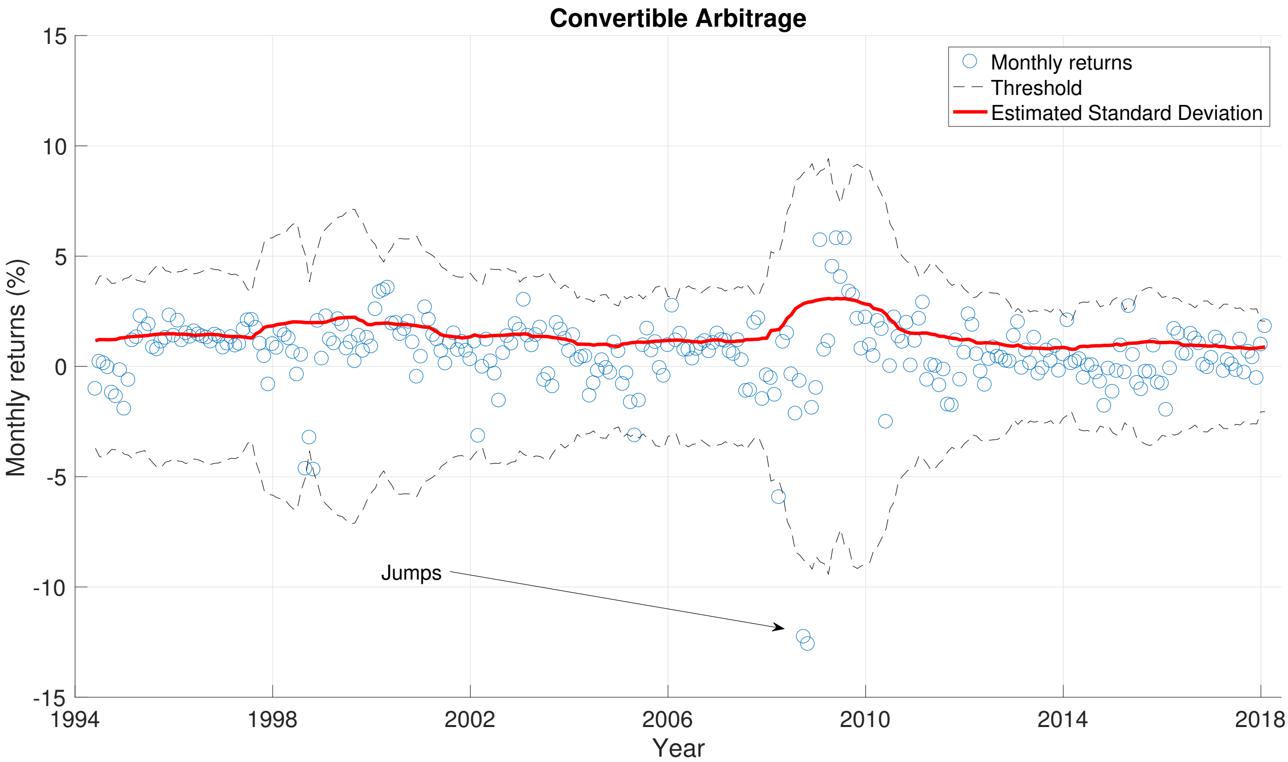

Figure 1: Monthly returns for the Convertible Arbitrage (CA) index, together with the calibrated threshold used to separate continuous movements from jumps and the corresponding estimated standard deviation.

(4.1) represents a weighted average of local volatility, estimated with the bipower term $\frac{\pi}{2}|r_{t'}^{(i)}||r_{t'+1}^{(i)}|$ to reduce the impact of jumps, multiplied by 3. The estimated jump time series for the i-th index is then simply defined as

$$\widehat{J}_t = r_t^{(i)} \cdot \mathbf{1}_{\{|r_t^{(i)}|>\theta_t^{(i)}\}},$$

that is a return is considered a jump if its absolute value exceeds the local standard deviation by a factor 3. We then estimate the dynamics of the covariance between the i-th and the j-th hedge fund index to be:

$$\widehat{V}_t^{(i,j)} = \frac{\displaystyle\sum_{t'=1}^{T-1} K\left(\frac{t-t'}{L}\right) r_{t'}^{(i)} \cdot r_{t'}^{(j)} \mathbf{1}_{\{|r_{t'}^{(i)}|\le\theta_{t'}^{(i)}\}} \mathbf{1}_{\{|r_{t'}^{(j)}|\le\theta_{t'}^{(i)}\}}}{\displaystyle\sum_{t'=1}^{T-1} K\left(\frac{t-t'}{L}\right) \mathbf{1}_{\{|r_{t'}^{(i)}|\le\theta_t^{(i)}\}} \mathbf{1}_{\{|r_{t'}^{(j)}|\le\theta_{t'}^{(i)}\}}} \qquad t = 1,\ldots,T, \tag{4.2}$$

with the same L and $K(\cdot)$ used before, that is we compute the spot covariance matrix using only returns which are smaller than the threshold. Figure 1 illustrates the procedure in one case, the CA index. Table 2 reports average covariances and correlations between returns on the hedge fund indeces, revealing a complex linear structure in hedge funds returns. The co-precision at month t is then defined as the spot inverse of the covariance matrix at month t.

