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Algorithms for generalized potential games with mixed-integer variables

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Abstract We consider generalized potential games, that constitute a fundamental subclass of generalized Nash equilibrium problems. We propose different methods to compute solutions of generalized potential games with mixed-integer variables, i.e., games in which some variable are continuous while the others are discrete. We investigate which types of equilibria of the game can be computed by minimizing a potential function over the common feasible set. In particular, for a wide class of generalized potential games, we characterize those equilibria that can be computed by minimizing potential functions as Pareto solutions of a particular multi-objective problem, and we show how different potential functions can be used to select equilibria. We propose a new Gauss-Southwell algorithm to compute approximate equilibria of any generalized potential game with mixed-integer variables. We show that this method converges in a finite number of steps and we also give an upper bound on this number of steps. Moreover, we make a thorough analysis on the behaviour of approximate equilibria with respect to exact ones. Finally, we make many numerical experiments to show the viability of the proposed approaches.

Keywords Generalized Nash equilibrium problem · Generalized potential game · Mixed-integer nonlinear problem · Parametric optimization

1 Introduction

Generalized Nash equilibrium problems (GNEPs) are widely used tools to model multi agent systems in many fields [16]. In the past two decades, sev-

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eral algorithms have been proposed for the numerical solution of GNEPs, see e.g. [1,6,12–14,17–19,21,25,27]. But, in spite of the fact that there are many applications in which some or all the variables of the players must be assumed to be integers, all the above-mentioned methods for GNEPs work only if the variables of all the players are continuous. Very recently, some methods have been proposed that can deal with integer variables [30,31], but they are designed only for standard Nash equilibrium problems (NEPs), that are GNEPs in which the feasible region of any player is independent on the other players' variables.

We consider generalized potential games, that constitute a fundamental subclass of GNEPs, see e.g. [7,20,24,32,33,38]. In particular we focus on potential games in which the feasible region of any player depends on the other players' variables, that is, games that are real GNEPs and not simple NEPs. All the proposed methods for these potential GNEPs assume that all the variables are continuous. In particular, in [20] the authors do not assume any convexity of the feasible sets of the players, but a basic condition for the convergence of their algorithms is the inner semicontinuity of the feasible mappings of the players, which is a very strong assumption in the mixed-integer setting, see section 4. We investigate different and new methods to compute solutions of generalized potential games with mixed-integer variables, i.e., games in which some variable are continuous while the others are discrete.

In section 2 we define the generalized potential game with mixed-integer variables and the concept of approximate equilibrium. Moreover, we describe two general applications of generalized potential games in the mixed-integer setting. In particular, in section 2.1 we propose a class of linear games that, thanks to the presence of integer variables, can model alternative choices, fixed costs, precedence constraints, and disjunctive constraints. In section 3 we investigate which types of equilibria of the game can be computed by minimizing a potential function over the common feasible set. Particular attention is given to the subclass of games in which the objective function of any player is independent on the other players' variables (note that all linear games belong to this class). For this subclass of games, we characterize those equilibria that can be computed by minimizing potential functions as Pareto solutions of a particular multi-objective problem. And, moreover, we show how different potential functions can be defined to select equilibria. In section 4 we propose a new Gauss-Southwell algorithm to compute approximate equilibria of the generalized potential game with mixed-integer variables. We show that this method converges in a finite number of steps and we also give an upper bound on this number of steps. Moreover, we make a thorough analysis on the behaviour of approximate equilibria with respect to exact ones. We prove that any sequence of approximate equilibria, as the approximating parameter goes to zero, always converges to an exact equilibrium in the completely continuous (if some mild assumptions hold) and the completely discrete settings, and also in the mixed-integer setting when the game is a NEP. This analysis is valid for all GNEPs with mixed-integer variables, and not only for generalized potential games. Finally, in section 5 we make many numerical experiments. In section

5.1 we consider an application in economics and we compute different equilibria by using different potential functions to show how an effective equilibrium selection can be obtained in potential games. In section 5.2 we consider the counter-example given in [20] and we show that, in spite of the fact that our Gauss-Southwell algorithm does not exploit any regularization, it effectively computes approximate equilibria of the potential game. In section 5.3 we use our Gauss-Southwell algorithm to compute equilibria of a discrete flow control problem on networks.

2 Problem description and examples

Consider a GNEP with N players. Let ν be a generic player of the game. We denote by $x^\nu \in \mathbb{R}^{n_\nu}$ the vector representing the private strategies of player ν and by $\mathbf{x}^{-\nu} \triangleq (x^{\nu'})_{\nu \neq \nu'=1}^N$ the vector of all the other players strategies. We write $\mathbb{R}^n \ni \mathbf{x} \triangleq (x^\nu, \mathbf{x}^{-\nu})$, where $n \triangleq n_1+\cdots+n_N$, to indicate the vector of all the strategies of the game. Any player ν must solve the following parametric optimization problem

$$\begin{aligned} &\min_{x^\nu} \theta_\nu(x^\nu, \mathbf{x}^{-\nu}) \\ &(x^\nu, \mathbf{x}^{-\nu}) \in X \\ &x_j^\nu \in \mathbb{Z}, \quad j = 1, \ldots, i_\nu, \end{aligned} \tag{1}$$

where $\theta_\nu : \mathbb{R}^n \to \mathbb{R}$ is continuously differentiable and convex with respect to x^ν, $X \subseteq \mathbb{R}^n$ is convex and compact, $i_\nu \leq n_\nu$ is a nonnegative integer, and the feasible set

$$\Omega \triangleq \{\mathbf{x} \in X \,:\, x_j^\nu \in \mathbb{Z},\, j = 1, \ldots, i_\nu,\, \nu = 1, \ldots, N\}$$

is nonempty. Moreover, we assume that a continuous (not necessarily convex) function $P : \mathbb{R}^n \to \mathbb{R}$ exists such that for all $\nu \in \{1, \ldots, N\}$ and all $(x^\nu, \mathbf{x}^{-\nu}), (y^\nu, \mathbf{x}^{-\nu}) \in X$:

$$\theta_\nu(x^\nu, \mathbf{x}^{-\nu}) < \theta_\nu(y^\nu, \mathbf{x}^{-\nu}) \qquad \Longleftrightarrow \qquad P(x^\nu, \mathbf{x}^{-\nu}) < P(y^\nu, \mathbf{x}^{-\nu}).$$

We say that P is an ordinal potential function for the game, and we say that the described GNEP is a generalized potential game with mixed-integer variables. Here the term "generalized" must be intended to be related to the fact that we are considering real GNEPs and not simple NEPs.

All in all, the GNEP considered in this work enjoys some peculiarities: (i) since P exists, then it is an ordinal potential game [23], (ii) all the variables must satisfy the so-called "Rosen's law" [29], that is, $\mathbf{x} \in X$, and (iii) some variables are subjected to integrality constraints.

Roughly speaking, "a generalized potential game is a GNEP where the players are (unknowingly) minimizing the same function and where the feasible set of each player is the section of a larger set in the product space $\mathbb{R}^n$", cit.

[20]. For such games, a (continuous) exact potential function $\overline{P} : \mathbb{R}^n \to \mathbb{R}$ may exist: for all $\nu \in \{1, \dots, N\}$ and all $(x^\nu, \mathbf{x}^{-\nu}), (y^\nu, \mathbf{x}^{-\nu}) \in X$:

$$\theta_\nu(x^\nu, \mathbf{x}^{-\nu}) - \theta_\nu(y^\nu, \mathbf{x}^{-\nu}) = \overline{P}(x^\nu, \mathbf{x}^{-\nu}) - \overline{P}(y^\nu, \mathbf{x}^{-\nu}).$$

Clearly any exact potential function is an ordinal potential one.

Let us introduce the best response set for player ν at $\overline{\mathbf{x}} \in \Omega$:

$$\widehat{x}^\nu(\overline{\mathbf{x}}^{-\nu}) \triangleq \arg\min_{x^\nu} \theta_\nu(x^\nu, \overline{\mathbf{x}}^{-\nu}), \quad \text{s.t. } (x^\nu, \overline{\mathbf{x}}^{-\nu}) \in \Omega. \tag{2}$$

Computing an element of the best response set requires, in general, the solution of a mixed-integer nonlinear problem (MINLP), see e.g. [3,4,26,36].

Given $\varepsilon \geq 0$, we say that $\overline{\mathbf{x}} \in \Omega$ is an ε-approximate equilibrium if, for all $\nu \in \{1, \dots, N\}$, it holds that

$$\theta_\nu(\overline{x}^\nu, \overline{\mathbf{x}}^{-\nu}) - \theta_\nu(\widehat{x}^\nu, \overline{\mathbf{x}}^{-\nu}) \leq \varepsilon, \quad \text{with } \widehat{x}^\nu \in \widehat{x}^\nu(\overline{\mathbf{x}}^{-\nu}). \tag{3}$$

If $\varepsilon = 0$ the concept of ε-approximate equilibrium reduces to the classical concept of equilibrium [16]. When $\overline{\mathbf{x}}$ satisfies (3) with $\varepsilon = 0$, we say that it is a 0-approximate equilibrium of the game.

To better understand the modelling power of the mixed-integer setting in potential games, now, we describe two general applications.

2.1 Jointly convex linear GNEPs with mixed-integer variables

Linear GNEPs in a completely continuous setting have been widely studied in [8,9,15,35]. In our mixed-integer framework, the problem solved by any player ν is the following

$$\begin{aligned} &\min\nolimits_{x^\nu} (c^\nu)^T x^\nu \\ &\sum_{\mu=1}^{N} A^\mu x^\mu \leq b \\ &l^\nu \leq x^\nu \leq u^\nu \\ &x_j^\nu \in \mathbb{Z}, \quad j = 1, \dots, i_\nu, \end{aligned} \tag{4}$$

where $c^\nu \in \mathbb{R}^{n_\nu}$, $A^\mu \in \mathbb{M}_{t \times n_\mu}$, for all $\mu = 1, \dots, N$, $b \in \mathbb{R}^t$, and $l^\nu \leq u^\nu \in \mathbb{R}^{n_\nu}$. We observe that in this generalized Nash game all the players share t linear constraints (since the game is a jointly convex GNEP), and that the function $\overline{P}(\mathbf{x}) = \sum_{\mu=1}^N (c^\mu)^T x^\mu$ is an exact potential function for the game.

