The Polynomial Approach to the LQ non-Gaussian Regulator Problem through Output Injection

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***Abstract*—In this paper, an improved approach for the solution of the regulator problem for linear discrete-time dynamical systems with non-Gaussian disturbances and quadratic cost functional is proposed. It is known that a sub-optimal recursive control can be derived from the classical LQG solution by substituting the linear filtering part with a quadratic, or in general polynomial, filter. However, we show that when the system is not asymptotically stable the polynomial control does not improve over the classical LQG solution, due to the lack of the internal stability of the polynomial filter. In order to enlarge the class of systems that can be controlled, we propose a new method based on a suitable rewriting of the system by means of an output injection term. We show that this allows to overcome the problem and to design a polynomial optimal controller also for non asymptotically stable systems. Numerical results show the effectiveness of the method.**

***Index Terms*—Stochastic systems, Stochastic optimal control, Kalman filtering, Nonlinear filters.**

I. Introduction

The importance of the optimal control policy in the engineering applications is well known. The problem is usually modeled as a nonlinear programming problem that involves finding the minimum of a function with dynamical constraints. A typical case is that the system to be controlled is linear and the performance criterion is a quadratic form in state and control. The aim is to find a minimum energy feedback control law that keeps the state of the system close to the state-space origin. In the Linear Quadratic Gaussian (LQG) Regulator problem the noise statistics are assumed to be Gaussian.

A well-known property of the LQG Regulator problem with partial state information is that the optimal regulator, synthesized by the LQ optimal technique, is generated from the optimal linear estimate of the state [22]. For linear Gaussian systems, the Kalman Filter (KF) is the optimal recursive estimator in the minimum mean-square error sense. On the other hand, for linear non-Gaussian systems the KF is the best affine estimator but it is yet possible to develop estimators that are more accurate. The same holds true for the Kalman Predictor (KP).

In many important technical areas the widely used Gaussian assumption cannot be accepted as a realistic statistical description of the random quantities involved. As a consequence, increasing attention has been paid to non-Gaussian systems in control engineering [2], [20], [23], [26], [29], [30]. In particular, non-Gaussian problems often arise in digital communications when the noise interference includes noise components that are essentially non-Gaussian [25], in problems concerning fault estimation [17], sensor or actuator faults [16], intermittent observations [7], stochastic output matrices [8], multiplicative noises and bilinear systems [11], [12].

In the case of non-Gaussian noises the conditional expectation, which gives the optimal minimum variance estimation, is the solution of an infinite dimensional problem [31] that can be solved by numerical approximate solutions, with high computational burden. Therefore, in order to design any control strategy, it is important to find satisfactory sub-optimal estimates of the state variables that are actually computable. Methods to approximate the state conditional probability density function include Monte Carlo methods [2], sums of Gaussian densities [1] and weighted sigma points [21] among others. These general solutions can cope with nonlinearities and/or with the presence of noise outliers or unknown parameters [26], and they generally have high computational cost.

From the point of view of the optimal control problem the same issue arises, as it is known since the work of [24]. The original incomplete-information stochastic optimal control problem with nonlinearities or non Gaussian noises is equivalent to a complete-information but infinite-dimensional one. Although the results in [13] show that in some cases finite-dimensional sufficient statistics are available and allow to reduce the original incomplete information optimal control problem to a finite-dimensional and complete-information one, the general case remains challenging.

As a consequence, a sensible alternative is to look for sub-optimal and more easily computable solutions. This is done by first improving the state prediction provided by the KP. Since this prediction is a linear function of the measurements, a natural development in the minimum variance framework is to design predictors that make use of quadratic or polynomial functions of the observations in order to improve the estimation accuracy and preserve easy computability and recursion. This idea was first proposed in [27] and [15] where an augmented system contains the original observations and their squares. The suboptimal quadratic estimate (prediction) of the state for the original system is obtained by applying the KF (KP) to the augmented system. This approach has been extended in [10] to polynomial estimates. The prediction provided by quadratic or polynomial predictors can be exploited in the finite horizon control problem of non-Gaussian systems, as described in [9], [18], [19]. The resulting quadratic optimal controller yields better performance in terms of the standard quadratic cost functional with respect to the standard linear optimal controller and it is obtained by replacing the linear optimal prediction of the KF with the polynomial optimal one, in virtue of the separation principle proved in [18].

These approaches have a serious drawback in the case of non asymptotically stable systems. Indeed, the state and measurement noises of the augmented polynomial system depend on the state, and their variance becomes unbounded when the original system is not asymptotically stable. Moreover, the internal stability of the Polynomial Filter (PF) and Polynomial Predictor (PP) is guaranteed only for asymptotically stable systems. In this paper we show that when this condition is not met the performance of the PF tends asymptotically to the KF, and the same happens to the controller based on the PP. A solution to this problem has been proposed in [6] that describes a quadratic filter, named Feedback Quadratic Filter (FQF) that improves over the KF also in the case of non-asymptotically stable systems. The corresponding predictor - named Feedback Quadratic Predictor (FQP) - has been introduced in the conference paper [4]. The main contributions of the present paper are the following.

- We solve the *polynomial* case, *i.e.* we prove a separation result for a generic order ν of the regulator and we describe the complete algorithm for the Feedback Polynomial Predictor (FPP) (Section III).
- We present a complete theoretical analysis in Section IV. We prove formal properties concerning the performance of the controllers based on the FPP and PP. In particular we show that the output injection gain is essential to improve over the standard LQG Regulator in the case of unstable plants. We discuss the optimal choice of the output injection term and we prove the non intuitive property that the best controller does not necessarily make use of the best filter.
- We extend the approach to the case of non-stationary noise sequences (Section IV-E).
- We present in Section V extensive numerical simulations for the quadratic and cubic case to illustrate the superiority of the FPP Regulator and validate the theoretical conclusions. Specifically we show that the FPP Regulator is superior to the PP Regulator of [19] also for stable systems and discuss the choice of the output injection gain for FPP Regulator of different orders.
- We consider an optimal planar positioning problem (Section V-D) to show the application and effectiveness of the proposed method to a wider class of systems, namely systems with nonlinear output function and gaussian noises.

In this paper it is proven that the optimal controller based on polynomial estimates remains linear in the state. In other words, an optimal recursive polynomial controller is straightforwardly obtained by using the proposed predictor instead of the Kalman predictor together with the same feedback control law as in the linear optimal regulator (LQG) for the Gaussian case.

Extensions of this approach to time-varying systems, continuous-time systems and random system matrices (see [8]) are future possible developments.

Notation. In a probability space $(\Omega, \mathcal{F}, \mathbb{P})$ let $\mathbb{E}$ denotes the expectation. If X and Y are two random variables (r.v.) with the same distribution we write $X \sim Y$. If X is a Gaussian r.v. with mean μ and variance σ^2 we write $X \sim \mathcal{N}(\mu, \sigma^2)$. We denote with L^2 the Hilbert space of the n-dimensional $\mathcal{F}$ measurable random variables with finite second order moment. Let $\mathcal{G}$ be a sub σ-algebra of $\mathcal{F}$, we denote with $L^2(\mathcal{G}, n)$ the Hilbert space of the n-dimensional, $\mathcal{G}$ measurable random variables with finite second order moment. We write $L^2(X, n)$ to denote $L^2(\sigma(X), n)$ where $\sigma(X)$ is the σ-algebra generated by X. Given a random vector X in $(\Omega, \mathcal{F}, \mathbb{P})$ $X^\top$ denotes the transpose and the norm in L^2 is defined as usual, *i.e.* $\|X\|_{L^2}^2 = \int_\Omega X^\top X d\mathbb{P} = \mathbb{E}[X^\top X]$. We denote with $\Pi\left[\cdot \mid \mathcal{M}\right]$ the orthogonal projection onto a given closed subspace $\mathcal{M}$ of a given Hilbert space. Let A be a matrix, then $||A||$ is a norm of A and $(A)_{r,s}$ denotes the element of the row r and column s. If $A > 0$ the matrix A is positive definite and $A > B$ means $A - B > 0$. The Kronecker product of A and B is $A \otimes B$. The i-th Kronecker power of A is $A^{[i]}$, $\mathrm{st}\{A\}$ denote the vectorization (or stack) function of A and $\mathrm{st}^{-1}\{\cdot\}$ its inverse. The trace of a square matrix A is $\mathrm{tr}\{A\}$ and $v = \mathrm{col}(v_1, \dots, v_n)$ denotes the column vector $v = [v_1, \dots, v_n]^\top$. We denote with $I_{n,i}$ the identity matrix in $\mathbb{R}^{n^i \times n^i}$, I_n the identity matrix in $\mathbb{R}^{n \times n}$ and with I the identity matrix of appropriate dimension when it is clear from the context. The spectrum of a square matrix A is defined as $\sigma(A)$, while $\tilde{\sigma}(A) \subset \sigma(A)$ indicates the set of the eigenvalues corresponding to the observable part of A. See [5], chapter 12 for details on matrices and Kronecker Algebra. The open unit ball of the complex plane is denoted with $\mathbb{C}_\odot$. If $\sigma(A) \subset \mathbb{C}_\odot$, then A is said to be Schur stable. If $X \in \mathbb{R}^n$ then $X_{1:p}$ denotes the vector in $\mathbb{R}^p$ of the first p components of X, where $p < n$. Throughout the paper we will use the symbol 0 to denote the zero scalar, vector or matrix of appropriate dimension.

II. Problem Formulation and Preliminaries

The control problem we solve concerns the class of linear, detectable and stabilizable systems driven by non-Gaussian additive noise described by the following equations:

$$x(k+1) = Ax(k) + Bu(k) + F\omega(k), \tag{1}$$

$$y(k) = Cx(k) + G\omega(k), \tag{2}$$

with $x(0) = x_0$ and the associated cost functional

$$J = \frac{1}{2}\mathbb{E}\Big[x^\top(\mathsf{N})Sx(\mathsf{N}) + \sum_{k=0}^{\mathsf{N}-1} x^\top(k)Qx(k) + u^\top(k)Ru(k)\Big] \tag{3}$$

where $\mathsf{N} \in \mathbb{N}$ is the time-horizon. For $k \geq 0$, $x(k) \in \mathbb{R}^n$ is the state, $\omega(k) \in \mathbb{R}^r$ is a non-Gaussian noise, $u(k) \in \mathbb{R}^p$ is the control input, $y(k) \in \mathbb{R}^q$ is the measurement output and the matrices A, B, C, F, G, S, Q, R are of appropriate dimensions. As usual, the covariance matrix $GG^\top$ is full rank and the matrices Q, S and R are symmetric non-negative definite (strictly positive definite in the case of R). Moreover, we assume the pairs (A,B), (A,F) stabilizable and (A,C), (A,Q) detectable.

Denoting $\bar{x}_0 = \mathbb{E}[x_0]$, assume there exists $\nu \in \mathbb{N}$ such that the initial state x_0 and the random sequence $\{\omega(k)\}$ satisfy the following conditions for $k \geq 0$:

(i) $\{\omega(k)\}$ is a zero mean i.i.d. sequence;

(ii) $\{F\omega(k)\}$, $\{G\omega(k)\}$ and x_0 have uncorrelated moments up to the 2ν order;

(iii) for $i = 1, 2, \ldots, 2\nu$ there exist finite and known vectors $\Psi_{\omega,i} \doteq \mathbb{E}[\omega^{[i]}(k)]$, $\Psi_{x_0,i} \doteq \mathbb{E}[(x_0 - \bar{x}_0)^{[i]}]$.

Clearly $\Psi_{x_0,1} = 0$ and (i) implies $\Psi_{\omega,1} = 0$. When the sequences $\{F\omega(k)\}$, $\{G\omega(k)\}$ and x_0 are mutually independent then (ii) is verified. We assume without loss of generality that $\mathrm{st}^{-1}\{\Psi_{\omega,2}\} = \mathbb{E}[\omega(k)\omega^\top(k)] = I$ and we denote with $\Psi_{x_0} = \mathrm{st}^{-1}\{\Psi_{x_0,2}\}$ the covariance matrix of the initial condition. In the non-Gaussian case, assuming the knowledge of the first 2ν moments of the state and output noise sequences is weaker than assuming the availability of the whole distributions.

