On symplectic semifield spreads of $PG(5, q^2)$, q odd

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August 29, 2016

Abstract

Let $q^2 > 2 \cdot 3^8$ be odd. We prove that there exist exactly three non-equivalent symplectic semifield spreads of $PG(5, q^2)$ whose associated semifield has center containing $\mathbb{F}_q$ by proving that there exist exactly three non-equivalent $\mathbb{F}_q$-linear sets of rank 6 in $PG(5, q^2)$ disjoint from the secant variety of a Veronese surface $\mathcal{V}$ under the action of the automorphism group of $\mathcal{V}$.

1 Introduction

Let $PG(r-1, q)$ be the projective space of dimension $r-1$ over the finite field $\mathbb{F}_q$ of order q. A *planar spread* $\mathcal{S}$ of $PG(2n-1, q)$, which we will call simply *spread* from now on, is a partition of the point-set in $(n-1)$-dimensional subspaces. With any spread $\mathcal{S}$ it is associated a translation plane $A(\mathcal{S})$ of order q^n in the following way: embed $PG(2n-1, q)$ in $PG(2n, q)$ as a hyperplane section, then the points of $A(\mathcal{S})$ are the points of $PG(2n, q) \setminus PG(2n-1, q)$, the lines are the n-dimensional subspaces of $PG(2n, q)$ intersecting $PG(2n-1, q)$ in an element of $\mathcal{S}$ and the incidence is containment (see e.g. [9, Sect.5.1]). Translation planes associated with different spreads are isomorphic if and only if there is a collineation of $PG(2n-1, q)$ mapping one spread to the other (see e.g. [21]), and in such a case we say that the two spreads are *equivalent*. Let $M(n, q)$ be the vector space of the matrices of $n \times n$ with entries in $\mathbb{F}_q$. Without loss of generality, we may always assume that $S(\infty) := \{(\mathbf{0}, \mathbf{y}), \mathbf{y} \in \mathbb{F}_q^n\}$ and $S(0) := \{(\mathbf{x}, \mathbf{0}), \mathbf{x} \in \mathbb{F}_q^n\}$ belong to $\mathcal{S}$, hence we may write $\mathcal{S} = \{S(A), A \in \mathbb{C}\} \cup S(\infty)$, with $S(A) := \{(\mathbf{x}, \mathbf{x}A), \mathbf{x} \in \mathbb{F}_q^n\}$ and

*This work was supported by the Research Project of MIUR (Italian Ministry for University and Research) and by the Research group GNSAGA of INDAM.

$\mathbb{C} \subset M(n,q)$ such that $|\mathbb{C}| = q^n$, $\mathbb{C}$ contains the zero matrix and $A - B$ is non-singular for every $A, B \in \mathbb{C}$. The set $\mathbb{C}$ is called *the spread set* associated with $\mathcal{S}$.

A spread $\mathcal{S}$ is said to be *Desarguesian* if $A(\mathcal{S})$ is isomorphic to $\mathrm{AG}(2, q^n)$ and hence a plane coordinatized by the field of order q^n. The spread $\mathcal{S}$ is said to be a semifield spread if $A(\mathcal{S})$ is a plane of Lenz-Barlotti class V and this is equivalent to saying that $A(\mathcal{S})$ is coordinatized by a semifield and it is called semifield plane. A finite semifield $(\mathbb{S}, +, *)$ is a finite nonassociative division algebra. If $(\mathbb{S}, +, *)$ satisfies all the axioms for a semifield except, possibly, the existence of an identity element for the multiplication, then it is called a presemifield (see, e.g., [16] Chapter 6 for definitions and notations on finite semifields). The left nucleus $\mathbb{N}_l$ and the center $\mathbb{K}$ of a semifield $\mathbb{S}$ are fields contained in $\mathbb{S}$ as substructures ($\mathbb{K}$ subfield of $\mathbb{N}_l$) and $\mathbb{S}$ is a vector space over $\mathbb{N}_l$ and over $\mathbb{K}$. Semifields are studied up to an equivalence relation called *isotopy* (which corresponds to the study of semifield planes up to isomorphisms and hence to the study of semifield spreads up to equivalences) and the dimensions of a semifield over its left nucleus and over its center are invariant up to isotopy.

A semifield spread $\mathcal{S}$ is such that there exists an elementary abelian subgroup G of $PGL(2n, q)$ of order q^n fixing an element $X \in \mathcal{S}$ point-wise and acting regularly on $\mathcal{S}$. If we set $X = S(\infty)$, then $\mathbb{C}$ turns out to be a subgroup of the additive group of $M(n,q)$ ([9, Sect.5.1]) and hence it is a vector space over some subfield of $\mathbb{F}_q$. Also, $\mathbb{C}$ is a set of q^n $n \times n$ matrices over $\mathbb{F}_q$, containing the zero matrix and any nonzero matrix is invertible. This has led to the following geometric interpretation (see [18] as it first appeared for $n = 2$ and [15] for the general case). Let $\mathbb{P}_M := PG(n^2 - 1, q) = PG(M(n,q), \mathbb{F}_q)$ be the projective space induced by $M(n,q)$. The Segre variety $\mathcal{S} = \mathcal{S}_{n,n}(q)$ of $\mathbb{P}_M$ is the set of all points $\langle X \rangle$ of $PG(n^2-1, q)$ such that X is a matrix of rank 1. The variety $\mathcal{S}_{n,n}$ contains two systems $\mathcal{R}_1$ and $\mathcal{R}_2$, all elements of which are $(n-1)-$subspaces with the following properties: (a) the subspaces of $\mathcal{R}_1$ (resp. $\mathcal{R}_2$) are mutually disjoint; (b) if A and B are $(n-1)-$subspaces belonging to different systems of $\mathcal{S}_{n,n}$, then $A \cap B$ is a point; (c) each point of $\mathcal{S}_{n,n}$ belongs to a unique element of $\mathcal{R}_1$ and to a unique element of $\mathcal{R}_2$.

A k–dimensional subspace S of $PG(n^2-1, q)$ is a *k–th secant subspace* to $\mathcal{S}_{n,n}$ when $S = \langle P_1, P_2, \ldots, P_{k+1} \rangle$ and $\{P_1, P_2, \ldots, P_{k+1}\} \subset \mathcal{S}_{n,n}$. The $(n-2)$–th *secant variety* $\Omega(\mathcal{S}_{n,n})$ of $\mathcal{S}_{n,n}$ is the set of all points of $PG(n^2-1, q)$ which belong to an $(n-2)$–th secant subspace to $\mathcal{S}_{n,n}$. Note that

$$\Omega(\mathcal{S}_{n,n}) = \{\langle X \rangle \mid X \in M(n, \mathbb{F}_q), detX = 0\},$$

i.e. $\Omega(\mathcal{S}_{n,n})$ is an algebraic variety, also called *determinantal hypersurface*, defined by the non–invertible matrices of $M(n,q)$ (see, e.g., [11, Section 25.5] and [10, Ch.9] for further details).

Define $\Gamma := \Gamma_{A\sigma B}$ the collineation of $\mathbb{P}_M$ induced by the semlinear map $\gamma = \gamma_{A\sigma B}$ of $M(n,q)$ whose rule is

$$\gamma : X \in M(n,q) \mapsto AX^{\sigma}B \in M(n,q) \tag{1}$$

(where A and B are invertible matrices of $M(n,q)$ and $\sigma \in Aut(\mathbb{F}_q)$). The collineation Γ leaves invariant the Segre variety $\mathcal{S}_{n,n}$ and the set

$$\mathcal{H}(\mathcal{S}_{n,n}) := \{\Gamma_{A\sigma B} : \ \sigma \in Aut(\mathbb{F}_q), A \text{ and } B \text{ invertible matrices of } M(n,q)\}$$

turns out to be the automorphism group of $\mathcal{S}_{n,n}$ preserving the systems $\mathcal{R}_1$ and $\mathcal{R}_2$ of $\mathcal{S}_{n,n}$ (see [11]). The group $\mathcal{H}(\mathcal{S}_{n,n})$ has index 2 in the automorphism group $Aut(\mathcal{S}_{n,n})$.

If $\mathbb{C}$ is an $\mathbb{F}_s$-vector space, $q = s^t$, then $\dim_{\mathbb{F}_s} \mathbb{C} = nt$ and it defines a subset $L(\mathbb{C})$ of $PG(n^2-1,q)$ which is an $\mathbb{F}_s$-linear set of rank nt. So finding a semifield spread of $PG(2n-1,q)$ (and hence a semifield plane of order q^n) is equivalent to finding an $\mathbb{F}_s$-linear set of $PG(n^2-1,q)$, $q = s^t$, of rank nt disjoint from $\mathcal{S}_{n,n}$. More precisely:

Theorem 1. ([15, Cor. 5.3]) *Let $\mathbb{S}_1$ and $\mathbb{S}_2$ be two semifields of order q^n, with left nucleus containing $\mathbb{F}_q$ and center containing $\mathbb{F}_s$ ($q = s^t$) and let $\mathbb{C}_1$ and $\mathbb{C}_2$ be the associated semifield spread sets, respectively. Then $\mathbb{S}_1$ and $\mathbb{S}_2$ are isotopic if and only if there is a collineation Γ of $\mathcal{H}(\mathcal{S}_{n,n})$ such that $L(\mathbb{C}_2) = L(\mathbb{C}_1)^{\Gamma} = L(\mathbb{C}_1^{\gamma})$, with $\mathbb{C}_1^{\gamma} = \mathbb{C}_2$.*

By [19, Lemma 2], any collineation of the group $\mathcal{H}(\mathcal{S}_{n,n})$ interchanging the two systems of the variety defines another presemifield $\mathbb{S}^t$, which is the so called *transpose* of $\mathbb{S}$, introduced by Knuth in [14].

