

**$L^\infty$ -NORM AND ENERGY QUANTIZATION  
FOR THE PLANAR LANE-EMDEN PROBLEM  
WITH LARGE EXPONENT**

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ABSTRACT. For any smooth bounded domain  $\Omega \subset \mathbb{R}^2$  we consider positive solutions to

$$\begin{cases} -\Delta u = u^p & \text{in } \Omega \\ u = 0 & \text{on } \partial\Omega \end{cases}$$

which satisfy the uniform energy bound

$$p\|\nabla u\|_\infty \leq C$$

for  $p > 1$ . We prove convergence to  $\sqrt{e}$  as  $p \rightarrow +\infty$  of the  $L^\infty$ -norm of any solution. We further deduce quantization of the energy to multiples of  $8\pi e$ , thus completing the analysis performed in [5].

1. INTRODUCTION

This paper focuses on the asymptotic analysis, as  $p \rightarrow +\infty$ , of families of solutions to the Lane-Emden problem

$$\begin{cases} -\Delta u = u^p & \text{in } \Omega \\ u = 0 & \text{on } \partial\Omega \\ u > 0 & \text{in } \Omega \end{cases} \quad (\mathcal{P}_p)$$

where  $\Omega$  is any smooth bounded planar domain.

This line of investigation started in [11, 12] for families  $u_p$  of least energy solutions, for which a one-point concentration behavior in the interior of  $\Omega$  is proved, as well as the  $L^\infty$ -bounds

$$\sqrt{e} \leq \lim_{p \rightarrow +\infty} \|u_p\|_\infty \leq C \quad (1.1)$$

and the following estimate

$$\lim_{p \rightarrow +\infty} p\|\nabla u_p\|_2^2 = 8\pi e.$$

The bound in (1.1) was later improved in [1], where it was shown that for families of least energy solutions it holds:

$$\lim_{p \rightarrow +\infty} \|u_p\|_\infty = \sqrt{e}. \quad (1.2)$$

Moreover in [1], [2] and [8] the Liouville equation in the whole plane

$$\begin{cases} -\Delta U = e^U & \text{in } \mathbb{R}^2 \\ \int_{\mathbb{R}^2} e^U dx = 8\pi \end{cases} \quad (1.3)$$

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was identified to be a *limit problem* for the Lane-Emden equation. Indeed in [1] it was proved that suitable rescalings around the maximum point of any least energy solution to  $(\mathcal{P}_p)$  converge, in  $C_{loc}^2(\mathbb{R}^2)$ , to the regular solution

$$U(x) = \log \left( \frac{1}{1 + \frac{1}{8}|x|^2} \right)^2 \quad (1.4)$$

of (1.3). Hence least energy solutions exhibit only one concentration point and the local limit profile is given by (1.4). More general solutions having only one peak have been recently studied in [7], where their Morse index is computed and connections with the question of the uniqueness of positive solutions in convex domains are shown.

Observe that when  $\Omega$  is a ball any solution to  $(\mathcal{P}_p)$  is radial by the result of Gidas, Ni and Nirenberg ([10]) and so the least energy is the unique solution for any  $p > 1$ .

In general in non-convex domains there may be families of solutions to  $(\mathcal{P}_p)$  other than the least energy ones. This is the case, for instance, of those found in [9] when the domain  $\Omega$  is not simply connected, which have higher energy, precisely

$$\lim_{p \rightarrow +\infty} p \|\nabla u_p\|_2^2 = 8\pi e \cdot k,$$

for any fixed integer  $k \geq 1$ . These solutions exhibit a concentration phenomenon at  $k$  distinct points in  $\Omega$  as  $p \rightarrow +\infty$  and their  $L^\infty$ -norm satisfies the same limit as in (1.2).