These generalized potential games with mixed-integer variables considerably extend the classes of systems that can be modeled with continuous linear GNEPs. In that, as it is well known, by using binary variables we can model, e.g., alternative choices, fixed costs, precedence constraints, and disjunctive constraints. Thus the problem solved by any player could be a scheduling, a partitiong, a covering, a facility location, or a p-center problem, just to name a few. Finally, more simply, discrete variables can represent indivisible quantities such as quantity of houses, cars or machines.

In the following example we describe a general production application of jointly convex linear GNEPs with mixed-integer variables.

Example 1 Let us consider a market with N firms. Any firm ν produces n^g_ν different goods and must decide their quantities $q^\nu \in \mathbb{R}^{n^g_\nu}$, with $0 \leq q^\nu \leq u^\nu$. We indicate with $p^\nu, m^\nu, f^\nu \in \mathbb{R}^{n^g_\nu}$ the prices, the marginal costs, and the fixed costs of the goods, respectively. The private part of the feasible set of the firm, which can be defined by technological or economic constraints, is defined by linear inequalities

$$B^\nu q^\nu \leq d^\nu,$$

where $B^\nu \in \mathbb{M}_{l_\nu \times n^g_\nu}$ and $d^\nu \in \mathbb{R}^{l_\nu}$. We suppose that all the firms share a set of r common resources, and then they have the following common constraints

$$\sum_{\mu=1}^{N} D^\mu q^\mu \leq h,$$

where $D^\mu \in \mathbb{M}_{r \times n^g_\mu}$ (D^μ_{ij} indicates the unitary consumption of resource i relative to the jth good of player μ) and $h \in \mathbb{R}^r$ (h_i is the amount of resource i available in the market). Moreover, we assume that an authority imposes k quality constraints on the problem of each firm (e.g. constraints to control the environmental pollution)

$$E^\nu q^\nu \leq s^\nu, \tag{5}$$

where $E^\nu \in \mathbb{M}_{k \times n^g_\nu}$ and $s^\nu \in \mathbb{R}^k$, but only $\overline{k} \in \{1, \ldots, k-1\}$ of these constraints must be satisfied (disjunctive constraints).

To model the fixed costs, we introduce, for each firm ν, the binary variables $\delta^\nu \in \{0,1\}^{n^g_\nu}$, whose values are decided by the firm itself. And we add the following constraints

$$q^\nu - M\delta^\nu \leq 0,$$

where M is a large constant.

To model disjunctive constraints (5), we introduce, for each firm ν, the binary variables $\gamma^\nu \in \{0,1\}^k$, whose values are decided by the firm itself. Then we rewrite constraints (5) as follows

$$E^\nu q^\nu - M\gamma^\nu \leq s^\nu, \qquad \sum_{j=1}^{k} \gamma^\nu_j \leq k - \overline{k},$$

where, again, M is a large constant. Therefore, the optimization problem solved by each firm ν is the following

$$\begin{aligned}
&\min_{\delta^\nu,\gamma^\nu,q^\nu} (f^\nu)^T\delta^\nu + (m^\nu - p^\nu)^T q^\nu \\
&\sum_{\mu=1}^N D^\mu q^\mu \le h \\
&B^\nu q^\nu \le d^\nu \\
&q^\nu - M\delta^\nu \le 0 \\
&E^\nu q^\nu - M\gamma^\nu \le s^\nu, \qquad \sum_{j=1}^k \gamma_j^\nu \le k - \overline{k} \\
&\delta^\nu \in \{0,1\}^{n_\nu^g}, \quad \gamma^\nu \in \{0,1\}^k, \quad 0 \le q^\nu \le u^\nu.
\end{aligned} \tag{6}$$

We observe that problem (6) is a particular instance of problem (4).

As a possible generalization, we can also consider the case in which the k authority constraints are shared by all the firms: $\sum_{\mu=1}^N E^\mu q^\mu \le \sum_{\mu=1}^N s^\mu$. To model these new disjunctive constraints, we introduce, for each firm ν, the binary variables $\gamma^\nu \in \{0,1\}^k$ and $\tau^\nu \in \{0,1\}^k$, whose values are decided by the firm itself. And we rewrite the constraints as follows

$$\begin{aligned}
&\sum_{\mu=1}^N E^\mu q^\mu - M\sum_{\mu=1}^N \tau^\mu \le \sum_{\mu=1}^N s^\mu, \\
&\tau^\nu - \tfrac{1}{N}\textstyle\sum_{\mu=1}^N \gamma^\mu \le 0, \qquad \displaystyle\sum_{j=1}^k \gamma_j^\nu \le k - \overline{k}, \quad \forall\, \nu = 1,\dots,N.
\end{aligned} \tag{7}$$

Roughly speaking, any firm ν can play $k - \overline{k}$ bonuses (variables γ^ν), and only if all the firms play the bonus on the same constraint j then the corresponding variable τ_j^ν, of any firm ν, can be active and then constraint j is omitted.

Certainly, if we assume that all the firms act rationally and have complete information, and there is no explicit collusion, then the Nash equilibrium paradigm fits well within this framework, see e.g. [37]. Moreover, assuming that the objectives are measured in euros or dollars, any approximating parameter $\varepsilon < 0.01$ models a realistic behaviour of the firms. $\square$

2.2 Discrete flow control problems on networks

A general transmission network model based on fluid approximation was presented in [20], here we consider the more realistic case in which the flows are discrete.

In general, we consider a network on which different users independently and simultaneously route discrete flows. Let $\mathcal{V} = \{1,\dots,V\}$ be the set of vertices, and let $\mathcal{L} = \{1,\dots,L\}$ be the set of links of the network. For each user $\nu \in \{1, \dots, N\}$, there is a predetermined path $\mathcal{P}_\nu \subseteq \mathcal{L}$, which is the subset of the links that form the path on which the user route its flow. The discrete flow that the user sends through $\mathcal{P}_\nu$ is denoted by $x^\nu \in \mathbb{Z}$. We introduce the $L \times N$ routing matrix A (defined by $A_{l\nu} = 1$ if $l \in \mathcal{P}_\nu$ and 0 otherwise), the vector of the capacities of the links $c \in \mathbb{R}^L$, and the upper bounds of the flows $\mathbf{u} \in \mathbb{Z}^N$. The optimization problem solved by each user is the following

$$\begin{aligned} \min_{x^\nu} & \sum_{l \in \mathcal{P}_\nu} C_l \left(\sum_{\nu \,:\, l \in \mathcal{P}_\nu} x^\nu \right) - U_\nu(x^\nu) \\ & A\mathbf{x} \leq c \\ & 0 \leq x^\nu \leq u^\nu, \quad x^\nu \in \mathbb{Z}, \end{aligned} \tag{8}$$

where C_l is a cost function relative to the congestion on link l, and U_ν is a utility function for the user. For example, to have the objective functions convex, we can define the functions as follows

$$C_l \left(\sum_{\nu \,:\, l \in \mathcal{P}_\nu} x^\nu \right) = \frac{a_l}{b_l + c_l - \sum_{\nu \,:\, l \in \mathcal{P}_\nu} x^\nu}, \quad U_\nu(x^\nu) = d_\nu \log\left(e_\nu(1 + x^\nu)\right),$$

where a_l, b_l, d_ν, and e_ν are positive constants. We observe that in this multi agent framework the Nash equilibrium concept can be effectively used to model the system. Moreover an exact potential function for the GNEP is the following

$$\overline{P}(\mathbf{x}) = \sum_{l \in \mathcal{L}} C_l \left(\sum_{\nu \,:\, l \in \mathcal{P}_\nu} x^\nu \right) - \sum_{\nu=1}^{N} U_\nu(x^\nu).$$

3 Methods based on potential functions

In this section, we investigate which types of equilibria of the potential game can be computed by minimizing a potential function P over the common feasible set Ω. In particular, our aim is to understand how to use potential functions to effectively select equilibria of the game.

A simple method to find equilibria of the generalized potential game consists in computing global minima of any ordinal potential function for the game.

Theorem 1 *Let P be any ordinal potential function for the GNEP defined by* (1)*. Any* 0*-approximate global solution of the following optimization problem*

$$\min_{\mathbf{x}} P(\mathbf{x}), \quad \text{s.t. } \mathbf{x} \in \Omega, \tag{9}$$

that is, any $\overline{\mathbf{x}} \in \Omega$ such that

$$P(\overline{\mathbf{x}}) \leq P(\mathbf{x}), \quad \forall\, \mathbf{x} \in \Omega, \tag{10}$$

is a 0*-approximate equilibrium of the GNEP.*

Proof Clearly, (10) implies (3) with $\varepsilon = 0$. □

This fact was already observed, e.g., in [20]. A direct consequence of Theorem 1 is that the generalized potential games considered in this paper always admit at least one 0-approximate equilibrium.

On the other hand, if we are interested in computing approximate equilibria, we can use exact potential functions.

Theorem 2 *Let $\overline{P}$ be any exact potential function for the GNEP defined by* (1)*. Given $\varepsilon \geq 0$, any ε-approximate global solution of the following optimization problem*

$$\min_{\mathbf{x}} \overline{P}(\mathbf{x}), \quad s.t.\ \mathbf{x} \in \Omega, \tag{11}$$

that is, any $\overline{\mathbf{x}} \in \Omega$ such that

$$\overline{P}(\overline{\mathbf{x}}) \leq \overline{P}(\mathbf{x}) + \varepsilon, \quad \forall\, \mathbf{x} \in \Omega, \tag{12}$$

is an ε-approximate equilibrium of the GNEP.

Proof Clearly, (12) implies (3). □

We observe that problems (9) and (11) are MINLPs. The converse of Theorem 1, and of Theorem 2, is not necessarily true as it is witnessed by the following example.

