# Block Thresholding on the Sphere

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#### Abstract

The aim of this paper is to study nonparametric regression estimators on the sphere based on needlet block thresholding. The block thresholding procedure proposed here follows the method introduced by Hall, Kerkyacharian and Picard in [27], [28], which we modify exploit the properties of spherical needlets. We establish convergence rates, and we show that they attain adaptivity over Besov balls in the regular region. This work is strongly motivated by issues arising in Cosmology and Astrophysics, concerning in particular the analysis of Cosmic rays.

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#### 1 Introduction

Over the last years, wavelet techniques have been used to achieve remarkable results in the field of statistics, in particular in the framework of minimax estimation in nonparametric settings. The pioneering work in this area was provided by Donoho et al. in [13], where authors proved that nonlinear wavelet estimators based on thresholding techniques attain nearly optimal minimax rates, up to logarithmic terms, for a large class of unknown density and regression functions. Since then, this research area has been deeply investigated and extended - we suggest for instance [26] as a textbook reference. In this paper, we shall focus on block thresholding procedure; loosely speaking, this method keeps or annihilates blocks of wavelet coefficients on each given level (for more details, see [26]), hence representing an intermediate way between local and global thresholding, which fix a threshold respectively for each coefficient and for all of them. Block thresholding was initially suggested in [18] for orthogonal series estimators and later applied by [27] for both wavelet and kernel density estimation on  $\mathbb{R}$  (see also [28]); it was also used in [6] in the framework of Oracle

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inequalities, while overlapping block thresholding estimators were studied in [8]. Block thresholding was also applied to study adaptivity in density estimation in [11], a data-driven block thresholding procedure for wavelet regression is investigated in [9], while wavelet-based block thresholding rules on maxisets are proposed by [1].

A huge number of results concerns estimation within the thresholding paradigm in standard Euclidean frameworks, such as  $\mathbb{R}$  or  $\mathbb{R}^d$ ; more recently general settings, such as spherical data or more general manifolds have been considered. Here we focus on a second-generation wavelet system on the sphere, the so-called needlets. Needlets were introduced by Narcowich, Petrushev and Ward in [39], [40]; their stochastic properties, when exploited on spherical random fields, were studied in [2], [3], [35] and [36]. This approach has been extended to more general manifolds by [23], [24], [25], while their generalization to spin fiber bundles on the sphere were described in [21], [22]. Most of these researches can be motivated by applications to Cosmology and Astrophysics: for instance, a huge amount of spherical data, concerning the Cosmic Microwave Background radiation, are being provided by satellite missions WMAP and Planck, see [42], [38], [43], [19], [44], [45], [12], [46], [16] and [17] for more details. The applications mentioned here, however, do not concern thresholding estimation, but rather they can be related to the study of random fields on the sphere, such as angular power spectrum estimation, higher-order spectra, testing for Gaussianity and isotropy, and several others (see also [10]). Of more direct interest here are experiments concerning incoming directions of Ultra High Energy Cosmic Rays, such as the AUGER Observatory (http://www.auger.org). Ultra-High Energy Cosmic Rays are particles with energy above  $10^{18}$  eV reaching the Earth. Even if they were discovered almost a century ago, their origin, their mechanisms of acceleration and propagation are still unknown. As described in [4], see also [20], an efficient nonparametric estimation of the density function of these data would explain the origin of the High Energy Cosmic Rays, i.e. if it is uniform, they are generated by cosmological effects, such as the decay of the massive particles generated during the Big Bang, or, on the other hand, if it is highly non-uniform and, moreover, strongly correlated with the local distribution of nearby Galaxies, it implies that the they are generated by astrophysical phenomena, as for instance the acceleration into Active Galactic Nuclei. Massive amount of data in this area are expected to be available in the next few years. Also in view of this application, the needlet approach was recently applied within the thresholding paradigm to the estimation of the directional data: the seminal contribution in this field is due to [4], see also [31], [30], while applications to astrophysical data is still under way, see for instance [19], [20] and [29] (the latter related to Gamma Rays, another major field where these ideas have proved extremely fruitful). Minimax estimators for spherical data, outside the needlets approach, were also studied by Kim and coauthors (see [33], [32], [34]). Furthermore, adaptive nonparametric regression estimators of spin-functions, based on spin pure and mixed needlets defined in [21], [22], were investigated in [15]. In this case, the needlet nonparametric regression estimators were built on spin fiber bundles on the sphere, i. e. the function to be estimated does not take as its values scalars but algebraic curves living on the tangent plane for each point of the sphere.

This work aims to extend the results established in [4] and [15] towards the needlet block thresholding procedure following two main directions. First of all, we will suggest a construction of blocks of needlet coefficients, exploiting the Voronoi cells based on the geodesic distance on the sphere. Then, we will define the needlet block thresholding estimator, whose we will achieve a near optimal convergence rate. In view of this purpose, we will use both the needlet properties established in [39], [40] (see also [37]) and a set of well consolidated standard techniques, introduced by [13] (see also [26]), remarking that this kind of approach has been also applied within the needlet framework, just considering local thresholding, by [4] and [15]. We also remark that we will describe the nonparametric regression problem in terms of the so-called Gaussian white noise model, able to give suitable approximation of discrete nonparametric regression model, already commonly used in problems over  $\mathbb{R}$  (see for instance [47] and Section 3) and here used over the *d*-dimensional sphere for the first time, at least at our knowledge.

Indeed, consider  $f \in L^p(S^d)$ , the needlet frame  $\{\psi_{jk}\}_{j,k}$ , whose main properties will be described in Section 2, and the corresponding needlet coefficients  $\{\beta_{jk}\}_{j,k}$  given as

$$\beta_{jk} := \int_{\mathbb{S}^d} f\left(x\right) \psi_{jk}\left(x\right) dx \; .$$

As shown in Section 2, from the reconstruction formula (5), we can describe f in terms of needlet decomposition as

$$f(x) = \sum_{j=0}^{+\infty} \sum_{k=1}^{N_j} \beta_{jk} \psi_{jk}(x) ,$$

where the equality holds in the  $L^2$ -sense. Consider now  $X_n$ , a sample path of an isonormal Gaussian process with common mean f (see Section 3), equivalent to the available dataset, where the random element  $X_n(\psi_{jk})$  can be described as

$$X_n\left(\psi_{jk}\right) = \beta_{jk} + \varepsilon_{jk;n} =: \widehat{\beta}_{jk} ,$$

so that  $\varepsilon_{jk;n}$  is the noise with the properties described in Section 3. For any given resolution level j, we therefore build  $S_j$  blocks, labeled as  $R_{j;s}$ ,  $s = 1, ..., S_j$ , each of them containing  $\ell_j$  cubature points. We define

$$\widehat{A}_{js;p} := \frac{1}{\ell_j} \sum_{k \in R_{j;s}} \widehat{\beta}_{jk}^p ,$$

and the corresponding weight function

$$w_{js;p} := I\left(\left|\widehat{A}_{js;p}\right| > \kappa t_n^p\right)$$

(more details on  $\kappa$  and  $t_n$  are in Section 3). Hence we build the needlet block thresholding estimator for f

$$\widehat{f} = \sum_{j=0}^{J_n} \sum_{s=0}^{S_j} w_{js;p} \left( \sum_{k \in R_{j;s}} \widehat{\beta}_{jk} \psi_{jk} \right) \ .$$

We will show that, under some regularity conditions (cfr. Theorem 1, Section 4), there exists  $c_p > 0$  so that

$$\sup_{f \in \mathcal{B}_{\pi_q}^r(M)} \mathbb{E} \left\| \widehat{f} - f \right\|_{L^p(\mathbb{S}^d)}^p \le c_p n^{-\alpha(r,\pi,p)}$$

where  $\alpha(r, \pi, p)$  corresponds to the optimal rate in the regular zone (for definition, see Section 4) and it attains almost the optimal rate in the sparse zone (recall that in the soft thresholding procedure the minimax rate is  $(n/\log n)^{-\alpha(r,\pi,p)}$ ). The improvement achieved by block thresholding in the regular zone can be explained by the better trade-off between bias and variance; the latter is due to the information in nearby coefficients. Note that adaptivity is conditional upon a very careful choice of the block sizes (see Section 3 and also [7], [27] and [28]). For what concerns the sparse zone, the choice of the block size itself will lead us to a not optimal result, as motivated in Sections 3 and 5.

The plan of the paper is as follows: Section 2 will recall some preliminary notions, as needlets, their main properties and the Besov spaces. Section 3 will describe the block thresholding procedure we build for needlet regression estimation , while Section 4 will present the main minimax results. Section 5 will collect some auxiliary probabilistic results, while Section 6 will exploit the proof of the main result of this work, named as Theorem 1. Finally, Section 7 will compare our results with the others in literature concerning needlet thresholding.

### 2 Background results

In this Section, we will review briefly a few of well-known features about the Voronoi cells on the sphere, the spherical needlet construction and the Besov spaces.

For what concerns the definition of Voronoi cells, we are following strictly [3]: further details can be found for instance in the textbook [37], see also [2] and [40]. From now on, given two positive sequences  $\{a_j\}$  and  $\{b_j\}$ , we write that  $a_j \approx b_j$  if there exists a constant c > 1 so that  $c^{-1}a_j \leq b_j \leq ca_j$  for all j. Let us call  $\mathbb{S}^d$  the unit sphere of  $\mathbb{R}^{d+1}$ . Furthermore,  $B_{x_0}(\alpha) = \{x \in \mathbb{S}^d : d(x, x_0) < \alpha\}$ , where  $d(\cdot, \cdot)$  is the natural geodesic distance over the sphere, denotes the standard open ball on  $\mathbb{S}^d$  around  $x_0 \in \mathbb{S}^d$ , while |A| is the surface measure of a general subset  $A \subset \mathbb{S}^d$ : let us recall that this is the unique positive measure invariant by rotation, with total mass  $\omega_d = (2\pi)^{(d+1)/2} / \Gamma((d+1)/2)$ . Given  $\varepsilon > 0$ , the set  $\Xi_{\varepsilon} = \{x_1, ..., x_N\}$  of points on  $\mathbb{S}^d$ , such that for  $i \neq j$  we have  $d(x_i, x_j) > \epsilon$ , is called a maximal  $\varepsilon$ -net if it satisfies  $d(x, \Xi_{\varepsilon}) < \varepsilon$  for  $x \in \mathbb{S}^d$ ,  $\bigcup_{x_i \in \Xi_{\varepsilon}} B_{x_i}(\varepsilon) = \mathbb{S}^d$  and  $B_{x_i}(\varepsilon/2) \cap B_{x_j}(\varepsilon/2) = \emptyset$ , for  $i \neq j$ . For all  $x_i \in \Xi_{\varepsilon}$ , a family of Voronoi cells is defined as

$$\mathcal{V}(x_i) = \left\{ x \in \mathbb{S}^d : \text{for } j \neq i, \ d(x, x_i) < d(x, x_j) \right\}.$$
(1)

In [3] it is proved that:

$$B_{x_i}\left(\frac{\varepsilon}{2}\right) \subset \mathcal{V}\left(x_i\right) \subset B_{x_i}\left(\varepsilon\right)$$
.