In this framework, we consider the finite-horizon suboptimal control problem for non-Gaussian discrete-time linear systems with partial state information. More precisely, our aim is to compute the control law in a class of recursively computable polynomial feedback which minimizes (3). It is well known (see [22]) that, for the finite-horizon regulator problem (1)–(3) with the assumptions above, the output feedback optimal control $\check{u}(k)$ among the affine transformations of $y(0), y(1), \ldots, y(k-1)$ is given by

$$\check{u}(k) = -M(k)\check{x}(k|k-1), \tag{4}$$

where

$$M(k) = R^{-1}B^\top P_c(k+1) \cdot \cdot \big(I + BR^{-1}B^\top P_c(k+1)\big)^{-1}A, \tag{5}$$

$P_c(k)$ is the solution of the backward Riccati equation

$$P_c(k) = A^\top P_c(k+1)\big(I + BR^{-1}B^\top P_c(k+1)\big)^{-1}A + Q \tag{6}$$

$$P_c(\mathsf{N}) = S \tag{7}$$

and $\check{x}(k|k-1)$ is the prediction of $x(k)$ provided by the Kalman predictor (see [3]), namely

$$\check{x}(k|k-1) = A\check{x}(k-1|k-2) + AK(k-1) \cdot \big(y(k-1) - C\check{x}(k-1|k-2)\big) + B\check{u}(k-1), \tag{8}$$

$$K(k) = P(k)C^\top\big(GG^\top\big)^{-1}, \tag{9}$$

$$P(k) = \Big(I + H(k-1)C^\top\big(GG^\top\big)^{-1}C\Big)^{-1}H(k-1), \tag{10}$$

$$H(k-1) = AP(k-1)A^\top + FF^\top. \tag{11}$$

A. The Geometric Approach to Filtering and Prediction

We recall the basic notions on polynomial filtering and recursive polynomial estimates through a geometric approach. Consider the system (1)–(2) and let

$$Y_k \doteq \mathrm{col}(y(0), \cdots, y(k)) \tag{12}$$

be the output sequence. The minimum variance estimate of the state $x(k)$ of system (1)–(2) can be defined as the orthogonal projection of $x(k)$ onto the Hilbert space $L^2(Y_k, n)$:

$$\tilde{x}(k) = \mathbb{E}\left[x(k)|\sigma(Y_k)\right] = \Pi\left[x(k)|L^2(Y_k,n)\right]. \tag{13}$$

Defining the auxiliary vector $Y'_k = \mathrm{col}(1, Y_k) \in \mathbb{R}^{1+l}$, with $l = (k+1)q$, if the sequences $\{x(k)\}$ and $\{y(k)\}$ are jointly Gaussian, the projection above is equivalent to the projection onto the subspace $\mathcal{L}_y^k \subset L^2(Y_k, n)$ of the affine functions of Y_k, *i.e.*

$$\mathcal{L}_y^k = \{z : \Omega \to \mathbb{R}^n : \exists T \in \mathbb{R}^{n\times 1+l} : z = TY'_k\}.$$

The KF recursively computes the projection $\Pi[x(k)|\mathcal{L}_y^k]$ which is the best affine estimate of $x(k)$ in the minimum variance sense. In the Gaussian case, this coincides with $E\left[x(k)|\sigma(Y_k)\right]$. When $\{x(k)\}$ and/or $\{y(k)\}$ are non-Gaussian, the computation of (13) is highly difficult.

Since the best affine estimate is obtained by projecting onto $\mathcal{L}_y^k$, better suboptimal estimates can be obtained by projecting the state $x(k)$ onto larger subspaces. We will consider the space of ν-order polynomial transformations of Y_k. Let us define the aggregate vector

$$Y_k^{(\nu)} = \mathrm{col}(Y'_k, Y_k^{[2]}, Y_k^{[3]}, \ldots, Y_k^{[\nu]}) \in \mathbb{R}^{l_\nu} \tag{14}$$

where $l_\nu = \sum_{i=0}^{\nu} l^i$, then

$$\mathcal{P}_y^k(\nu) \doteq \{z : \Omega \to \mathbb{R}^n : \exists T \in \mathbb{R}^{n\times l_\nu} : z = TY_k^{(\nu)}\}. \tag{15}$$

The computation of the *optimal polynomial estimate* $\Pi[x(k)|\mathcal{P}_y^k(\nu)]$ requires a growing filter size because $Y_k^{(\nu)}$ contains powers of the output at different time instants. For example $Y_k^{[2]}$ contains terms of the form $y(k_1) \otimes y(k_2)$ with $k_1, k_2 \leq k$. In general, for $i \in \{2, \ldots, \nu\}$, the vector $Y_k^{[i]}$ contains terms of the form $y^{[p_1]}(k_1) \otimes y^{[p_2]}(k_2) \otimes \cdots \otimes y^{[p_i]}(k_i)$ for any $p_1, \ldots, p_i$ such that $\sum_{j=1}^{i} p_j = i$ and $k_1, k_2, \ldots, k_i \leq k$.

In order to obtain a filter which can be written recursively, we replace $Y_k^{(\nu)}$ of (14) with

$$\overline{Y}_k^{(\nu)} \doteq \mathrm{col}(Y'_k, \widetilde{Y}_k^{(2)}, \widetilde{Y}_k^{(3)}, \ldots, \widetilde{Y}_k^{(\nu)}) \in \mathbb{R}^{\bar{l}_\nu} \tag{16}$$

where $\widetilde{Y}_k^{(i)} \doteq \mathrm{col}(y^{[i]}(0), \ldots, y^{[i]}(k))$ with $i \in \{2, \ldots, \nu\}$ and $\bar{l}_\nu = 1 + l\sum_{i=0}^{\nu-1} q^i$. The difference is that $\widetilde{Y}_k^{(i)}$ contains only the i-th power of each output $y(j)$, $j \leq k$, but not the cross products of the output at different time instants. Consequently, $\widetilde{Y}_k^{(i)}$ grows linearly in time, rather than polynomially. The associated subspace is

$$\overline{\mathcal{P}}_y^k(\nu) \doteq \{z : \Omega \to \mathbb{R}^n : \exists T \in \mathbb{R}^{n\times \bar{l}_\nu} : z = T\overline{Y}_k^{(\nu)}\}. \tag{17}$$

Thus, we shall refer to $\Pi[x(k)|\overline{\mathcal{P}}_y^k(\nu)]$ as the *optimal recursive polynomial estimate*. Analogous definitions can be given for the *optimal polynomial prediction*, *i.e.* $\Pi\left[x(k)|\mathcal{P}_y^{k-1}(\nu)\right]$ and for the *optimal recursive polynomial prediction*, *i.e.* $\Pi[x(k)|\overline{\mathcal{P}}_y^{k-1}(\nu)]$.

It is well known that conditional expectation minimizes the covariance matrix of the estimation error. This is true even for projections as it is clarified by the following known lemmas for which the proofs are provided in the Appendix A.

Lemma 1: Let $\mathcal{K}$ and $\mathcal{M}$ be two closed subspaces of L^2 such that $\mathcal{K} \subseteq \mathcal{M}$. Let $v \in L^2$, we set $v_{\mathcal{K}} = \Pi[v|\mathcal{K}]$ and $v_{\mathcal{M}} = \Pi[v|\mathcal{M}]$. If $\Psi_{\mathcal{K}}$ and $\Psi_{\mathcal{M}}$ are the covariance matrices associated to the errors $e_{\mathcal{K}} = v - v_{\mathcal{K}}$ and $e_{\mathcal{M}} = v - v_{\mathcal{M}}$ respectively, then

$$\Psi_{\mathcal{K}} \geq \Psi_{\mathcal{M}}, \tag{18}$$

$$\operatorname{tr}\{\Psi_{\mathcal{K}}\} \geq \operatorname{tr}\{\Psi_{\mathcal{M}}\}, \tag{19}$$

$$\operatorname{tr}\{\Psi_{\mathcal{K}}\Lambda\} \geq \operatorname{tr}\{\Psi_{\mathcal{M}}\Lambda\} \tag{20}$$

with Λ any positive semidefinite matrix.

Lemma 2: Let $P_\nu(k)$ and $H_\nu(k)$ be the covariance matrices of the estimation and prediction errors $x(k) - \Pi[x(k)|\overline{\mathcal{P}}^i_y(\nu)]$ for $i = k$ and $i = k-1$ respectively. Let $P(k)$ and $H(k)$ be the covariance matrices of the estimation and prediction errors $x(k) - \Pi[x(k)|\mathcal{L}^i_y]$ for $i = k$ and $i = k-1$ respectively, then

$$\left\| x(k) - \Pi[x(k)|\overline{\mathcal{P}}^i_y(\nu)] \right\|^2_{L^2} \leq \left\| x(k) - \Pi\left[x(k)|\mathcal{L}^i_y\right] \right\|^2_{L^2}.$$

B. Output Injection

The state equation (1) is transformed using the output equation (2):

$$\begin{aligned} x(k+1) &= Ax(k) + Bu(k) + F\omega(k) \\ &= \tilde{A}x(k) + Bu(k) + F\omega(k) + Ly(k) - LG\omega(k) \\ &= \tilde{A}x(k) + Bu(k) + Ly(k) + W\omega(k), \end{aligned} \tag{21}$$

where $\tilde{A} = A - LC$, the output injection gain matrix $L \in \mathbb{R}^{n \times q}$ is arbitrarily chosen and $W = F - LG$. As the second step, taking advantage of linearity, the state sequence is split into two sequences, $\{x_p(k)\}$ and $\{x_s(k)\}$. The control-dependent predictable component $x_p(k)$ is the solution of

$$x_p(k+1) = \tilde{A}x_p(k) + Bu(k) + Ly(k), \tag{22}$$

$$x_p(0) = \bar{x}_0 \doteq x^p_0, \tag{23}$$

namely

$$x_p(k) = \tilde{A}^k x^p_0 + \sum_{\tau=0}^{k-1} \tilde{A}^{k-\tau-1}\left(Bu(\tau) + Ly(\tau)\right), \tag{24}$$

which can be computed at time k, since the deterministic initial value x^p_0 and the output sequence Y_{k-1} are available.

The stochastic component $x_s(k)$ is the solution of

$$x_s(k+1) = \tilde{A}x_s(k) + W\omega(k), \quad x_s(0) = x^s_0, \tag{25}$$

with $x^s_0 \sim x_0 - \bar{x}_0$ where $\Psi_{x^s_0,i} \doteq E\left[x^{s\,[i]}_0\right] = \Psi_{x_0,i}$. From (22) and (25) trivially follows that

$$x(k) = x_p(k) + x_s(k) \quad \forall k \geq 0. \tag{26}$$

The output map (2) is arranged as

$$y_s(k) = y(k) - Cx_p(k) = Cx_s(k) + G\omega(k), \tag{27}$$

where $y_s(k)$ is an available information at time k. Denoting the corresponding sequence vector as

$$Y_{s,k} \doteq \operatorname{col}(y_s(0), y_s(1), \ldots, y_s(k)) \tag{28}$$

we can define $Y^{(\nu)}_{s,k}$, $\mathcal{P}^k_{y_s}(\nu)$, $\overline{Y}^{(\nu)}_{s,k}$ and $\overline{\mathcal{P}}^k_{y_s}(\nu, L)$ as in (14), (15), (16) and (17) respectively, using $y_s(k)$ instead of $y(k)$. Note that, in particular, the set $\overline{\mathcal{P}}^k_{y_s}(\nu, L)$ depends explicitly on the output injection gain L. We use the notation $\overline{\mathcal{P}}^k_{y_s}(\nu)$ when it is not necessary to specify L.

Remark 1: It has been shown (see [6], Theorem 4) that for any value of the output injection gain matrix L, we have $\mathcal{P}^k_y(\nu) \equiv \mathcal{P}^k_{y_s}(\nu)$. However, this is not true for the *recursive* quadratic transformations, *i.e.* for $L \neq 0$, we have $\overline{\mathcal{P}}^k_y(2) \not\equiv \overline{\mathcal{P}}^k_{y_s}(2)$ (see [6], Theorem 5). This extends to recursive polynomial transformations $\nu \geq 2$, for which $\overline{\mathcal{P}}^k_y(\nu) \not\equiv \overline{\mathcal{P}}^k_{y_s}(\nu)$. Roughly speaking, this is due to the fact that recursive polynomial estimates neglect the powers of the output at different time instants, and the correlation of these terms depend on the dynamic matrix of the stochastic component, and therefore on the gain L.