Example 2 There are two players each controlling one variable. The players' problems are

$$\begin{aligned} \min_{x^1}\ \theta_1(x^1, x^2) &= \frac{9}{2}(x^1)^2 + 8x^1x^2 - \frac{153}{2}x^1 \\ & 0 \leq x^1 \leq 9 \end{aligned}$$

$$\begin{aligned} \min_{x^2}\ \theta_2(x^1, x^2) &= \frac{9}{2}(x^2)^2 + 8x^1x^2 - \frac{153}{2}x^2 \\ & 0 \leq x^2 \leq 9 \\ & x^2 \in \mathbb{Z}. \end{aligned}$$

It is easy to see that

$$\overline{P}(x^1, x^2) = \frac{9}{2}(x^1)^2 + 8x^1x^2 + \frac{9}{2}(x^2)^2 - \frac{153}{2}x^1 - \frac{153}{2}x^2,$$

is an exact potential function for the game. The points $A = \left(\frac{19}{6}, 6\right)$ and $E = \left(\frac{73}{18}, 5\right)$ are 0-approximate equilibria of the game (see figure 1). Let us consider the following sequence of points from A to E and the corresponding values of θ_1, θ_2, and $\overline{P}$:

	A		B		C		D		E
	$\left(\frac{19}{6}, 6\right)$	$\rightarrow$	$\left(\frac{19}{6}, \frac{11}{2}\right)$	$\rightarrow$	$\left(4, \frac{11}{2}\right)$	$\rightarrow$	$(4, 5)$	$\rightarrow$	$\left(\frac{73}{18}, 5\right)$
θ_1 :	-45.125		-57.792		-58.000		-74.000		-74.014
θ_2 :	-145.000		-145.292		-108.625		-110.000		-107.778
$\overline{P}$:	-342.125		-342.417		-342.625		-344.000		-344.014

This sequence is decreasing for any ordinal potential function and E is feasible, then the converse of Theorem 1 cannot hold. Moreover, for any $\varepsilon < \overline{P}\left(\frac{19}{6}, 6\right) - \overline{P}\left(\frac{73}{18}, 5\right) = 1.889$, A is not an ε-approximate global solution of problem (11), then also the converse of Theorem 2 cannot hold. $\square$

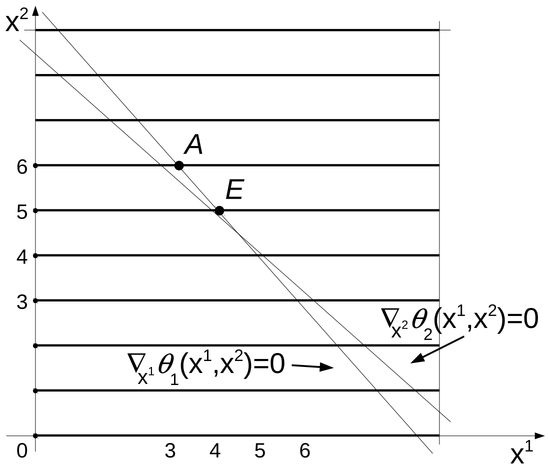

Fig. 1 A sketch of Example 2. Points A and E are 0-approximate equilibria of the generalized potential game.

Therefore we can conclude that not all the equilibria of the generalized potential game can be computed by minimizing a potential function. Nonetheless, following the techniques described in Theorems 1 and 2, potential functions can be used to compute different equilibria of the game. In particular, if we are solving the game by addressing problem (9) or (11), then we are computing a point that has two important features: (i) it is an (approximate) equilibrium of the game, and (ii) it is an (approximate) optimal solution of a specific merit function, i.e., the potential function used. This is equivalent to selecting the equilibrium that best fits with the merit function. It is then interesting to define wide classes of ordinal and exact potential functions for different types of potential games, since any potential function is a specific merit function with which we can select the equilibria.

Now we focus our attention on a particular subclass of generalized potential games for which we can give stronger results. Let us consider problems in which the objective function of any player ν depends only on the private variables x^ν, i.e., $\theta_\nu : \mathbb{R}^{n_\nu} \to \mathbb{R}$. We observe that linear problems described in section 2.1 belong to this class of problems. The function $\overline{P}(\mathbf{x}) = \sum_{\nu=1}^N \theta_\nu(x^\nu)$ is an exact potential for any game in this class. Given any continuous function $f : \mathbb{R}^N \to \mathbb{R}$ strictly increasing relative to $\mathbb{R}^N_+$, i.e., $f(y) < f(y+d)$ for all $y \in \mathbb{R}^N$ and all $d \in \mathbb{R}^N_+$, the function $P(\mathbf{x}) = f(\theta_1(x^1), \ldots, \theta_N(x^N))$ is an ordinal potential function for any generalized potential game of this class. We describe three emblematic examples of function f:

(i) $f(\theta_1(x^1), \ldots, \theta_N(x^N)) = \sum_{\nu=1}^N \alpha_\nu \theta_\nu(x^\nu)$, with weights $\alpha_\nu > 0$ for all ν: this potential function selects those equilibria that favour the most the players with heavier weights α;
(ii) $f(\theta_1(x^1), \ldots, \theta_N(x^N)) = \alpha \min \{\theta_\nu(x^\nu)\}_{\nu=1}^N + \sum_{\nu=1}^N \theta_\nu(x^\nu)$, with $\alpha >> 0$: this choice "promotes evolution", in that it is suitable if one wants to compute the equilibria for which one single player (no matter who) reaches its best possible objective;
(iii) $f(\theta_1(x^1), \ldots, \theta_N(x^N)) = \alpha \max \left\{\theta_\nu(x^\nu) - \overline{\theta}_\nu\right\}_{\nu=1}^N + \sum_{\nu=1}^N \theta_\nu(x^\nu)$, with $\alpha >> 0$: this choice "promotes parity", in that it selects the equilibria at which all the players have objective distance from the reference values $\overline{\theta}_\nu$ as similar as possible.

In all these three cases, problem (9) can be remodeled as a MINLP with a convex and continuously differentiable objective function and with convex and continuously differentiable constraints. In case (i), P is trivially convex and continuously differentiable. In case (ii), P is neither convex nor continuously differentiable, but we can introduce one continuous variable z and N binary variables δ, and thus rewrite problem (9) by adding disjunctive constraints as follows

$$\begin{aligned}
\min_{\mathbf{x},z,\delta} \quad & \alpha z + \sum_{\nu=1}^N \theta_\nu(x^\nu) \\
& \theta_\nu(x^\nu) - z - (1-\delta_\nu)M \leq 0, \quad \nu = 1, \ldots, N \\
& \sum_{\nu=1}^N \delta_\nu \geq 1 \\
& \mathbf{x} \in X \\
& x_j^\nu \in \mathbb{Z}, \quad j = 1, \ldots, i_\nu, \quad \nu = 1, \ldots, N, \\
& \delta \in \{0,1\}^N, \quad M >> 0.
\end{aligned} \tag{13}$$

In case (iii), P is not continuously differentiable, but problem (9) can be solved by addressing the following problem

$$\begin{aligned} \min_{\mathbf{x},z} \quad & \alpha z + \sum_{\nu=1}^{N} \theta_\nu(x^\nu) \\ & \theta_\nu(x^\nu) - \overline{\theta}_\nu - z \leq 0, \quad \nu = 1,\dots,N \\ & \mathbf{x} \in X \\ & x_j^\nu \in \mathbb{Z}, \quad j = 1,\dots,i_\nu, \quad \nu = 1,\dots,N. \end{aligned} \tag{14}$$

Keeping the focus on the class of potential games with independent objective functions, to better understand which equilibria can be computed by optimizing potential functions and which can not, let us consider the following multi-objective optimization problem

$$\min_{\mathbf{x}} \left(\theta_1(x^1),\dots,\theta_N(x^N)\right), \quad \text{s.t. } \mathbf{x} \in \Omega. \tag{15}$$

A point $\overline{\mathbf{x}} \in \Omega$ is a Pareto optimum of problem (15) if a point $\widetilde{\mathbf{x}} \in \Omega$ does not exist such that $\theta_\nu(\widetilde{x}^\nu) \leq \theta_\nu(\overline{x}^\nu)$ for all $\nu \in \{1,\dots,N\}$, and at least one of these inequalities is strict.

It is not difficult to see that any Pareto optimum of problem (15) is a 0-approximate equilibrium of the generalized potential game. In that, this is a direct consequence of the definition of Pareto optimality. Moreover, we can prove that any solution of problem (9) is a Pareto optimum of problem (15).

Theorem 3 *Let $\overline{\mathbf{x}}$ be a 0-approximate global solution of problem (9), then $\overline{\mathbf{x}}$ is a Pareto optimum of problem (15).*

Proof Let us suppose by contradiction that $\overline{\mathbf{x}}$ is not a Pareto optimum of problem (15). Then a point $\widetilde{\mathbf{x}} \in \Omega$ exists such that $\theta_\nu(\widetilde{x}^\nu) \leq \theta_\nu(\overline{x}^\nu)$ for all $\nu \in \{1,\dots,N\}$, and at least one of these inequalities is strict. This fact implies that

$$\begin{aligned} & P(\overline{\mathbf{x}}) \geq P(\widetilde{x}^1, \overline{x}^2, \dots, \overline{x}^N) \geq \dots \geq P(\widetilde{x}^1, \dots, \widetilde{x}^{\nu-1}, \overline{x}^\nu, \dots, \overline{x}^N) \geq \dots \geq \\ & \geq P(\widetilde{x}^1, \dots, \widetilde{x}^{N-1}, \overline{x}^N) \geq P(\widetilde{\mathbf{x}}), \end{aligned}$$

and at least one of these inequalities is strict. But this contradicts the fact that $\overline{\mathbf{x}}$ is optimal for problem (9). □

Therefore, considering any generalized potential game with independent objective functions, by solving problem (9), we can compute only those equilibria that are also Pareto optima of problem (15). The following example shows that equilibria of generalized potential games of this class may exist that are not Pareto optima of problem (15), and, therefore, they are not computable by optimizing any potential function.

Example 3 There are two players each controlling one variable. The players' problems are

$$\begin{array}{ll} \min_{x^1} \theta_1(x^1) = -x^1 & \min_{x^2} \theta_2(x^2) = -x^2 \\ 0 \leq x^1 \leq 4 & x^2 \in \mathbb{Z} \\ -\frac{1}{2} \leq x^1 - x^2 \leq 0 & -\frac{1}{2} \leq x^1 - x^2 \leq 0. \end{array}$$

All the points $A = (0,0)$, $B = (1,1)$, $C = (2,2)$, $D = (3,3)$, and $E = (4,4)$ are 0-approximate equilibrium of the game (see figure 2). But only E is a Pareto optimum of problem (15), since E dominates all the other points. Therefore E is the unique equilibrium that can be computed by optimizing a potential function. In particular, E can be computed by minimizing, e.g., $\overline{P}(x^1, x^2) = -x^1 - x^2$. □

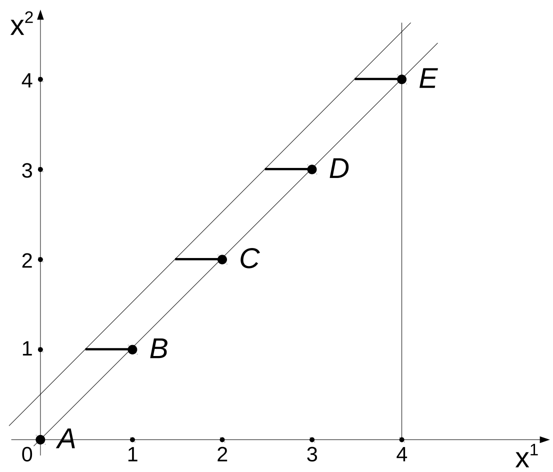

Fig. 2 A sketch of Example 3. Points A, B, C, D, and E are 0-approximate equilibria of the generalized potential game.