Now, we resume the construction of the scalar needlet framework, suggesting for a more detailed discussion [39], [40], see also [4] and [37]. A needlet system describes a well-localized tight frame on the sphere: it is a well-known fact (cfr. [39]) that any function belonging to  $L^2(\mathbb{S}^d)$  can be represented as a linear combination of the components of that frame, preserving some of the most relevant properties of needlets. Indeed, let us recall that the space  $L^2(\mathbb{S}^d)$  of square-integrable functions on the sphere can be decomposed as the direct sum of the spaces  $H_l$  of harmonic polynomials of degree l, spanned by spherical harmonics of degree l, whose definition and properties can be found in [48] and [4]; here we just recall that its dimension corresponds to  $g_{l,d} = \binom{l+d}{d} - \binom{l+d-2}{d}$ . For every  $f \in L^2(\mathbb{S}^d)$ , the following kernel operator describes the orthogonal projector onto  $H_l$ :

$$P_{H_{l}}f(x) = \int_{\mathbb{S}^{d}} L_{l}\left(\langle x, y \rangle\right) f(y) \, dy \, ,$$

where  $L_l$  is the Gegenbauer polyomial with parameter (d-1)/2 and degree l, normalized so that

$$\int_{\mathbb{S}^d} L_l(x) L_m(x) \left(1 - x^2\right)^{\frac{d}{2} - 1} dx = \frac{g_{l,d} \Gamma\left(\frac{d}{2}\right)^2}{\Gamma(d) \omega_d^2} \delta_{l,k} .$$

Following [39], [40] (see also [4]), if we consider

$$\Pi_l = \bigoplus_{l'=0}^l H_{l'},$$

the space of the restrictions to  $\mathbb{S}^d$  of the polynomials of degree less (and equal) to l, the following quadrature formula holds (see for instance [4]): given  $l \in \mathbb{N}$ , there exists a finite subset  $\chi_l$  such that a positive real number  $\lambda_{\xi}$  (the cubature weight) corresponds to each  $\xi \in \chi_l$  (the cubature point) and for all  $f \in \Pi_l$ ,

$$\int_{\mathbb{S}^d} f(x) \, dx = \sum_{\xi \in \chi_l} \lambda_{\xi} f(\xi) \, .$$

Given B > 1 and a resolution level j, we call  $\chi_{\left[B^{2(j+1)}\right]} = Z_j$ ,  $card(Z_j) = N_j$ ; since now any element of the set of cubature points and weights,  $\left\{\xi_{jk}, \lambda_{jk}\right\}$ , will be indexed by j, the resolution level, and k, the cardinality over j, belonging to  $Z_j$ . Furthermore, we choose  $\{Z_j\}_{j>1}$  to be nested so that

$$N_j \approx B^{dj}, \lambda_{jk} \approx B^{-dj}$$
 (2)

We consider a symmetric, real-valued, nonnegative function  $b(\cdot)$  (see again [4]) such that

- 1. it has compact support on  $[B^{-1}, B]$ ;
- 2.  $b \in C^{\infty}(\mathbb{R});$
- 3. the following unitary property holds for  $|\xi| \ge 1$ :

$$\sum_{j\geq 0} b^2 \left(\frac{\xi}{B^j}\right) = 1 \; .$$

For each  $\xi_{jk} \in \mathcal{Z}_j$ , given  $b(\cdot)$  and B, the scalar needlets are defined as:

$$\psi_{jk}\left(x\right) = \sqrt{\lambda_{jk}} \sum_{B^{j-1} < l < B^{j+1}} b\left(\frac{l}{B^{j}}\right) L_l\left(\left\langle x, \xi_{jk}\right\rangle\right)$$

The properties of the function  $b(\cdot)$  yield to three basic properties of the needlets. Indeed, from the infinite differentiability of  $b(\cdot)$ , we obtain a quasi-exponential localization property (see for instance [40]), which states that for  $k \in \mathbb{N}$ , there exists  $c_k$  such that for  $x \in \mathbb{S}^d$ 

$$\left|\psi_{jk}\left(x\right)\right| \leq \frac{c_k B^{\frac{a}{2}j}}{\left(1 + B^{\frac{d}{2}j}d\left(\xi_{jk}, x\right)\right)^k},$$
(3)

where  $d(\xi_{jk}, x)$  is the geodesic distance on the sphere. In view of this property, it is possible to fix a bound (upper and lower), for the norms of needlets on  $L^p(\mathbb{S}^d)$ , for  $1 \le p \le +\infty$ . Given p, there exist two positive constants  $c_p$  and  $C_p$ such that

$$c_p B^{jd\left(\frac{1}{2}-\frac{1}{p}\right)} \le \left\|\psi_{jk}\right\|_{L^p(\mathbb{S}^d)} \le C_p B^{jd\left(\frac{1}{2}-\frac{1}{p}\right)}$$
 (4)

Because the function  $b(\cdot)$  has compact support in  $[B^{-1}, B]$ , it follows that  $b(\frac{l}{B^{j}})$  has compact support in  $[B^{j-1}, B^{j+1}]$ , hence needlets have compact support in the harmonic domain. Finally, the unitary property leads to the following reconstruction formula (see again [39]): for  $f \in L^2(\mathbb{S}^d)$ , in the  $L^2$  sense,

$$f(x) = \sum_{j,k} \beta_{jk} \psi_{jk}(x) , \qquad (5)$$

$$\beta_{jk} := \left\langle f, \psi_{jk} \right\rangle_{L^2(\mathbb{S}^d)} = \int_{\mathbb{S}^d} \overline{\psi}_{jk} \left( x \right) f\left( x \right) dx , \qquad (6)$$

where  $\beta_{jk}$  are the so-called needlet coefficients.

Before concluding this Section, we recall the definition and some main properties of the Besov spaces, referring again to [4], [15] and [26] for further theoretical details and discussions. Let  $f \in L^{\pi}(\mathbb{S}^d)$ ; we define

$$G_k(f,\pi) = \inf_{H \in \mathcal{H}_k} \|f - H\|_{L^{\pi}(\mathbb{S}^d)} ,$$

which is the approximation error when replacing f by an element in  $\mathcal{H}_k$ . The Besov space  $\mathcal{B}_{\pi q}^r$  is therefore defined as the space of functions such that  $f \in L^{\pi}(\mathbb{S}^d)$  and

$$\left(\sum_{k=0}^{\infty} \frac{1}{k} \left(k^r G_k\left(f,\pi\right)\right)^q\right) < \infty$$

The last condition is equivalent to

$$\left(\sum_{j=0}^{\infty} \left(B^{jr}G_{B^{j}}\left(f,\pi\right)\right)^{q}\right) < \infty \; .$$

Moreover,  $F \in \mathcal{B}_{\pi q}^r$  if and only if, for every  $j = 1, 2, \ldots$ 

$$\left(\sum_{k} \left( \left| \beta_{jk} \right| \left\| \psi_{jk} \right\|_{L^{\pi}(\mathbb{S}^d)} \right)^{\pi} \right)^{\frac{1}{\pi}} = \varepsilon_j B^{-jr}$$

where  $\varepsilon_j \in \ell_q$  and B > 1. The Besov norm is defined as follows:

$$\|f\|_{\mathcal{B}_{\pi q}^{r}} = \begin{cases} \|f\|_{L^{\pi}(\mathbb{S}^{d})} + \left[\sum_{j} B^{jq\left(r+d\left(\frac{1}{2}-\frac{1}{\pi}\right)\right)} \left\{\sum_{k} |\beta_{jk}|^{\pi}\right\}^{\frac{q}{\pi}}\right]^{\frac{1}{q}} & q < \infty \\ \|f\|_{L^{\pi}(\mathbb{S}^{d})} + \sup_{j} B^{j\left(r+d\left(\frac{1}{2}-\frac{1}{\pi}\right)\right)} \left\|\left(\beta_{jk}\right)_{k}\right\|_{\ell^{\pi}} & q = \infty \end{cases}$$

As shown for instance in [4], if  $\max(0, 1/\pi - 1/q) < r$  and  $\pi, q > 1$ , then we have

$$f \in \mathcal{B}_{\pi q}^r \Leftrightarrow \|f\|_{\mathcal{B}_{\pi q}^r} < \infty$$

The Besov spaces present, among their properties, some embeddings which will be pivotal in our proofs below. As proven in [4] and [15], we have that, for  $\pi_1 \leq \pi_2$ ,  $q_1 \leq q_2$ 

$$\mathcal{B}_{\pi q_1}^r \subset \mathcal{B}_{\pi q_2}^r , \ \mathcal{B}_{\pi_2 q}^r \subset \mathcal{B}_{\pi_1 q}^r , \ \mathcal{B}_{\pi_1 q}^r \subset \mathcal{B}_{\pi_2 q}^{r-d\left(\frac{1}{\pi_1} - \frac{1}{\pi_2}\right)}.$$
(7)

## 3 Needlet Block Thresholding on the Sphere

In this Section we will discuss the needlet estimators for nonparametric regression problems and, then, we will suggest a procedure to fix blocks for any given resolution level j and, consequently, we will define the so-called needlet block

threshold estimator. The construction of the needlet estimators is close to the one described in [4], [15] for local thresholding, in turn an adaptation to the sphere of the procedure developed on  $\mathbb{R}$  in [27], [28], see also [26].

We start by introducing the Gaussian white noise model over the line segment, then we will extend it to the *d*-dimensional sphere using the so-called uncentered isonormal Gaussian processes.