In the second step, we fit model parameters to the first and second moments of the co-precision matrix. In total, we calibrate $2 \cdot N(N+1)/2$ moments, corresponding to the first and second moment of the diagonal and upper diagonal of the co-precision matrix. The moments implied by the model are obtained via simulation. We minimize the sum of the squared differences between estimated moments

Table 2: The table reports average covariances (for returns expressed in percentages and in monthly units) and correlations (in bold) among the hedge funds indexes. We use monthly returns from May, 1994 to January, 2018 for a total of $T = 285$ months. The hedge funds are: Convertible Arbitrage (CA), Emerging Markets (EM), Equity Market Neutral (EMN), Event Driven (ED), Event Driven Distressed (EDD), Event Driven Multi-Strategy (EDMS), Event Driven Risk Arbitrage (EDRA), Fixed Income (FI), Global Macro (GM), Long/Short (L/S), Managed Futures (MF), and Multi-Strategy (MS).

Covariances/**Correlations**												
	CA	EM	EMN	ED	EDD	EDMS	EDRA	FI	GM	L/S	MF	MS
CA	2.41	2.82	1.32	2.41	2.12	2.26	1.24	1.70	1.71	2.22	−0.25	1.53
		0.56	**0.71**	**0.93**	**0.80**	**0.81**	**0.68**	**0.90**	**0.45**	**0.55**	**−0.05**	**0.71**
EM	–	10.37	2.07	3.72	3.31	4.06	1.92	1.88	4.30	6.08	0.99	1.88
			0.54	**0.69**	**0.60**	**0.71**	**0.51**	**0.48**	**0.54**	**0.73**	**0.09**	**0.42**
EMN	–	–	1.43	1.73	1.98	1.61	0.80	1.45	0.91	1.98	0.22	1.65
				0.87	**0.96**	**0.75**	**0.57**	**1.00**	**0.31**	**0.64**	**0.06**	**1.00**
ED	–	–	–	2.78	2.74	2.99	1.39	1.60	2.30	3.35	0.58	1.82
					0.96	**1.00**	**0.71**	**0.79**	**0.56**	**0.78**	**0.11**	**0.79**
EDD	–	–	–	–	2.94	2.64	1.28	1.36	2.17	3.10	0.53	1.62
						0.86	**0.64**	**0.65**	**0.51**	**0.70**	**0.10**	**0.68**
EDMS	–	–	–	–	–	3.20	1.45	1.65	2.42	3.44	0.59	1.77
							0.69	**0.76**	**0.55**	**0.74**	**0.10**	**0.72**
EDRA	–	–	–	–	–	–	1.37	0.72	1.07	1.87	0.22	0.88
								0.51	**0.37**	**0.62**	**0.06**	**0.55**
FI	–	–	–	–	–	–	–	1.47	1.49	1.74	0.03	1.12
									0.50	**0.56**	**0.01**	**0.67**
GM	–	–	–	–	–	–	–	–	6.12	3.32	3.12	1.17
										0.52	**0.39**	**0.34**
L/S	–	–	–	–	–	–	–	–	–	6.68	1.51	2.10
											0.18	**0.59**
MF	–	–	–	–	–	–	–	–	–	–	10.58	0.65
												0.15
MS	–	–	–	–	–	–	–	–	–	–	–	1.91

and simulated moments. We perform three different calibrations, corresponding to the absence of jumps, the absence of volatility jumps, and the full model respectively. The parameter estimates obtained by using this method are reported in Table 3, and they correspond to monthly returns expressed in percentages. The point estimate of the jump intensity parameter λ is 0.012, which corresponds to one co-jump every 6.94 years, on average. Thus, event risk in hedge funds is rare but not negligible.

To get optimal portfolio weights implied by our calibration, we use the Runge-Kutta method to solve the system of ODEs (3.10), by using parameters values obtained through the calibration exercise previously proposed and shown in Table 3. We use a risk-aversion parameter $\gamma = 3$, an investment horizon of 1 year, and Y_t equal to the average co-precision in our sample. Optimal weights are reported in Table 4. The column "No Jumps" displays the optimal weights when the investor uses a model without jumps. The column "Jumps in Price Only" describes the optimal allocation for an investor

Table 3: Parameter estimates for the co-jumps model (Panel A, $\psi = 146.9339$), the jumps-in-price model (Panel B, $\psi = 48.3789$), and the no-jumps model (Panel C, $\psi = 62.0173$). The jump intensity λ is equal to 0.0120 when jumps occur, or is equal to 0 otherwise, $diag(Q)$, $diag(K)$, $diag(\xi)$ represent the main diagonal of matrices Q, K, ξ, respectively. Estimates are for returns expressed in percentages and in monthly units.

