MULTI-VALUE MICROSTRUCTURAL DESCRIPTORS FOR FINER SCALE MATERIAL COMPLEXITY: ANALYSIS OF GROUND STATES

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Abstract. A multi-scale and multi-field description of material microstructures which distinguishes at every point multiple microstructural individuals is analyzed paying attention to the determination of equilibrium configurations. Rigid bodies with microstructure represented by means of $Q-$valued maps from the reference place to the manifold of microstructural shapes are considered first. In that case, just microenergetics is considered and macroscopic motion neglected. Then, the stage is enlarged up to include deformable bodies suffering large strains, endowed with energies of Ginzburg-Landau type with respect to the microstructural descriptor fields. In both cases, conditions for semicontinuity of the relevant energies and the existence of ground states are provided.

0. Representation of material morphologies: individuals in microstructural families

To account for the influence of microscopic events on the mechanical behavior of deformable bodies, we often find necessary or appropriate the introduction of variables, say ν, describing material features that we believe essential for the phenomena under analysis. ν can be referred to a single microstructure or it is a sort of average (in some sense) over a family of microstructures, depending on the spatial scales involved. This is the starting point of the general model building framework of what we call mechanics of complex materials. Here we want to enlarge that view by allowing a detailed description of local families of microstructures made of a given number of unordered fellows. The conceptual path addressing us toward the approach that we pursue here is outlined in the rest of this section.

We have a body placed in an environment and we want to describe its macroscopic behavior under conditions prescribed by the environment itself. This one is the basic aim of continuum mechanics – everybody knows that. In going toward this target, the first step is to fix what we consider to be a body. To be precise, the object of our sensorial perception – the body, indeed – is an intricate rich crop of entangled molecules and/or ordered atomic lattices. Its structure develops along a cascade of spatial and/or temporal characteristic scales. The construction of a mechanical model, then, needs the selection of specific features that we consider essential (the judgement is a priori) for what we aim to describe.

Traditional continuum mechanics has a minimalistic approach to the problem (see [40], [39], [37]). The morphology of a body is described by a region $\mathcal{B}$ which can be at least in principle occupied by the body under scrutiny, and the description of small scale geometries is neglected. $\mathcal{B}$ represents then the gross shape of a body. A certain regularity is prescribed to $\mathcal{B}$: it is consider a regularly open subset of the Euclidean point space with piecewise Lipschitz surface-like boundary. The prescription is motivated by the need of having shapes which can include what we intuitively consider a body on the basis of our daily experience and allow us the use of technical tools like divergence theorem. After fixing $\mathcal{B}$ as a reference place and considering it in a copy of the physical space[1], we evaluate, traditionally, just changes in macroscopic shapes. Every geometric point is then considered as a material point, an indistinct patch of atoms, a sort of black box, indeed, since no information on its peculiar finer scale geometry is given. The velocity field is just a vector field defined over $\mathcal{B}$, with values on the translation space of the ambient space, that specifies the rate at which material elements tend to be crowded and/or sheared. Then, interactions are defined by the power that they develop in the rate of change of the material morphology. In this scheme the sole mechanism considered is just the relative displacement of neighbouring material points, so that the interactions are just covectors (fellows of the dual space where the velocity lives) and the local power density is measured by the values these covectors (tractions and bulk forces) take on the velocity. Just later, in assigning state functions, we try to take into account, although indirectly in a sense, the effective material structures – its inner entanglements – by means of constitutive structures: the state functions.

For classes of materials such as ferroelectrics, quasicrystals, liquid crystals, polymers etc., the approach, beyond being minimalistic, appears even simplistic. There are, in fact, phenomena driven by actions that are not completely described when we consider interactions given by the sole tractions associated with the relative displacement of material points. Paradigmatic examples are the local alignment of stick molecules in case of liquid crystals, the atomic rearrangements in quasicrystals, the polarization in ferroelectrics etc. A refined representation of the material morphology seems then necessary in appropriate cases, in order to take into account the effects on the macroscopic behavior of phenomena developing at micro-scales in space. Besides the placement of a body into the ambient space, we can then consider variables bringing at macroscopic scales information on at least some features of the material morphology at finer spatial scales (we use the world *microstructure* along the paper). So, we have, for example, the polarization vector for ferroelectrics, the degrees of freedom exploited by atoms at low scales to rearrange themselves in quasicrystals, the peculiar direction of stick molecules with head-to-tail symmetry in liquid crystals, etc. A unified view on the matter foresees that, besides the transplacement field through which we reach from a placement taken as reference, say $\mathcal{B}$, other macroscopic shapes $\mathcal{B}_a$, we have a field, defined over $\mathcal{B}$ itself, which takes values on a differentiable manifold $\mathcal{M}$ (see [4], [25], [30]). And it is appropriate, then, to call $\mathcal{M}$ *the manifold of microstructural shapes*. The common assumption that $\mathcal{M}$ be endowed with finite dimension covers cases we know in solid-state physics. Exception is the choice to describe crack paths in a solid by means of Radon measures over the natural Grassmanian constructed over $\mathcal{B}$, taking into account

[1] Reasons for the choice of distinguishing the space hosting the reference place from that where we evaluate the actual ones are associated with the notion of observer, the definitions of classes of changes in observers and their rôle in deriving balance equations, and investigating material symmetries (see further remarks and related results in [27]).

this way at every point the possibility that a crack could occur there along some direction (see [17]). In any case, however, the representation of the morphology of a body becomes multi-field and, intrinsically, multi-scale because the additional field taking values over $\mathcal{M}$ transfers at macroscopic scale information on what is the intricate inner geometry at some finer scale that hosts events influencing even drastically the macroscopic behavior.

The attribution of geometrical structure to $\mathcal{M}$ has to be handled with care. Metric and connection bring with them physical meaning. Metric, in fact, is associated with the representation of possible microstructural kinetic energy, relative to the macroscopic motion – there are reasons to foresee such a kind of additional kinetics, at least in appropriate special circumstances (see [7], [32]). Moreover, connection is involved in the representation of first-neighbour interactions. Sometimes a physically significant connection seems to be not available (see [7]). So, we find convenient to endow $\mathcal{M}$ with as skeletal as possible geometric structure, unless technical instances impose us the choice of additional properties. In this case, however, we have to state clearly the consequent limitations in the ability to describe physical events.

Since we have chosen $\mathcal{M}$ with finite dimension, it could be natural to suggest embedding into a linear space, with the consequent non-trivial advantages of having at disposal the linear algebraic structure. The embedding is always available by Whitney theorem. It is even isometric by Nash theorems in case $\mathcal{M}$ be Riemannian. The choice would then save the representation of microstructural kinetic energy when it is available. However, the embedding is not unique, so its choice would become a structural ingredient of the modeling procedure.

For this reason, with the aim of furnishing results as general as possible, *we avoid in the sequel to embed $\mathcal{M}$ into some linear space, although we shall restrict ourselves to the case in which $\mathcal{M}$ is a smooth, complete, and connected Riemannian manifold, endowed with its geodesic distance*[2].

At $\nu \in \mathcal{M}$, fellows of the tangent space $T_\nu \mathcal{M}$ indicate rates – let us write $\dot{\nu}$ for them – of change in the geometric microscopic features represented at a certain $x \in \mathcal{B}$ by ν, which is then $\nu(x)$. Elements of the cotangent space $T_\nu^* \mathcal{M}$ express the power performed in developing microstructural changes, when they are evaluated over a certain $\dot{\nu}$. So, fellows of $T_\nu^* \mathcal{M}$ represent then microstructural (microscopic if you want) actions, which have different nature: (i) contact actions of first-neighbour type exerted between pairs of material points when there is inhomogeneity in the field $x \longmapsto \nu(x)$, namely there is non-zero spatial derivative of ν, (ii) external bulk actions working directly on the microstructure (and this is essentially the case of electromagnetic fields on microstructures which are sensitive to them, such as local polarization or magnetization), (iii) microstructural self-actions. For the last class, the typical example is the one of ferroelectrics where the polarization at a point generates a local electric field and a consequent self-action. Such a class of actions does not appear in the standard format of continuum mechanics. The free energy density of a deforming body, in fact, cannot depend on the transplacement field but just on its spatial

[2]In the dynamic case (not treated here), the Riemannian structure would imply a quadratic form for the kinetic microenergy, when it would be available.

derivative (of first or higher rank), if we want that the energy density itself be invariant with respect to changes in the atlas over the ambient space, determined by isometries. The restriction is strictly induced by the linear structure of the ambient space which is, in fact, the $m-$dimensional Euclidean point space. Such a restriction is barely available in general over $\mathcal{M}$ which is taken as an abstract non-linear manifold when we construct general structures of the mechanics of materials the representation of which falls within the scheme just sketched, materials that we call *complex* just to distinguish them from those falling within the traditional scheme of continuum mechanics.

In this view, every material point is no more a representative at macroscopic scale of an indistinct patch of matter, rather it is considered as a *system*. When we select $\mathcal{M}$, we are assuming implicitly that there is a sort of homogeneity in the *type* of microstructure, or better, we are affirming that we want to account for some specific features of the microstructure everywhere, irrespective of possible fluctuations. The system placed at x can be made by a number of individuals. A paradigmatic example is the one of liquid crystals for which we imagine that at x there is a family of stick molecules. In this case the assignment of ν implies at least some form of average over the family. The computation of the average implies a number of difficulties. In fact, if $\mathcal{M}$ is a non-linear manifold, the integral of a field taking values over $\mathcal{M}$ is in general not defined. Rather, in common cases $\mathcal{M}$ itself is chosen to be a sort of manifold of averages. To be precise, in the case of liquid crystals in nematic order, $\mathcal{M}$ is naturally selected as the projective plane $\mathbb{P}^2$, so ν is just a direction as the head-to-tail symmetry of the stick molecules composing liquid crystals imposes. Then, to us, $\nu(x)$ is at x the 'average' direction (the prevailing one, then) along which the molecules tend to be aligned [13]. Of course, the representation can be not always satisfactory, according to the type of phenomenon we want to describe. We can find appropriate, in fact, to account for some other elements characterizing the local distribution of molecules, information on which we want to attribute to the point x. A choice can be the addition to the direction ν of another parameter representing the degree of orientation [14] or, in case we want to analyze events connected with optical biaxiality, the degree of prolation and the one of triaxiality [6]. Beyond liquid crystals, higher moments of the distribution of microstructures have been found to be useful descriptors of the local state of the matter in the cases of microcracked materials [28] and for granular assemblies in agitation [5]. Tentatives of extending such a view to the general setting of the mechanics of complex materials have been pursued along different paths also in [38] and [26], in the latter case taking into account the possibility of migration of fellows of a local family of microstructures, the one pertaining to the patch of matter that we imagine placed at x in a multi-scale representation of the material complexity.

We do not know approaches where at every point *we consider a microstructure made of a family of indistinguishable individuals* (let say a finite number of polymeric molecules, for example) *that interacts among them*. And this is exactly what we try to do here.

- In what follows, to us a material point is a system containing *a family of microstructures composed by a finite number of individuals which cannot be distinguished one from the other, and then cannot be ordered and labelled accordingly*. An appropriate example is to consider the generic material point as the representative of a patch of matter containing a finite number of polymers. The system is assumed to be canonical in the standard sense of statistical mechanics. So, we do not consider migration of microstructures.

- The manifold of microstructural shapes $\mathcal{M}$ contains *descriptors of each single individual* of the microstructural family. For example, in the case of multiple polymeric chains, we can consider $\nu \in \mathcal{M}$ as an head-to-tail vector if the polymer is linear (meaning it is not a star polymer).
- *The field assigning the microstructural descriptor $\nu(x)$ at every point x in the macroscopic place $\mathcal{B}$, that we take as a reference, is a Q-valued map*, that is a map taking Q unordered values. *The multiple values correspond then each one to a single individual in the microstructural family at x.*

The last item in the list above marks the difference with previous work on the matter. We are essentially interested in finding properties of semicontinuity and existence of ground states in two circumstances:

(1) materials with rigid macroscopic behaviour and microscopic energetics,
(2) elastic complex materials with decomposed energies being the sum of a part depending on x, ν, and the spatial derivative of the macroscopic transplacement, and a part which is quadratic with respect to the derivative of the $Q-$valued map.

The framework in which we develop our analyses is conservative.

Our setting is not only pertinent to the case of liquid crystals and microcracked materials already mentioned. Another example can be the one of polymeric bodies. Once we decide a model spatial scale at which the generic material element identified with a point can be thought of as a patch of matter including a certain amount of molecules even in a melt, the Q values of ν describe Q polymers imagined to be entangled at a given point.

In a different setting, other examples can be called upon even by thinking of direct models of elastic structures. For shells constituted by different layers, we can consider their 'middle' surface and attach at each point of it a vector for each layer. The scheme applies also to thin films considered to be generated by the superposition of layers with atom thickness (see [16]).

Coming back to the example of liquid crystals and polymeric bodies, we stress the importance in our model of considering *unordered* families of microstructures. When we choose to consider a material element as a patch of matter containing a number of molecules, and we assign to the generic one a descriptor of peculiar aspects of its shape, the same choice for all molecules, we are no more able to distinguish one molecule from the other. Such an indetermination applies obviously more in general and we translate it in our scheme by affirming that ν takes Q-values modulo permutations. So, we are not superposing the description of Q molecules, or, generally speaking, Q microstructures, rather, the elements of our general representation of the material texture are strongly coupled.