Usually, in the mathematical statistics literature, a nonparametric regression problem over the line segment [0, 1] is defined by the following Gaussian white noise model, e.g., the stochastic differential equation (see for instance [47])

$$dY_t = f(t) dt + \varepsilon dW(t) , t \in [0,1] , \qquad (8)$$

where W is a standard Wiener process on [0,1], f is an unknown function over [0,1] and  $\varepsilon = n^{-1/2}$ , for n a growing sequence of integers. It is assumed that a sample path  $X = \{Y(t), 0 \le t \le 1\}$  is observed; the statistical problem regards the estimation of the unknown function  $f \in \mathcal{F}$ , where  $\mathcal{F}$  is a given nonparametric class of functions. For  $x \in [0,1]$ , the function  $x \mapsto f_n(x,X)$ , defined on [0,1] and measurable with respect X, is the estimator of f.

**Remark 1** As proven in [5], see also for instance [47] Section 1.10 and the references therein, the nonparametric linear regression model and linear regression in terms of the Gaussian white noise model are asymptotically equivalent. Indeed, consider the process Y in (8). If  $\Delta > 0$ , we have

$$\frac{Y\left(t+\Delta\right)-Y\left(t\right)}{\Delta} = \frac{1}{\Delta} \int_{t}^{t+\Delta} f\left(s\right) ds + \frac{\varepsilon}{\Delta} \left(W\left(t+\Delta\right) - W\left(t\right)\right).$$

Now, if

$$y(t) = \frac{Y(t + \Delta) - Y(t)}{\Delta}, \ z(t) = \frac{\varepsilon}{\Delta} \left( W(t + \Delta) - W(t) \right)$$

for any  $t \in [0,1]$ , z(t) is a centered Gaussian with variance  $\varepsilon^2/\Delta$ . Taking  $\varepsilon = 1/\sqrt{n}$  and  $\Delta = 1/n$ ,  $z(t) \sim \mathcal{N}(0,1)$ . Up to deterministic residuals, for sufficient small  $\Delta$  and sufficient smooth f,

$$\frac{1}{\Delta} \int_{t}^{t+\Delta} f(s) \, ds - f(t) \to 0 \, ,$$

hence

$$y\left(t\right)=f\left(t\right)+z\left(t\right) \ .$$

For i = 1, ..., n, we take  $X_i = i/n$ ,  $Y_i = Y(X_i)$ ,  $z_i = z(X_i)$ , so that

$$Y_i = f\left(X_i\right) + z_i \; ,$$

which corresponds to the nonparametric regression model with regular design and i.i.d. errors  $z_i$  distributed as  $\mathcal{N}(0,1)$ . Further details can be found in [47]. If we consider the *d*-dimensional sphere, we can describe the same problem in terms of the so-called uncentered isonormal Gaussian processes with mean f. Following [41], an isonormal Gaussian process over  $\mathfrak{H}$  is defined as  $X = \{X(h) : h \in \mathfrak{H}\}$ , where  $\mathfrak{H}$  is a real separable Hilbert space, with inner product  $\langle \cdot, \cdot \rangle$ . Hence, we assume that X describes a family of (uncentered) Gaussian variables, defined on some probability space  $(\Omega, \mathfrak{F}, P)$  such that for all  $h_1, h_2 \in \mathfrak{H}$ ,  $\{X(h_1), X(h_2)\}$  are jointly Gaussian with mean

$$\mathbb{E}X(h) = \langle h, f \rangle = \int_{\mathbb{S}^d} f(x)h(x)dx$$

and covariance

$$\mathbb{E} \left( X(h_1) - \mathbb{E} X(h_1) \right) \left( X(h_2) - \mathbb{E} X(h_2) \right) = \langle h_1, h_2 \rangle$$

In our case,  $\Omega := \mathbb{S}^d$  and  $\mathfrak{F}$  is the  $\sigma$ -algebra generated by X. We will use  $L^2(\mathbb{S}^d)$  instead of  $L^2(\mathbb{S}^d, \mathfrak{F}, P)$  to simplify the notation. We shall in fact be concerned with  $X_n = \{Y(x), x \in \mathbb{S}^d\}$ , the observed sample path associated to the process, where we assume that

$$\mathbb{E}X_n(h) = \langle h, f \rangle = \int_{\mathbb{S}^d} f(x)h(x)dx$$

and covariance

$$\mathbb{E}\left(X_n(h_1) - \mathbb{E}X_n(h_1)\right)\left(X_n(h_2) - \mathbb{E}X_n(h_2)\right) = \frac{1}{n}\left\langle h_1, h_2\right\rangle \ .$$

In other words, in order to estimate the unknown function f, on a proper class of function (in our case, the Besov ball), we will study the estimator of f which is a function  $x \mapsto \hat{f}(x) = \hat{f}(x, X_n)$  defined on the *d*-dimensional sphere and measurable with respect to the observation  $X_n$ , see again [47] and cfr. Remark 1.

Consider now the usual needlet system  $\{\psi_{jk}\}_{j,k}$  and let  $f \in L^p(\mathbb{S}^d)$ ; we have the following:

$$\beta_{jk} = \mathbb{E}X_n(\psi_{jk}) = \langle \psi_{jk}, f \rangle = \int_{\mathbb{S}^d} f(x)\psi_{jk}(x)dx ,$$
  
$$\widehat{\beta}_{jk} = X_n(\psi_{jk}) = \beta_{jk} + \varepsilon_{jk;n} , \qquad (9)$$

where

$$\mathbb{E}\varepsilon_{jk;n} = \mathbb{E}\left(X_n(\psi_{jk}) - \mathbb{E}X_n(\psi_{jk})\right) = 0, \\
E\varepsilon_{jk;n}^2 = \frac{1}{n} \langle \psi_{jk}, \psi_{jk} \rangle_{L^2(\mathbb{S}^d)} = \frac{1}{n} \left\|\psi_{jk}\right\|_{L^2(\mathbb{S}^d)}^2, \quad (10) \\
E\varepsilon_{jk_1;n}\varepsilon_{jk_2;n} = \frac{1}{n} \langle \psi_{jk_1}, \psi_{jk_2} \rangle_{L^2(\mathbb{S}^d)} \\
= \frac{1}{n} \frac{\sum_l b^2(\frac{l}{2^j}) L_l(\langle \xi_{jk_1}, \xi_{jk_2} \rangle)}{\sum_l b^2(\frac{l}{2^j}) L_l(\langle \xi_{jk_1}, \xi_{jk_2} \rangle)} \\
= \frac{1}{n} \frac{\sum_l b^2(\frac{l}{2^j}) L_l(\langle \xi_{jk_1}, \xi_{jk_2} \rangle)}{\sum_l b^2(\frac{l}{2^j}) \frac{g_{l,d}}{\omega_d}}.$$

In a formal sense, one could consider the Gaussian white noise measure on the sphere such that for all  $A, B \subset \mathbb{S}^d$ , we have

$$\mathbb{E}W(A)W(B) = \int_{A\cap B} dx \; ,$$

so that

$$\varepsilon_{jk;n} = \frac{1}{n} \int_{\mathbb{S}^d} \psi_{jk}(x) W(dx) \ ,$$

as in the Gaussian white noise model on [0, 1], described by [47].

**Remark 2** Following the same reasoning illustrated in Remark 1, asymptotic equivalence holds between our Gaussian white noise model over  $\mathbb{S}^d$  and the discrete nonparametric regression model

$$Y_i = f(X_i) + \varepsilon_i$$
,  $i = 1...n$ ,

where in this case  $\{X_i\}_{i=1}^n$  are uniform random locations over  $\mathbb{S}^d$  and  $\{Y_i\}_{i=1}^n$  are the corresponding observations. By the practical point of view, considering the Remarks 1 and 2, (cfr. also [15]), given the dataset  $\{X_i\}_{i=1}^n$ , the needlets estimator defined in (9) corresponds to

$$\widehat{\beta}_{jk} := \frac{1}{n} \sum_{i=1}^{n} Y_i \psi_{jk} \left( X_i \right) = \frac{1}{n} \sum_{i=1}^{n} \left[ \psi_{jk} \left( X_i \right) f \left( X_i \right) + \psi_{jk} \left( X_i \right) \varepsilon_i \right] .$$

Furthermore, the unbiasedness is easily verified :

$$E\left(\widehat{\beta}_{jk}\right) = \frac{1}{n} \sum_{i=1}^{n} E\left[\psi_{jk}\left(X_{i}\right) f\left(X_{i}\right) + \psi_{jk}\left(X_{i}\right)\varepsilon_{i}\right]$$
$$= \int_{\mathbb{S}^{d}} \psi_{jk}\left(x\right) f\left(x\right) dx = \beta_{jk} . \tag{11}$$

As described above (see also [4], [15]), f can be described in terms of needlet coefficients, up to a constant, as

$$f = \sum_{j \ge 0} \sum_{k=1}^{N_j} \beta_{jk} \psi_{jk} \ .$$

Let us now define the threshold blocks: as anticipated in the Introduction, differently from [27], the structure itself of the needlet framework suggests a quite intuitive way to be followed. Let us fix j > 0: recall that for each resolution level j, we have  $N_j \approx B^{dj}$  cubature points. Given the size of the blocks, i.e. the number of cubature points belonging to each of them - let us say  $\ell_j$  - we will build using (1) a set of Voronoi cells, containing  $\ell_j$  cubature points. For each cell, we choose a cubature point  $\xi_{js}$  to index it: we define  $S_j$  ( $\ell_j$ ) as the number

of Voronoi cells obtained to split cubature points into groups of cardinality  $\ell_j.$  Let us define the set

$$R_{j;s} = \left\{ k : \xi_{jk} \in \mathcal{V}\left(\xi_{js}\right) \right\}, \ s = 1, ..., S_j.$$
(12)

From (1), it is immediate to see that each cubature point  $\xi_{jk}$  belongs to a unique Voronoi cell. Obviously,  $S_j\cdot\ell_j=N_j$ .

Let us call, for any integer  $p \ge 1$ ,

$$A_{js;p} := \frac{1}{\ell_j} \sum_{k \in R_{j;s}} \beta_{jk}^p ,$$

hence we can define the corresponding estimator

$$\widehat{A}_{js;p} = \frac{1}{\ell_j} \sum_{k \in R_{j;s}} \widehat{\beta}_{jk}^p ,$$

similar to the ones suggested in [27], Remark 4.7.