Panel A: full model (with price/volatility co-jumps)												
	CA	EM	EMN	ED	EDD	EDMS	EDRA	FI	GM	L/S	MF	MS
α	0.68	0.72	0.51	0.76	0.85	0.69	0.56	0.55	0.89	0.89	0.41	0.68
$diag(Q)$	0.05	0.06	0.10	0.02	0.03	0.02	0.05	0.05	0.15	0.07	0.00	0.07
$diag(K)$	-0.10	-0.33	-0.43	-0.02	-0.03	-0.02	-0.11	-0.10	-1.47	-0.26	-0.28	-0.22
J	-8.48	-2.47	-7.48	-7.40	-7.51	-4.99	-6.15	-8.25	-1.45	-9.62	3.18	-4.04
$diag(\xi)$	-0.33	-0.73	-0.46	-0.46	-0.44	-0.37	-0.10	-0.13	-1.18	-0.77	-0.01	-0.42
Panel B: only jumps in price												
	CA	EM	EMN	ED	EDD	EDMS	EDRA	FI	GM	L/S	MF	MS
α	0.68	0.72	0.51	0.76	0.85	0.69	0.56	0.55	0.89	0.89	0.41	0.68
$diag(Q)$	0.05	0.09	0.08	0.08	0.06	0.05	0.09	0.07	0.06	0.08	0.08	0.07
$diag(K)$	-0.04	-0.26	-0.09	-0.09	-0.07	-0.05	-0.11	-0.07	-0.08	-0.14	-0.48	-0.08
J	-8.48	-2.47	-7.48	-7.40	-7.51	-4.99	-6.15	-8.25	-1.45	-9.62	3.18	-4.04
$diag(\xi)$	–	–	–	–	–	–	–	–	–	–	–	–
Panel C: No jumps												
	CA	EM	EMN	ED	EDD	EDMS	EDRA	FI	GM	L/S	MF	MS
α	0.57	0.69	0.42	0.67	0.76	0.63	0.48	0.45	0.87	0.77	0.45	0.63
$diag(Q)$	0.13	0.20	0.13	0.11	0.10	0.14	0.08	0.10	0.14	0.18	0.19	0.13
$diag(K)$	-0.40	-1.01	-0.40	-0.27	-0.22	-0.43	-0.20	-0.24	-0.47	-0.70	-1.17	-0.34
J	–	–	–	–	–	–	–	–	–	–	–	–
$diag(\xi)$	–	–	–	–	–	–	–	–	–	–	–	–

which adds jumps in price, but not jumps in volatility. The column "Price/Volatility Co-Jumps" reports the optimal weights for an investor which uses a model with contemporaneous jumps in prices and volatility. The three modeling choices induce a substantial difference in the optimal allocation. As more risks are introduced, the total optimal allocation in risky funds decreases. As it can be observed from the last row of Table 4, the total optimal allocation decreases if the presence of jumps (in price, in volatility or in both of them) is involved in the model. This is not necessarily the case for individual funds. For example, when introducing jumps in volatility, the allocation in the Global Macro index spikes at 17.5%, at the expense of allocation in the remaining funds, which is not surprising since Global Macro is the only hedge fund with positive skewness (see Table 1). Also notice how the optimal allocation in the Equity Market Neutral index is annihilated by the introduction of jumps in the model. This shows that, in the case of hedge funds, modeling risk in an accurate way has a crucial impact on the investor's optimal selection.

The optimal allocation can be separated in three components: a myopic component, defined as $\frac{(\alpha - r\mathbf{1}) + \lambda \mathbb{E}[J]}{\gamma}$ (we compensate for jumps in the definition of the myopic component so that it does not change across the three different models), a hedging component defined as $\frac{2F_t Q' \rho}{\gamma}$, and a jumps component defined as $\frac{\lambda \mathbb{E}\left[(e^{Tr(F_t \xi)} - 1)J)\right]}{\gamma}$. Figure 2 shows the decomposition of the total optimal allocation in risky funds into the three components for the three considered models. As expected, the myopic

Table 4: Optimal portfolio allocation w^* for the model with no jumps (second column), the model with only jumps in price (third column), and the model with co-jumps (fourth column). The maturity of the investment is 1 year, the risk aversion parameter is $\gamma = 3$, the jump intensity is equal to $\lambda = 0.0120$ for models with jumps, and 0 otherwise. Y_t is the average co-precision in our sample. The remaining parameter estimates are provided in Table 3.

	No Jumps	Jumps in Price Only	Price/Volatility Co-Jumps
Hedge Fund	$w^*(\%)$	$w^*(\%)$	$w^*(\%)$
CA	3.4448	1.1447	-0.1919
EM	-3.0224	-1.6955	-0.7815
EMN	2.4403	1.9312	-0.1230
ED	-1.3438	0.1826	-3.1292
EDD	3.3878	3.7078	2.9552
EDMS	-2.0740	-3.7390	-2.4163
EDRA	4.5276	5.5862	3.5086
FI	3.2185	3.3614	-1.6721
GM	11.6001	8.6216	18.0327
L/S	3.0813	2.1369	-0.0903
MF	-0.3379	0.1323	-1.0131
MS	6.0363	5.1134	9.2400
Total allocation (%)	30.9587	26.4837	24.3191

component dominates the hedging component. When introducing jumps in price only, the hedging components becomes larger to compensate the extra-risk, and the total allocation is reduced. When introducing jumps in volatility as well, the hedging component shrinks, and the jumps component is the main correction factor. Total allocation is further reduced. This result suggests that jumps in volatility are an important driver of event risk for hedge funds.

We finally evaluate the importance of jumps in volatility using an economic metric, the Wealth Equivalent Loss (WEL), which is defined as the relative loss in wealth due to following the suboptimal strategy implied by not introducing co-jumps in the model when they are actually there (see Das and Uppal, 2004 and Liu, Longstaff, and Pan, 2003). More precisely, the WEL is computed as follows. First we assume that no jumps occur in volatility, and we calibrate the model as described above, obtaining a parameter set that implies the weights $\widehat{w}^*$. The value function in this case is

$$\widehat{V}(t, Y_t, W_t) = \exp\{Tr(\widehat{F}_t Y_t) + \widehat{G}_t\} \frac{W_t^{1-\gamma}}{1-\gamma} \ ,$$

where $\widehat{F}_t$, $\widehat{G}_t$ are solutions of the ODEs (3.10) when $\xi = 0$, with Y_t being, as before, the average co-precision estimated in our sample, and W_t is the initial wealth. Then, we consider the true case in which co-jumps take place, and we evaluate the value function

$$V(t, Y_t, \widetilde{W}_t) = \exp\{Tr(F_t Y_t) + G_t\} \frac{\widetilde{W}_t^{1-\gamma}}{1-\gamma} \ ,$$

where $\widetilde{W}_t$ represents the (unknown) initial wealth the investor would need in t to get the same utility if she followed the optimal strategy. To quantify the loss in adopting the sub-optimal allocation, we equate the value function associated with the optimal strategy with initial wealth unknown to the