If $\mathbb{S}$ is a symplectic semifield, we may assume that $\mathbb{S}$ is defined by a spread set $\mathbb{C}$ of symmetric matrices (see, e.g., [13] and [23]). In this case the associated linear set $L(\mathbb{C})$ is contained in the projective subspace $\Delta = PG(Sym_n(q), \mathbb{F}_q) = PG(\frac{n(n+1)}{2}-1,q)$ of $\mathbb{P}_M$ defined by the symmetric matrices of order n over $\mathbb{F}_q$, and it is disjoint from the variety $\mathcal{M}$ defined by the non–invertible $n \times n$ symmetric matrices over $\mathbb{F}_q$. By [10, p.99], $\mathcal{M} = \mathcal{M}(\mathcal{V}_n)$ is the $(n-2)$-th secant variety of the quadric Veronesean $\mathcal{V}_n = \Delta \cap \mathcal{S}_{n,n}$ defined by the $n \times n$ symmetric matrices of rank 1. Hence we have

Theorem 2. *([19, Thm. 2]) If $\mathbb{S}$ is a symplectic semifield of order q^n, left nucleus containing $\mathbb{F}_q$ and with center containing $\mathbb{F}_s$ ($q = s^t$), then the associated $\mathbb{F}_s$–linear set L of $\mathbb{P} = PG(n^2-1,q)$ is contained in an $(\frac{n(n+1)}{2}-1)$–dimensional subspace of $\mathbb{P}$ intersecting $\mathcal{S}_{n,n}$ in a quadric Veronesean $\mathcal{V}_n$.*

The converse of the previous theorem holds true as well (see [19, Remark 1]). The automorphism group $\mathcal{G}$ of $\mathcal{V}_n$ consists of the collineations of $PG(Sym_n(q), \mathbb{F}_q)$ induced by the semilinear maps $\gamma = \gamma_{A\sigma A^T}$ of $Sym_n(q)$ with rule

$$\gamma : X \in Sym_n(q) \mapsto AX^\sigma A^T \in Sym_n(q) \tag{2}$$

(where A is an invertible matrix of $M(n, q)$, A^T denotes its transpose matrix and $\sigma \in Aut(\mathbb{F}_q)$). Since any element of $Aut(\mathcal{V}_n)$ can be extended to an element of $Aut(\mathcal{S}_{n,n})$, fixing or interchanging the systems, and since any symplectic semifield is self–transpose (i.e. the isotopy classes of $\mathbb{S}$ and $\mathbb{S}^t$ coincide), from Theorems 1 and 2, we have the following result.

Corollary 1. *Let $\mathbb{S}_1$ and $\mathbb{S}_2$ be two symplectic semifields of order q^n, with left nucleus containing $\mathbb{F}_q$ and center containing $\mathbb{F}_s$ ($q = s^t$) and let $\mathbb{C}_1$ and $\mathbb{C}_2$ be the associated semifield spread sets, respectively. Then $\mathbb{S}_1$ and $\mathbb{S}_2$ are isotopic if and only if there is a collineation Γ of $\mathcal{G}$ such that $L(\mathbb{C}_2) = L(\mathbb{C}_1)^\Gamma = L(\mathbb{C}_1^\gamma)$, with $\mathbb{C}_1^\gamma = \mathbb{C}_2$.*

Symplectic semifield spreads and commutative semifields are related via the cubical array ([14]) and, as noted in [13], we know few examples of such structures. After Kantor's paper was published, some examples of commutative semifields in odd characteristic have been constructed, all obtained by using perfect nonlinear Dembowski–Ostrom polynomials (see [26], [6], [3], [19], [4], [5] and [27]). For a complete list of proper commutative presemifields of odd order see [22, Table 1].

In this paper we focus on the case $n = 3$, i.e. on the symplectic semifield spreads of $PG(5, q^2)$, q odd, whose associated semifield has center containing $\mathbb{F}_q$. When q is even the only possibility is that the symplectic spread is Desarguesian ([8]).

About the case q odd, so far, the known symplectic semifield spreads of $PG(5, q^2)$ whose associated semifield has center containing $\mathbb{F}_q$ are:

- $\mathcal{D}$ The Desarguesian spread;
- $\mathcal{A}$ The symplectic semifield spreads associated with the twisted fields ([1], [2]).
- $\mathcal{BH}$ The symplectic semifield spreads associated with the commutative presemifields constructed in [6];
- $\mathcal{LMPT}$ The symplectic semifield spreads constructed in [19];
- $\mathcal{ZP}$ The symplectic semifield spreads associated with the commutative presemifields constructed in [27].

Here, using the connection with the associated linear sets, we get the following classification result.

Theorem 3. *Let $q^2 > 2 \cdot 3^8$ be odd. Then there exist only three non-equivalent symplectic semifield spreads of $PG(5, q^2)$ whose associated semifield has center containing $\mathbb{F}_q$: the Desarguesian spread $\mathcal{D}$, the symplectic semifield spread associated with a twisted field $\mathcal{A}$ and a spread comprising the constructions $\mathcal{BH}, \mathcal{LMPT}, \mathcal{ZP}$.*

2 Preliminary results

As we have observed in the introduction the study, up to equivalences, of symplectic semifield spreads of $PG(5, q^2)$ whose associated semifield has center containing $\mathbb{F}_q$ corresponds to the study, up to isotopisms, of symplectic semifields of order q^6, whose left nucleus contains $\mathbb{F}_{q^2}$ and whose center contains $\mathbb{F}_q$. Also, from Corollary 1, two $\mathbb{F}_q$–linear sets of rank 6 in $\mathbb{P} := PG(5, q^2)$ disjoint from the secant variety of the Veronese surface $\mathcal{V}_3$ of $\mathbb{P}$, which are not equivalent under the action of the automorphism group $\mathcal{G}$ of $\mathcal{V}_3$, produce non isotopic semifields.

From Theorems 1 and 2 this is equivalent to study up to the action of the collineation group $\mathcal{G}$ of $P\Gamma L(6, q^2)$ fixing $\mathcal{V}_3$.

2.1 Linear sets

The set L of $\mathbb{P} = PG(V, \mathbb{F}_{q^t}) = PG(r-1, q^t)$, V r-dimensional vector space over $\mathbb{F}_{q^t}$, is said to be an $\mathbb{F}_q$*–linear* set of rank m if it is defined by the non-zero vectors of an $\mathbb{F}_q$-vector subspace U of V of dimension m, i.e.

$$L = L_U = \{\langle \mathbf{u} \rangle_{\mathbb{F}_{q^t}} : \mathbf{u} \in U \setminus \{\mathbf{0}\}\}.$$

We point out that different vector subspaces can define the same linear set. For this reason a linear set and the vector space defining it must be considered as coming in pair. A point $P = \langle v \rangle_{\mathbb{F}_{q^t}} \in PG(r-1, q^t)$ has *weight* i in L_U if $\dim_{\mathbb{F}_q}(\langle v \rangle_{\mathbb{F}_{q^t}} \cap U) = i$, hence a point belongs to L_U if and only if it has weight at least 1. If $L_U \neq \emptyset$, we have $|L_U| \leq q^{m-1} + q^{m-2} + \cdots + q + 1$ and $|L_U| \equiv 1 \,(mod\, q)$. An $\mathbb{F}_q$–linear set L_U of $\mathbb{P}$ of rank m is *scattered* if all of its points have weight 1, or equivalently, if L_U has maximum size $q^{m-1} + q^{m-2} + \cdots + q + 1$. If $dim_{\mathbb{F}_q} U = dim_{\mathbb{F}_{q^t}} V = r$ and $\langle U \rangle_{q^t} = V$, then the $\mathbb{F}_q$–linear set L_U is a *subgeometry* of $\mathbb{P}$ isomorphic to $PG(r-1, q)$. If $t = 2$, then L_U is a Baer subgeometry of $PG(r-1, q^2)$. (For further details on linear sets see [25].) Recall that a subgeometry of $\mathbb{P} = PG(r-1, q^t)$ isomorphic to $PG(r-1, q)$ can be also defined as the set of points of $\mathbb{P}$ fixed by a semilinear collineation σ of order t. A subspace S of $\mathbb{P}$ intersects the subgeometry in a subspace of the same dimension if and only if $S = S^\sigma$ and if two subspaces S_1 and S_2 fixed by σ meet in a subspace S_3, then also S_3 is obviously fixed by σ and hence it intersects the subgeometry in a subspace of the same dimension.

In [20], the authors give the following characterization of $\mathbb{F}_q$–linear sets. Let $\Sigma = PG(m-1,q)$ be a subgeometry of $\Sigma^* = PG(m-1,q^t)$, let Γ be an $(m-r-1)$–dimensional subspace of Σ^* disjoint from Σ and let $\mathbb{P} = PG(r-1,q^t)$ be an $(r-1)$–dimensional subspace of Σ^* disjoint from Γ. Denote by

$$L = \{\langle \Gamma, P\rangle_{q^t} \cap \mathbb{P} \,:\, P \in \Sigma\}$$

the projection of Σ from Γ to $\mathbb{P}$. We call Γ and $\mathbb{P}$, respectively, the *center* and the *axis* of the projection. Denote by $p_{\Gamma,\mathbb{P}}$ the map from Σ to L defined by $P \mapsto \langle \Gamma, P\rangle_{q^t} \cap \mathbb{P}$ for each point P of Σ. By definition $p_{\Gamma,\mathbb{P}}$ is surjective and $L = p_{\Gamma,\mathbb{P}}(\Sigma)$.

Theorem 4. [20, Theorems 1 and 2] *If L is a projection of $\Sigma = PG(m-1,q)$ to $\mathbb{P} = PG(r-1,q^t)$, then L is an $\mathbb{F}_q$–linear set of $\mathbb{P}$ of rank m and $\langle L\rangle_{q^t} = \mathbb{P}$. Conversely, if L is an $\mathbb{F}_q$–linear set of $\mathbb{P}$ of rank m and $\langle L\rangle_{q^t} = \mathbb{P}$, then either L is a subgeometry of $\mathbb{P}$ or for each $(m-r-1)$–dimensional subspace Γ of $\Sigma^* = PG(m-1,q^t)$ disjoint from Λ there exists a subgeometry Σ of Σ^* disjoint from Γ such that $L = p_{\Gamma,\mathbb{P}}(\Sigma)$.*

2.1.1 Linear sets in $PG(5,q^2)$ of rank 6

Let us focus on the case $t = 2$. In such a case any point of L_U has weight either 1 or 2. In the following preliminary result we prove which are the possible geometric configurations for an $\mathbb{F}_q$–linear set of $PG(5,q^2)$ of rank 6.

Theorem 5. *If L_U is an $\mathbb{F}_q$–linear set of rank 6 of $\mathbb{P} := PG(5,q^2)$, then one of the following configurations occurs:*

(1) L_U is a plane of $\mathbb{P}$.

(2) L_U is a union of $q+1$ planes of $\mathbb{P}$ passing through a line and contained in a 3–dimensional space of $\mathbb{P}$.

(3) L_U is a union of q^3+q^2+q+1 lines passing through a point and contained in a hyperplane of $\mathbb{P}$.

(4) L_U is a Baer subgeometry of $\mathbb{P}$.

Proof. Let U_1 be the maximal $\mathbb{F}_{q^2}$-subspace contained in U and let U_2 be such that $U = U_1 \oplus U_2$. Then L_U is a cone with vertex L_{U_1} and base L_{U_2} and L_{U_2} is a Baer subgeometry of the projective subspace $\langle L_{U_2}\rangle_{q^2}$ of $\mathbb{P}^*$. We have $\dim_{\mathbb{F}_q} U = 6 = 2\dim_{\mathbb{F}_{q^2}} U_1 + \dim_{\mathbb{F}_q} U_2$ and $\dim_{\mathbb{F}_{q^2}} U_1 \in \{0,1,2,3\}$. Hence the configurations listed in the statement according to $\dim_{\mathbb{F}_{q^2}} U_1$. □

Since the previous configurations are invariant under the action of the collineation group of the space, from Theorem 1 it follows that

Theorem 6. *Two symplectic semifields spreads of $PG(5,q^2)$ whose associated $\mathbb{F}_q$–linear sets have two different configurations of Theorem 5 are not equivalent.* $\square$

2.2 Quadric Veronesean and its secant variety

In this section we want to focus on the Veronese surface and its secant variety.

