CONCENTRATION AND HOMOGENIZATION IN ELECTRICAL CONDUCTION IN HETEROGENEOUS MEDIA INVOLVING THE LAPLACE-BELTRAMI OPERATOR

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Abstract. We study a concentration and homogenization problem modelling electrical conduction in a composite material. The novelty of the problem is due to the specific scaling of the physical quantities characterizing the dielectric component of the composite. This leads to the appearance of a peculiar displacement current governed by a Laplace-Beltrami pseudo-parabolic equation. This pseudo-parabolic character is present also in the homogenized equation, which is obtained by the unfolding technique.

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1. Introduction

The continuous development of new nano-engineered materials, as well as the search for new diagnostic devices in medicine and biology, has given big momentum to new studies in the field of composite media, in particular with regard to their physical behavior especially from the electric conduction and heat conduction point of view. Typically, these materials are composed by a hosting medium in which nano-particles inclusions (having different physical properties) are present, commonly arranged in a periodic lattice.

The thermal behavior of these composite materials plays a fundamental role, e.g., in the design of heat dissipating fillers (e.g. for electronic devices) and in the creation of new generation motor coolants (see [24, 26, 27, 28]). Some of the authors have conducted researches in this direction ([16, 17, 18]) in which surface heat conduction on the interface Γ separating the inclusions from the host material plays a fundamental role. The case of electrical conduction was considered in [15]. In the models thus obtained, a non-standard surface conduction appears, governed by a Laplace-Beltrami operator.

In this article, we deal with the electric behavior of nano-composites, thus pursuing a research started by some of the authors years ago ([4]–[14]).

In that framework, the composite was used to model a biological tissue in which the hosting medium is the extracellular material, while the inclusions are the cells separated from the surrounding medium by the lipidic cell membrane which behaves like a dielectric and, therefore, exhibits a capacitive behavior.

In those papers both the case of thick N-dimensional membranes and the one in which the membranes are regarded as electrically active, $(N-1)$-dimensional surfaces are considered. Namely, the first case is reduced to the second one via a concentration technique. Obviously, this requires specific scalings of the relevant physical quantities. The scalings used in that case were consistent with the peculiar physical situation under examination, namely we assumed that the relative dielectric constant of the thick membrane scaled as η (where η is the relative width of the membrane with respect to the diameter of the cell). This choice reduces the limit problem to a system of partial differential equations in which the current is continuous across the $(N-1)$-dimensional membranes and the time derivative of the jump of the potential across the membranes is proportional to the current. Such a model reproduces the standard equations satisfied, in electric conduction, by capacitors and therefore appears to be the natural model to describe the capacitive behavior of cell membranes. Finally, the $(N-1)$-dimensional model was homogenized taking into account the large number of cells present in the medium, i.e. a limit for ε going to zero of the solution was performed (ε being the typical length scale of a cell). The homogenized limit thus obtained solves an elliptic equation with memory (the memory being a direct consequence of the presence of capacitors).

In this paper, we investigate the same electrical conduction model, but with a relative dielectric constant scaling as $1/\eta$ in the thick interface. This is consistent therefore with a strongly insulating coating as made possible by new materials (for instance, the barium titanate) in the construction of electronic devices.

Hence, in this paper, a concentration limit is performed using the aforementioned scaling. Such a concentration limit relies, as usual, on suitable choices of test functions in the weak formulation to select the quantities appearing in the limit equations satisfied on the membranes.

Going into details, after concentration, we obtain a system of partial differential equations made of two elliptic equations satisfied in the hosting medium and in the inclusions with the potential being continuous across the interfaces, while the jump of the currents plays the role of the source term for a Laplace-Beltrami equation satisfied by the time derivative of the potential on the interfaces.

This problem is highly non-standard and, as far as we know, mathematically new. Existence and uniqueness for the previous problem are mathematically interesting and were proved by the authors in [15].
Here, the concentration limit equations are homogenized letting ε go to zero (assuming a periodic arrangement of the inclusions), via an unfolding method. It is worthwhile noticing that the unfolding limits on Γ are quite technical and of some mathematical interest (see also [17]).
Owing to the linearity of the problem, the first corrector can be factorized even if the structure of the factorization is neither standard, nor simple (as was already the case in the problem studied in [7, 8, 12]). For this reason, we are able to produce a limit equation for the macroscopic variable u, which contains a memory term, as in the previously quoted papers. The main and more relevant difference with respect to the previous case, is that now the partial differential equation with memory is not always an elliptic equation, but it substantially depends on the underlying geometry. In particular, when both the hosting medium and the inclusion are connected, we obtain a pseudo-parabolic equation. The class of such homogenized problem will be studied by the authors in a forthcoming paper. It is worthwhile noticing that in the present case of the dielectric constant scaling as $1/\eta$ we expect that the two steps of concentration and homogenization commute, that is they lead to the same problem, if applied in different order. This is not expected for the problem with dielectric constant scaling as η. An additional peculiarity of the present problem is the appearance of compatibility conditions needed to solve the cell equations.
The results quoted above concern the case in which the permeability in the concentrated problem scales as ε; however, in order to present a complete study of the problem, we consider also all the other possible scalings ε^k, with $k \in \mathbb{R}$. In these cases, the resulting homogenized equation is a standard elliptic partial differential equation, not containing any memory term (see Theorems 2.17 and 2.18). Note that the concentration procedure is independent of k. Moreover, for certain scalings and a particular geometry (i.e. if both the hosting medium and the inclusion are connected), the sequence of the solutions u_ε of the approximating problems tends to 0, independently of the presence of a non-zero source and non-zero initial datum (see Theorem 2.16).

The paper is organized as follows. In Subsection 2.1 we briefly recall the definition and the main properties of the tangential operators (gradient, divergence, Laplace-Beltrami operator), in Subsection 2.2 we state our geometrical setting, in Subsection 2.3 we recall the main properties of the unfolding operator and, finally, in Subsection 2.4 we state the problem and our main results. Section 3 is devoted to the derivation of the concentrated problem. In Section 4, we prove the homogenization result for the microscopic problem (2.32)–(2.36) (i.e., the scaling $k = 1$). Finally, in Section 5, we consider the other scalings ε^k, for $k < 1$ and $k > 1$.

2. Preliminaries

2.1. **Laplace-Beltrami derivatives.** Let ϕ be a $\mathcal{C}^2$-function, $\mathbf{\Phi}$ be a $\mathcal{C}^2$-vector function and S a smooth surface with normal unit vector n. We recall that the tangential gradient of ϕ is given by

$$\nabla^B \phi = \nabla\phi - (n\cdot\nabla\phi)n \tag{2.1}$$

and the tangential divergence of Φ is given by

$$\begin{aligned}\operatorname{div}^B \mathbf{\Phi} &= \operatorname{div}\mathbf{\Phi} - (n\cdot\nabla\mathbf{\Phi}_i)n_i - (\operatorname{div} n)(n\cdot\mathbf{\Phi})\\ &= \operatorname{div}^B(\mathbf{\Phi} - (n\cdot\mathbf{\Phi})n) = \operatorname{div}(\mathbf{\Phi} - (n\cdot\mathbf{\Phi})n)\,,\end{aligned} \tag{2.2}$$

where, taking into account the smoothness of S, the normal vector n can be naturally defined in a small neighborhood of S as a regular field. Moreover, we define the Laplace-Beltrami operator as

$$\Delta^B\phi = \operatorname{div}^B(\nabla^B\phi)\,, \tag{2.3}$$

so that, by (2.1) and (2.2), we get that the Laplace-Beltrami operator can be written as

$$\begin{aligned}\Delta^B\phi &= \Delta\phi - n^t\nabla^2\phi n - (n\cdot\nabla\phi)\operatorname{div} n\\ &= (\delta_{ij} - n_i n_j)\partial^2_{ij}\phi - n_j\partial_j\phi\partial_i n_i = (I - n\otimes n)_{ij}\partial^2_{ij}\phi - (n\cdot\nabla\phi)\operatorname{div} n\,.\end{aligned} \tag{2.4}$$

Finally, we recall that on a regular surface S with no boundary (i.e. when $\partial S = \emptyset$) we have

$$\int_S \operatorname{div}^B \mathbf{\Phi}\,\mathrm{d}\sigma = 0\,. \tag{2.5}$$

2.2. **Geometrical setting.** The typical periodic geometrical setting is displayed in Figure 1 and Figure 2. Here we give, for the sake of clarity, its detailed formal definition.

Let $N\geq 3$. Introduce a periodic open subset E of $\mathbb{R}^N$, so that $E+z=E$ for all $z\in\mathbb{Z}^N$. We employ the notation $Y=(0,1)^N$ and $E_{\text{int}} = E\cap Y$, $E_{\text{out}} = Y\setminus\overline{E}$, $\Gamma = \partial E\cap\overline{Y}$. We assume that E_{out} is connected, while E_{int} may be connected or not. As a simplifying assumption, we stipulate that $|\Gamma\cap\partial Y|_{N-1}=0$.

Let Ω be an open connected bounded subset of $\mathbb{R}^N$; we assume that Ω and E are of class $\mathcal{C}^\infty$, though this assumption can be weakened. For all $\varepsilon>0$, define $\Omega^\varepsilon_{\text{int}} = \Omega\cap\varepsilon E$, $\Omega^\varepsilon_{\text{out}} = \Omega\setminus\overline{\varepsilon E}$, so that $\Omega = \Omega^\varepsilon_{\text{int}}\cup\Omega^\varepsilon_{\text{out}}\cup\Gamma^\varepsilon$, where $\Omega^\varepsilon_{\text{int}}$ and $\Omega^\varepsilon_{\text{out}}$ are two disjoint open subsets of Ω, and $\Gamma^\varepsilon = \partial\Omega^\varepsilon_{\text{int}}\cap\Omega = \partial\Omega^\varepsilon_{\text{out}}\cap\Omega$. The region $\Omega^\varepsilon_{\text{out}}$ [respectively, $\Omega^\varepsilon_{\text{int}}$] corresponds to the outer phase [respectively, the inclusions], while Γ^ε is the interface; in fact, these definitions are slightly modified below. We will consider two different cases: in the first one (to which we will refer as the *connected/disconnected case*, see Fig.1) we will assume $\Gamma\cap\partial Y=\emptyset$; moreover, we stipulate that all the cells which intersect $\partial\Omega$ do not contain any inclusion.

In the second case (to which we will refer as the *connected/connected case*, see Fig.2) we will assume that E_{int}, E_{out}, $\Omega^\varepsilon_{\text{int}}$ and $\Omega^\varepsilon_{\text{out}}$ are connected and, without loss of generality, that they have Lipschitz continuous boundary. In this last case, we stipulate that there exist $\gamma_1>\gamma_0>0$ such that the picture just described is actually valid in

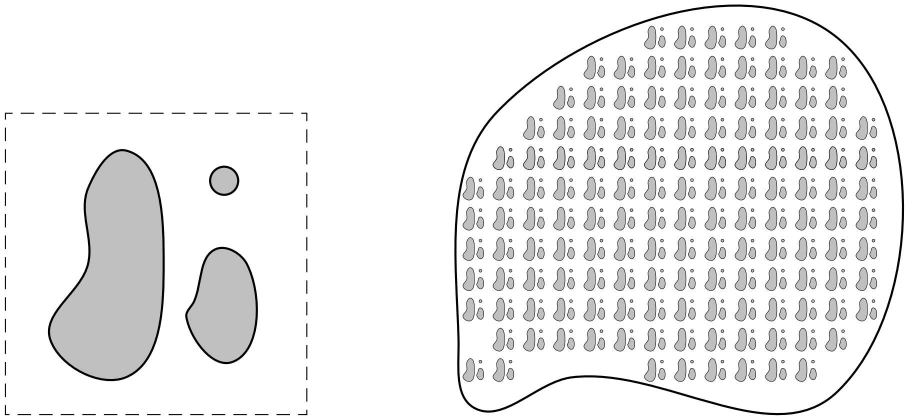

FIGURE 1. *Left*: the periodic cell Y. E_{int} is the shaded region and E_{out} is the white region. *Right*: the region Ω.

the set $\{x \in \Omega \ : \ \mathrm{dist}(x, \partial\Omega) \geq \gamma_1\varepsilon\}$, while in the layer $\gamma_0\varepsilon \leq \mathrm{dist}(x, \partial\Omega) \leq \gamma_1\varepsilon$, the geometry of Γ^ε is modified in a standard way to keep it regular and preserve the topological properties of $\Omega^\varepsilon_{\mathrm{int}}$ and $\Omega^\varepsilon_{\mathrm{out}}$, as well as to obtain that $\mathrm{dist}(\Gamma^\varepsilon, \partial\Omega) \geq \gamma_0\varepsilon$.

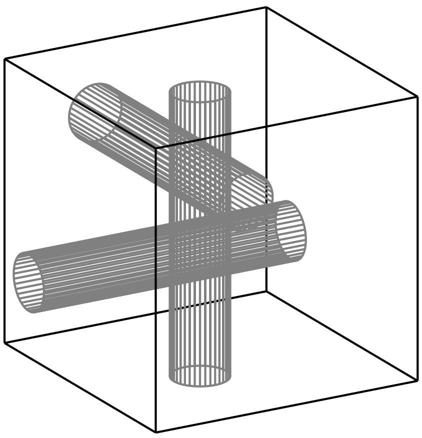

FIGURE 2. The periodic cell Y. E_{int} is the shaded region and E_{out} is the white region.

Finally, let ν denote the normal unit vector to Γ pointing into E_{out}, extended by periodicity to the whole of $\mathbb{R}^N$, so that $\nu_\varepsilon(x) = \nu(x/\varepsilon)$ denotes the normal unit vector to Γ^ε pointing into $\Omega^\varepsilon_{\mathrm{out}}$.

Actually, in a more realistic framework, the isolating interface is not an $(N-1)$-dimensional surface, but it has a very small positive thickness. Hence, we consider also a more physical geometric setting where a small parameter $\eta > 0$ represents the small ratio between the thickness of the physical interface and the characteristic dimension of the microstructure. To this purpose, for $\eta > 0$, let us write Ω also as $\Omega = \Omega^{\varepsilon,\eta} \cup \Gamma^{\varepsilon,\eta} \cup \partial\Gamma^{\varepsilon,\eta}$, where $\Omega^{\varepsilon,\eta}$ and $\Gamma^{\varepsilon,\eta}$ are two disjoint open subsets of Ω,

$\Gamma^{\varepsilon,\eta}$ is the tubular neighborhood of Γ^ε with thickness $\varepsilon\eta$, and $\partial\Gamma^{\varepsilon,\eta}$ is the boundary of $\Gamma^{\varepsilon,\eta}$. Moreover, we also assume that $\Omega^{\varepsilon,\eta} = \Omega^{\varepsilon,\eta}_{\text{int}} \cup \Omega^{\varepsilon,\eta}_{\text{out}}$, where $\Omega^{\varepsilon,\eta}_{\text{out}} \subset \Omega^{\varepsilon}_{\text{out}}$, $\Omega^{\varepsilon,\eta}_{\text{int}} \subset \Omega^{\varepsilon}_{\text{int}}$ and $\partial\Gamma^{\varepsilon,\eta} = (\partial\Omega^{\varepsilon,\eta}_{\text{int}} \cup \partial\Omega^{\varepsilon,\eta}_{\text{out}}) \cap \Omega$. We notice that, for $\eta \to 0$ and $\varepsilon > 0$ fixed, $|\Gamma^{\varepsilon,\eta}| \sim \varepsilon\eta|\Gamma^\varepsilon|_{N-1}$. Set also $Y = E^\eta_{\text{int}} \cup E^\eta_{\text{out}} \cup \Gamma^\eta \cup \partial\Gamma^\eta$ where E^η_{int}, E^η_{out} and Γ^η are disjoint open subsets of Y, Γ^η is the tubular neighborhood of Γ with thickness η, and $\partial\Gamma^\eta = (\partial E^\eta_{\text{int}} \cup \partial E^\eta_{\text{out}}) \cap Y$ (see Figure 3). For the sake of brevity, we will denote by E^η the union $E^\eta_{\text{int}} \cup E^\eta_{\text{out}}$. Finally, for $\eta \to 0$, $|\Gamma^\eta| \sim \eta|\Gamma|_{N-1}$.