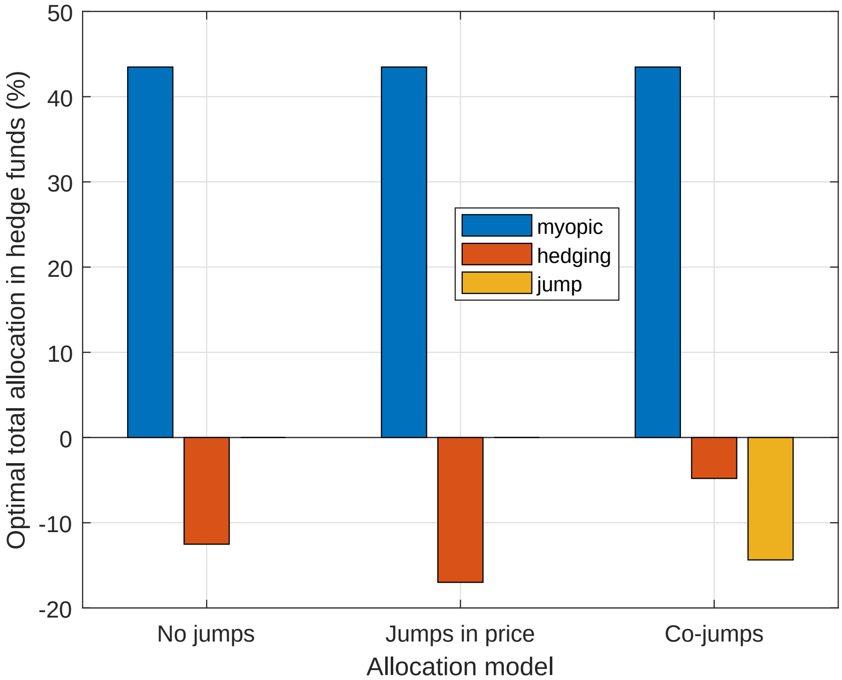

Figure 2: Optimal allocation for each component and for each model (no jumps, only jumps in price, co-jumps). The maturity of the investment is 1 year, the risk aversion parameter is $\gamma = 3$, the jump intensity λ is equal to 0.0120 when jumps occur, or is equal to 0 otherwise. Y_t is the average co-precision in our sample. The remaining parameter estimates are provided in Table 3.

value function associated with the non-optimal strategy with initial unit wealth

$$V(t, Y_t, \widetilde{W}_t) = \widehat{V}(t, Y_t, 1) .$$

This yields

$$\widetilde{W}_t = \left[\frac{\exp\{Tr(\widehat{F}_t Y_t) + \widehat{G}_t\}}{\exp\{Tr(F_t Y_t) + G_t\}} \right]^{\frac{1}{1-\gamma}} ,$$

and the Wealth Equivalent Loss (WEL) is calculated as

$$\text{WEL} := 1 - \widetilde{W}_t .$$

Figure 3 displays the WEL as a function of the risk-aversion parameter. The loss is minor for higher volatility because the allocation in risky funds decreases. Most importantly, the Figure shows that not including volatility jumps in the model would result in non-trivial losses in terms of wealth. For example, with a risk-aversion parameter $\gamma = 3$, the investor would suffer a loss of roughly 12% in monetary terms. This result reinforces the point that jumps in volatility play a crucial role in determining the optimal behavior of an investor in a fund of hedge funds.

5 Conclusions

We consider a dynamic portfolio problem on a set of risky assets and a riskless bond when prices are subject to volatility and co-jump risk, with the purpose of providing an optimal allocation strategy