Remark 1 Let us consider the following ordinal potential function $P(\mathbf{x}) = \max \left\{\theta_\nu(x^\nu) - \overline{\theta}_\nu\right\}_{\nu=1}^N + \frac{1}{\alpha}\sum_{\nu=1}^N \theta_\nu(x^\nu)$, which is similar to one of the potential functions described above. We observe that, by suitably setting the parameters α and $\overline{\theta}_\nu$ for all ν, then any Pareto optimum $\widehat{\mathbf{x}}$ of problem (15) is an ε-approximate global solution of problem (9), given any $\varepsilon > 0$. Let $\Delta = \min_{\mathbf{x}\in\Omega} \sum_{\nu=1}^N (\theta_\nu(x^\nu) - \theta_\nu(\widehat{x}^\nu))$, note that Δ is a nonpositive and finite quantity. If we set $\alpha = -\frac{\Delta}{\varepsilon}$ and $\overline{\theta}_\nu = \theta_\nu(\widehat{x}^\nu)$ for all ν, then, for any $\mathbf{x} \in \Omega$, we get the following chain of inequalities:

$$
\begin{aligned}
P(\mathbf{x}) - P(\widehat{\mathbf{x}}) &= \max\left\{\theta_\nu(x^\nu) - \theta_\nu(\widehat{x}^\nu)\right\}_{\nu=1}^N + \frac{1}{\alpha}\sum_{\nu=1}^N \left(\theta_\nu(x^\nu) - \theta_\nu(\widehat{x}^\nu)\right) \\
&\geq \frac{1}{\alpha}\sum_{\nu=1}^N \left(\theta_\nu(x^\nu) - \theta_\nu(\widehat{x}^\nu)\right) \geq \frac{\Delta}{\alpha} = -\varepsilon,
\end{aligned}
$$

where the first inequality is due to the fact that $\max\left\{\theta_\nu(x^\nu) - \theta_\nu(\widehat{x}^\nu)\right\}_{\nu=1}^N \geq 0$ being $\widehat{\mathbf{x}}$ a Pareto optimum. Therefore, $P(\widehat{\mathbf{x}}) \leq P(\mathbf{x}) + \varepsilon$, for all $\mathbf{x} \in \Omega$.

With the following example we show that Theorem 3 may not hold when the objective functions depend on the variables of the other players.

Example 4 There are two players each controlling one variable. The first player solves

$$
\begin{aligned}
&\min_{x^1}\ \theta_1(\mathbf{x}) = (x^1)^2 + x^1x^2 + (x^2)^2 + x^1 \\
&\quad -1 \leq x^1 \leq 1,
\end{aligned}
$$

while the second player solves

$$
\begin{aligned}
&\min_{x^2}\ \theta_2(\mathbf{x}) = (x^1)^2 + x^1x^2 + (x^2)^2 + x^2 \\
&\quad -1 \leq x^2 \leq 1.
\end{aligned}
$$

Certainly, $P(\mathbf{x}) = (x^1)^2 + x^1x^2 + (x^2)^2 + x^1 + x^2$ is an ordinal potential function. In this case, the unique (global) solution of problem (9) is $\left(-\frac{1}{3}, -\frac{1}{3}\right)$. Anyway, we obtain $\theta_1\left(-\frac{1}{3}, -\frac{1}{3}\right) = 0 > -\frac{1}{12} = \theta_1\left(-\frac{1}{6}, -\frac{1}{6}\right)$ and $\theta_2\left(-\frac{1}{3}, -\frac{1}{3}\right) = 0 > -\frac{1}{12} = \theta_2\left(-\frac{1}{6}, -\frac{1}{6}\right)$. Therefore, $\left(-\frac{1}{3}, -\frac{1}{3}\right)$ is not a Pareto optimum of problem (15). □

4 Methods based on best responses

In this section we define a method to compute equilibria of the generalized potential game, defined by (1), that does not directly solve the optimization problem (9), but uses instead the players' best responses as simple steps. This can be convenient for different reasons: (i) problem (9) can be hard to solve because it may be either nonconvex or huge dimensional, on the other hand the best response problems are convex and, in general, easier (see e.g. sections 5.2 and 5.3), (ii) as shown in examples 2 and 3, the set of approximate global solutions of problem (9) in general does not contain all approximate equilibria of the game, and (iii) as thoroughly discussed in [20], in many practical situations, the solution of problem (9) cannot be conceptually considered, e.g.

if a centralized algorithm is not implementable since it would require an ineffective high degree of coordination among the players (see e.g. the application described in section 2.2).

To compute approximate equilibria of the generalized potential GNEP in its whole generality, we propose the Gauss-Southwell method described in Algorithm 1.

Algorithm 1: Gauss-Southwell method

choose a starting point $\mathbf{x}^0 \in \Omega$, and set $k := 0$;
while $\mathbf{x}^k$ *is not an* ε*-approximate equilibrium* **do**
  choose a player $\overline{\nu}^k \in \{1, \dots, N\}$;
  compute a best response $\widehat{x}^{k,\overline{\nu}^k} \in \widehat{x}^{\overline{\nu}^k}(\mathbf{x}^{k,-\overline{\nu}^k})$;
  if $\theta_{\overline{\nu}^k}(x^{k,\overline{\nu}^k}, \mathbf{x}^{k,-\overline{\nu}^k}) - \theta_{\overline{\nu}^k}(\widehat{x}^{k,\overline{\nu}^k}, \mathbf{x}^{k,-\overline{\nu}^k}) > \varepsilon$ **then**
    set $x^{k+1,\overline{\nu}^k} := \widehat{x}^{k,\overline{\nu}^k}$;
  else
    set $x^{k+1,\overline{\nu}^k} := x^{k,\overline{\nu}^k}$;
  end
  forall $\nu \in \{1, \dots, N\} \setminus \overline{\nu}^k$ **do**
    set $x^{k+1,\nu} := x^{k,\nu}$;
  end
  set $k := k + 1$;
end
Result: an ε-approximate equilibrium $\mathbf{x}^k$

With the following theorem we give a simple condition to ensure that Algorithm 1 returns an ε-approximate equilibrium of the potential GNEP, with a given $\varepsilon > 0$, in a finite number of steps.

Theorem 4 *Assume that, in Algorithm 1, every h iterations at least one best response of any player ν is computed, that is $\nu \in \left\{\overline{\nu}^k, \dots, \overline{\nu}^{k+h}\right\}$ for each player ν and each iterate k. Given any $\varepsilon > 0$, Algorithm 1 stops in a finite number of steps and returns an ε-approximate equilibrium of the potential GNEP defined by* (1).

Proof By the assumption of the theorem, if $\overline{k}$ exists such that $\mathbf{x}^{\overline{k}} = \dots = \mathbf{x}^{\overline{k}+h}$, then $\mathbf{x}^{\overline{k}}$ is an ε-approximate equilibrium of the game.

By contradiction assume that Algorithm 1 generates an infinite sequence of points $\left\{\mathbf{x}^k\right\} \subseteq X$. Let P be any ordinal potential function. By the assumptions, for all k, it holds that $P(\mathbf{x}^k) \geq P(\mathbf{x}^{k+1}) \geq P^* > -\infty$. Therefore, by the continuity of P, we obtain

$$\lim_{k \to \infty} P(\mathbf{x}^k) = P(\mathbf{z}) = \widetilde{P} \geq P^*, \tag{16}$$

where $\mathbf{z}$ is any accumulation point of $\left\{\mathbf{x}^k\right\}$.

By the assumption of the theorem, a player ν and an infinite subset of indices K exist such that

$$\theta_\nu(\mathbf{x}^k) - \theta_\nu(\mathbf{x}^{k+1}) = \theta_\nu(\mathbf{x}^k) - \theta_\nu(x^{k+1,\nu}, \mathbf{x}^{k,-\nu}) > \varepsilon, \quad \forall k \in K.$$

Taking the limit we obtain

$$\lim_{k\to\infty,k\in K}\left(\theta_\nu(\mathbf{x}^k)-\theta_\nu(\mathbf{x}^{k+1})\right)=\lim_{k\to\infty,k\in K}\left(\theta_\nu(\mathbf{x}^k)-\theta_\nu(x^{k+1,\nu},\mathbf{x}^{k,-\nu})\right)\geq\varepsilon. \tag{17}$$

By the compactness of X an infinite subset of indices $\overline{K}\subseteq K$ exists such that

$$\lim_{k\to\infty,k\in\overline{K}}\mathbf{x}^k=\overline{\mathbf{x}}\in X,$$

and an infinite subset of indices $\widetilde{K}\subseteq\overline{K}$ exists such that

$$\lim_{k\to\infty,k\in\widetilde{K}}\mathbf{x}^{k+1}=\lim_{k\to\infty,k\in\widetilde{K}}(x^{k+1,\nu},\mathbf{x}^{k,-\nu})=(\widetilde{x}^\nu,\overline{\mathbf{x}}^{-\nu})\in X.$$

By (17) and the continuity of θ_ν we obtain

$$\theta_\nu(\overline{\mathbf{x}})-\theta_\nu(\widetilde{x}^\nu,\overline{\mathbf{x}}^{-\nu})\geq\varepsilon.$$

This implies that

$$P(\overline{\mathbf{x}})-P(\widetilde{x}^\nu,\overline{\mathbf{x}}^{-\nu})>0,$$

and thus, by using (16), we get

$$0<P(\overline{\mathbf{x}})-P(\widetilde{x}^\nu,\overline{\mathbf{x}}^{-\nu})=\widetilde{P}-\widetilde{P}=0.$$

But this is impossible, and the thesis holds. □

Therefore, Algorithm 1 always produces an ε-approximate equilibrium of the GNEP. In principle, choosing a specific sequence of players' problems to optimize during the iterations, all the approximate equilibria are computable by the algorithm. However, it is easy to show that equilibria could exist such that the algorithm can compute them only by starting from them.

To give an upper bound on the number of steps required by Algorithm 1 to return an ε-approximate equilibrium of the potential GNEP, we make the following non-demanding assumption.