We build the following weight function as follows

$$w_{js;p} = I\left(\left|\widehat{A}_{js;p}\right| > \kappa t_n^p\right)$$

we can define the function estimator as:

$$\widehat{f} = \sum_{j=0}^{J_n} \sum_{s=1}^{S_j} \left( \sum_{k \in R_{j;s}} \widehat{\beta}_{jk} \psi_{jk} \right) w_{js;p} , \qquad (13)$$

where:

•  $J_n$  is the highest resolution level considered, taken such that

$$B^{J_n} = n^{\frac{1}{d}} aga{14a}$$

consistent with the existent literature (see for instance [26], [4])

- $\kappa$  is the threshold constant (for more discussions see for instance [4], [15], and [26]). As suggested in [4],  $\kappa$  has to be proportional to M, the bound of  $||f||_{\infty}$ , multiplied by a constant  $\kappa_0$  that can be made explicit with an iterative procedure to count the blocks not annihilated by the threshold;
- the scaling factor  $t_n$ , depends on the size of the sample. We will fix

$$t_n = n^{-\frac{1}{2}}$$

This choice is motivated by two main facts. On one hand, it allows  $\hat{f}$  to attain the optimal rate of convergence in the regular zone (cfr. Theorem 1 and Remark 3). On the other hand, this choice is consistent with the literature related to thresholding procedures in needlet frameworks, see [4] and [15]).

• The block size will be chosen so that

$$\ell_j = [N_j]^{\eta} ,$$

where  $[\cdot]$  denotes the integer part and  $0 < \eta < \frac{1}{2}$ . The size of the block has to be chosen also considering, on one hand, the value of the threshold (see above the point related to the choice of  $\kappa$ ) and, on the other hand, the number of cubature points at a fixed resolution level j. More details will be given in the Section 7.

By the practical point of view, given the size of the sample n and the scale parameter B,  $J_n$  and  $t_n$  are easily computed. Therefore, the experimenter should test, for different sizes of the blocks, chosen taking on account the whole number of cubature points, and for different values of  $\kappa_0$  the number of blocks not annihilated by the procedure.

#### 4 Minimax risk rates of convergence

This Section aims to describe the performance of the procedure in terms of the optimality of its convergence rates with respect to general  $L^p(\mathbb{S}^d)$ -loss functions: this result is established in the Theorem 1. First of all, we recall the definition of optimal rate of convergence from [26]. We say that an estimator  $\hat{f}$  attains the optimal rate of convergence  $R_n(V, p)$  on the class V for the  $L_p$ -risk if

$$\sup_{f \in V} \mathbb{E} \left\| \widehat{f} - f \right\|_{L^p}^p \simeq R_n \left( V, p \right) \; .$$

In our case V corresponds to the Besov ball  $\mathcal{B}_{\pi q}^{r}(M)$  and  $R_{n}(V,p)$  will assume the form of  $n^{-\alpha(r,\pi,p)}$ .

**Theorem 1** Let  $f \in \mathcal{B}^{r}_{\pi q}(M)$ , the Besov ball so that  $||f||_{\mathcal{B}^{r}_{\pi q}(M)} \leq M < +\infty$ ,  $r - \frac{d}{\pi} > 0$ . Consider  $\hat{f}$  as defined by (13). For  $p \in \mathbb{N}$ , there exists a constant  $c_{p} = c_{p}(p, r, q, M, B)$  such that

$$\sup_{f \in \mathcal{B}^{r}_{\pi_{q}}(M)} \mathbb{E} \left\| \widehat{f} - f \right\|_{L^{p}(\mathbb{S}^{d})}^{p} \le c_{p} n^{-\alpha(r,\pi,p)}$$

where

$$\alpha\left(r,\pi,p\right) = \begin{cases} \frac{rp}{2r+d} & \text{for } \pi \ge \frac{dp}{2r+d} \\ \frac{p\left(r-d\left(\frac{1}{\pi}-\frac{1}{p}\right)\right)}{2\left(r-d\left(\frac{1}{\pi}-\frac{1}{2}\right)\right)} - \delta & \text{for } \pi < \frac{dp}{2r+d} \end{cases}$$

where  $\delta = \delta(\eta, d, \pi, p, r) = \frac{\eta d \left(1 - \frac{\pi}{p}\right)}{2\left(r - d\left(\frac{1}{\pi} - \frac{1}{2}\right)\right)}$ . If  $p = +\infty$ , there exists a constant  $c_{\infty} = c_{\infty}(r, q, M, B)$ 

$$\sup_{f \in \mathcal{B}_{\pi_q}^r(M)} \mathbb{E} \left\| \widehat{f} - f \right\|_{\infty} \le c_{\infty} n^{-\alpha(r,\pi,p)} ,$$

where

$$\alpha(r,\pi,p) = \frac{\left(r - \frac{d}{\pi}\right)}{2\left(r - d\left(\frac{1}{\pi} - \frac{1}{2}\right)\right)} \ .$$

Let us recall that in literature (cfr. for instance [26]) the case  $\pi \ge dp/(2r+d)$  is named *regular case*, the other being referred as *sparse case*.

**Remark 3** Our results achieve the minimax rates provided in [4] and [15], see also [26] just in the regular zone, while in the sparse zone the rate is worsened by the term  $n^{\delta}$ . Note that for convenience our arguments are implemented for integer values  $p \in \mathbb{N}$ . Other real values can be dealt with by interpolation, but we omit to do it for brevity's sake. Of course, the most relevant case for practitioners is p = 2, in which case the function certainly belongs to the regular zone, where our rates are optimal, see also [7], [14].

As in [4] and [15], the minimax rates are not affected by the construction over the sphere, which instead is pivotal in the development of statistical procedures. As mentioned in the Introduction, the proof of this Theorem makes extensive use of standard techniques (see for instance [26]), modified to exploit the properties of the needlets described in Section 2. The procedure is therefore close to the ones employed in [4] and [15], the main differences concerning the probabilistic inequalities in Section 5 and some of the pivotal steps of the proof remarked in Section 6.

#### 5 Auxiliary Results

This Section collects the probabilistic inequalities necessary to prove Theorem 1.

**Lemma 2** Consider  $\widehat{\beta}_{jk}$  as described in 9. There exist constants  $C_p, C_{\infty}, C_A$  such that, for  $B^j \leq n^{\frac{1}{2}}, j = 0, ..., J_n$ ,

$$\mathbb{E}\left[\left|\widehat{\beta}_{jk} - \beta_{jk}\right|^{p}\right] \le C_{p} n^{-p/2} , \ p \ge 1$$
(15)

$$\mathbb{E}\left[\sup_{k=1,\dots,N_j} \left|\widehat{\beta}_{jk} - \beta_{jk}\right|^p\right] \le C_{\infty} \left(j+1\right)^p n^{-p/2}, \ p \ge 1 \ , \tag{16}$$

and for all  $\gamma > 0, p \in \mathbb{N}$ , there exists  $\kappa > 0$  such that

$$\mathbb{P}\left(\left|\widehat{A}_{js;p} - A_{js;p}\right| > \kappa t_n^p\right) \le C_{p,\gamma} \frac{1}{n^{\gamma}} .$$
(17)

**Proof.** First of all, consider that, from Equations (9) and (10), we have

$$\begin{split} \mathbb{E}\left(\left|\widehat{\beta}_{jk} - \beta_{jk}\right|^{p}\right) &= \mathbb{E}\left(|\varepsilon_{jk;n}|^{p}\right) \\ &= \left(Var\left(\varepsilon_{jk;n}\right)\right)^{\frac{p}{2}} \frac{2^{\frac{p}{2}}\Gamma\left(\frac{p+1}{2}\right)}{\sqrt{\pi}} \\ &= \frac{1}{n^{\frac{p}{2}}} \left\|\psi_{jk}\right\|_{L^{2}(\mathbb{S}^{d})}^{p} \frac{2^{\frac{p}{2}}\Gamma\left(\frac{p+1}{2}\right)}{\sqrt{\pi}} \\ &= O\left(n^{-\frac{p}{2}}\right) \,, \end{split}$$

to obtain (15). Now, for Mill's inequality, if  $Z \sim N(0, 1)$ , we have  $\mathbb{P}(|Z| \ge x) \le \sqrt{2/\pi} \exp\left(-x^2/2\right)/x$ . Hence, from (10), we obtain

$$\mathbb{P}\left(|\varepsilon_{jk;n}| \ge x\right) = 2\mathbb{P}\left(|Z| \ge \sqrt{nx}\right)$$
$$\le \sqrt{\frac{2}{\pi}} \frac{e^{-\frac{nx^2}{2}}}{\sqrt{nx}}$$
$$\le C_{\varepsilon} e^{-\frac{nx^2}{2}}.$$

On the other hand, we have

$$\mathbb{E}\left[\sup_{k=1,\dots,N_{j}}\left|\widehat{\beta}_{jk}-\beta_{jk}\right|^{p}\right] = \int_{\mathbb{R}^{+}} x^{p-1} \mathbb{P}\left(\sup_{k=1,\dots,N_{j}}\left|\widehat{\beta}_{jk}-\beta_{jk}\right| \ge x\right) dx$$
$$= \int_{\mathbb{R}^{+}} x^{p-1} \mathbb{P}\left(\sup_{k=1,\dots,N_{j}}\left|\varepsilon_{jk;n}\right| \ge x\right) dx$$
$$= E_{1} + E_{2},$$

where

$$E_1 = \int_{0 \le x \le \frac{2\sqrt{2}}{\sqrt{n}}j} x^{p-1} dx ,$$
  

$$E_2 = C \int_{x > \frac{2\sqrt{2}}{\sqrt{n}}j} x^{p-1} B^{2j} \max_k \mathbb{P}(|\varepsilon_{jk;n}| \ge x) dx .$$

We can easily see that

$$E_1 = C_1 j^p n^{-\frac{p}{2}} ,$$

while on the other hand, considering that for  $x > 2\sqrt{2/n}j$ 

$$B^{2j}e^{-\frac{nx^2}{2}} \le e^{-\frac{nx^2}{4} - \frac{nx^2}{4} + 2j} \le e^{-\frac{nx^2}{4}},$$

we obtain

$$E_{2} \leq C \int_{x > \frac{2\sqrt{2}}{\sqrt{n}}j} x^{p-1} B^{2j} e^{-\frac{nx^{2}}{2}} dx$$
$$\leq C_{2} n^{-\frac{p}{2}},$$

so we achieve (16). In order to prove (17), we write

$$\mathbb{P}\left(\left|\widehat{A}_{js;p} - A_{js;p}\right| > \kappa t_n^p\right)$$
$$= \mathbb{P}\left\{\left(\frac{1}{\ell_j} \sum_{k=1}^{\ell_j} \left(\widehat{\beta}_{jk}^p - \mathbb{E}\widehat{\beta}_{jk}^p\right)\right)^{1/p} > \frac{\kappa}{\sqrt{n}}\right\}.$$