The Veronese surface is the variety $\mathcal{V} := \mathcal{V}_3$ of $PG(5,q)$ image of the *Veronese embedding* of $PG(2,q)$:

$$v : \langle(x_0,x_1,x_2)\rangle \in PG(2,q) \mapsto \langle(x_0^2,x_1^2,x_2^2,x_0x_1,x_0x_2,x_1x_2)\rangle \in PG(5,q). \qquad (3)$$

We stress some important properties of the Veronese surface $\mathcal{V}$ (for further details about the Veronese surface over finite fields see [11, Ch. 25]). If we use the so-called determinantal representation of the Veronese variety of degree 2 (see [10, Example 2.6]), then $PG(5,q) = PG(Sym_3(q),q)$, where $Sym_3(q)$ is the space of symmetric matrices of order 3, and the points of $\mathcal{V}$ correspond to the matrices of rank 1, i.e. $P \in \mathcal{V}$ if and only if $P = \langle \begin{pmatrix} x_0^2 & x_0x_1 & x_0x_2 \\ x_0x_1 & x_1^2 & x_1x_2 \\ x_0x_2 & x_1x_2 & x_2^2 \end{pmatrix} \rangle$ for some $(x_0,x_1,x_2) \neq (0,0,0)$. The image of a line ℓ of $PG(2,q)$ is a conic, intersection of $\mathcal{V}$ with a suitable plane, which is then called a *conic plane* of $\mathcal{V}$. The variety $\mathcal{V}$ is smooth, hence every point $P \in \mathcal{V}$ has a *tangent plane* π_P such that $\pi_P \cap \mathcal{V} = \{P\}$; the tangent lines of the conics of $\mathcal{V}$ at P are all contained in π_P. By [11, Thms 25.1.11, 25.1.16], every two conic planes and every two tangent planes meet in a point. A section $H \cap \mathcal{V}$, where H is a hyperplane of $PG(5,q)$, consists of the points of $v(\mathcal{C})$, where $\mathcal{C}$ is a conic of $PG(2,q)$. Then, if $\mathcal{C}$ is a non–degenerate conic of $PG(2,q)$, then H meets $\mathcal{V}$ along a rational quartic curve. If $\mathcal{C}$ is a pair of distinct lines of $PG(2,q)$, then H meets $\mathcal{V}$ at two non–degenerate conics with exactly one point in common. If the conic $\mathcal{C}$ of $PG(2,q)$ is a repeated line, then the corresponding hyperplane H of $PG(5,q)$ meets $\mathcal{V}$ at a non–degenerate conic. Finally, $\mathcal{C}$ can consist of two conjugated lines of $PG(2,q^2)$ intersecting in an $\mathbb{F}_q$–rational point, hence $H \cap \mathcal{V}$ exactly consists of one $\mathbb{F}_q$–rational point. It follows that a hyperplane H contains a conic plane or a tangent plane if and only if $H \cap \mathcal{V} = v(\mathcal{C})$ and $\mathcal{C}$ is a reducible (in $PG(2,q)$ or in $PG(2,q^2)$) conic.

The automorphism group $\mathcal{G}$ of $\mathcal{V}$ is the lifting of the group $P\Gamma L(3,q)$ acting in the obvious way: $v(P)^g = v(P^g)$, for each element $g \in P\Gamma L(3,q)$. The linear part of $\mathcal{G}$ is given by the matrices:

$$\begin{pmatrix} a_{11}^2 & a_{12}^2 & a_{13}^2 & 2a_{11}a_{12} & 2a_{11}a_{13} & 2a_{12}a_{13} \\ a_{21}^2 & a_{22}^2 & a_{23}^2 & 2a_{21}a_{22} & 2a_{21}a_{23} & 2a_{22}a_{23} \\ a_{31}^2 & a_{32}^2 & a_{33}^2 & 2a_{31}a_{32} & 2a_{31}a_{33} & 2a_{32}a_{33} \\ a_{11}a_{21} & a_{12}a_{22} & a_{13}a_{23} & a_{11}a_{22}+a_{12}a_{21} & a_{11}a_{23}+a_{13}a_{21} & a_{12}a_{23}+a_{13}a_{22} \\ a_{11}a_{31} & a_{12}a_{32} & a_{13}a_{33} & a_{11}a_{32}+a_{12}a_{31} & a_{11}a_{33}+a_{13}a_{31} & a_{12}a_{33}+a_{13}a_{32} \\ a_{21}a_{31} & a_{22}a_{32} & a_{23}a_{33} & a_{21}a_{32}+a_{22}a_{31} & a_{21}a_{33}+a_{23}a_{31} & a_{22}a_{33}+a_{23}a_{32} \end{pmatrix} (\star)$$

such that $a_{ij} \in \mathbb{F}_q$ and the matrix $\begin{pmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{pmatrix}$ is non-singular.

The secant variety $\mathcal{M}$ of $\mathcal{V}$ is the variety of $PG(5, q)$ consisting of the points lying on a line secant to $\mathcal{V}$. It is a determinantal variety, i.e., it consists of the points determined by the symmetric matrices of order 3 $\left\{ \begin{pmatrix} Y_0 & Y_3 & Y_4 \\ Y_3 & Y_1 & Y_5 \\ Y_4 & Y_5 & Y_2 \end{pmatrix} : \ Y_i \in \mathbb{F}_q, i = 0, 1, \dots, 5, \text{ not all zero} \right\}$ with zero determinant. So, if we denote by $(Y_0, Y_1, \dots, Y_5)$ the homogeneous projective coordinates in $PG(5, q)$, $\mathcal{M}$ is the hypersurface with equation

$$\mathcal{M}: \ Y_0Y_1Y_2 - Y_0Y_5^2 - Y_1Y_4^2 - Y_2Y_3^2 + 2Y_3Y_4Y_5 = 0.$$

The hypersurface $\mathcal{M}$ has Veronesean $\mathcal{V}$ as double surface and the automorphism group $\mathcal{G}$ of $\mathcal{V}$ is the automorphism group of $\mathcal{M}$ as well.

In the following, we describe the intersection of a tangent hyperplane to $\mathcal{M}$ with $\mathcal{V}$.

Proposition 7. *Each tangent hyperplane to $\mathcal{M}$ at a point on $\mathcal{M} \setminus \mathcal{V}$ intersects $\mathcal{V}$ in $v(\ell)$, where ℓ is a line of $PG(2, q)$.*

Proof. Let $P \in \mathcal{M} \setminus \mathcal{V}$, then the tangent hyperplane T_P of $\mathcal{M}$ at a point $P = (u_0, u_1, u_2, u_3, u_4, u_5)$ has equation $(u_1u_2 - u_5^2)Y_0 + (u_0u_2 - u_4^2)Y_1 + (u_0u_1 - u_3^2)Y_2 + 2(u_4u_5 - u_2u_3)Y_3 + 2(u_3u_5 - u_1u_4)Y_4 + 2(u_3u_4 - u_0u_5)Y_5 = 0$. By [11, Thms. 25.1.11, 25.1.12], P is contained in a unique conic plane $\pi = \langle v(\ell) \rangle$, with ℓ a line of $PG(2, q)$. As $PGL(3, q)$ acts transitively on the lines of $PG(2, q)$, so does $\mathcal{G}$ on the conics planes, hence we can assume that ℓ is the line with equation $x_2 = 0$ and $\pi = \{(u_0, u_1, 0, u_3, 0, 0,), u_i \in \mathbb{F}_q\}$. So, for each point of π which is not on the Veronese surface the tangent hyperplane has equation $Y_2 = 0$ and such hyperplane intersects $\mathcal{V}$ in $v(\ell)$. □

When q is even, the hypersurface $\mathcal{M}$ contains the plane $\pi_N : Y_0 = Y_1 = Y_2 = 0$, which is disjoint from $\mathcal{V}$. Such a plane is called the *nucleus* of $\mathcal{V}$, and consists of all nuclei of conics of $\mathcal{V}$. The conic planes, the tangent plane and the nucleus (in even characteristic) are the maximal projective subspaces contained in the variety $\mathcal{M}$ (see [8, Prop.1]).
By the Chevalley-Warning Theorem, the maximal projective subspaces disjoint by $\mathcal{M}$ are planes. We conclude this paragraph with the following result on the planes disjoint from $\mathcal{M}$, which will be very useful in the sequel.

Proposition 8. *A plane π of $PG(5,q)$ is disjoint from the secant variety $\mathcal{M}$ of the Veronese surface $\mathcal{V}$ if and only if in the cubic extension the intersection $\pi \cap \mathcal{M}$ consists of three nonconcurrent lines conjugated under the $\mathbb{F}_q$–linear collineation induced by $Gal(\mathbb{F}_{q^3}/\mathbb{F}_q)$. Moreover, if q is odd, then there are exactly two orbits of such planes under the action of the automorphism group $\mathcal{G}$ of $\mathcal{V}$.*

Proof. For any projective subspace, say S, and for any algebraic variety, say $\mathcal{W}$, of $PG(5,q)$ we will denote by $S(q^t)$ and $\mathcal{W}(q^t)$, respectively, their extensions over $\mathbb{F}_{q^t}$.

The intersection $\mathcal{C} = \mathcal{M} \cap \pi$ is a curve of degree 3. By the Hasse-Weil bound, it easy follows that an absolutely irreducible cubic defined over $\mathbb{F}_q$ has at least one $\mathbb{F}_q$–rational points. Then $\mathcal{C}$ does not have any $\mathbb{F}_q$–rational points if and only if it consists of three non–concurrent lines over $\mathbb{F}_{q^3}$, say $\ell, \ell^\tau, \ell^{\tau^2}$, where τ is the semilinear collineation of $PG(5,q^3)$ induced by the automorphism $x \mapsto x^q$. Then $PG(5,q) = Fix\,\tau$ and obviously $\mathcal{V}^\tau = \mathcal{V}$ and $\mathcal{M}^\tau = \mathcal{M}$. Let P be $\ell \cap \ell^\tau$, hence the points P, P^τ, P^{τ^2} are singular for $\mathcal{C}$. Then either P, P^τ and P^{τ^2} belong to $\mathcal{V}(q^3)$ (hence they are singular for $\mathcal{M}(q^3)$) or the points are non singular for $\mathcal{M}(q^3)$ and $\pi(q^3)$ is contained in the intersection of their tangent hyperplanes.