We conclude the introduction by describing briefly the contents of the various sections in the paper. We start off in Section 1 to settle the notations and definitions of metric space valued Sobolev maps, considering in particular $\mathcal{M}$-valued, $\mathcal{M}$ a manifold, and $\mathcal{A}_Q(\mathcal{M})$-valued maps in subsections 1.2 and 1.3, respectively. Various approximate differentiability properties of $\mathcal{A}_Q(\mathcal{M})$-valued maps are investigated in Section 2. Thanks to those properties we define and study the lower semicontinuity of energies in the model case of rigid bodies with microstructures in Section 3 (see Theorem 3.2). The existence of ground states in the elastic case is addressed finally in Section 4 (see Theorem 4.2). Eventually, in Appendix A, we prove a technical lemma instrumental for our approach.

1. Function spaces

Throughout the paper, $\mathcal{B}$ will always be a bounded, regularly open subset of the Euclidean space $\mathbb{R}^m$ endowed with canonical base $\mathrm{e}_1, \ldots, \mathrm{e}_m$. $\mathcal{B}$ is considered to be the place that the body under scrutiny could in principle occupy having a shape that we take as a reference for comparing changes in lengths, areas, volume. $\mathcal{B}$ is the domain where we define two types of maps: (i) **transplacements** describing changes of places, (ii) **morphological descriptor fields** bringing information on the architecture of the matter at finer spatial scales. The choice of functional classes for these types of maps has constitutive nature. Roughly speaking, to belong to a space, a map should have some properties which bring with them physical meaning for they allow the description of some aspects of the physical phenomena they are referred to and exclude others.

Here, we first discuss the case of *rigid bodies endowed with* (active – meaning that their changes contribute to the energetic landscape) *microstructure* composed by a number of *unordered* individual substructures. For them we neglect macroscopic transplacements and consider just microenergetics. We have then to characterize maps assigning to each point x in $\mathcal{B}$ descriptors of the microstructure individuals (they are finite in number – say Q) at x. These maps take then multiple values over the manifold of microstructural shapes where each microstructure individual is described. In particular we shall speak then about *metric space* valued, *manifold* valued and *multiple* valued Sobolev functions.

In what follows the letter C will denote generically a positive constant, leaving understood that the nature of the constant might change from line to line. The parameters on which each constant C depends will be explicitly highlighted.

1.1. Metric space valued Sobolev maps.

Let (X, d) be a complete, separable and locally compact metric space – the metric is indicated by d. Different definitions of weakly differentiable functions with values in a metric space have been proposed in the literature (see, e.g., [2, 22, 23, 34]). In the case of metric spaces with the properties above, all such definitions give rise to the same space of functions (see [8]). To our purposes, the most convenient definition is the one proposed in [2] and then generalized in [34, 35, 36].

Definition 1.1. For $p \in [1, +\infty]$, we say that a map ν belongs to $W^{1,p}(\mathcal{B}, X)$ if there exists $h \in L^p(\mathcal{B})$ such that, for every $\nu_0 \in X$,

(i) the real valued function $x \mapsto d(\nu(x), \nu_0)$ is $W^{1,p}(\mathcal{B})$;
(ii) and the distributional gradient satisfies $|D\big(d(\nu(\cdot), \nu_0)\big)| \leq h(\cdot)$ $\mathcal{L}^m$-a.e. in $\mathcal{B}$.

Remark 1.2. Maps $\nu \in W^{1,p}(\mathcal{B}, X)$ are stable under composition with Lipschitz functions $\varphi : (X, d) \to \mathbb{R}^k$, i.e. $\varphi \circ \nu \in W^{1,p}(\mathcal{B}, \mathbb{R}^k)$ and $|D(\varphi \circ u)| \leq \mathrm{Lip}(\varphi)\, h$ (see, for example, [34]).

Loosely speaking, in such a general framework only the definition of modulus of the gradient is possible. By following [2] (see also [8, 24, 36]), in the previous definition of $W^{1,p}(\mathcal{B}, X)$, we are interested in finding the smallest function h for which the requirement (ii) above is fulfilled. Such a function is realized by fixing a dense and denumerable set $\{\nu_i\}_{i \in \mathbb{N}}$ in X and setting

$$|D\nu| := \sup_{i \in \mathbb{N}} \left|D\big(d(\nu(\cdot), \nu_i)\big)\right|. \tag{1.1}$$

On the other hand, in case (X, d) be either a smooth, complete and connected Riemannian manifold endowed with its geodesic distance $(\mathcal{M}, d_{\mathcal{M}})$, or the corresponding space of Q-valued maps $(\mathcal{A}_Q(\mathcal{M}), \mathcal{G}_{\mathcal{M}})$ (see below for the definition), an approximate differential can be introduced $\mathcal{L}^m$-a.e. on $\mathcal{B}$ by exploiting the linear structure of the tangent bundle to $\mathcal{M}$.

Before doing this, we recall the definition of weak convergence in $W^{1,p}(\mathcal{B}, X)$.

Definition 1.3. For $p \in [1, \infty]$ and $\nu \in W^{1,p}(\mathcal{B}, X)$, a sequence $\{\nu_k\}_{k \in \mathbb{N}} \in W^{1,p}(\mathcal{B}, X)$ converges weakly to ν for $k \to \infty$ in $W^{1,p}(\mathcal{B}, X)$ – and we write $\nu_k \rightharpoonup \nu$ in this case – if

(i) $\|d(\nu_k, \nu)\|_{L^p(\mathcal{B})} \to 0$ as $k \to \infty$;
(ii) $\sup_k \||D\nu_k|\|_{L^p(\mathcal{B})} < \infty$;
(iii) in case $p = 1$, $(|D\nu_k|)_{k \in \mathbb{N}}$ is equi-integrable.[3]

1.2. **Manifold constrained Sobolev maps.** In the general model building framework in which we describe material microstructures, we need to define spaces of maps taking values on a manifold $\mathcal{M}$. To establish the relevant definitions, we shall use previous concepts but we need also to recall some standard notions and results in Riemannian geometry (the reader can refer to [12] for further details).

In what follows, $(\mathcal{M}^n, g)$ will always denote a connected, $n-$dimensional, complete, C^k Riemannian manifold with $k \geq 2$, indicated simply by $\mathcal{M}$. It is understood that $\mathcal{M}$ satisfies Hausdorff and countable basis axioms. Since $\mathcal{M}$ is assumed always to be complete – it means that the exponential map $\exp_\nu$ is defined for every $\nu \in \mathcal{M}$ – by Hopf-Rinow's theorem, $\mathcal{M}$ and its geodesic distance $d_{\mathcal{M}}$ constitute a complete metric space. In particular, for $\nu \in \mathcal{M}$ we shall denote with $B_r(\nu) \subseteq \mathcal{M}$ the open ball of radius r, defined with respect to the metric $d_{\mathcal{M}}$. With a slight abuse of notation, the Euclidean ball in $\mathbb{R}^m$, centered at x, with radius $r > 0$, will be denoted also by $B_r(x)$, and the m-dimensional Euclidean cube with side r and center x by $C_r(x)$, i.e. $C_r(x) := x + [-\frac{r}{2}, \frac{r}{2}]^m \subset \mathbb{R}^m$.

As usual, $T\mathcal{M}$ denotes the tangent bundle: points of $T\mathcal{M}$ are couples (ν, v), where ν is in $\mathcal{M}$ and v is a tangent vector to $\mathcal{M}$ at ν, in symbols $\mathrm{v} \in T_\nu\mathcal{M}$. In addition, we consider the vector bundle with base space $\mathcal{M}$ and total space the one of linear homomorphisms $\mathrm{Hom}(\mathbb{R}^m, T\mathcal{M})$, the points of which are couples (ν, A) with ν in $\mathcal{M}$ and $A : \mathbb{R}^m \to T_\nu\mathcal{M}$ a linear map. For this bundle, $\pi : \mathrm{Hom}(\mathbb{R}^m, T\mathcal{M}) \to \mathcal{M}$ denotes the projection map over $\mathcal{M}$.

[3] This definition has been introduced in [11, Definition 1.3], where condition (iii) was erroneously forgotten – although implicitly used in the proofs.

With fixed ν in $\mathcal{M}$, $\mathrm{Hom}(\mathbb{R}^m, T_\nu\mathcal{M})$ can be identified with $(T_\nu\mathcal{M})^m$ through the identification

$$A \simeq (\mathrm{v}_1, \dots, \mathrm{v}_m) \quad \text{with} \quad \mathrm{v}_i = A\,\mathrm{e}_i \in T_\nu\mathcal{M}, \quad \text{for} \quad i = 1, \dots, m.$$

Since in the following we shall consider continuous functionals on such bundles, we specify that we endow $T\mathcal{M}$ with the induced Riemannian metric (see, for instance, [12, Chapter 3, exercise 2]). For $(p, \mathrm{v}), (q, \mathrm{w}) \in T\mathcal{M}$, and $\gamma(t)$ a path – t the parameter defining it – which connects p and q, the distance $d_{T\mathcal{M}}\big((p, \mathrm{v}), (q, \mathrm{w})\big)$ between the two elements of $\mathcal{M}$ is given by

$$d_{T\mathcal{M}}\big((p, \mathrm{v}), (q, \mathrm{w})\big) := \inf_{\vartheta=(\gamma,Y)} \int_0^1 \sqrt{|\dot\gamma(t)|^2_{g(\gamma(t))} + |\nabla_{\dot\gamma(t)} Y(t)|^2_{g(\gamma(t))}}\, dt,$$

where the infimum is taken among all smooth curves

$$[0, 1] \ni t \mapsto \vartheta(t) = (\gamma(t), Y(t)) \in T\mathcal{M},$$

such that $\vartheta(0) = (p, \mathrm{v})$ and $\vartheta(1) = (q, \mathrm{w})$. Above, ∇ indicates Levi-Civita connection.

With this metric at disposal, we define a metric structure on $\mathrm{Hom}(\mathbb{R}^m, T\mathcal{M})$ simply specifying the distance

$$D\big((p, A), (q, B)) := \sqrt{\sum_{i=1}^m d_{T\mathcal{M}}\big((p, \mathrm{v}_i), (q, \mathrm{w}_i)\big)^2}, \tag{1.2}$$

where $A \simeq (\mathrm{v}_1, \dots, \mathrm{v}_m)$ and $B \simeq (\mathrm{w}_1, \dots, \mathrm{w}_m)$ with the above identification. Being arbitrary, such a choice is however equivalent to any reasonable metric which is compatible with the one on $T\mathcal{M}$ in the case $m = 1$.

An approximate differentiability property has been established in [15, Corollary 0.3].

Proposition 1.4. *Every map $\nu \in W^{1,p}(\mathcal{B}, \mathcal{M})$ is approximately differentiable $\mathcal{L}^m$-a.e. on $\mathcal{B}$, i.e. for $\mathcal{L}^m$-a.e. $x_0 \in \mathcal{B}$, there exists a unique linear map $d\nu_{x_0} : \mathbb{R}^m \to T_{\nu(x_0)}\mathcal{M}$ such that, for all $\varepsilon > 0$,*

$$\lim_{r\to 0^+} r^{-m}\mathcal{L}^m\left(\left\{x \in C_r(x_0) : d_{\mathcal{M}}\left(\nu(x), \exp_{\nu(x_0)}(d\nu_{x_0}(x - x_0))\right) \geq \varepsilon\,|x - x_0|\right\}\right) = 0. \tag{1.3}$$

It is possible to prove that for some dimensional constant $C_m > 0$

$$C_m^{-1}\|d\nu_x\|_{g(\nu(x))} \leq |D\nu|(x) \leq C_m\|d\nu_x\|_{g(\nu(x))} \quad \mathcal{L}^m \text{ a.e. in } \mathcal{B},$$

where $|D\nu|$ is the norm of the differential defined in the metric setting (cp. with (1.1)) and $\|\cdot\|_{g(\nu(x))}$ denotes the operatorial norm of $d\nu_x$, namely

$$\|d\nu_x\|_{g(\nu(x))} := \sup_{v\in\mathbb{R}^m,\, |v|=1} |d\nu_x(v)|_{g(\nu(x))},$$

with $|\cdot|_{g(\nu)}$ the norm in $T_\nu\mathcal{M}$ induced by the metric g (see [15, Remark 1.7]). The metric definition of Sobolev maps coincides with the extrinsic one obtained by means of an isometric embedding $i : \mathcal{M} \to \mathbb{R}^N$, and moreover the corresponding differentials are related by a chain rule formula (see [15, Remark 0.4]).

1.3. **Multiple valued Sobolev maps.** With the expression in the title of this section we mean maps valued in the complete metric space of unordered sets of Q points in $\mathcal{M}$. This notion has been introduced by Almgren [1] in connection with the regularity theory of minimizing surfaces. It has been also revisited and exploited in different contexts (see [10, 11] for a more detailed bibliography on the subject).

Definition 1.5. We denote by $(\mathcal{A}_Q(\mathcal{M}), \mathcal{G}_{\mathcal{M}})$ the metric space of unordered Q-tuples of points in $\mathcal{M}$ given by

$$\mathcal{A}_Q(\mathcal{M}) := \left\{ \sum_{i=1}^{Q} [\![P_i]\!] \, : \, P_i \in \mathcal{M} \; \text{ for every } i = 1, \ldots, Q \right\},$$

where $[\![P_i]\!]$ denotes the Dirac mass in $P_i \in \mathcal{M}$ and

$$\mathcal{G}_{\mathcal{M}}(T_1, T_2) \;:=\; \min_{\pi \in \mathscr{P}_Q} \sqrt{\sum_i d^2_{\mathcal{M}}(P_i, S_{\pi(i)})},$$

with $T_1 = \sum_i [\![P_i]\!]$ and $T_2 = \sum_i [\![S_i]\!] \in \mathcal{A}_Q(\mathcal{M})$, and $\mathscr{P}_Q$ denotes the group of permutations of $\{1, \ldots, Q\}$.