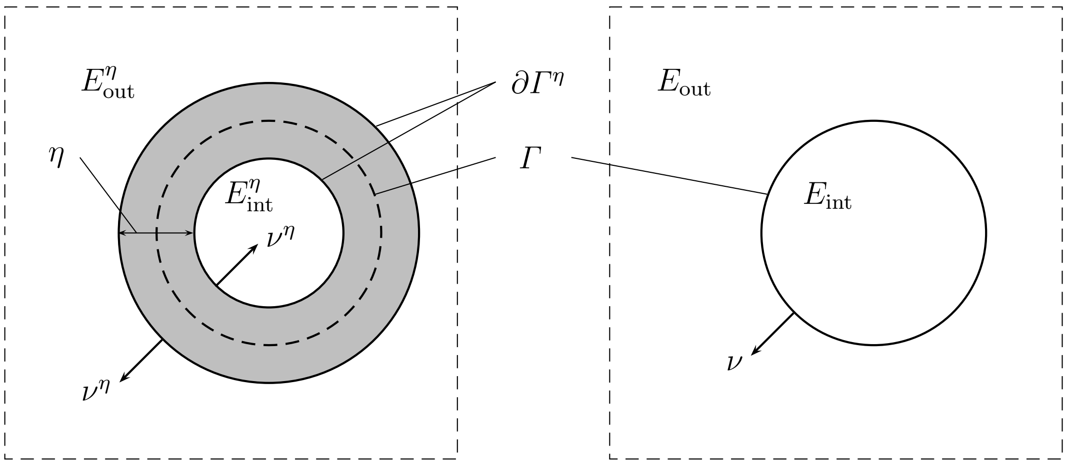

FIGURE 3. The periodic cell Y. *Left*: before concentration; Γ^η is the shaded region, and $E^\eta = E^\eta_{\text{int}} \cup E^\eta_{\text{out}}$ is the white region. *Right*: after concentration; Γ^η shrinks to Γ as $\eta \to 0$.

We stress the fact that the appearance of the two small parameters η and ε calls for two different limit procedures: we will first perform a concentration of the thin membranes, in order to simplify the geometrical setting of the microstructure, and then we will perform a homogenization limit, in order to obtain a macroscopic model.

2.3. **Definition and main properties of the unfolding operator.** In this subsection, we define and collect some properties of a space-time version of the space unfolding operator introduced and developed in [19, 20, 22, 23] (see also [17]).
A space-time version of the unfolding operator in a more general framework, in which also a time-microscale is actually present, has been introduced in [2] and [3], to which we also refer for a survey on this topic.
However, in the unfolding technique used here, the time variable does not play any special role and can be treated essentially as a parameter, hence most of the properties of this operator can be proven essentially as in the above quoted papers and are therefore only recalled here. An analogous remark is valid for the other operators which will be introduced in the following.
For the sake of simplicity, for any spatial domain G, we will denote by $G_T = G \times (0, T)$ the corresponding space–time cylindrical domain over the time interval $(0, T)$.

Let us set

$$\Xi_\varepsilon = \big\{\xi \in \mathbb{Z}^N\,, \quad \varepsilon(\xi + Y) \subset \Omega\big\}\,, \quad \widehat{\Omega}_\varepsilon = \text{interior}\left\{\bigcup_{\xi\in\Xi_\varepsilon} \varepsilon(\xi + \overline{Y})\right\}\,,$$

$$\Omega_T = \Omega \times (0,T)\,, \quad \Lambda^\varepsilon_T = \widehat{\Omega}_\varepsilon \times (0,T)\,.$$

Denoting by $[r]$ the integer part of $r \in \mathbb{R}$, we define for $x \in \mathbb{R}^N$

$$\left[\frac{x}{\varepsilon}\right]_Y = \left(\left[\frac{x_1}{\varepsilon}\right], \dots, \left[\frac{x_N}{\varepsilon}\right]\right)\,, \quad \text{so that} \qquad x = \varepsilon\left(\left[\frac{x}{\varepsilon}\right]_Y + \left\{\frac{x}{\varepsilon}\right\}_Y\right)\,.$$

Then, we introduce the space cell containing x as $Y_\varepsilon(x) = \varepsilon\Big(\left[\frac{x}{\varepsilon}\right]_Y + Y\Big)$. Finally, set $y_M = y - \fint_Y y\,\mathrm{d}y$.

Definition 2.1. For a Lebesgue-measurable w on Ω_T, the (time-depending) periodic unfolding operator $\mathcal{T}_\varepsilon$ is defined as

$$\mathcal{T}_\varepsilon(w)(x,t,y) = \begin{cases} w\left(\varepsilon\left[\frac{x}{\varepsilon}\right]_Y + \varepsilon y, t\right)\,, & (x,t,y) \in \Lambda^\varepsilon_T \times Y; \\ 0\,, & \text{otherwise.} \end{cases}$$

For a Lebesgue-measurable w on Γ^ε_T, the (time-depending) boundary unfolding operator $\mathcal{T}^b_\varepsilon$ is defined as

$$\mathcal{T}^b_\varepsilon(w)(x,t,y) = \begin{cases} w\left(\varepsilon\left[\frac{x}{\varepsilon}\right]_Y + \varepsilon y, t\right)\,, & (x,t,y) \in \Lambda^\varepsilon_T \times \Gamma; \\ 0\,, & \text{otherwise.} \end{cases}$$

□

Clearly, for w_1, w_2 as in Definition 2.1

$$\mathcal{T}_\varepsilon(w_1 w_2) = \mathcal{T}_\varepsilon(w_1)\mathcal{T}_\varepsilon(w_2)\,, \tag{2.6}$$

and the same property holds for the boundary unfolding operator. Note that $\mathcal{T}^b_\varepsilon(w)$ is the trace of the unfolding operator on $\Lambda^\varepsilon_T \times \Gamma$, when both operators are defined.

Definition 2.2. For a Lebesgue-measurable w on Ω_T, the (time-depending) local average operator is defined as

$$\mathcal{M}^\varepsilon(w)(x,t) = \begin{cases} \dfrac{1}{\varepsilon^N} \displaystyle\int_{Y_\varepsilon(x)} w(\zeta,t)\,\mathrm{d}\zeta\,, & \text{if } (x,t) \in \Lambda^\varepsilon_T\,, \\ 0\,, & \text{otherwise.} \end{cases} \tag{2.7}$$

□

By a change of variable, it is not difficult to see that

$$\mathcal{M}^\varepsilon(w)(x,t) = \mathcal{M}_Y\big(\mathcal{T}_\varepsilon(w)\big)\,,$$

where we denote by $\mathcal{M}_Y(\cdot)$ the integral average on Y.
We collect here some properties of the operators defined above.

Proposition 2.3. *The operator* $\mathcal{T}_\varepsilon : L^2(\Omega_T) \to L^2(\Omega_T \times Y)$ *is linear and continuous. In addition, we have*

$$\|\mathcal{T}_\varepsilon(w)\|_{L^2(\Omega_T\times Y)} \le \|w\|_{L^2(\Omega_T)}\,, \tag{2.8}$$

and

$$\left| \int\limits_{\Omega_T} w \,\mathrm{d}x\,\mathrm{d}t - \iint\limits_{\Omega_T\times Y} \mathcal{T}_\varepsilon(w)\,\mathrm{d}y\,\mathrm{d}x\,\mathrm{d}t \right| \le \int\limits_{\Omega_T\setminus\Lambda_T^\varepsilon} |w|\,\mathrm{d}x\,\mathrm{d}t\,. \tag{2.9}$$

Proposition 2.4. *Let* $\{w_\varepsilon\}$ *be a sequence of functions in* $L^2(\Omega_T)$.
If $w_\varepsilon \to w$ *strongly in* $L^2(\Omega_T)$ *as* $\varepsilon \to 0$, *then*

$$\mathcal{T}_\varepsilon(w_\varepsilon) \to w\,, \quad \textit{strongly in}\, L^2(\Omega_T \times Y)\,. \tag{2.10}$$

If $w_\varepsilon \to w$ *strongly in* $L^2(0,T;H^1(\Omega))$ *as* $\varepsilon \to 0$, *then*

$$\frac{\mathcal{T}_\varepsilon(w_\varepsilon) - \mathcal{M}^\varepsilon(w_\varepsilon)}{\varepsilon} \to y_M \cdot \nabla w\,, \quad \textit{strongly in}\, L^2(\Omega_T)\,. \tag{2.11}$$

If w_ε *is a bounded sequence of functions in* $L^2(\Omega_T)$, *then, up to a subsequence*

$$\mathcal{T}_\varepsilon(w_\varepsilon) \rightharpoonup \widehat{w}\,, \quad \textit{weakly in}\, L^2\left(\Omega_T \times Y\right)\,, \tag{2.12}$$

and

$$w_\varepsilon \rightharpoonup \mathcal{M}_Y(\widehat{w})\,, \quad \textit{weakly in}\, L^2(\Omega_T)\,. \tag{2.13}$$

Remark 2.5. In particular, if $w \in L^2(\Omega_T)$, we get that $\mathcal{T}_\varepsilon(w) \to w$, for $\varepsilon \to 0$, strongly in $L^2(\Omega_T \times Y)$. □

Remark 2.6. We note that the only cases in which (2.10) it is known to hold without assuming the strong convergence of the sequence $\{w_\varepsilon\}$ is when $w_\varepsilon(x,t) = \phi(x,t,\varepsilon^{-1}x)$ where ϕ corresponds to one of the following cases (or sum of them): $\phi(x,t,y) = f_1(x,t)f_2(y)$, with $f_1 f_2 \in L^1(\Omega_T \times Y)$, $\phi \in L^1(Y;\mathcal{C}(\Omega_T))$, $\phi \in L^1(\Omega_T;\mathcal{C}(Y))$. In all such cases we have $\mathcal{T}_\varepsilon(w_\varepsilon) \to \phi$ strongly in $L^2(\Omega_T \times Y)$ (see, for instance, [1, 19, 20] and [3, Remark 2.9]). □

Proposition 2.7. *The operator* $\mathcal{T}^b_\varepsilon : L^2(\Gamma^\varepsilon_T) \to L^2(\Omega_T \times \Gamma)$ *is linear and continuous. In addition, we have*

$$\|\mathcal{T}^b_\varepsilon(w)\|_{L^2(\Omega_T\times \Gamma)} \le \sqrt{\varepsilon}\|w\|_{L^2(\Gamma^\varepsilon_T)}\,, \tag{2.14}$$

and

$$\int\limits_{\Gamma^\varepsilon_T} w\,\mathrm{d}\sigma\,\mathrm{d}t = \frac{1}{\varepsilon}\int\limits_{\Omega_T\times\Gamma} \mathcal{T}^b_\varepsilon(w)\,\mathrm{d}\sigma\,\mathrm{d}x\,\mathrm{d}t\,. \tag{2.15}$$

Note that (2.15) holds since we can choose γ_0 in Subsection 2.2 in such a way that $\Gamma^\varepsilon_T \setminus \Lambda^\varepsilon_T = \emptyset$.

Proposition 2.8. [17, Proposition 5] *Assume that* $w_\varepsilon \rightharpoonup w$ *weakly in* $L^2(0,T;H^1_0(\Omega))$. *Then,*

$$\mathcal{T}^b_\varepsilon(w_\varepsilon) \rightharpoonup w\,, \qquad \textit{weakly in } L^2(\Omega_T \times \Gamma).$$

Finally, we state some results which will be mainly used when we deal with testing functions.

Proposition 2.9. [17, Corollary 1] *Let w be a function belonging to $L^2\big(0,T;H^1(\Omega)\big)$. Then, as $\varepsilon \to 0$,*

$$\mathcal{T}_\varepsilon^b(w) \to w\,, \quad \textit{strongly in}\, L^2\left(\Omega_T \times \Gamma\right)\,. \tag{2.16}$$

Proposition 2.10. *Let $\phi : Y \to \mathbb{R}$ be a function extended by Y-periodicity to the whole of $\mathbb{R}^N$ and define the sequence*

$$\phi^\varepsilon(x) = \phi\left(\frac{x}{\varepsilon}\right)\,, \qquad x \in \mathbb{R}^N\,. \tag{2.17}$$

If ϕ is measurable on Y, then

$$\mathcal{T}_\varepsilon(\phi^\varepsilon)(x,y) = \begin{cases} \phi(y)\,, & (x,y) \in \widehat{\Omega}_\varepsilon \times Y\,, \\ 0\,, & \textit{otherwise.} \end{cases} \tag{2.18}$$

Analogously, if ϕ is measurable on Γ, then

$$\mathcal{T}_\varepsilon^b(\phi^\varepsilon)(x,y) = \begin{cases} \phi(y)\,, & (x,y) \in \widehat{\Omega}_\varepsilon \times \Gamma\,, \\ 0\,, & \textit{otherwise.} \end{cases} \tag{2.19}$$

Moreover, if $\phi \in L^2(Y)$, as $\varepsilon \to 0$,

$$\mathcal{T}_\varepsilon(\phi^\varepsilon) \to \phi\,, \qquad \textit{strongly in}\, L^2(\Omega \times Y)\,; \tag{2.20}$$

if $\phi \in L^2(\Gamma)$, as $\varepsilon \to 0$,

$$\mathcal{T}_\varepsilon^b(\phi^\varepsilon) \to \phi\,, \qquad \textit{strongly in}\, L^2(\Omega \times \Gamma)\,; \tag{2.21}$$

if $\phi \in H^1(Y)$, as $\varepsilon \to 0$,

$$\nabla_y(\mathcal{T}_\varepsilon(\phi^\varepsilon)) \to \nabla_y \phi\,, \quad \textit{strongly in}\, L^2(\Omega \times Y)\,. \tag{2.22}$$

Now, let us state some properties concerning the behavior of the unfolding operator with respect to gradients. To this aim, we denote by $H^1_\#(Y)$ the space of those Y-periodic functions belonging to $H^1_{loc}(\mathbb{R}^N)$.

Theorem 2.11. *Let $\{w_\varepsilon\}$ be a sequence converging weakly to w in $L^2\big(0,T;H^1_0(\Omega)\big)$. Then, up to a subsequence, there exists $\widetilde{w} = \widetilde{w}(x,y,t) \in L^2\big(\Omega_T; H^1_\#(Y)\big)$, $\mathcal{M}_Y(\widetilde{w}) = 0$, such that, as $\varepsilon \to 0$,*

$$\mathcal{T}_\varepsilon(w_\varepsilon) \rightharpoonup w\,, \qquad \textit{weakly in}\, L^2(\Omega_T \times Y)\,, \tag{2.23}$$

$$\mathcal{T}_\varepsilon(\nabla w_\varepsilon) \rightharpoonup \nabla_x w + \nabla_y \widetilde{w}\,, \quad \textit{weakly in}\, L^2(\Omega_T \times Y)\,. \tag{2.24}$$

Proof. Whereas the convergence in (2.24) is a well-known property (see for instance [17, 25]), in order to prove (2.23), we proceed as follows. Since $w_\varepsilon \rightharpoonup w$ weakly in $L^2\big(0,T;H^1_0(\Omega)\big)$, we have that, for every test function $\theta \in L^2(0,T)$,

$$\int_0^T w_\varepsilon \theta(t)\,\mathrm{d}t \to \int_0^T w\theta(t)\,\mathrm{d}t\,, \qquad \text{strongly in } L^2(\Omega). \tag{2.25}$$

Therefore, by [20, Theorem 3.5],

$$\mathcal{T}_\varepsilon\left(\int_0^T w_\varepsilon\theta(t)\,\mathrm{d}t\right) \to \int_0^T w\theta(t)\,\mathrm{d}t\,, \qquad \text{strongly in } L^2(\Omega\times Y). \tag{2.26}$$

Moreover, by (2.12), there exists $\widehat{w} \in L^2(\Omega_T \times Y)$, such that $\mathcal{T}_\varepsilon(w) \rightharpoonup \widehat{w}$ weakly in $L^2(\Omega_T \times Y)$ and $\mathcal{M}_Y(\widehat{w}) = w$. Hence, for every $\phi \in L^2(\Omega)$ and every $\psi \in L^2_\#(Y)$,

$$\int_{\Omega_T\times Y} \widehat{w}\phi(x)\psi(y)\theta(t)\,\mathrm{d}x\,\mathrm{d}y\,\mathrm{d}t \leftarrow \int_{\Omega_T\times Y} \mathcal{T}_\varepsilon(w_\varepsilon)\phi(x)\psi(y)\theta(t)\,\mathrm{d}x\,\mathrm{d}y\,\mathrm{d}t$$

$$= \int_{\Omega\times Y} \mathcal{T}_\varepsilon\left(\int_0^T w_\varepsilon\theta(t)\,\mathrm{d}t\right)\phi(x)\psi(y)\,\mathrm{d}x\,\mathrm{d}y \to \int_{\Omega\times Y}\left(\int_0^T w\theta(t)\,\mathrm{d}t\right)\phi(x)\psi(y)\,\mathrm{d}x\,\mathrm{d}y\,,$$

which implies that $\widehat{w} = w$ a.e. in $\Omega_T \times Y$. Therefore, $\mathcal{T}_\varepsilon(w_\varepsilon) \rightharpoonup w$ weakly in $L^2(\Omega_T \times Y)$ and (2.23) is proven. □

Proposition 2.12. [17, Proposition 8] *Let $\{w_\varepsilon\}$ be a sequence in $L^2\big(0,T;\mathcal{X}^\varepsilon_0(\Omega)\big)$ (see* (2.38) *below, for the definition of the space $\mathcal{X}^\varepsilon_0(\Omega)$) converging weakly to w in $L^2\big(0,T;H^1_0(\Omega)\big)$, as $\varepsilon \to 0$, and such that*

$$\varepsilon\int_0^T\int_{\Gamma^\varepsilon} |\nabla^B w_\varepsilon|^2\,\mathrm{d}\sigma\,\mathrm{d}t \le \gamma\,, \tag{2.27}$$

where $\gamma > 0$ is a constant independent of ε. Then, for the same function $\widetilde{w}$ as in (2.24)*, we have that $\nabla^B_y\widetilde{w} \in L^2(\Omega_T\times\Gamma)$ does exist and*

$$\mathcal{T}^b_\varepsilon(\nabla^B w_\varepsilon) \rightharpoonup \nabla^B_x w + \nabla^B_y\widetilde{w}\,, \quad \textit{weakly in}\, L^2(\Omega_T\times\Gamma)\,. \tag{2.28}$$

2.4. **Setting of the problem.** In this subsection, we will present both the physical problem involving thick membranes and the concentrated version involving only $(N-1)$-dimensional interfaces. It will be the purpose of the next section to show that the concentration limit ($\eta \to 0$) of the physical model actually gives rise to the mathematical microscopic scheme.