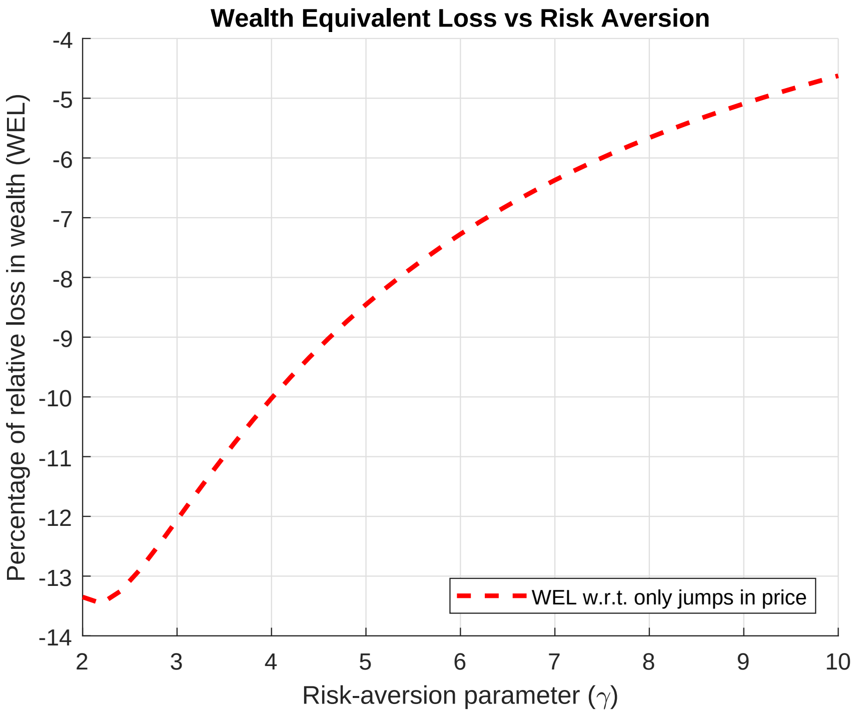

Figure 3: Percentage of relative loss in wealth as a function of the risk aversion parameter. The maturity investment is 1 year, the jump intensity is equal to $\lambda = 0.0120$. Y_t is the average co-precision in our sample. The remaining parameter estimates are provided in Table 3.

which depends on the jump characteristics and, at the same time, on the (inverse) covariance matrix.

We argue that this result cannot be obtained in affine models, so that we propose a Wishart model for the inverse of the covariance matrix, dubbed co-precision. We provide an exact solution to this problem in the absence of jumps, and an approximated solution in the presence of co-jumps. The solution can be dissected in the myopic and hedging components, as in the existing literature. The hedging component has two terms: the first corresponding to Brownian shocks, and the second corresponding to co-jumps. The approximation in the co-jump case is proved to be sensible, and providing accurate solutions in special cases. Our optimal strategy has several implications. One is that investors will rebalance their portfolio after a market crash, decreasing their investment in risky assets. Moreover, the inclusion of jumps in volatility implies a decrease of the optimal allocation in the risky asset, as well as a sizable negative impact in the wealth of the investor who does not include volatility co-jumps in her model. We apply the methodology to the problem of optimal selection of a portfolio of hedge fund indexes. Using a realistic calibration of the market, we show that the co-jumps component is crucial, and in particular that the addition of jumps in volatility is an indispensable statistical channel to model event risk for hedge funds.

This paper represents a first step. Further contributions along our line consist in providing a mathematical justification for the approximation, evaluating the impact of jumps in the covariance matrix which are uncorrelated to jumps in price, and considering the case of non-separable preferences. In particular, it appears interesting to use our approximation to evaluate utility gains in trading derivatives, especially in the empirically important problem of optimal hedging of co-jump risk, along the lines of Liu and Pan (2003). These additions are left for future work.

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A Proofs

Proof of Proposition 3.1

It follows from the same arguments of the proof of Proposition 3.2 below with $\lambda = 0$.

Proof of Proposition 3.2.

The proof follows standard arguments of dynamic programming. The HJB equation for the investment problem is:

$$0 = \max_{w_t} \left\{ \frac{\partial V}{\partial t} + [w_t'(\alpha - r\mathbf{1}) + r] W_t \frac{\partial V}{\partial W} + Tr\left([\Omega\Omega' + KY_t + Y_t K']\nabla V\right) + \frac{1}{2} W_t^2 w_t' Y_t^{-1} w_t \frac{\partial^2 V}{\partial W^2} \right.$$
$$\left. + \left(2w_t'\nabla Q'\rho \frac{\partial V}{\partial W}\right) W + \frac{1}{2} Tr\left(4Y_t\nabla Q'Q\nabla\right) V + \lambda \mathbb{E}[V(W_t + w'JW_t, Y_t + \xi) - V(W_t, Y_t)] \right\} , \quad \text{(A.1)}$$

where we write

$$\nabla := \left(\frac{\partial}{\partial Y_{ij}}\right)_{1 \le i,j \le N} .$$

For the approximated HJB equation, we use the solution (3.9). Simple algebra yields

$$\begin{aligned}
\frac{\partial V}{\partial t} &= (Tr(\dot{F}_t Y_t) + \dot{G}_t)V , \\
\nabla V &= F_t V , \\
\frac{\partial V}{\partial W} &= \exp\{Tr(F_t Y_t) + G_t\} W_t^{-\gamma} = V(1-\gamma)W_t^{-1} , \\
\frac{\partial^2 V}{\partial W^2} &= -\gamma \exp\{Tr(F_t Y_t) + G_t\} W_t^{-\gamma - 1} = -V\gamma(1-\gamma)W_t^{-2} , \\
Tr\left(Y_t \nabla Q'Q\nabla\right) V &= Tr\left(Y_t F_t Q'QF_t\right) V , \\
\nabla Q'\rho \frac{\partial V}{\partial W} &= \nabla \frac{\partial V}{\partial W} Q'\rho = F_t Q'\rho V(1-\gamma)W_t^{-1} , \\
V(W_t + w'JW_t, Y_t + \xi) - V(W_t, Y_t) &= V\left(e^{Tr(F_t\xi)}(1 + w_t'J)^{1-\gamma} - 1\right)
\end{aligned}$$

where $\dot{F}_t = \frac{\partial F}{\partial t}$, and $\dot{G}_t = \frac{\partial G}{\partial t}$. Substituting the value function in the approximated HJB equation we get

$$0 \approx \max_{w_t} \left\{ Tr(\dot{F}_t Y_t) + \dot{G}_t + (1-\gamma)[w_t'(\alpha - r\mathbf{1}) + r] + Tr\left([\Omega\Omega' + KY_t + Y_t K']F_t\right) - \frac{\gamma(1-\gamma)}{2} w_t' Y_t^{-1} w_t \right.$$
$$\left. + 2(1-\gamma) w_t' F_t Q'\rho + 2Tr(Y_t F_t Q'QF_t) + \lambda \mathbb{E}\left[e^{Tr(F_t\xi)}(1 + (1-\gamma)w_t'J) - 1\right] \right\} .$$

