

**TOPOLOGY OF REPRESENTATION SPACES OF SURFACE GROUPS
IN $\mathrm{PSL}_2(\mathbb{R})$ WITH ASSIGNED BOUNDARY MONODROMY AND
NONZERO EULER NUMBER**

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ABSTRACT. In this paper we complete the topological description of the space of representations of the fundamental group of a punctured surface in $\mathrm{SL}_2(\mathbb{R})$ with prescribed behavior at the punctures and nonzero Euler number, following the strategy employed by Hitchin in the unpunctured case and exploiting Hitchin-Simpson correspondence between flat bundles and Higgs bundles in the parabolic case. This extends previous results by Boden-Yokogawa and Nasatyr-Steer. A relevant portion of the paper is intended to give an overview of the subject.

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1. INTRODUCTION

Representations $\rho : \pi_1(M) \rightarrow G$ of the fundamental group of a manifold M inside a Lie group G naturally arise as monodromies of $(G, G/H)$ -geometric structures à la Ehresmann [17] [18] on M (see also [24] and [26]).

From a differential-geometric point of view, the datum of such a representation is equivalent to that of a flat principal G -bundle on M , or of a vector bundle of rank N endowed with a flat connection and with monodromy in G , in case $G \subset \mathrm{GL}_N$ is a linear group: flatness is somehow the counterpart of the homogeneity of G/H .

Conversely, a way of “understanding” such a representation ρ is to *geometrize* it, namely to find a geometric structure on M with monodromy ρ .

1.1. Closed surfaces. Let S be a compact connected oriented surface of genus $g(S) \geq 2$.

1.1.1. Hyperbolic structures. A remarkable example of geometric structure on S is given by hyperbolic structures, that is hyperbolic metrics on S up to isotopy. Indeed, a hyperbolic metric is locally isometric to the upper half-plane \mathbb{H}^2 , and so it induces a $(\mathrm{PSL}_2(\mathbb{R}), \mathbb{H}^2)$ -structure on S with monodromy representation $\rho : \pi_1(S) \rightarrow \mathrm{PSL}_2(\mathbb{R}) \cong \mathrm{Isom}_+(\mathbb{H}^2)$. A result credited to Fricke-Klein [19] (see also Vogt [59]) states that hyperbolic structures on S are in bijective correspondence with a connected component of the space $\mathrm{Rep}(S, G)$ of conjugacy classes of representations $\pi_1(S) \rightarrow G$ with $G = \mathrm{PSL}_2(\mathbb{R})$.

1.1.2. Euler number of a representation in $\mathrm{PSL}_2(\mathbb{R})$. The connected components of the whole space $\mathrm{Rep}(S, \mathrm{PSL}_2(\mathbb{R}))$ can be classified according to a topological invariant of the \mathbb{RP}^1 -bundle over S associated to each such representation ρ : the Euler number $\mathrm{eu}(\rho) \in \mathbb{Z}$. The bound $|\mathrm{eu}(\rho)| \leq -\chi(S)$ was proven by Milnor [43] and Wood [66]; then Goldman [25] showed that each admissible value corresponds exactly to a connected component of the representation space and that monodromies of hyperbolic structures correspond to the component with $\mathrm{eu} = -\chi(S)$. It is easy to see that $\rho : \pi_1(S) \rightarrow \mathrm{PSL}_2(\mathbb{R})$ can be lifted to $\mathrm{SL}_2(\mathbb{R})$ if and only if $\mathrm{eu}(\rho)$ is even.

Remark 1.1. More refined invariants of a representation are given by *bounded* characteristic classes. The bounded Euler class for topological S^1 -bundles was investigated by Matsumoto [40] and the analogous Toledo invariant for G/H of Hermitian type by Toledo [57]. Bounded Euler and Toledo classes were used by Burger-Iozzi-Wienhard [10] to characterize maximal representations.

1.1.3. Local study of the representation spaces. Traces of a local study of $\mathrm{Rep}(S, G)$ are already in Weil [61] [62]. A more general treatment of the tangent space at a point $[\rho]$ and the determination of the smooth locus of $\mathrm{Rep}(S, G)$ can be found in Goldman [22], Lubotzky-Magid [39] and in the lectures notes [23] by Goldman and [37] by Labourie. A deeper analysis of the singularities of such moduli space can be found in Goldman-Millson [27].