The question of characterizing the behavior of any family  $u_p$  of positive solutions to  $(\mathcal{P}_p)$  naturally arises. This issue was studied in [5] in any general smooth bounded domain  $\Omega$ , under the uniform energy bound assumption

$$p \|\nabla u_p\|_2^2 \leq C \quad (1.5)$$

(see also [3], where the asymptotic analysis started for solutions of any sign, and the related papers [4, 6]). The results in [5] show that under the assumption (1.5) the solutions to  $(\mathcal{P}_p)$  are necessarily *spike-like*. More precisely in [5, Theorem 1.1] it is proved that, up to a subsequence, there exists an integer  $k \geq 1$  and  $k$  distinct points  $x_i \in \Omega$ ,  $i = 1, \dots, k$ , such that, setting

$$\mathcal{S} = \{x_1, \dots, x_k\},$$

one has

$$\lim_{p \rightarrow +\infty} \sqrt{p} u_p = 0 \quad \text{in } C_{loc}^2(\bar{\Omega} \setminus \mathcal{S}) \quad (1.6)$$

and the energy satisfies

$$\lim_{p \rightarrow +\infty} p \|\nabla u_p\|_2^2 = 8\pi \sum_{i=1}^k m_i^2, \quad (1.7)$$

where  $m_i$ 's are positive constants given by

$$m_i = \lim_{\delta \rightarrow 0} \lim_{p \rightarrow +\infty} \max_{B_\delta(x_i)} u_p \quad (1.8)$$

which satisfy

$$m_i \geq \sqrt{e}. \quad (1.9)$$

Furthermore the location of the concentration points is shown to depend on the Dirichlet Green function  $G$  of  $-\Delta$  in  $\Omega$  and on its regular part  $H$

$$H(x, y) = -G(x, y) + \frac{1}{2\pi} \log \frac{1}{|x - y|}. \quad (1.10)$$

according to the following system

$$-m_i \nabla_x H(x_i, x_i) + \sum_{\ell \neq i} m_\ell \nabla_x G(x_i, x_\ell) = 0,$$

and moreover

$$\lim_{p \rightarrow +\infty} p u_p = 8\pi \sum_{i=1}^k m_i G(\cdot, x_i) \text{ in } C_{loc}^2(\bar{\Omega} \setminus \mathcal{S}).$$

In [5, Lemma 4.1] it is also proved that a suitable rescaling of  $u_p$  around each concentration point converges to the regular solution  $U$  in (1.4).

The convergence (1.6) and the inequality (1.9) immediately imply the following bound on the  $L^\infty$ -norm:

$$\sqrt{e} \leq \lim_{p \rightarrow +\infty} \|u_p\|_\infty \leq C. \quad (1.11)$$

In [5] it was conjectured that for all solutions to  $(\mathcal{P}_p)$ , under the assumption (1.5), one should have the equality in (1.9) and hence also the equality in the left hand side of (1.11).

As far as we know the only case where the equality has been proved is when  $u_p$  are least-energy solutions (see [1]), as recalled before. The proof of [1] strongly uses the minimality property of the solutions and does not apply to general families of solutions.

Here we answer this question by computing the exact value of the  $m_i$ 's for any solution. Our result is the following:

**Theorem 1.1.** *Let  $u_p$  be a family of solutions to  $(\mathcal{P}_p)$  satisfying (1.5). Then*

- (i)  $m_i = \sqrt{e}, \quad \forall i = 1, \dots, k$
- (ii)  $\lim_{p \rightarrow +\infty} \|u_p\|_\infty = \sqrt{e}.$

This theorem shows that, for  $p$  large, all peaks of the solutions have essentially the same height, namely  $\sqrt{e}$ . Moreover Theorem 1.1 allows to improve the analysis of the asymptotic behavior of the solutions to  $(\mathcal{P}_p)$  performed in [5], in particular implying, by (1.7), a *quantization of the energy* to integer multiples of  $8\pi e$  as  $p$  goes to infinity:

**Corollary 1.2.** *Let  $u_p$  be a family of solutions to  $(\mathcal{P}_p)$  and assume that (1.5) holds. Then there exist a number  $k \in \mathbb{N}$  and a sequence  $p_n \rightarrow +\infty$  as  $n \rightarrow +\infty$  such that one has*

$$\lim_{n \rightarrow \infty} p_n \int_{\Omega} |\nabla u_{p_n}(x)|^2 dx = 8\pi e \cdot k. \quad (1.12)$$

This shows that each concentration point carries the same mass and implies that solutions to  $(\mathcal{P}_p)$  can exist, for  $p$  large, only at the level of energies given by multiples of  $8\pi e$ . Hence this number plays in the two dimensional case the same role as the best Sobolev constant  $S$  in dimension  $N \geq 3$ .