Let $\lambda_{\mathrm{int}}, \lambda_{\mathrm{out}}, \alpha$ be strictly positive constants. We give here a complete formulation of the problems stated in the Introduction (the operators div and ∇, as well as div^B and ∇^B, act only with respect to the space variable x).

We first state the physical problem with thick membranes. To this purpose, we set $A^{\eta\varepsilon}(x) = \lambda_{\mathrm{int}}$ in $\Omega^{\varepsilon,\eta}_{\mathrm{int}}$, $A^{\eta\varepsilon}(x) = \lambda_{\mathrm{out}}$ in $\Omega^{\varepsilon,\eta}_{\mathrm{out}}$, $A^{\eta\varepsilon}(x) = 0$ in $\Gamma^{\varepsilon,\eta}$, $B^{\eta\varepsilon}(x) = 0$ in $\Omega^{\varepsilon,\eta}_{\mathrm{int}} \cup \Omega^{\varepsilon,\eta}_{\mathrm{out}}$, $B^{\eta\varepsilon}(x) = \alpha/\eta$ in $\Gamma^{\varepsilon,\eta}$. Moreover, let $\overline{u}_0 \in H^1_0(\Omega)$ be a given function. For every $\varepsilon, \eta > 0$, we consider the problem for $u^\eta_\varepsilon \in L^2\big(0,T;H^1_0(\Omega)\big)$ given by

$$-\,\mathrm{div}(A^{\eta\varepsilon}\nabla u^\eta_\varepsilon + B^{\eta\varepsilon}\nabla u^\eta_{\varepsilon t}) = 0\,, \qquad \text{in } \Omega_T; \tag{2.29}$$

$$\nabla u^\eta_\varepsilon(x,0) = \nabla\overline{u}_0(x)\,, \qquad \text{in } \Gamma^{\varepsilon,\eta}, \tag{2.30}$$

which has the following weak formulation

$$\int_0^T\int_\Omega A^{\eta\varepsilon}\nabla u_\varepsilon^\eta\cdot\nabla\Phi\,\mathrm{d}x\,\mathrm{d}t-\int_0^T\int_\Omega B^{\eta\varepsilon}\nabla u_\varepsilon^\eta\cdot\nabla\Phi_t\,\mathrm{d}x\,\mathrm{d}t=\int_\Omega B^{\eta\varepsilon}\nabla\overline{u}_0\cdot\nabla\Phi(0)\,\mathrm{d}x\,, \quad (2.31)$$

for every test function $\Phi\in\mathcal{C}^\infty(\overline{\Omega}_T)$ such that Φ has compact support in Ω for every $t\in[0,T)$ and $\Phi(\cdot,T)=0$ in Ω. By [15, Theorem 2.3], for any given $\eta>0$, problem (2.29)–(2.30) (or (2.31)) has a unique solution $u_\varepsilon^\eta\in L^2\big(0,T;H_0^1(\Omega)\big)$.

Now, let us state the concentrated problem. To this purpose, we define $\lambda^\varepsilon:\Omega\to\mathbb{R}$ as

$$\lambda^\varepsilon=\lambda_{\mathrm{int}}\quad\text{in }\Omega^\varepsilon_{\mathrm{int}},\qquad \lambda^\varepsilon=\lambda_{\mathrm{out}}\quad\text{in }\Omega^\varepsilon_{\mathrm{out}}.$$

For every $\varepsilon>0$, we consider the problem for $u_\varepsilon(x,t)$ given by

$$-\operatorname{div}(\lambda^\varepsilon\nabla u_\varepsilon)=0\,,\qquad \text{in } (\Omega^\varepsilon_{\mathrm{int}}\cup\Omega^\varepsilon_{\mathrm{out}})\times(0,T); \quad (2.32)$$

$$[u_\varepsilon]=0\,,\qquad \text{on } \Gamma^\varepsilon_T; \quad (2.33)$$

$$-\varepsilon\alpha\Delta^B u_{\varepsilon t}=[\lambda^\varepsilon\nabla u_\varepsilon\cdot\nu_\varepsilon]\,,\qquad \text{on } \Gamma^\varepsilon_T; \quad (2.34)$$

$$u_\varepsilon(x,t)=0\,,\qquad \text{on } \partial\Omega\times(0,T); \quad (2.35)$$

$$\nabla^B u_\varepsilon(x,0)=\nabla^B\overline{u}_0(x)\,,\qquad \text{on } \Gamma^\varepsilon, \quad (2.36)$$

where we denote

$$[u_\varepsilon]=u_\varepsilon^{\mathrm{out}}-u_\varepsilon^{\mathrm{int}}\,, \quad (2.37)$$

and the same notation is employed also for other quantities.

Since problem (2.32)–(2.36) is not standard, in order to define a proper notion of weak solution, we need to introduce some suitable function spaces. To this purpose and for later use, we denote by $H^1(\Gamma^\varepsilon)$ the space of Lebesgue measurable functions $u:\Gamma^\varepsilon\to\mathbb{R}$ such that $u\in L^2(\Gamma^\varepsilon)$, $\nabla^B u\in L^2(\Gamma^\varepsilon)$. Let us also set

$$\mathcal{X}_0^\varepsilon(\Omega):=\{u\in H_0^1(\Omega)\ :\ u_{|\Gamma^\varepsilon}\in H^1(\Gamma^\varepsilon)\}\,. \quad (2.38)$$

Definition 2.13. Assume that $\overline{u}_0\in\mathcal{X}_0^\varepsilon(\Omega)$. We say that $u_\varepsilon\in L^2\big(0,T;\mathcal{X}_0^\varepsilon(\Omega)\big)$ is a weak solution of problem (2.32)–(2.36) if

$$\int_0^T\int_\Omega \lambda^\varepsilon\nabla u_\varepsilon\cdot\nabla\Phi\,\mathrm{d}x\,\mathrm{d}t-\varepsilon\alpha\int_0^T\int_{\Gamma^\varepsilon}\nabla^B u_\varepsilon\cdot\nabla^B\Phi_t\,\mathrm{d}\sigma\,\mathrm{d}t=\varepsilon\alpha\int_{\Gamma^\varepsilon}\nabla^B\overline{u}_0\cdot\nabla^B\Phi(x,0)\,\mathrm{d}\sigma\,, \quad (2.39)$$

for every test function $\Phi\in\mathcal{C}^\infty(\overline{\Omega}_T)$ such that Φ has compact support in Ω for every $t\in[0,T)$ and $\Phi(\cdot,T)=0$ in Ω. □

For every $\varepsilon>0$, by [15, Theorem 2.8] the problem (2.32)–(2.36) admits a unique solution $u_\varepsilon\in L^2\big(0,T;\mathcal{X}_0^\varepsilon(\Omega)\big)$.

If u_ε is smooth, by (2.4) it follows that equation (2.34) can be written in the form

$$-\varepsilon\alpha\left(\Delta u_{\varepsilon t}-\nu_\varepsilon^t\nabla^2 u_{\varepsilon t}\nu_\varepsilon-(\nu_\varepsilon\cdot\nabla u_{\varepsilon t})\operatorname{div}\nu_\varepsilon\right)=[\lambda\nabla u_\varepsilon\cdot\nu_\varepsilon]\,,\qquad \text{on } \Gamma^\varepsilon, \quad (2.40)$$

where $\nabla^2 u_{\varepsilon t}$ stands for the Hessian matrix of $u_{\varepsilon t}$.

Finally, it will be useful in the sequel to define also $\lambda : Y \to \mathbb{R}$ as

$$\lambda = \lambda_{\mathrm{int}} \quad \text{in } E_{\mathrm{int}}, \qquad \lambda = \lambda_{\mathrm{out}} \quad \text{in } E_{\mathrm{out}}.$$

Remark 2.14. We have assumed, for the sake of simplicity, that equations (2.29) and (2.32) are homogeneous, but essentially in the same way we can treat also the case where a source $f \in L^2(\Omega_T)$ appears in (2.29), so that it occurs, after the concentration, also in (2.32). However, in this case, if f does not satisfy some stronger regularity condition, we cannot expect that the solutions u^η_ε and u_ε admit a time-derivative belonging to L^2, so that the second term in (2.29), as well as the left-hand side in (2.34), should be considered in a weak sense as above (see [15, Remark 2.9]). In the homogeneous case, actually, one could show that a stronger formulation is possible. □

The main result of the paper is the following.

Theorem 2.15. *Assume that $\overline{u}_0 \in H^1_0(\Omega) \cap H^2(\Omega)$. The unique solution u_ε of the variational problem* (2.39) *converges, in the sense of Lemma 4.3, to $(u, w) \in \mathcal{V}$ ($\mathcal{V}$ is the function space defined in Theorem 4.4), where u is the unique solution of the homogenized problem*

$$\begin{aligned} &-\operatorname{div}\Big(C^0\nabla u_t + (\lambda_0 I + A^0)\nabla u + \int_0^t B^0(t-\tau)\nabla u(x,\tau)\,\mathrm{d}\tau\Big) = F, && \text{in } \Omega_T; \\ &u = 0\,, && \text{on } \partial\Omega \times (0,T); \\ &-\operatorname{div}\big(C^0\nabla u(x,0)\big) = -\operatorname{div}\big(C^0\nabla\overline{u}_0(x)\big)\,, && \text{in } \Omega. \end{aligned} \tag{2.41}$$

The corrector w in (4.3) *can be factorized as*

$$w(x,y,t) = \chi^i_0(y)\,\partial_i u(x,t) + \int_0^t \chi^i_1(y, t-\tau)\,\partial_i u(x,\tau)\,\mathrm{d}\tau + W(x,y,t). \tag{2.42}$$

The matrices A^0, B^0 and C^0 are given by (4.35)- (4.37), *the functions χ^i_0 and χ^i_1 are defined in* (4.17)–(4.21) *and, respectively, in* (4.22)–(4.27), *while the function W is given by* (4.28)–(4.32). *The source term F is defined by* (4.38) *and*

$$\lambda_0 = \lambda_{\mathrm{int}}|E_{\mathrm{int}}| + \lambda_{\mathrm{out}}|E_{\mathrm{out}}|.$$

Notice that the problem (2.41) must be intended in the following weak sense

$$\begin{aligned} -\int_0^T\!\!\int_\Omega C^0\nabla u\cdot\nabla\Phi_t\,\mathrm{d}x\,\mathrm{d}t + \int_0^T\!\!\int_\Omega (\lambda_0 I + A^0)\nabla u\cdot\nabla\Phi\,\mathrm{d}x\,\mathrm{d}t \\ + \int_0^T\!\!\int_\Omega \left(\int_0^t B^0(t-\tau)\nabla u(x,\tau)\,\mathrm{d}\tau\right)\cdot\nabla\Phi\,\mathrm{d}x\,\mathrm{d}t \\ = \int_0^T\!\!\int_\Omega F\Phi\,\mathrm{d}x\,\mathrm{d}t + \int_\Omega C^0\nabla\overline{u}_0\cdot\nabla\Phi(0)\,\mathrm{d}x\,, \end{aligned} \tag{2.43}$$

for every test function $\Phi \in \mathcal{C}^\infty(\overline{\Omega_T})$ such that Φ has compact support in Ω for every $t \in [0, T)$ and $\Phi(\cdot, T) = 0$ in Ω. In particular, the initial condition in (2.41) comes from the proof of the Theorem 2.15 (see (4.16)); moreover, we remark that the order of derivation in (2.41) (as well as in (2.34) above or (2.46) below) is not trivial and that $-\operatorname{div}(C^0 \nabla u_t)$ should be rewritten in the form $-\operatorname{div}\left((C^0 \nabla u)_t\right)$ or $-\partial_t \left(\operatorname{div}(C^0 \nabla u)\right)$ (see [15, Remark 2.9]). However, the weak formulation (2.43) is rigorous.

When different scalings with respect to ε are present, we consider, for $k \in \mathbb{R}$, the problem

$$-\operatorname{div}(\lambda^\varepsilon \nabla u_\varepsilon) = f\,, \qquad \text{in } (\Omega^\varepsilon_{\text{int}} \cup \Omega^\varepsilon_{\text{out}}) \times (0, T); \tag{2.44}$$

$$[u_\varepsilon] = 0\,, \qquad \text{on } \Gamma^\varepsilon_T; \tag{2.45}$$

$$-\varepsilon^k \alpha \Delta^B u_{\varepsilon t} = [\lambda^\varepsilon \nabla u_\varepsilon \cdot \nu_\varepsilon]\,, \qquad \text{on } \Gamma^\varepsilon_T; \tag{2.46}$$

$$u_\varepsilon(x, t) = 0\,, \qquad \text{on } \partial\Omega \times (0, T); \tag{2.47}$$

$$\nabla^B u_\varepsilon(x, 0) = \nabla^B \overline{u}_{0\varepsilon}(x)\,, \qquad \text{on } \Gamma^\varepsilon, \tag{2.48}$$

where we are interested in keeping a nonzero source $f \in L^2(\Omega_T)$, in order to show that the following results are non trivial.

Theorem 2.16. *Assume that $f \in L^2(\Omega)$ and $\overline{u}_{0\varepsilon} = \varepsilon^{(1-k)/2}\overline{u}_0$, with $\overline{u}_0 \in H^1_0(\Omega) \cap H^2(\Omega)$. Then, if $k < 1$ and we are in the connected/connected case, the unique solution u_ε of the problem (2.44)–(2.48) converges to 0, weakly in $L^2\big(0, T; \mathcal{X}^\varepsilon_0(\Omega)\big)$, for $\varepsilon \to 0$.*

Theorem 2.17. *Assume that $f \in L^2(\Omega)$ and $\overline{u}_{0\varepsilon} = \varepsilon^{(1-k)/2}\overline{u}_0$, with $\overline{u}_0 \in H^1_0(\Omega) \cap H^2(\Omega)$. Then, if $k < 1$ and we are in the connected/disconnected case, the unique solution u_ε of the problem (2.44)–(2.48) weakly converges to $u \in L^2\big(0, T; H^1_0(\Omega)\big)$, for $\varepsilon \to 0$, where u is the unique solution of the problem*

$$-\operatorname{div}\left(\Big(\int\limits_{E_{\text{out}}} \lambda_{\text{out}}(I + \nabla_y \chi_0)\,\mathrm{d}y + \int\limits_{\Gamma} \lambda_{\text{out}}\big((\nu + \nu \nabla_y \chi_0^{\text{out}}) \otimes y_M\big)\,\mathrm{d}\sigma\Big)\nabla u\right) = f, \quad \textit{in } \Omega_T\,;$$

$$u = 0, \qquad \textit{on } \partial\Omega \times (0, T),$$

where $\chi^j_0 \in H^1_\#(Y)$, $j = 1, \dots, N$, are Y-periodic functions with null mean average satisfying (4.17)–(4.21) and $y_M = y - \int_Y y\,\mathrm{d}y$. The homogenized matrix $A^{\text{hom}} := \int_{E_{\text{out}}} \lambda_{\text{out}}(I + \nabla_y \chi_0)\,\mathrm{d}y + \int_\Gamma \lambda_{\text{out}}\big((\nu + \nu\nabla_y \chi_0^{\text{out}}) \otimes y_M\big)\,\mathrm{d}\sigma$ is symmetric and positive definite.