In the first case, we have that P, P^τ, P^{τ^2} are points of $\mathcal{V}(q^3) \setminus \mathcal{V}$. Then, using the Veronese map defined in (3), it follows that $P^{\tau^i} = v(R^{\tau^i})$, $R \in PG(2,q^3) \setminus PG(2,q)$, $i = 0,1,2$, where by abuse of notation we have denoted by τ also the $\mathbb{F}_q$–linear collineation of $PG(2,q^3)$ induced by $Gal(\mathbb{F}_{q^3}/\mathbb{F}_q)$. If the points R, R^τ, R^{τ^2} were collinear, then π would contain the image under v of the subline through them and contained in $PG(2,q)$, i.e. π would contain a conic of $\mathcal{V}$ and hence $\pi \subset \mathcal{M}$, a contradiction. It follows that R, R^τ, R^{τ^2} are not collinear. Let $R = \langle (x,y,z) \rangle \in PG(2,q^3)$, then R, R^τ, R^{τ^2} are not collinear if and only if $\begin{pmatrix} x & y & z \\ x^q & y^q & z^q \\ x^{q^2} & y^{q^2} & z^{q^2} \end{pmatrix}$ is nonsingular and by [17, Lemma 3.51] this is equivalent to having $\{x,y,z\}$ independent over $\mathbb{F}_q$. Let $R' = \langle (x',y',z') \rangle \in PG(2,q^3)$ be another point such that R', R'^τ, R'^{τ^2} are not collinear, hence $\{x,y,z\}$ and $\{x',y',z'\}$ are two bases of $\mathbb{F}_{q^3}$ considered as $\mathbb{F}_q$-vector space, so there exists an element $g \in G = PGL(3,q)$ such that $\{x,y,z\}^g = \{x',y',z'\}$. As g and τ commute, $\forall g \in G$, we have that G is transitive on the sets $\{R, R^\tau, R^{\tau^2}\}$ with $R \in PG(2,q^3)$ such that R, R^τ, R^{τ^2} are not collinear. This implies that the lifting $\mathcal{G}$ of G fixing $\mathcal{V}$ acts transitively on the planes of $PG(5,q)$ disjoint from $\mathcal{M}$ and containing three $\mathbb{F}_{q^3}$–rational points of $\mathcal{V}(q^3)$.

Suppose that P is not a singular point for $\mathcal{M}(q^3)$, then P, P^τ, P^{τ^2} are three $\mathbb{F}_{q^3}$–rational points of $\mathcal{M}(q^3)\setminus\mathcal{V}(q^3)$ and $\pi(q^3) \subseteq T_P \cap T_{P^\tau} \cap T_{P^{\tau^2}}$, where $T_{P^{\tau^i}}$ is the tangent hyperplane to $\mathcal{M}(q^3)$ at P^{τ^i}. Then, by Proposition 7, T_P intersects $\mathcal{V}(q^3)$ in $v(\ell)$, with ℓ a line of $PG(2,q^3)$. By [11, pag. 154]), a hyperplane intersecting $\mathcal{V}$ just in $v(m)$, m a line, contains the tangent plane to $\mathcal{V}$ at every point of $v(m)$. So suppose that the line ℓ contains an $\mathbb{F}_q$-rational point R, then $\pi_{v(R)} \subseteq T_P \cap T_{P^\tau} \cap T_{P^{\tau^2}}$. Since $T_P \cap T_{P^\tau} \cap T_{P^{\tau^2}}$ and $\pi_{v(R)}$ are both fixed by τ, then $\pi_{v(R)}(q) = \pi_{v(R)} \cap Fix\,\tau$ is a subplane isomorphic to $PG(2,q)$ and $\pi_{v(R)}$ and π are both contained in the subspace $T_P \cap T_{P^\tau} \cap T_{P^{\tau^2}} \cap Fix\,\tau$ which has at most dimension 3. Then $\pi_{v(R)}$ and π intersect in at least a line. But $\pi_{v(R)} \subset \mathcal{M}$ and $\pi \cap \mathcal{M} = \emptyset$, a contradiction. Hence ℓ does not contain $\mathbb{F}_q$-rational points and the three lines $\ell, \ell^\tau, \ell^{\tau^2}$ of $PG(2,q^3)$ are nonconcurrent. Arguing as in the previous case (here we have just the dual case), we have again only one orbit under the action of $\mathcal{G}$. □

Remark 1. By the proof of the last proposition, we see that if a plane π disjoint from $\mathcal{M}$ such that $\pi \subset T_P$, with T_P a hyperplane tangent to $\mathcal{M}(q^3)$ in a point $P \in \mathcal{M}(q^3)\setminus\mathcal{M}$, then $T_P \cap \mathcal{V}(q^3) = v(\ell)$ with ℓ a line of $PG(2,q^3)$ with no $\mathbb{F}_q$–rational points. So $\ell, \ell^\tau, \ell^{\tau^2}$ are nonconcurrent and hence $T_P, T_P^\tau, T_P^{\tau^2}$ are independent hyperplanes, i.e. $\pi(q^3) = T_P \cap T_P^\tau \cap T_P^{\tau^2} = T_P \cap T_{P^\tau} \cap T_{P^{\tau^2}}$.

2.3 A non–canonical embedding of $PG(5,q)$ in $PG(5,q^3)$

Let $Y_0, Y_1, \ldots, Y_5$ be homogeneous projective coordinates in $\mathbb{P}^* := PG(5,q^3)$, $\mathcal{V}^*$ the Veronese surface of $\mathbb{P}^*$

$$\mathcal{V}^* = \{\langle(x^2, y^2, z^2, xy, xz, yz)\rangle :\ x, y, z \in \mathbb{F}_{q^3}, (x,y,z) \neq (0,0,0)\} \tag{4}$$

and

$$\mathcal{M}^* :\ Y_0Y_1Y_2 - Y_0Y_5^2 - Y_1Y_4^2 - Y_2Y_3^2 + 2Y_3Y_4Y_5 = 0 \tag{5}$$

its secant variety. The geometric setting that we will adopt for the proof of the main result is the following cyclic representation of $\mathbb{P} \cong PG(5,q)$ inside $\mathbb{P}^*$ (in a non–canonical position):

$$\mathbb{P} = \{\langle(a, a^q, a^{q^2}, b^{q^2}, b^q, b)\rangle :\ a, b \in \mathbb{F}_{q^3}, (a,b) \neq (0,0)\},$$

hence $\mathbb{P}$ is fixed by the $\mathbb{F}_q$-linear collineation of order 3

$$\tau : \langle(y_0, y_1, y_2, y_3, y_4, y_5)\rangle \in \mathbb{P}^* \mapsto \langle(y_2^q, y_0^q, y_1^q, y_4^q, y_5^q, y_3^q)\rangle \in \mathbb{P}^*.$$

The Veronese surface $\mathcal{V}^*$ of $\mathbb{P}^*$ is set-wise fixed by τ and it turns out

$$\mathcal{V} = \mathcal{V}^* \cap \mathbb{P} = \{\langle (x^2, x^{2q}, x^{2q^2}, x^{q+1}, x^{q^2+1}, x^{q+q^2}) \rangle :\ x \in \mathbb{F}_{q^3}^*\}, \tag{6}$$

whereas its secant variety $\mathcal{M}$ is represented by the equation

$$\mathcal{M} :\ N_{q^3/q}(x) - Tr_{q^3/q}(xy^2) + 2N_{q^3/q}(y) = 0 \ (^1). \tag{7}$$

Again, let v be the Veronese embedding of $PG(2, q^3)$ in $\mathbb{P}^* = PG(5, q^3)$ defined in (3). By abuse of notation, denote by τ the following semilinear collineation of order 3 of $PG(2, q^3)$: $\langle (x_0, x_1, x_2) \rangle \mapsto \langle (x_2^q, x_0^q, x_1^q) \rangle$. Then it is easy to see that $\mathcal{V}$ turns out to be the image under v of $Fix\,\tau \cong PG(2, q) = \{\langle (x, x^q, x^{q^2}) \rangle,\ x \in \mathbb{F}_{q^3}^*\}$. The subgroup of $PGL(3, q^3)$ fixing such a $PG(2, q)$ consists of the collineations commuting with τ, hence by the non–singular matrices of the form $\begin{pmatrix} a & b & c \\ c^q & a^q & b^q \\ b^{q^2} & c^{q^2} & a^{q^2} \end{pmatrix}$ and then the linear part of $\mathcal{G}$, in this representation, consists of the lifting of such matrices.

In this model we have an easy description of the linear set disjoint from $\mathcal{M}$, for q **odd**, corresponding to the spreads $\mathcal{D}, \mathcal{A}, \mathcal{LMPT}$ (see [19]):

$\mathcal{D}$ corresponds, up to equivalence, to the plane of $\mathbb{P}$

$$\pi_1 = \{\langle (a, a^q, a^{q^2}, 0, 0, 0) \rangle :\ a \in \mathbb{F}_{q^3}^*\}; \tag{8}$$

$\mathcal{A}$ corresponds, up to equivalence, to the plane of $\mathbb{P}$

$$\pi_2 = \{\langle (0, 0, 0, b^{q^2}, b^q, b) \rangle :\ b \in \mathbb{F}_{q^3}^*\}; \tag{9}$$

$\mathcal{LMPT}$ corresponds, up to equivalence, to the Baer subgeometry of $\mathbb{P}$

$$\{\langle (a, a^{s^2}, a^s, b^{q^2}, b^q, b) \rangle : a \in \mathbb{F}_{s^3}, b \in \mathbb{F}_{q^3} |\ T_{q^3/s^3}(b) = 0 \text{ and } (a, b) \neq (0, 0)\} \tag{10}$$

isomorphic to $PG(5, s)$, $q = s^2$.

Remark 2. Let π_i^*, $i \in \{1, 2\}$, denote the projective plane of $\mathbb{P}^*$ generated by π_i. Then,

$$\pi_1^* :\ Y_3 = Y_4 = Y_5 = 0 \quad \text{and} \quad \pi_2^* :\ Y_0 = Y_1 = Y_2 = 0.$$

[1]Here $N_{q^3/q}$ and $T_{q^3/q}$ denote the norm and the trace function from $\mathbb{F}_{q^3}$ to $\mathbb{F}_q$, respectively

If $i=1$, then

$$\pi_1^* \cap \mathcal{M}^* = \ell \cup \ell^\tau \cup \ell^{\tau^2},$$

where $\ell : Y_0 = Y_3 = Y_4 = Y_5 = 0$ and the triangles determined by the three lines $\{\ell, \ell^\tau, \ell^{\tau^2}\}$ is $\{P, P^\tau, P^{\tau^2}\}$, with $P := \langle(0,0,1,0,0,0)\rangle$, which are points of $\mathcal{V}^*$ not in $\mathcal{V}$.

If $i=2$, then

$$\pi_2^* \cap \mathcal{M}^* = \ell \cup \ell^\tau \cup \ell^{\tau^2},$$

where $\ell : Y_0 = Y_1 = Y_2 = Y_3 = 0$ and the triangles determined by the three lines $\{\ell, \ell^\tau, \ell^{\tau^2}\}$ is $\{P, P^\tau, P^{\tau^2}\}$, with $P := \langle(0,0,0,0,1,0)\rangle$, which are points of $\mathcal{M}^*$ not in $\mathcal{V}^*$.