Since (24) is an affine transformation of Y_k, the projection of $x_p(k)$ onto $\overline{\mathcal{P}}^k_{y_s}(\nu)$ trivially corresponds to itself while $\Pi[x_s(k)|\overline{\mathcal{P}}^k_{y_s}(\nu)]$ can be obtained by processing the system (25)-(27). Thus, we can obtain $\hat{x}(k)$ as the sum of the two terms

$$\hat{x}(k) \doteq \Pi[x(k)|\overline{\mathcal{P}}^k_{y_s}(\nu)] = x_p(k) + \Pi[x_s(k)|\overline{\mathcal{P}}^k_{y_s}(\nu)] \tag{29}$$

Analogously, we can obtain $\hat{x}(k|k-1)$ as the sum of $x_p(k)$ and $\Pi[x_s(k)|\overline{\mathcal{P}}^{k-1}_{y_s}(\nu)]$. Since the projection subspace $\overline{\mathcal{P}}^k_{y_s}(\nu)$ depends on L the estimate $\hat{x}(k)$ and the prediction $\hat{x}(k|k-1)$ will depend on L.

Figure 1 gives a representation of the sets of polynomial transformations of the output. The outer solid line circle represents the set of polynomial transformations $\mathcal{P}^k_y(\nu)$ which is independent of the output injection gain L, the inner dashed line circle represents the set of recursive polynomial transformations for all the values of L. The two inner solid line circles represent the sets of recursive polynomial transformations for two specific values L_1 and L_2, namely $\overline{\mathcal{P}}^k_{y_s}(\nu, L_1)$ and $\overline{\mathcal{P}}^k_{y_s}(\nu, L_2)$. The intersection of these two sets is non-empty and contains the linear transformations $\mathcal{L}^k_y \equiv \mathcal{L}^k_{y_s}$, that are independent of L. Note that if $L = 0$ then $\overline{\mathcal{P}}^k_{y_s}(\nu, 0) \equiv \overline{\mathcal{P}}^k_y(\nu)$. We prove in Section IV that we can improve the performance by projecting the state onto the L-parametrized space $\overline{\mathcal{P}}^k_{y_s}(\nu, L)$. This argument can be extended to predictions, since the recursive polynomial estimate and the recursive polynomial prediction belong to the set $\overline{\mathcal{P}}^k_{y_s}(\nu, L)$ and $\overline{\mathcal{P}}^{k-1}_{y_s}(\nu, L)$, respectively. Section IV-B discusses the choice of the parameter L.

III. Separation Principle and Recursive Polynomial Control

Let us rewrite the difference equation (22) of the predictable component $x_p(k)$ and consider the system

$$\begin{aligned} x_p(k+1) = Ax_p(k) + Bu(k) + & \\ + LCx_s(k) + LG\omega(k), & \end{aligned} \tag{30}$$

$$x_s(k+1) = \tilde{A}x_s(k) + W\omega(k), \tag{31}$$

$$y_s(k) = Cx_s(k) + G\omega(k) \tag{32}$$

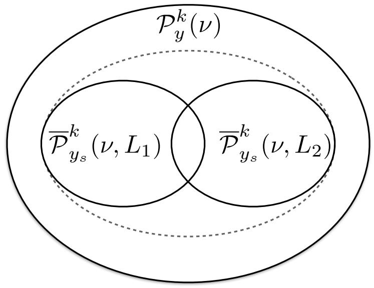

Fig. 1. Sketch of the sets of different types of polynomial transformations of the output.

with the initial conditions $x_p(0) = \bar{x}_0$ and $x_s(0) \sim x_0 - \bar{x}_0$. With the notation introduced in the previous section, the output feedback optimal control among the affine transformations of output sequence $\{y(k)\}$ is in the set $\mathcal{L}_{y_s}^{k-1}$ while, for a given output injection gain matrix L, the recursive polynomial control of (1)–(3), belongs to the set $\overline{\mathcal{P}}_{y_s}^{k-1}(\nu)$ and can be expressed as an affine transformation of the zero-mean vector $\mathrm{col}(Y_e(0), Y_e(1), \ldots, Y_e(k-1))$ where

$$Y_e(k) \doteq \begin{bmatrix} y_s(k) \\ y_s^{[2]}(k) - G^{[2]}\Psi_{\omega,2} \\ y_s^{[3]}(k) - G^{[3]}\Psi_{\omega,3} \\ \vdots \\ y_s^{[\nu]}(k) - G^{[\nu]}\Psi_{\omega,\nu} \end{bmatrix}.$$

Definition 1: A discrete-time stochastic process ξ taking values in $\mathbb{R}^n$ is said to be a *second-order convergent process* if there exist two constant vectors $m_1 \in \mathbb{R}^n$ and $m_2 \in \mathbb{R}^{n^2}$ such that $\lim_{k\to\infty} \mathbb{E}[\xi(k)] = m_1$ and $\lim_{k\to\infty} \mathbb{E}[\xi^{[2]}(k)] = m_2$.

Notice that since $\mathbb{E}[\xi^{[2]}(k)] = \mathrm{st}\{\mathbb{E}[\xi(k)\xi^\top(k)]\}$, the covariance matrix of a second-order convergent process tends asymptotically to $\mathrm{st}^{-1}\{m_2\}$. Moreover, condition (iii) in Section II implies that for $i = 1, \ldots, \nu$ the process $\omega^{[i]}(k)$ is second-order convergent.

Lemma 3: If the output injection gain matrix L is such that $\sigma(\tilde{A}) = \sigma(A - LC) \subset \mathbb{C}_\odot$ and conditions (i)–(iii) in Section II hold true then the process $x_s^{[i]}$, $i = 1, \ldots, \nu$, where x_s is defined in (31), is second-order convergent.

Proof: The hypothesis $\sigma(\tilde{A}) \subset \mathbb{C}_\odot$ implies $\sigma(\tilde{A}^{[i]}) \subset \mathbb{C}_\odot$, *i.e.* $\tilde{A}^{[i]}$ is Schur stable, for all $i \geq 1$. The statements is easily proved by induction. Since $\mathbb{E}[x_s(k+1)] = \tilde{A}\mathbb{E}[x_s(k)]$ with a null initial condition, $\mathbb{E}[x_s(k)]$ is identically null. For $i = 2$ the statement holds true since

$$\mathbb{E}[x_s^{[2]}(k+1)] = \tilde{A}^{[2]}\mathbb{E}[x_s^{[2]}(k)] + W^{[2]}\Psi_{\omega,2}, \tag{33}$$

$\tilde{A}^{[2]}$ is Schur stable and consequently under assumption (iii) $\mathbb{E}[x_s^{[2]}(k)]$ tends to a finite limit. For $i = 3, \ldots, 2\nu$ we use the following representation of the i-th Kronecker power of the binomial (see Appendix B, Proposition 5) to write

$$\begin{aligned} x_s^{[i]}(k+1) =& \left(\tilde{A}x_s(k) + W\omega(k)\right)^{[i]} \\ =& \sum_{j=0}^{i} \Theta_j^i(n) \left(\tilde{A}^{[j]} x_s^{[j]}(k) \otimes W^{[i-j]}\omega^{[i-j]}(k)\right) \\ =& \tilde{A}^{[i]}x_s^{[i]}(k) + \sum_{j=0}^{i-1} \Theta_j^i(n) \left(\tilde{A}^{[j]} x_s^{[j]}(k) \otimes W^{[i-j]}\omega^{[i-j]}(k)\right), \end{aligned} \tag{34}$$

where $\Theta_j^i(n)$ are suitably defined coefficient matrices of dimension $n \times n$. Taking expectations yields

$$\begin{aligned} \mathbb{E}\left[x_s^{[i]}(k+1)\right] =& \tilde{A}^{[i]}\mathbb{E}\left[x_s^{[i]}(k)\right] + \\ &+ \sum_{j=0}^{i-1} \Theta_j^i(n) \left(\tilde{A}^{[j]}\mathbb{E}\left[x_s^{[j]}(k)\right] \otimes W^{[i-j]}\mathbb{E}\left[\omega^{[i-j]}(k)\right]\right). \end{aligned} \tag{35}$$

Since $\tilde{A}^{[i]}$ is Schur stable, $\mathbb{E}\left[\omega^{[i-j]}(k)\right]$ is constant and $\mathbb{E}\left[x_s^{[j]}(k)\right]$, $j < i$ tends to a limit because of the inductive hypothesis, $\mathbb{E}\left[x_s^{[i]}(k)\right]$, $i = 1, \ldots, 2\nu$ tends to a finite limit and consequently $x_s^{[i]}$ is second-order convergent for $i = 1, \ldots, \nu$. ■

Lemma 4: The sequence $\{Y_e(k)\}$ is generated by the following stochastic system

$$X_e(k+1) = \mathcal{A}X_e(k) + \mathcal{B}u(k) + d_e + h_e(k) \tag{36}$$
$$Y_e(k) = \mathcal{C}X_e(k) + g_e(k) \tag{37}$$

where

$$X_e(k) = \begin{bmatrix} x_p(k) \\ x_s(k) \\ x_s^{[2]}(k) \\ \vdots \\ x_s^{[\nu]}(k) \end{bmatrix} = \begin{bmatrix} x_p(k) \\ X_s(k) \end{bmatrix}, \quad h_e = \begin{bmatrix} LG\omega(k) \\ w_e(k) \end{bmatrix} \tag{38}$$

$$d_e = \begin{bmatrix} 0 \\ W\Psi_{\omega,1} \\ W^{[2]}\Psi_{\omega,2} \\ \vdots \\ W^{[\nu]}\Psi_{\omega,\nu} \end{bmatrix} = \begin{bmatrix} 0 \\ d_s \end{bmatrix} \quad d_s = \begin{bmatrix} 0 \\ W^{[2]}\Psi_{\omega,2} \\ \vdots \\ W^{[\nu]}\Psi_{\omega,\nu} \end{bmatrix} \tag{39}$$

where $w_e = \mathrm{col}(w_1, \ldots, w_\nu)$ and $g_e = \mathrm{col}(g_1, \ldots, g_\nu)$ with $w_i(k)$, $g_i(k)$ for $i \in \{1, \ldots, \nu\}$ defined in (43)-(44) and the matrices $\mathcal{A}$, $\mathcal{B}$, $\mathcal{C}$ defined in (45)–(47). The sequences $\{w_e(k)\}$ and $\{g_e(k)\}$ are zero-mean, white and such that $\mathbb{E}[w_e(k)g_e^\top(j)] = 0$ for any $k \neq j$. Moreover the expression of the auto-covariance and cross-covariance matrices of the noise sequences

$$\Psi_w(k) \doteq \mathbb{E}\left[w_e(k)w_e^\top(k)\right], \tag{40}$$
$$\Psi_g(k) \doteq \mathbb{E}\left[g_e(k)g_e^\top(k)\right], \tag{41}$$
$$\Upsilon(k) \doteq \mathbb{E}\left[w_e(k)g_e^\top(k)\right] \tag{42}$$

are given in (49)–(51), where the $\Xi_{j,i}$ terms are suitably defined commutation matrices (see Appendix B, Proposition 3).

Proof: By considering Lemma 3.3.1 of [10] the proof is obtained. By applying twice this Lemma with reference to (31) and (32) and taking into account the predictable component of the state given by (30) the proposition is proved. ■

Lemma 5: If L is such that $\sigma(A-LC) \subset \mathbb{C}_\odot$ and conditions (i)–(iii) in Section II hold true then the noise processes $\{w_e(k)\}$ and $\{g_e(k)\}$ are second-order convergent.