1.1.4. Symplectic structure on the representation space. When G is reductive, a natural symplectic structure on the smooth locus of $\mathrm{Rep}(S, G)$ is defined by Atiyah-Bott [2] by using the equivalence between representations of the fundamental group of S in G and flat G -bundles on S .

In the case of the Fricke-Klein component of $\mathrm{Rep}(S, \mathrm{PSL}_2(\mathbb{R}))$, such a symplectic structure was seen by Goldman [22] to agree with the Hermitian pairing defined by Weil [60] using Petersson’s work [50] on automorphic forms. Ahlfors [1] showed that such a Weil-Petersson pairing defines a Kähler form, which is rather ubiquitous when dealing with deformations of hyperbolic structures (see for instance [65], [56], [9]).

1.1.5. Flat unitary bundles and holomorphic bundles. Consider the case $G = \mathrm{U}_N$ and fix a complex structure I on S . Representations of $\pi_1(S)$ in the unitary group U_N were object of a classical theorem by Narasimhan-Seshadri [48], in which a real-analytic correspondence is established between irreducible representations $\pi_1(S) \rightarrow \mathrm{U}_N$ and stable holomorphic vector bundles of rank N and degree 0 on the Riemann surface (S, I) . One direction is easy, since every flat complex bundle is I -holomorphic; for the other direction, the authors show that stable bundles of degree 0 admit a flat Hermitian metric: their argument works by

continuity method; an analytic proof of this statement was found later by Donaldson [15] proving the convergence of the hermitian Yang-Mills flow, as suggested in the fundamental work of Atiyah-Bott [2].

1.1.6. *Flat bundles and Higgs bundles.* The celebrated paper [29] by Hitchin treated the case of representations in $G = \mathrm{SL}_2$ and established a real-analytic correspondence between irreducible $\rho : \pi_1(S) \rightarrow \mathrm{SL}_2(\mathbb{C})$ and stable Higgs bundles (E, Φ) , namely holomorphic vector bundles $E \rightarrow (S, I)$ of rank N and trivial determinant endowed with a holomorphic $\mathrm{End}_0(E)$ -valued $(1, 0)$ -form Φ on (S, I) and subject to a suitable stability condition. Compared to Narasimhan-Seshadri's, the correspondence is less intuitive, since the holomorphic structure on E does not agree with the underlying holomorphic structure on the flat complex vector bundle $V \rightarrow S$ determined by the representation ρ (coming from the fact that locally constant functions are holomorphic) but it is twisted: the exact amount of such twisting is determined by the aid of the harmonic metric on V , whose existence was shown by Donaldson [16]. For $G = \mathrm{GL}_N$ or $G = \mathrm{SL}_N$, the existence of the harmonic metric was proven (on any compact manifold) by Corlette [12] (and later Labourie [36]) and the correspondence (in any dimension) was proven by Simpson [52], who also clarified the general picture by showing [54] [55] that the fundamental objects to consider are local systems (classified by a ‘‘Betti’’ moduli space), vector bundles with a flat connection (classified by a ‘‘de Rham’’ moduli space) and holomorphic Higgs bundles (classified by a ‘‘Dolbeault’’ moduli space) and by constructing their moduli spaces.

1.1.7. *Correspondence for $\mathrm{SL}_2(\mathbb{R})$.* Back to the rank 2 case, among the many results contained in [29], Hitchin could determine which Higgs bundles correspond to monodromies of hyperbolic metrics, thus parametrizing Teichmüller space by holomorphic quadratic differentials on (S, I) and making connection with Wolf's result [63] (namely, the Higgs field in Hitchin's work identifies to the Hopf differential of the harmonic map in Wolf's parametrization). Moreover, the space of isomorphism classes of Higgs bundles (E, Φ) carries a natural S^1 -action $u \cdot (E, \Phi) = (E, u\Phi)$, which is also rather ubiquitous when dealing with harmonic maps with a two-dimensional domain (for instance [8]); in rank 2, the locus fixed by the (-1) -involution $[(E, \Phi)] \leftrightarrow [(E, -\Phi)]$ is identified to the locus of unitary (if $\Phi = 0$) or real (if $\Phi \neq 0$) representations. This allows Hitchin to fully determine the topology of the connected components of $\mathrm{Rep}(S, \mathrm{PSL}_2(\mathbb{R}))$ with non-zero Euler number as that of a complex vector bundle over a symmetric product of copies of S . The real component with Euler number zero seems slightly subtler to treat, since it contains classes of reducible representations (or, equivalently, of strictly semi-stable Higgs bundles) for which the correspondence does not hold.