Theorem 2.18. *Assume that $f \in L^2(\Omega)$ and $\overline{u}_{0\varepsilon} = \varepsilon^{(1-k)/2}\overline{u}_0$, with $\overline{u}_0 \in H^1_0(\Omega) \cap H^2(\Omega)$. Then, if $k > 1$, the unique solution u_ε of the problem (2.44)–(2.48) weakly converges to $u \in L^2\big(0, T; H^1_0(\Omega)\big)$, for $\varepsilon \to 0$, where u is the unique solution of the problem*

$$-\operatorname{div}\left(\Big(\int\limits_Y \lambda(I + \nabla_y \widetilde{\chi}_0)\,\mathrm{d}y\Big)\nabla u\right) = f\,, \qquad \textit{in } \Omega_T\,;$$

$$u = 0\,, \qquad \textit{in } \partial\Omega \times (0, T)\,,$$

where $\widetilde{\chi}_0^j \in H^1_\#(Y)$, $j = 1, \dots, N$, are Y-periodic functions with null mean average satisfying (5.7)–(5.8). *The homogenized matrix $A^{\rm hom} := \int_Y \lambda(I + \nabla_y \widetilde{\chi}_0)\,{\rm d}y$ is symmetric and positive definite.*

3. Derivation of the concentrated problem

In this section, using a local parametrization of the regular surface Γ^ε as in [18, Section 3] (see also [8, Section 3]) and following the outline of [18, Theorem 3.1], we prove the following result.

Theorem 3.1. *Assume that $\overline{u}_0 \in W^{1,\infty}_0(\Omega) \cap H^2(\Omega)$. Let u^η_ε be the solution of* (2.31). *Then,*

$$u^\eta_\varepsilon \rightharpoonup u_\varepsilon\,, \quad \nabla u^\eta_\varepsilon \rightharpoonup \nabla u_\varepsilon\,, \qquad \text{weakly in } L^2(\Omega_T), \tag{3.1}$$

where u_ε is the solution of (2.39).

Proof. We first note that, as proven in [15, Proposition 2.2], one can derive from (2.31) the energy inequality

$$\begin{aligned}&\int_0^T \int_{\Omega^{\varepsilon,\eta}_{\rm out} \cup \Omega^{\varepsilon,\eta}_{\rm int}} |\nabla u^\eta_\varepsilon|^2 \,{\rm d}x\,{\rm d}t + \frac{1}{\eta} \sup_{t \in (0,T)} \int_{\Gamma^{\varepsilon,\eta}} |\nabla u^\eta_\varepsilon|^2(t)\,{\rm d}x \\ \le& \frac{\alpha}{\eta \min(\lambda_{\rm int}, \lambda_{\rm out}, \alpha)} \int_{\Gamma^{\varepsilon,\eta}} |\nabla \overline{u}_0|^2\,{\rm d}x \le \gamma \frac{\alpha\,\eta |\Gamma^\varepsilon|}{\eta \min(\lambda_{\rm int}, \lambda_{\rm out}, \alpha)} \|\nabla \overline{u}_0\|^2_{L^\infty(\Omega)} \le \gamma\,,\end{aligned} \tag{3.2}$$

where γ depends on $\varepsilon, \lambda_{\rm int}, \lambda_{\rm out}, \alpha, \|\nabla \overline{u}_0\|_{L^\infty(\Omega)}$, but not on η. As a consequence of (3.2), as $\eta \to 0$, we may assume, extracting a subsequence if needed, that

$$u^\eta_\varepsilon \rightharpoonup u_\varepsilon\,, \quad \nabla u^\eta_\varepsilon \rightharpoonup \nabla u_\varepsilon\,, \qquad \text{weakly in } L^2(\Omega_T),$$

where, for every $\varepsilon > 0$, $u_\varepsilon \in L^2\big(0, T; H^1_0(\Omega)\big)$.

In order to proceed with the concentration of problem (2.29)–(2.30), we need to choose a suitable testing function in the weak formulation (2.31), before passing to the limit for $\eta \to 0$. To this purpose, we recall that there exists an $\eta_0 > 0$, such that for $\eta < \eta_0$, the application

$$\psi : \Gamma^\varepsilon \times [-\varepsilon\eta, \varepsilon\eta] \to \Gamma^{\varepsilon,2\eta}\,, \qquad \psi(y_{\Gamma^\varepsilon}, r) = y_{\Gamma^\varepsilon} + r\nu_\varepsilon(y_{\Gamma^\varepsilon}) = y \in \Gamma^{\varepsilon,2\eta}$$

is a diffeomorfism onto its image, where we denote by $\Gamma^{\varepsilon,2\eta}$ the tubular neighborhood of Γ^ε with thickness $2\varepsilon\eta$. Clearly, $\Gamma^{\varepsilon,2\eta}$ can be considered as the union of surfaces denoted by Γ^ε_r parallel to Γ^ε and at distance $|r|$ from it, when r varies in $[-\varepsilon\eta, \varepsilon\eta]$. Hence, for $y \in \Gamma^{\varepsilon,2\eta}$, there exists a unique $(y_{\Gamma^\varepsilon}, r) \in \Gamma^\varepsilon \times [-\varepsilon\eta, \varepsilon\eta]$ such that $y = y_{\Gamma^\varepsilon} + r\nu_\varepsilon(y_{\Gamma^\varepsilon})$ and, then, $y \in \Gamma^\varepsilon_r$ and $\nu_\varepsilon(y_{\Gamma^\varepsilon})$ coincides with the normal to the surface Γ^ε_r at y. Moreover, we can locally parametrize Γ^ε in such a way that there exist $\widehat{\Gamma}^\varepsilon \subset \mathbb{R}^{N-1}$ and $y_{\Gamma^\varepsilon} : \widehat{\Gamma}^\varepsilon \to \Gamma^\varepsilon$ such that $\Gamma^\varepsilon \ni y_{\Gamma^\varepsilon} = y_{\Gamma^\varepsilon}(\xi)$, where $\xi = (\xi_1, \dots, \xi_{N-1}) \in \widehat{\Gamma}^\varepsilon$ and, if we set ${\rm d}\sigma = \sqrt{g(\xi)}\,{\rm d}\xi$, we may assume that $\gamma_1 \le \sqrt{g(\xi)} \le \gamma_2$, for every $\xi \in \widehat{\Gamma}^\varepsilon$, where γ_1, γ_2 are suitable strictly positive constants. As a consequence, we have obtained a change of coordinates in $\mathbb{R}^N$, whose Jacobian matrix will be denoted by $J(\xi,r)$, defined by

$$\Gamma^{\varepsilon,2\eta} \ni y = (y_1,\dots,y_N) \longleftrightarrow (\xi,r) = (\xi_1,\dots,\xi_{N-1},r) \in \widehat{\Gamma}^{\varepsilon} \times [-\varepsilon\eta,\varepsilon\eta]\,.$$

By the assumed regularity of Γ^{ε}, it follows that $J(\xi,r) = J(\xi,0) + M_\eta$, where M_η denotes a suitable matrix such that $|M_\eta| \leq \gamma\eta$, so that $|detJ(\xi,r)| = |detJ(\xi,0)| + R_\eta$, where $|R_\eta| \leq \gamma\eta$; moreover, by the choice of the coordinates (ξ,r), we have that $|detJ(\xi,0)| = \sqrt{g(\xi)}$ (recall that the volume element $\mathrm{d}y = |detJ(\xi,r)|\,\mathrm{d}\xi\,\mathrm{d}r$, for $r=0$, i.e. on Γ^{ε}, becomes $\mathrm{d}y = |detJ(\xi,0)|\,\mathrm{d}\xi\,\mathrm{d}r = \mathrm{d}\sigma\,\mathrm{d}r = \sqrt{g(\xi)}\,\mathrm{d}\xi\,\mathrm{d}r$).
Finally, we define $\pi_0(y) = y_{\Gamma^{\varepsilon}}$ as the orthogonal projection of $y \in \Gamma^{\varepsilon,2\eta}$ on Γ^{ε} and $\rho(y) = r$ as the signed distance of $y \in \Gamma^{\varepsilon,2\eta}$ from Γ^{ε}. Note that $|\nabla\rho(y)|$ is bounded. In the sequel, we assume without loss of generality that the support of our testing functions is sufficiently small to allow for the representation introduced above. The general case can then be recovered by means of a standard partition of unity argument. Moreover, for the sake of brevity, we will use the same symbol for the same function even if written with respect to different variables.
Let now $\Phi \in \mathcal{C}^\infty(\overline{\Omega_T})$ be any given testing function for the concentrated problem (2.32)–(2.36) as in Definition 2.13, with the additional assumption on its support stated above. Starting from Φ, we construct a suitable test function $\Phi^{\eta\varepsilon}$ for problem (2.31) in such a way that it does not depend on the transversal coordinate inside $\Gamma^{\varepsilon,\eta}$ (thus it is equal to its value on Γ^{ε}) and it is linearly connected with Φ in $\Omega^{\varepsilon,\eta}_{\mathrm{int}}$ and $\Omega^{\varepsilon,\eta}_{\mathrm{out}}$ along the r-direction. It is crucial in order to develop the concentration procedure to make this gluing where the diffusivity in equation (2.29) is stable with respect to η, i.e. inside the set $\Gamma^{\varepsilon,2\eta} \setminus \Gamma^{\varepsilon,\eta} \subset \Omega^{\varepsilon,\eta}_{\mathrm{int}} \cup \Omega^{\varepsilon,\eta}_{\mathrm{out}}$. To this purpose, define

$$\Phi^{\eta\varepsilon}(y,t) = \begin{cases} \Phi(y,t) & \text{if } (y,t) \in (\Omega^{\varepsilon,\eta}_{\mathrm{out}} \setminus \Gamma^{\varepsilon,2\eta}) \times (0,T); \\ \Phi^{\eta\varepsilon}_{\mathrm{out}}(y,t) & \text{if } (y,t) \in (\Omega^{\varepsilon,\eta}_{\mathrm{out}} \cap \Gamma^{\varepsilon,2\eta}) \times (0,T); \\ \Phi\big(\pi_0(y),t\big) & \text{if } (y,t) \in \Gamma^{\varepsilon,\eta} \times (0,T); \\ \Phi^{\eta\varepsilon}_{\mathrm{int}}(y,t) & \text{if } (y,t) \in (\Omega^{\varepsilon,\eta}_{\mathrm{int}} \cap \Gamma^{\varepsilon,2\eta}) \times (0,T); \\ \Phi(y,t) & \text{if } (y,t) \in (\Omega^{\varepsilon,\eta}_{\mathrm{int}} \setminus \Gamma^{\varepsilon,2\eta}) \times (0,T), \end{cases} \tag{3.3}$$

where

$$\Phi^{\eta\varepsilon}_{\mathrm{out}}(y,t) = \Big[\Phi\big(\pi_0(y) + \varepsilon\eta\nu_\varepsilon(\pi_0(y)),t\big) - \Phi\big(\pi_0(y),t\big)\Big]\frac{2\rho(y) - \varepsilon\eta}{\varepsilon\eta} + \Phi\big(\pi_0(y),t\big)$$

and

$$\Phi^{\eta\varepsilon}_{\mathrm{int}}(y,t) = \Big[\Phi\big(\pi_0(y),t\big) - \Phi\big(\pi_0(y) - \varepsilon\eta\nu_\varepsilon(\pi_0(y)),t\big)\Big]\frac{2\rho(y) + \varepsilon\eta}{\varepsilon\eta} + \Phi\big(\pi_0(y),t\big).$$

By a density argument, we can use the Lipschitz continuous function $\Phi^{\eta\varepsilon}$ as a testing function in (2.31); then, it follows that

$$\int_0^T \int_{(\Omega_{\rm int}^{\varepsilon,\eta}\cup\Omega_{\rm out}^{\varepsilon,\eta})\setminus\Gamma^{\varepsilon,2\eta}} \lambda^\varepsilon \nabla u_\varepsilon^\eta \cdot \nabla\Phi \,\mathrm{d}y\,\mathrm{d}t - \frac{\alpha}{\eta}\int_0^T\int_{\Gamma^{\varepsilon,\eta}} \nabla u_\varepsilon^\eta \cdot \nabla\Big(\Phi_t\big(\pi_0(y),t\big)\Big)\,\mathrm{d}y\,\mathrm{d}t$$
$$+\int_0^T\int_{\Omega_{\rm int}^{\varepsilon,\eta}\cap\Gamma^{\varepsilon,2\eta}} \lambda^\varepsilon\nabla u_\varepsilon^\eta\cdot\nabla\Phi_{\rm int}^{\eta\varepsilon}\,\mathrm{d}y\,\mathrm{d}t + \int_0^T\int_{\Omega_{\rm out}^{\varepsilon,\eta}\cap\Gamma^{\varepsilon,2\eta}} \lambda^\varepsilon\nabla u_\varepsilon^\eta\cdot\nabla\Phi_{\rm out}^{\eta\varepsilon}\,\mathrm{d}y\,\mathrm{d}t$$
$$=\frac{\alpha}{\eta}\int_{\Gamma^{\varepsilon,\eta}}\nabla\overline{u}_0\cdot\nabla\Big(\Phi\big(\pi_0(y),0\big)\Big)\,\mathrm{d}y\,. \quad (3.4)$$

We take into account that

$$\nabla\Phi_{\rm out}^{\eta\varepsilon}(y,t) = \Im(\eta) + \Big[\Phi\big(\pi_0(y)+\varepsilon\eta\nu_\varepsilon(\pi_0(y)),t\big) - \Phi\big(\pi_0(y),t\big)\Big]\frac{2\nabla\rho(y)}{\varepsilon\eta}\,,$$

where with $\Im(\eta)$ we denote a bounded quantity with respect to η, and that

$$\Big|\Phi\big(\pi_0(y)+\varepsilon\eta\nu_\varepsilon(\pi_0(y)),t\big) - \Phi\big(\pi_0(y),t\big)\Big| \le \gamma\varepsilon\eta\,, \quad (3.5)$$

with γ independent of η. Clearly, similar estimates hold for $\Phi_{\rm int}^{\eta\varepsilon}$. Owing to (3.2), it is easy to see that, when $\eta\to 0$, the second line in the equality (3.4) tends to 0. In addition obviously

$$\int_0^T\int_{(\Omega_{\rm int}^{\varepsilon,\eta}\cup\Omega_{\rm out}^{\varepsilon,\eta})\setminus\Gamma^{\varepsilon,2\eta}} \lambda^\varepsilon\nabla u_\varepsilon^\eta\cdot\nabla\Phi\,\mathrm{d}y\,\mathrm{d}t \to \int_0^T\int_{\Omega_{\rm int}^{\varepsilon}\cup\Omega_{\rm out}^{\varepsilon}} \lambda^\varepsilon\nabla u_\varepsilon\cdot\nabla\Phi\,\mathrm{d}y\,\mathrm{d}t\,.$$

Hence, the crucial limits are the second and the fifth ones in (3.4). Let us deal with the second limit; the fifth can be treated in a similar and even simpler way. In order to do this, we pass to the new coordinates (ξ,r) defined above, recalling that $J(\xi,r)$ denotes the Jacobian matrix of such a change of coordinates. Moreover, denoting by $\nabla^B_{\Gamma_r^\varepsilon}$ the tangential gradient with respect to the surface Γ_r^ε and recalling that the normal vector at $y\in\Gamma_r^\varepsilon$ coincides with the normal at $\pi_0(y)\in\Gamma^\varepsilon$, we have $\nabla^B_{\Gamma_r^\varepsilon}u_\varepsilon^\eta = \nabla u_\varepsilon^\eta - (\nu_\varepsilon(\pi_0(y))\cdot\nabla u_\varepsilon^\eta)\nu_\varepsilon(\pi_0(y))$, with $r=\rho(y)$. Also, since the test function does not depend on the normal coordinate r in $\Gamma^{\varepsilon,\eta}$, we have that $\nabla\big(\Phi(\pi_0(y),t)\big) = \nabla^B\big(\Phi(\pi_0(y),t)\big)$ and hence $\nabla u_\varepsilon^\eta\cdot\nabla\Phi = \nabla^B_{\Gamma^\varepsilon_{\rho(y)}}u_\varepsilon^\eta\cdot\nabla^B\Phi$.