Proof: From (52) it is evident that the time varying part of $\Psi_w(k)$ and $\Psi_g(k)$ is $\mathbb{E}[x_s^{[i]}(k)]$, with $i \leq 2\nu$. Since Lemma 3 guarantees that $\mathbb{E}[x_s^{[i]}(k)]$ tends to a finite limit, the same holds for $\Psi_w(k)$ and $\Psi_g(k)$. ■

It is now possible to provide the solution to the polynomial optimal regulator problem of (1)–(3).

Theorem 1: Let L be an output injection gain matrix such that $\sigma(A-LC) \subset \mathbb{C}_\odot$ and conditions (i)–(iii) in Section II hold true. The recursive optimal polynomial regulator for the system (1)–(3) is given by the linear controller

$$\hat{u}(k) = -M(k)\hat{x}(k|k-1) \tag{53}$$

where $M(k)$ is the gain of the LGQ Regulator defined in (5) with the associated solution to the backward Riccati equation (6)–(7) and the prediction $\hat{x}(k|k-1)$ is computed with the expressions of the Recursive Polynomial Filter and Predictor (54)–(65).

Proof: The optimal linear control of the extended system (36)–(37), endowed with the same cost functional written in the new extended state

$$J = \frac{1}{2}\mathbb{E}\Big[X_e^\top(\mathsf{N})\mathcal{S}X_e(\mathsf{N}) + \sum_{k=0}^{\mathsf{N}-1} X_e^\top(k)\mathcal{Q}(k)X_e(k) + u^\top(k)R(k)u(k)\Big] \tag{66}$$

with

$$\mathcal{S} = \begin{bmatrix} S & S & 0 & \cdots & 0 \\ S & S & 0 & \cdots & 0 \\ 0 & 0 & 0 & \cdots & 0 \\ \vdots & \vdots & \ddots & \ddots & \vdots \\ 0 & 0 & \cdots & \cdots & 0 \end{bmatrix}, \mathcal{Q} = \begin{bmatrix} Q & Q & 0 & \cdots & 0 \\ Q & Q & 0 & \cdots & 0 \\ 0 & 0 & 0 & \cdots & 0 \\ \vdots & \vdots & \ddots & \ddots & \vdots \\ 0 & 0 & \cdots & \cdots & 0 \end{bmatrix}, \tag{67}$$

corresponds to the optimal polynomial control of the original problem (1)–(3), since it is obtained by applying the LQG solution (4)–(11) to the extended system (36)-(37) with the cost functional (66). Note that, due to the presence of the forcing term d_e, the optimal linear control of the extended system (36)-(37) is given by

$$\hat{u}(k) = -(\Pi(k)d_e + \mathcal{M}(k)X_e(k|k-1)) \tag{68}$$

with

$$\Pi(k) = \mathcal{R}^{-1}\mathcal{B}^\top\mathcal{P}_c(k+1)(I + \mathcal{B}\mathcal{R}^{-1}\mathcal{B}^\top\mathcal{P}_c(k+1))^{-1},$$
$$\mathcal{M}(k) = \Pi(k)\mathcal{A},$$

where $\mathcal{P}_c(k+1)$ is the solution of the backward Riccati equation associated to the extended system (36)-(37) and the cost (66). Note that, at each k, the matrix $\mathcal{P}_c(k)$ has the same structure of $\mathcal{S}$ and $\mathcal{Q}$ of (67). Now, by noticing from (39) that the first two blocks of d_e are equal to zero, *i.e.* $d_{e1:2n} = 0$, it is easy to see that the control law in (68) simplifies in (53) and the algorithm (54)–(65) provides the polynomial prediction $\hat{x}(k|k-1)$. ■

We name the predictor (57) Feedback Polynomial Predictor (FPP).

Remark 2: Theorem 1 shows the validity of the separation principle for the polynomial case. Hence, the same control system design tools can be used as in the LQG case.

IV. Properties of the Recursive Polynomial Control

In this section we provide additional properties and performance results of the recursive polynomial control with the FPP (FPP Regulator for short) described in Section III. In particular we discuss the case of unstable system, the choice of the output injection gain matrix L, we compare the proposed algorithm with the standard LQG Regulator, and with the recursive polynomial predictor of [19] (PP Regulator for short), we extend the method to non-stationary noise sequences and we discuss some observability and implementation issues.

For comparison purpose, we consider the stationary problem (infinite-horizon). Let us denote H_∞^{LQG}, H_∞^{PP} and $H_\infty^{\mathrm{FPP}}(L)$ the steady-state covariance matrices of the prediction errors of the KP, PP and FPP respectively. Note that $H_\infty^{\mathrm{FPP}}(L)$ depends on L and corresponds to the limit of the first $n \times n$ block of $H_r(k)$ in (65), whereas the H_∞^{PP} does not depend on L and in particular we have $H_\infty^{\mathrm{PP}} = H_\infty^{\mathrm{FPP}}(0)$.

Our conclusions, motivated in the following, can be summarized as follows. Consider the system (1)-(2) with the cost functional (3) and (A, C) a detectable pair.

- When A is not Schur stable and for any L such that $\sigma(\tilde{A}) \subset \mathbb{C}_\odot$. $H_\infty^{\mathrm{LQG}} \equiv H_\infty^{\mathrm{PP}} \geq H_\infty^{\mathrm{FPP}}$, and the same relationship holds for the related cost functionals J.
- When A is Schur stable it is possible to find L such that $H_\infty^{\mathrm{LQG}} \geq H_\infty^{\mathrm{PP}} \geq H_\infty^{\mathrm{FPP}}$ and it is possible to find L such that the same relationship holds for the related cost functionals J. The two gains L may in general be different.

A. Unstable systems

The main point motivating the introduction of the output injection term is the following. The PP Regulator is obtained by setting $L = 0$ for which $\tilde{A}^{[i]} = A^{[i]}$ for any $i = 1, \ldots, \nu$. Consider the case of an unstable system, *i.e.* A is not Schur stable. Note that even $A^{[i]}$ for $i = 2, \ldots, \nu$ is not Schur stable. By the (45)–(47) it is immediate to see that $x_s(k)$ is in the observable but not controllable part of the extended system (36)-(37). Reasoning as in Lemma 3 and Lemma 5 we can conclude that $\mathbb{E}[x_s^{[i]}(k)]$ and the extended noise sequences are not stable for $i \neq 1$. The blocks of the state and measurement noise covariance matrices $(\Psi_w(k))_{r,s}$, $(\Psi_g(k))_{r,s}$ experience unbounded grow except for $r = 1$ or $s = 1$. As a consequence, $P_r(k)$ and $H_r(k)$ of (59)–(65) do not admit a steady-state value except for the first $n \times n$ block, the one relative to $x_s(k)$, for which the noise covariance matrices $FF^\top$ and $GG^\top$ do not depend on the state. Since the blocks of the polynomial part of the covariance matrix in (65) go to infinity, The extended system

$$w_i(k) = \sum_{j=0}^{i-1} \Theta^i_{i-j}(n) \left(W^{[i-j]} \otimes \tilde{A}^{[j]}\right) \left(\left(\omega^{[i-j]}(k) - \Psi_{\omega,i-j}\right) \otimes I_{n,j}\right) x_s^{[j]}(k) \tag{43}$$

$$g_i(k) = \sum_{j=0}^{i-1} \Theta^i_{i-j}(q) \left(G^{[i-j]} \otimes C^{[j]}\right) \left(\left(\omega^{[i-j]}(k) - \Psi_{\omega,i-j}\right) \otimes I_{n,j}\right) x_s^{[j]}(k) \tag{44}$$

$$\mathcal{A} = \begin{bmatrix} A & LC & \cdots & \cdots & \cdots & 0 \\ 0 & \tilde{A} & 0 & \cdots & \cdots & 0 \\ 0 & Q_{2,1} & \tilde{A}^{[2]} & 0 & \cdots & 0 \\ \vdots & \vdots & \ddots & \ddots & \ddots & \vdots \\ \vdots & \vdots & \ddots & \ddots & \ddots & 0 \\ 0 & Q_{\nu,1} & Q_{\nu,2} & \cdots & Q_{\nu,\nu-1} & \tilde{A}^{[\nu]} \end{bmatrix} \qquad \mathcal{A}_s = \begin{bmatrix} \tilde{A} & 0 & \cdots & \cdots & 0 \\ Q_{2,1} & \tilde{A}^{[2]} & 0 & \cdots & 0 \\ \vdots & \ddots & \ddots & \ddots & \vdots \\ \vdots & \ddots & \ddots & \ddots & 0 \\ Q_{\nu,1} & Q_{\nu,2} & \cdots & Q_{\nu,\nu-1} & \tilde{A}^{[\nu]} \end{bmatrix} \tag{45}$$

$$\mathcal{B} = \begin{bmatrix} B^\top & 0 & \cdots & 0 \end{bmatrix}^\top \tag{46}$$

$$\mathcal{C} = \begin{bmatrix} 0 & C & 0 & \cdots & \cdots & 0 \\ 0 & O_{2,1} & C^{[2]} & 0 & \cdots & 0 \\ \vdots & \vdots & \ddots & \ddots & \ddots & \vdots \\ \vdots & \vdots & \ddots & \ddots & \ddots & 0 \\ 0 & O_{\nu,1} & O_{\nu,2} & \cdots & O_{\nu,\nu-1} & C^{[\nu]} \end{bmatrix} \qquad \mathcal{C}_s = \begin{bmatrix} C & 0 & \cdots & \cdots & 0 \\ O_{2,1} & C^{[2]} & 0 & \cdots & 0 \\ \vdots & \ddots & \ddots & \ddots & \vdots \\ \vdots & \ddots & \ddots & \ddots & 0 \\ O_{\nu,1} & O_{\nu,2} & \cdots & O_{\nu,\nu-1} & C^{[\nu]} \end{bmatrix} \tag{47}$$

$$Q_{i,j} = \Theta^i_{i-j}(n) \left(W^{[i-j]} \otimes \tilde{A}^{[j]}\right) \left(\Psi_{\omega,i-j} \otimes I_{n,j}\right) \quad O_{i,j} = \Theta^i_{i-j}(q) \left(G^{[i-j]} \otimes C^{[j]}\right) \left(\Psi_{\omega,i-j} \otimes I_{n,j}\right). \tag{48}$$

Covariance matrices of the extended system

$$\left(\Psi_w(k)\right)_{r,s} = \sum_{i=0}^{r-1} \sum_{j=0}^{s-1} \Theta^r_{r-i}(n) \left(W^{[r-i]} \otimes \tilde{A}^{[i]}\right) P^{r,s}_{i,j}(k) \left(W^{[s-j]} \otimes \tilde{A}^{[j]}\right)^\top \left(\Theta^s_{s-j}(n)\right)^\top \tag{49}$$

$$\left(\Psi_g(k)\right)_{r,s} = \sum_{i=0}^{r-1} \sum_{j=0}^{s-1} \Theta^r_{r-i}(q) \left(G^{[r-i]} \otimes C^{[i]}\right) P^{r,s}_{i,j}(k) \left(G^{[s-j]} \otimes C^{[j]}\right)^\top \left(\Theta^s_{s-j}(q)\right)^\top \tag{50}$$

$$\left(\Upsilon(k)\right)_{r,s} = \sum_{i=0}^{r-1} \sum_{j=0}^{s-1} \Theta^r_{r-i}(n) \left(W^{[r-i]} \otimes \tilde{A}^{[i]}\right) P^{r,s}_{i,j}(k) \left(G^{[s-j]} \otimes C^{[j]}\right)^\top \left(\Theta^s_{s-j}(q)\right)^\top \tag{51}$$

$$P^{r,s}_{i,j}(k) = \mathrm{st}^{-1}\left\{ (I_{n,s-j} \otimes \Xi^\top_{n^{r-l},n^j} \otimes I_{n,i}) \big((\Psi_{\omega,s+r-i-j} - \Psi_{\omega,s-j} \otimes \Psi_{\omega,r-i}) \otimes \Xi_{1,q^j} \otimes I_{q,i} \big) \mathbb{E}[x_s^{[i+j]}(k)] \right\} \tag{52}$$

$H^{\mathrm{PP}}(k) \to H^{\mathrm{PP}}_\infty \equiv H^{\mathrm{LQG}}_\infty$. Figure 4 of Section V-B shows an experimental validation of this fact. It is worth noticing that the overall prediction error is still bounded and has a limit. In other words, the performance of the PP Regulator tends to the standard LQG Regulator although the covariance matrices (59)–(65) are not stable and cannot be implemented. Conversely, in the case of the FPP Regulator if $\sigma(\tilde{A}) \subset \mathbb{C}_\odot$ the sequences $\{w_e(k)\}$, $\{g_e(k)\}$, $\{x_s(k)\}$ become second-order convergent and thus the covariance matrices (59)–(65) remain bounded with a consequent improvement of the performance. As before, for a choice of L such that $\sigma(\tilde{A})$ is not in $\mathbb{C}_\odot$ $H^{\mathrm{FPP}}_L(k) \to H^{\mathrm{FPP}}_\infty(L) \equiv H^{\mathrm{LQG}}_\infty$.