1.2. **Surfaces with punctures.** Let S be a compact connected oriented surface and let $P = \{p_1, \dots, p_n\} \subset S$ be a subset of n distinct marked points. Denote by \dot{S} the punctured surface $S \setminus P$ and assume $\chi(\dot{S}) < 0$.

1.2.1. *Absolute and relative representation space.* The space $\mathrm{Rep}(\dot{S}, G)$ of conjugacy classes of representations $\rho : \pi_1(\dot{S}) \rightarrow G$ can be partitioned according to the boundary behavior of ρ . More explicitly, fix an n -uple $\mathbf{c} = (c_1, \dots, c_n)$ of conjugacy classes in G and define $\mathrm{Rep}(\dot{S}, G, \mathbf{c})$ as the space of conjugacy classes of representations $\rho : \pi_1(\dot{S}) \rightarrow G$ that send a loop positively winding about the puncture p_i to an element of $c_i \subset G$.

1.2.2. *Spherical and hyperbolic structures.* Similarly to the case of a closed surface, isotopy classes of metrics of constant curvature K are the easiest examples of geometric structures on \dot{S} ; a standard requirement is to ask that the completion of such metrics has either conical singularities or geodesic boundary of finite length (or cusps, if $K < 0$) at the punctures. Monodromies of spherical structures ($K = 1$) naturally take values in $\mathrm{PSU}_2 \cong \mathrm{SO}_3(\mathbb{R})$ but can be lifted to SU_2 : if S has genus 0, such liftability imposes restrictions on the angles of the conical points of a spherical metric [46]. Monodromies of hyperbolic structures ($K = -1$)

determine conjugacy classes of representations $\rho : \pi_1(\dot{S}) \rightarrow \mathrm{PSL}_2(\mathbb{R})$, which are also liftable to $\mathrm{SL}_2(\mathbb{R})$.

1.2.3. *Euler number of a representation in $\mathrm{PSL}_2(\mathbb{R})$.* The Euler number of a representation $\rho : \pi_1(\dot{S}) \rightarrow \mathrm{PSL}_2(\mathbb{R})$ and a generalized Milnor-Wood inequality $|\mathrm{eu}(\rho)| \leq -\chi(\dot{S})$ are treated by Burger-Iozzi-Wienhard in [10], who also show that all values in the interval $[\chi(\dot{S}), -\chi(\dot{S})]$ are attained and that representations ρ with $\mathrm{eu}(\rho) = -\chi(\dot{S})$ correspond to monodromies of complete hyperbolic metrics on \dot{S} . Since $\mathrm{eu} : \mathrm{Rep}(\dot{S}, \mathrm{PSL}_2(\mathbb{R})) \rightarrow \mathbb{R}$ is continuous and its restriction to the locus $\mathrm{Rep}(\dot{S}, \mathrm{PSL}_2(\mathbb{R}), \mathbf{c})$ is locally constant, it is an invariant of the connected components of $\mathrm{Rep}(\dot{S}, \mathrm{PSL}_2(\mathbb{R}), \mathbf{c})$.

1.2.4. *Local structure and Poisson structure.* Similarly to the closed case, representations $\pi_1(\dot{S}) \rightarrow G$ (possibly with prescribed boundary values) correspond to flat G -bundles with the same boundary monodromy; the deformation theory is also analogous. If G is reductive, a natural Poisson structure [28] can be defined on the smooth locus of $\mathrm{Rep}(\dot{S}, G)$, which restricts to a symplectic structure on the smooth locus of the spaces $\mathrm{Rep}(\dot{S}, G, \mathbf{c})$ (see for instance [45] [44] for its link with Weil-Petersson structure when $G = \mathrm{PSL}_2(\mathbb{R})$ and for explicit formulae in the case of surfaces with conical points or with boundary geodesics).