Define

$$\widetilde{\boldsymbol{\beta}}_{jk} := \sqrt{n} \boldsymbol{\beta}_{jk} + \sqrt{n} \boldsymbol{\varepsilon}_{jk;n} = \sqrt{n} \boldsymbol{\beta}_{jk} + \boldsymbol{\varepsilon}_{jk} \ ,$$

where

$$\varepsilon_{jk} := \sqrt{n} \varepsilon_{jk;n} ;$$

our aim is hence to study the behaviour of the terms of the form

$$\left(\frac{1}{\ell_j}\sum_{k=1}^{\ell_j}\varepsilon_{jk}^p + \frac{p\sqrt{n}}{\ell_j}\sum_{k=1}^{\ell_j}\beta_{jk}\varepsilon_{jk}^{p-1} + \dots + \frac{pn^{(p-1)/2}}{\ell_j}\sum_{k=1}^{\ell_j}\beta_{jk}^{p-1}\varepsilon_{jk}\right)^{1/p}.$$
 (18)

Observe that:

$$\frac{1}{\ell_j} \sum_{k=1}^{\ell_j} \beta_{jk}^{p-1} \varepsilon_{jk} \le \left( \frac{1}{\ell_j} \sum_{k=1}^{\ell_j} \beta_{jk}^{2p-2} \right)^{1/2} \left( \frac{1}{\ell_j} \sum_{k=1}^{\ell_j} \varepsilon_{jk}^2 \right)^{1/2} ;$$

we have that

$$\begin{split} \sum_{k=1}^{\ell_j} \beta_{jk}^{2p-2} &\leq \sum_{k=1}^{N_j} \beta_{jk}^{2p-2} = O\left(B^{-js} B^{-j\frac{d}{2}\left(1-\frac{1}{p-1}\right)}\right) = O\left(B^{-js} B^{-j\frac{d}{2}\left(\frac{p-2}{p-1}\right)}\right) \\ & O\left(B^{-js} B^{-j\frac{d}{2}\left(\frac{p-2}{p-1}\right)}\right) \end{split}$$

On the other hand, by Lemma 3, for all  $p,\gamma>0,$  there exists  $\kappa>0$  such that

$$\mathbb{P}\left\{\frac{1}{\ell_j}\sum_{k=1}^{\ell_j} \left|\varepsilon_{jk}\right|^p > \kappa\right\} \le \frac{C_{p,\gamma}}{\ell_j^{\gamma/2}} \ .$$

Hence, we obtain

$$\frac{pn^{(p-1)/2}}{\ell_j} \sum_{k=1}^{\ell_j} \beta_{jk}^{p-1} \varepsilon_{jk} \le C \frac{n^{(p-1)/2}}{\ell_j^{\frac{\gamma+2}{2}}} B^{-j(\frac{d}{2}\frac{p-2}{p-1}+s)} .$$

By choosing suitable s and  $\gamma,$  we have

$$\frac{pn^{(p-1)/2}}{\ell_j} \sum_{k=1}^{\ell_j} \beta_{jk}^{p-1} \varepsilon_{jk} = o\left(\frac{1}{\ell_j} \sum_{k=1}^{\ell_j} \varepsilon_{jk}^p\right) \,.$$

The same holds for all the other mixed terms in Equation (18).  $\blacksquare$ 

**Lemma 3** Assume that  $\mathbb{E}\varepsilon_{jk} = 0$ ,  $\mathbb{E}\varepsilon_{jk}^2 = 1$ , and

$$\mathbb{E}\varepsilon_{jk_1}\varepsilon_{jk_2} \leq \frac{C_M}{\left\{1 + B^{\frac{d}{2}j}d(\xi_{jk_1}, \xi_{jk_2})\right\}^M} \text{ , for all } M > 0 \text{ .}$$

For all  $p \in \mathbb{N}$ ,  $\gamma > 0$  there exists  $\kappa > 0$  such that

$$\mathbb{P}\left\{\frac{1}{\ell_j}\sum_{k=1}^{\ell_j}|\varepsilon_{jk}|^p>\kappa\right\}\leq \frac{C_{p,\gamma}}{\ell_j^{\gamma/2}}\ .$$

**Proof.** Without loss of generality we can take p to be even; note indeed that

$$\mathbb{P}\left\{\frac{1}{\ell_j}\sum_{k=1}^{\ell_j}|\varepsilon_{jk}|^p > \kappa\right\} \le \mathbb{P}\left\{\frac{1}{\ell_j}\sum_{k=1}^{\ell_j}\varepsilon_{jk}^{2p} > \kappa^2\right\} .$$

Let us rewrite

$$\frac{1}{\ell_j} \sum_{k=1}^{\ell_j} \varepsilon_{jk}^p = \mathbb{E}\varepsilon_{jk}^p + \sum_{\tau=1}^p c_\tau H_\tau(\varepsilon_{jk}) ,$$

whence

$$\mathbb{P}\left\{\frac{1}{\ell_j}\sum_{k=1}^{\ell_j}\varepsilon_{jk}^p > p(\kappa + E\varepsilon_{jk}^p)\right\} \le \sum_{\tau=1}^p \mathbb{P}\left\{\frac{1}{\ell_j}\sum_{k=1}^{\ell_j}H_\tau(\varepsilon_{jk}) > \frac{\kappa}{c_\tau}\right\}$$

By the Markov's inequality, the result will hence follow if we prove that

$$\mathbb{E}\left[\frac{1}{\ell_j}\sum_{k=1}^{\ell_j}H_\tau(\varepsilon_{jk})\right]^{\gamma} \leq \frac{C}{\ell_j^{\gamma/2}} \ .$$

Now let us take for notational simplicity  $\tau = 2$ ; the argument for the other terms is identical. We have

$$\mathbb{E}\left[\frac{1}{\ell_j}\sum_{k=1}^{\ell_j} H_{\tau}(\varepsilon_{jk})\right]^{\gamma} = \frac{1}{\ell_j^{\gamma}}\sum_{k_1,\dots,k_{\gamma}=1}^{\ell_j} \mathbb{E}\left\{H_{\tau}(\varepsilon_{jk_1})\dots H_{\tau}(\varepsilon_{jk_{\gamma}})\right\}$$
$$= \frac{1}{\ell_j^{\gamma}}\left\{\sum_{k_1k_2}^{\ell_j} \left[\mathbb{E}(\varepsilon_{jk_1}\varepsilon_{jk_2})\right]^2\right\}^{\gamma/2}$$
$$+ \frac{1}{\ell_j^{\gamma}}\left\{\sum_{k_1k_2}^{\ell_j} \left[\mathbb{E}(\varepsilon_{jk_1}\varepsilon_{jk_2})\right]^2\right\}^{\frac{\gamma}{2}-2} \left\{\sum_{k_1\dots,k_4}^{\ell_j} \mathbb{E}(\varepsilon_{jk_1}\varepsilon_{jk_2})\mathbb{E}(\varepsilon_{jk_2}\varepsilon_{jk_3})\mathbb{E}(\varepsilon_{jk_3}\varepsilon_{jk_4})\mathbb{E}(\varepsilon_{jk_4}\varepsilon_{jk_1})\right\}$$

$$+\frac{1}{\ell_j^{\gamma}} \left\{ \sum_{k_1 k_2}^{\ell_j} \left[ \mathbb{E}(\varepsilon_{jk_1} \varepsilon_{jk_2}) \right]^2 \right\}^{\frac{\gamma}{2} - 4} \left\{ \sum_{k_1 \dots k_6}^{\ell_j} \mathbb{E}(\varepsilon_{jk_1} \varepsilon_{jk_2}) \dots \mathbb{E}(\varepsilon_{jk_6} \varepsilon_{jk_1}) \right\} \\ + \frac{1}{\ell_j^{\gamma}} \left\{ \sum_{k_1 \dots k_{\gamma}}^{\ell_j} \mathbb{E}(\varepsilon_{jk_1} \varepsilon_{jk_2}) \dots \mathbb{E}(\varepsilon_{jk_{\gamma}} \varepsilon_{jk_1}) \right\} \\ = O(\ell_j^{-\gamma/2}) + O(\ell_j^{-\frac{\gamma}{2} - 1}) + \dots + O(\ell_j^{-\gamma + 1}) ,$$

because

$$\sum_{k_1...k_{\gamma}}^{\ell_j} E(\varepsilon_{jk_1}\varepsilon_{jk_2})...E(\varepsilon_{jk_{\gamma}}\varepsilon_{jk_1}) \leq \sum_{k_1...k_q}^{\ell_j} |E(\varepsilon_{jk_1}\varepsilon_{jk_2})|...|E(\varepsilon_{jk_{\gamma-1}}\varepsilon_{jk_{\gamma}})|$$
$$\leq \ell_j \left\{ \sum_{k_2}^{\ell_j} |E(\varepsilon_{jk_1}\varepsilon_{jk_2})| \right\}^{\gamma-1} = O(\ell_j) .$$

# 6 Proof of Theorem 1 (upper bound)

This Section will describe in details the proof of the Theorem 1. As previously mentioned, some of the passages of this proof will be very close to those developed for local thresholding described in [4] and [15], hence we will omit them. First of all, observe that

$$\sum_{s=1}^{S_j} \sum_{k \in R_{j;s}} = \sum_{k=1}^{N_j} .$$

Standard calculations (see for instance [26]) lead to:

$$\begin{split} \mathbb{E} \left\| \widehat{f} - f \right\|_{L^{p}(\mathbb{S}^{d})}^{p} &= \mathbb{E} \left\| \sum_{j=0}^{J_{n}} \sum_{s=1}^{S_{j}} \left( \sum_{k \in R_{j;s}} \widehat{\beta}_{jk} \psi_{jk} \right) w_{js;p} - \sum_{j \ge 0} \sum_{k=1}^{N_{j}} \beta_{jk} \psi_{jk} \right\|_{L^{p}(\mathbb{S}^{d})}^{p} \\ &= \mathbb{E} \left\| \sum_{j=0}^{J_{n}} \sum_{s=1}^{S_{j}} \sum_{k \in R_{j;s}} \left( w_{js;p} \widehat{\beta}_{jk} - \beta_{jk} \right) \psi_{jk} - \sum_{j > J_{n}} \sum_{k=1}^{N_{j}} \beta_{jk} \psi_{jk} \right\|_{L^{p}(\mathbb{S}^{d})}^{p} \\ &\leq 2^{p-1} \left( \mathbb{E} \left\| \sum_{j=0}^{J_{n}} \sum_{s=1}^{S_{j}} \sum_{k \in R_{j;s}} \left( w_{js;p} \widehat{\beta}_{jk} - \beta_{jk} \right) \psi_{jk} \right\|_{L^{p}(\mathbb{S}^{d})}^{p} \\ &+ \left\| \sum_{j > J_{n}} \sum_{k=1}^{N_{j}} \beta_{jk} \psi_{jk} \right\|_{L^{p}(\mathbb{S}^{d})}^{p} \\ &= : I + II . \end{split}$$

Consider now the two different cases mentioned in Section 4.