B. Optimal choice of the output injection gain L

As we have shown in Section II-B the projection subspace $\overline{\mathcal{P}}^{k-1}_{y_s}(\nu, L)$ depends on the output injection gain matrix L, and the control $\hat{u}(k)$ defined in (53) depends on L through $\hat{x}(k|k-1)$ of the FPP. The control solution (53) is the same as it would be obtained in the absence of the additive noise sequences, *i.e.* the *certainty equivalence principle*. However, this time the resulting cost functional depends on the design parameter L.

Hence, in this sub-optimal setting, it is no longer true the property that *the optimal L with respect to the prediction error is also optimal with respect to the cost functional*. In fact, it is well known (see [22]) that, if the control law (4) is used then the steady state cost $J^{\mathrm{LQG}} = \lim_{\mathsf{N}\to\infty} J/\mathsf{N}$ of (3) for system (1)–(2) is

$$J^{\mathrm{LQG}} = \mathrm{tr}\{F P_{c,\infty} F^\top\} + \mathrm{tr}\{H^{\mathrm{LQG}}_\infty M^\top_\infty B^\top P_{c,\infty} A\} \tag{69}$$

where $P_{c,\infty}$ is the solution of the discrete time algebraic Riccati equation and M_∞ is the corresponding gain. The cost

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$$x_p(0) = \bar{x}_0, \qquad \widehat{X}_s(0|-1) = \mathrm{col}(\Psi_{x_0,1}, \Psi_{x_0,2}, \dots, \Psi_{x_0,\nu}) \tag{54}$$

$$H_r(0) = \begin{bmatrix} \mathrm{st}^{-1}\{\Psi_{x_0,2}\} & \mathrm{st}^{-1}\{\Psi_{x_0,3}\} & \cdots & \mathrm{st}^{-1}\{\Psi_{x_0,\nu+1}\} \\ \mathrm{st}^{-1}\{\Psi_{x_0,3}\}^\top & (P_r(0))_{2,2} & \cdots & (P_r(0))_{2,\nu} \\ \vdots & \vdots & \ddots & \vdots \\ \mathrm{st}^{-1}\{\Psi_{x_0,\nu+1}\}^\top & (P_r(0))_{\nu,2} & \cdots & (P_r(0))_{\nu,\nu} \end{bmatrix} \tag{55}$$

$$(H_r(0))_{i,j} = (P_r(0))_{j,i} = \mathrm{st}^{-1}\{\Psi_{x_0,i+j}\} - \Psi_{x_0,i}\Psi_{x_0,j}^\top \qquad i,j = 2,\dots,\nu \tag{56}$$

$$\hat{x}(k|k-1) = x_p(k) + \hat{x}_s(k|k-1), \quad \hat{x}_s(k|k-1) = \widehat{X}_s(k|k-1)_{1:n} \tag{57}$$

$$K_r(k) = H_r(k)\mathcal{C}_s^\top \left(\mathcal{C}_s H_r(k)\mathcal{C}_s^\top + \Psi_g(k)\right)^{-1} \tag{58}$$

$$P_r(k) = H_r(k) - K_r(k)\mathcal{C}_s H_r(k) \tag{59}$$

$$\widehat{X}_s(k) = \widehat{X}_s(k|k-1) + K_r(k)\left(Y_e(k) - \mathcal{C}_s\widehat{X}_s(k|k-1)\right) \tag{60}$$

$$\hat{x}(k) = x_p(k) + \hat{x}_s(k), \quad \hat{x}_s(k) = \widehat{X}_s(k)_{1:n} \tag{61}$$

$$x_p(k+1) = Ax_p(k) + B\hat{u}(k) + Ly_s(k) \tag{62}$$

$$\Gamma(k) = \Upsilon(k)\left(\mathcal{C}_s H_r(k)\mathcal{C}_s^\top + \Psi_g(k)\right)^{-1} \tag{63}$$

$$\widehat{X}_s(k+1|k) = \left(\mathcal{A}_s - \left(\mathcal{A}_s K_r(k) + \Gamma(k)\right)\mathcal{C}_s\right)\widehat{X}_s(k|k-1) + \left(\mathcal{A}_s K_r(k) + \Gamma(k)\right)Y_e(k) + d_s \tag{64}$$

$$H_r(k+1) = \mathcal{A}_s P_r(k)\mathcal{A}_s^\top + \Psi_w(k) - \Gamma(k)\Upsilon^\top(k) - \mathcal{A}_s K_r(k)\Upsilon^\top(k) - \Upsilon(k)K_r^\top(k)\mathcal{A}_s^\top \tag{65}$$

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(69) is the optimal one when the noise sequences are Gaussian. In the non-Gaussian case and for a given output injection gain L, we have for the FPP regulator

$$J^{\mathrm{FPP}}(L) = \mathrm{tr}\{FP_{c,\infty}F^\top\} + \mathrm{tr}\{H_\infty^{\mathrm{FPP}}(L)M_\infty^\top B^\top P_{c,\infty}A\}. \tag{70}$$

Thus, it is reasonable to find the optimal gain L_c^* as

$$L_c^* = \underset{\substack{\lambda\in\tilde{\sigma}(\tilde{A}) \\ |\lambda|\le 1}}{\arg\min}\ \mathrm{tr}\left\{H_\infty^{\mathrm{FPP}}(L)M_\infty^\top B^\top P_{c,\infty}A\right\}. \tag{71}$$

On the other hand, the optimal gain L_p^* from the point of view of the variance of the prediction error is

$$L_p^* = \underset{\substack{\lambda\in\tilde{\sigma}(\tilde{A}) \\ |\lambda|\le 1}}{\arg\min}\ \mathrm{tr}\left\{H_\infty^{\mathrm{FPP}}(L)\right\}. \tag{72}$$

We see that in general $L_c^* \neq L_p^*$, since $\mathrm{tr}\{H_\infty^{\mathrm{FPP}}(L_1)\} < \mathrm{tr}\{H_\infty^{\mathrm{FPP}}(L_2)\}$ does not imply that $H_\infty^{\mathrm{FPP}}(L_1) - H_\infty^{\mathrm{FPP}}(L_2)$ is positive definite. Note that L_p^* minimizes also $\mathrm{tr}\{P_\infty^{\mathrm{FPP}}(L)\}$, *i.e.* the limit for $k\to\infty$ of the trace of the first $n\times n$ block of $P_r(k)$ of (59).

The computation of (71) and (72) is a constrained polynomial optimization problem that in general may be hard to solve. However, the computation needs to be done just *once* and *off-line*. In the numerical examples of Section V we use standard tools of Matlab® to solve that problem without difficulty. We remark that when the optimization problems (71) and (72) are really intractable due to the high dimension of the system and of the output, one can resort to a suboptimal choice or even choose any L that makes $A-LC$ Schur stable. This implies giving up optimality in $\overline{\mathcal{P}}_{y_s}^k(\nu)$, but in any case it yields better performance than existing methods (see Section V).

C. FPP Regulator vs LQG Regulator

Theorem 1 shows that the structure of the gain of the control law (53) is the same as the standard LQG Regulator. In essence, the difference stems uniquely in the computation of the prediction. We remind that when the noise sequences are non-Gaussian, the cost (69) is not optimal.

Proposition 1: Consider the system (1)-(2) with the cost functional (3). For any L consider the cost $J^{\mathrm{FPP}}(L)$ in (70) obtained by using the FPP Regulator, *i.e.* the control law (53), and let J^{LQG} be the cost in (69) obtained by using the standard LQG Regulator, *i.e.* the control law (4). Then

$$J^{\mathrm{FPP}}(L) \le J^{\mathrm{LQG}} \tag{73}$$

Proof: From equations (69), (70) we can write the difference $\Delta J = J^{\mathrm{LQG}} - J^{\mathrm{FPP}}(L)$ as

$$\Delta J = \mathrm{tr}\left\{\left(H_\infty^{\mathrm{LQG}} - H_\infty^{\mathrm{FPP}}(L)\right)M_\infty^\top B^\top P_{c,\infty}A\right\}.$$

Noting that the control gain (5) can be written as

$$M_\infty = \left(B^\top P_{c,\infty}B + R\right)^{-1}B^\top P_{c,\infty}A$$

it is clear that the matrix $M_\infty^\top B^\top P_{c,\infty}A$ is positive semidefinite. By Lemma 1 the matrix $H_\infty^{\mathrm{LQG}} - H_\infty^{\mathrm{FPP}}(L)$ is positive semidefinite too. Hence by inequality (20) it follows that $\Delta J \ge 0$ and the proof is complete. ■

Note that if L is such that $\tilde{A} = A - LC$ is not Schur stable, then $J^{\mathrm{FPP}}(L) = J^{\mathrm{LQG}}$ and the predictor (54)–(65) is not implementable in the long run. Thus, it is enough to set L such that $\sigma(\tilde{A}) \subset \mathbb{C}_\odot$ to obtain, in general, the inequality (73).

D. FPP Regulator vs PP Regulator

We recall that, for any $k \geq 0$, we have $H_\infty^{\text{PP}} = H_\infty^{\text{FPP}}(0)$. Defining J^{PP} as in (70), we can state the following

Proposition 2: Consider the system (1)-(2) with the cost functional (3). The cost $J^{\text{FPP}}(L_c^*)$ obtained by using the control law (53) with the output injection gain L_c^* defined in (71) is smaller or equal than the cost J^{PP} obtained by using the control law (53) with $L = 0$, namely

$$J^{\text{FPP}}(L_c^*) \leq J^{\text{PP}}. \tag{74}$$

Moreover, if the matrix A is not Schur stable, then for any value of L such that $\sigma(A - LC) \subset \mathbb{C}_\odot$, we have

$$J^{\text{FPP}}(L) \leq J^{\text{PP}} = J^{\text{LQG}}. \tag{75}$$

The proof descends immediately from the definition of L_c^* in (71) and from Proposition 1.

E. Extension to non-stationary noise sequences

A straightforward extension of this method is the case of non-stationary noise sequences. Let us consider the system

$$x(k+1) = Ax(k) + Bu(k) + F_k\omega(k), \tag{76}$$
$$y(k) = Cx(k) + G_k\omega(k), \tag{77}$$

with $x(0) = x_0$ and the same cost functional (3). In (76)-(77) the matrices F_k and G_k are time-varying and uniformly bounded, *i.e.* $\sup_k ||F_k|| + ||G_k|| < \infty$. Following the same line of Section III, we can split the state into two parts, namely

$$\begin{aligned} x_p(k+1) &= Ax_p(k) + Bu(k) + \\ &\qquad + LCx_s(k) + LG_k\omega(k), \end{aligned} \tag{78}$$
$$x_s(k+1) = \tilde{A}x_s(k) + W_k\omega(k), \tag{79}$$
$$y_s(k) = Cx_s(k) + G_k\omega(k) \tag{80}$$

where $W_k \doteq F_k - LG_k$. Consequently, we can state the following Definition and Lemmas.

Definition 2: A discrete-time stochastic process ξ taking values in $\mathbb{R}^n$ is said to be a *second-order asymptotically bounded process* if there exist two constant vectors $m_1 \in \mathbb{R}^n$ and $m_2 \in \mathbb{R}^{n^2}$ such that $\limsup_{k\to\infty} \mathbb{E}[\xi(k)] = m_1$ and $\limsup_{k\to\infty} \mathbb{E}[\xi^{[2]}(k)] = m_2$.