## 2. PROOF OF THEOREM 1.1

Let  $k \geq 1$  and  $x_i \in \Omega$ ,  $i = 1, \dots, k$ , be as in the introduction and let us keep the notation  $u_p$  to denote the corresponding subsequence of the family  $u_p$  for which the results in [5] hold true.

In particular (see [5, Theorem 1.1 & Lemma 4.1]) for  $r > 0$  such that  $B_{3r}(x_j) \subset \Omega$ , for any  $j = 1, \dots, k$ , and  $B_{3r}(x_j) \cap B_{3r}(x_i) = \emptyset$ , for any  $i, j = 1, \dots, k$ ,  $j \neq i$ , letting  $y_{i,p} \in \Omega$  be a sequence defined as

$$u_p(y_{i,p}) := \max_{B_{2r}(x_i)} u_p \quad (2.13)$$

it follows that

$$\lim_{p \rightarrow +\infty} y_{i,p} = x_i, \quad (2.14)$$

$$\lim_{p \rightarrow +\infty} u_p(y_{i,p}) = m_i, \quad (2.15)$$

$$\lim_{p \rightarrow +\infty} \varepsilon_{i,p} \left( := [p u_p(y_{i,p})^{p-1}]^{-1/2} \right) = 0 \quad (2.16)$$

and setting

$$w_{i,p}(y) := \frac{p}{u_p(y_{i,p})} (u_p(y_{i,p} + \varepsilon_{i,p} y) - u_p(y_{i,p})), \quad y \in \Omega_{i,p} := \frac{\Omega - y_{i,p}}{\varepsilon_{i,p}}, \quad (2.17)$$

then

$$\lim_{p \rightarrow +\infty} w_{i,p} = U \quad \text{in } C_{loc}^2(\mathbb{R}^2), \quad (2.18)$$

where  $U$  is as in (1.4).

Furthermore by the result in [5, Proposition 4.3 & Lemma 4.4] we have that for any  $\gamma \in (0, 4)$  there exists  $R_\gamma > 1$  such that

$$w_{i,p}(z) \leq (4 - \gamma) \log \frac{1}{|z|} + \tilde{C}_\gamma, \quad \forall i = 1, \dots, k \quad (2.19)$$

for some  $\tilde{C}_\gamma > 0$ , provided  $R_\gamma \leq |z| \leq \frac{r}{\varepsilon_{i,p}}$  and  $p$  is sufficiently large.

The pointwise estimate (2.19) implies the following uniform bound, which will be the key to use the dominated convergence theorem in the proof of Theorem 1.1:

**Lemma 2.1.**

$$0 \leq \left( 1 + \frac{w_{j,p}(z)}{p} \right)^p \leq \begin{cases} 1 & \text{for } |z| \leq R_\gamma \\ C_\gamma \frac{1}{|z|^{4-\gamma}} & \text{for } R_\gamma \leq |z| \leq \frac{r}{\varepsilon_{j,p}} \end{cases}. \quad (2.20)$$

*Proof.* Observe that by (2.14)

$$B_r(y_{i,p}) \subset B_{2r}(x_i), \quad \text{for } p \text{ sufficiently large,}$$

as a consequence

$$w_{i,p} \leq 0, \quad \text{in } B_{\frac{r}{\varepsilon_{i,p}}}(0) (\subset \Omega_{i,p}), \quad \text{for } p \text{ large,} \quad (2.21)$$

which implies the first bound in (2.20).