Next we denote by $\widetilde{J}(\xi,r)$ the $N\times(N-1)$ rectangular matrix such that, for every function $v(y)$, $\widetilde{J}(\xi,r)\nabla_\xi v(\xi,r) = \nabla^B_{\Gamma^\varepsilon_{\rho(y)}}v(y)$, and we let, for the sake of simplicity,

$\widetilde{J}(\xi) := \widetilde{J}(\xi, 0)$. Then, we can rewrite

$$\frac{\alpha}{\eta}\int_0^T\int_{\Gamma^{\varepsilon,\eta}} \nabla u_\varepsilon^\eta \cdot \nabla\Big(\Phi_t\big(\pi_0(y),t\big)\Big)\,\mathrm{d}y\,\mathrm{d}t = \frac{\alpha}{\eta}\int_0^T\int_{\Gamma^{\varepsilon,\eta}} \nabla^B_{\Gamma^\varepsilon_{\rho(y)}}\Phi_t\big(\pi_0(y),t\big)\cdot\nabla^B_{\Gamma^\varepsilon_{\rho(y)}} u_\varepsilon^\eta\,\mathrm{d}y\,\mathrm{d}t$$
$$= \frac{\alpha}{\eta}\int_0^T\int_{\widehat{\Gamma}^\varepsilon}\int_{-\varepsilon\eta/2}^{\varepsilon\eta/2} (\widetilde{J}(\xi,r)\nabla_\xi\Phi_t(\xi,0,t))^T\widetilde{J}(\xi,r)\nabla_\xi u_\varepsilon^\eta |detJ(\xi,r)|\,\mathrm{d}\xi\,\mathrm{d}r\,\mathrm{d}t$$
$$= \alpha\varepsilon\int_0^T\int_{\widehat{\Gamma}^\varepsilon}\Big(\widetilde{J}(\xi)\nabla_\xi\Phi_t\big(\xi,0,t\big)\Big)^T\left(\frac{1}{\varepsilon\eta}\int_{-\varepsilon\eta/2}^{\varepsilon\eta/2}\widetilde{J}(\xi)\nabla_\xi u_\varepsilon^\eta\big(\xi,r,t\big)\,\mathrm{d}r\right)\sqrt{g(\xi)}\,\mathrm{d}\xi\,\mathrm{d}t + I_2(\eta)$$
$$=: I_1 + I_2(\eta)\,,$$

where the superscript ^{T} denotes the transposed vector. Obviously, due to the regularity of Γ^ε, also the matrix $\widetilde{J}$ is regular, so that $\widetilde{J}(\xi,r) = \widetilde{J}(\xi,0) + O(\eta)$.
Clearly, using the energy estimate (3.2), we obtain

$$|I_2(\eta)| \le \gamma\frac{\eta}{\sqrt{\eta}}\left(\frac{1}{\eta}\int_0^T\int_{\Gamma^{\varepsilon,\eta}}|\nabla u_\varepsilon^\eta|^2\,\mathrm{d}y\,\mathrm{d}t\right)^{1/2}\sqrt{\eta} \le \gamma\eta \to 0 \qquad \text{as } \eta\to 0.$$

On the other hand, again by the energy estimate (3.2), it follows that there exists a vector function $\mathbf{V} \in L^2(0,T;L^2(\widehat{\Gamma}^\varepsilon))$ such that, up to a subsequence,

$$\frac{1}{\varepsilon\eta}\int_{-\varepsilon\eta/2}^{\varepsilon\eta/2}\widetilde{J}(\xi)\nabla_\xi u_\varepsilon^\eta(\xi,r,t)\,\mathrm{d}r \rightharpoonup \mathbf{V}\,, \qquad \text{weakly in } L^2(0,T;L^2(\widehat{\Gamma}^\varepsilon)),$$

so that

$$I_1 \to \alpha\varepsilon\int_0^T\int_{\widehat{\Gamma}^\varepsilon}\Big(\widetilde{J}(\xi)\nabla_\xi\Phi_t(\xi,0,t)\Big)^T\mathbf{V}\sqrt{g(\xi)}\,\mathrm{d}\xi\,\mathrm{d}t = \alpha\varepsilon\int_0^T\int_{\Gamma^\varepsilon}\nabla^B\Phi_t\cdot\mathbf{V}\,\mathrm{d}\sigma\,\mathrm{d}t\,.$$

It remains to identify $\mathbf{V}$ as the tangential gradient of the limit u_ε; i.e., $\mathbf{V} = \nabla^B u_\varepsilon$ on Γ^ε. To this aim, we consider a vector test function $\Psi \in \mathcal{C}^1_c(\Omega_T)$; we obtain

$$\int_0^T \int_{\Gamma^\varepsilon} \operatorname{div}^B \Psi \, u_\varepsilon \, \mathrm{d}\sigma \, \mathrm{d}t \longleftarrow \int_0^T \int_{\Gamma^\varepsilon} \operatorname{div}^B \Psi \left(\frac{1}{\varepsilon\eta} \int_{-\varepsilon\eta/2}^{\varepsilon\eta/2} u^\eta_\varepsilon(y_{\Gamma^\varepsilon} + r\nu_\varepsilon(y_{\Gamma^\varepsilon}), t) \, \mathrm{d}r \right) \mathrm{d}\sigma \, \mathrm{d}t$$

$$= - \int_0^T \int_{\Gamma^\varepsilon} \Psi \cdot \nabla^B \left(\frac{1}{\varepsilon\eta} \int_{-\varepsilon\eta/2}^{\varepsilon\eta/2} u^\eta_\varepsilon(y_{\Gamma^\varepsilon} + r\nu_\varepsilon(y_{\Gamma^\varepsilon}), t) \, \mathrm{d}r \right) \mathrm{d}\sigma \, \mathrm{d}t$$

$$= - \int_0^T \int_{\widehat{\Gamma}^\varepsilon} \Psi \cdot \left(\frac{1}{\varepsilon\eta} \int_{-\varepsilon\eta/2}^{\varepsilon\eta/2} \widetilde{J}(\xi) \nabla_\xi u^\eta_\varepsilon(\xi, r, t) \, \mathrm{d}r \right) \sqrt{g(\xi)} \, \mathrm{d}\xi \, \mathrm{d}t \longrightarrow$$

$$- \int_0^T \int_{\widehat{\Gamma}^\varepsilon} \Psi \cdot \mathbf{V} \sqrt{g(\xi)} \, \mathrm{d}\xi \, \mathrm{d}t = - \int_0^T \int_{\Gamma^\varepsilon} \Psi \cdot \mathbf{V} \, \mathrm{d}\sigma \, \mathrm{d}t \,,$$

which implies that $\mathbf{V} = \nabla^B u_\varepsilon$. Similarly, we can prove that

$$\frac{\alpha}{\eta} \int_{\Gamma^{\varepsilon,\eta}} \nabla \overline{u}_0 \cdot \nabla \Big(\Phi\big(\pi_0(y), 0\big) \Big) \, \mathrm{d}y \to \alpha\varepsilon \int_{\Gamma^\varepsilon} \nabla^B \overline{u}_0 \cdot \nabla^B \Phi(y, 0) \, \mathrm{d}\sigma \,.$$

This proves that the limit for $\eta \to 0$ of equality (3.4) yields (2.39); i.e., the concentration limit of u^η_ε is the weak solution of system (2.32)–(2.36).
By the uniqueness of solutions to the limit problem (2.39), the whole sequence $\{u^\eta_\varepsilon\}$ converges. □

4. Homogenization of the microscopic problem

Our goal in this section is to describe the asymptotic behavior, as $\varepsilon \to 0$, of the solution $u_\varepsilon \in L^2\big(0, T; \mathcal{X}^\varepsilon_0(\Omega)\big)$ of problem (2.32)–(2.36).
From [15, Theorem 2.8], we obtain the following result.

Theorem 4.1. *If $\overline{u}_0 \in H^1_0(\Omega) \cap H^2(\Omega)$, then, for any $\varepsilon \in (0, 1)$, the variational problem (2.39) has a unique solution $u_\varepsilon \in L^2\big(0, T; \mathcal{X}^\varepsilon_0(\Omega)\big)$. Moreover, there exists a constant $\gamma > 0$, independent of ε, such that*

$$\int_0^T \int_\Omega |\nabla u_\varepsilon|^2 \, \mathrm{d}x \, \mathrm{d}\tau + \varepsilon \sup_{t \in (0,T)} \int_{\Gamma^\varepsilon} |\nabla^B u_\varepsilon|^2 \, \mathrm{d}\sigma \le \gamma. \tag{4.1}$$

Remark 4.2. Notice that, as proven in [17, Section 3], the assumption $\overline{u}_0 \in H^1_0(\Omega) \cap H^2(\Omega)$ implies that, for every $\varepsilon > 0$, $\overline{u}_0 \in H^1(\Gamma^\varepsilon)$ and

$$\varepsilon \int_{\Gamma^\varepsilon} |\nabla^B \overline{u}_0|^2 \, \mathrm{d}\sigma \le \gamma \,, \tag{4.2}$$

with γ independent of ε. This is the crucial tool in order to obtain (4.1) above. □

The convergence results stated in the next lemma are a consequence of the *a priori* estimates (4.1) and of the general compactness results obtained in Subsection 2.3 (see Theorem 2.11 and Proposition 2.12).

Lemma 4.3. *Let* $u_\varepsilon \in L^2\big(0,T;\mathcal{X}_0^\varepsilon(\Omega)\big)$ *be the unique solution of problem* (2.39). *Then, up to a subsequence, still denoted by* ε, *there exist* $u \in L^2(0,T;H_0^1(\Omega))$ *and* $w \in L^2(\Omega_T;H^1_\#(Y))$ *with* $\mathcal{M}_Y(w)=0$ *and* $\nabla_y^B w \in L^2(\Omega_T\times\Gamma)$ *such that*

$$
\begin{aligned}
&u_\varepsilon \rightharpoonup u && \textit{weakly in } L^2(0,T;H_0^1(\Omega)),\\
&\mathcal{T}_\varepsilon(u_\varepsilon) \rightharpoonup u && \textit{weakly in } L^2\left(\Omega_T\times Y\right),\\
&\mathcal{T}_\varepsilon(\nabla u_\varepsilon) \rightharpoonup \nabla u + \nabla_y w && \textit{weakly in } L^2(\Omega_T\times Y),\\
&\mathcal{T}^b_\varepsilon(\nabla^B u_\varepsilon) \rightharpoonup \nabla^B u + \nabla^B_y w && \textit{weakly in } L^2(\Omega_T\times\Gamma).
\end{aligned}
$$

Theorem 4.4. *Let us set*

$$
\begin{aligned}
\mathcal{W}_\#(Y) &= \{v\in H^1_\#(Y)\,|\,\mathcal{M}_Y(v)=0,\ \nabla^B_y v\in L^2(\Omega_T\times\Gamma)\}\\
\mathcal{V} &= L^2(0,T;H_0^1(\Omega))\times L^2(\Omega_T;\mathcal{W}_\#(Y)).
\end{aligned}
$$

The unique solution $u_\varepsilon \in L^2\big(0,T;\mathcal{X}_0^\varepsilon(\Omega)\big)$ *of the variational problem* (2.39) *converges, in the sense of Lemma 4.3, to the unique solution* $(u,w)\in\mathcal{V}$ *of the following unfolded limit problem*

$$
\begin{aligned}
&\int_0^T\!\!\int_{\Omega\times Y} \lambda\,(\nabla u+\nabla_y w)\cdot(\nabla\varphi+\nabla_y\Psi)\,\mathrm{d}x\,\mathrm{d}y\,\mathrm{d}t\\
&-\alpha\int_0^T\!\!\int_{\Omega\times\Gamma} (\nabla^B u+\nabla^B_y w)\cdot(\nabla^B\varphi_t+\nabla^B_y\Psi_t)\,\mathrm{d}x\,\mathrm{d}\sigma\,\mathrm{d}t\\
&=\alpha\int_{\Omega\times\Gamma}\nabla^B\overline{u}_0\cdot(\nabla^B\varphi(x,0)+\nabla^B_y\Psi(x,y,0))\,\mathrm{d}x\,\mathrm{d}\sigma,
\end{aligned}
\tag{4.3}
$$

for all $\varphi\in H^1\big(0,T;H^1_0(\Omega)\big)$ *and* $\Psi\in L^2(\Omega_T;H^1_\#(Y))\cap H^1\left(0,T;L^2(\Omega,H^1(\Gamma))\right)$ *with* $\varphi(\cdot,T)=0$ *in* Ω, $\Psi(\cdot,\cdot,T)=0$ *in* $\Omega\times Y$.

Proof. Preliminarily, we note that equation (4.3) admits at most one solution. Indeed, setting $U=u_2-u_1$ and $\mathcal{W}=w_2-w_1$, where (u_i,w_i), $i=1,2$, are two solutions of (4.3), we obtain in a standard way from the equation written for $(U,\mathcal{W})$

$$
\int_0^T\!\!\int_{\Omega\times Y}\lambda|\nabla U+\nabla_y\mathcal{W}|^2\,\mathrm{d}x\,\mathrm{d}y\,\mathrm{d}t+\frac{\alpha}{2}\int_{\Omega\times\Gamma}|\nabla^B U+\nabla^B_y\mathcal{W}|^2(T)\,\mathrm{d}x\,\mathrm{d}\sigma=0\,.
$$

Hence, dropping the last integral (which is nonnegative) and using the Y-periodicity of $\mathcal{W}$, we obtain

$$
\begin{aligned}
&\int_0^T\int_{\Omega\times Y} (|\nabla U|^2 + |\nabla_y \mathcal{W}|^2)\,\mathrm{d}x\,\mathrm{d}y\,\mathrm{d}t \\
&\quad = \int_0^T\int_{\Omega\times Y} (|\nabla U|^2 + |\nabla_y \mathcal{W}|^2)\,\mathrm{d}x\,\mathrm{d}y\,\mathrm{d}t + 2\int_0^T\int_\Omega \left(\nabla U\cdot\int_Y \nabla_y\mathcal{W}\,\mathrm{d}y\right)\mathrm{d}x\,\mathrm{d}t \\
&\quad = \int_0^T\int_{\Omega\times Y} |\nabla U + \nabla_y\mathcal{W}|^2\,\mathrm{d}x\,\mathrm{d}y\,\mathrm{d}t \le 0\,.
\end{aligned}
$$

Therefore, taking into account that U vanishes on $\partial\Omega$ and $\mathcal{W}$ has null mean average in Y, we get $U = \mathcal{W} = 0$; i.e., the asserted uniqueness.
In order to obtain the limit problem (4.3), we choose in the variational formulation (2.39) the admissible test function

$$\Phi(x,t) = \varphi(x,t) + \varepsilon\phi(x,t)\psi\left(\frac{x}{\varepsilon}\right), \tag{4.4}$$

with $\varphi,\phi \in \mathcal{C}^\infty([0,T];\mathcal{C}_c^\infty(\Omega))$, $\varphi(\cdot,T) = \phi(\cdot,T) = 0$ in Ω and $\psi\in\mathcal{C}^\infty_\#(Y)$.
Then, by unfolding each term with the corresponding operator, we get

$$
\begin{aligned}
&\int_0^T\int_{\Omega\times Y} \mathcal{T}_\varepsilon(\lambda^\varepsilon)\mathcal{T}_\varepsilon(\nabla u_\varepsilon)\cdot\mathcal{T}_\varepsilon(\nabla\Phi)\,\mathrm{d}y\,\mathrm{d}x\,\mathrm{d}t - \alpha\int_0^T\int_{\Omega\times\Gamma} \mathcal{T}^b_\varepsilon(\nabla^B u_\varepsilon)\cdot\mathcal{T}^b_\varepsilon(\nabla^B\Phi_t)\,\mathrm{d}\sigma\,\mathrm{d}x\,\mathrm{d}t \\
&\qquad = \alpha\int_{\Omega\times\Gamma} \mathcal{T}^b_\varepsilon(\nabla^B\overline{u}_0)\cdot\mathcal{T}^b_\varepsilon(\nabla^B\Phi(x,0))\,\mathrm{d}\sigma\,\mathrm{d}x + R_\varepsilon,
\end{aligned}
\tag{4.5}
$$

where $R_\varepsilon = o(1)$ as $\varepsilon\to 0$. Our goal now is to pass to the limit with $\varepsilon\to 0$ in (4.5). By Remark 2.5, Proposition 2.9 and Proposition 2.10, we obviously have

$$\mathcal{T}_\varepsilon(\nabla\Phi)\to\nabla\varphi+\nabla_y\Psi \qquad \text{strongly in } L^2(\Omega_T\times Y); \tag{4.6}$$

$$\mathcal{T}^b_\varepsilon(\nabla^B\Phi_t)\to\nabla^B\varphi_t+\nabla^B_y\Psi_t \qquad \text{strongly in } L^2(\Omega_T\times \Gamma); \tag{4.7}$$

$$\mathcal{T}^b_\varepsilon(\nabla^B\Phi(x,0))\to\nabla^B\varphi(x,0)+\nabla^B_y\Psi(x,0) \qquad \text{strongly in } L^2(\Omega_T\times \Gamma), \tag{4.8}$$

where Φ is given in (4.4) and $\Psi(x,y,t) = \phi(x,t)\psi(y)$. Therefore, by using the convergence results stated in Lemma 4.3 and recalling again (2.20) and Proposition 2.9, we obtain

$$\int_0^T\int_{\Omega\times Y} \lambda\,(\nabla u+\nabla_y w)\cdot(\nabla\varphi+\nabla_y\Psi)\,\mathrm{d}x\,\mathrm{d}y\,\mathrm{d}t$$

$$-\alpha\int_0^T\int_{\Omega\times\Gamma}(\nabla^B u+\nabla^B_y w)\cdot(\nabla^B\varphi_t+\nabla^B_y\Psi_t)\,\mathrm{d}x\,\mathrm{d}\sigma\,\mathrm{d}t=$$

$$\alpha\int_{\Omega\times\Gamma}\nabla^B\overline{u}_0\cdot(\nabla^B\varphi(x,0)+\nabla^B_y\Psi(x,y,0))\,\mathrm{d}x\,\mathrm{d}\sigma.$$