In case k P_{i_j}'s are all equal to some P_0, with a slight abuse of notation we shall write $k\,[\![P_0]\!]$ for $\sum_{j=1}^{k} [\![P_0]\!]$. Continuous, Hölder, Lipschitz, Lebesgue measurable maps from $\mathcal{B}$ into $\mathcal{A}_Q(\mathcal{M})$ are defined in the usual way, while Sobolev maps are defined according to Definition 1.1.

Due to the fact that the values are not ordered, for a map ν taking values in $\mathcal{A}_Q(\mathcal{M})$, the existence of *selections*, that are functions $\nu_i : \mathcal{B} \to \mathcal{M}$, $i = 1, \ldots, Q$, with the same regularity and

$$\nu(x) = \sum_i [[\nu_i(x)]]$$

fails in general. One exception is discussed in Proposition 0.4 of [10]. Here we prove a selection lemma for L^p maps as a simple consequence of the measurable selection analyzed in Proposition 0.4 of [10].

Lemma 1.6. *Let ξ and ν be in $L^p(\mathcal{B}, \mathcal{A}_Q(\mathcal{M}))$. There exist selections in $L^p(\mathcal{B}, \mathcal{M})$ such that*

$$\int_{\mathcal{B}} \mathcal{G}^p_{\mathcal{M}}(\xi, \nu)\, dx = \int_{\mathcal{B}} \left(\sum_i d^2_{\mathcal{M}}(\xi_i, \nu_i) \right)^{p/2} dx.$$

Proof. Proposition 0.4 of [10] ensures that there exist measurable selections $\xi_1, \ldots, \xi_Q$ for ξ and $\nu_1, \ldots, \nu_Q$ for ν. For every $\pi \in \mathscr{P}_Q$ consider the set

$$\mathcal{B}_\pi = \left\{ x \in \mathcal{B} : \sum_i d^2_{\mathcal{M}} \left(\xi_{\pi(i)}(x), \nu_i(x) \right) \leq \mathcal{G}^2_{\mathcal{M}}(\xi(x), \nu(x)) \right\}.$$

Clearly $\mathcal{B}_\pi$ is measurable and $\mathcal{L}^m(\mathcal{B} \setminus \cup_\pi \mathcal{B}_\pi) = 0$. Set

$$\widetilde{\xi}_i(x) = \xi_{\pi(i)}(x) \quad \text{if } x \in \mathcal{B}_\pi,\ \pi \in \mathscr{P}_Q,$$

then $\widetilde{\xi}_1, \dots, \widetilde{\xi}_Q$ is a $L^p(\mathcal{B}, \mathcal{M}^Q)$ selection of ξ and the thesis follows. □

2. Differentiability of $\mathcal{A}_Q(\mathcal{M})$-valued maps

As for classical $\mathcal{M}$-valued and $\mathcal{A}_Q$-valued Sobolev maps, a feature of Sobolev $\mathcal{A}_Q(\mathcal{M})$-valued functions is the existence of an approximate differential almost everywhere.

Definition 2.1. Let $\nu \in W^{1,p}(\mathcal{B}, \mathcal{A}_Q(\mathcal{M}))$, $x_0 \in \mathcal{B}$ be a Lebesgue point for ν and $\nu(x_0) = \sum_{i=1}^Q [\![\nu_i(x_0)]\!]$. We say that ν is approximately differentiable at x_0 if there exist linear maps $L_i : \mathbb{R}^m \to T_{\nu_i(x_0)}\mathcal{M}$, $i = 1, \dots, m$, such that $L_i = L_j$ if $\nu_i(x_0) = \nu_j(x_0)$, and, for all $\varepsilon > 0$, it holds

$$\lim_{r \to 0^+} r^{-m} \mathcal{L}^m \left(\{ x \in C_r(x_0) : \mathcal{G}_{\mathcal{M}} \left(\nu(x), T_{x_0}\nu(x) \right) \geq \varepsilon |x - x_0| \} \right) = 0, \tag{2.1}$$

with

$$T_{x_0}\nu(x) := \sum_{i=1}^{Q} [\![\exp_{\nu_i(x_0)}(L_i(x - x_0))]\!] \,.$$

When defined, the linear maps L_i are uniquely determined; in such a case we shall denote them respectively by $(d\nu_i)_{x_0}$. This way the first-order approximation $T_{x_0}\nu$ is then unambiguously determined.

Below we show that multiple valued Sobolev maps with target a manifold are almost everywhere differentiable and satisfy a L^p-approximate differentiability estimate.

2.1. Approximate differentiability.

Proposition 2.2. *Every map $\nu \in W^{1,p}(\mathcal{B}, \mathcal{A}_Q(\mathcal{M}))$ is approximately differentiable $\mathcal{L}^m$-a.e. on $\mathcal{B}$.*

We follow the proof of Rademacher's theorem for $\mathcal{A}_Q(\mathbb{R}^n)$ maps in [10, Theorem 1.13], despite in the current setting no extension theorem for Lipschitz maps is in general available. Moreover, as discussed in the introduction, we always work with the abstract definition of manifold $\mathcal{M}$, without exploiting any isometric embedding on a linear space.

We start showing that Lipschitz multiple valued maps are approximately differentiable almost everywhere.

Proposition 2.3. *Let $B \subseteq \mathbb{R}^m$ be a Borel set and $\nu : B \to \mathcal{A}_Q(\mathcal{M})$ be Lipschitz. Then, ν is approximately differentiable $\mathcal{L}^m$ a.e. on B.*

Proof. We prove the result by induction on Q. We first notice that the case $Q = 1$, that is when $\nu : B \to \mathcal{M}$ is a Lipschitz map, follows by (an inspection of) the proof of Corollary 0.3 in [15].

Next, we assume the result to be true for $1 \leq Q < Q^*$ and prove its validity for Q^*. To this aim, consider a measurable selection of ν, i.e. $\nu(x) = \sum_{i=1}^{Q^*} [\![\nu_i(x)]\!]$, and set

$$\widetilde{B} := \{ x \in B : \nu(x) = Q\, [\![\nu_1(x)]\!] \}.$$

We argue differently for $B \setminus \widetilde{B}$ and $\widetilde{B}$.

Given a point $x_0 \in B \setminus \widetilde{B}$, we may find a neighborhood U of x_0 and Lipschitz functions $\nu_K : B \cap U \to \mathcal{A}_K(\mathcal{M})$, $\nu_H : B \cap U \in \mathcal{A}_H(\mathcal{M})$ such that $\nu(x) = [\![\nu_K(x)]\!] + [\![\nu_H(x)]\!]$ on $B \cap U$ (see [10, Proposition 1.6]). By inductive hypothesis, the map ν is then differentiable almost everywhere in U.

On $\widetilde{B}$, we claim that ν is approximately differentiable if $x_0 \in \widetilde{B}$ is a point of density one and ν_1 is approximately differentiable at x_0 (both conditions satisfied almost everywhere in $\widetilde{B}$). In this case, the linear approximation is given by

$$T_{x_0}\nu(x) = Q\,[\![\exp_{\nu_1(x_0)}(d\nu_1)_{x_0}(x - x_0)]\!]\,.$$

In fact, fix $\gamma \in (0,1)$ and take $x \in B \setminus \widetilde{B}$, $r = |x - x_0|$ and $x^* \in \widetilde{B} \cap \overline{B_{2r}(x_0)}$ satisfying $|x - x^*| < (1+\gamma)\,d(x, \widetilde{B} \cap \overline{B_{2r}(x_0)})$. Then, we find a positive constant $C = C(Q, \nu_1(x_0), \mathrm{Lip}(\nu_1), \mathcal{M})$ such that

$$\begin{aligned}
\mathcal{G}_{\mathcal{M}}(\nu(x), T_{x_0}\nu(x)) &\le \mathcal{G}_{\mathcal{M}}(\nu(x), \nu(x^*)) + \mathcal{G}_{\mathcal{M}}(\nu(x^*), T_{x_0}\nu(x^*)) + \mathcal{G}_{\mathcal{M}}(T_{x_0}\nu(x^*), T_{x_0}\nu(x)) \\
&\le C|x - x^*| + \mathcal{G}_{\mathcal{M}}(Q\,[\![\nu_1(x^*)]\!]\,, Q\,[\![\exp_{\nu_1(x_0)}(d\nu_1)_{x_0}(x^* - x_0)]\!]) \\
&\le C|x - x^*| + Q\,d_{\mathcal{M}}(\nu_1(x^*), \exp_{\nu_1(x_0)}(d\nu_1)_{x_0}(x^* - x_0)) \\
&\le C|x - x^*| + o(|x^* - x_0|).
\end{aligned}$$

Since $|x^* - x_0| \le 2\,r = 2\,|x - x_0|$, to conclude it suffices to show that $|x - x^*| = o(|x - x_0|)$ as $x \to x_0$. By construction

$$B_\rho(x) \subseteq B_{2\,r}(x_0) \quad\text{and}\quad B_\rho(x) \cap \widetilde{B} = \emptyset, \quad\text{with}\quad \rho := \frac{|x^* - x|}{1+\gamma}.$$

In turn, the inequality $\mathcal{L}^m(B_\rho(x)) \le \mathcal{L}^m(B_{2\,r}(x_0) \setminus \widetilde{B})$ is implied. Eventually, by taking into account that $\widetilde{B}$ has density one in x_0, we infer property $|x^* - x| = o(|x - x_0|)$ as $x \to x_0$ from

$$\limsup_{r\to 0^+} r^{-m}\rho^m \le \limsup_{r\to 0^+} r^{-m}\mathcal{L}^m(B_{2\,r}(x_0) \setminus \widetilde{B}) = 0. \qquad \square$$

The approximate differentiability property for $\mathcal{A}_Q(\mathcal{M})$-valued Sobolev maps is now a straightforward consequence of Proposition 2.3 and the following lemma.

Lemma 2.4. *Let ν be in $W^{1,p}(\mathcal{B}, \mathcal{A}_Q(\mathcal{M}))$, $p \in [1, +\infty]$. Then, there exists an increasing family of compactly supported Borel sets $\mathcal{B}_\lambda \subseteq \mathcal{B}$ such that $\mathcal{L}^m(\mathcal{B} \setminus \mathcal{B}_\lambda) \to 0$ as $\lambda \uparrow \infty$ and $\nu|_{\mathcal{B}_\lambda}$ is Lipschitz continuous.*

The proof of the lemma is an application of standard arguments for the maximal function operator (see, e.g., [15, Lemma 1.1]). We leave the details to the reader.

2.2. **L^p-approximate differentiability.** A more refined differentiability result can be proven: the approximate differentiability property holds in the stronger sense of integral averages rather than only for the measure of superlevel sets.

As above, we prove first it for Lipschitz functions.

Proposition 2.5. *Take $\nu \in \mathrm{Lip}(\mathcal{B}, \mathcal{A}_Q(\mathcal{M}))$ and $p \in [1, \infty[$. Then, for $\mathcal{L}^m$-a.e. $x_0 \in \mathcal{B}$ we get*

$$\lim_{r \to 0} \fint_{C_r(x_0)} \frac{\mathcal{G}^p_{\mathcal{M}}\left(\nu(x), T_{x_0}\nu(x)\right)}{|x - x_0|^p} dx = 0. \tag{2.2}$$

Proof. Consider the family of real valued functions $\{w_x\}_{x \in \mathcal{B}}$, with $w_x := \mathcal{G}_{\mathcal{M}}\left(\nu(\cdot), T_x\nu(\cdot)\right)$. Clearly w_x is Lipschitz continuous on $\mathcal{B}$ for all x, and moreover, for $\mathcal{L}^m$ a.e. point $x \in \mathcal{B}$, the function w_x turns out to be approximately differentiable at x with $w_x(x) = 0$ and $|Dw_x(x)| = 0$ by Proposition 2.2 and equation (2.1) in Definition 2.1, .

Fix a point x_0 for which Proposition 2.2 applies, and denote by $I_r(x_0)$ the integral on the left hand side in (2.2) and by L_{x_0} the Lipschitz constant of w_{x_0}. Since w_{x_0} is positive and $w_{x_0}(x_0) = 0$, we obtain

$$\begin{aligned} I_r(x_0) &= r^{-m} \left(\int_{\{x \in C_r(x_0) \colon w_{x_0}(x) \geq \varepsilon |x - x_0|\}} + \int_{\{x \in C_r(x_0) \colon w_{x_0}(x) < \varepsilon |x - x_0|\}} \right) \frac{w^p_{x_0}(x)}{|x - x_0|^p} \, dx \\ &\leq L^p_{x_0} \, r^{-m} \, \mathcal{L}^m(\{x \in C_r(x_0) : \, w_{x_0}(x) \geq \varepsilon |x - x_0|\}) + \varepsilon^p = o(1) + \varepsilon^p \quad \text{as } r \downarrow 0^+. \end{aligned}$$

The conclusion then follows by letting first $r \downarrow 0^+$ and then $\varepsilon \downarrow 0^+$. □

To extend the previous statement to Sobolev Q-valued maps we need a characterization of standard Sobolev functions in terms of the corresponding maximal function of (the modulus of) the gradient. This characterization has been employed to define Sobolev mappings on metric measure spaces. Actually, we shall exploit only the sufficiency part of such a result.

In what follows, given $w \in W^{1,p}(\mathcal{B})$, $\mathcal{B}$ an extension domain, by $m(|Dw|)$ we denote the maximal function of an extension of w to an open set $\mathcal{B}' \supset\supset \mathcal{B}$ (see [21, Theorem 1]).

Theorem 2.6. *Let $\mathcal{B}$ be a bounded domain with the extension property. There exists a constant $C = C(\mathcal{B}) > 0$ such that if $w \in W^{1,p}(\mathcal{B})$, then*

$$|w(x) - w(x_0)| \leq C \left(m(|Dw|)(x) + m(|Dw|)(x_0) \right) |x - x_0|, \tag{2.3}$$

for all x and x_0 Lebesgue points of w.