1.2.5. *Flat unitary bundles and holomorphic parabolic bundles.* Unitary representations $\pi_1(\dot{S}) \rightarrow \mathrm{U}_N$ determine a complex vector bundle $\dot{V} \rightarrow \dot{S}$ of rank N endowed with a flat connection and a parallel Hermitian metric. Such a vector bundle admits a canonical extension $V \rightarrow S$ (Deligne [13]), in such a way that the connection may have at worst simple poles at P with eigenvalues of the residues in $[0, 1)$ and the natural parallel Hermitian metric H vanishes at P of order in $[0, 1)$. Mehta-Seshadri [42] introduced the important notion of a parabolic structure on V at P , namely a filtration of the fibers of V over P by order of growth with respect to H , and established the analogue of Narasimhan-Seshadri's result: for every complex structure I on S , there is a correspondence between irreducible flat unitary bundles on \dot{S} of rank N with prescribed monodromy at the punctures and stable holomorphic bundles of rank N and (parabolic) degree 0 on (S, I) with parabolic structure at P of prescribed type. As in the closed case, going from a flat bundle to a holomorphic parabolic bundle is easy; conversely, the existence of a flat Hermitian metric on a stable holomorphic bundle of degree 0 with prescribed polynomial growth at the parabolic points p_i was achieved in [42] by continuity method, and then proved by Biquard [3] by analytic techniques.

1.2.6. *Flat bundles and parabolic Higgs bundles.* In a fundamental article [53] Simpson established the correspondence between irreducible representations $\rho : \pi_1(\dot{S}) \rightarrow \mathrm{GL}_N(\mathbb{C})$ and parabolic I -holomorphic vector bundles E_\bullet of rank N and degree 0 endowed with a Higgs field $\Phi \in H^0(S, K_S \otimes \mathrm{End}(E_\bullet))$ subject to a suitable stability condition, the weights of E_\bullet and the residues of Φ at P being determined by the values of ρ on peripheral loops. The real-analytic nature of Simpson's correspondence was proven by Konno [34] and Biquard-Boalch [4]. The case of a general algebraic reductive group G was recently treated by Biquard, García-Prada and Mundet i Riera [5].

1.2.7. *Topological study of moduli spaces of Higgs bundles.* Following Hitchin's ideas, Boden-Yokogawa [7] analyzed some aspects of the case of $G = \mathrm{SL}_2(\mathbb{C})$ and in particular the Betti numbers of the moduli space using Morse theory. Their result was then extended by Logares [38] to the case of $\mathrm{U}(2, 1)$ -Higgs bundles and by García-Prada, Gothen and Muñoz [20] to the $\mathrm{SL}_3(\mathbb{C})$ and $\mathrm{GL}_3(\mathbb{C})$ cases; on the other hand, García-Prada, Logares and Muñoz [21] established the a Milnor-Wood inequality and determined the connected components of the moduli space of $\mathrm{U}(p, q)$ -Higgs bundles.

A different approach via orbifold structures was taken by Nasatyr-Steer [49], who implemented Hitchin's ideas in rank 2 and also determined the topology of the relative $\mathrm{SL}_2(\mathbb{R})$ -representation space in the case of positive Euler number and elliptic boundary monodromy of finite order.

1.2.8. *Content of the paper.* After giving an overview of the subject and of the fundamental results of the theory mentioned above, we focus on the topology of the real locus of the moduli space of parabolic SL_2 -Higgs bundles, and in particular on what happens as the parabolic structure degenerates, or equivalently on the topology of $\mathrm{SL}_2(\mathbb{R})$ -representation spaces as some of the boundary monodromies cease to be strictly elliptic.

Fix a complex structure I on S . Given conjugacy classes $\mathbf{c}_i \subset \mathfrak{sl}_2(\mathbb{C})$, numbers $w_1(p_i) \in [0, \frac{1}{2}]$ and a line bundle \mathcal{D} on S , we consider the moduli space $\mathrm{Higgs}^s(S, \mathbf{w}, 2, \mathcal{D}, \mathbf{c})$ of stable parabolic Higgs bundles (E_\bullet, Φ) on S of rank 2 with parabolic weights $w_1(p_1), 1 - w_1(p_i)$ (briefly, of type \mathbf{w}), $\det(E_\bullet) \cong \mathcal{D}$ and residue $\mathrm{Res}_{p_i}(\Phi)$ in \mathbf{c}_i .