For  $p$  sufficiently large, by (2.21) and (2.19), we also get the second bound in (2.20):

$$0 \leq \left( 1 + \frac{w_{j,p}(z)}{p} \right)^p = e^{p \log \left( 1 + \frac{w_{j,p}(z)}{p} \right)} \leq e^{w_{j,p}(z)} \leq C_\gamma \frac{1}{|z|^{4-\gamma}}$$

for  $R_\gamma \leq |z| \leq \frac{r}{\varepsilon_{j,p}}$ . □

*Proof of Theorem 1.1.* Observe that by the assumption (1.5) and Hölder inequality

$$\begin{aligned} (0 \leq) p \int_{\Omega} u_p^p(x) dx &\leq p^{\frac{1}{p+1}} |\Omega|^{\frac{1}{p+1}} \left[ p \int_{\Omega} |\nabla u_p|^2 dx \right]^{\frac{p}{p+1}} \\ &= p \int_{\Omega} |\nabla u_p|^2 dx (1 + o_p(1)) \\ &\stackrel{(1.5)}{\leq} C + o_p(1), \end{aligned}$$

so that, by the properties of the Green function  $G$ ,

$$\begin{aligned} \int_{\Omega \setminus B_{2r}(x_j)} G(y_{j,p}, x) u_p^p(x) dx &\leq C_r \int_{\Omega \setminus B_{2r}(x_j)} u_p^p(x) dx \\ &\leq C_r \int_{\Omega} u_p^p(x) dx = O\left(\frac{1}{p}\right) \end{aligned} \quad (2.22)$$

and similarly, observing that for  $p$  large enough the points  $y_{j,p} \in B_{\frac{r}{2}}(x_j)$  by (2.14) and  $B_{\frac{r}{2}}(x_j) \subset B_r(y_{j,p}) \subset B_{2r}(x_j)$ , also

$$\begin{aligned} \int_{B_{2r}(x_j) \setminus B_r(y_{j,p})} G(y_{j,p}, x) u_p^p(x) dx &\leq \int_{\{\frac{r}{2} < |x-x_j| < 2r\}} G(y_{j,p}, x) u_p^p(x) dx \\ &\leq C_{\frac{r}{2}} \int_{\Omega} u_p^p(x) dx = O\left(\frac{1}{p}\right). \end{aligned} \quad (2.23)$$

By the Green representation formula, using the previous estimates, we then get

$$\begin{aligned} u_p(y_{j,p}) &= \int_{\Omega} G(y_{j,p}, x) u_p^p(x) dx \\ &= \int_{B_{2r}(x_j)} G(y_{j,p}, x) u_p^p(x) dx + \int_{\Omega \setminus B_{2r}(x_j)} G(y_{j,p}, x) u_p^p(x) dx \\ &\stackrel{(2.22)}{=} \int_{B_r(y_{j,p})} G(y_{j,p}, x) u_p^p(x) dx + o_p(1) \\ &\stackrel{(2.23)}{=} \int_{B_{\frac{r}{\varepsilon_{j,p}}}(0)} G(y_{j,p}, y_{j,p} + \varepsilon_{j,p} z) \left(1 + \frac{w_{j,p}(z)}{p}\right)^p dz + o_p(1) \\ &\stackrel{(1.10)}{=} -\frac{u_p(y_{j,p})}{p} \int_{B_{\frac{r}{\varepsilon_{j,p}}}(0)} H(y_{j,p}, y_{j,p} + \varepsilon_{j,p} z) \left(1 + \frac{w_{j,p}(z)}{p}\right)^p dz \\ &\quad -\frac{u_p(y_{j,p})}{2\pi p} \int_{B_{\frac{r}{\varepsilon_{j,p}}}(0)} \log |z| \left(1 + \frac{w_{j,p}(z)}{p}\right)^p dz \\ &\quad -\frac{u_p(y_{j,p}) \log \varepsilon_{j,p}}{2\pi p} \int_{B_{\frac{r}{\varepsilon_{j,p}}}(0)} \left(1 + \frac{w_{j,p}(z)}{p}\right)^p dz + o_p(1) \\ &= A_p + B_p + C_p + o_p(1). \end{aligned} \quad (2.24)$$