Remark 3. By [12, Thm. 25.1.18] when q is odd there is a polarity in $PG(5,q)$ which maps the set of all conic planes of the Veronese surface $\mathcal{V}$ to the set of all tangent planes of $\mathcal{V}$. By [19, Thm. 4] such a polarity interchanges the planes π_1 and π_2.

By abuse of notation, let $\mathcal{D}$ be the $\mathcal{G}$-orbit of planes containing π_1 and $\mathcal{A}$ be the $\mathcal{G}$-orbit of planes containing π_2, where recall $\mathcal{G}$ is the automorphism group of the Veronese surface $\mathcal{V}$ of $\mathbb{P}$ (cf. Prop. 8).

We conclude this section with the following lemmas that will be very useful in the sequel.

Lemma 1. *Let $\mathcal{G}^*$ be the automorphism group of the Veronese surface $\mathcal{V}^*$ of $\mathbb{P}^* = PG(5,q^3)$, q odd, and let g be a linear collineation of $\mathcal{G}^*$ leaving invariant the subplane π_i, $i=1,2$, and fixing pointwise the vertices of the triangle arising from $\pi_i^* \cap \mathcal{M}^*$.*

1. *If $i=1$, then g is determined by a matrix of type*

   $$Diag(a^2, a^{2q}, a^{2q^2}, \epsilon_1 a^{q+1}, \epsilon_2 a^{1+q^2}, \epsilon_1\epsilon_2 a^{q+q^2}),$$

   with $\epsilon_i = \pm 1$.

2. *If $i=2$, then g is determined by a matrix of type*

   $$Diag(a^2, a^{2q}, a^{2q^2}, a^{q+1}, a^{1+q^2}, a^{q+q^2})$$

   and hence $g \in \mathcal{G}$.

Proof. If $i=1$, by Remark 2, it is clear that g fixes the points $\langle(1,0,0,0,0,0)\rangle = v(\langle(1,0,0)\rangle)$, $\langle(0,1,0,0,0,0)\rangle = v(\langle(0,1,0)\rangle)$ and $\langle(0,0,1,0,0,0)\rangle = v(\langle(0,0,1)\rangle)$ of $\mathcal{V}^*$, where v is the Veronese embedding of $PG(2,q^3)$ in $\mathbb{P}^*$ as defined in (3). Thus, by ($\star$), g is obviously determined by the lifting of a matrix

$$A = \begin{pmatrix} a_{11} & 0 & 0 \\ 0 & a_{22} & 0 \\ 0 & 0 & a_{33} \end{pmatrix},$$

with $a_{ii} \in \mathbb{F}_{q^3}^*$. This means that, g is represented by the matrix

$$Diag(a_{11}^2, a_{22}^2, a_{33}^2, a_{11}a_{22}, a_{11}a_{33}, a_{22}a_{33}).$$

Since g fixes the subplane π_1 of $\mathbb{P}^*$, by (8), we get $a_{22} = \pm a_{11}^q$ and $a_{33} = \pm a_{11}^{q^2}$ and we get the first statement.

Let $i = 2$. Since g fixes π_2^*, by (9) and by ($\star$), we have that g is represented by a matrix which is the lifting of a matrix $A = (a_{ij})$, $i,j \in \{1,2,3\}$ such that, for each $i \in \{1,2,3\}$, $a_{ij}a_{ih} = 0\ \forall j \neq h$. Also, it is easy to see that g fixes pointwise the triangle $\langle(0,0,0,1,0,0)\rangle$, $\langle(0,0,0,0,1,0)\rangle$ and $\langle(0,0,0,0,0,1)\rangle$ only if $A = Diag(a_{11}, a_{22}, a_{33})$. The subplane π_2 is then fixed if and only if $a_{11}a_{33} = a_{22}^q a_{33}^q$ and $a_{11}a_{22} = a_{22}^{q^2} a_{33}^{q^2}$. It follows that $\frac{a_{33}}{a_{22}} = \frac{a_{22}^q a_{33}^q}{a_{22}^{q^2} a_{33}^{q^2}}$ and hence $a_{33}^{1-q+q^2} = a_{22}^{-q^2+q+1}$. Raising to the $q+1$-th power, we get $a_{33}^2 = a_{22}^{2q}$ and so $a_{33} = \pm a_{22}^q$, but only $a_{33} = a_{22}^q$ fulfills $a_{33}^{1-q+q^2} = a_{22}^{-q^2+q+1}$. Then $a_{11}a_{33} = a_{22}^q a_{33}^q \Leftrightarrow a_{11}a_{22}^q = a_{22}^{q+q^2} \Leftrightarrow a_{11} = a_{22}^{q^2}$. Hence $a_{33} = a_{22}^q$ and $A = Diag(a, a^q, a^{q^2})$, with $a \in \mathbb{F}_{q^3}^*$. By (6), it is clear that g fixes the Veronese surface $\mathcal{V}$, i.e. $g \in \mathcal{G}$. □

Let ρ be another $\mathbb{F}_q$–linear collineation of order 3, different from τ, of $\mathbb{P}^* = PG(5,q^3)$, then $Fix\,\rho \cong PG(5,q)$. If ρ fixes $\mathcal{M}^*$, then it is the lifting of an $\mathbb{F}_q$–linear collineation of order 3 of $PG(2,q^3)$ and $\mathcal{M}_\rho := Fix\,\rho \cap \mathcal{M}^* \cong \mathcal{M}$. If π is a plane of $\mathbb{P}^*$ fixed by ρ, then $\pi_\rho := \pi \cap Fix\,\rho$ is a subplane of the subgeometry $Fix\,\rho$. If $\pi_\rho \cap \mathcal{M}_\rho = \emptyset$, then we extend the definition of families $\mathcal{D}$ and $\mathcal{A}$ to $Fix\,\rho$ in the obvious way.

Lemma 2. *Let π be a plane of $\mathbb{P}^* = PG(5,q^3)$ fixed by ρ such that $\pi_\rho \in \mathcal{A}$ with respect to $\mathcal{M}_\rho$. Then the action of an element $g \in \mathcal{G}^*$ fixing π_ρ and the vertices of the triangle $\mathcal{M}^* \cap \pi$ is uniquely determined by its action on π.*

Proof. Any automorphism of $\mathbb{F}_{q^3}$ leaves invariant the subplane π_2 and fixes pointwise the vertices of the triangle arising from $\pi_2^* \cap \mathcal{M}^*$, hence an element g of $\mathcal{G}^*$ with the same property must have the linear part as in Lemma 1.2. So the action of g on π_2^* is given by $(0,0,0,y_3,y_4,y_5) \mapsto (0,0,0,a^{1+q}y_3^\phi, a^{q^2+1}y_4^\phi, a^{q+q^2}y_5^\phi)$, where ϕ is an automorphism of $\mathbb{F}_{q^3}$. So the collineation g' of $\mathbb{P}^*$ with linear part given by $Diag(b^2, b^{2q}, b^{2q^2}, b^{q+1}, b^{1+q^2}, b^{q+q^2})$

and accompanying automorphism $\psi \in Aut(\mathbb{F}_{q^3})$ is such that $g' \equiv g$ on π_2^* if and only if $\psi = \phi$ and $\frac{b^{q+1}}{a^{q+1}} = \frac{b^{q^2+1}}{a^{q^2+1}} \Rightarrow \frac{b^q}{a^q} = \frac{b^{q^2}}{a^{q^2}} \Rightarrow \frac{b}{a} \in \mathbb{F}_q$ and hence $g = g'$.

Let Π_1 and Π_2 be two subplanes of $PG(2, q^3)$ whose Veronese embedding in P^* are the Veronese surface $\mathcal{V}$ defined in (6) and the Veronese surface $\mathcal{V}_\rho$ underlying $\mathcal{M}_\rho$, respectively. Since the subplanes of $PG(2, q^3)$ are all equivalent under the action of $P\Gamma L(3, q^3)$, the lifting φ of the collineation of the $PG(2, q^3)$ mapping Π_2 to Π_1 is an element of $\mathcal{G}^*$ such that $\mathcal{V}_\rho^\varphi = \mathcal{V}$. Hence $\varphi(Fix\,\rho) = Fix\,\tau$ and π_ρ^φ is a subplane belonging to the orbit $\mathcal{A}$ of the planes of $Fix\,\tau$, disjoint from $\mathcal{M}$, under the action the automorphism group of $\mathcal{G} = Aut(\mathcal{V})$. Then there exists an element $\psi \in \mathcal{G}$ such that $\pi_\rho^\eta = \pi_2$, where $\eta := \psi \circ \varphi$. It follows that η is an element of $\mathcal{G}^*$ such that $\pi_\rho^\eta = \pi_2$ and hence $\pi^\eta = \pi_2^*$.

Let now h and h' be two collineations of $\mathbb{P}^*$ satisfying the assumptions of the statement and such that $h \equiv h'$ on π. Then $g \circ h \circ g^{-1}$ and $g \circ h' \circ g^{-1}$ are two elements of $\mathcal{G}^*$ both fixing π_2 and the vertices of the triangle $\pi_2^* \cap \mathcal{M}^*$ and such that $g \circ h \circ g^{-1} \equiv g \circ h' \circ g^{-1}$ on π_2^*. Hence, by the previous arguments $g \circ h \circ g^{-1} = g \circ h' \circ g^{-1}$ in the whole space $\mathbb{P}^*$, i.e. $h = h'$. □

3 Classification result

Let L be an $\mathbb{F}_q$–linear set of $\mathbb{P} := PG(5, q^2)$ of rank 6 disjoint from the secant variety of the Versonese surface $\mathcal{V}$ of $\mathbb{P}$:

$$\mathcal{M}: \; Y_0Y_1Y_2 - Y_0Y_5^2 - Y_1Y_4^2 - Y_2Y_3^2 + 2Y_3Y_4Y_5 = 0. \tag{11}$$

As in the proof of Proposition 8, for any projective subspace, say S, and for any algebraic variety, say $\mathcal{W}$, of $PG(r-1, q)$ we will denote by $S(q^t)$ and $\mathcal{W}(q^t)$, respectively, their extensions over $\mathbb{F}_{q^t}$ and by $\bar{S}$ and $\bar{\mathcal{W}}$ their extensions over $\mathbb{F}$. Let $\mathbb{F}$ denote the algebraic closure of $\mathbb{F}_q$.

Then L has one of the configuration of Theorem 5.

3.1 Case (1)

By Proposition 8, there are two orbits of planes disjoint from the secant variety $\mathcal{M}$ of $\mathbb{P}$ under the action of the automorphism group of the Veronese surface $\mathcal{V}$, which we have denoted by $\mathcal{D}$ and $\mathcal{A}$ (cf. arguments after Remark 3). By Remark 2, the planes of the family $\mathcal{D}$ are the ones whose $\mathbb{F}_{q^6}$–extension intersects $\mathcal{V}(q^6) \setminus \mathcal{V}$ in three points which are the images under the Veronese embedding v of three non–collinear points of

$PG(2,q^6)\setminus PG(2,q^2)$, whereas the planes of the family $\mathcal{A}$ are the ones contained in three tangent hyerplanes of $\mathcal{M}(q^6)$.