Lemma 6: If the output injection gain matrix L is such that $\sigma(\tilde{A}) = \sigma(A - LC) \subset \mathbb{C}_\odot$ and conditions (i)–(iii) in Section II hold true then for $1 \leq i \leq \nu$ the processes $x_s^{[i]}$, where x_s is defined in (31), are second-order asymptotically bounded.

Lemma 7: If the output injection gain matrix L is such that $\sigma(\tilde{A}) = \sigma(A - LC) \subset \mathbb{C}_\odot$ and conditions (i)–(iii) in Section II hold true, then the noise sequences $\{w_e(k)\}$ and $\{g_e(k)\}$ are second-order asymptotically bounded.

The proofs of Lemma 6 and 7 follow the same lines of Lemma 3 and Lemma 5. As a consequence, Theorem 1 continue to hold with the only exception of considering at each step k the time-varying matrices F_k and G_k instead of F and G in the extend system expressions (43)–(48), in the covariance matrices of the extended system (49)–(52) and in the recursive polynomial filter and predictor (54)–(65). For, all the considerations of Section IV-A, IV-B, IV-C, IV-D continue to hold in the case of uniformly bounded non-stationary noise sequences. We shall take advantage of this extension in the Example V-D.

From the results of this section it descends that in the case of non-stationary and non-Gaussian noise sequences the FPP Regulator yields better performance than the LQG regulator.

Finally, in the case of time-varying system, *i.e.* when also the matrices A, B and C depend on time, the main difference is that $L(k)$ needs to uniformly stabilize $A(k) - L(k)C(k)$. Note that the time dependence of the matrix B does not add any difficulties to the control problem.

F. Observability and implementation issues

So far we have not explicitly mentioned the observability properties of the extended system (36)–(37). Theorem 1 requires (A, C) to be detectable. However, detectability of $(\mathcal{A}, \mathcal{C})$ is needed in order for the FPF and FPP to be internally stable. To this aim, in the first place we recall that detectability of (A, C) is equivalent to detectability of $(\tilde{A}, C) = (A - LC, C)$, see [28] (pag. 79, Section 3.10 as a dual result of Lemma 2.1). In the second place, since $x_p(k)$ is known we can restrict the detectability analysis to the pair $(\mathcal{A}_s, \mathcal{C}_s)$ of the stochastic part (see (45)–(47)). Since the eigenvalues of Kronecker powers of a matrix are the product of eigenvalues of the base matrix, if the output injection gain L is such that $\sigma(\tilde{A}) \subset \mathbb{C}_\odot$ all the $\tilde{A}^{[i]}$ are Schur stable, hence $(\mathcal{A}_s, \mathcal{C}_s)$ is detectable because $\mathcal{A}_s$ is Schur stable.

Notice that, due to the redundancies of Kronecker powers, $(\mathcal{A}_s, \mathcal{C}_s)$ is never observable. The noise covariance matrices $\Psi_w(k)$, $\Psi_g(k)$ and $\Upsilon(k)$ of the extended system as well as $P_r(k)$, $H_r(k)$ are singular, and the computation of (54)–(65) requires the use of the pseudo-inverse. The computation of redundant terms can be avoided by using tools such as *reduced Kronecker algebra* ([5] ch 12, [10]). In this case an advantage is that the resulting FPP has smaller size. However, such a reduction is not necessary in principle since the extended system is inherently stable when L is designed to make $\tilde{A}$ Schur stable. On the contrary, the PP of [19] can be implemented in the long run only when $\sigma(A) \subset \mathbb{C}_\odot$, since the extended system is not observable and the unobservable part is not stable when A is not Schur stable.

V. Numerical examples

We test in this section the effectiveness of the proposed approach with four numerical examples. In the examples of Section V-A, V-B, V-C the time-horizon is $\mathsf{N} = 10^4$ and we compute $\mathsf{N}_r = 50$ realizations, while in the example of Section V-D the time-horizon is $\mathsf{N} = 10^3$ and $\mathsf{N}_r = 100$. The cost functionals are characterized by $S = Q = I_n$, and $R = I_p$. We consider polynomial filters and polynomial regulators of order $\nu = 2, 3$, *i.e.* the Feedback Quadratic Filter (FQF), Feedback Quadratic Predictor (FQP) Regulator, the Feedback Cubic Filter (FCF) and the Feedback Cubic Predictor (FCP) Regulator. When possible we compare them with the Quadratic and Cubic Filters (QF, CF) and Predictor Regulators (QP and CP) of [19].

state noise			output noise		
$f_1(k)$	1/25	-4/25	$g(k)$	1	-5/20
$\mathbb{P}[f_1(k)]$	0.8	0.2	$\mathbb{P}[g(k)]$	0.2	0.8
$f_2(k)$	-1/100	9/100			
$\mathbb{P}[f_2(k)]$	0.9	0.1			

TABLE I
EXAMPLE 1. PROBABILITY MASS FUNCTIONS OF THE NOISE SEQUENCES.

Filter/ Regulator	$\mathrm{tr}\{P_\infty\}$	MSE	α_{MSE}	J	J_{emp}	α_J
KF/ LGQ R.	0.0296	0.0300	-	0.0432	0.0436	-
QF/ QP R.	0.0190	0.0195	35.8%	0.0365	0.0370	15.4%
CF/ CP R.	0.0152	0.0156	48.7%	0.0343	0.0348	20.5%
FQF/ FQP R.	0.0120	0.0123	59.5%	0.0321	0.0326	25.6%
FCF/ FCP R.	0.0057	0.0061	80.7%	0.0285	0.0291	34.0%

TABLE II
EXAMPLE 1. A PRIORI AND ACTUAL MSE, PERCENTAGE REDUCTION OF THE ACTUAL MSE, A PRIORI AND ACTUAL VALUE OF THE COST FUNCTIONAL AND PERCENTAGE REDUCTION OF THE ACTUAL COST.

The performance of the algorithms is reported in terms of the a priori and actual state filtering errors and in terms of the a priori and actual cost functional, which is related to the state prediction error. The steady-state covariance of the estimation error is denoted with P_∞ and, for polynomial filters, it corresponds to the limit as $k \to \infty$ of the first block of the matrix $P_r(k)$ of (59). We denote J^{P} and $J^{\text{P}}_{\text{emp}}$, where P is one of LQG, QP, FQP, CP, FCP, the a priori and actual cost functional calculated as in (69) and (3), respectively. In particular, the actual cost $J^{\text{P}}_{\text{emp}}$ is obtained by replacing in (3) the expected valued with the average across realizations when the P Regulator is used to control the system. If the variables $x_i(j)$ and $\hat{x}_i(j)$ are the state and the estimated state of the i-th realization at time j. MSE^{F}, where F is one of KF, QF, FQF, CF, FCF, is the MSE when the filter F is used to compute $\hat{x}_i(j)$, defined as usual as

$$\text{MSE} = \frac{1}{\mathsf{N}_r}\frac{1}{\mathsf{N}}\sum_{i=0}^{\mathsf{N}_r}\sum_{j=0}^{\mathsf{N}}\|x_i(j) - \hat{x}_i(j)\|^2. \tag{81}$$

The performance index is defined as the percentage improvement with respect to the Kalman Filter for the filtering analysis and with respect to the LQG Regulator for the control analysis,

$$\alpha_{\text{MSE}} = 10^2 \cdot (\text{MSE}^{\text{KF}} - \text{MSE}^{\text{F}})/\text{MSE}^{\text{KF}} \tag{82}$$

$$\alpha_J = 10^2 \cdot (J^{\text{LQG}}_{\text{emp}} - J^{\text{P}}_{\text{emp}})/J^{\text{LQG}}_{\text{emp}}. \tag{83}$$

A. Example 1: Stable Dynamics

In order to compare the different regulators, we consider in this first example a stable system $n = 2$ characterized by

$$A = \begin{bmatrix} 0.85 & 0.25 \\ 0.2 & 0.5 \end{bmatrix}, \quad B = \begin{bmatrix} 0 \\ 1 \end{bmatrix}, \quad C = \begin{bmatrix} 1 & 0 \end{bmatrix}$$

and the state and output noise sequences

$$\begin{aligned} F\omega(k) &= \mathrm{col}\left(f_1(k), f_2(k)\right) \in \mathbb{R}^2, \\ G\omega(k) &= g(k) \in \mathbb{R}. \end{aligned} \tag{84}$$

The components of $F\omega(k)$ and $G\omega(k)$ are i.i.d. sequences with probability mass functions shown in Table I and the initial condition is $x_0 = \mathcal{N}(0, 0.6I)$.

Since the spectrum of the dynamic matrix A is $\sigma(A) = \{0.9589, 0.3911\}$ it is possible to implement the QF and the CF of [19], which correspond to the FQF and FCF when the output injection is set to $L = 0$, and control the system by using the corresponding QP and CP Regulators. We compare the QF and CF with FQF and FCF where the optimal output injection gain L is chosen by finding the numerical solution of (72). This value of L_p^* is such that $\sigma(\tilde{A}) = \{0.3526, 0.8519\}$ for the FQF and $\sigma(\tilde{A}) = \{0.3470, 0.8584\}$ for the FCF.

state noise			output noise		
$f_1(k)$	1/25	-4/25	$g(k)$	1	-5/20
$\mathbb{P}[f_1(k)]$	0.8	0.2	$\mathbb{P}[g(k)]$	0.2	0.8
$f_2(k)$	-1/100	9/100			
$\mathbb{P}[f_2(k)]$	0.9	0.1			

TABLE III
EXAMPLE 2. PROBABILITY MASS FUNCTIONS OF THE NOISE SEQUENCES.

Moreover, to compare the control performance we compute the optimal output injection gain L_c^* for the FQP Regulator according to (71) and we find that $\sigma(\tilde{A}) = \{0.3524, 0.8500\}$ whereas for the FCP Regulator the optimal L_c^* is such that $\sigma(\tilde{A}) = \{0.3429, 0.8555\}$. In this example the pair (A, C) is observable, thus the minimization over the whole spectrum of $\tilde{A}$ makes sense.

Figure 2 (left) plots the a priori MSE, *i.e.* $\mathrm{tr}\{P_\infty\}$, of the KF, QF, FQF. The marked lines refer to the related actual MSE. Figure 2 (right) plots the a priori and actual MSE of the KF, CF, FCF. Figure 3 shows the corresponding a priori and actual cost functionals J^{P} and $J^{\text{P}}_{\text{emp}}$ (marked lines) for the quadratic and cubic filters.

We summarize the results in Table II. These results show a good agreement between a priori and actual values as well as a staggering 80.7% reduction of the MSE with respect to the KF for the FCF and a 34% percentage reduction of the cost functional of the FCP Regulator with respect to the LQG Regulator. This example shows that, even in the case of a stable system, the choice of the output injection gain L dramatically improves the performance.

B. Example 2: Unstable Dynamics

In this example we consider an unstable system characterized by

$$A = \begin{bmatrix} 0.9 & 0.25 \\ 0.2 & 1 \end{bmatrix}, \quad B = \begin{bmatrix} 0 \\ 1 \end{bmatrix}, \quad C = \begin{bmatrix} 1 & 0 \end{bmatrix} \tag{85}$$

and the state and output noise sequences as in (84) with probability mass functions shown in Table III.