Assumption 1 *Assume that an exact potential function $\overline{P}$ exists such that $\nabla\overline{P}$ is Lipschitz continuous on X with constant $L_{\overline{P}}$, i.e.,*

$$\|\nabla\overline{P}(\mathbf{x})-\nabla\overline{P}(\mathbf{y})\|_2\leq L_{\overline{P}}\|\mathbf{x}-\mathbf{y}\|_2,\qquad\forall\,\mathbf{x},\mathbf{y}\in X.$$

Let us denote

$$\psi\triangleq\max_{\mathbf{x}\in X}\left\|\nabla\overline{P}(\mathbf{x})\right\|_2,\qquad\phi\triangleq\max_{\mathbf{x},\mathbf{y}\in X}\|\mathbf{x}-\mathbf{y}\|_2\,.$$

By the assumptions done and if Assumption 1 holds, ψ and ϕ are well defined and finite quantities.

Theorem 5 *Assume that, in Algorithm 1, every h iterations at least one best response of any player ν is computed, that is $\nu\in\left\{\overline{\nu}^k,\dots,\overline{\nu}^{k+h}\right\}$ for each player ν and each iterate k. Suppose that Assumption 1 holds. Therefore, given any $\varepsilon>0$, Algorithm 1 returns an ε-approximate equilibrium of the potential GNEP defined by (1) in at most $h\left(\frac{\psi\phi+\frac{L_{\overline{P}}}{2}\phi^2}{\varepsilon}+1\right)$ iterations.*

Proof Let $\mathbf{x}^*$ be a 0-approximate global solution of problem (11). By Assumption 1 and the descent lemma, see e.g. [5], we can write

$$\overline{P}(\mathbf{x}^0) - \overline{P}(\mathbf{x}^*) \leq \nabla \overline{P}(\mathbf{x}^*)^T(\mathbf{x}^0 - \mathbf{x}^*) + \frac{L_{\overline{P}}}{2}\|\mathbf{x}^0 - \mathbf{x}^*\|_2^2 \leq \psi\phi + \frac{L_{\overline{P}}}{2}\phi^2.$$

We observe that for any iterate k it holds that $\overline{P}(\mathbf{x}^k) \geq \overline{P}(\mathbf{x}^{k+1})$, and that, unless the algorithm stops, $\overline{P}(\mathbf{x}^k) - \overline{P}(\mathbf{x}^{k+h}) \geq \varepsilon$. Therefore, after at most $h\left(\frac{\psi\phi + \frac{L_{\overline{P}}}{2}\phi^2}{\varepsilon} + 1\right)$ iterations, it holds that $\overline{P}(\mathbf{x}^k) = \overline{P}(\mathbf{x}^*)$, i.e., by Theorem 2, $\mathbf{x}^k$ is an ε-approximate equilibrium of the GNEP. □

Note that, despite the rule to choose the player $\overline{\nu}^k$ is left free in Algorithm 1, the upper bound on the iterations in Theorem 5 suggests to use the minimum $h = N$, which is achieved by cyclic choosing the players. We use this strategy in the numerical experiments reported in section 5.

Moreover, we observe that, considering the linear GNEPs in which any player solves (4), Assumption 1 holds and the upper bound defined in Theorem 5 is simply equal to

$$h\left(\frac{\|\mathbf{c}\|_2\,\|\mathbf{u}-\mathbf{l}\|_2}{\varepsilon} + 1\right), \quad \text{with} \quad \mathbf{c} = \begin{pmatrix} c^1 \\ \vdots \\ c^N \end{pmatrix}, \ \mathbf{u} = \begin{pmatrix} u^1 \\ \vdots \\ u^N \end{pmatrix}, \ \mathbf{l} = \begin{pmatrix} l^1 \\ \vdots \\ l^N \end{pmatrix}.$$

Theorems 4 and 5 show that Algorithm 1 converges, within a finite number of steps, to an ε-approximate equilibrium whenever the given ε is strictly positive. However, it is well-known, see [20] and section 5.2, that it may fail if $\varepsilon = 0$. So it is interesting to understand if any accumulation point of any sequence of ε-approximate equilibria, as ε goes to 0, is a 0-approximate equilibrium of the game. To do this, we distinguish three cases.

The first one is the case in which $i_\nu = 0$ for all ν, i.e., $\Omega = X$, which is the completely continuous setting. First of all it is useful to recall some basical definitions of nonsmooth analysis, see e.g. [28].

Definition 1 A function $f : \mathcal{F} \to \mathbb{R}$, with $\mathcal{F} \subseteq \mathbb{R}^n$, is upper semicontinuous (usc) at $\overline{x} \in \mathcal{F}$ relative to $\mathcal{F}$ if

$$\limsup_{x^k \to \overline{x},\, \{x^k\} \subseteq \mathcal{F}} f(x^k) = f(\overline{x}),$$

i.e., for any sequence $\{x^k\} \subseteq \mathcal{F}$ that goes to $\overline{x}$, the limit value of f is not greater than $f(\overline{x})$.

Definition 2 A set-valued mapping $S : \mathcal{D} \rightrightarrows \mathbb{R}^m$, with $\mathcal{D} \subseteq \mathbb{R}^n$, and which is closed-valued, is inner semicontinuous (isc) at $\overline{x} \in \mathcal{D}$ relative to $\mathcal{D}$ if

$$\liminf_{x^k \to \overline{x},\, \{x^k\} \subseteq \mathcal{D}} S(x^k) = S(\overline{x}),$$

i.e., for any sequence $\{x^k\} \subseteq \mathcal{D}$ that goes to $\overline{x}$, and for any $\overline{u} \in S(\overline{x})$, a sequence $\{u^k\}$ exists such that $u^k \in S(x^k)$ and $u^k \to \overline{u}$.

Now we are ready to give a technical result.

Lemma 1 *Let* $\overline{\mathbf{x}}$ *be any accumulation point of any sequence of* ε*-approximate equilibria* $\{\mathbf{x}_\varepsilon\}$ *as* ε *goes to* 0*, and let, for any* ν*,*

$$\Omega_\nu(\mathbf{x}^{-\nu}) \triangleq \left\{x^\nu \in \mathbb{R}^{n_\nu} \,:\, (x^\nu, \mathbf{x}^{-\nu}) \in \Omega\right\}$$

be isc at $\overline{\mathbf{x}}^{-\nu}$ *relative to its domain, i.e., the set*

$$\operatorname{dom} \Omega_\nu \triangleq \{\mathbf{x}^{-\nu} \in \mathbb{R}^{n-n_\nu} : \Omega_\nu(\mathbf{x}^{-\nu}) \neq \emptyset\}.$$

Then $\overline{\mathbf{x}}$ *is a* 0*-approximate equilibrium of the game.*

Proof Let us define, for any player ν, the value function

$$\varphi_\nu(\mathbf{x}^{-\nu}) \triangleq \min_{x^\nu \in \Omega_\nu(\mathbf{x}^{-\nu})} \theta_\nu(x^\nu, \mathbf{x}^{-\nu}).$$

By the fact that all the mappings Ω_ν are isc at $\overline{\mathbf{x}}^{-\nu}$ relative to $\operatorname{dom}\Omega_\nu$, and by using [2, Theorem 4.2.2 (1)], we obtain that all the functions φ_ν are usc at $\overline{\mathbf{x}}^{-\nu}$ relative to $\operatorname{dom}\Omega_\nu$.

By using optimality conditions (3), we have, for any ε-approximate equilibrium $\mathbf{x}_\varepsilon$,

$$\theta_\nu(\mathbf{x}_\varepsilon) \leq \varphi_\nu(\mathbf{x}_\varepsilon^{-\nu}) + \varepsilon, \quad \forall\, \nu \in \{1, \ldots, N\}.$$

Taking the limit $\mathbf{x}_\varepsilon \to \overline{\mathbf{x}}$, subsequencing if necessary, and exploiting the continuity properties of the functions, we obtain

$$\theta_\nu(\overline{\mathbf{x}}) \leq \varphi_\nu(\overline{\mathbf{x}}^{-\nu}), \quad \forall\, \nu \in \{1, \ldots, N\},$$

i.e., the thesis holds. □

Let us assume that $\Omega = X$ is defined by continuous convex inequalities satisfying some constraint qualification. Although a formal proof is not given in the literature, no examples are known in which the conditions of Lemma 1 do not hold in this case, see the discussion in [34]. In particular, if X is polyhedral as in all the examples of this work, then all the conditions of the lemma are certainly satisfied, and, therefore, any accumulation point of any sequence of ε-approximate equilibria, with $\varepsilon \to 0$, is a 0-approximate equilibrium of the game.

The second case we consider is that in which all the variables must be integers, i.e., $i_\nu = n_\nu$ for all ν.

Theorem 6 *(discrete setting) Let* $i_\nu = n_\nu$ *for all* ν*. Then any accumulation point of any sequence of* ε*-approximate equilibria, as* ε *goes to* 0*, is a* 0*-approximate equilibrium of the game.*

Proof Let us define the strictly positive quantity

$$\overline{\delta} \triangleq \min \{\delta \in \mathbb{R} : \delta = |\theta_\nu(x^\nu, \mathbf{x}^{-\nu}) - \theta_\nu(y^\nu, \mathbf{x}^{-\nu})|, \, (x^\nu, \mathbf{x}^{-\nu}) \in \Omega, \\ (y^\nu, \mathbf{x}^{-\nu}) \in \Omega, \, \nu \in \{1, \ldots, N\}, \, \delta > 0\big\},$$

which is well defined since Ω contains a finite number of points. Trivially, any ε-approximate equilibrium, with $\varepsilon < \overline{\delta}$, is a 0-approximate equilibrium of the game. Thus the proof readily follows. □

Finally, in the last case in which a player ν exists such that $0 < i_\nu < n_\nu$ (the mixed-integer setting), the desired result cannot be proved in general. Certainly one can resort to Lemma 1 to check if a specific sequence of ε-approximate equilibria leads to a specific 0-approximate equilibrium, but the following example shows that there is no hope to obtain a general result even if X is polyhedral.