The first-order condition reads:

$$0 \approx (\alpha - r\mathbf{1}) - \gamma Y_t^{-1} w_t^* + 2F_t Q'\rho + \lambda \mathbb{E}[e^{Tr(F_t\xi)} J] ,$$

which implies Eq. (3.8). Plugging w_t^* in the approximated HJB equation we get

$$0 \approx Tr(\dot{F}_t Y_t) + \dot{G}_t + (1-\gamma)r + (1-\gamma)B_t' Y_t(\alpha - r\mathbf{1}) + Tr\left([\Omega\Omega' + KY_t + Y_t K']F_t\right) - \frac{\gamma(1-\gamma)}{2} B_t' Y_t B_t$$
$$+ 2(1-\gamma)B_t' Y_t F_t Q'\rho + 2Tr(Y_t F_t Q'QF_t) + \lambda \mathbb{E}[e^{Tr(F_t\xi)} - 1] + \lambda(1-\gamma)B_t' Y_t \mathbb{E}[e^{Tr(F_t\xi)} J] .$$

which implies (3.10) and ends the proof.

B A critique to affine stochastic volatility models for optimal portfolio allocation

Why optimal allocation for a power utility investor does not depend on volatility (loosely speaking) when the stochastic volatility model for the risky assets is affine?

To answer this question, we analyze the case with a single risky asset following the Heston (1993) model, that is assuming that the price dynamics is given by:

$$
\begin{aligned}
dS_t &= S_t \mu(v_t)dt + \sqrt{v_t} dZ_t^{(1)} \\
dv_t &= k(\theta - v_t)dt + \eta \sqrt{v_t} dZ_t^{(2)}
\end{aligned}
$$

with $corr(dZ_t^{(1)}, dZ_t^{(2)}) = \rho$ and k, θ, η, ρ are real constants. We leave the conditional expected return $\mu(v_t)$ unspecified for the moment being.

In this model, volatility v_t has mean-reversion level θ, which also coincides with the long-run mean since there are no jumps. The investors solves the problem (3.3), and the associated HJB equation for the value function $V(t, v_t, W_t)$, where W_t is wealth at time t, and the optimal allocation in the risky assets w_t in this case reads:

$$
\begin{aligned}
0 = \max_w \Bigg\{ & \frac{\partial V}{\partial t} + [w_t(\mu(v_t) - r) + r] W_t \frac{\partial V}{\partial W} + k(\theta - v_t)\frac{\partial V}{\partial v} + \frac{1}{2} W_t^2 w_t^2 v_t \frac{\partial^2 V}{\partial W^2} \\
& + \eta \rho W_t w_t v_t \frac{\partial^2 V}{\partial W \partial v} + \frac{1}{2}\eta^2 v_t \frac{\partial^2 V}{\partial W^2} \Bigg\} \ .
\end{aligned} \tag{B.1}
$$

As usual, the solution for this problem has to take the exponential-affine form

$$
V(t, v_t, W_t) = e^{F_t v_t + G_t} \frac{W_t^{1-\gamma}}{1-\gamma}, \tag{B.2}
$$

which leads to the equation:

$$
\begin{aligned}
0 = \max_w \Bigg\{ & \dot{F}_t v_t + \dot{G}_t + (1-\gamma)\left[w_t(\mu(v_t) - r) + r \right] + k(\theta - v_t) F_t - \frac{1}{2}\gamma(1-\gamma) w_t^2 v_t \\
& + (1-\gamma)\eta\rho w_t v_t F_t + \frac{1}{2}\eta^2 v_t F_t^2 \Bigg\} \ ,
\end{aligned} \tag{B.3}
$$

and the first-order condition yields

$$
\mu(v_t) - r - \gamma w_t v_t + \eta \rho v_t F_t = 0 \ ,
$$

which gives our (traditional) optimal strategy

$$
w_t^* = \frac{\mu(v_t) - r}{\gamma v_t} + \frac{\eta \rho F_t}{\gamma}, \tag{B.4}
$$

which is composed by the usual myopic (mean-variance) term plus the hedging component. The myopic component thus depends on γ (the risk-aversion parameter), on $\mu(v_t) - r$, the excess return of the risky asset, and on v_t, that is the variance level. The hedging component further depends on $\eta\rho F_t$, which is proportional to the price-volatility covariance. Notice that here the long-run variance mean θ, which determines the average volatility level, does not appear in the myopic component of the optimal allocation equation (B.4). It could appear in the hedging component, but as we will show in what follows, it does not, see Eq. (B.7) below. However, this is not a problem at this stage since these parameters determine the dynamic of v_t, which, in turn, determines the optimal strategy (B.4).