Since the spectrum of the dynamic matrix A is $\sigma(A) = \{1.1791, 0.7209\}$ we cannot implement the QF and the CF or the corresponding QP and the CP regulators of [19] in the long run, since the covariance of the estimation and prediction error tends to the one of the Kalman Filter while the polynomial part of the covariance of the estimation and prediction error tends to infinity. Hence, we control the system with the proposed regulators of order $\nu = 2$, *i.e.* the FQP Regulator, and $\nu = 3$,

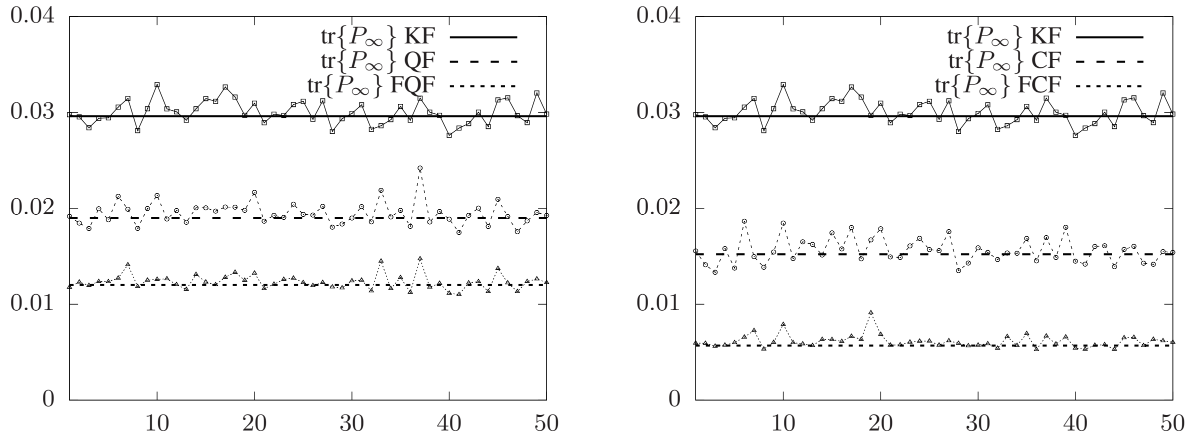

Fig. 2. Example 1. Trace of the steady-state covariance matrix of the estimation error $\mathrm{tr}\{P_\infty\}$ and the corresponding MSE across realizations for the quadratic case(left) and cubic case (right).

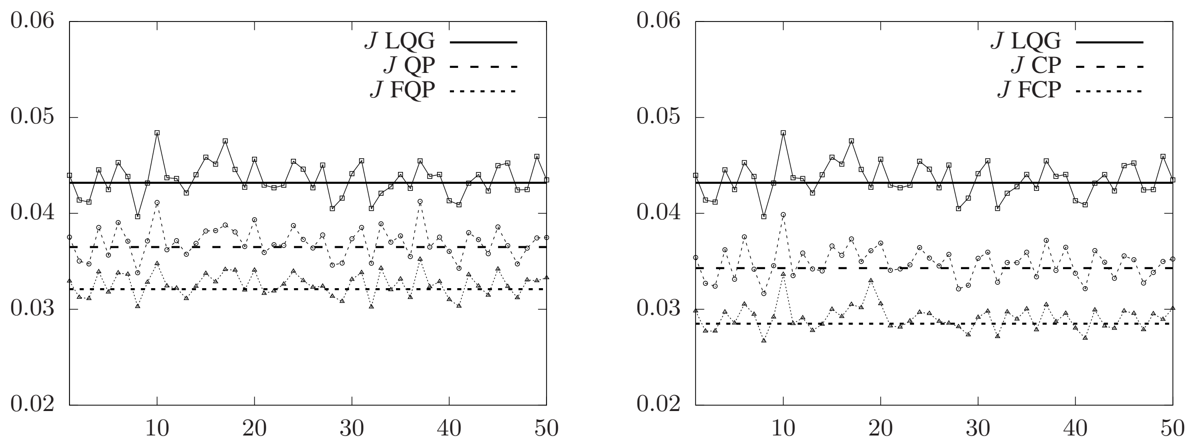

Fig. 3. Example 1. A priori and actual cost functionals of the controlled dynamics across realizations for the quadratic case(left) and cubic case (right).

i.e. the FCP Regulator. The optimal output injection gain L_c^* sets the eigenvalues $\sigma(\tilde{A}) = \{0.6371, 0.8079\}$.

Figure 4 shows the expected behavior of the trace of the estimation error covariance matrix, with and without output injection. In particular, as time goes by, the trace of the covariance matrices of the estimation error $\mathrm{tr}\{P_\infty\}$ of the QF and CF tends to the steady-state trace of the estimation error covariance matrix of the KF. The same holds for the covariance of the prediction error $\mathrm{tr}\{H_\infty\}$ (not shown). This causes a performance degradation of the QP and CP Regulators. Conversely, by using the output injection, the performance are definitively improved as highlighted by the percentage reductions in Table IV. We have a percentage reduction of the MSE of 81% with respect to the KF for the FCF and a percetage reduction of the cost functional of 80% with respect to the LQG Regulator for the FCP Regulator.

C. Example 3: Dependence on L

For a scalar system with $A = B = 1$, $C = 5 \cdot 10^{-3}$ and $x_0 \sim \mathcal{N}(0, 0.5)$ and non-Gaussian noise sequences with probability mass functions shown in Table V we highlight the dependence $L \mapsto J(L)$ of the cost functional for the filter and regulator of order $\nu = 2, 3$. Note that in the scalar case both $\mathrm{tr}\{P_\infty(L)\}$ and $J(L)$ attain the minimum for the same L since the minimizations (71) and (72) are equivalent. In that case the optimum output injection is $L_c^* = L_p^* \doteq L^* = 0.9082$ for the FQF and FQP Regulator and we have $L_c^* = L_p^* \doteq L^* = 0.9067$ for the FCF and FCP Regulator.

Filter/ Regulator	$\mathrm{tr}\{P_\infty\}$	MSE	α_{MSE}	J	J_{emp}	α_J
KF/ LGQ R.	0.1741	0.1737	-	0.6446	0.6429	-
FQF/ FQP R.	0.0445	0.0449	74%	0.1712	0.1726	73%
FCF/ FCP R.	0.0325	0.0328	81%	0.1287	0.1300	80%

TABLE IV
EXAMPLE 2. A PRIORI AND ACTUAL MSE, PERCENTAGE REDUCTION OF THE ACTUAL MSE, A PRIORI AND ACTUAL VALUE OF THE COST FUNCTIONAL AND PERCENTAGE REDUCTION OF THE ACTUAL COST.

In Figure 5 we compare the value J^{LQG} of the LQG Regulator (which is insensitive to the output injection) with the values of J^{FQP} and J^{FCP} of the FQP and FCP Regulator. Table VI summarizes the results. For $L = L^*$, we note in particular the remarkable 93% percentage reduction of the a priori $\mathrm{tr}\{P_\infty(L)\}$ of FCF with respect to the KF, and the

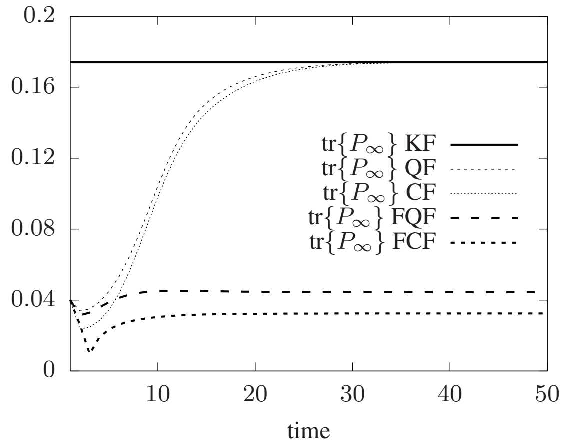

Fig. 4. Example 2. Trace of the covariance of the estimation error without output injection, *i.e.* $\mathrm{tr}\{P_\infty\}$ QF/CF, and with output injection, *i.e.* $\mathrm{tr}\{P_\infty\}$ FQF/FCF.

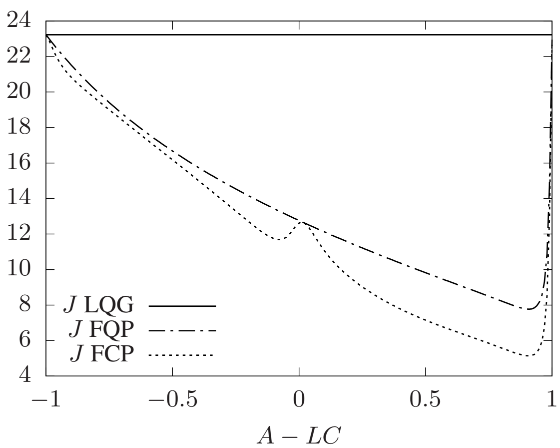

Fig. 5. Example 3. Dependence on the output injection gain L of the cost functional J for the LQG, FQP and FCP Regulators.

77.9% percentage reduction of the a priori cost functional J of the FCP Regulator with respect to the LQG Regulator.

D. Example 4: Planar positioning

In this section we consider an application of the FPP Regulator beyond the class of linear systems. A planar positioning is the control of an actuated object that must be moved at some point in the plane. The system is described by a continuous-time state equation and discrete-time measurement equations of distance and angular position. After the discretization step of the state equation and assuming that the input is a feedback from the output, constant in the discretization interval τ,

$$x(k+1) = Ax(k) + Bu(k) + f_k, \tag{86}$$

$$z(k) = \begin{bmatrix} z_1(k) \\ z_2(k) \end{bmatrix} = \begin{bmatrix} \rho(k) \\ \theta(k) \end{bmatrix} + \begin{bmatrix} \epsilon_\rho(k) \\ \epsilon_\theta(k) \end{bmatrix} \tag{87}$$

state noise			output noise		
$F\omega(k)$	2/50	-18/50	$G\omega(k)$	4	-60/85
$\mathbb{P}[F\omega(k)]$	0.9	0.1	$\mathbb{P}[G\omega(k)]$	0.15	0.85

TABLE V
EXAMPLE 3. PROBABILITY MASS FUNCTIONS OF THE NOISE SEQUENCES.

Predictor	$\mathrm{tr}\{P_\infty\}$	$\%\,\mathrm{tr}\{P_\infty\}$	J	$\%J$
KF	19.46	-	23.22	-
FQF	4.00	79.5%	7.77	66.5%
FCF	1.37	93.0%	5.14	77.9%

TABLE VI
EXAMPLE 3. A PRIORI ESTIMATION ERRORS, A PRIORI PERCENTAGE REDUCTION OF THE ESTIMATION ERRORS, A PRIORI VALUES OF THE COST FUNCTIONALS AND A PRIORI PERCENTAGE REDUCTIONS OF THE COST FUNCTIONALS.

where $\rho(k)$ and $\theta(k)$ are the distance and angular position at time k, namely

$$\rho(k) = \sqrt{x_1^2(k\tau) + x_3^2(k\tau)}, \tag{88}$$

$$\theta(k) = \arctan 2\left(\frac{x_3(k\tau))}{x_1(k\tau)}\right) \tag{89}$$

with $x(k) = \mathrm{col}\,(x_1(k), x_3(k)), x_3(k)), x_4(k))$. Since $(x_1(k),\, x_3(k))$ is the position in the plane and $(x_2(k),\, x_4(k))$ represents the velocity vector, the system dynamics is characterized by

$$A = \begin{bmatrix} 1 & \tau & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & \tau \\ 0 & 0 & 0 & 1 \end{bmatrix}, \quad B = \begin{bmatrix} \tau^2/2 & 0 \\ \tau & 0 \\ 0 & \tau^2/2 \\ 0 & \tau \end{bmatrix}. \tag{90}$$

We assume $x(0) = \mathrm{col}(10, 1, 10, 1)$. The system and the sensors are affected by Gaussian i.i.d. sequences, *i.e.* $f_k \sim \mathcal{N}(0, \sigma_f^2 I_4)$, $\epsilon_\rho(k) \sim \mathcal{N}(0, \sigma_\rho^2)$ and $\epsilon_\rho(k) \sim \mathcal{N}(0, \sigma_\theta^2)$ for each k, where $\sigma_f^2 = 5 \cdot 10^{-2}$, $\sigma_\rho^2 = 4$ (error amplitude of 2 units on the distance) and three values for σ_θ^2, namely 10^{-2}, $2 \cdot 10^{-2}$ and $4 \cdot 10^{-2}$. As usual, the sequences $\{f_k\}$, $\{\epsilon_\rho(k)\}$ and $\{\epsilon_\theta(k)\}$ are assumed mutually independent.