CASE I: Regular Case

Consider  $p < +\infty$ . For  $p \le \pi$ , we have  $\mathcal{B}_{\pi q}^r \subset \mathcal{B}_{pq}^r$ : we therefore take  $\pi = p$ . Consider instead the case  $p > \pi$ : we use the embedding  $\mathcal{B}_{\pi q}^r \subset \mathcal{B}_{pq}^{r-d(\frac{1}{p}-\frac{1}{\pi})}$ , and moreover we assume

$$r\geq \frac{d}{p}, \frac{r}{2r+d}=\frac{rp}{\left(2r+d\right)p}\leq \frac{r\pi}{dp}\ ,$$

we have as in [4], [15], that

$$II \le O\left(n^{-\frac{pr}{2r+d}}\right) \;,$$

as claimed.

About the variance term, from the Loève's inequality we have

$$I \leq C\mathbb{E} \left\| \sum_{j=0}^{J_n} \sum_{s=1}^{S_j} \sum_{k \in R_{j;s}} \left( w_{js;p} \widehat{\beta}_{jk} - \beta_{jk} \right) \psi_{jk} \right\|_{L^p(\mathbb{S}^d)}^p$$
$$\leq CJ_n^{p-1} \sum_{j \leq J_n} \mathbb{E} \left\| \sum_{s=1}^{S_j} \sum_{k \in R_{j;s}} \left( w_{js;p} \widehat{\beta}_{jk} - \beta_{jk} \right) \psi_{jk} \right\|_{L^p(\mathbb{S}^d)}^p$$

As described in [4], see also [37], we have the following needlet property:

$$\mathbb{E} \left\| \sum_{k} \alpha_{k} \psi_{jk} \right\|_{L^{p}(\mathbb{S}^{d})}^{p} = \left\| \psi_{jk} \right\|_{L^{p}(\mathbb{S}^{d})}^{p} \sum_{k} \mathbb{E} \left\| \alpha_{k} \right\|_{L^{p}(\mathbb{S}^{d})}^{p} .$$

Hence, we obtain

=

$$\sum_{j \leq J_n} \mathbb{E} \left\| \sum_{s=1}^{S_j} \sum_{k \in R_{j;s}} \left( w_{js;p} \widehat{\beta}_{jk} - \beta_{jk} \right) \psi_{jk} \right\|_{L^p(\mathbb{S}^d)}^p \tag{19}$$

$$\sum_{j \leq J_n} \mathbb{E} \left\| \sum_{s=1}^{S_j} \sum_{k \in R_{j;s}} \left( w_{js;p} \widehat{\beta}_{jk} - \beta_{jk} \right) \psi_{jk} I \left( \left| \widehat{A}_{js;p} \right| \geq t_n^p \right) I \left( \left| A_{js;p} \right| \geq \frac{t_n^p}{2} \right) \right\|_{L^p(\mathbb{S}^d)}^p$$

$$+ \sum_{j \leq J_n} \mathbb{E} \left\| \sum_{s=1}^{S_j} \sum_{k \in R_{j;s}} \left( w_{js;p} \widehat{\beta}_{jk} - \beta_{jk} \right) \psi_{jk} I \left( \left| \widehat{A}_{js;p} \right| \geq t_n^p \right) I \left( \left| A_{js;p} \right| < \frac{t_n^p}{2} \right) \right\|_{L^p(\mathbb{S}^d)}^p$$

$$+ \sum_{j \leq J_n} \mathbb{E} \left\| \sum_{s=1}^{S_j} \sum_{k \in R_{j;s}} \left( w_{js;p} \widehat{\beta}_{jk} - \beta_{jk} \right) \psi_{jk} I \left( \left| \widehat{A}_{js;p} \right| < t_n^p \right) I \left( \left| A_{js;p} \right| \geq 2t_n^p \right) \right\|_{L^p(\mathbb{S}^d)}^p$$

$$+ \sum_{j \leq J_n} \mathbb{E} \left\| \sum_{s=1}^{S_j} \sum_{k \in R_{j;s}} \left( w_{js;p} \widehat{\beta}_{jk} - \beta_{jk} \right) \psi_{jk} I \left( \left| \widehat{A}_{js;p} \right| < t_n^p \right) I \left( \left| A_{js;p} \right| < 2t_n^p \right) \right\|_{L^p(\mathbb{S}^d)}^p$$

$$\leq C \left\{ \sum_{j \leq J_n} \sum_{s=1}^{S_j} \sum_{k \in R_{j;s}} \left\| \psi_{jk} \right\|_{L^p(\mathbb{S}^d)}^p \mathbb{E} \left[ \left( \widehat{\beta}_{jk} - \beta_{jk} \right)^p I\left( \left| \widehat{A}_{js;p} \right| \geq t_n^p \right) I\left( \left| A_{js;p} \right| \geq \frac{t_n^p}{2} \right) \right] \right. \\ \left. + \sum_{j \leq J_n} \sum_{s=1}^{S_j} \sum_{k \in R_{j;s}} \left\| \psi_{jk} \right\|_{L^p(\mathbb{S}^d)}^p \mathbb{E} \left[ \left( \widehat{\beta}_{jk} - \beta_{jk} \right)^p I\left( \left| \widehat{A}_{js;p} \right| \geq t_n^p \right) I\left( \left| A_{js;p} \right| < \frac{t_n^p}{2} \right) \right] \right. \\ \left. + \sum_{j \leq J_n} \sum_{s=1}^{S_j} \sum_{k \in R_{j;s}} \left\| \psi_{jk} \right\|_{L^p(\mathbb{S}^d)}^p \left| \beta_{jk} \right|^p \mathbb{E} \left[ I\left( \left| \widehat{A}_{js;p} \right| < t_n^p \right) I\left( \left| A_{js;p} \right| \geq 2t_n^p \right) \right] \right. \\ \left. + \sum_{j \leq J_n} \sum_{s=1}^{S_j} \sum_{k \in R_{j;s}} \left\| \psi_{jk} \right\|_{L^p(\mathbb{S}^d)}^p \left| \beta_{jk} \right|^p \mathbb{E} \left[ I\left( \left| \widehat{A}_{js;p} \right| < t_n^p \right) I\left( \left| A_{js;p} \right| < 2t_n^p \right) \right] \right. \\ \left. + \left. Aa + Au + Ua + Uu \right. \right] \right\}$$

The procedure follows these guidelines: we have to split (19) into four terms: in one of them, Aa, both the  $\hat{A}_{js;p}$  and  $A_{js;p}$  are supposed to be bigger than the respective threshold; in another one, Uu, they are both smaller and in the last two of them, Au and Ua, the distance between  $\hat{A}_{js;p}$  and  $A_{js;p}$  is shown to be bigger than a suitable threshold. In the first two cases, in order to achieve the minimax rate of convergence, we will split these terms into two parts and we will show the convergence of each part by using mainly (4), (15) and (16). The convergence of the last two terms will be instead proved by applying (17).

Observe that

$$Aa \leq C \sum_{j \leq J_n} \sum_{s=1}^{S_j} \sum_{k \in R_{j;s}} \left\| \psi_{jk} \right\|_{L^p(\mathbb{S}^d)}^p \mathbb{E} \left[ \left| \widehat{\beta}_{jk} - \beta_{jk} \right|^p \right] I \left( |A_{js;p}| \geq \frac{t_n^p}{2} \right)$$
$$\leq C \sum_{j \leq J_n} \sum_{s=1}^{S_j} \sum_{k \in R_{j;s}} B^{dj\left(\frac{p}{2}-1\right)} I \left( |A_{js;p}| \geq \frac{t_n^p}{2} \right) \mathbb{E} \left[ \left| \widehat{\beta}_{jk} - \beta_{jk} \right|^p \right] .$$

As in [4], [15], we fix  $J_{1n}$  such that

$$B^{J_{1n}} = O\left(n^{\frac{1}{2r+d}}\right) \;;$$

simple calculations show that

$$\sum_{j=J_{1n}}^{J_n} \ell_j B^{dj\left(\frac{p}{2}-1\right)} \sum_{s=1}^{S_j} I\left(|A_{js;p}| \ge \frac{t_n^p}{2}\right)$$

$$\leq \sum_{j=J_{1n}}^{J_n} \ell_j B^{dj\left(\frac{p}{2}-1\right)} \sum_{s=1}^{S_j} |A_{js;p}| \left(\frac{t_n^p}{2}\right)^{-1} \\ \leq C t_n^{-p} \sum_{j=J_{1n}}^{J_n} B^{dj\left(\frac{p}{2}-1\right)} \sum_{k=1}^{N_j} |\beta_{jk}|^p \\ \leq C n^{\frac{p}{2}} \sum_{j=J_{1n}}^{J_n} \sum_{k_1=1}^{N_j} |\beta_{jk_1}|^p B^{dj\left(\frac{p}{2}-1\right)} .$$

Because  $f \in \mathcal{B}_{pq}^r$ , we have

$$\sum_{k=1}^{N_j} \left| \beta_{jk_1} \right|^p B^{dj\left(\frac{p}{2}-1\right)} = C \sum_{k=1}^{N_j} \left| \beta_{jk_1} \right|^p \left\| \psi_{jk} \right\|_p^p \le C B^{-prj} ,$$

and, as in [4], [15]

$$n^{\frac{p}{2}} \sum_{j=J_{1n}}^{J_n} \sum_{k_1=1}^{N_j} \left|\beta_{jk_1}\right|^p B^{dj\left(\frac{p}{2}-1\right)} \le C n^{\frac{p}{2r+d}} \le B^{pJ_{1n}} .$$

so that

$$\sum_{j=J_{1n}}^{J_n} \ell_j B^{dj\left(\frac{p}{2}-1\right)} \sum_{s=1}^{S_j} I\left(|A_{js;p}| \ge \frac{t_n^p}{2}\right) \le B^{pJ_{1n}}.$$