The problem of affine models we want to highlight here originates from the need of an analytical solution for the value function (B.2). Indeed, when we substitute back the optimal strategy (B.4) in Eq. (B.3), in order to determine F_t, we find:

$$\begin{aligned}0 &= \dot{F}_t v_t + \dot{G}_t + (1-\gamma)\left[\left(\frac{\mu(v_t)-r}{\gamma v_t} + \frac{\eta\rho F_t}{\gamma}\right)(\mu(v_t)-r) + r\right] + k(\theta - v_t)F_t \\ &- \frac{1}{2}\gamma(1-\gamma)\left(\frac{\mu(v_t)-r}{\gamma v_t} + \frac{\eta\rho F_t}{\gamma}\right)^2 v_t \\ &+ (1-\gamma)\eta\rho\left(\frac{\mu(v_t)-r}{\gamma v_t} + \frac{\eta\rho F_t}{\gamma}\right) v_t F_t + \frac{1}{2}\eta^2 v_t F_t^2,\end{aligned} \tag{B.5}$$

which has to be affine in v_t to identify F_t and G_t as solutions of suitable ODEs. However, inspection of Eq. (B.5) reveals that the *only* way to get an affine equation is to set

$$\mu(v_t) - r = \psi v_t \tag{B.6}$$

with a constant ψ, that is in assuming that expected excess returns are linear in the variance. After assuming Eq. (B.6), the equation for F_t (which enters the optimal allocation through the hedging component) is

$$0 = \dot{F}_t + (1-\gamma)\left(\frac{\psi}{\gamma} + \frac{\eta\rho F_t}{\gamma}\right)\psi - kF_t - \frac{1}{2}\gamma(1-\gamma)\left(\frac{\psi}{\gamma} + \frac{\eta\rho F_t}{\gamma}\right)^2 + (1-\gamma)\eta\rho\left(\frac{\psi}{\gamma} + \frac{\eta\rho F_t}{\gamma}\right)F_t + \frac{1}{2}\eta^2 F_t^2, \tag{B.7}$$

with terminal condition $F_T = 0$.

The assumption (B.6) is not terribly restrictive from an economic point of view, since standard models (like CAPM or APT) postulate that expected stock return should be monotone in the stock's variance. As discussed in Liu (2007), page 30, it makes perfect economic sense that the Sharpe ratio depends on variance. The problem of this restriction is in the implication in the optimal strategy, which after assumption (B.6) becomes:

$$w_t^* = \frac{\psi}{\gamma} + \frac{\eta\rho F_t}{\gamma}. \tag{B.8}$$

Now, optimal allocation in Eq. (B.8) does not depend on the volatility level *at all*, that is neither on the instantaneous level v_t, nor on the long-run level θ. Indeed, θ is absent even in the hedging component, determined by Eq. (B.7). For the myopic component, only the ratio of the two risk-premia ψ/γ counts. Of course, ψ can be interpreted as a mean/variance ratio, so that optimal allocation is still numerically very similar to the one obtained in our case. However, even assuming Eq. (B.6) is the correct specification for expected returns, the value of ψ depends, in equilibrium, on the risk aversion of the aggregate investor, and not on the stock dynamics. This implies, for example, that if we double θ, the average value of v_t would change, as well as the average value of $\mu(v_t)$, but not the value of ψ, so that the allocation in the risky assets would not change. Further, the parameter ψ is notoriously nasty to be estimated, given the inherent noise of stock returns (while average returns can be easily estimated by equilibrium models like CAPM and/or adjusted using the Black-Litterman technique).

In the multivariate case the story is the very same, as discussed in Liu (2007). For example, Bäuerle and Li (2013) show that only with a linear drift, that is the multivariate extension of Eq. (B.6), they can find a closed-form solution for a matrix affine stochastic volatility model. This solution is still static and volatility independent (in the sense described above). This implies that if an investor has to allocate wealth in two stocks which are uncorrelated, the optimal allocation does not depend on the relative variance of the stocks, but only on the risk premium parameters. If we add jumps, as in Liu, Longstaff, and Pan (2003), the same reasoning leads to the same conclusion, and for the same reasons we need to assume that the intensity of jumps is linear in v_t. The same applies to multivariate models with jump, as in Branger et al. (2017).

Thus, we conclude that affine stochastic volatility models, which are popular since they provide closed-form solutions for the continuos-time dynamic programming model, imply economically improbable allocations since the optimal strategy they recommend does not depend on the average level of the covariance matrix.

C Reliability of the approximation

Here we assume the co-precision matrix Y_t to be constant over time. In this case, we can compute the closed-form solution for the optimal strategy, and we can compare it with our approximated solution to evaluate the goodness of the approximation itself. We further assume, for simplicity, the jump size J be deterministic. The model dynamics simplifies to

$$\begin{cases} \mathrm{d}S_t = diag(S_t)\left[\alpha \mathrm{d}t + \sqrt{Y_t^{-1}}\mathrm{d}Z_t^{(1)} + J\mathrm{d}N(\lambda)_t\right] \\ \mathrm{d}Y_t = \mathbf{0} \end{cases} , \tag{C.1}$$

where $\mathbf{0}$ represents the $N \times N$ matrix of zeros. Given previous assumptions, the optimal weights when linearizing the jump component of the HJB equation become

$$w_t^* \approx Y_t \left[\frac{(\alpha - r\mathbf{1}) + \lambda J}{\gamma}\right] . \tag{C.2}$$

However, the dynamics described by Eq. (C.1) is equivalent to a Merton's Jump-Diffusion model, whose corresponding HJB equation results to be simplified. The associated first order condition equation reads as follows:

$$0 = (\alpha - r\mathbf{1}) - \gamma Y_t^{-1} w + \lambda(1 + w'J)^{-\gamma} J . \tag{C.3}$$