Example 5 There are two players each controlling one variable. The players' problems are

$$\begin{array}{ll} \min\limits_{x^1} \theta_1(x^1) = x^1 & \qquad \min\limits_{x^2} \theta_2(x^2) = -x^2 \\ \quad x^1 \geq 0 & \qquad \quad x^2 \geq 0, \quad x^2 \in \mathbb{Z} \\ x^1 + x^2 \leq 2 & \qquad x^1 + x^2 \leq 2. \end{array}$$

Any point $\mathbf{x}_\varepsilon = (\varepsilon, 1)$, with $\varepsilon > 0$, is an ε-approximate equilibrium of the game. But, as ε goes to 0, $\mathbf{x}_\varepsilon \to B = (0, 1)$, which is not a 0-approximate equilibrium of the game since point $A = (0, 2)$ is better for player 2 (see figure 3). Notice that, as expected, Ω_2 is not isc at 0 relative to dom $\Omega_2 = [0, 2]$. To see this, let us consider $\{x^k\} = \{\frac{1}{k^2+1}\} \subseteq [0, 2]$, which is a sequence converging to $\overline{x} = 0$ as $k \to \infty$, and let us consider $\overline{u} = 2 \in \Omega_2(\overline{x})$. Since $\{0, 1\} = \Omega_2(x^k)$ for all k, therefore a sequence $\{u^k\}$ cannot exist such that $u^k \in \Omega_2(x^k)$ and $u^k \to \overline{u}$. □

However, notice that the result can be trivially obtained if the set X is separable, i.e., if the game is a NEP with mixed-integer variables.

Theorem 7 *(separable mixed-integer setting) Let $X = \prod_{\nu=1}^N X_\nu$, with $X_\nu \subseteq \mathbb{R}^{n_\nu}$. Then any accumulation point of any sequence of ε-approximate equilibria, as ε goes to 0, is a 0-approximate equilibrium of the game.*

Proof By exploiting Lemma 1, we only need to show that, for any player ν, the feasible mapping Ω_ν is isc at any point in dom Ω_ν relative to dom Ω_ν. In this case all Ω_ν are fixed sets, therefore the thesis trivially holds. □

Example 5 also shows that the inner semicontinuity of the feasible mappings Ω_ν cannot be assumed to be true in mixed-integer games. This fact makes the methods proposed in [20] not effective in order to solve generalized potential games with mixed-integer variables.

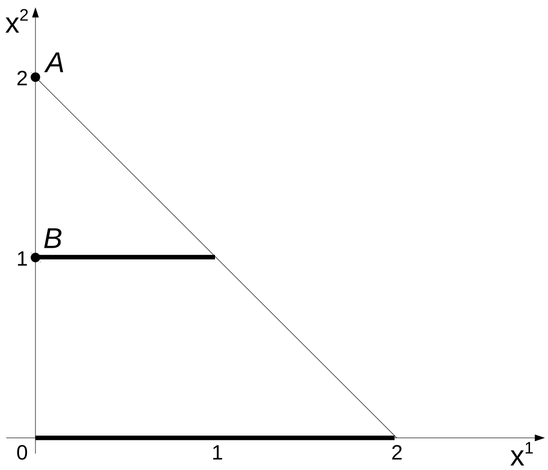

Fig. 3 A sketch of Example 5. Point A is the unique 0-approximate equilibrium of the generalized potential game.

Moreover, notice that this last analysis (Lemma 1, and Theorems 6-7) is valid for all GNEPs with mixed-integer variables, and not only for generalized potential games.

Remark 2 Algorithm 1 can be used to define a procedure to compute 0-approximate equilibria of the GNEP. Specifically, given a sequence $\{\varepsilon^k\} > 0$ such that $\varepsilon^k \to 0$, such procedure produces a sequence of ε^k-approximate equilibria in which the generic kth equilibrium is computed by using Algorithm 1 starting from the $(k-1)$th equilibrium. This procedure always converges to a 0-approximate equilibrium in all the cases (discussed above) in which any accumulation point of any sequence of ε-approximate equilibria, as ε goes to 0, is a 0-approximate equilibrium of the game. In particular, it provably produces a 0-approximate equilibrium in the following cases: (i) in all the cases in which the algorithms proposed in [20] work, (ii) in the completely discrete setting, and (iii) if the game is a NEP. In the general mixed-integer setting, this procedure computes a 0-approximate equilibrium if an accumulation point of the sequence of ε^k-approximate equilibria exists such that all the maps Ω_ν are isc at it relative to their domain, see Lemma 1.

However, we remark that Algorithm 1 could compute 0-approximate equilibria even if it is running with a given $\varepsilon > 0$. In particular, this occurs in our numerical tests in sections 5.2 and 5.3.

Remark 3 In [10,11,22] some local error bound conditions are proposed for GNEPs in a totally continuous setting. Such good results are strongly related to the possibility of reformulating the GNEP as a quasi-variational inequality, and, then, as a constrained system of equations. As described in [30], these reformulations cannot be used whenever, in the GNEP, there are discrete variables. Therefore, it is very difficult to obtain similar local error bound results in our mixed-integer framework. On the other hand, the concept of ε-approximate equilibrium considered in this work (and also used by other authors), referring to the objective values of the players, seems to fit rather well with this mixed-integer setting. In that, it is directly related to the definition of equilibrium.

Moreover, we underline that an ε-approximate equilibrium can be arbitrarily far from any other 0-approximate equilibrium. Consider for simplicity the case of a single player ($N = 1$) that minimizes $\theta_1(x^1) = (4x^1_1 + x^1_2 - 4)^2 + \delta(x^1_2 - (2 - \delta))^2$ over $\Omega = \{x^1 \in \mathbb{Z}^2 : 0 \leq x^1_1 \leq 1, 0 \leq x^1_2 \leq 4\}$, where δ is a small positive parameter (e.g., $\delta = 1e-4$). The point $(1, 0)$ is the unique global solution, i.e., it is the unique 0-approximate equilibrium. The point $(0, 4)$ is an ε-approximate equilibrium for any $\varepsilon \geq 8\delta^2$. But, the euclidean distance between $(0, 4)$ and $(1, 0)$ is $\sqrt{17}$, and then it does not depend on δ or ε. Thus, the concept of ε-approximate equilibrium seems not linked with any error bound result.

5 Numerical experiments

In this section we make numerical experiments to show the practical effectiveness of the proposed methods. In particular, in section 5.1 we consider the market described in Example 1, and we compute different equilibria by using different potential functions. Our aim is to show how an effective equilibrium selection can be obtained in potential games. In section 5.2 we consider the counter-example given in [20] and we show that, in spite of the fact that our Gauss-Southwell algorithm does not exploit any regularization, it effectively computes approximate equilibria of the potential game in a finite number of iterations. In section 5.3 we use our Gauss-Southwell algorithm to compute equilibria of the discrete flow control problem on networks described in section 2.2.

All the experiments were carried out on an Intel Core i7-4702MQ CPU @ 2.20GHz x 8 with Ubuntu 14.04 LTS 64-bit and by using AMPL. As optimization solver we used CPLEX 12.6.0.1 with default options. We never report CPU time consumption since, in all our tests, AMPL returns a solution in less than one second.

5.1 Experiments on the market described in Example 1

We consider the jointly convex linear GNEP with mixed-integer variables defined by problems (6). This generalized potential game is particularly relevant if the number of the firms is small, and challenging if the number of decision variables of each firm is large. Thus, we assume that in the market there are $N = 3$ firms each producing $n^g_\nu = 30$ goods. The goods are divided into three groups: high quality (HQ), medium quality (MQ), and low quality (LQ) goods. Firm 1 produces 10 HQ, 10 MQ, and 10 LQ goods. Firm 2 produces 8 HQ, 14 MQ, and 8 LQ goods. Firm 3 produces 12 HQ, 6 MQ, and 12 LQ goods.

Moreover, there are $r = 10$ common constraints, $k = 5$ authority constraints whereof only $\overline{k} = 3$ must be satisfied, and each firm ν has $l_\nu = 5$ private constraints.

The upper bounds, the prices, the marginal costs, and the fixed costs of the goods of any firm ν were randomly generated by using the uniform distribution: $u_i^\nu \in [200, 400]$ for any good i; $p_i^\nu \in [200, 400]$, $m_i^\nu \in [0.2p_i^\nu, 0.4p_i^\nu]$, $f_i^\nu \in [5000, 10000]$ if the ith good of firm ν is of HQ; $p_i^\nu \in [150, 300]$, $m_i^\nu \in [0.2p_i^\nu, 0.3p_i^\nu]$, $f_i^\nu \in [2000, 5000]$ if it is of MQ; $p_i^\nu \in [100, 250]$, $m_i^\nu \in [0.1p_i^\nu, 0.3p_i^\nu]$, $f_i^\nu \in [1000, 2000]$ if it is of LQ.

The first six common constraints state that: the total quantity of HQ goods in the market must be in $[300, 1000]$, that of MQ goods in $[1000, 2000]$, and that of LQ goods in $[500, 1500]$. The parameters in the last four common constraints were randomly generated by using the uniform distribution: $h_j \in [1500, 2500]$, $D_{ji}^\nu \in [0.5, 0.8]$ if the ith good of firm ν is of HQ, $D_{ji}^\nu \in [0.4, 0.7]$ if it is of MQ, and $D_{ji}^\nu \in [0.4, 0.6]$ if it is of LQ (in order to relate the quality of a good with its consumption of resources). Also the parameters in the private constraints, for any firm ν, were randomly generated by using the uniform distribution: $d_j^\nu \in [1000, 3000]$, $B_{ji}^\nu \in [0.4, 0.6]$ if the ith good of firm ν is of HQ, $B_{ji}^\nu \in [0.3, 0.5]$ if it is of MQ, and $B_{ji}^\nu \in [0.2, 0.4]$ if it is of LQ.

The five authority constraints are the following:

1) the total quantity of LQ goods produced by the firm must be no more than 30% of its production;
2) the total quantity of LQ goods produced by the firm must be no more than 80% of the total quantity of HQ goods produced by the firm;
3) the total quantity of LQ and MQ goods produced by the firm must be no more than 60% of its production;
4) the total quantity of LQ goods produced by the firm must be no more than 300;
5) the total quantity of HQ goods produced by the firm must be at least 300.

We generated 3 different instances of the game. They are denoted by A, B, and C. We set $M = 1e6$.

We computed 0-approximate equilibria of the game by solving problem (9) with different ordinal potential functions (see section 3). In particular, we denote with $\overline{\mathbf{P}}$ the solution obtained with the exact potential function $\overline{P}(\mathbf{x}) = \sum_{\nu=1}^N \theta_\nu(x^\nu)$, with **Pmin** the solution obtained by solving problem (13) with $\alpha = 1e4$, and with **Pmax** the solution obtained by solving problem (14) with $\alpha = 1e4$ and $\overline{\theta}_1 = \overline{\theta}_2 = \overline{\theta}_3 = 0$. We observe that in all the cases we solved a mixed-integer linear problem, and we used CPLEX. We report the results for instances A-C in tables 1-3, where HQ_ν indicates the total quantity of HQ goods produced by firm ν, MQ_ν that of MQ goods, and LQ_ν that of LQ goods. Moreover, notice that $\gamma_i^\nu = 1$ means that the ith authority constraint is not satisfied by firm ν.