We can now prove the L^p-differentiability for any Sobolev function.

Proposition 2.7. *Let $\nu \in W^{1,p}(\mathcal{B}, \mathcal{A}_Q(\mathcal{M}))$. Then, for $\mathcal{L}^m$-a.e. $x_0 \in \mathcal{B}$, we get*

$$\lim_{r \to 0} \fint_{C_r(x_0)} \frac{\mathcal{G}^p_{\mathcal{M}}\left(\nu(x), T_{x_0}\nu(x)\right)}{|x - x_0|^p} dx = 0. \tag{2.4}$$

Proof. Consider the family $\{w_x\}_{x \in \mathcal{B}}$, with $w_x := \mathcal{G}_{\mathcal{M}}\left(\nu(\cdot), T_x\nu(\cdot)\right)$. Then, Corollaries 1 and 2 in [34] and Proposition 2.2 imply that for $\mathcal{L}^m$ a.e. point $x \in \mathcal{B}$ each map w_x is in the standard Sobolev space $W^{1,p}(\mathcal{B})$, x is a Lebesgue point for w_x, and w_x is approximately differentiable at x with $w_x(x) = 0$ and $|Dw_x(x)| = 0$.

We then select points x_0 in $\mathcal{B}$ satisfying the following requirements:

(i) $x_0 \in \cup_i \mathcal{B}_{\lambda_i}$, $\mathcal{B}_{\lambda_i}$ being the sets in Lemma 2.4 with $\lambda_i \downarrow 0^+$, and actually x_0 is of density one for some $\mathcal{B}_{\lambda_i}$ (and then for all $\mathcal{B}_{\lambda_k}$ for $k \geq i$);

(ii) if $w_{x_0} \in W^{1,p}(\mathcal{B})$, w_x is approximately differentiable at x_0, with $w_{x_0}(x_0) = 0$ and $|Dw_{x_0}(x_0)| = 0$;

(iii) x_0 is a p-Lebesgue point for ν, $|D\nu|$ and for $m(|D\nu|)\chi_{\mathcal{B}\setminus\mathcal{B}_{\lambda_i}}$ for all $i \in \mathbb{N}$.

The points x satisfying previous requirements constitute a set of full measure in $\mathcal{B}$.

Since $w|_{\mathcal{B}_{\lambda_i}}$ is Lipschitz continuous, by arguing as in Proposition 2.5, to get the conclusion it suffices to show that

$$\limsup_{r\to 0} \fint_{C_r(x_0)\setminus\mathcal{B}_{\lambda_i}} \frac{w^p_{x_0}(x)}{|x-x_0|^p} dx = 0. \tag{2.5}$$

Let us now show that under the conditions above x_0 is actually a p-Lebesgue point for w_{x_0}, that is

$$\lim_{r\downarrow 0} \fint_{C_r(x_0)} |w_{x_0}(x)|^p dx = 0.$$

Indeed, we have

$$\begin{aligned}\limsup_{r\downarrow 0} &\fint_{C_r(x_0)} |w_{x_0}(x)|^p \\ &\le 2^{p-1} \limsup_{r\downarrow 0} \fint_{C_r(x_0)} \Big(\mathcal{G}^p_{\mathcal{M}}(\nu(x),\nu(x_0)) + \mathcal{G}^p_{\mathcal{M}}(\nu(x_0), T_{x_0}\nu(x))\Big) = 0,\end{aligned}$$

since x_0 is a p-Lebesgue point of ν, and $T_{x_0}\nu$ is a Lipschitz map.

Then, note that $|Dw_{x_0}(x)| \le C\,|D\nu|(x) + C\,|D\nu|(x_0)$ for $\mathcal{L}^m$ a.e. $x \in \mathcal{B}$. By applying (2.3) to any Lebesgue point of w_{x_0} in $C_r(x_0)\setminus\mathcal{B}_{\lambda_i}$, we get

$$\begin{aligned}\limsup_{r\downarrow 0} &\fint_{C_r(x_0)\setminus\mathcal{B}_{\lambda_i}} \frac{w^p_{x_0}(x)}{|x-x_0|^p} dx \\ &\le C \limsup_{r\downarrow 0} \fint_{C_r(x_0)} \left(|m(|Dw_{x_0}|)(x)|^p + |m(|Dw_{x_0}|)(x_0)|^p\right) \chi_{\mathcal{B}\setminus\mathcal{B}_{\lambda_i}}(x)\, dx \\ &\le C \limsup_{r\downarrow 0} \fint_{C_r(x_0)} \left(m(|D\nu|)^p(x) + |D\nu|(x_0)^p\right) \chi_{\mathcal{B}\setminus\mathcal{B}_{\lambda_i}}(x)\, dx = 0.\end{aligned}$$ □

For $\nu \in W^{1,p}(\mathcal{B}, \mathcal{A}_Q(\mathcal{M}))$ and $\mathcal{L}^m$ a.e. $x \in \mathcal{B}$, we set

$$\|d\nu\|^p_{g(\nu(x))} := \sum_{i=1}^{Q} |(d\nu_i)_x|^p_{g(\nu_i(x))} \tag{2.6}$$

and

$$\|d\nu\|^p_p := \int_{\mathcal{B}} \|d\nu\|^p_{g(\nu(x))}\, dx < +\infty \tag{2.7}$$

3. Quasiconvexity and lower semicontinuity: the rigid case

Let $\mathcal{B} \subset \mathbb{R}^m$ be a bounded open set. By following [11], we say that a measurable map $e_M : \mathcal{B} \times (\mathrm{Hom}(\mathbb{R}^m, T\mathcal{M}))^Q \to [0, +\infty)$ is a *Q-integrand* if, for every permutation π of $\{1, \ldots, Q\}$, we get

$$e_M\big(x, \nu_1, \ldots, \nu_Q, N_1, \ldots, N_Q\big) = e_M\big(x, \nu_{\pi(1)}, \ldots, \nu_{\pi(Q)}, N_{\pi(1)}, \ldots, N_{\pi(Q)}\big), \tag{3.1}$$

where $(\nu_i, N_i) \in \mathrm{Hom}(\mathbb{R}^m, T\mathcal{M})$ for each i.

Given any Sobolev Q-valued function ν, the expression

$$e_M(x, \nu(x), d\nu_x) = e_M(x, \nu_1(x), \ldots, \nu_Q(x), (d\nu_1)_x, \ldots, (d\nu_Q)_x)$$

is well defined almost everywhere in $\mathcal{B}$. We choose e_M as the integrand of microscopic energy of a rigid body with microstructure that we call *active* imagining that it may have changes in the energy landscape, induced by external agencies, such as electric fields. We write $\mathcal{E}(\nu)$ for such a microscopic energy which is then defined by

$$\mathcal{E}(\nu) = \int_{\mathcal{B}} e_M\big(x, \nu(x), (d\nu)_x\big) dx. \tag{3.2}$$

For $\mathcal{E}(\nu)$ we assume quasiconvexity as constitutive prescription, a choice that we should take with care in case the body under scrutiny would undergo finite strain, because quasi-convexity with respect to the deformation gradient would not allow us to assure the orientation preserving nature of the macroscopic transplacements minimizing, together with the microstructural descriptor fields, the energy pertaining to that case.

An extension of the notion of quasi-convexity to the case of multiple valued functions with values on a manifold can be proposed (see [11] for the flat case).

Definition 3.1 (Quasi-convexity)**.** Let $e_M : (\mathrm{Hom}(\mathbb{R}^m, T\mathcal{M}))^Q \to \mathbb{R}$ be a locally bounded Q-integrand. We say that e_M is *quasi-convex* if for every

(i) affine Q-valued function $\nu : \mathbb{R}^m \to \mathcal{A}_Q(\mathcal{M})$ given by $\nu(x) = \sum_{j=1}^J q_j \left[\!\left[\exp_{\bar\nu_j}(L_j x) \right]\!\right]$, with $\bar\nu_i \neq \bar\nu_j \in \mathcal{M}$ for $i \neq j$,

(ii) and collection of maps $w^j \in W^{1,\infty}(C_1, \mathcal{A}_{q_j}(T_{\bar\nu_j}\mathcal{M}))$ with $w^j|_{\partial C_1} = q_j \left[\!\left[L_j|_{\partial C_1} \right]\!\right]$,

the inequality

$$e_M\big(\nu(0), (d\nu)_0\big) \leq \int_{C_1} e_M\big(\underbrace{\bar\nu_1, \ldots, \bar\nu_1}_{q_1}, \ldots, \underbrace{\bar\nu_J, \ldots, \bar\nu_J}_{q_J}, dw^1_x, \ldots, dw^J_x\big) dx \tag{3.3}$$

holds, where, as usual, we identify the tangent space $T_{w^j_i(x)}(T_{\bar\nu_j}\mathcal{M})$ with $T_{\bar\nu_j}\mathcal{M}$ itself.

This definition generalizes the notion of quasi-convexity introduced by Morrey, which characterizes sequentially lower semicontinuous functionals in Sobolev spaces. The main result here is to show that such a standard property holds also for $\mathcal{A}_Q(\mathcal{M})$-valued maps.

Theorem 3.2. *Let $p \in [1,\infty[$ and $e_M : \mathcal{B} \times (\mathrm{Hom}(\mathbb{R}^m, T\mathcal{M}))^Q \to \mathbb{R}$ be a continuous Q-integrand. If $e_M(x,\cdot,\cdot)$ is quasiconvex for every $x \in \mathcal{B}$ and*

$$0 \le e_M(x,\nu,N) \le C\left(1 + \mathcal{G}^q_{\mathcal{M}}(\nu,\nu_0) + \sum_{i=1}^{Q} |N_i|^p_{g(\nu_i)}\right) \quad \text{for some constant } C>0,$$

where $q = 0$ if $p > m$, $q = p^*$ if $p < m$ and $q \ge 1$ is any exponent if $p = m$, then the functional $\mathcal{E}$ in (3.2) is weakly lower semicontinuous in $W^{1,p}(\mathcal{B},\mathcal{A}_Q(\mathcal{M}))$. Conversely, if $\mathcal{E}$ is weakly$-*$ lower semicontinuous in $W^{1,\infty}(\mathcal{B},\mathcal{A}_Q(\mathcal{M}))$, then $e_M(x,\cdot,\cdot)$ is quasiconvex for every $x \in \mathcal{B}$.

The proof of Theorem 3.2 is provided below. For it, we follow the intrinsic approach developed in [11] and [15]. We avoid any embedding of the manifold $\mathcal{M}$ into a linear space for the reasons underlined in the introduction.

3.1. **Necessity of quasiconvexity.** Here we show that if $\mathcal{E}$ is weakly$-*$ lower semicontinuous in $W^{1,\infty}(\mathcal{B},\mathcal{A}_Q(\mathcal{M}))$, then $e_M(x,\cdot,\cdot)$ is quasiconvex for every $x \in \mathcal{B}$.

To this aim, let ν and w^j be as in (i) and (ii) of Definition 3.1. We consider the functions $z^j : C_1 \to \mathcal{A}_{q_j}(T_{\bar{\nu}_j}\mathcal{M})$ given by

$$z^j(x) := \sum_{i=1}^{q_j} \left[\!\left[w^j_i(x) - L_j\, x \right]\!\right].$$

By (ii) we get $z^j|_{\partial C_1} \equiv q_j\,[\![0]\!]$. Therefore, there is no loss of generality in assuming that z^j is defined in the entire space $\mathbb{R}^m$ by a C_1-periodic extension.

Let now $x_0 \in \mathcal{B}$ be fixed. For every $r \in (0, \mathrm{dist}(x_0,\partial\mathcal{B}))$ and $k \in \mathbb{N}$, we consider the functions $u_{k,r} : \mathcal{B} \to \mathcal{A}_Q(\mathcal{M})$ given by

$$u_{k,r}(x) := \begin{cases} \displaystyle\sum_{j=1}^{J}\sum_{i=1}^{q_j} \left[\!\left[\exp_{\bar{\nu}_j}\left(L_j\,x + \frac{r}{k}\, z^j_i\left(\frac{k\,(x-x_0)}{r}\right)\right)\right]\!\right] & \text{for } x \in C_r(x_0), \\ \nu(x) & \text{for } x \in \mathcal{B}\setminus C_r(x_0). \end{cases}$$

The following two conclusions hold: for every fixed $r \in (0, \mathrm{dist}(x_0,\partial\mathcal{B}))$,

$$u_{k,r} \to \nu \quad \text{in } L^\infty(\mathcal{B}) \quad \text{as} \quad k \to +\infty,$$

$$\||Du_{k,r}|\|_{L^\infty(\mathcal{B})} \le C \sum_{j=1}^{J} \big(|L_j| + r\,C\,\|Dz^j\|_{L^\infty(\mathcal{B})}\big).$$

The results imply that $u_{k,r} \overset{*}{\rightharpoonup} \nu$ in $W^{1,\infty}(\mathcal{B},\mathcal{A}_Q(\mathcal{M}))$. Therefore, by assumption of semicontinuity we infer that

$$\mathcal{E}(\nu, C_r(x_0)) \le \liminf_{k\to+\infty} \mathcal{E}(u_{k,r}, C_r(x_0)). \tag{3.4}$$

We pass now to estimate the two sides of (3.4) separately. For what concerns the left hand side, it is simple to see that, by the continuity of the integrand, we have

$$\lim_{r\to 0} r^{-m}\,\mathcal{E}(\nu, C_r(x_0)) = e_M(\nu(x_0),(d\nu)_{x_0}). \tag{3.5}$$

The right hand side of (3.4) can be estimated by using a change of coordinates and the chain rule for multiple valued functions as proven in [10, Proposition 1.12]. So, we get

$$\begin{aligned} r^{-m}\,\mathcal{E}(u_{k,r}, C_r(x_0)) &= \int_{C_1} e_M\Big(x_0 + r\,y, \exp_{\bar\nu_j}\Big(r\,L_j\,y + \frac{r}{k} z_i^j(k\,y)\Big), \ldots \\ &\qquad \ldots, (d\exp_{\bar\nu_j})_{r\,L_j\,y+\frac{r}{k}z_i^j(k\,y)} \circ (L_j + Dz_i^j(ky))\Big)\,dy \\ &= \int_{C_1} e_M\big(\underbrace{\bar\nu_1,\ldots,\bar\nu_1}_{q_1},\ldots,\underbrace{\bar\nu_J,\ldots,\bar\nu_J}_{q_J},(dw^1)_y,\ldots,(dw^J)_y\big)\,dy + \omega(r), \end{aligned} \tag{3.6}$$

where we have used the periodicity of z^j, and we have noticed that $\omega(r)$ is a modulus of continuity (uniform in k), which is clearly infinitesimal as $r \to 0$, because all the functions involved are continuous and $(d\exp_{\bar\nu_j})_0 = \mathrm{Id}$.