Thus, Eq. (C.3) can be easily solved numerically, providing the exact optimal allocation. Such a particular situation allows us to execute a sanity check, by comparing the exact solution with the approximate one. We use $\gamma = 3$ and we assume that Y_t is the average co-precision in our sample. The estimates of the remaining parameters are given in Table 3. The results are illustrated in Figure 4. The top plot of the figure gives the comparison between the true solution (given in Eq. (C.3)) and the approximate solution (as in Eq. (C.2)), when we consider the total allocation of our fund of hedge funds, when the jump intensity parameter is either equal to the estimated value $\lambda = 0.0120$ (left plot) and to $\lambda = 0.04$ (right plot), implying that one co-jump occurs every 2.08 years, on average, thus higher frequency of crashes. The lower plot of the figure represents an analogue comparison in one case, the CA index. Hence, Figure 4 shows that the approximated optimal allocation is a reasonable approximation of the true optimal allocation. Importantly, the monotonicity of the allocation with respect to these parameters is preserved.

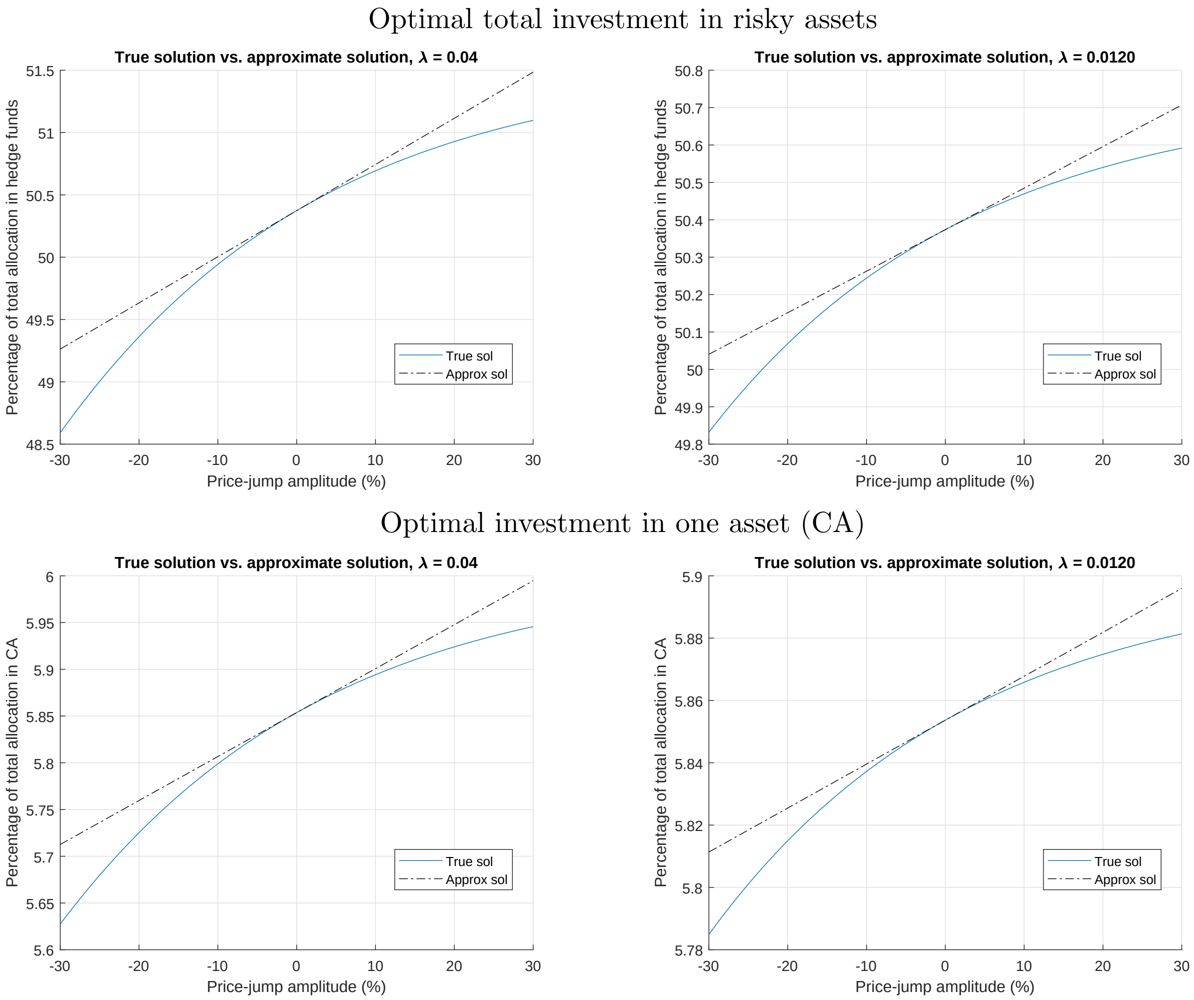

Figure 4: Comparison between true and approximate optimal portfolio weights, as functions of the price-jump size for different jump frequencies, when the instantaneous precision is constant over time. The risk-aversion parameter is $\gamma = 3$, the maturity of the investment is 1 year, the remaining parameter estimates are provided in Table 3. The top charts give the comparison between the true solution and the approximate for total allocation, the lower charts represent the comparison between the true solution and the approximate one in the CA index case. The left plots are related to $\lambda = 0.04$, the right ones are evaluated with $\lambda = 0.0120$. Y_t is the average co-precision in our sample.