As expected, tables 1-3 show that: with $\overline{\mathbf{P}}$ we obtain the 0-approximate equilibrium that gives the maximum market outcome, with **Pmin** we obtain the 0-approximate equilibrium in which we get the maximum possible outcome

Table 1 Market equilibria computed by solving problem (9), instance A.

	$\overline{\mathbf{P}}$	**Pmin**	**Pmax**
θ_1	-275033	-602161	-217243
HQ_1	251.8	860.3	222.3
MQ_1	1041.9	1317	766.3
LQ_1	0	688.2	0
γ^1	$(0,0,1,0,1)^T$	$(0,0,1,1,0)^T$	$(0,0,1,0,1)^T$
θ_2	-136158	0	-217243
HQ_2	385.5	0	385.5
MQ_2	0	0	432.3
LQ_2	265.2	0	305.4
γ^2	$(1,0,0,0,0)^T$	$(0,0,0,0,1)^T$	$(0,0,1,1,0)^T$
θ_3	-252180	-6.6	-217243
HQ_3	320.6	0	243.3
MQ_3	654.3	12.5	654.3
LQ_3	291.7	0	194.6
γ^3	$(0,1,1,0,0)^T$	$(0,0,1,0,1)^T$	$(0,0,1,0,1)^T$

Table 2 Market equilibria computed by solving problem (9), instance B.

	$\overline{\mathbf{P}}$	**Pmin**	**Pmax**
θ_1	-322057	0	-264526
HQ_1	303.9	0	194.2
MQ_1	1004.8	0	937.2
LQ_1	300	0	155.3
γ^1	$(0,1,1,0,0)^T$	$(0,0,0,0,1)^T$	$(0,0,1,0,1)^T$
θ_2	-237265	-687431	-264526
HQ_2	300	1000	300
MQ_2	552.8	2000	704.1
LQ_2	300	768	300
γ^2	$(0,1,1,0,0)^T$	$(0,0,1,1,0)^T$	$(0,1,1,0,0)^T$
θ_3	-243205	0	-264526
HQ_3	396.1	0	505.8
MQ_3	442.4	0	358.7
LQ_3	311.2	0	400.1
γ^3	$(0,0,1,1,0)^T$	$(0,0,0,0,1)^T$	$(1,0,0,1,0)^T$

earned by a single firm, and with **Pmax** we obtain the 0-approximate equilibrium in which all the firms gain the same. Moreover, as stated in Theorem 3, any computed equilibrium is not dominated, in terms of outcome, by any other equilibrium since it is also a Pareto solution of problem (15).

Remark 4 Consider games in which the players act rationally and simultaneously, they have complete information, and there is no collusion. If there is a unique Nash equilibrium, then it is the best decision for any player. On the other hand, whenever the Nash equilibrium set is not a singleton (this is a typical situation in generalized games), it is crucial to select one single equilibrium that best fits with some rule that achieves consensus of all the players. This selected equilibrium is then the best decision for any player. The equilibrium obtained by solving problem (9) with $\overline{\mathbf{P}}$ or **Pmax**, assuming that it is the unique solution of the optimization problem, could be a suitable choice in this sense.

Table 3 Market equilibria computed by solving problem (9), instance C.

	$\overline{\mathbf{P}}$	**Pmin**	**Pmax**
θ_1	-285883	0	-212960
HQ_1	378.5	0	293.8
MQ_1	750.6	0	547.7
LQ_1	294.8	0	235
γ^1	$(0,0,1,0,1)^T$	$(0,0,0,0,1)^T$	$(0,0,1,0,1)^T$
θ_2	-389529	-637350	-212960
HQ_2	393	880.8	0
MQ_2	1249.4	1742.2	1118.4
LQ_2	314.4	684.891	0
γ^2	$(0,0,1,1,0)^T$	$(0,0,1,1,0)^T$	$(0,0,1,0,1)^T$
θ_3	0	0	-212960
HQ_3	0	0	529.6
MQ_3	0	0	253.5
LQ_3	0	0	296.7
γ^3	$(0,0,0,0,1)^T$	$(0,0,0,0,1)^T$	$(0,1,0,1,0)^T$

Now we consider the case in which the k authority constraints are shared by all the firms (see (7)). We report the results for instances A-C in tables 4-6. Notice that $\tau_i^1 + \tau_i^2 + \tau_i^3 \geq 1$ means that the ith authority constraint is not satisfied by the market.

Table 4 The case of shared authority constraints: market equilibria computed by solving problem (9), instance A.

	$\overline{\mathbf{P}}$	**Pmin**	**Pmax**
θ_1	-350286	-608561	-219680
HQ_1	603.5	912.6	230.5
MQ_1	1041.9	1102.2	766.3
LQ_1	0	874.9	0
γ^1	$(0,0,1,0,0)^T$	$(0,1,1,0,0)^T$	$(0,0,1,0,1)^T$
τ^1	$(0,0,1,0,0)^T$	$(0,0,0,0,0)^T$	$(0,0,1,0,0)^T$
θ_2	-129989	0	-219680
HQ_2	385.5	0	694.1
MQ_2	0	0	215.8
LQ_2	229.7	0	150.8
γ^2	$(0,0,1,0,0)^T$	$(0,1,1,0,0)^T$	$(0,1,1,0,0)^T$
τ^2	$(0,0,1,0,0)^T$	$(0,0,1,0,0)^T$	$(0,0,1,0,0)^T$
θ_3	-188319	-2752.4	-219680
HQ_3	0	0	0
MQ_3	654.3	26.6	654.3
LQ_3	291.7	0	489.3
γ^3	$(0,0,1,0,0)^T$	$(0,1,1,0,0)^T$	$(0,0,1,0,0)^T$
τ^3	$(0,0,0,0,0)^T$	$(0,1,0,0,0)^T$	$(0,0,0,0,0)^T$

The outcomes obtained by the firms in this different situation are always better than those obtained in tables 1-3. In fact, in this case, even if each firm loses the possibility to drop the authority constraints independently, this different version of the authority constraints is a relaxation of the previous one.

Table 5 The case of shared authority constraints: market equilibria computed by solving problem (9), instance B.

	$\overline{\mathbf{P}}$	**Pmin**	**Pmax**
θ_1	-306711	0	-274576
HQ_1	201.3	0	0
MQ_1	1004.8	0	1089.8
LQ_1	303.5	0	303.5
γ^1	$(0,1,1,0,0)^T$	$(0,1,1,0,0)^T$	$(0,1,1,0,0)^T$
τ^1	$(0,1,1,0,0)^T$	$(0,0,0,0,0)^T$	$(0,0,1,0,0)^T$
θ_2	-238747	-691355	-274576
HQ_2	0	947.9	255
MQ_2	759.2	2000	680.7
LQ_2	389.9	900	379.5
γ^2	$(0,1,1,0,0)^T$	$(0,1,1,0,0)^T$	$(0,1,1,0,0)^T$
τ^2	$(0,0,0,0,0)^T$	$(0,1,0,0,0)^T$	$(0,0,0,0,0)^T$
θ_3	-285100	0	-274576
HQ_3	768.2	0	745
MQ_3	229.5	0	229.5
LQ_3	203.3	0	182.5
γ^3	$(0,1,1,0,0)^T$	$(0,1,1,0,0)^T$	$(0,1,1,0,0)^T$
τ^3	$(0,0,0,0,0)^T$	$(0,0,1,0,0)^T$	$(0,1,0,0,0)^T$

Table 6 The case of shared authority constraints: market equilibria computed by solving problem (9), instance C.

	$\overline{\mathbf{P}}$	**Pmin**	**Pmax**
θ_1	-264974	0	-214703
HQ_1	378.5	0	378.5
MQ_1	888.3	0	575.9
LQ_1	0	0	88.8
γ^1	$(0,0,1,0,1)^T$	$(0,0,1,0,1)^T$	$(0,0,1,0,1)^T$
τ^1	$(0,0,1,0,1)^T$	$(0,0,0,0,0)^T$	$(0,0,0,0,1)^T$
θ_2	-369258	-637350	-214703
HQ_2	393	880.8	0
MQ_2	1107	1742.2	879.7
LQ_2	351.2	684.9	225.4
γ^2	$(0,0,1,0,1)^T$	$(0,0,1,0,1)^T$	$(0,0,1,0,1)^T$
τ^2	$(0,0,1,0,0)^T$	$(0,0,0,0,0)^T$	$(0,0,0,0,1)^T$
θ_3	-42872.5	0	-214703
HQ_3	0	0	366.2
MQ_3	0	0	508.8
LQ_3	219	0	281.5
γ^3	$(0,0,1,0,1)^T$	$(0,0,1,0,1)^T$	$(0,0,1,0,1)^T$
τ^3	$(0,0,1,0,0)^T$	$(0,0,1,0,1)^T$	$(0,0,1,0,0)^T$

5.2 Experiments on the example discussed in [20, Sec. 3.1]

In [20, Sec. 3.1] an example is given for which Algorithm 1 does not work if $\varepsilon = 0$. In particular, there are three players, each moving one variable in

$[-10, 10]$, that minimize the same continuous function

$$\begin{aligned}\overline{P}(\mathbf{x}) = -x^1x^2 - x^1x^3 - x^2x^3 + \\ \max\{0, x^1 - 1\}^2 + \max\{0, -x^1 - 1\}^2 + \\ \max\{0, x^2 - 1\}^2 + \max\{0, -x^2 - 1\}^2 + \\ \max\{0, x^3 - 1\}^2 + \max\{0, -x^3 - 1\}^2,\end{aligned}$$

and assume the classical Gauss-Seidel iterations in which the players take turns to move their variables. As shown in [20], if $\mathbf{x}^0 = \left(-1-\epsilon, 1+\frac{1}{2}\epsilon, -1-\frac{1}{4}\epsilon\right)^T$, with $\epsilon \in (0, 9]$, then the algorithm with $\varepsilon = 0$ produces an infinite sequence:

$$\begin{pmatrix} -1-\epsilon \\ 1+\frac{1}{2}\epsilon \\ -1-\frac{1}{4}\epsilon \end{pmatrix} \to \begin{pmatrix} 1+\frac{1}{8}\epsilon \\ 1+\frac{1}{2}\epsilon \\ -1-\frac{1}{4}\epsilon \end{pmatrix} \to \begin{pmatrix} 1+\frac{1}{8}\epsilon \\ -1-\frac{1}{16}\epsilon \\ -1-\frac{1}{4}\epsilon \end{pmatrix} \to \begin{pmatrix} 1+\frac{1}{8}\epsilon \\ -1-\frac{1}{16}\epsilon \\ 1+\frac{1}{32}\epsilon \end{pmatrix}$$
$$\to \begin{pmatrix} -1-\frac{1}{64}\epsilon \\ -1-\frac{1}{16}\epsilon \\ 1+\frac{1}{32}\epsilon \end{pmatrix} \to \begin{pmatrix} -1-\frac{1}{64}\epsilon \\ 1+\frac{1}{128}\epsilon \\ 1+\frac{1}{32}\epsilon \end{pmatrix} \to \begin{pmatrix} -1-\frac{1}{64}\epsilon \\ 1+\frac{1}{128}\epsilon \\ -1-\frac{1}{256}\epsilon \end{pmatrix} \to \cdots$$

This sequence has six limit points

$$\begin{pmatrix} 1 \\ 1 \\ -1 \end{pmatrix}, \quad \begin{pmatrix} 1 \\ -1 \\ -1 \end{pmatrix}, \quad \begin{pmatrix} 1 \\ -1 \\ 1 \end{pmatrix}, \quad \begin{pmatrix} -1 \\ -1 \\ 1 \end{pmatrix}, \quad \begin{pmatrix} -1 \\ 1 \\ 1 \end{pmatrix}, \quad \begin{pmatrix} -1 \\ 1 \\ -1 \end{pmatrix},$$

and none of them is a 0-approximate equilibrium of the potential game.