Hence, we obtain

$$Aa \leq Cn^{-p/2} \left( \sum_{j \leq J_{1n}} \sum_{s=1}^{S_j} \ell_j B^{dj\left(\frac{p}{2}-1\right)} I\left( |A_{js;p}| \geq \frac{t_n^p}{2} \right) \right. \\ \left. + \sum_{j=J_{1n}}^{J_n} \sum_{s=1}^{S_j} \ell_j B^{dj\left(\frac{p}{2}-1\right)} I\left( |A_{js;p}| \geq \frac{t_n^p}{2} \right) \right) \right. \\ \leq Cn^{-p/2} \left( \sum_{j \leq J_{1n}} B^{j\frac{d}{2}p} + B^{pJ_{1n}} \right) \\ \leq Cn^{-p/2} B^{pJ_{1n}} = Cn^{\frac{-pr}{2r+d}} .$$

Consider now the term Uu. We have that

$$\begin{aligned} Uu &\leq C \sum_{j \leq J_n} \sum_{s=1}^{S_j} \sum_{k \in R_{j;s}} \left\| \psi_{jk} \right\|_{L^p(\mathbb{S}^d)}^p \left| \beta_{jk} \right|^p I\left( |A_{js;p}| < 2t_n^p \right) \\ &\leq C \sum_{j \leq J_n} l_j B^{dj\left(\frac{p}{2} - 1\right)} \sum_{s=1}^{S_j} A_{js;p} I\left( |A_{js;p}| < 2t_n^p \right) \\ &\leq C \left[ \sum_{j \leq J_{1n}} N_j B^{dj\left(\frac{p}{2} - 1\right)} 2t_n^p + \sum_{j=J_{1n}} \sum_{k=1}^{N_j} \left| \beta_{jk} \right|^p \left\| \psi_{jk} \right\|_{L^p(\mathbb{S}^d)}^p \right] \\ &\leq C \left[ n^{-\frac{p}{2}} B^{pJ_{1n}} + B^{-prJ_{1n}} \right] = O\left( n^{-\frac{pr}{2(r+1)}} \right) \,. \end{aligned}$$

Let us study now Au and Ua. As in [4], [15], we have

$$Au \leq \sum_{j \leq J_n} \sum_{s=1}^{S_j} \sum_{k \in R_{j;s}} \left\| \psi_{jk} \right\|_{L^p(\mathbb{S}^d)}^p \left( \mathbb{E} \left[ \left| \widehat{\beta}_{jk} - \beta_{jk} \right|^{2p} \right] \right)^{\frac{1}{2}} \\ \times \left( \mathbb{P} \left( \left| \widehat{A}_{js;p} - A_{js;p} \right| \geq \frac{\kappa n^{-\frac{p}{2}}}{2} \right) \right)^{\frac{1}{2}} \\ \leq CB^{pJ_n} n^{-\frac{p}{2}} n^{-\gamma} \leq Cn^{-\gamma} ; \\ Ua \leq \sum_{j \leq J_n} \sum_{s=1}^{S_j} \sum_{k \in R_{j;s}} \left\| \psi_{jk} \right\|_{L^p(\mathbb{S}^d)}^p \left| \beta_{jk} \right|^p \left( \mathbb{P} \left( \left| \widehat{A}_{js;p} - A_{js;p} \right| \geq \kappa n^{-\frac{p}{2}} \right) \right) \\ \leq Cn^{-\gamma} \left\| F \right\|_p^p .$$

$$(20)$$

Because for  $r \ge 1$ , we have

$$n^{-\gamma} \le n^{-\frac{1}{2}} \le n^{\frac{-r}{2r+d}}$$
,

the result is proved.

Consider now  $p = +\infty$ : we assume now  $f \in \mathcal{B}^r_{\infty,\infty}$ , to obtain

$$\mathbb{E} \left\| \widehat{f} - f \right\|_{\infty} \leq \mathbb{E} \left\| \sum_{j=0}^{J_n} \sum_{k=1}^{N_j} \left( w_{j;p} \widehat{\beta}_{jk} - \beta_{jk} \right) \psi_{jk} \right\|_{L^{\infty}(\mathbb{S}^d)} + \left\| \sum_{j>J_n} \sum_{k=1}^{N_j} \beta_{jk} \psi_{jk} \right\|_{L^{\infty}(\mathbb{S}^d)}$$
$$= : I + II .$$

As in [4], [15], we have:

$$II = O\left(n^{-\frac{r}{2r+d}}\right) \;.$$

For what concerns I, we have instead

$$I \leq \sum_{j=0}^{J_n} \mathbb{E} \left\| \sum_{k=1}^{N_j} \left( w_{j;p} \widehat{\beta}_{jk} - \beta_{jk} \right) \psi_{jk} \right\|_{L^{\infty}(\mathbb{S}^d)} \leq C \sum_{j=0}^{J_n} B^j \mathbb{E} \left[ \sup_k \left( w_{j;p} \widehat{\beta}_{jk} - \beta_{jk} \right) \right]$$

$$\leq C \sum_{j=0}^{J_n} B^j \mathbb{E} \left[ \sup_k \left( \widehat{\beta}_{jk} - \beta_{jk} \right) \right] I \left( |A_{js;p}| \geq \frac{\kappa n^{-\frac{p}{2}}}{2} \right)$$

$$+ C \sum_{j=0}^{J_n} B^j \mathbb{E} \left[ \sup_k \left( \widehat{\beta}_{jk} - \beta_{jk} \right) I \left( \left| \widehat{A}_{js;p} - A_{js;p} \right| \geq \frac{\kappa n^{-\frac{p}{2}}}{2} \right) \right]$$

$$+ C \sum_{j=0}^{J_n} B^j \sup_k |\beta_{jk}| \mathbb{E} \left[ I \left( \left| \widehat{A}_{js;p} - A_{js;p} \right| \geq \kappa n^{-\frac{p}{2}} \right) \right]$$

$$+ C \sum_{j=0}^{J_n} B^j \sup_k |\beta_{jk}| I \left( |A_{js;p}| < 2\kappa n^{-\frac{p}{2}} \right)$$

$$= Aa + Au + Ua + Uu .$$

Again, we choose  $J_{1,n}$  such that

$$B^{J_{1,n}} = \kappa' n^{\frac{1}{2r+d}} ; \ I\left(|A_{js;p}| \ge \frac{\kappa n^{-\frac{p}{2}}}{2}\right) = 0 \text{ for } j > J_{1,n} ,$$

and, similarly to [4], [15], we obtain

$$\begin{aligned} Aa &\leq CJ_{1,n}n^{-\frac{1}{2}}B^{J_{1,n}} \leq Cn^{-\frac{r}{2(r+1)}};\\ Uu &\leq C\left\{B^{-J_{1,n}(r+1)} + B^{-J_{1,n}}\right\} \leq Cn^{-\frac{r}{2(r+1)}}. \end{aligned}$$

The other two terms Au and Ua are similar to the case previously described. For general  $\pi$  and q, we observe that  $\mathcal{B}_{\pi q}^r \subset \mathcal{B}_{\infty \infty}^{r'}$ ,  $r' = r - 2/\pi$ . Hence we obtain

$$\mathbb{E} \left\| \widehat{f} - f \right\|_{L^{\infty}(\mathbb{S}^d)} \le C J_n n^{-\frac{r'}{2r'+d}} = C J_n n^{-\frac{r-d/\pi}{2(r-d(1/\pi-1/2))}} ,$$

as claimed.

**CASE II:** Sparse Case

The proof follows the same procedure of the regular case. Indeed, recalling that we have  $\mathcal{B}_{\pi q}^r \subset \mathcal{B}_{pq}^{r-d(\frac{1}{\pi}-\frac{1}{p})}$ , we have

$$\mathbb{E} \left\| \widehat{f} - f \right\|_{L^p(\mathbb{S}^d)}^p$$

$$\leq 2^{p-1} \mathbb{E} \left\| \sum_{j=0}^{J_n} \sum_{s=1}^{S_j} \sum_{k \in R_{j;s}} \left( w_{js;p} \widehat{\beta}_{jk} - \beta_{jk} \right) \psi_{jk} \right\|_{L^p(\mathbb{S}^d)}^p + \left\| \sum_{j>J_n} \sum_{k=1}^{N_j} \beta_{jk} \psi_{jk} \right\|_{L^p(\mathbb{S}^d)}^p \\ = :I + II \ .$$

Also in this case, as in [4], [15], because  $r-d\left(\frac{1}{\pi}-\frac{1}{p}\right) \ge \left(r-d\left(\frac{1}{\pi}-\frac{1}{p}\right)\right)/2\left(r-d\left(\frac{1}{\pi}-\frac{1}{2}\right)\right)$ , we have for the bias term:

$$II = O\left(n^{-p\left(r - d\left(\frac{1}{\pi} - \frac{1}{p}\right)\right)/2\left(r - d\left(\frac{1}{\pi} - \frac{1}{2}\right)\right)}\right) \;.$$

On the other hand, we split  ${\cal I}$  again into four terms as above. On one hand, we obtain

$$Au \leq \sum_{j \leq J_n} \sum_{s=1}^{S_j} \sum_{k \in R_{j;s}} \left\| \psi_{jk} \right\|_{L^p(\mathbb{S}^d)}^p \left( \mathbb{E} \left[ \left| \widehat{\beta}_{jk} - \beta_{jk} \right|^{2p} \right] \right)^{\frac{1}{2}} \\ \times \left( \mathbb{P} \left( \left| \widehat{A}_{js;p} - A_{js;p} \right| \geq \frac{\kappa t_n^p}{2} \right) \right)^{\frac{1}{2}} \\ Ua \leq \sum_{j \leq J_n} \sum_{s=1}^{S_j} \sum_{k \in R_{j;s}} \left\| \psi_{jk} \right\|_{L^p(\mathbb{S}^d)}^p \left| \beta_{jk} \right|^p \left( \mathbb{P} \left( \left| \widehat{A}_{js;p} - A_{js;p} \right| \geq \kappa t_n^p \right) \right) \right),$$

whose upper bounds recall exactly the same procedure developed in regular zone. On the other hand, consider initially:

$$Aa \le Cn^{-\frac{p}{2}} \sum_{j \le J_n} \sum_{s=1}^{S_j} \sum_{k \in R_{j;s}} B^{jd\left(\frac{p}{2}-1\right)} I\left(|A_{js;p}| \ge \frac{\kappa n^{-\frac{p}{2}}}{2}\right) .$$