Standard density arguments lead us to (4.3). Moreover, due to the uniqueness of $(u,w)\in\mathcal{V}$, all the above convergences hold true for the whole sequence. □

Remark 4.5. Our goal now is to obtain the factorized formulation of the unfolded problem (4.3). Let us point out that this formulation involves the time derivatives of the solution and of its corrector (in the distributional sense). Due to the presence of these terms generated by the dynamical boundary condition on Γ^ε in the microscopic problem, we have to introduce a non-standard type of cell functions containing memory terms. Our limit model can be compared with [5, 7, 11, 17]. □

Proof of Theorem 2.15 By taking first $\varphi=0$ and suitable test functions Ψ, in the unfolded limit problem (4.3), and then $\Psi=0$ with suitable choices of φ, we obtain formally

$$-\operatorname{div}\Big(\lambda_0\nabla u+\int_Y\lambda\nabla_y w\,\mathrm{d}y\Big)-\operatorname{div}\Big(\alpha\Big(\int_\Gamma\nabla^B u_t\,\mathrm{d}\sigma+\int_\Gamma\nabla^B_y w_t\,\mathrm{d}\sigma\Big)\Big)=0,\ \text{in }\Omega_T;\qquad(4.9)$$

$$-\operatorname{div}_y(\lambda(\nabla u+\nabla_y w))=0\,,\qquad \text{in }\Omega_T\times(E_{\text{int}}\cup E_{\text{out}});\qquad(4.10)$$

$$-\operatorname{div}^B_y\big(\alpha(\nabla^B u_t+\nabla^B_y w_t)\big)=[\lambda(\nabla u+\nabla_y w)\cdot\nu]\,,\qquad \text{on }\Omega_T\times\Gamma;\qquad(4.11)$$

$$u=0\,,\qquad \text{on }\partial\Omega\times(0,T);\qquad(4.12)$$

complemented with the initial conditions

$$-\operatorname{div}\left(\int_\Gamma(\nabla^B u(x,0)+\nabla^B_y w(x,y,0))\,\mathrm{d}\sigma\right)=-\operatorname{div}\left(\int_\Gamma\nabla^B\overline{u}_0\,\mathrm{d}\sigma\right),\ \text{in }\Omega.\qquad(4.13)$$

$$-\operatorname{div}^B_y\big(\nabla^B u(x,0)+\nabla^B_y w(x,y,0)\big)=-\operatorname{div}^B_y\big(\nabla^B\overline{u}_0\big),\qquad \text{on }\Omega\times\Gamma,\qquad(4.14)$$

Note that (4.13) is the initial condition associated to (4.9), while (4.14) is the initial condition associated to (4.11).

Equation (4.14) for w is uniquely solvable on each connected component Γ_i, $i=1,\dots,m$, of Γ in terms of u and $\overline{u}_0$, up to an additive y-constant function $C_i(x)$, depending on the i^{th} connected component. The functions $C_i(x)$ can be chosen identically equal to 0, since they do not play any role in (4.14). Moreover, it is easy to prove that the function $w(x,y,0)=-\chi_0(y)\cdot[\nabla\overline{u}_0(x)-\nabla u(x,0)]$, with χ_0 defined by

(4.17)–(4.20) below, is the required solution. In particular, if $\Omega^\varepsilon_{\rm out}$ is connected and $\Omega^\varepsilon_{\rm int}$ is disconnected, then $\chi_0(y) = -y + c_i$, on each connected component Γ_i, so that

$$\nabla^B_y w(x,y,0) = \nabla^B_y \left(y \cdot [\nabla \overline{u}_0(x) - \nabla u(x,0)]\right) = \nabla^B \overline{u}_0(x) - \nabla^B u(x,0)\,, \tag{4.15}$$

and equation (4.13) becomes simply an identity. On the contrary, if $\Omega^\varepsilon_{\rm out}$ and $\Omega^\varepsilon_{\rm int}$ are both connected, then (4.13) becomes

$$-\operatorname{div}\left(C^0 \nabla u(x,0)\right) = -\operatorname{div}\left(C^0 \nabla \overline{u}_0\right), \tag{4.16}$$

where C^0 is the matrix defined in (4.37) below.
The presence of the time derivatives in (4.9) and (4.11) suggests us looking for the corrector w in the non-standard form given in (2.42). The factorization in terms of the cell function $\chi_0 : Y \to \mathbb{R}^N$ is rather standard, though the problem which defines χ_0 is not (see (4.17)–(4.21)). However, due to the dynamical boundary condition on Γ^ε, apart from this function, we need to introduce two new cell functions, $\chi_1 : Y \times (0,T) \to \mathbb{R}^N$ and $W : \Omega \times Y \times (0,T) \to \mathbb{R}$ (see (4.22)–(4.27) and (4.28)–(4.32)). More precisely, introducing (2.42) in (4.10)–(4.14), we are led to the following local problems for χ_0 and, respectively, χ_1:

$$-\operatorname{div}_y \left(\lambda \nabla_y (y_j + \chi_0^j)\right) = 0\,, \qquad \text{in } E_{\rm int} \cup E_{\rm out}; \tag{4.17}$$

$$[\chi_0^j] = 0\,, \qquad \text{on } \Gamma; \tag{4.18}$$

$$-\operatorname{div}^B_y \left(\alpha \nabla^B_y (y_j + \chi_0^j)\right) = 0\,, \qquad \text{on } \Gamma; \tag{4.19}$$

$$\int_{\Gamma_i} \left(\nabla_y \chi_0^j\right)^{(\rm out)} \cdot \nu \,\mathrm{d}\sigma = 0\,,\ i = 1,\dots,m \tag{4.20}$$

$$\int_Y \chi_0^j \,\mathrm{d}y = 0\,, \tag{4.21}$$

and

$$-\operatorname{div}_y \left(\lambda \nabla_y \chi_1^j\right) = 0\,, \qquad \text{in } (E_{\rm int} \cup E_{\rm out}) \times (0,T); \tag{4.22}$$

$$[\chi_1^j] = 0\,, \qquad \text{on } \Gamma \times (0,T); \tag{4.23}$$

$$-\alpha \operatorname{div}^B_y \left(\nabla^B_y \chi^j_{1t}\right) = \left[\lambda \nabla_y \chi_1^j \cdot \nu\right]\,, \qquad \text{on } \Gamma \times (0,T); \tag{4.24}$$

$$\nabla^B_y \chi_1^j(y,0) = \nabla^B_y v_j(y)\,, \qquad \text{on } \Gamma, \tag{4.25}$$

$$\int_Y \chi_1^j \,\mathrm{d}y = 0\,, \tag{4.26}$$

where the initial data $\nabla^B_y v_j = \nabla^B_y v_j(y)$ is the solution of the problem

$$-\alpha \operatorname{div}^B_y \left(\nabla^B_y v_j(y)\right) = \left[\lambda \nabla_y (y_j + \chi_0^j) \cdot \nu\right], \qquad \text{on } \Gamma = \bigcup_{i=1}^m \Gamma_i. \tag{4.27}$$

The scalar function $W = W(x,y,t)$ is defined as the solution of the problem

$$-\operatorname{div}_y\,(\lambda\,\nabla_y W) = 0\,, \qquad \text{in } (E_{\text{int}} \cup E_{\text{out}}) \times (0,T); \tag{4.28}$$

$$[W] = 0\,, \qquad \text{on } \Gamma \times (0,T); \tag{4.29}$$

$$-\alpha \operatorname{div}^B_y \left(\nabla^B_y W_t\right) = [\lambda(\nabla_y W \cdot \nu)]\,, \qquad \text{on } \Gamma \times (0,T); \tag{4.30}$$

$$\nabla^B_y W(x,y,0) = -\nabla^B_y \chi_0(y) \nabla \overline{u}_0(x)\,, \qquad \text{on } \Gamma; \tag{4.31}$$

$$\int_Y W\,\mathrm{d}y = 0\,. \tag{4.32}$$

The cell problem (4.17)–(4.21) admits a unique solution $\chi^j_0 \in H^1_\#(Y) \cap H^1(\Gamma)$, for $j = 1,\dots,N$. Indeed, it can be solved starting from (4.19), which gives the boundary conditions on each connected component Γ_i of Γ, up to an additive constant c_i on Γ_i, $i = 1,\dots,m$. Then, we solve the problem (4.17) in E_{out}, and we use [15, Proposition 2.6] in order to satisfy also (4.20), properly choosing the constants c_i appearing in the first step. In the next step, we solve (4.17)–(4.18) in each connected component of E_{int}. In the last step, we can add a global suitable constant to the solution thus obtained, in order to satisfy also (4.21). Notice that the conditions (4.20) are crucial, since they are required in order that the right-hand side of (4.27) satisfies the compatibility conditions needed for the existence of a unique solution (up to an additive constant, which plays no role in (4.25), so that it can be chosen equal to zero). Finally, the cell problems (4.22)–(4.26) and (4.28)–(4.32) have unique solutions belonging to $L^2\big(0,T; H^1_\#(Y) \cap H^1(\Gamma)\big)$ by [15, Theorem 2.8 and Remark 4.7]. Actually, by standard bootstrap arguments, it follows that $\chi_0 \in \mathcal{C}^\infty_\#(Y)$ and $\chi_1 \in \mathcal{C}^\infty\big(0,T; \mathcal{C}^\infty_\#(Y)\big)$. By introducing the particular form of the corrector (2.42) in the equation (4.9), we get, rearranging the terms,

$$\begin{aligned}
&-\operatorname{div}\Big(\alpha \int_\Gamma \nabla^B u_t\,\mathrm{d}\sigma + \alpha \int_\Gamma \nabla^B_y \chi_0 \nabla u_t\,\mathrm{d}\sigma\Big) \\
&\qquad -\operatorname{div}\Big(\lambda_0 \nabla u + \int_Y \lambda \nabla_y \chi_0 \nabla u\,\mathrm{d}y + \alpha \int_\Gamma \nabla^B_y \chi_1(y,0) \nabla u(x,t)\,\mathrm{d}\sigma\Big) \\
&-\operatorname{div}\Big(\int_0^t\!\!\int_Y \lambda \nabla_y \chi_1(y,t-\tau) \nabla u(x,\tau)\,\mathrm{d}\tau\,\mathrm{d}y + \alpha \int_0^t\!\!\int_\Gamma \nabla^B_y \chi_{1t}(y,t-\tau) \nabla u(x,\tau)\,\mathrm{d}\tau\,\mathrm{d}\sigma\Big) \\
&\qquad\qquad = \operatorname{div}\Big(\int_Y \lambda \nabla_y W\,\mathrm{d}y + \alpha \int_\Gamma \nabla^B_y W_t\,\mathrm{d}\sigma\Big).
\end{aligned} \tag{4.33}$$

The principal part of the equation in the first line of (4.33) can be written as follows

$$
\begin{aligned}
-\operatorname{div}\Big(\alpha\int_{\Gamma}\nabla^B u_t\,\mathrm{d}\sigma+\alpha\int_{\Gamma}\nabla^B_y\chi_0\nabla u_t\,\mathrm{d}\sigma\Big) &= \\
-\alpha\operatorname{div}\left(\Big(\int_{\Gamma}(I-\nu\otimes\nu)\Big)\nabla u_t+\Big(\int_{\Gamma}\nabla^B_y\chi_0\,\mathrm{d}\sigma\Big)\nabla u_t\right) &= \\
-\alpha\operatorname{div}\left(\Big(\int_{\Gamma}(I-(\nu\otimes\nu)+\nabla^B_y\chi_0)\,\mathrm{d}\sigma\Big)\nabla u_t\right). &\qquad (4.34)
\end{aligned}
$$

We define

$$
A^0=\int_Y\lambda\nabla_y\chi_0\,\mathrm{d}y+\alpha\int_{\Gamma}\nabla^B_y\chi_1(y,0)\,\mathrm{d}\sigma=\int_{\Gamma}\big\{\alpha\nabla^B_y\chi_1(y,0)-[\lambda]\chi_0\otimes\nu\big\}\,\mathrm{d}\sigma\,, \quad (4.35)
$$

$$
B^0(t)=\alpha\int_{\Gamma}\nabla^B_y\chi_{1t}(y,t)\,\mathrm{d}\sigma+\int_Y\lambda\nabla_y\chi_1(y,t)\,\mathrm{d}y=\int_{\Gamma}\big\{\alpha\nabla^B_y\chi_{1t}(y,t)-[\lambda]\chi_1(y,t)\otimes\nu\big\}\mathrm{d}\sigma \quad (4.36)
$$

and

$$
C^0=\alpha\int_{\Gamma}\big(I-\nu\otimes\nu+\nabla^B_y\chi_0\big)\,\mathrm{d}\sigma=\alpha\int_{\Gamma}\nabla^B_y(\chi_0+y)\,\mathrm{d}\sigma. \quad (4.37)
$$

Obviously, the matrix B^0 is of class $\mathcal{C}^\infty(0,T)$. Also, we set

$$
F=\operatorname{div}\left(\int_Y\lambda\nabla_y W\,\mathrm{d}y+\alpha\int_{\Gamma}\nabla^B_y W_t\,\mathrm{d}\sigma\right)=\operatorname{div}\int_{\Gamma}\big(\alpha\nabla^B_y W_t-[\lambda]W\nu\big)\,\mathrm{d}\sigma. \quad (4.38)
$$

The matrices A^0, B^0 and C^0 are well-defined. By using the definition of the matrices A^0, B^0 and C^0, we obtain that the two-scale system (4.9)–(4.14) can be decoupled and we get immediately the homogenized equation appearing in problem (2.41). Concerning the boundary and the initial condition, we note that the first one is simply a direct consequence of the fact that the pair $(u,w)\in\mathcal{V}$, while the second one depends on the geometry. Indeed, in the connected/disconnected case, taken into account that $C^0=0$, as follows from Lemma 4.8 below, the homogenized problem does not require any initial condition, according to the fact that the last equation in (2.41) disappears. As observed at the beginning of the proof, this corresponds to the fact that (4.13) becomes, in this case, an identity. On the contrary, in the connected-connected case C^0 is positive definite, as a consequence of Lemma 4.7 below, so that an initial condition in (2.41) is in fact required.

□

Remark 4.6. Assume that $\Omega^\varepsilon_{\mathrm{out}}$ and $\Omega^\varepsilon_{\mathrm{int}}$ are both connected and that the solution u of (4.3) is sufficiently regular so that, up to time $t=0$, we have that $u=0$ on $\partial\Omega$.

Then, any solution $\big(u(\cdot,0), w(\cdot,\cdot,0)\big)$ of the system (4.13)–(4.14), complemented with the condition $u(x,0)=0$ on $\partial\Omega$, satisfies

$$u(x,0)=\overline{u}_0(x)\,, \qquad \nabla_y^B w(x,y,0)=0\,. \tag{4.39}$$

Indeed, setting $U(x)=u_2(x,0)-u_1(x,0)$ and $\mathcal{W}(x,y)=w_2(x,y,0)-w_1(x,y,0)$, where (u_i,w_i), $i=1,2$, are two solutions of the system (4.13)–(4.14) satisfying $u_{1|\partial\Omega}=0=u_{2|\partial\Omega}$, using U and $\mathcal{W}$ as testing functions in (4.13) and (4.14) (written for (u_i,w_i)), respectively, integrating by parts and subtracting the two equations, it follows

$$\int\limits_{\Omega\times\Gamma} |\nabla_y^B\mathcal{W}|^2 \,\mathrm{d}x\,\mathrm{d}\sigma = -\int\limits_{\Omega\times\Gamma} \nabla_y^B\mathcal{W}\cdot\nabla^B U \,\mathrm{d}x\,\mathrm{d}\sigma \tag{4.40}$$

and

$$\int\limits_{\Omega\times\Gamma} |\nabla^B U|^2 \,\mathrm{d}x\,\mathrm{d}\sigma = \int\limits_{\Omega}\left(\int\limits_{\Gamma}\nabla^B U\,\mathrm{d}\sigma\right)\cdot\nabla U\,\mathrm{d}x = \\ -\int\limits_{\Omega}\left(\int\limits_{\Gamma}\nabla_y^B \mathcal{W}\,\mathrm{d}\sigma\right)\cdot\nabla U\,\mathrm{d}x = -\int\limits_{\Omega\times\Gamma}\nabla_y^B\mathcal{W}\cdot\nabla^B U\,\mathrm{d}x\,\mathrm{d}\sigma\,. \tag{4.41}$$

Summing (4.40) and (4.41), we obtain

$$\int\limits_{\Omega\times\Gamma} \left|\nabla^B U+\nabla_y^B\mathcal{W}\right|^2 \,\mathrm{d}x\,\mathrm{d}\sigma = 0\,.$$

This implies $\nabla_y^B\mathcal{W}=-(I-\nu\otimes\nu)\nabla U=-\nabla_y^B(y\cdot\nabla U)$, with U depending only on x and where we used the fact that $\nabla_y^B y = I-\nu\otimes\nu$. Hence, there exists a y-constant function $C(x)$ such that $\mathcal{W}(x,y)+y\cdot\nabla U(x)=C(x)$ on Γ; but exploiting the Y-periodicity of $\mathcal{W}$ and taking into account the geometrical setting, we get $\nabla U=0$ in Ω and $W(x,y)=C(x)$ on Γ. Therefore, since U vanishes on $\partial\Omega$, it follows that $U=0$ in Ω. Moreover, $\nabla_y^B\mathcal{W}=0$ on Γ. Hence, taking into account that $\overline{u}_{0|\partial\Omega}=0$ and that a pair satisfying (4.39) (that is a pair $(\overline{u}_0,w)$, with $\nabla_y^B w=0$) is a solution of (4.13)–(4.14), the assertion follows.