By (3.5) and (3.6), taking the limit as $r \to 0$ in (3.4), we get (3.3), thus showing the quasiconvexity of the integrand.

3.2. **Sufficiency of quasiconvexity.** Here, we assume quasiconvexity of $e_M(x,\cdot,\cdot)$ for every $x \in \mathcal{B}$ and prove that the functional $\mathcal{E}$ in (3.2) is then weakly lower semicontinuous on $W^{1,p}(\mathcal{B},\mathcal{A}_Q(\mathcal{M}))$. We want to prove that, given $\nu_k \rightharpoonup \nu$,

$$\mathcal{E}(\nu) \le \liminf_{k\to\infty} \mathcal{E}(\nu_k).$$

Without loss of generality (up to extracting a subsequence which will *never* be renamed in the sequel) we assume that the inferior limit above is in fact a limit. Moreover, in view of the growth hypothesis on e_M, we can assume that there exists a finite positive measure μ on $\mathcal{B}$ such that

$$e_M(x,\nu_k(x),(d\nu_k)_x)\,\mathcal{L}^m \llcorner \mathcal{B} \overset{*}{\rightharpoonup} \mu.$$

Hence, it suffices to show that

$$e_M(x,\nu(x),d\nu_x) \le \frac{d\mu}{d\mathcal{L}^m}(x) \;\text{ for }\; \mathcal{L}^m\text{-a.e. } \; x \in \mathcal{B}. \tag{3.7}$$

According to Lemma A.1 in [15], without relabeling the subsequence, there exist sets $\mathcal{B}_l$, $l \in \mathbb{N}$, such that

(i) $\mathcal{B}_l \subseteq \mathcal{B}_{l+1}$ for every $l \in \mathbb{N}$,
(ii) $\mathcal{L}^m(\mathcal{B}\setminus\mathcal{B}_l) = o(1)$ as $l \uparrow \infty$,
(iii) $((\mathcal{G}^p_{\mathcal{M}}(\nu_0,\nu_k(x)) + \|d(\nu_k)_x\|^p_{g(\nu_k(x))})\chi_{\mathcal{B}_l})_{k\in\mathbb{N}}$ is equi-integrable uniformly in $l \in \mathbb{N}$, i.e. there exists a superlinear function φ such that, for all $l \in \mathbb{N}$,

$$\sup_{k\in\mathbb{N}} \int_{\mathcal{B}_l} \varphi\Big(\mathcal{G}^p_{\mathcal{M}}(\nu_0,\nu_k(x)) + \|d(\nu_k)_x\|^p_{g(\nu_k(x))}\Big)\,dx < +\infty,$$

for some $\nu_0 \in \mathcal{A}_Q(\mathcal{M})$.

Therefore, for every $l \in \mathbb{N}$, up to subsequences, we may assume the existence of a positive measure μ_l on $\mathcal{B}$ such that

$$\varphi\left(\mathcal{G}^p_{\mathcal{M}}(\nu_0, \nu_k(x)) + \|d(\nu_k)_x\|^p_{g(\nu_k(x))}\right) \chi_{\mathcal{B}_l}(x)\, \mathcal{L}^m \llcorner \mathcal{B} \overset{*}{\rightharpoonup} \mu_l.$$

Finally, from the equi-boundedness $\sup_k \|d\nu_k\|_p < +\infty$, we assume that there exists a measure $\tilde{\mu}$ such that

$$\|d(\nu_k)_x\|_{g(\nu_k(x))}\, \mathcal{L}^m \llcorner \mathcal{B} \overset{*}{\rightharpoonup} \tilde{\mu}.$$

We are now able to specify the points x for which we prove inequality (3.7), that is the subset $\mathcal{B}'_l$ of points $x \in \mathcal{B}_l$ such that

(a) the function ν is L^p-differentiable in x according to (2.4);
(b) $\mathcal{B}_l$ has density one in x;
(c) $\dfrac{d\mu}{d\mathcal{L}^m}(x) + \dfrac{d\mu_l}{d\mathcal{L}^m}(x) + \dfrac{d\tilde{\mu}}{d\mathcal{L}^m}(x) < +\infty$.

Clearly $\mathcal{L}^m(\mathcal{B}_l \setminus \mathcal{B}'_l) = 0$, so that $\mathcal{B}' := \cup_l \mathcal{B}'_l$ is a set of full measure in $\mathcal{B}$. We shall prove that inequality (3.7) is satisfied by all points belonging to $\mathcal{B}'$.

To this aim we modify the sequence $(\nu_k)_{k\in\mathbb{N}}$ in two steps.

3.2.1. *Truncation.* We fix $l \in \mathbb{N}$ and a point $x_0 \in \mathcal{B}'_l$, and choose radii $\rho_k \to 0$ such that

$$\mu(\partial C_{\rho_k}(x_0)) = \mu_l(\partial C_{\rho_k}(x_0)) = \tilde{\mu}(\partial C_{\rho_k}(x_0)) = 0.$$

By item (c), we can extract a further subsequence (as usual not renamed) such that

$$\fint_{C_{\rho_k}(x_0)} \mathcal{G}^p_{\mathcal{M}}(\nu_k(x), \nu(x))\, dx = o(\rho_k^p), \tag{3.8}$$

$$\lim_{k\uparrow\infty} \fint_{C_{\rho_k}(x_0)} e_M(x, \nu_k(x), d(\nu_k)_x)\, dx = \frac{d\mu}{d\mathcal{L}^m}(x_0) < +\infty. \tag{3.9}$$

$$\sup_k \rho_k^{-m} \int_{C_{\rho_k}(x_0)\cap\mathcal{B}_l} \varphi\left(\mathcal{G}^p_{\mathcal{M}}(\nu_0, \nu_k(x)) + \|d(\nu_k)_x\|^p_{g(\nu_k(x))}\right) dx < +\infty, \tag{3.10}$$

$$\sup_k \fint_{C_{\rho_k}(x_0)} \|d(\nu_k)_x\|^p_{g(\nu_k(x))}\, dx < +\infty. \tag{3.11}$$

In particular, from item (a) and (3.8) we get

$$\fint_{C_{\rho_k}(x_0)} \mathcal{G}^p_{\mathcal{M}}\left(\nu_k(x), T_{x_0}\nu(x)\right)\, dx = o(\rho_k^p). \tag{3.12}$$

Claim 1. *Let* $\nu(x_0) = \sum_{j=1}^J q_j [\![a_j]\!]$, *with* $a_i \neq a_j$ *for* $i \neq j$. *Then, there exist* $w_k \in W^{1,\infty}(C_{\rho_k}(x_0), \mathcal{A}_Q(\mathcal{M}))$ *such that* $w_k = \sum_{j=1}^J [\![w_k^j]\!]$ *with* $w_k^j \in W^{1,\infty}(C_{\rho_k}(x_0), \mathcal{A}_{q_j}(\mathcal{M}))$,

$$\|\mathcal{G}_{\mathcal{M}}(w_k, \nu(x_0))\|_{L^\infty(C_{\rho_k}(x_0))} = o(1), \tag{3.13}$$

$$\mathcal{G}^2_{\mathcal{M}}(w_k(x), \nu(x_0)) = \sum_{j=1}^J \mathcal{G}^2_{\mathcal{M}}(w_k^j(x), q_j [\![a_j]\!]) \quad \text{for every } x \in C_{\rho_k}(x_0), \tag{3.14}$$

$$\fint_{C_{\rho_k}(x_0)} \mathcal{G}^p_{\mathcal{M}}(w_k, T_{x_0}\nu)\, dx = o(\rho_k^p), \tag{3.15}$$

$$\sup_k \fint_{C_{\rho_k}(x_0)} \|d(w_k)_x\|^p_{g(w_k(x))}\, dx < +\infty, \tag{3.16}$$

$$\lim_{k\uparrow\infty} \rho_k^{-m} \int_{C_{\rho_k}(x_0)\cap\mathcal{B}_l} e_M\big(x, w_k(x), (dw_k)_x\big)\, dx \leq \frac{d\mu}{d\mathcal{L}^m}(x_0). \tag{3.17}$$

Proof. Let $r_k \downarrow 0$ be radii such that $\rho_k/r_k \to 0$ and consider the retraction maps Θ_{r_k} constructed in Lemma A.1. We show that $w_k := \Theta_{r_k}(\nu_k)$ satisfy the conclusions of the claim. Set

$$H_k := \big\{x \in C_{\rho_k}(x_0) : \nu_k(x) \neq w_k(x)\big\}.$$

Note that $H_k = \big\{x \in C_{\rho_k}(x_0) : \mathcal{G}_{\mathcal{M}}(\nu_k(x), \nu(x_0)) > r_k\big\}$. So, we deduce that

$$\begin{aligned} r_k^p\, \mathcal{L}^m(H_k) &\leq \int_{H_k} \mathcal{G}^p_{\mathcal{M}}(\nu_k(x), \nu(x_0))\, dx \\ &\leq C \int_{C_{\rho_k}(x_0)} \mathcal{G}^p_{\mathcal{M}}\left(\nu_k(x), T_{x_0}\nu(x)\right)\, dx + C \int_{H_k} \mathcal{G}^p_{\mathcal{M}}\left(T_{x_0}\nu(x), \nu(x_0)\right)\, dx \\ &\overset{(3.12)}{\leq} o(\rho_k^{p+m}) + C\, \rho_k^p\, \mathcal{L}^m(H_k). \end{aligned} \tag{3.18}$$

The latter estimate implies that

$$\rho_k^{-m}\, \mathcal{L}^m(H_k) \leq \frac{o(\rho_k^p)}{r_k^p\left(1 - C\, \rho_k^p r_k^{-p}\right)}; \tag{3.19}$$

hence, by recalling the choice of r_k, we infer that

$$\mathcal{L}^m(H_k) = o(\rho_k^m). \tag{3.20}$$

In turn, the previous inequality inserted in (3.18) implies also that

$$\int_{H_k} \mathcal{G}^p_{\mathcal{M}}(\nu_k(x), \nu(x_0))\, dx = o(\rho_k^{p+m}). \tag{3.21}$$

Therefore, the Lipschitz continuity of Θ_{r_k}, the locality of the approximate differentials and the growth hypothesis on e_M, together with (3.10), (3.20) and Lemma A.2 in [15] imply that

$$\rho_k^{-m}\int_{C_{\rho_k}(x_0)\cap\mathcal{B}_l}\big(e_M(x,\nu_k(x),d(\nu_k)_x)-e_M(x,w_k(x),d(w_k)_x)\big)dx\le$$
$$\le C\,\rho_k^{-m}\int_{H_k\cap\mathcal{B}_l}\Big(\mathcal{G}^p_{\mathcal{M}}(\nu(x_0),\nu_k)+\|d(\nu_k)_x\|^p_{g(\nu_k(x))}\Big)\,dx=o(1),$$

from which (3.17) follows. Moreover, by definition of w_k and H_k, we find

$$\begin{aligned}\fint_{C_{\rho_k}(x_0)}\mathcal{G}^p_{\mathcal{M}}(w_k(x),\nu(x))\,dx &\overset{(3.8)}{\le} C\fint_{C_{\rho_k}(x_0)}\mathcal{G}^p_{\mathcal{M}}(w_k(x),\nu_k(x))\,dx+o(\rho_k^p)\\ &\le C\,\rho_k^{-m}\int_{H_k}\mathcal{G}^p_{\mathcal{M}}(w_k(x),\nu(x_0))\,dx\\ &\quad+C\,\rho_k^{-m}\int_{H_k}\mathcal{G}^p_{\mathcal{M}}(\nu_k(x),\nu(x_0))\,dx+o(\rho_k^p)\\ &\le C\,r_k^p\,\rho_k^{-m}\,\mathcal{L}^m(H_k)+C\,\rho_k^{-m}\int_{H_k}\mathcal{G}^p_{\mathcal{M}}(\nu_k(x),\nu(x_0))\,dx+o(\rho_k^p)\\ &\overset{(3.19),\,(3.21)}{\le} o(\rho_k^p).\end{aligned}$$

Then, (3.15) follows from (3.12). Finally, (3.13), (3.14) and (3.16) follow easily since Θ_{r_k} takes values in $B_{r_k}(\nu(x_0))$ and it is Lipschitz continuous. □

3.2.2. *Reduction to the flat case.* Since the w_k^j's take values into $B_{r_k}(a_j)$, a set contained in a normal coordinate chart, we are able to reduce ourselves to the case of maps with values in a fixed tangent space, namely,

$$v_k^j:=\sum_{i=1}^{q_j}\left[\!\left[\exp_{a_j}^{-1}\circ(w_k^j)_i\right]\!\right]:C_{\rho_k}(x_0)\to\mathcal{A}_{q_j}(T_{a_j}\mathcal{M}).$$

In what follows we shall regard the map v_k as taking values in $\mathcal{A}_Q(\mathbb{R}^n)$ endowed with the metric $\mathcal{G}$. In particular, $\mathcal{G}$ is equivalent to the metric induced by $g(\nu(x_0))$ on $\Pi_j^J\mathcal{A}_{q_j}(T_{a_j})$.