3.2 Case (2)

Let L be a linear set consisting of $q+1$ planes in $\mathbb{P}=PG(5,q^2)$ through a line, say ℓ, disjoint from $\mathcal{M}$, then by the previous case they belong to $\mathcal{D}$ or $\mathcal{A}$. Suppose that two of these planes, namely π_1 and π_2, belong to $\mathcal{A}$. Denoting by $\perp$ the polarity of $PG(5,q^2)$ defined in [19, Thm. 4] interchanging $\mathcal{D}$ and $\mathcal{A}$, it follows that $\pi_1^\perp$ and $\pi_2^\perp$ are contained in $\ell^\perp$ and $\pi_1^\perp$ and $\pi_2^\perp\in\mathcal{D}$. So this configuration always implies that there exist two planes, say α_1 and α_2, in $\mathcal{D}$ spanning a solid, say T. Such a solid intersects the Veronese surface $\mathcal{V}$ in $v(\mathcal{C}_1)\cap v(\mathcal{C}_2)$, where $\mathcal{C}_i$ are conics. By Case (1), the cubic extension $T(q^6)$ of the solid T must contain 6 points of $\mathcal{V}(q^6)$, which implies that $\mathcal{C}_1$ and $\mathcal{C}_2$ are reducible and have a common linear component m. It follows that $T(q^6)\cap\mathcal{V}(q^6)$ consists either of just $v(m)$ or $v(m)$ and a point, but in both cases we cannot have 6 points, 3 of them in $\alpha_1(q^6)$ and 3 of them in $\alpha_2(q^6)$.

3.3 Case (4)

Let Σ be a Baer subgeometry of $\mathbb{P}=PG(5,q^2)$ and let σ be the semilinear collineation of $\mathbb{P}$ of order 2 such that $\Sigma=Fix\,\sigma$. Set $\mathcal{M}:=\mathcal{M}(q^2)$, it turns out $\Sigma\cap\mathcal{M}\subset\mathcal{M}\cap\mathcal{M}^\sigma$. The variety $\overline{\mathcal{M}}\cap\overline{\mathcal{M}^\sigma}$ in $PG(5,\mathbb{F})$ has dimension 3 (see, e.g., [10, Exercise 11.6]) and $\deg\overline{\mathcal{M}}\cap\overline{\mathcal{M}^\sigma}=\deg\overline{\mathcal{M}}\deg\overline{\mathcal{M}^\sigma}=9=\sum_i\mu_i\deg(\mathcal{W}_i)$, where the sum is over all the irreducible components $\mathcal{W}_i$ of $\overline{\mathcal{M}}\cap\overline{\mathcal{M}^\sigma}$ of dimension 3 and μ_i is the multiplicity of $\mathcal{W}_i$ (see [10, Theorem 18.4]). As $\overline{\mathcal{M}}$ contains subspaces of dimension at most 2, we have $\deg\mathcal{W}_i\geq 2$. If $\overline{\mathcal{M}}\cap\overline{\mathcal{M}^\sigma}$ was absolutely irreducible, then it would have at least one point in Σ, for q big enough; by [7, Cor. 7.4] we get the bound $q^2>2{\cdot}9^4$. So $\overline{\mathcal{M}}\cap\overline{\mathcal{M}^\sigma}$ is reducible. Suppose that there is a component in $PG(5,q^2)$, say $\mathcal{W}$, then $\overline{\mathcal{M}}\cap\overline{\mathcal{M}^\sigma}=\mathcal{W}\cup\mathcal{W}^\sigma\cup Y$, with Y a component not over $\mathbb{F}_{q^2}$ of dimension 3. Again, Y must be reducible over some extension of $\mathbb{F}_{q^2}$, so $Y=Z\cup Z^\tau\cup\cdots\cup Z^{\tau^{n-1}}$, where τ is the collineation induced by $Gal(\mathbb{F}_{q^{2n}}|\mathbb{F}_{q^2})$. As $\deg\mathcal{W}=\deg\mathcal{W}^\sigma$ and $\deg Z=\deg Z^{\tau^i}$, $\deg Y$ must be an odd composite number less or equal to 5, a contradiction. So $\overline{\mathcal{M}}\cap\overline{\mathcal{M}^\sigma}$ does not have $\mathbb{F}_{q^2}$–components. Let m be the smallest integer such that $\overline{\mathcal{M}}\cap\overline{\mathcal{M}^\sigma}=\bigcup_{i=0}^{m-1}\mathcal{W}^{\tau^i}$, with $\mathcal{W}$ possibly reducible, $m\mu\deg\mathcal{W}=9$ and μ the sum of the multiplicities of all the irreducible components of $\mathcal{W}$. As $\deg\mathcal{W}>1$ and $m>1$, the only possibility is that $m=\deg\mathcal{W}=3$. Again, as we cannot have components of degree 1, $\mathcal{W}$ must be irreducible. This means that $\overline{\mathcal{M}}\cap\overline{\mathcal{M}^\sigma}$

is reducible in $PG(5,q^6)$ and it turns out

$$\mathcal{M}(q^6)\cap\mathcal{M}^{\sigma}(q^6)=\mathcal{W}\cup\mathcal{W}^{\tau}\cup\mathcal{W}^{\tau^2}, \tag{12}$$

where τ is the $\mathbb{F}_{q^2}$–linear collineation of $PG(5,q^6)$ of order 3 such that $\mathbb{P}=PG(5,q^2)=Fix\,\tau$. The collineation σ has order 6 when it acts on $PG(5,q^6)$ and since σ^2 is the identity in $Fix\,\tau$, we have that $Fix\tau=Fix\sigma^2$. Then, taking into account that τ has order 3, either $\sigma^2=\tau$ or $\sigma^2=\tau^2$. Since $Fix\tau=Fix\tau^2$, without loss of generality, we can assume that $\sigma^2=\tau$. Since $\mathcal{M}^{\sigma}$ is a variety defined in $PG(5,q^2)=Fix\,\tau$, $\mathcal{M}^{\sigma}(q^6)=\mathcal{M}^{\sigma\circ\tau}(q^6)$ and hence we can replace $\mathcal{M}^{\sigma}(q^6)$ by $\mathcal{M}^{\sigma^*}(q^6)$, where $\sigma^*=\sigma\circ\tau$. As in $PG(5,q^6)$ the accompanying automorphisms of σ and τ have orders 6 and 3, respectively, it turns out σ^* is a semilinear collineation of $PG(5,q^6)$ of order 2. We remark that $\sigma\equiv\sigma^*$ on $PG(5,q^2)$ and observe that $\mathcal{M}^{\sigma^*}(q^6)=\mathcal{M}(q^6)^{\sigma^*}$.

Since $\mathcal{W}$ is a variety of dimension 3 and since the maximal subspaces contained in $\mathcal{M}(q^6)$ are planes, the projective space spanned by $\mathcal{W}$ has dimension either 4 or 5.

If $\langle\mathcal{W}\rangle=PG(5,q^6)$, then $\mathcal{W}$ is a variety of minimal degree and, by [10, Theorem 19.9], it is necessarily a Segre variety product of a line and a plane. Such a Segre variety consists of disjoint planes, but the planes contained in $\mathcal{M}(q^6)$ pairwise meet in at least one point, a contradiction.

Hence $\langle\mathcal{W}\rangle=H$, where H is a hyperplane of $PG(5,q^6)$. So $\mathcal{W}$ is an irreducible hypersurface of degree 3 contained in $H\cap\mathcal{M}(q^6)$, hence $\mathcal{W}=H\cap\mathcal{M}(q^6)=H\cap\mathcal{M}^{\sigma^*}(q^6)$. The semilinear collineation σ^* has order 2 on $\mathcal{M}(q^6)$, hence $\mathcal{M}(q^6)\cap\mathcal{M}^{\sigma^*}(q^6)$ is fixed by σ^*. Taking into account that the decomposition (12) is unique and that $\sigma^*\circ\tau=\tau\circ\sigma^*$ (since σ^* arises from a collineation of $Fix\,\tau$), we have $\mathcal{W}=\mathcal{W}^{\sigma^*}$ and hence $H=H^{\sigma^*}$. Let $\pi=H\cap H^{\tau}\cap H^{\tau^2}$ and denote by π_τ and π_σ the intersection between π and $PG(5,q^2)=Fix\,\tau$ and $\Sigma=Fix\,\sigma\cong PG(5,q)$, respectively. Since $\pi=\pi^{\tau}$ and $\pi=\pi^{\sigma^*}=\pi^{\sigma\circ\tau}=\pi^{\sigma}$, then π_σ, π_τ and π have the same projective dimension, which is 2 or 3. Now observe that on the hyperplane H (as on H^{τ} and H^{τ^2}) the varieties $\mathcal{M}(q^6)$ and $\mathcal{M}^{\sigma^*}(q^6)$ coincide with $\mathcal{W}$ (with $\mathcal{W}^{\tau}$ and $\mathcal{W}^{\tau^2}$, respectively), then the same happens on the subspace π. Hence $\pi\cap\mathcal{M}(q^6)$ is a hypersurface of degree 3 fixed by σ^* and by τ and such that $\pi_\sigma\cap\mathcal{M}=\pi\cap Fix\,\tau\cap Fix\,\sigma^*\cap\mathcal{M}\subseteq\Sigma\cap\mathcal{M}=\emptyset$, hence $\pi\cap\mathcal{M}$ must be reducible. As σ^* has order 2, τ has order 3 and they commute, we easily get that $\pi\cap\mathcal{M}=\ell\cup\ell^{\tau}\cup\ell^{\tau^2}$, ℓ, ℓ^{τ} and ℓ^{τ^2} are hyperplanes of π fixed by σ^*. If $S:=\ell\cap\ell^{\tau}\cap\ell^{\tau^2}\neq\emptyset$, then S turns out to be a subspace of $\pi(q)=\pi\cap Fix\,\tau\cap Fix\,\sigma^*\subset\Sigma$ of the same dimension contained in $\mathcal{M}$, a contradiction. It follows that $\ell\cap\ell^{\tau}\cap\ell^{\tau^2}=\emptyset$, hence necessarily π is a plane of $PG(5,q^6)$, ℓ is a line and also $\pi_\tau\cap\mathcal{M}=\emptyset$.

Hence, by Case (1), $\pi_\tau\in\mathcal{D}$ or $\mathcal{A}$. Let $P:=\ell\cap\ell^{\tau}$, then P, P^{τ} and P^{τ^2} are fixed by σ^*. As $\mathcal{M}(q^6)$ and $\mathcal{M}^{\sigma^*}(q^6)$ coincide on H,H^{τ},H^{τ^2}, on these hyperplanes σ^* acts as an element of the automorphism group of $\mathcal{M}(q^6)$ (and hence of $\mathcal{V}(q^6)$). Also σ^* fixes $\pi_\tau = H \cap H^\tau \cap H^{\tau^2}$.