So the question is: what happens when $\varepsilon > 0$? By Theorem 4 we know that the sequence produced by Algorithm 1 is finite and the returned point is an ε-approximate equilibrium of the potential game. To strengthen this conviction, we show the sequence produced by Algorithm 1 with $\varepsilon = 1e-3$ and $\mathbf{x}^0 = (-1.01, 1.005, -1.0025)^T$, see table 7. At iteration 4 the algorithm with $\varepsilon = 1e-3$ deviates from the path described in [20], in that, $\mathbf{x}^4$ is set equal to $\mathbf{x}^3$ and not equal to $(-1.000156, -1.000625, 1.000312)$ since $\overline{P}(1.00125, -1.000625, 1.000312) - \overline{P}(-1.000156, -1.000625, 1.000312) = 1.001253 - 1.000625 \leq \varepsilon$. This simple and practical rule is sufficient to produce a finite sequence converging to an ε-approximate equilibrium of the potential game. In particular, we observe that the computed point is a 0-approximate equilibrium.

5.3 Experiments on discrete flow control problems on networks

Let us consider the potential GNEP described in section 2.2 on the network depicted in figure 4. The order of the links is the following

$(1 \to 2)$, $(1 \to 6)$, $(1 \to 5)$, $(2 \to 3)$, $(2 \to 7)$, $(2 \to 6)$, $(3 \to 4)$, $(3 \to 8)$, $(3 \to 7)$, $(4 \to 8)$, $(5 \to 6)$, $(5 \to 10)$, $(5 \to 9)$, $(6 \to 7)$, $(6 \to 11)$, $(6 \to 10)$, $(7 \to 8)$, $(7 \to 12)$, $(7 \to 11)$, $(8 \to 12)$, $(9 \to 10)$, $(10 \to 11)$, $(11 \to 12)$.

Table 7 The sequence produced by Algorithm 1 when solving the example discussed in [20, Sec. 3.1].

	$\overline{P}$	x^1	x^2	x^3
0	1.010169	-1.01	1.005	-1.0025
1	1.005042	1.00125	1.005	-1.0025
2	1.002511	1.00125	-1.000625	-1.0025
3	1.001253	1.00125	-1.000625	1.000312
4	1.001253	1.00125	-1.000625	1.000312
5	-4.004687	1.00125	2.000781	1.000312
6	-6.256797	1.00125	2.000781	2.501016
7	-11.317715	3.250898	2.000781	2.501016
⋮		⋮		
17	-51.00595	9.223106	9.889853	8.556601
18	-54.695792	9.223106	9.889853	10
19	-56.767574	10	9.889853	10
20	-57	10	10	10

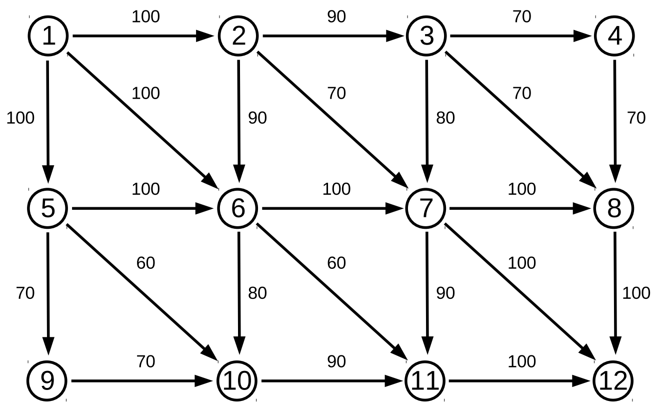

Fig. 4 A network with capacities.

We assume that there are $N = 10$ users whose data are:

$$A = \begin{pmatrix} 1&1&1&1&1&0&0&0&0&0 \\ 0&0&0&0&0&1&1&0&0&0 \\ 0&0&0&0&0&0&0&1&1&1 \\ 1&1&1&0&0&0&0&0&0&0 \\ 0&0&0&1&0&0&0&0&0&0 \\ 0&0&0&0&1&0&0&0&0&0 \\ 1&0&0&0&0&0&0&0&0&0 \\ 0&1&0&0&0&0&0&0&0&0 \\ 0&0&1&0&0&0&0&0&0&0 \\ 1&0&0&0&0&0&0&0&0&0 \\ 0&0&0&0&0&0&0&0&1&0 \\ 0&0&0&0&0&0&0&0&0&1 \\ 0&0&0&0&0&0&0&1&0&0 \\ 0&0&0&0&0&0&1&0&1&0 \\ 0&0&0&0&1&1&0&0&0&0 \\ 0&0&0&0&0&0&0&0&0&0 \\ 0&0&1&1&0&0&0&0&0&0 \\ 0&0&0&0&0&0&1&0&0&0 \\ 0&0&0&0&0&0&0&0&1&0 \\ 1&1&1&1&0&0&0&0&0&0 \\ 0&0&0&0&0&0&0&1&0&0 \\ 0&0&0&0&0&0&0&1&0&1 \\ 0&0&0&0&1&1&0&1&1&1 \end{pmatrix}, \qquad c = \begin{pmatrix} 100 \\ 100 \\ 100 \\ 90 \\ 70 \\ 90 \\ 70 \\ 70 \\ 80 \\ 70 \\ 100 \\ 60 \\ 70 \\ 100 \\ 60 \\ 80 \\ 100 \\ 100 \\ 90 \\ 100 \\ 70 \\ 90 \\ 100 \end{pmatrix},$$

and $a_l = 20$, $b_l = 1$, for all l, $u^\nu = 30$, $d_\nu = 10$, $e_\nu = 100$, for all ν.

In general, computing a global solution of problem (9) can be a demanding task, e.g. if $N >> 0$. For this reason, to compute an ε-approximate equilibrium of this generalized potential game, we used Algorithm 1, and, for simplicity, we assumed the order of play 1, 2, ..., 10. It is important to say that, by using this simple procedure, we were able to compute all the best responses by performing simple enumerations on $[0, u^\nu] \cap \mathbb{Z}$, which are extremely fast. We set $\varepsilon = 1e-3$ and $\mathbf{x}^0 = 0$. We report the sequence produced by the algorithm in table 8. The algorithm converged in 140 iterations, and took less than one second to compute the equilibrium. In particular, it is easy to check that the computed point is a 0-approximate equilibrium of the game.

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Table 8 The sequence produced by Algorithm 1 when solving the discrete flow control problem.

	$\overline{P}$	x^1	x^2	x^3	x^4	x^5	x^6	x^7	x^8	x^9	x^{10}
0	-455	0	0	0	0	0	0	0	0	0	0
1	-488.7	30	0	0	0	0	0	0	0	0	0
2	-522.1	30	30	0	0	0	0	0	0	0	0
3	-550.6	30	30	23	0	0	0	0	0	0	0
4	-571.9	30	30	23	11	0	0	0	0	0	0
5	-584.1	30	30	23	11	4	0	0	0	0	0
6	-617.9	30	30	23	11	4	30	0	0	0	0
7	-651.8	30	30	23	11	4	30	30	0	0	0
8	-685.3	30	30	23	11	4	30	30	30	0	0
9	-716.8	30	30	23	11	4	30	30	30	29	0
10	-730.4	30	30	23	11	4	30	30	30	29	5
11	-735.3	23	30	23	11	4	30	30	30	29	5
12	-735.3	23	29	23	11	4	30	30	30	29	5
13	-735.4	23	29	24	11	4	30	30	30	29	5
14	-736.3	23	29	24	14	4	30	30	30	29	5
15	-736.3	23	29	24	14	4	30	30	30	29	5
16	-738.9	23	29	24	14	4	25	30	30	29	5
17	-738.9	23	29	24	14	4	25	30	30	29	5
18	-738.9	23	29	24	14	4	25	30	30	29	5
19	-738.9	23	29	24	14	4	25	30	30	29	5
20	-741.4	23	29	24	14	4	25	30	30	29	9
⋮						⋮					
121	-750.9	19	21	21	22	11	18	30	18	25	22
122	-750.9	19	21	21	22	11	18	30	18	25	22
123	-750.9	19	21	21	22	11	18	30	18	25	22
124	-750.9	19	21	21	22	11	18	30	18	25	22
125	-750.9	19	21	21	22	11	18	30	18	25	22
126	-750.9	19	21	21	22	11	18	30	18	25	22
127	-750.9	19	21	21	22	11	18	30	18	25	22
128	-750.9	19	21	21	22	11	18	30	18	25	22
129	-751	19	21	21	22	11	19	30	18	24	22
130	-751	19	21	21	22	11	19	30	18	24	22
131	-751	19	21	21	22	11	19	30	18	24	22
132	-751	19	21	21	22	11	19	30	18	24	22
133	-751	19	21	21	22	11	19	30	18	24	22
134	-751	19	21	21	22	11	19	30	18	24	22
135	-751	19	21	21	22	11	19	30	18	24	22
136	-751	19	21	21	22	11	19	30	18	24	22
137	-751	19	21	21	22	11	19	30	18	24	22
138	-751	19	21	21	22	11	19	30	18	24	22
139	-751	19	21	21	22	11	19	30	18	24	22
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