Since  $H$  is smooth and  $x_j \notin \partial\Omega$ , by (2.14) and (2.16) we get

$$\lim_{p \rightarrow +\infty} H(y_{j,p}, y_{j,p} + \varepsilon_{j,p} z) = H(x_j, x_j), \text{ for any } z \in \mathbb{R}^2,$$

so by (2.15), the convergence (2.18) and the uniform bounds in (2.20) we can apply the dominated convergence theorem, and since the function  $z \mapsto \frac{1}{|z|^{4-\gamma}}$  is integrable in  $\{|z| > R_\gamma\}$  choosing  $\gamma \in (0, 2)$  we deduce

$$\begin{aligned} \lim_{p \rightarrow +\infty} u_p(y_{j,p}) \int_{B_{\frac{r}{\varepsilon_{j,p}}}(0)} H(y_{j,p}, y_{j,p} + \varepsilon_{j,p} z) \left(1 + \frac{w_{j,p}(z)}{p}\right)^p dz \\ = m_j H(x_j, x_j) \int_{\mathbb{R}^2} e^U \stackrel{(1.3)}{=} 8\pi m_j H(x_j, x_j), \end{aligned}$$

from which

$$A_p := -\frac{u_p(y_{j,p})}{p} \int_{B_{\frac{r}{\varepsilon_{j,p}}}(0)} H(y_{j,p}, y_{j,p} + \varepsilon_{j,p} z) \left(1 + \frac{w_{j,p}(z)}{p}\right)^p dz = o_p(1). \quad (2.25)$$

For the second term in (2.24) we apply again the dominated convergence theorem, using (2.20) and observing now that the function  $z \mapsto \frac{\log|z|}{|z|^{4-\gamma}}$  is integrable in  $\{|z| > R_\gamma\}$  and that  $z \mapsto \log|z|$  is integrable in  $\{|z| \leq R_\gamma\}$ . Hence we get

$$\lim_{p \rightarrow +\infty} u_p(y_{j,p}) \int_{B_{\frac{r}{\varepsilon_{j,p}}}(0)} \log|z| \left(1 + \frac{w_{j,p}(z)}{p}\right)^p dz = m_j \int_{\mathbb{R}^2} \log|z| e^{U(z)} dz < +\infty$$

and this implies that

$$B_p := -\frac{u_p(y_{j,p})}{2\pi p} \int_{B_{\frac{r}{\varepsilon_{j,p}}}(0)} \log|z| \left(1 + \frac{w_{j,p}(z)}{p}\right)^p dz = o_p(1). \quad (2.26)$$

Finally for the last term in (2.24) let us observe that by the definition of  $\varepsilon_{j,p}$  in (2.16)

$$\log \varepsilon_{j,p} = -\frac{(p-1)}{2} \log u_p(y_{j,p}) - \frac{1}{2} \log p, \quad (2.27)$$

again by the dominated convergence theorem

$$\lim_{p \rightarrow +\infty} \int_{B_{\frac{r}{\varepsilon_{j,p}}}(0)} \left(1 + \frac{w_{j,p}(z)}{p}\right)^p dz = \int_{\mathbb{R}^2} e^U \stackrel{(1.3)}{=} 8\pi, \quad (2.28)$$

and it follows

$$\begin{aligned} C_p &:= -\frac{u_p(y_{j,p}) \log \varepsilon_{j,p}}{2\pi p} \int_{B_{\frac{r}{\varepsilon_{j,p}}}(0)} \left(1 + \frac{w_{j,p}(z)}{p}\right)^p dz \\ &\stackrel{(2.28)}{=} -\frac{u_p(y_{j,p}) \log \varepsilon_{j,p}}{2\pi p} (8\pi + o_p(1)) \\ &\stackrel{(2.27)}{=} u_p(y_{j,p}) \left[ \frac{(p-1)}{p} \log u_p(y_{j,p}) + \frac{\log p}{p} \right] (2 + o_p(1)). \end{aligned} \quad (2.29)$$

Substituting (2.25), (2.26) and (2.29) into (2.24) we get

$$u_p(y_{j,p}) = u(y_{j,p}) \left[ \frac{(p-1)}{p} \log u_p(y_{j,p}) + \frac{\log p}{p} \right] (2 + o_p(1)) + o_p(1),$$

passing to the limit as  $p \rightarrow +\infty$  and using (2.15) we conclude that

$$\log m_j = \frac{1}{2}.$$

From this we immediately deduce statement (i). Then, due to the property (1.6), also (ii) follows.  $\square$

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