Let us first notice that (3.15) and item (a) in the definition of $\mathcal{B}_l'$ imply the estimate

$$\begin{aligned}\sum_{j=1}^J\fint_{C_{\rho_k}(x_0)}\mathcal{G}^p(v_k^j,\exp_{a_j}^{-1}(T_{x_0}\nu^j))\,dx&\le C\fint_{C_{\rho_k}(x_0)}\mathcal{G}^p_{\mathcal{M}}(w_k,T_{x_0}\nu)\,dx\\ &\le C\fint_{C_{\rho_k}(x_0)}\big(\mathcal{G}^p_{\mathcal{M}}(w_k,\nu)+\mathcal{G}^p_{\mathcal{M}}(\nu,T_{x_0}\nu)\big)dx=o(\rho_k^p).\end{aligned}\tag{3.22}$$

Next, we show that the continuity of the integrand e_M leads to

$$\lim_{k\uparrow\infty}\rho_k^{-m}\left|\int_{C_{\rho_k}(x_0)\cap\mathcal{B}_l}\Big(e_M(x,w_k(x),d(w_k)_x)-e_M(x_0,\nu(x_0),d(v_k)_x)\Big)\,dx\right|=0,\tag{3.23}$$

where, for every $x \in C_{\rho_k}(x_0)$ we identify, as usual, the tangent space to $T_{a_j}\mathcal{M}$ at $v_k(x)$ with $T_{a_j}\mathcal{M}$ itself.

To this aim, we notice that, for every $t > 0$, the integral on the left hand side of (3.23) is dominated by the sum of the two terms in the sequel:

$$I_t^k := C\,\rho_k^{-m} \int_{\{x\in C_{\rho_k}(x_0)\cap\mathcal{B}_l\colon \|d(w_k)_x\|_{g(w_k(x))}\geq t\}} \big(1 + \|d(w_k)_x\|^p_{g(w_k(x))}\big)\,dx,$$

and

$$J_t^k :=\ \rho_k^{-m} \int_{\{x\in C_{\rho_k}(x_0)\cap\mathcal{B}_l\colon \|d(w_k)_x\|_{g(w_k(x))}<t\}} \big|e_M(x, w_k(x), d(w_k)_x) - e_M(x_0, \nu(x_0), d(v_k)_x)\big|\,dx.$$

Moreover, by Lemma A.2 in [15] and the equi-integrability of dw_k in $\mathcal{B}_l$, which easily follows from (3.10) and the definition of w_k itself, we have that

$$\lim_{t\uparrow\infty}\sup_k I_t^k = 0.$$

Hence, to derive (3.23), it is enough to show that for every $t > 0$ the term J_t^k is infinitesimal as $k \uparrow \infty$.

For this result, the uniform continuity of the integrand e_M on compact sets provides us with a modulus of continuity $\omega_{f,t}$ such that

$$J_t^k \leq \omega_{f,t}\left(\rho_k + \sum_{j=1}^{J}\sum_{i=1}^{q_j} \|D\big((a_j, d(v_k^j)_i), ((w_k^j)_i, d(w_k^j)_i)\big)\|_{L^\infty(C_{\rho_k}(x_0))}\right),$$

where the distance D appearing on the right hand side is the one introduced in (1.2) for $\mathrm{Hom}(\mathbb{R}^m, T\mathcal{M})$. To clarify the previous inequality, we remark that for $\nu(x_0)$ and $w_k(x)$, $x \in C_{\rho_k}(x_0)$, we have chosen the order giving the L^∞ distance between them. Therefore, if we show that for all $1 \leq j \leq J$ and $1 \leq i \leq q_j$, we have

$$\|D\big((a_j, d(v_k^j)_i), ((w_k^j)_i, d(w_k^j)_i)\big)\|_{L^\infty(C_{\rho_k}(x_0))} \leq C\, r_k, \tag{3.24}$$

so we get (3.23). The proof of (3.24) follows easily from the definition of the distance D. We refer to [15, Section 2.2.2] for all the details.

3.2.3. *Conclusion of the proof.* Since the functions v_k have full multiplicity at x_0, we can blow-up them. Let then $z_k := \sum_{j=1}^J [\![z_k^j]\!]$, the maps $z_k^j \in W^{1,\infty}(C_1, \mathcal{A}_{q_j})$ defined by

$$z_k^j(x) := \tau_{-a_j}\big(\rho_k^{-1}\,\tau_{a_j}(v_k^j)(x_0 + \rho_k\cdot)\big)(x),$$

τ_{-a_j} being the usual translation in $\mathcal{A}_{q_j}$, i.e.

$$\tau_{-a_j}\left(\sum_{i=1}^{q_j} [\![p_i]\!]\right) := \sum_{i=1}^{q_j} [\![p_i - a_j]\!]$$

A simple change of variables together with estimate (3.22) gives

$$z_k^j \to \exp_{a_j}^{-1}(T_{x_0}\nu^j) \quad \text{in} \quad L^p(C_1, \mathcal{A}_{q_j}), \tag{3.25}$$

while together with (3.16) yields

$$\sup_k \int_{C_1} \|d(z_k)_x\|^p_{g(\nu(x_0))}\, dx < +\infty. \tag{3.26}$$

In addition, formulas (3.17) and (3.23) give

$$\lim_{k\uparrow\infty} \int_{C_1\cap\rho_k^{-1}(\mathcal{B}_l-x_0)} e_M\big(x_0,\nu(x_0),d(z_k)_x\big)\, dx \le \frac{d\mu}{d\mathcal{L}^m}(x_0). \tag{3.27}$$

By taking into account (3.26), Lemma 1.5 in [11] provides a subsequence $(\zeta_k)_{k\in\mathbb{N}} \subset W^{1,p}(\mathcal{B},\mathcal{A}_Q)$ such that

(i) $\mathcal{L}^m(\{z_{j_k} \neq \zeta_k\}) = o(1)$ and $\zeta_k^j \rightharpoonup T_{x_0}\nu^j$ in $W^{1,p}(\mathcal{B},\mathcal{A}_{q_j})$;
(ii) $(\|(d\zeta_k)_x\|^p)_{k\in\mathbb{N}}$ is equi-integrable;
(iii) if $p \in [1,m)$, $(\mathcal{G}^{p^*}(\zeta_k,\nu_0))_{k\in\mathbb{N}}$ is equi-integrable and, if $p = m$, $(\mathcal{G}^q(\zeta_k,\nu_0))_{k\in\mathbb{N}}$ is equi-integrable for any $q \ge 1$.

Eventually, since x_0 is a point of density of $\mathcal{B}_l$ we have that $\mathcal{L}^m(C_1 \setminus \rho_k^{-1}(\mathcal{B}_l - x_0)) = o(1)$ as $k \uparrow \infty$. Thus, by taking into account the equi-integrability of $(\zeta_k)_{k\in\mathbb{N}}$, Theorem 0.2 in [11] implies that

$$\begin{aligned}
\liminf_{k\uparrow\infty} & \int_{C_1\cap\rho_k^{-1}(\mathcal{B}_l-x_0)} e_M(x_0,\nu(x_0),d(z_k)_x)\, dx \\
&= \liminf_{k\uparrow\infty} \int_{C_1\cap\rho_k^{-1}(\mathcal{B}_l-x_0)} e_M(x_0,\nu(x_0),d(\zeta_k)_x)\, dx \\
&= \liminf_{k\uparrow\infty} \int_{C_1} e_M(x_0,\nu(x_0),d(\zeta_k)_x) dx \ge e_M(x_0,\nu(x_0),d\nu_{x_0}).
\end{aligned}$$

This inequality, together with (3.27), concludes the proof of (3.7).

4. An elastic case: existence of ground states

4.1. **Kinematics and representation of the actions: geometrical issues.** Here we apply the results proved in Section 3 to the case of elastic materials for which microstructural events are coupled with macroscopic strain – materials that are then called complex just to remind these features.

As anticipated in introducing our analyses, we restrict our attention to a class of energies of Ginzburg-Landau type with respect to the spatial derivative of the microstructural descriptor ν, which is also here, as above, a manifold $Q-$valued map.

Since we want to include macroscopic strain, we have to account for the **transplacement**

$$x \longmapsto y := u(x) \in \tilde{\mathbb{R}}^3, \quad x \in \mathcal{B},$$

which maps the reference shape $\mathcal{B}$ in $\mathbb{R}^3$ onto the current ones in $\tilde{\mathbb{R}}^3$.

We maintain distinguished the space hosting the reference place and the one where the current macroscopic shapes are evaluated – the distinction being determined by an isomorphism, the identification, in fact – for questions connected with the definition of observers, their changes and their use in deriving from invariance requirements the balances of standard, microstructural, and configurational actions, a question not tackled here (the reader can find details in [27]).

In standard treatments, the transplacement map is assumed to be (i) one-to-one, (ii) differentiable, and (iii) orientation preserving. We write F for the spatial derivative of u at x, namely $Du(x)$ and call it **deformation gradient**, by following the traditional terminology, although there is difference between Du and the gradient ∇u given by the metric in the reference place – namely the material metric g in $\mathcal{B}$ – and the relation $\nabla u(x) = Du(x)\, g^{-1}$ holds. By definition, then, F is a fellow of $\mathrm{Hom}\,(T_x\mathcal{B}, T_y u(\mathcal{B}))$. F itself, the relevant cofactor $\mathrm{cof}F$, and the determinant $\det F$, are the essential ingredients determining strain measures connected respectively with the stretch of lines, surfaces, and variations in volumes – the remark is standard, indeed, and the details can be found in basic treatises on traditional continuum mechanics. The requirement that u be orientation preserving is a constraint imposing $\det F > 0$. The components of F, $\mathrm{cof}F$, and $\det F$ can be put together in a unique geometric object: a fully contravariant third-rank tensor, the so-called 3−vector. The construction of it starts from the selection at $x \in \mathcal{B}$ of three linearly independent vectors a_1, a_2, a_3. We then realize that[4] maps of the type

$$\begin{aligned}
a_1 \wedge a_2 \wedge a_3 &\longmapsto Fa_1 \wedge a_2 \wedge a_3, \\
a_1 \wedge a_2 \wedge a_3 &\longmapsto Fa_1 \wedge Fa_2 \wedge a_3, \\
a_1 \wedge a_2 \wedge a_3 &\longmapsto Fa_1 \wedge Fa_2 \wedge Fa_3,
\end{aligned}$$

have as values linearly independent skew-symmetric tensors, 3−vectors indeed, elements of a space commonly indicated by $\Lambda_3(\mathbb{R}^3 \times \tilde{\mathbb{R}}^3)$. We can then define a 3−vector $M(F) \in \Lambda_3(\mathbb{R}^3 \times \tilde{\mathbb{R}}^3)$ by

$$\begin{aligned}
M(F) :&= a_1 \wedge a_2 \wedge a_3 + Fa_1 \wedge a_2 \wedge a_3 + a_1 \wedge Fa_2 \wedge a_3 + a_1 \wedge a_2 \wedge Fa_3 \\
&+ Fa_1 \wedge Fa_2 \wedge a_3 + Fa_1 \wedge a_2 \wedge Fa_3 + a_1 \wedge Fa_2 \wedge Fa_3 + Fa_1 \wedge Fa_2 \wedge Fa_3 \\
&= (a_1, Fa_1) \wedge (a_2, Fa_2) \wedge (a_3, Fa_3) \in \Lambda_3(\mathbb{R}^3 \times \tilde{\mathbb{R}}^3).
\end{aligned}$$

$M(F)$ has 20 components, which are 1 and the entries of F, $\mathrm{cof}F$, and $\det F$, namely the entities determining, at the point where F is evaluated, the strain measures. The space of 3−vectors, namely $\Lambda_3(\mathbb{R}^3 \times \tilde{\mathbb{R}}^3)$, does not contain, in fact, just elements of the type $M(F)$. With (e_1, e_2, e_3) a basis in $\mathbb{R}^3$ and $(\tilde{e}_1, \tilde{e}_2, \tilde{e}_3)$ another basis in $\tilde{\mathbb{R}}^3$, we can write

[4]Given two linear spaces $\mathcal{L}_1$ and $\mathcal{L}_2$, by $\wedge$ we indicate the map $\wedge : \mathcal{L}_1 \times \mathcal{L}_2 \longrightarrow Skw\,(\mathcal{L}_2^*, \mathcal{L}_1)$, where the target space collects skew-symmetric tensors mapping covectors in $\mathcal{L}_2^*$ to vectors in $\mathcal{L}_1$.

every 3−vector $M \in \Lambda_3(\mathbb{R}^3 \times \tilde{\mathbb{R}}^3)$ in the form

$$M = \zeta e_1 \wedge e_2 \wedge e_3 + \sum_{i,J}^{3} (-1)^{J-1} F^{iJ} e_{\bar{J}} \wedge \tilde{e}_i + \sum_{i,J}^{3} (-1)^{i-1} A^{iJ} e_J \wedge \tilde{e}_{\bar{\imath}} + a\, \tilde{e}_1 \wedge \tilde{e}_2 \wedge \tilde{e}_3,$$

where $\bar{J}$ is the complementary multi-index to J with respect to $(1,2,3)$ and $\bar{\imath}$ has an analogous relation with i (for example, if $J = 1$, then $\bar{J} = (2,3)$ and $e_{\bar{J}} = e_2 \wedge e_3$, and the same holds for the index i and its pertinent $\bar{\imath}$). In previous expression, F and A are second-rank, skew-symmetric tensors while ζ and a are scalars.