Lemma 4.7. *The matrix C^0 is symmetric. Moreover, if we are in the connected/connected case, then the matrix C^0 is also positive definite.*

Proof. Taking into account (4.19), let us compute

$$\alpha \int_\Gamma \nabla_y^B(\chi_0^h + y_h)\cdot\nabla_y^B(\chi_0^j + y_j)\,\mathrm{d}\sigma$$
$$= \alpha\int_\Gamma \nabla_y^B(\chi_0^h + y_h)\cdot\nabla_y^B\chi_0^j\,\mathrm{d}\sigma + \alpha\int_\Gamma \nabla_y^B(\chi_0^h + y_h)\cdot\nabla_y^B y_j\,\mathrm{d}\sigma$$
$$= \alpha\int_\Gamma \nabla_y^B\chi_0^h\cdot(\mathbf{e}_j - \nu_j\nu)\,\mathrm{d}\sigma + \alpha\int_\Gamma(\mathbf{e}_h - \nu_h\nu)\cdot(\mathbf{e}_j - \nu_j\nu)\,\mathrm{d}\sigma$$
$$= \alpha\int_\Gamma \left(\nabla_y^B\chi_0^h\right)_j\,\mathrm{d}\sigma + \alpha\int_\Gamma(\delta_{hj} - \nu_h\nu_j)\,\mathrm{d}\sigma = C^0_{hj}\,, \quad (4.42)$$

where the last equality follows from definition (4.37). Hence, the symmetry of the matrix C^0 is proved. In order to prove the positive definiteness, we calculate

$$\sum_{h,j=1}^N C^0_{hj}\xi_h\xi_j = \alpha\int_\Gamma \sum_{h,j=1}^N \nabla_y^B(\chi_0^h\xi_h + y_h\xi_h)\cdot\nabla_y^B(\chi_0^j\xi_j + y_j\xi_j)\,\mathrm{d}\sigma$$
$$= \alpha\int_\Gamma \Big|\sum_{h=1}^N \nabla_y^B(\chi_0^h\xi_h + y_h\xi_h)\Big|^2\,\mathrm{d}\sigma = \alpha\int_\Gamma\left|\nabla_y^B\left(\sum_{h=1}^N(\chi_0^h\xi_h + y_h\xi_h)\right)\right|^2\,\mathrm{d}\sigma \ge 0\,. \quad (4.43)$$

Assume, by contradiction, that the last integral is equal to zero for a nonzero vector $\xi = (\xi_1, \dots, \xi_N)$; this implies that, a.e. on Γ,

$$\nabla_y^B\left(\sum_{h=1}^N(\chi_0^h\xi_h + y_h\xi_h)\right) = 0\,, \qquad \text{which implies that} \qquad \sum_{h=1}^N(\chi_0^h(y)\xi_h + y_h\xi_h) = C\,,$$

for a suitable constant C. However, in this case we have

$$C - \sum_{h=1}^N \chi_0^h(y)\xi_h = \sum_{h=1}^N y_h\xi_h\,, \quad (4.44)$$

which leads to a contradiction, since the left-hand side of (4.44) is a periodic function on Γ while the right-hand side is not because of our geometrical assumptions. Hence, the last inequality in (4.43) is actually strict and, by standard arguments, this is enough to prove that the homogenized matrix is positive definite. □

Lemma 4.8. *If we are in the connected/disconnected case, then the matrix $C^0 = 0$.*

Proof. Equation (4.17) implies that, up to an additive constant, $\chi_0^j = -y_j$ on each connected component of $E_{\mathrm{int}} \cup \Gamma$ and hence the matrix $C^0 = 0$. □

Remark 4.9. In the connected/disconnected case, the previous Lemma 4.8 implies that the principal part of equation (4.33) becomes

$$-\operatorname{div}\left(\Big(\lambda_0 I + \int_Y \lambda \nabla_y \chi_0 \,\mathrm{d}y + \alpha \int_\Gamma \nabla_y^B \chi_1(y,0)\,\mathrm{d}\sigma\Big)\nabla u\right) = -\operatorname{div}\left(\Big(\lambda_0 I + A^0\Big)\nabla u\right),$$

while this is not the case when $\Omega^\varepsilon_{\mathrm{int}}$ and $\Omega^\varepsilon_{\mathrm{out}}$ are both connected. Moreover, since $\nabla_y \chi_0^j = -\boldsymbol{e}_j$ on E_{int}, we may rewrite the homogenized matrix in the form

$$\begin{aligned}
&\lambda_0 I + \int_Y \lambda \nabla_y \chi_0 \,\mathrm{d}y + \alpha \int_\Gamma \nabla_y^B \chi_1(y,0)\,\mathrm{d}\sigma \\
&= \lambda_{\mathrm{int}}|E_{\mathrm{int}}|I + \lambda_{\mathrm{out}}|E_{\mathrm{out}}|I + \lambda_{\mathrm{out}} \int_{E_{\mathrm{out}}} \nabla_y \chi_0 \,\mathrm{d}y - \lambda_{\mathrm{int}}|E_{\mathrm{int}}|I + \alpha \int_\Gamma \nabla_y^B \chi_1(y,0)\,\mathrm{d}\sigma \\
&= \int_{E_{\mathrm{out}}} \lambda_{\mathrm{out}}(I + \nabla_y \chi_0)\,\mathrm{d}y + \alpha \int_\Gamma \nabla_y^B \chi_1(y,0)\,\mathrm{d}\sigma \\
&= \lambda_{\mathrm{out}}|E_{\mathrm{out}}|I + \lambda_{\mathrm{out}} \int_\Gamma y \otimes \nu \,\mathrm{d}\sigma + \alpha \int_\Gamma \nabla_y^B \chi_1(y,0)\,\mathrm{d}\sigma\,. \qquad (4.45)
\end{aligned}$$

□

Remark 4.10. We note that in the case of a layered geometry, where the layers are for instance transversal to the direction $\boldsymbol{e}_h$, a similar argument as the one in Lemma 4.8 leads to prove that the matrix C^0 is not identically equal to zero, but it degenerates in the direction $\boldsymbol{e}_h$, since in this case we have $\chi_0^h(y) = -y_h$ in E_{int} (up to an additive constant). □

Lemma 4.11. *The matrix $\lambda_0 I + A^0$ is symmetric and positive definite.*

Proof. Taking into account (4.17), let us compute

$$0 = -\int_Y \operatorname{div}_y\big(\lambda \nabla_y (\chi_0^j + y_j)\big)\chi_0^h \,\mathrm{d}y = \int_Y \lambda \nabla_y (\chi_0^j + y_j)\cdot \nabla_y \chi_0^h \,\mathrm{d}y + \int_\Gamma [\lambda \nabla_y (\chi_0^j + y_j)\cdot \nu]\chi_0^h \,\mathrm{d}\sigma\,. \qquad (4.46)$$

Moreover, by (4.19), it follows that

$$0 = -\alpha \int_\Gamma \Delta_y^B (\chi_0^h + y_h)\chi_1^j(y,0)\,\mathrm{d}\sigma = \alpha \int_\Gamma \nabla_y^B (\chi_0^h + y_h)\cdot \nabla_y^B \chi_1^j(y,0)\,\mathrm{d}\sigma\,,$$

i.e.

$$
\begin{aligned}
-\alpha \int_{\Gamma} \nabla_y^B \chi_1^j(y,0) \cdot \nabla_y^B \chi_0^h \, \mathrm{d}\sigma &= \alpha \int_{\Gamma} \nabla_y^B \chi_1^j(y,0) \cdot \nabla_y^B y_h \, \mathrm{d}\sigma \\
&= \alpha \int_{\Gamma} \nabla_y^B \chi_1^j(y,0) \cdot (\mathbf{e}_h - \nu_h \nu) \, \mathrm{d}\sigma = \alpha \int_{\Gamma} \Big(\nabla_y^B \chi_1^j(y,0) \Big)_h \, \mathrm{d}\sigma \, . \quad (4.47)
\end{aligned}
$$

Next, by (4.27), we get

$$
\begin{aligned}
0 = -\alpha \int_{\Gamma} \Delta_y^B \chi_1^j(y,0) \chi_0^h \, \mathrm{d}\sigma - \int_{\Gamma} [\lambda \nabla_y (\chi_0^j + y_j) \cdot \nu] \chi_0^h \, \mathrm{d}\sigma \\
= \alpha \int_{\Gamma} \nabla_y^B \chi_1^j(y,0) \cdot \nabla_y^B \chi_0^h \, \mathrm{d}\sigma + \int_{Y} \lambda \nabla_y (\chi_0^j + y_j) \cdot \nabla_y \chi_0^h \, \mathrm{d}y \quad (4.48)
\end{aligned}
$$

where, in the last equality, we used (4.46). Finally, we check directly

$$
\int_{Y} \lambda \nabla_y (\chi_0^j + y_j) \cdot \nabla_y y_h \, \mathrm{d}y = \int_{Y} \lambda \left(\frac{\partial \chi_0^j}{\partial y_h} + \delta_{hj} \right) \mathrm{d}y \, . \quad (4.49)
$$

Then, by (4.47)–(4.49), we obtain

$$
\begin{aligned}
\int_{Y} \lambda \nabla_y (\chi_0^j + y_j) \cdot \nabla_y (\chi_0^h + y_h) \, \mathrm{d}y &= \int_{Y} \lambda \nabla_y (\chi_0^j + y_j) \cdot \nabla_y \chi_0^h \, \mathrm{d}y + \int_{Y} \lambda \nabla_y (\chi_0^j + y_j) \cdot \nabla_y y_h \, \mathrm{d}y \\
&= -\alpha \int_{\Gamma} \nabla_y^B \chi_1^j(y,0) \cdot \nabla_y^B \chi_0^h \, \mathrm{d}\sigma + \lambda_0 \delta_{hj} + \int_{Y} \lambda \frac{\partial \chi_0^j}{\partial y_h} \, \mathrm{d}y \\
&= \alpha \int_{\Gamma} \Big(\nabla_y^B \chi_1^j(y,0) \Big)_h \, \mathrm{d}\sigma + \lambda_0 \delta_{hj} + \int_{Y} \lambda \frac{\partial \chi_0^j}{\partial y_h} \, \mathrm{d}y = \lambda_0 \delta_{hj} + A_{jh}^0 \, , \quad (4.50)
\end{aligned}
$$

where, in the last equality, we recall (4.35). Therefore, the symmetry of the matrix $\lambda_0 I + A^0$ is proved.

In order to prove its positive definiteness, we proceed as follows: setting $\lambda_{min} = \min(\lambda_{\text{int}}, \lambda_{\text{out}})$ and using Jensen's inequality, we obtain

$$
\begin{aligned}
&\sum_{h,j=1}^{N} (\lambda_0 I + A^0)_{hj}\xi_h\xi_j = \int_Y \lambda \sum_{h,j=1}^{N} \nabla_y(\chi_0^h\xi_h + y_h\xi_h)\cdot\nabla_y(\chi_0^j\xi_j + y_j\xi_j)\,\mathrm{d}y \\
&\geq \lambda_{min}\int_Y \Big|\sum_{h=1}^{N}\nabla_y(\chi_0^h\xi_h + y_h\xi_h)\Big|^2 \mathrm{d}y \geq \lambda_{min}\left|\int_Y \sum_{h=1}^{N}\nabla_y(\chi_0^h\xi_h + y_h\xi_h)\,\mathrm{d}y\right|^2 \\
&= \lambda_{min}\sum_{j=1}^{N}\left(\sum_{h=1}^{N}(\xi_h\int_Y \frac{\partial\chi_0^h}{\partial y_j}\,\mathrm{d}y + \delta_{hj}\xi_h)\right)^2 \\
&= \lambda_{min}\sum_{j=1}^{N}\left(\sum_{h=1}^{N}\xi_h\int_{\partial Y}\chi_0^h\, n_j\,\mathrm{d}\sigma + \xi_j\right)^2 = \lambda_{min}|\xi|^2
\end{aligned}
\tag{4.51}
$$

where we have denoted by $n = (n_1, \dots, n_N)$ the outward unit normal to ∂Y. Indeed, we remark that the last integral vanishes because of the periodicity of the cell function χ_0^h. This proves that the homogenized matrix $\lambda_0 I + A^0$ is positive definite and concludes the lemma. □

Remark 4.12. Notice that in the connected-connected or connected-disconnected geometry, since C^0 and/or A^0 are positive definite matrices, problem (2.41) is well-posed (see [6] for the case $C^0 = 0$, while the case $C^0 \neq 0$ will be treated in a forthcoming paper). □

5. Other scalings

In this section, we will consider the homogenization of our microscopic problem (2.44)–(2.48), whose weak formulation is the following

$$
\begin{aligned}
&\int_0^T\!\!\int_\Omega \lambda^\varepsilon \nabla u_\varepsilon\cdot\nabla\Phi\,\mathrm{d}x\,\mathrm{d}t - \varepsilon^k\alpha\int_0^T\!\!\int_{\Gamma^\varepsilon}\nabla^B u_\varepsilon\cdot\nabla^B\Phi_t\,\mathrm{d}\sigma\,\mathrm{d}t \\
&\qquad = \varepsilon^k\alpha\int_{\Gamma^\varepsilon}\nabla^B\overline{u}_{0\varepsilon}\cdot\nabla^B\Phi(x,0)\,\mathrm{d}\sigma + \int_0^T\!\!\int_\Omega f\Phi\,\mathrm{d}x\,\mathrm{d}t\,,
\end{aligned}
\tag{5.1}
$$

for every test function $\Phi \in \mathcal{C}^\infty(\overline{\Omega}_T)$ such that Φ has compact support in Ω for every $t \in [0, T)$ and $\Phi(\cdot, T) = 0$ in Ω. The corresponding energy estimate is

$$
\int_0^T\!\!\int_\Omega |\nabla u_\varepsilon|^2\,\mathrm{d}x\,\mathrm{d}\tau + \varepsilon^k \sup_{t\in(0,T)}\int_{\Gamma^\varepsilon}|\nabla^B u_\varepsilon|^2\,\mathrm{d}\sigma \leq \gamma\,,
\tag{5.2}
$$

where we used the assumption $\overline{u}_{0\varepsilon} = \varepsilon^{(1-k)/2}\overline{u}_0$, with $\overline{u}_0 \in H_0^1(\Omega)\cap H^2(\Omega)$.

In Section 4 we have studied the case $k = 1$, which seems to be the most physical one, since it is the only case where the limit problem keeps memory of the physical properties of the active membranes. Here, we will consider the two cases $k > 1$ and $k < 1$, respectively, where no memory of α remains in the homogenized equation. However, in some cases, memory of the geometry or of $\lambda_{\rm int}$ is kept. In particular, when $k < 1$ and we are in the connected/connected case, the limit solution is identically equal to 0, even if the initial datum and the source are not null; notice that this is not the case in the connected/disconnected geometry.