In order to avoid to introduce too much notation at this point, we prefer not to give here complete statements of the main results contained in this paper but rather to list them in an informal way:

- (1) stable parabolic Higgs bundles in $\mathrm{Higgs}^s(S, \mathbf{w}, 2, \mathcal{D}, \mathbf{c})$ that correspond to representations $\rho : \pi_1(\dot{S}) \rightarrow \mathrm{SL}_2(\mathbb{R})$ with $\mathrm{eu}(\rho) > 0$ are characterized in Lemma 3.11 and Theorem 4.14;
- (2) a classification of the connected and the irreducible components of the locus of $\mathrm{Higgs}^s(S, \mathbf{w}, 2, \mathcal{D}, \mathbf{c})$ mentioned in (1) and of its closure and the determination of their topology is obtained in Proposition 3.14 and Proposition 3.16; in particular, these results combine in Theorem 4.14 to show that the closure (for the classical topology) of each connected component of $\mathrm{Rep}(\dot{S}, \mathrm{SL}_2(\mathbb{R}), \mathbf{c})$ with $\mathrm{eu} > 0$ is homeomorphic either to a complex affine space or to an $H^1(S; \mathbb{Z}/2\mathbb{Z})$ -cover of a complex affine bundle over a symmetric product of $S \setminus P_{hyp}$ with $P_{hyp} = \{p_i \in P \mid c_i \text{ hyperbolic}\}$;
- (3) again by Proposition 3.16 and Theorem 4.14, the closure of each connected component of $\mathrm{Rep}(\dot{S}, \mathrm{PSL}_2(\mathbb{R}), \mathbf{c})$ with $\mathrm{eu} > 0$ is homeomorphic to a complex affine bundle over a symmetric product of $S \setminus P_{hyp}$;
- (4) the connected components of $\mathrm{Rep}(\dot{S}, \mathrm{PSL}_2(\mathbb{R}), \mathbf{c})$ are classified by their Euler number by Corollary 4.15;
- (5) the connected components of $\mathrm{Rep}(\dot{S}, \mathrm{PSL}_2(\mathbb{R}), \mathbf{c})$ that can host monodromies of hyperbolic structures are determined in Proposition 2.20 and their topology is deduced in Corollary 4.16.

The precise topological description of such representation spaces in the punctured case can be compared to the one in the closed case by looking at Theorem 2.18 and Theorem 2.19.

Remark 1.2. In the special case of all elliptic boundary monodromies, the above results (1-4) are already in [7] and, for elliptic boundary monodromies of finite order, the same topological description was obtained in [49]. The case of monodromy representations of hyperbolic structures with cusps (and so maximal Euler number) was analyzed by Biswas, Arés-Gastesi and Govindarajan [6]. Similarly to what happens with closed surfaces, the work of Wolf [64] on harmonic maps that “open the node” from nodal Riemann surfaces to smooth hyperbolic surfaces relates to the case of an $\mathrm{SL}_2(\mathbb{R})$ -Higgs bundle with imaginary residues at the punctures.

As an example of a by-product of our analysis, we describe the topology of components of the representation space that can contain monodromies of hyperbolic structure (see also Corollary 4.16).

Corollary 1.3 (Topology of uniformization irreducible components). Let ρ be the monodromy representation of a hyperbolic metric on \dot{S} of area $2\pi e > 0$ whose completion has conical singularities of angles $\vartheta_1, \dots, \vartheta_k > 0$ at p_1, \dots, p_k , cusps at p_{k+1}, \dots, p_r and geodesic boundaries of lengths $\ell_{r+1}, \dots, \ell_n > 0$. Let c_i be the conjugacy class of the monodromy ρ about the i -th end of \dot{S} and let $s_0 = \#\{i \in \{1, \dots, k\} \mid \vartheta_i \in 2\pi\mathbb{N}_+\}$.