With dx^1, dx^2, dx^3 and dy^1, dy^2, dy^3 the bases in the dual spaces $\mathbb{R}^{3*}$ and $\tilde{\mathbb{R}}^{3*}$, respectively, any element ω of the dual space of $\Lambda_3(\mathbb{R}^3 \times \tilde{\mathbb{R}}^3)$, commonly indicated by $\Lambda^3(\mathbb{R}^3 \times \tilde{\mathbb{R}}^3)$, can be expressed as the sum

$$\omega := \beta\, dx^1 \wedge dx^2 \wedge dx^3 + \sum_{i,j=1}^{3} (-1)^{J-1} r_{iJ} dx^{\bar{J}} \wedge dy^i + \sum_{i,j=1}^{3} (-1)^{i-1} s_{iJ} dx^J \wedge dy^{\bar{\imath}} + \varsigma\, dy^1 \wedge dy^2 \wedge dy^3,$$

where, as in the formula defining M, $\bar{J}$ is the complementary multi-index to J with respect to $(1,2,3)$ and there is the same relation between i and $\bar{\imath}$. For example, if $J = 1$ then $\bar{J} = (2,3)$ and we have $dx^{\bar{J}} = dx^2 \wedge dx^3$. Moreover, if $i = 3$ then $\bar{\imath} = (1,2)$ and we write $dy^{\bar{\imath}} = dy^1 \wedge dy^2$, and so on. β and ς are scalars. r and s are linear operators with covariant components.

A special ω can be constructed by using the stress tensor, in particular the first Piola-Kirchhoff stress. A few preliminary notions are, however, necessary to pave the way. Consider a point y in the actual place $u(\mathcal{B})$, reached by means of the transplacement u. Imagine a smooth surface, a plane indeed, crossing y, and oriented by a normal n_a, considered as a covector. The plane divides ideally $u(\mathcal{B})$ in two different pieces. These two parts of the body interact across the plane at the point y: in a sense the interaction is what maintains integer the material. An essential question is then the representation of such an interaction. The answer is suggested by the geometrical representation of the morphology of a body, for an action is something that furnished the power needed in changing the shape of the body when it is multiplied by the rates of the descriptors of the body morphology. When such rates are just velocity fields, namely vector fields in $\tilde{\mathbb{R}}^3$, to get the power developed in these rates it is then necessary that the interactions be covectors, say $\mathtt{t}$, each one depending in principle on the place y and the normal n_a, with the proviso that $\mathtt{t}(y, n_a) = -\mathtt{t}(y, -n_a)$, the action-reaction principle. Once we have derived integral balances from a requirement of invariance just of the external power of actions over a generic part of $\mathcal{B}$ with respect to classical changes in observers, i.e. changes induced by the Euclidean group $\tilde{\mathbb{R}}^3 \ltimes SO(3)$ on the atlas of the ambient space where current shapes are evaluated, we may use these integral balances to realize that there is a linear operator $\sigma \in \mathrm{Hom}(T^*_y u(\mathcal{B}), T^*_y u(\mathcal{B}))$ such that $\mathsf{t}(y, n_a) = \sigma(y) n_a$ – it is the standard Cauchy stress tensor, and the proof of the construction of it is matter of basic treatises. By pulling back, point by point, in the reference place the contravariant component of σ, we find a vector field with values mapping the normal to the plane crossing $\mathcal{B}$ at the pre-image x of the point y where we are considering σ, namely

$$x \longmapsto P := P(x) = (\det F)\, \sigma(x)\, F^{-*} \in \mathrm{Hom}(T^*_x \mathcal{B}, T^*_y u(\mathcal{B})).$$

With P we can act like with F in the sense that we can choose at $x \in \mathcal{B}$ three linearly independent covectors, say the covector basis dx_1, dx_2, and dx_3, and construct maps of the type

$$\begin{aligned} dx_1 \wedge dx_2 \wedge dx_3 &\longmapsto dx_1 \wedge dx_2 \wedge P dx_3, \\ dx_1 \wedge dx_2 \wedge dx_3 &\longmapsto dx_1 \wedge P dx_2 \wedge P dx_3, \\ dx_1 \wedge dx_2 \wedge dx_3 &\longmapsto P dx_1 \wedge P dx_2 \wedge P dx_3, \end{aligned}$$

which reach linearly independent values. With these premises, we can define

$$\begin{aligned} \omega(P) : &= dx^1 \wedge dx^2 \wedge dx^3 \\ &+ P dx^1 \wedge dx^2 \wedge dx^3 + dx^1 \wedge P dx^2 \wedge dx^3 + dx^1 \wedge dx^2 \wedge P dx^3 \\ &+ P dx^1 \wedge P dx^2 \wedge dx^3 + P dx^1 \wedge dx^2 \wedge P dx^3 + dx^1 \wedge P dx^2 \wedge P dx^3 \\ &+ P dx^1 \wedge P dx^2 \wedge P dx^3 \\ &= \left(dx^1, P dx^1\right) \wedge \left(dx^2, P dx^2\right) \wedge \left(dx^3, P dx^3\right) \in \Lambda^3(\mathbb{R}^3 \times \tilde{\mathbb{R}}^3), \end{aligned}$$

which is an element of the dual space of $\Lambda_3(\mathbb{R}^3 \times \tilde{\mathbb{R}}^3)$ including information on stresses along lines and surfaces (expressed by the components of the terms of the type $dx^1 \wedge P dx^2 \wedge P dx^3$), and the pressure associated with volume changes (the term $P dx^1 \wedge P dx^2 \wedge P dx^3$).

When $M = M(F)$ – in this case the first constant appearing in the definition of M, namely ζ, is equal to 1, and the linear contravariant operator A is equal to the fully contravariant expression of $\mathrm{cof} F$; the linear operator F itself should not necessarily be the spatial derivative of a transplacement map so that $M(F)$ could be not necessarily $M(Du)$ – the value that ω takes over M has then a clear physical significance. To see it, write $\Sigma_{1,+}$ for the subset of $\Lambda_3(\mathbb{R}^3 \times \tilde{\mathbb{R}}^3)$ containing Ms characterized by $\zeta = 1$, and $a > 0$, which is $\det F > 0$ when $M = M(F)$. Such a set includes, then, all $M(F)$s but not just them. For any piecewise-C^1 curve $\gamma : [-1, 1] \longrightarrow \Sigma_{1,+}$, consider a continuous map $t \longmapsto \omega(t) \in \Sigma^*_{1,+}$, a form indeed, with $t \in [-1, 1]$. Along γ we define a functional $\mathsf{w}(\omega, \gamma)$ by

$$\mathsf{w}(\omega, \gamma) := \int_0^1 \omega(\gamma(t)) \cdot \dot{\gamma}(t)\, dt.$$

It has the meaning of a *generalized internal* (*inner* if you want) *work*. In fact, since

$$\omega = \sum_{k=0}^{3} \omega_{(k)}, \quad \gamma(t) = \sum_{k=0}^{3} \gamma_{(k)}(t),$$

when $\gamma(t)$ is of the type $M(F)$, the product $\omega(\gamma(t)) \cdot \dot{\gamma}(t) = \sum_{k=0}^{3} \omega_{(k)}(\gamma(t)) \cdot \dot{\gamma}_{(k)}(t)$ involves

(1) the power density $\omega_{(1)} \cdot \dot{\gamma}_{(1)}$ produced along line strain,
(2) a power determined by volume changes, given by $\omega_{(3)} \cdot \dot{\gamma}_{(3)}(t) = (-p\,\dot{\gamma}_{(3)})(t)$, with $-p$ the scalar appearing in $\omega_{(3)} = -p\,dy^1 \wedge dy^2 \wedge dy^3$, and having the meaning of a pressure,
(3) terms given by the multiplication of the components of $\omega_{(2)}$ with the ones of $\dot{\gamma}_{(2)}(t)$, which represent the power over coordinate planes in a local frame, and for every local frame.

From the definition of $\mathsf{w}(\omega, \gamma)$ we infer that the value of ω over a given M is exactly a density of inner work when $M = M(F)$, generalized in the sense explained in the items above. The (generalized) overall inner work over the whole $\mathcal{B}$ is then the integral

$$\int_{\mathcal{B}} \omega \cdot M(F)\ dx.$$

In particular, when $M = M(Du)$, the functional

$$G_u(\omega) := \int_{\mathcal{B}} \omega \cdot M(Du)\, dx,$$

obtained by fixing $M(Du)$ and allowing ω to vary in $\Lambda^3(\mathbb{R}^3 \times \tilde{\mathbb{R}}^3)$ – which is a generalized virtual work obtained by testing on a given deformation virtual stresses – has a clear physical meaning in case u belongs to the Sobolev space $W^{1,1}$: it is the **current** associated with u. We can relate with it a notion of boundary by calling **boundary current** the functional ∂G_u, defined by

$$\partial G_u(\omega) := G_u(d\omega)$$

over the space of 2−forms compactly supported over $\mathcal{B}$. For $u \in W^{1,1}$, with $|M(Du)| \in L^1(\mathcal{B})$, in general the boundary current ∂G_u does not vanish. However, if u is smooth, $\partial G_u(\omega) = 0$ for all 2−forms as indicated above. In particular, it is possible to prove (see [19], and also [20]) that $\partial G_u = 0$ on 2−forms if $u \in W^{1,3}$. Once we select the transplacement u in spaces such as $W^{1,1}$ or $W^{1,2}$, the notion of current allows us to determine the subset including maps which can represent what we have intuitively in mind when we talk about elasticity, at least ideally: the possibility of straining essentially at will a body with the possibility of recovering the deformation without cavitation, nucleation of fractures, and dissipation. Such a set is the space of weak diffeomorphisms.

Definition 4.1 ([19])**.** $u \in W^{1,1}(\mathcal{B}, \tilde{\mathbb{R}}^3)$ is said a *weak diffeomorphism* (and we write in this case $u \in \mathrm{dif}^{1,1}(\mathcal{B}, \tilde{\mathbb{R}}^3)$) if

(1) $|M(Du(x))| \in L^1(\mathcal{B})$,
(2) $\partial G_u = 0$ on $\mathcal{D}_c^2(\mathcal{B}, \tilde{\mathbb{R}}^3)$,

(3) $\det Du(x) > 0$ a.e. $x \in \mathcal{B}_0$,
(4) for any $f \in C_c^\infty(\mathcal{B} \times \tilde{\mathbb{R}}^3)$

$$\int_{\mathcal{B}} f(x, u(x)) \det Du(x)\, dx \leq \int_{\mathbb{R}^3} \sup_{x\in\mathcal{B}} f(x,y) dy.$$

In previous definition, is the space of compactly supported $2-$forms. The last inequality is a condition allowing self-contact of the boundary without self-penetration of the matter. In particular, later we shall write $\mathrm{dif}^{r,1}(\mathcal{B}, \tilde{\mathbb{R}}^3)$ when $|M(Du(x))| \in L^r(\mathcal{B})$, with $|M(Du(x))|$ the square root of the product of $M(Du(x))$ by itself.

The analytical notion of weak diffeomorphisms has been introduced in [19], while the physical significance of the path leading to the definition has been interpreted in [18]. What emerges from the last work is that the choice of a function space where we decide to check whether some boundary value problems admit solutions has constitutive nature. To be fellows of some space, maps have to be endowed with certain properties; their presence allows us to describe certain phenomena excluding others.

4.2. **Energy.** The requirements that w be non-negative along closed paths and any path in $\Sigma_{1,+}$ be physically realizable are tantamount to impose that w vanishes along all closed paths, which is, in fact, equivalent to the existence of a function $\hat{e} : \Sigma_{1,+} \longrightarrow \mathbb{R}$ such that

$$\omega(M) = d\hat{e}(M), \quad \forall M \in \Sigma_{1,+}.$$

This is the way we can follow in non-linear elasticity (see [18]) when we do not consider the energy density as a function of F, $\mathrm{cof}F$, and $\det F$ taken as separated entries and put all together in a unique $3-$vector, as suggested first in [19]. The need of including $\mathrm{cof}F$ and $\det F$ together with F in the list of entries of the energy density is evidenced by the incompatibility of the physical requirement of objectivity for the energy – it is its invariance with respect to changes produced by the action of the Euclidean group in the atlas of the ambient space hosting all possible current shapes of the body – with its possible convex structure with respect to F. The incompatibility has suggested to adopt polyconvexity of the energy density with respect to F, namely to consider it as a convex function of F, $\mathrm{cof}F$, and $\det F$, or M, in an enlarged sense that allow one to determine the physically significant function space, the ones of weak diffeomorphisms.

However, here we are interested in the enlarged setting including the representation of material morphology at finer spatial scales by means of multi-value descriptor maps ν, along the path followed in previous sections. So, the energy we are interested in involves not only F as state variable but also ν and its spatial derivative. In case ν be single-valued, a general expression of the elastic energy and the existence of relevant minimizers – they are couples (u, ν), indeed – have been analyzed in [29]. Here we consider an energy with less general form. However, in contrast to [29], it includes multi-valued maps ν. The energy that we consider has the following decomposed structure:

$$\mathcal{E}(u,\nu) := \int_{\mathcal{B}} (e_E(x, u, Du, \nu) + e_M(x, \nu, (d\nu)_x))\, dx. \tag{4.1}$$

A number of constitutive assumptions apply and are listed below.