5.1. **Case $k < 1$: proof of Theorems 2.16 and 2.17.** When $k < 1$, from the energy estimate (5.2), we obtain that the estimate (4.1) is satisfied as well, so that all the results in Lemma 4.3 hold. In particular, we still obtain that $\mathcal{T}_\varepsilon(\nabla u_\varepsilon) \rightharpoonup \nabla u + \nabla_y w$ weakly in $L^2(\Omega_T \times Y)$ and $\mathcal{T}^b_\varepsilon(\nabla^B u_\varepsilon) \rightharpoonup \nabla^B u + \nabla^B_y w$ weakly in $L^2(\Omega_T \times \Gamma)$.
Moreover, let us take in the weak formulation (5.1) a test function of the type $\Phi(x,t) = \varepsilon\varphi(x,t)\psi(\varepsilon^{-1}x)$, with $\varphi \in \mathcal{C}^\infty(\overline{\Omega}_T)$ such that φ has compact support in Ω for every $t \in [0,T)$, $\varphi(\cdot,T) = 0$ in Ω and $\psi \in \mathcal{C}^\infty_\#(Y)$ with $supp(\psi) \subset\subset E_{\rm int} \cup E_{\rm out}$. Unfolding and then passing to the limit for $\varepsilon \to 0$, we obtain

$$\int_0^T \int_{\Omega\times Y} \lambda(\nabla u + \nabla_y w)\cdot \nabla_y\psi\varphi \,\mathrm{d}y\,\mathrm{d}x\,\mathrm{d}t = 0\,,$$

i.e.,

$$-\operatorname{div}_y\big(\lambda(\nabla u + \nabla_y w)\big) = 0\,, \qquad \text{in } E_{\rm int}\cup E_{\rm out}, \tag{5.3}$$

with $w \in L^2(\Omega_T; H^1_\#(Y))$. Next, take in the weak formulation (5.1) a test function of the type $\Phi(x,t) = \varepsilon^{2-k}\varphi(x,t)\psi(\varepsilon^{-1}x)$, with φ as before and $\psi \in \mathcal{C}^\infty_\#(Y)$. Unfolding and passing to the limit for $\varepsilon \to 0$, we obtain, owing to our assumption $\overline{u}_{0\varepsilon} = \varepsilon^{(1-k)/2}\overline{u}_0$,

$$\alpha \int_0^T \int_{\Omega\times\Gamma} (\nabla^B u + \nabla^B_y w)\cdot\nabla^B_y\psi\varphi_t \,\mathrm{d}\sigma\,\mathrm{d}x\,\mathrm{d}t = 0\,,$$

i.e.,

$$\begin{aligned} &-\alpha\operatorname{div}^B_y(\nabla^B u_t + \nabla^B_y w_t) = 0\,, &&\text{on } \Gamma\times(0,T),\\ &-\alpha\operatorname{div}^B_y(\nabla^B u(0) + \nabla^B_y w(0)) = 0\,, &&\text{on } \Gamma,\end{aligned}$$

which gives

$$-\alpha\operatorname{div}^B_y(\nabla^B u + \nabla^B_y w) = 0\,, \qquad \text{on } \Gamma\times(0,T).$$

This implies that we can factorize $w(x,t,y) = \chi^j_0(y)\partial_j u(x,t)$, where χ^j_0 satisfies (4.17)–(4.19) and (4.21). Moreover,condition (4.20) is automatically satisfied in the connected/connected case, since Γ has only one connected component, while it is satisfied in the connected/disconnected case thanks to [15, Proposition 2.6].

Now we have to proceed separately in the two cases, since we have to consider different test functions. Indeed, the test function which we will use in the connected/connnected case will not give any information in the connected/disconnected case (see Remark 5.2), while the test function which we will use in this last case cannot be constructed in the connected/connected case.

Proof of Theorem 2.16. Let us take in the weak formulation (5.1) a test function of the type $\Phi(x,t) = \varepsilon^{1-k}\varphi(x,t)$, with $\varphi \in \mathcal{C}^\infty(\overline{\Omega}_T)$ such that φ has compact support in Ω for every $t \in [0,T)$, $\varphi(\cdot,T) = 0$ in Ω. Unfolding and passing to the limit for $\varepsilon \to 0$, we obtain

$$-\alpha \int_0^T \int_{\Omega\times\Gamma} (\nabla^B u + \nabla_y^B w) \cdot \nabla^B \varphi_t \,\mathrm{d}x\,\mathrm{d}\sigma\,\mathrm{d}t = 0\,,$$

i.e.,

$$\begin{aligned} &-\alpha \operatorname{div} \int_\Gamma (\nabla^B u_t + \nabla_y^B w_t)\,\mathrm{d}\sigma = 0\,, && \text{on } \Omega_T, \\ &-\alpha \operatorname{div} \int_\Gamma (\nabla^B u(0) + \nabla_y^B w(0)) = 0\,, && \text{on } \Omega, \end{aligned}$$

which gives

$$-\alpha \operatorname{div}\left(\int_\Gamma (\nabla^B u + \nabla_y^B w)\,\mathrm{d}\sigma \right) = 0\,, \qquad \text{on } \Omega_T.$$

Inserting in the previous equation the factorization of w in terms of the cell functions, we obtain

$$-\operatorname{div}\left(\big(\alpha \int_\Gamma (I - \nu\otimes\nu + \nabla_y^B \chi_0)\,\mathrm{d}\sigma\big)\nabla u \right) = 0\,, \tag{5.4}$$

which, recalling (4.37), can be rewritten as $-\operatorname{div}(C^0\nabla u) = 0$. Taking into account that $u_{|\partial\Omega} = 0$ and that in the connected/connected case the matrix C^0 is positive definite by Lemma 4.7, it follows that $u \equiv 0$ in Ω, so that the whole sequence $\{u_\varepsilon\}$ converges to zero.

□

Remark 5.1. Notice that the previous result holds true though a non-zero source term appears in (2.44). Moreover, according to (2.14) and from the energy estimate (5.2), we also get

$$\int_0^T \int_{\Omega\times\Gamma} |\mathcal{T}_\varepsilon^b(\nabla^B u_\varepsilon)|^2 \,\mathrm{d}\sigma\,\mathrm{d}t \le \varepsilon \int_0^T \int_{\Gamma^\varepsilon} |\nabla^B u_\varepsilon|^2 \,\mathrm{d}\sigma\,\mathrm{d}t \le \gamma\varepsilon^{1-k} \to 0\,, \qquad \text{for } \varepsilon \to 0,$$

which gives a rate of convergence to zero of $\|\mathcal{T}_\varepsilon^b(\nabla^B u_\varepsilon)\|_{L^2(\Omega_T\times\Gamma)}$. Hence, passing to the limit and taking into account the lower semicontinuity of the norm with respect to the weak convergence, it follows that

$$\|\nabla^B u + \nabla_y^B w\|_{L^2(\Omega_T\times\Gamma)} \le \liminf_{\varepsilon\to 0} \|\mathcal{T}_\varepsilon^b(\nabla^B u_\varepsilon)\|_{L^2(\Omega_T\times\Gamma)} = 0\,. \tag{5.5}$$

□

Remark 5.2. Notice that when we are in the connected/disconnected geometrical setting, as pointed out in Lemma 4.8, $\chi_0^j = -y_j$, up to an additive constant on each connected component of $E_{\text{int}} \cup \Gamma$, so that the matrix $C^0 = 0$. Hence, equations (5.4) and (5.5) do not give any information on u. □

Proof of Theorem 2.17. As in [21, Proof of Lemma 4.1], let us take in the weak formulation (5.1) a test function of the type $\Phi(x,t) = \varphi(x,t)\big(1-\psi(\varepsilon^{-1}x)\big)+\mathcal{M}^\varepsilon(\varphi)\psi(\varepsilon^{-1}x)$, with $\varphi \in \mathcal{C}^\infty(\overline{\Omega}_T)$ such that φ has compact support in Ω for every $t \in [0,T)$, $\varphi(\cdot,T) = 0$ in Ω, and $\psi \in \mathcal{C}_c^\infty(Y)$, with $\psi = 1$ on $E_{\text{int}} \cup \Gamma$. Thus we obtain

$$\int_0^T\int_\Omega \lambda^\varepsilon \nabla u_\varepsilon \cdot \big\{\nabla\varphi(1-\psi) + \frac{1}{\varepsilon}(\mathcal{M}^\varepsilon(\varphi)-\varphi)\nabla_y\psi\big\}\,\mathrm{d}x\,\mathrm{d}t = \int_0^T\int_\Omega f\big\{\varphi(1-\psi)+\mathcal{M}^\varepsilon(\varphi)\psi\big\}\,\mathrm{d}x\,\mathrm{d}t\,.$$

Then, unfolding and passing to the limit for $\varepsilon \to 0$, it follows

$$\int_0^T\int_\Omega\int_Y \lambda(\nabla u + \nabla_y w)\cdot\big\{\nabla\varphi(1-\psi) - (y_M\cdot\nabla\varphi)\nabla_y\psi\big\}\,\mathrm{d}x\,\mathrm{d}y\,\mathrm{d}t = \int_0^T\int_\Omega f\varphi\,\mathrm{d}x\,\mathrm{d}t\,, \tag{5.6}$$

where we have taken into account that $\varphi(1-\psi)+\mathcal{M}^\varepsilon(\varphi)\psi \to \varphi$, strongly in $L^2(\Omega_T)$ and (2.11) holds. Notice that, by the identity $\nabla\varphi\psi+(y_M\cdot\nabla\varphi)\nabla_y\psi = \nabla_y\big((y_M\cdot\nabla\varphi)\psi\big)$, equality (5.6) can be rewritten in the form

$$\int_0^T\int_\Omega\int_Y \lambda(\nabla u+\nabla_y w)\cdot\nabla\varphi\,\mathrm{d}x\,\mathrm{d}y\,\mathrm{d}t - \int_0^T\int_\Omega\int_Y \lambda(\nabla u+\nabla_y w)\nabla_y\big((y_M\cdot\nabla\varphi)\psi\big)\,\mathrm{d}x\,\mathrm{d}y\,\mathrm{d}t = \int_0^T\int_\Omega f\varphi\,\mathrm{d}x\,\mathrm{d}t\,,$$

which becomes

$$\int_0^T\int_\Omega\int_Y \lambda(\nabla u+\nabla_y w)\cdot\nabla\varphi\,\mathrm{d}x\,\mathrm{d}y\,\mathrm{d}t + \int_0^T\int_\Omega\int_\Gamma [\lambda(\nabla u+\nabla_y w)\cdot\nu](y_M\cdot\nabla\varphi)\,\mathrm{d}x\,\mathrm{d}\sigma\,\mathrm{d}t = \int_0^T\int_\Omega f\varphi\,\mathrm{d}x\,\mathrm{d}t\,,$$

as a consequence of (5.3). Finally, taking into account the factorization of $w(x,t,y) = \chi_0^j(y)\partial_j u(x,t)$, with χ_0 satisfying (4.17)–(4.21) and recalling that in the connected/disconnected case, $\nabla_y \chi_0(y) = -I$ on E_{int}, we obtain the homogenized equation

$$-\operatorname{div}\left(\Big(\int_{E_{\mathrm{out}}}\lambda_{\mathrm{out}}(I+\nabla_y\chi_0)\,\mathrm{d}y + \int_{\Gamma}\lambda_{\mathrm{out}}\Big((\nu+(\nabla_y\chi_0)^{\mathrm{out}}\nu)\otimes y_M\Big)\,\mathrm{d}\sigma\Big)\nabla u\right) = f\,.$$

In order to prove that the homogenized matrix

$$A^{\mathrm{hom}} := \int_{E_{\mathrm{out}}}\lambda_{\mathrm{out}}(I+\nabla_y\chi_0)\,\mathrm{d}y + \int_{\Gamma}\lambda_{\mathrm{out}}\big((\nu+(\nabla_y\chi_0)^{\mathrm{out}}\nu)\otimes y_M\big)\,\mathrm{d}\sigma$$

is symmetric and positive definite, we proceed as follows. By (4.46), we get

$$\begin{aligned}
&\int_Y \lambda\nabla_y(\chi_0^j+y_j)\cdot\nabla_y(\chi_0^h+y_h)\,\mathrm{d}y \\
&\qquad= \int_Y \lambda\nabla_y(\chi_0^j+y_j)\cdot\nabla_y\chi_0^h\,\mathrm{d}y + \int_Y \lambda\nabla_y(\chi_0^j+y_j)\cdot\nabla_y y_h\,\mathrm{d}y \\
&\qquad= -\int_\Gamma [\lambda\nabla_y(\chi_0^j+y_j)\cdot\nu]\chi_0^h\,\mathrm{d}\sigma + \int_Y \lambda\Big(\frac{\partial\chi_0^j}{\partial y_h}+\delta_{hj}\Big)\,\mathrm{d}y \\
&= \sum_{i=1}^m \int_{\Gamma_i}\lambda_{\mathrm{out}}\big((\nabla_y\chi_0^j)^{\mathrm{out}}+\boldsymbol{e}_j\big)\cdot\nu(y_h+c_i)\,\mathrm{d}\sigma + \int_{E_{\mathrm{out}}}\lambda_{\mathrm{out}}\Big(\frac{\partial\chi_0^j}{\partial y_h}+\delta_{hj}\Big)\,\mathrm{d}y \\
&= \int_\Gamma \lambda_{\mathrm{out}}\big((\nabla_y\chi_0^j)^{\mathrm{out}}\cdot\nu+\nu_j\big)(y_M)_h\,\mathrm{d}\sigma + \int_{E_{\mathrm{out}}}\lambda_{\mathrm{out}}\Big(\frac{\partial\chi_0^j}{\partial y_h}+\delta_{hj}\Big)\,\mathrm{d}y = A^{\mathrm{hom}}_{jh}\,,
\end{aligned}$$

where Γ_i, $i=1,\dots,m$, are the connected component of Γ and, in the fourth equality, y_h+c_i has been replaced with $(y_M)_h$, since we have taken into account that $\int_\Gamma \nu_j\,\mathrm{d}\sigma = 0$ and (4.20) holds. This proves the symmetry; the positive definiteness now follows directly from (4.51). Hence, u is uniquely determined, which implies that the whole sequence $\{u_\varepsilon\}$ converges.

□

Remark 5.3. Notice that the homogenized solution u does not depend on α nor on λ_{int}; i.e., it does not depend on the physical properties of the interface and of the inclusions, being affected only by the physical properties of the surrounding matrix. □

5.2. **Case $k>1$: proof of Theorem 2.18.** Setting $v_\varepsilon = \varepsilon^{(k-1)/2}u_\varepsilon$, from the energy estimate (5.2) it follows that $\mathcal{T}_\varepsilon^b(\nabla^B v_\varepsilon)$ is bounded in $L^2(\Omega_T\times\Gamma)$ and, up to a subsequence, $\mathcal{T}_\varepsilon(\nabla u_\varepsilon)\rightharpoonup \nabla u+\nabla_y w$ weakly in $L^2(\Omega_T\times Y)$. Then, taking in the weak formulation (5.1) a test function of the type $\Phi(x,t)=\varphi(x,t)+\varepsilon\phi(x,t)\psi(x,\varepsilon^{-1}x,t)$ with $\varphi,\phi\in\mathcal{C}^\infty(\Omega_T)$ such that φ,ϕ have compact support in Ω for every $t\in[0,T)$,

$\varphi(\cdot,T)=\phi(\cdot,T)=0$ in Ω and $\psi\in\mathcal{C}^\infty_\#(Y)$, unfolding and then passing to the limit, using also the boundedness of $\mathcal{T}^b_\varepsilon(\nabla^B v_\varepsilon)$, we obtain

$$\int_0^T\int_{\Omega\times Y}\lambda(\nabla u+\nabla_y w)\cdot(\nabla\varphi+\nabla_y\psi\,\phi)\,\mathrm{d}x\,\mathrm{d}y\,\mathrm{d}t=\int_0^T\int_{\Omega\times Y}f\varphi\,\mathrm{d}x\,\mathrm{d}t\,,$$

which gives

$$-\operatorname{div}\left(\int_Y\lambda(\nabla u+\nabla_y w)\,\mathrm{d}y\right)=f\,,$$
$$-\operatorname{div}_y\big(\lambda(\nabla u+\nabla_y w)\big)=0\,.$$

Hence, we are led to the standard homogenized two-scale system which can be obtained in the case of perfect contact (i.e., when $\alpha=0$ in (2.46)). Here, the time-dependence is only parametric through the source f.
Factorizing $w(x,t,y)=\widetilde{\chi}^j_0(y)\partial_j u(x,t)$ and inserting in the previous set of equations, it follows that the homogenized function u is the solution of the problem

$$-\operatorname{div}\left(\big(\int_Y\lambda(I+\nabla_y\widetilde{\chi}_0)\,\mathrm{d}y\big)\nabla u\right)=f\,,\qquad\text{in }\Omega_T\,;$$
$$u=0\,,\qquad\text{in }\partial\Omega\times(0,T)\,,$$

where $\widetilde{\chi}^j_0\in H^1_\#(Y)$, $j=1,\dots,N$, are Y-periodic functions with null mean average satisfying

$$-\operatorname{div}_y\big(\lambda\nabla_y(y_j+\widetilde{\chi}^j_0)\big)=0\,,\qquad\text{in }Y,$$

which can be rewritten also in the form

$$-\operatorname{div}_y\big(\lambda\nabla_y(y_j+\widetilde{\chi}^j_0)\big)=0\,,\qquad\text{in }E_{\text{int}}\cup E_{\text{out}};\tag{5.7}$$
$$[\lambda\nabla_y(y_j+\widetilde{\chi}^j_0)\cdot\nu]=0\,,\qquad\text{on }\Gamma.\tag{5.8}$$

Notice that, by standard results, the homogenized matrix is symmetric and positive definite, so that u is uniquely determined, which implies that the whole sequence $\{u_\varepsilon\}$ converges; moreover, the geometrical setting does not play any role and the result holds both in the connected/connected and in the connected/disconnected case.

□

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