Omitting the time dependence for brevity, we exploit the relations $x_1 = \rho\cos(\theta)$ and $x_3 = \rho\sin(\theta)$ obtaining

$$x_1 = (z_1 - \epsilon_\rho)\cos(z_2 - \epsilon_\theta), \tag{91}$$

$$x_3 = (z_1 - \epsilon_\rho)\sin(z_2 - \epsilon_\theta). \tag{92}$$

We define

$$y_1 = z_1 \mathbb{E}[\cos(z_2 - \epsilon_\theta)], \tag{93}$$

$$y_2 = z_1 \mathbb{E}[\sin(z_2 - \epsilon_\theta)]. \tag{94}$$

Since the vector $z = \mathrm{col}(z_1, z_2)$ is measured at each k, we can compute y_1 and y_2 at each time by using the moments of ϵ_θ and the actual measurements z. Equations (91)–(94) imply

$$\begin{aligned} y_1 = x_1 + z_1\big(\mathbb{E}[\cos(z_2 - \epsilon_\theta)] - \cos(z_2 - \epsilon_\theta)\big) + \\ + \epsilon_\rho \cos(z_2 - \epsilon_\theta) = x_1 + g_1, \end{aligned} \tag{95}$$

$$\begin{aligned} y_2 = x_3 + z_1\big(\mathbb{E}[\sin(z_2 - \epsilon_\theta)] - \sin(z_2 - \epsilon_\theta)\big) + \\ + \epsilon_\rho \sin(z_2 - \epsilon_\theta) = x_3 + g_2, \end{aligned} \tag{96}$$

where

$$g_1 = z_1\left(\mathbb{E}[\cos(z_2-\epsilon_\theta)] - \cos(z_2-\epsilon_\theta)\right) + \\ + \epsilon_\rho \cos(z_2-\epsilon_\theta), \tag{97}$$

$$g_2 = z_1\left(\mathbb{E}[\sin(z_2-\epsilon_\theta)] - \sin(z_2-\epsilon_\theta)\right) + \\ + \epsilon_\rho \sin(z_2-\epsilon_\theta). \tag{98}$$

Summarizing, system (86)-(87) can be rewritten with the new measurements y_1 and y_2 as

$$x(k+1) = Ax(k) + Bu(k) + f_k, \tag{99}$$

$$y(k) = Cx(k) + g_k \tag{100}$$

where

$$C = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{bmatrix}.$$

Note that the system is marginally stable and it is in the form (1)-(2) with $\{g_k\}$ a zero-mean non-Gaussian non-stationary noise sequence and $\{f_k\}$ a Gaussian noise sequence, and hypotheses (i)-(ii)-(iii) of Section II are satisfied. We call upon Section IV-E for details to the case of non-stationary noise sequences.

We apply the proposed FQP Regulator to this problem and we compare it with the QP Regulator and the standard LGQ Regulator. For, at each k, we need the moments of g_k up to the fourth order that can be obtained with tedious but simple computations as in [14], where a planar tracking problem has been discussed. We require the moving object to reach the target final point on the plane $(-100, 100)$ with null final velocity. Thus, the target state is $x^\circ = \mathrm{col}(-100, 0, 100, 0)$. Consequently, we need to slightly modify the cost index J as

$$J = \frac{1}{2}\mathbb{E}\Big[(x(\mathsf{N})-x^\circ)^\top S(x(\mathsf{N})-x^\circ) + \\ + \sum_{k=0}^{\mathsf{N}-1}(x(k)-x^\circ)^\top Q(x(k)-x^\circ) + u^\top(k)Ru(k)\Big] \tag{101}$$

which yields with easy passages to the control law

$$\hat{u}(k) = -M(k)\hat{x}(k|k-1) + \pi(k)x^\circ \tag{102}$$

where $M(k)$ is the usual gain of (5), $\pi(k) = R^{-1}B^\top P_c(k+1)(I + BR^{-1}B^\top P_c(k+1))^{-1}$ and $\hat{x}(k|k-1)$ is the prediction given by the FQP of the algorithm (54)–(65).

Since the covariance of the transformed output noises is time-varying and it depends on the realization, the system does not have a steady state and it is not possible to find the output injection gain L by solving (71). Instead, a suboptimal L is chosen such that $\sigma(\tilde{A}) = \{0.880, 0.902, 0.924, 0.946\}$.

Table VII summarizes the results for $\tau = 1$ and three values of the variance $\sigma^2_{\epsilon_\theta}$. In particular, in the case of $\sigma^2_{\epsilon_\theta} = 4\cdot 10^{-2}$ we have a reduction of 13.18% of the MSE with respect to the KF for the FQF and a 11.35% percentage reduction of the cost functional of the FQP Regulator with respect to the LQG Regulator. Since the system is mariginally stable, in this case the QF/QP algorithms of [19] show no improvement.

We can draw two conclusions from this example. The use of the output injection gain L is essential to improve the performance of the control even for marginally stable systems. Moreover, the FPP-based regulator can be successfully applied to a nonlinear system with Gaussian noises that can be transformed into a linear one with non-Gaussian noises. Finally, we remark that for any choice of $\tau \geq 0.2$ a standard Extended Kalman Filter for the original system (86)-(87) exhibits a large frequency of diverging estimates and thus, it is not suitable for control purposes.

$\sigma^2_{\epsilon_\theta} = 10^{-2}$	MSE	α_{MSE}	J_{emp}	α_J
KF/ LGQ R.	39.95	-	$1.22\cdot 10^5$	-
QF/ QP R.	39.96	-0.02%	$1.22\cdot 10^5$	0.00%
FQF/ FQP R.	39.67	0.7%	$1.22\cdot 10^5$	1.03%
$\sigma^2_{\epsilon_\theta} = 2\cdot 10^{-2}$	MSE	α_{MSE}	J_{emp}	α_J
KF/ LGQ R.	72.95	-	$1.70\cdot 10^5$	-
QF/ QP R.	73.07	-0.17%	$1.70\cdot 10^5$	0.00%
FQF/ FQP R.	69.55	4.65%	$1.62\cdot 10^5$	4.55%
$\sigma^2_{\epsilon_\theta} = 4\cdot 10^{-2}$	MSE	α_{MSE}	J_{emp}	α_J
KF/ LGQ R.	130.42	-	$2.49\cdot 10^5$	-
QF/ QP R.	130.83	-0.32%	$2.49\cdot 10^5$	0.00%
FQF/ FQP R.	113.23	13.18%	$2.21\cdot 10^5$	11.35%

TABLE VII
EXAMPLE 4. ACTUAL MSE AND PERCENTAGE REDUCTION WITH RESPECT TO THE KF, MEAN VALUE OF THE COST FUNCTIONAL J AND PERCENTAGE REDUCTION WITH RESPECT TO THE LQG REGULATOR FOR QF/QP AND FQF/FQP.

VI. CONCLUSIONS

This paper describes the recursive optimal polynomial regulator for discrete-time linear systems with additive non-Gaussian noises. The existing polynomial regulator has been extended to non asymptotically stable systems by introducing an output injection term. It has been proven that the optimal polynomial regulator can still be obtained by replacing the Kalman Predictor with the Feedback Polynomial Predictor and by using the same feedback control law as in the linear optimal regulator.

An important property of recursive polynomial filters is that the projection space depends on the output gain parameter L. A somewhat surprising conceptual consequence is that the optimal LQ regulator is not designed from the optimal filter (in the minimum variance sense), in contrast with the intuition and the usual design choice. Even if in most practical cases this difference is very small, the notion of "optimality" for recursive polynomial filters turns out to be more complex than initially thought.

Numerical simulations show that when noise moments are known, the use of polynomial filters on the system, rewritten by means of an output injection term, may deliver a very meaningful improvement of the performance for both the estimation and optimal control problems also in the case of systems that are not asymptotically stable.

The approach based on polynomial filtering and output injection can be extended to systems for which a polynomial filter can be designed, such as discrete and continuous-time systems with multiplicative state noise [11], [12], systems with intermittent observations [7] and systems with random matrices [8].

Further points deserve to be investigated, for example the extension of the output injection approach to time-varying systems and the analytical expression of the optimal value for the output injection gain L.

Appendix A

Proof of Lemma 1: Let us denote with $\widetilde{e} = v_{\mathcal{M}} - v_{\mathcal{K}}$ and write the covariance matrix of $e_{\mathcal{K}}$ as

$$\begin{aligned}\Psi_{\mathcal{K}} &= \mathbb{E}\left[e_{\mathcal{K}} e_{\mathcal{K}}^\top\right] \\ &= \mathbb{E}\left[(v - v_{\mathcal{M}} + v_{\mathcal{M}} - v_{\mathcal{K}})(v - v_{\mathcal{M}} + v_{\mathcal{M}} - v_{\mathcal{K}})^\top\right] \\ &= \Psi_{\mathcal{M}} + \Psi' + \mathbb{E}\left[e_{\mathcal{M}}\ \widetilde{e}^\top\right] + \mathbb{E}\left[\widetilde{e}\ e_{\mathcal{M}}^\top\right] \end{aligned} \qquad (103)$$

where $\Psi' = \mathbb{E}[\widetilde{e}\ \widetilde{e}^\top]$ is a positive semidefinite matrix. Since $\mathcal{K} \subseteq \mathcal{M}$, it is evident from elementary geometry that the terms $e_{\mathcal{M}}$ and $\widetilde{e}$ are orthogonal, *i.e.* $v - v_{\mathcal{M}} \perp v_{\mathcal{M}} - v_{\mathcal{K}}$, thus the term $\mathbb{E}\left[e_{\mathcal{M}}\ \widetilde{e}^\top\right]$ vanishes. Hence, by (103) we conclude $\Psi_{\mathcal{K}} \geq \Psi_{\mathcal{M}}$. ■

Proof of Lemma 2: By noticing that for $i = k, k-1$

$$\mathcal{L}_y^i \equiv \overline{\mathcal{P}}_y^i(1) \subset \overline{\mathcal{P}}_y^i(2) \subset \cdots \subset \overline{\mathcal{P}}_y^i(\nu) \subset L^2(Y_i, n),$$

from Lemma 1 and inequality (19) we have

$$\begin{aligned}\mathrm{tr}\{P_\nu(k)\} &\leq \mathrm{tr}\{P(k)\}, \\ \mathrm{tr}\{H_\nu(k)\} &\leq \mathrm{tr}\{H(k)\}.\end{aligned}$$

■

Appendix B
Kroneker Algebra

For sake of completeness we expose here some useful facts on Kroneker Algebra proved in [5] and [10].

Proposition 3: For any given pair of matrices $A \in \mathbb{R}^{r\times s}, B \in \mathbb{R}^{n\times m}$, we have

$$B \otimes A = \Xi_{r,n}^\top (A \otimes B) \Xi_{s,m}, \qquad (104)$$

where $\Xi_{r,n}$ and $\Xi_{s,m}$ are suitable $0-1$ matrices.
In the vector case the commutation matrices satisfy the following recursive formula.

Proposition 4: For any $a, b \in \mathbb{R}^n$ and for natural l let $G_l = \Xi_{n,n^l}^\top$ be such that $b^{[l]} \otimes a = G_l(a \otimes b^{[l]})$. Then the sequence $\{G_l\}$ satisfies the following equations.

$$\begin{aligned}G_1 &= \Xi_{n,n}^\top, \\ G_l &= (I_n \otimes G_{l-1}) \cdot (G_1 \otimes I_{n,l-1}), \quad l > 1.\end{aligned}$$

Proposition 5: For $a, b \in \mathbb{R}^n$ and any integer $h \geq 0$ the matrix coefficients of the binomial power formula

$$(a+b)^{[h]} = \sum_{k=0}^{h} \Theta_k^h(n)(a^{[k]} \otimes b^{[h-k]})$$

are such that

$$\begin{aligned}\Theta_h^h(n) =& \Theta_0^h = I_{n,h}, \\ \Theta_j^h(n) =& (\Theta_j^{h-1}(n) \otimes I_n) \\ &+ (\Theta_{j-1}^{h-1}(n) \otimes I_n) \cdot (I_{n,j-1} \otimes G_{h-j}),\end{aligned}$$

where G_l is defined in Proposition 4 and $0 < j < h$.

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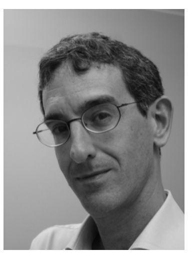

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