(H1) $e_E : \mathcal{B} \times \tilde{\mathbb{R}}^3 \times M^+_{3\times3} \times \mathcal{M}^Q \to [0, +\infty)$ is a Borel map such that

(a) for every $(x, u, F) \in \mathcal{B} \times \tilde{\mathbb{R}}^3 \times M^+_{3\times3}$ the function $e_E(x, u, F, \cdot)$ is invariant under the action of $\mathscr{P}_Q$, i.e. for every permutation π of $\{1, \dots, Q\}$, we have

$$e_E(x, u, F, \nu_1, \dots, \nu_Q) = e_E(x, u, F, \nu_{\pi(1)}, \dots, \nu_{\pi(Q)}); \tag{4.2}$$

(b) $e_E(x, u, \cdot, \nu)$ is a (standard) polyconvex integrand for any $(x, u, \nu) \in \mathcal{B} \times \tilde{\mathbb{R}}^3 \times \mathcal{A}_Q(\mathcal{M})$, i.e. there exists a Borel function $Pe : \mathcal{B} \times \tilde{\mathbb{R}}^3 \times \Lambda_3(\mathbb{R}^3 \times \hat{\mathbb{R}}^3) \times \mathcal{A}_Q(\mathcal{M}) \to [0, +\infty]$ such that

(i) $Pe(x, \cdot, \cdot, \cdot)$ is lower semicontinuous for $\mathcal{L}^3$ a.e. $x \in \mathcal{B}$,

(ii) $Pe(x, u, \cdot, \nu)$ is convex for all $(x, u, \nu) \in \mathcal{B} \times \tilde{\mathbb{R}}^3 \times \mathcal{A}_Q(\mathcal{M})$,

(iii) $Pe(x, u, M(F), \nu) = e_E(x, u, F, \nu)$ for all lists of entries with $\det F > 0$;

(c) for a constant $C_1 > 0$, an exponent $r > 1$ and a function $\vartheta : (0, +\infty) \to (0, +\infty)$ with $\vartheta(t) \uparrow +\infty$ as $t \downarrow 0^+$, the inequality

$$e_E(x, u, F, \nu) \geq C_1 \left(|M(F)|^r + \vartheta(\det F)\right)$$

holds for all $(x, u, F, \nu) \in \mathcal{B} \times \tilde{\mathbb{R}}^3 \times M^+_{3\times3} \times \mathcal{M}^Q$.

(H2) $e_M : \mathcal{B} \times (\mathrm{Hom}(\mathbb{R}^3, T\mathcal{M}))^Q \to [0, +\infty)$ is a continuous map such that

(a) there exists a regular field $\mathcal{B} \ni x \mapsto \Omega(x)$ taking as values fourth-rank tensors with major symmetry alone, such that

$$e_M(x, \nu, N) = \frac{1}{2} \sum_{i=1}^{Q} (\Omega(x) N_i) \cdot N_i$$

for all $(x, \nu, N) \in \mathcal{B} \times (\mathrm{Hom}(\mathbb{R}^m, T\mathcal{M}))^Q$, with $N = \sum_i^Q [\![N_i]\!]$;

(b) $e_M(x, \cdot)$ is quasiconvex for all $x \in \mathcal{B}$ according to Definition 3.1;

(c) for constants C_2, $C_3 > 0$, the inequality

$$C_2 \|N\|^2_{g(\nu)} \leq e_M(x, \nu, N) \leq C_3 \|N\|^2_{g(\nu)}$$

for all $(x, \nu, N) \in \mathcal{B} \times (\mathrm{Hom}(\mathbb{R}^3, T\mathcal{M}))^Q$.

In view of item (c) in (H2), for all $\varphi \in \mathrm{Lip}(\mathbb{R}^3, \mathcal{A}_Q(\mathcal{M}))$ with compact support, we have

$$\int_{\mathbb{R}^3} \sum_{i=1}^{Q} (\Omega(x) D\varphi_i(x)) \cdot D\varphi_i(x) dx \geq C_2 \int_{\mathbb{R}^3} \|d\varphi\|^2_{g(\varphi(x))} dx,$$

where $D\varphi_i$ is the $(m \times 3)$-matrix representing $d\varphi_i$, $i = 1, \dots, Q$. In particular, if the field Ω is constant, the quadratic integrand turns out to be Q–semielliptic in the sense of Mattila [31], i.e.

$$\int_{\mathbb{R}^3} \sum_{i=1}^{Q} (\Omega D\varphi_i(x)) \cdot D\varphi_i(x) \; dx \geq 0$$

for all $\varphi \in \mathrm{Lip}(\mathbb{R}^3, \mathcal{A}_Q(\mathcal{M}))$ with compact support. The equivalence of Q-semiellipticity and quasiconvexity has been addressed in [11, Remark 2.1].

4.3. **Existence of ground states.** To develop an existence theory via the direct methods of the calculus of variations we first look for conditions implying the lower semicontinuity of the relevant energies. In particular, we assume that the transplacement u varies in the class $\mathrm{dif}^{r,1}(\mathcal{B},\tilde{\mathbb{R}}^3)$ of *weak diffeomorphisms*, while the morphological descriptor ν is in $W^{1,2}(\Omega;\mathcal{A}_Q(\mathcal{M}))$. So, the space we are considering is

$$\mathcal{W}_{r,2} = \left\{(u,\nu) :\ u \in \mathrm{dif}^{r,1}(\mathcal{B},\hat{\mathbb{R}}^3),\ \nu \in W^{1,2}(\mathcal{B};\mathcal{A}_Q(\mathcal{M}))\right\},$$

with $\mathcal{B}$ a bounded open connected subset of $\mathbb{R}^3$.

The energy can be then considered as a map $\mathcal{E} : L^1(\mathcal{B};\hat{\mathbb{R}}^3) \times L^2(\mathcal{B};\mathcal{A}_Q(\mathcal{M})) \to [0,+\infty]$ defined by

$$\mathcal{E}(u,\nu) := \int_{\mathcal{B}} e\left(x, u(x), Du(x), \nu(x), (d\nu)_x\right)\, dx \tag{4.3}$$

for $(u,\nu) \in \mathcal{W}_{r,2}$, and $+\infty$ otherwise.

The results in Section 3 and the subsequent assumptions allow us to get a first conclusion.

Theorem 4.2. *Let $e : \mathcal{B} \times M^+_{3\times 3} \times (\mathrm{Hom}(\mathbb{R}^m, T\mathcal{M}))^Q \to [0,+\infty]$ be as in (4.1) with e_E and e_M satisfying (H1) and (H2) above. Then, the energy $\mathcal{E}$ in (4.3) is sequentially lower semicontinuous on $L^1(\mathcal{B},\hat{\mathbb{R}}^3) \times L^2(\mathcal{B},\mathcal{A}_Q(\mathcal{M}))$.*

Proof. On one hand the energy

$$\mathcal{E}_E(u,\nu) := \int_{\mathcal{B}} e_E\left(x, u(x), Du(x), \nu(x)\right)\, dx$$

is sequentially strongly lower semicontinuous on $L^1(\mathcal{B},\hat{\mathbb{R}}^3) \times L^1(\mathcal{B},\mathcal{A}_Q(\mathcal{M}))$ thanks to item (a) in (H1), Lemma 1.6 and Ioffe's classical weak-strong lower semicontinuity result.

On the other hand, the energy

$$\mathcal{E}_M(\nu) := \frac{1}{2}\int_{\mathcal{B}} \sum_i \left(\Omega(x) D\varphi_i(x)\right) \cdot D\varphi_i(x)\, dx$$

is sequentially strongly lower semicontinuous on $L^2(\mathcal{B},\mathcal{A}_Q(\mathcal{M}))$ thanks to Theorem 3.2 and assumption (c) in (H2).

The conclusion then follows. □

The existence of ground states for $\mathcal{E}$, if supplemented with suitable boundary conditions, follows immediately by using the closure of the space $\mathcal{W}_{r,2}$ in $L^1(\mathcal{B},\hat{\mathbb{R}}^3) \times L^2(\mathcal{B},\mathcal{A}_Q(\mathcal{M}))$ under the natural topology, and the coerciveness implied by the bounds in items (c) of (H1) and (H2).

Appendix A. A projection lemma

Finally we prove a truncation lemma in $\mathcal{A}_Q(\mathcal{M})$ that will be exploited in the proof of Theorem 3.2.

Lemma A.1. *For every $T = \sum_j^J q_j [\![a_j]\!]$, with $a_i \neq a_j$ for $i \neq j$ in $\mathcal{A}_Q(\mathcal{M})$, let $r_j > 0$ be the injectivity radius in a_j of $\mathcal{M}$, and $s(T) := \min_{i\neq j} d_{\mathcal{M}}(a_i, a_j)$.*

Then, there exists $r_0 > 0$ such that, for all $r < r_0$, there exists a Lipschitz map $\Theta_r : \mathcal{A}_Q(\mathcal{M}) \to B_r(T) \subset \mathcal{A}_Q(\mathcal{M})$ with $\Theta_r|_{B_r(T)} = \mathrm{Id}$, $\Theta_r|_{\mathcal{A}_Q(\mathcal{M})\setminus B_{2r}(T)} = T$ and $Lip(\Theta_r) \leq C$, for some positive constant $C = C(T, \mathcal{M})$.

Proof. Let $r_0 \leq \min\{r_j, s(T)\}/16$ be such that all the exponential maps $\exp_{a_j}$ are 2-Lipschitz in $B_{r_0} \subset T_{a_j}\mathcal{M}$. Note that for $S \in B_{4r}(T)$ there exists a unique decomposition $S = \sum_j^J [\![S_j]\!]$, with $S_j \in \mathcal{A}_{q_j}(\mathcal{M})$ equal to $\sum_{i=1}^{q_j} [\![S_j^i]\!]$, such that

$$\mathcal{G}^2_{\mathcal{M}}(S, T) = \sum_j^J \mathcal{G}^2_{\mathcal{M}}(S_j, q_j [\![a_j]\!]).$$

We define the map $\Theta_r : \mathcal{A}_Q(\mathcal{M}) \to \mathcal{A}_Q(\mathcal{M})$ as follows

$$\Theta_r(S) := \begin{cases} T & \text{if } S \notin B_{2r}(T) \\ \sum_j^J \sum_{i=1}^{q_j} \left[\!\!\left[\exp_{a_j}\left(F_r(\mathcal{G}_{\mathcal{M}}(T,S))\ \exp^{-1}_{a_j}(S^i_j)\right)\right]\!\!\right] & \text{if } S \in B_{2r}(T), \end{cases} \tag{A.1}$$

with $F_r(t) := \left(1 \wedge \left(\frac{2\,r-t}{t}\right)\right) \vee 0$, $t \in \mathbb{R}$. We only check the Lipschitz continuity of Θ_r since by construction all the other properties are straightforwardly verified.

Fix Q-points $S, R \in \mathcal{A}_Q(\mathcal{M})$, we distinguish three cases:

(a) $S, R \notin B_{2r}(T)$;
(b) $S \in B_{2r}(T)$ and $R \in B_{3r}(T)$;
(c) $S \in B_{2r}(T)$ and $R \notin B_{3r}(T)$.

Case (a) is immediate. Case (c) follows easily from

$$\mathcal{G}_{\mathcal{M}}(\Theta_r(S), \Theta_r(R)) = \mathcal{G}_{\mathcal{M}}(\Theta_r(S), T) \leq r \leq 3r - \mathcal{G}_{\mathcal{M}}(S, T) \leq \mathcal{G}_{\mathcal{M}}(S, R).$$

We show the more intricate case (b). We use the decomposition above $S = \sum_{j=1}^J \sum_{i=1}^{q_j} [\![S_j^i]\!]$, $R = \sum_{j=1}^J \sum_{i=1}^{q_j} [\![R_j^i]\!]$, and we assume that

$$\mathcal{G}^2_{\mathcal{M}}(S, R) = \sum_{j,i} d^2_{\mathcal{M}}(S^i_j, R^i_j).$$

By definition it is enough to estimate the following:

$$
\begin{aligned}
d_{\mathcal{M}} & \Big(\exp_{a_j}(F_r(\mathcal{G}_{\mathcal{M}}(T,S)) \exp^{-1}_{a_j}(S^i_j)), \exp_{a_j}(F_r(\mathcal{G}_{\mathcal{M}}(T,R)) \exp^{-1}_{a_j}(R^i_j))) \Big) \\
& \leq 2 \left| F_r(\mathcal{G}_{\mathcal{M}}(T,S)) \exp^{-1}_{a_j}(S^i_j) - F_r(\mathcal{G}_{\mathcal{M}}(T,R)) \exp^{-1}_{a_j}(R^i_j)) \right| \\
& \leq 2 \left| (F_r(\mathcal{G}_{\mathcal{M}}(T,S)) - F_r(\mathcal{G}_{\mathcal{M}}(T,R))) \exp^{-1}_{a_j}(S^i_j) \right| + \\
& \quad +2 \left| F_r(\mathcal{G}_{\mathcal{M}}(T,R)) \, (\exp^{-1}_{a_j}(S^i_j) - \exp^{-1}_{a_j}(R^i_j)) \right| \\
& \leq \frac{2}{r} \, \mathcal{G}_{\mathcal{M}}(S,R) \, | \exp^{-1}_{a_j}(S^i_j)| + 2 \, d_{\mathcal{M}}(R^i_j, S^i_j) \\
& \leq \frac{2}{r} \, \mathcal{G}_{\mathcal{M}}(S,R) \, \mathcal{G}_{\mathcal{M}}(S,T) + 2 \, d_{\mathcal{M}}(R^i_j, S^i_j) \leq 6 \, \mathcal{G}_{\mathcal{M}}(S,R). \qquad \text{(A.2)}
\end{aligned}
$$

Summing up all the contributions we finally get

$$
\mathcal{G}_{\mathcal{M}}(\Theta_r(S), \Theta_r(R)) \leq 6 \, Q \, \mathcal{G}_{\mathcal{M}}(S,R). \qquad \square
$$

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