In this case, we fix  $J_{2n}$  so that

$$B^{J_{2n}} = O\left(n^{\frac{1}{2\left(r-d\left(\frac{1}{\pi}-\frac{1}{2}\right)\right)}}\right) , I\left(|A_{js;p}| \ge \frac{t_n^p}{2}\right) \equiv 0 \text{ for } j \ge J_{2n} .$$

to obtain

$$Aa \leq Cn^{-\frac{p}{2}} \sum_{j \leq J_{2n}} \sum_{s=1}^{S_j} \sum_{k \in R_{j;s}} B^{jd(\frac{p}{2}-1)} I\left(|A_{js;p}| \geq \frac{t_n^p}{2}\right)$$
(21)  
$$\leq Cn^{-\frac{p}{2}} \sum_{j \leq J_{2n}} B^{jd(\frac{p}{2}-1)} \sum_{s=1}^{S_j} \ell_j I\left(|A_{js;p}| \geq \frac{t_n^p}{2}\right)$$
  
$$\leq Cn^{-\frac{p}{2}} t_n^{-p} \sum_{j \leq J_{2n}} B^{jd(\frac{p}{2}-1)} B^{-prj} B^{-jdp(\frac{1}{2}-\frac{1}{\pi})}$$
  
$$\leq CB^{J_{2n}} (-p(r-d(\frac{1}{\pi}-\frac{1}{p})))$$
  
$$\leq Cn^{-\frac{p(r-d(\frac{1}{\pi}-\frac{1}{p}))}{2(r-d(\frac{1}{\pi}-\frac{1}{2}))}}.$$

where we used the inequality

$$\sum_{k=1}^{N_j} |\beta_{jk}|^p \le \left( \sum_{k=1}^{N_j} |\beta_{jk}|^{\pi} \right)^{\frac{p}{\pi}} .$$

Consider now

$$Uu \leq C \sum_{j \leq J_n} B^{jd\left(\frac{p}{2}-1\right)} \ell_j \sum_{s=1}^{S_j} A_{js;p} I\left(|A_{js;p}| < 2t_n^p\right)$$
(22)  
$$= C \sum_{j \leq J_{2n}} B^{jd\left(\frac{p}{2}-1\right)} \ell_j \sum_{s=1}^{S_j} A_{js;p} I\left(|A_{js;p}| < 2t_n^p\right)$$
$$+ \sum_{j=J_{2n}}^{J_n} B^{jd\left(\frac{p}{2}-1\right)} \ell_j \sum_{s=1}^{S_j} A_{js;p} I\left(|A_{js;p}| < 2t_n^p\right)$$
$$= Uu_1 + Uu_2 .$$

As in [4], [15], fix

$$m = \frac{dp\left(\frac{1}{2} - \frac{1}{p}\right)}{r - d\left(\frac{1}{\pi} - \frac{1}{2}\right)} ,$$

so that

$$p - m = p \frac{r - d\left(\frac{1}{\pi} - \frac{1}{p}\right)}{r - d\left(\frac{1}{\pi} - \frac{1}{2}\right)} > 0 ;$$
  
$$m - \pi = \frac{\frac{d}{2}p - \pi\left(r + \frac{d}{2}\right)}{r - d\left(\frac{1}{\pi} - \frac{1}{2}\right)} > 0 .$$

Furthermore, consider that the following implication holds

$$\left(\left|A_{js}\right| < 2t_n^p\right) \to \forall k \in R_{js}, \ \left|\beta_{jk}\right|^p < 2\ell_j t_n^p \ ,$$

so that

$$\forall k \in R_{js}, \ \left|\beta_{jk}\right|^{p-\pi} < (2\ell_j)^{\frac{p-\pi}{p}} t_n^{p-\pi} .$$

Simple calculations lead to

$$Uu_{1} = C \sum_{j \leq J_{2n}} B^{jd\left(\frac{p}{2}-1\right)} \sum_{k=1}^{N_{j}} \left|\beta_{jk}\right|^{p} I\left(\left|A_{js}\right| < 2t_{n}^{p}\right)$$

$$\leq C \sum_{j \leq J_{2n}} B^{jd\left(\frac{p}{2}-1\right)} t \sum_{k=1}^{N_{j}} \left|\beta_{jk}\right|^{\pi} \ell_{j}^{1-\frac{\pi}{p}} t_{n}^{p-\pi}$$

$$\leq C n^{\frac{\pi-p}{2}} \sum_{j \leq J_{2n}} B^{j\eta d\left(1-\frac{\pi}{p}\right)} B^{j\left(p\frac{d}{2}-\pi\left(\frac{d}{2}+r\right)\right)}$$

$$= O\left(n^{-\frac{p\left(r+d\left(\frac{1}{p}-\frac{1}{\pi}\right)\right)}{2\left(r+d\left(\frac{1}{2}-\frac{1}{\pi}\right)\right)}+\delta}\right)$$

We have to study just the last term

$$Uu_{2} = C \sum_{j=J_{2n}}^{J_{n}} B^{j\eta\left(1-\frac{\pi}{p}\right)} B^{jd\left(\frac{p}{2}-1\right)} \sum_{k=1}^{N_{j}} \left|\beta_{jk}\right|^{p} I\left(|A_{js;p}| < 2t_{n}^{p}\right) .$$

Analogously, we have

$$\begin{aligned} Uu_2 &= C \sum_{j=J_{2n}}^{J_n} B^{jd\left(\frac{p}{2}-1\right)} \sum_{k=1}^{N_j} \left|\beta_{jk}\right|^p I\left(|A_{js;p}| < 2t_n^p\right) \\ &\leq C \sum_{j=J_{2n}}^{J_n} B^{jd\left(\frac{p}{2}-1\right)} \sum_{k=1}^{N_j} \left|\beta_{jk}\right|^m t^{p-m} \ell_j^{1-\frac{m}{p}} \\ &\leq C t_n^{p-m} \ell_{J_{2n}}^{1-\frac{m}{p}} \sum_{j=J_{2n}}^{J_n} B^{jd\left(\frac{p}{2}-\frac{m}{2}\right)} \left(\sum_{k=1}^{N_j} \left|\beta_{jk}\right|^m B^{jd\left(\frac{m}{2}-1\right)}\right) \\ &\leq C t_n^{p-m} \ell_{J_{2n}}^{1-\frac{m}{p}} \sum_{j=J_{2n}}^{J_n} B^{jd\left(\frac{p}{2}-1\right)} B^{-mj\left(r-d\left(\frac{1}{\pi}-\frac{1}{m}\right)\right)} \end{aligned}$$

We can easily see that

$$(p-m) - m\left(r - d\left(\frac{1}{\pi} + \frac{1}{m}\right)\right) = 0$$
.

Hence

$$Uu_{2} \leq C\ell_{J_{2n}}^{1-\frac{m}{p}} t_{n}^{p-n} = O\left(n^{-\frac{p\left(r+d\left(\frac{1}{p}-\frac{1}{\pi}\right)\right)}{2\left(r-d\left(\frac{1}{\pi}-\frac{1}{2}\right)\right)}+\delta}\right) ,$$

as claimed.

# 7 Conclusions

In this final Section we shall compare our results with those obtained by similar procedures, involving needlets, in [4] and [15].

While in [4] the authors established minimax results on density estimation by using local needlet thresholding (i.e., fixing a threshold for each coefficients), in [15] the authors attain the same minimax results for the nonparametric regression problem on sections of spin s fiber bundles defined on the sphere, which can be reduced to the scalar case taking s = 0 (for more details see [15]). In both cases, the convergence rates for the  $L^p$  (S<sup>d</sup>)-loss functions assume the form

$$\sup_{f \in \mathcal{B}^{r}_{\pi q}(M)} \mathbb{E} \left\| \widehat{f} - f \right\|_{L^{p}(\mathbb{S}^{d})}^{p} \leq c_{p} \left( \log n \right)^{p} \left( \frac{n}{\log n} \right)^{-\alpha_{1}(r,\pi,p)} ,$$

where

$$\alpha_1(r, \pi, p) = \begin{cases} \frac{rp}{2r+d} & \text{for } \pi \ge \frac{dp}{2r+d} \\ \frac{p(r-d(\frac{1}{\pi} - \frac{1}{p}))}{2(r-d(\frac{1}{\pi} - \frac{1}{2}))} & \text{for } \pi < \frac{dp}{2r+d} \end{cases}$$

In the regular zone, the block thresholding rate we established is faster, indeed the ratio with the local one is provided by

$$\frac{\left(\frac{n}{\log n}\right)^{-\alpha_1(r,\pi,p)}}{n^{-\alpha(r,\pi,p)}} = O\left((\log n)^{\frac{rp}{2r+d}}\right) \;;$$

on the other hand, in the sparse zone, we obtain worse results, because

$$\frac{\left(\frac{n}{\log n}\right)^{-\alpha_1(r,\pi,p)}}{n^{-\alpha(r,\pi,p)}} = O\left(n^{-\delta} \left(\log n\right)^{\frac{p\left(r-d\left(\frac{1}{\pi}-\frac{1}{p}\right)\right)}{2\left(r-d\left(\frac{1}{\pi}-\frac{1}{2}\right)\right)}}\right) \ .$$

This can be motivated by choice of the sample scaling factor  $t_n$ , fixed to allow optimality in the regular zone. In the sparse zone this is not possible also in view of the result in Lemma 2, where (17) is proportional to  $n^{-\gamma}$  and can not be improved. We indeed recall that in [4] and in [15] the corresponding inequality, related just on a coefficient instead of a sum of them, follows Bernstein inequality and, therefore, that probability decays as a negative exponential. As already mentioned in the Introduction, the best performance achieved by block thresholding in the regular zone can be explained by the better trade-off between bias and variance. The latter is due to the information provided by nearby coefficients, which allows the balance between variance and bias to be "adaptively smoothed" along the curve, given a suitable choice of the threshold  $t_n$ . On the other hand, the worse results obtained in the sparse regions are due to the balance between the choice of the size of threshold  $t_n$  and the size of the block. Indeed, given  $t_n$ , the probability inequality (17) is suitable to attain minimax rate in the regular zone if we choose blocks as described in (12). Fixing a smaller size, as for instance  $(\log N_i)^{\gamma}$  (see [27]), the convergence rate in the regular zone is worsened. Our suggestion is to fix the block sizes which ensure the minimax results in the regular zone; as explained in Remark 3, this warrants optimality in the most relevant case for practitioners, e.g., the case of a quadratic loss function.

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