Embed $\mathbb{P} = PG(5, q^2)$ in $\mathbb{P}^* := PG(5, q^6)$ as showed in Subsection 2.3. Using the same notations we have that $\mathcal{V}$ and $\mathcal{M}$ are defined as in (6) and (7), respectively, whereas $\mathcal{V}(q^6)$ and $\mathcal{M}(q^6)$ are the varieties $\mathcal{V}^*$ and $\mathcal{M}^*$ defined in (4) and in (5), respectively. Also, $\mathcal{G}$ and $\mathcal{G}^*$ denote the automorphism groups of $\mathcal{V}$ (and hence of $\mathcal{M}$) and $\mathcal{V}^*$ (and hence of $\mathcal{M}^*$), respectively. Since $\sigma \equiv \sigma^*$ in $Fix\tau \cong PG(5, q^2)$, and the accompanying automorphism of σ^* has order 2, we have that the accompanying automorphism of σ in $Fix\,\tau$ is

$$t_\sigma : \langle (Y_0, Y_1, Y_2, Y_3, Y_4, Y_5) \rangle \mapsto \langle (Y_1^q, Y_2^q, Y_0^q, Y_5^q, Y_3^q, Y_4^q) \rangle \tag{13}$$

and we can explicitly note that

$$\mathcal{M}^{t_\sigma} = \mathcal{M}. \tag{14}$$

Also, σ and σ^* have the same linear part g. We recall that the accompanying automorphism of σ^* is

$$t_{\sigma^*} : \langle (Y_0, Y_1, Y_2, Y_3, Y_4, Y_5) \rangle \mapsto \langle (Y_0^{q^3}, Y_1^{q^3}, Y_2^{q^3}, Y_3^{q^3}, Y_4^{q^3}, Y_5^{q^3}) \rangle \tag{15}$$

By Proposition 8, up to the action of $\mathcal{G}$, we can assume that π_τ is then

$$\pi_1 = \{\langle (a, a^{q^2}, a^{q^4}, 0, 0, 0) \rangle : \ a \in \mathbb{F}_{q^6}^*\}$$

or

$$\pi_2 = \{\langle (0, 0, 0, a^{q^4}, a^{q^2}, a) \rangle : \ a \in \mathbb{F}_{q^6}^*\}$$

(cf. (8) and (9)).

If $\pi_\tau = \pi_2$, then t_{σ^*} fixes π_τ and the points P, P^τ, P^{τ^2}, hence so does the linear part g of σ^*. Hence, by Lemma 1, on H^{τ^i} the map g corresponds to a linear collineation g_i represented by a matrix of type $Diag(a_i^2, a_i^{2q^2}, a_i^{2q^4}, a_i^{q^2+1}, a_i^{q^4+1}, a_i^{q^4+q^2}) \in \mathcal{G}$, with $a_i \in \mathbb{F}_{q^6}^*$. On $\pi_\tau = H \cap H^\tau \cap H^{\tau^2}$, $g_0 \equiv g_1 \equiv g_2 \equiv g$, hence a_0/a_1 and a_0/a_2 belong to $\mathbb{F}_{q^2}^*$. This means that $g \equiv g_0 \equiv g_1 \equiv g_2 \in \mathcal{G}$ on three hyperplanes of $PG(5, q^6)$ and hence the linear part g of σ^* turns out to be the linear automorphism of $\mathcal{M}$ represented by the matrix

$$Diag(a^2, a^{2q^2}, a^{2q^4}, a^{q^2+1}, a^{q^4+1}, a^{q^4+q^2}), \tag{16}$$

with $a \in \mathbb{F}_{q^6}^*$. Then g is a linear collineation of $Fix\,\tau \cong PG(5, q^2)$ fixing $\mathcal{M}$. Also, by (14) we have $\mathcal{M}^\sigma = \mathcal{M}^{g \circ t_\sigma} = \mathcal{M}$. By Chevalley–Warning Theorem, $\mathcal{M} \cap \Sigma \neq \emptyset$, a contradiction.

Hence, necessarily, $\pi_\tau = \pi_1$. By Remark 2 the plane $\pi_1^* = H \cap H^\tau \cap H^{\tau^2}$ generated by π_1 in $PG(5, q^6)$ intersects $\mathcal{M}^*$ in three lines ℓ, ℓ^τ and ℓ^{τ^2} and such lines form a triangle $P = \langle(0, 0, 1, 0, 0, 0)\rangle$, P^τ and P^{τ^2} of points of $\mathcal{V}^*$. Again, since t_{σ^*} fixes π_τ and the points P, P^τ, P^{τ^2}, so does g, hence by Lemma 1, on H^{τ^i}, $g \equiv g_i$ where g_i is induced by a matrix of type

$$Diag(a_i^2, a_i^{2q^2}, a_i^{2q^4}, \epsilon_1 a_i^{q^2+1}, \epsilon_2 a_i^{q^4+1}, \epsilon_1\epsilon_2 a_i^{q^4+q^2}),$$

with $a_i \in \mathbb{F}_{q^6}^*$ and $\epsilon_1, \epsilon_2 \in \{1, -1\}$.

If $g_i = g_j$ for some $i \neq j$, then $g = g_i = g_j$ on $PG(5, q^6)$. As g is the linear part of σ, it fixes $Fix\,\tau \cong PG(5, q^2)$ and hence $g \in \mathcal{G}$, yielding, as before, to a contradiction. Hence, $g_i \neq g_j$ for each $i \neq j$. On the plane $\pi_1^* = H \cap H^\tau \cap H^{\tau^2}$, $g_0 \equiv g_1 \equiv g_2$ and this implies $(\frac{a_0}{a_1})^2, (\frac{a_0}{a_2})^2 \in \mathbb{F}_{q^2}$.

From $(\frac{a_0}{a_1})^{2q^2} = (\frac{a_0}{a_1})^2$, we get $(\frac{a_0}{a_1})^{q^2} = \frac{a_0}{a_1}$, and hence again the matrices representing g_0 and g_1 coincide up to a nonzero scalar in $\mathbb{F}_{q^2}$. The same happens for g_0 and g_2. Hence, we can assume (up to a nonzero scalar in $\mathbb{F}_{q^2}$) that $a := a_0 = a_1 = a_2$.

On the 3–dimensional space $H \cap H^\tau$, $g \equiv g_i \equiv g_j$. Since $g_i \neq g_j$, it can be seen that the only 3–dimensional space (containing the plane π) where two such collineations coincide must have equations $Y_h = 0$, $Y_k = 0$ for some $h \neq k \in \{3, 4, 5\}$. Also, in the pencil of hyperplanes through that solid the only two that are one the image of the other under τ are the hyperplanes $Y_h = 0$ and $Y_k = 0$. So the hyperplanes H, H^τ, H^{τ^2} have necessarily equations $Y_j = 0$, $j \in \{3, 4, 5\}$.

Denoting by $A = (a_{ij})$, with $a_{ij} \in \mathbb{F}_{q^6}$, $i, j \in \{0, 1, \ldots, 5\}$, the nonsingular matrix representing the linear part g of σ^* (and hence of σ). Since g fixes the points P, P^τ and P^{τ^2} and the hyperplanes H, H^τ and H^{τ^2} the matrix A must be diagonal. Also, σ^* leaves invariant the subgeometry $PG(5, q^2) = Fix\,\tau$ (where it acts as σ) and hence it turns out

$$A = \begin{pmatrix} c & 0 & 0 & 0 & 0 & 0 \\ 0 & c^{q^2} & 0 & 0 & 0 & 0 \\ 0 & 0 & c^{q^4} & 0 & 0 & 0 \\ 0 & 0 & 0 & d & 0 & 0 \\ 0 & 0 & 0 & 0 & d^{q^4} & 0 \\ 0 & 0 & 0 & 0 & 0 & d^{q^2} \end{pmatrix}, \tag{17}$$

with $c, d \in \mathbb{F}_{q^6}^*$.

Since on each hyperplane H^{τ^i}, the matrix A must be equal (up to a nonzero scalar in $\mathbb{F}_{q^2}$) to a matrix of type (16), it follows that $\epsilon := \epsilon_1 = \epsilon_2$ and, up to a given nonzero scalar in $\mathbb{F}_{q^2}$, $c = a^2$ and $d = \epsilon a^{q^2+1}$, with $a \in \mathbb{F}_{q^6}^*$.

If $\epsilon = 1$, i.e. $d = a^{q^2+1}$, then

$$A = Diag(a^2, a^{2q^2}, a^{2q^4}, a^{q^2+1}, a^{q^4+1}, a^{q^4+q^2}),$$

and hence g leaves invariant the surface $\mathcal{V}$. This implies $g \in \mathcal{G}$ and hence, by (14), $\mathcal{M}^\sigma = \mathcal{M}$; a contradiction as before.

Then, $\epsilon = -1$, i.e. $d = -a^{q^2+1}$, and

$$A = Diag(a^2, a^{2q^2}, a^{2q^4}, -a^{q^2+1}, -a^{q^4+1}, -a^{q^4+q^2}).$$

Since σ has order 2 in $Fix\tau$, taking into account its accompanying automorphism (13), we have $\rho := a^{2(q^3+1)} \in \mathbb{F}_{q^2}$. Since $\rho^{q^3-1} = \rho^{q-1} = 1$, then $\rho \in \mathbb{F}_q$ and we can choose $\eta \in \mathbb{F}^*_{q^6}$ such that $a^{q^3+1} = \eta^{(q^3+1)(q^2+q+1)}$. Hence $a = \eta^{q^2+q+1}\delta$, where $\delta^{q^3+1} = 1$. Set $\delta = \gamma^{q^3-1}$, with $\gamma \in \mathbb{F}^*_{q^6}$, we have $a = (\eta\gamma^{q-1})^{q^2+q+1}$ and hence

$$a = \xi^{q^2+q+1},$$

for a given element $\xi := \eta\gamma^{q-1}$ of $\mathbb{F}^*_{q^6}$.

Straightforward computations show that the subgeometry $\Sigma = Fix\sigma$ is

$$\{\langle(\xi^{2q}u, \xi^{2q^3}u^{q^2}, \xi^{2q^5}u^q, \frac{v^{q^4}}{\xi^{1+q^4}}, \frac{v^{q^2}}{\xi^{q^4+q^2}}, \frac{v}{\xi^{q^2+1}})\rangle : \ u \in \mathbb{F}_{q^3}, v \in \mathbb{F}_{q^6}, (u,v) \neq (0,0), v^{q^3} = -v\}.$$

As the matrix $Diag(\xi^{2q}, \xi^{2q^3}, \xi^{2q^5}, \xi^{q^3+q}, \xi^{q^5+q}, a^{q^3+q^5})^{-1}$ induces a linear automorphism of the Veronese surface $\mathcal{V}$, up to the action of the group $\mathcal{G}$, we have Σ is the subgeometry

$$\{\langle(u, u^{q^2}, u^q, (mv)^{q^4}, (mv)^{q^2}, mv)\rangle : \ u \in \mathbb{F}_{q^3}, v \in \mathbb{F}_{q^6}, (u,v) \neq (0,0), v^{q^3} = -v\},$$

where $m := \xi^{-(q^2+1+q^3+q^5)}$. Taking into account that $m \in \mathbb{F}_{q^3}$, we have proved the following result.

Theorem 9. *If Σ is a Baer seubgeometry of $PG(5,q^2)$ disjoint from the secant variety $\mathcal{M}$ of the Veronese surface $\mathcal{V}$ of $PG(5,q^2)$, then Σ is equivalent, under the action of the automorphism group of $\mathcal{V}$, to the subgeometry*

$$\{\langle(x, x^{q^2}, x^q, y^{q^4}, y^{q^2}, y)\rangle : x \in \mathbb{F}_{q^3}, y \in \mathbb{F}_{q^6}, \ (x,y) \neq (0,0) \ \textit{and} \ T_{q^6/q^3}(y) = 0\}. \qquad (18)$$

3.4 Case (3)

Let L be an $\mathbb{F}_q$-linear set with exactly one point of weight 2, hence $\langle L\rangle_{q^2} = PG(4,q^2)$. The linear set L has rank 6 and $PG(4,q^2)$ has rank 10 over $\mathbb{F}_q$, hence in order to have $L \cap \mathcal{M} = \emptyset$, by Grassmann, $\mathcal{M}' := PG(4,q^2) \cap \mathcal{M}$ must contain at most lines (over $\mathbb{F}_{q^2}$). The planes contained in $\mathcal{M}$ are the conic planes and the tangent planes of $\mathcal{V}$ and they are contained in a hyperplane H if and only if $H \cap \mathcal{V} = V(\mathcal{C})$ and $\mathcal{C}$ is a reducible conic (cf. Section 2.2). So we must have $\mathcal{V}' := PG(4,q^2) \cap \mathcal{V} = v(\mathcal{C})$, with $\mathcal{C}$ a non–degenerate conic. By Theorem 4, L is the projection in $PG(5,q^2)$ of a subgeometry $\Sigma \cong PG(5,q)$ from a point $P \notin \Sigma$ and $P \notin PG(4,q^2)$. Let $\mathcal{K}$ be the cone of $PG(5,q^2)$ with vertex P and base $\mathcal{M}'$. It is easy to see that $L \cap \mathcal{M}' = \emptyset$ if and only if $\Sigma \cap \mathcal{K} = \emptyset$. Let σ be the $\mathbb{F}_q$-linear collineation of order 2 of $PG(5,q^2)$ such that $\Sigma = Fix\,\sigma$. Let τ be the collineation of $PG(5,q^6)$ of order 3 such that $PG(5,q^2) = Fix\,\tau$. Again, denote by $\sigma^* = \sigma \circ \tau$ the unique collineation of order 2 of $PG(5,q^6)$ such that $\sigma^* \equiv \sigma$ in $PG(5,q^2)$. This means that σ and σ^* have the same linear part g (which is a collineation of $PG(5,q^2)$).

Taking into account the previous arguments, we have $\mathcal{K}^\sigma(q^6) = \mathcal{K}^{\sigma^*}(q^6) = \mathcal{K}(q^6)^{\sigma^*}$. Hence, as in Case (4), for $q^2 > 2\cdot 3^8$,

$$\mathcal{K}(q^6) \cap \mathcal{K}^\sigma(q^6) = \mathcal{W} \cup \mathcal{W}^\tau \cup \mathcal{W}^{\tau^2},$$

where $\mathcal{W}$ is an irreducible variety of $PG(5,q^6)$ of dimension and degree 3 and each $\mathcal{W}^{\tau^i}$, $i \in \{0,1,2\}$, is fixed by σ^*. The cone $\mathcal{K}$ is a hypersurface of $PG(5,q^2)$ of degree 3 which does not contain solids (also in its extension over any field). Again, since $\mathcal{W}$ cannot be a solid, that $\mathcal{W}$ is either a Segre variety (product of a line with a plane) or a hypersurface of degree 3 contained in a hyperplane H of $PG(5,q^6)$.

Suppose that W is a Segre variety product of a line with a plane. Such a variety consists of disjoint planes. Since $\mathcal{M}'(q^6)$ does not contain planes, the planes of $\mathcal{W}$ are projected over lines of $\mathcal{M}'(q^6)$, hence they all contain the point P, a contradiction.

So $\mathcal{W}$ is a hyperpsurface of degree 3 of a hyperplane H of $PG(5,q^6)$. Then $H^{\tau^i\sigma^*} = H^{\tau^i}$ and on each hyperplane H^{τ^i} the two cones $\mathcal{K}(q^6)$ and $\mathcal{K}^\sigma(q^6)$ coincide with $\mathcal{W}^{\tau^i}$. Suppose that $P \in H^{\tau^i}$ for some $i \in \{0,1,2\}$, i.e. $P \in \mathcal{W}^{\tau^i}$, which implies $P^{\sigma^*} = P^\sigma \in \mathcal{W}$ (since $P \in Fix\,\tau$). Then either $P = P^\sigma$ or PP^σ is a line of the cone $\mathcal{K}$, i.e. either $P \in \Sigma$ or $\mathcal{K} \cap \Sigma \neq \emptyset$; in both cases we get a contradiction. So $P \notin H^{\tau^i}$ for each $i \in \{0,1,2\}$ and this implies that each hyperplane H^{τ^i} intersects the cone $\mathcal{K}(q^6)$ in a copy of $\mathcal{M}'(q^6)$, i.e. $\mathcal{W}$, $\mathcal{W}^\tau$ and $\mathcal{W}^{\tau^2}$ are copies of $\mathcal{M}'(q^6)$.

Also, by the same arguments as in Case (4), $\pi := H \cap H^\tau \cap H^{\tau^2}$ is a plane such that $\pi \cap \mathcal{W} = \ell \cup \ell^\tau \cup \ell^{\tau^2}$, with ℓ, ℓ^τ, ℓ^{τ^2} three nonconcurrent lines. Moreover π intersects $Fix\,\tau \simeq PG(5,q^2)$ in a subplane π_τ isomorphic to $PG(2,q^2)$, σ^* leaves invariant π_τ, fixes each line ℓ^{τ^i}, $i \in \{0,1,2\}$, and each point of the triangles determined by them. We observe that $\mathcal{W}$ contains a subvariety $\hat{\mathcal{M}}'$ that is a copy of $\mathcal{M}'$ obtained projecting $\mathcal{M}'$ from P over H. Also, $\hat{\mathcal{M}}'$ leaves in the projection from P over H of $Fix\,\tau$ (which is always isomorphic to $PG(5,q^2)$) also containing the subplane π_τ and $\pi_\tau \cap \hat{\mathcal{M}}' = \emptyset$. We clearly have the same for $\mathcal{W}^\tau$ and $\mathcal{W}^{\tau^2}$.

As σ^* fixes $\mathcal{W}$, on H $\sigma^* \equiv \sigma_0^*$, with σ_0^* a collineation of $\mathcal{W}$ and hence it can be seen as a collineation of the secant variety of a Veronese surface in $PG(5,q^6)$. Analogously, on H^τ $\sigma^* \equiv \sigma_1^*$, with σ_1^* a collineation of $\mathcal{W}^\tau$ and on $H \cap H^\tau$, $\sigma_0^* \equiv \sigma_1^* \equiv \sigma^*$. Let $\overline{\sigma^*}$ be the restriction of σ_0^* on $H \cap H^\tau$, then σ_1^* is an extension of $\overline{\sigma^*}$ fixing $\mathcal{W}^\tau$. Suppose that $\pi_\tau \in \mathcal{A}$ with respect to $\hat{\mathcal{M}}'^\tau$ (see the arguments before Lemma 2), then $\overline{\sigma^*}$ satisfies all the assumptions of Lemma 2 with respect to the subplane π_τ and hence there exists only one possible extension σ_1^* of $\overline{\sigma^*}$ in H^τ fixing $\mathcal{W}^\tau$. Suppose now that $\pi_\tau \in \mathcal{D}$ with respect to $\hat{\mathcal{M}}'^\tau$. A collineation fixing W^τ also fixes the locus of its singular points, i.e., the Veronese embedding of a non–singular conic. The collineation $\overline{\sigma^*}$ is defined over $H \cap H^\tau \supset \pi$, hence over the Veronese embedding of three points of such a conic. Hence, again, since the accompanying automorphism of σ_1^* is the same as σ^* (and it is uniquely determined) there is only one possible extension σ_1^* of $\overline{\sigma^*}$ in H^τ fixing W^τ.

Let us consider the collineation φ of H^τ defined as follows. For every $R \in H^\tau$, let $R' := PR \cap H$. Then $R^\varphi = PR'^{\sigma_0^*} \cap H^\tau$. If $R \in H \cap H^\tau$, then $R' = R$ and $R^\varphi = R^{\sigma_0^*} = R^{\sigma_1^*} = R^{\overline{\sigma^*}} \in H \cap H^\tau$, i.e. φ is an extension of $\overline{\sigma^*}$. Also, if $R \in \mathcal{W}^\tau$, the line PR belongs to the cone $\mathcal{K}(q^6)$ and hence $R' \in \mathcal{W}$. Since σ^* fixes $\mathcal{W}^{\tau^i}$, this implies that $R'^{\sigma_0^*} \in \mathcal{W}$ and hence $R^\varphi \in PR'^{\sigma_0^*} \subset \mathcal{K}(q^6)$, i.e. φ fixes W^τ. Let now ℓ be a line of the cone $\mathcal{K}(q^6)$ through P and let $R_0 := \ell \cap H$ and $R_1 := \ell \cap H^\tau$, with $R_0 \neq R_1$. Then $P \in \ell^{\sigma^*} = R_0^{\sigma_0^*} R_1^{\sigma_1^*} = R_0^{\sigma_0^*} R_1^\varphi$ and hence $P^{\sigma^*} = P^\sigma = P$, a contradiction.

3.5 Proof of Theorem 3

Let $\mathcal{S}$ be a symplectic semifield spread of $PG(5,q^2)$, q **odd**, whose associated semifield $\mathbb{S}$ has center containing $\mathbb{F}_q$ and let L be the associated $\mathbb{F}_q$–linear set of $\mathbb{P} := PG(5,q^2)$ of rank 6. Then, up to isotopism, $\mathbb{S}$ has order q^6, left nucleus containg $\mathbb{F}_{q^2}$ and center containing $\mathbb{F}_q$ and L has one of the configurations of Theorem 5 and must be disjoint from the secant variety of the Versonese surface $\mathcal{V}$ of $\mathbb{P}$. By the previous arguments Case (2) and, for $q^2 > 2 \cdot 3^8$, Case (3) cannot occur. If L is as in Case (1), then the associated symplectic semifield $\mathbb{S}$ is 3–dimensional over the center and hence by [24] it is isotopic either to a field or to a Generalized twisted field. If L is as in Case (4), then comparing (10) with (18), taking into account Corollary 1, Proposition 9 and the presemifields listed in Section 1, we get the assertion for $q^2 > 2 \cdot 3^8$. □

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