Then $[\rho]$ belongs to the irreducible component of $\mathrm{Rep}(\dot{S}, \mathrm{PSL}_2(\mathbb{R}), \mathbf{c})$ with Euler number

$e = -\chi(\dot{S}) - \sum_{i=1}^k \frac{\vartheta_i}{2\pi} > 0$, and such irreducible component is real-analytically diffeomorphic to a holomorphic affine bundle of rank $3g - 3 + n - m$ over $\text{Sym}^{m-s_0}(S \setminus \{p_{k+1}, \dots, p_n\})$, where $m = \sum_{i=1}^k \lfloor \frac{\vartheta_i}{2\pi} \rfloor$. Moreover, its closure $\text{Rep}(\dot{S}, \text{PSL}_2(\mathbb{R}), \bar{c})$ is homeomorphic to a holomorphic affine bundle of rank $3g - 3 + n - m$ over $\text{Sym}^{m-s_0}(S \setminus \{p_{r+1}, \dots, p_n\})$

1.3. Structure of the paper. In Section 2 we review the definition of representation space of the fundamental group of a punctured surface \dot{S} and the basic smoothness results. Then we recall the (Riemann-Hilbert) correspondence between representations of the fundamental group of a surface in an algebraic group G and flat principal G -bundles. In particular, we review the case of a linear group $G \subset \text{GL}_N$ and the flat vector bundle of rank N attached to a representation. Then we analyze the case of a $\text{PSL}_2(\mathbb{R})$ -representation ρ and we discuss the Euler number of ρ , some well-known fundamental results on the topology of the $\text{PSL}_2(\mathbb{R})$ -representation space and we compare our results with those. Finally, we briefly mention monodromy representations coming from hyperbolic structures possibly with cusps, conical singularities and geodesic boundaries.

In Section 3 we first recall the notion of parabolic bundle following Simpson and we show examples over the disk Δ , originating (à la Mehta-Seshadri) from representations $\pi_1(\dot{\Delta}) \rightarrow \text{U}_1$ and $\pi_1(\dot{\Delta}) \rightarrow \text{SU}_2$. Similarly, we introduce the definition of Higgs bundles and show examples coming from $\pi_1(\dot{\Delta}) \rightarrow \text{GL}_N(\mathbb{C})$ with $N = 1, 2$. Then we recall the notion of slope stability and of moduli space of stable parabolic Higgs bundles. Finally, we specialize to the case of $G = \text{SL}_2$ and we study the topology of the locus fixed by the involution $[(E_\bullet, \Phi)] \leftrightarrow [(E_\bullet, -\Phi)]$ and that corresponds to representations in $\text{SL}_2(\mathbb{R})$ with $\text{eu} > 0$.

In the final Section 4 we recall first the main correspondence results in the theory of representations of fundamental groups of surfaces and holomorphic (parabolic) (Higgs) bundles. Then we illustrate how the correspondence works for $\text{SL}_2(\mathbb{R})$ and for $\text{PSL}_2(\mathbb{R})$ and we describe the topology of the components of the representation space. We finally conclude with two corollaries about uniformization components.

1.4. Acknowledgements. The motivation for this work originated in conversations about monodromies of hyperbolic structures with conical points with Roberto Frigerio, whom I wish to thank.

I am grateful to Nicolas Tholozan (reporting a conversation with Olivier Biquard) for observing that the case of compact components with non-degenerate parabolic structure as in Corollary 3.20 corresponds to the class of “super-maximal” representations studied by Deroin-Tholozan in [14] and to Bertrand Deroin for explaining me that the analysis carried out in this paper leads to Corollary 4.17. I also thank Paul Seidel and Laurent Côté for pointing out some discrepancies in earlier versions on the paper, Johannes Huebschmann for precise remarks on the locally semialgebraic nature of real representation spaces and on their tangent spaces, Peter Gothen and an anonymous referee for drawing my attention to some relevant works in the field, and both referees for useful comments.

The author’s research was partially supported by FIRB 2010 national grant “Low-dimensional geometry and topology” (code: RBFR10GHHH.003) and by GNSAGA INdAM group.

1.5. Notation. Let S be compact, connected, oriented surface of genus $g(S)$ and $P = \{p_1, \dots, p_n\}$ be a subset of distinct points of S . Denote by \dot{S} the punctured surface $S \setminus P$ and assume $\chi(\dot{S}) < 0$.

Let π be the fundamental group $\pi_1(\dot{S}, b)$, where $b \in \dot{S}$ is a base point, and fix a universal cover $(\tilde{S}, \tilde{b}) \rightarrow (\dot{S}, b)$ on which π then acts by deck transformations.

Fix a standard set $\{\alpha_1, \beta_1, \dots, \alpha_{g(S)}, \beta_{g(S)}, \gamma_1, \dots, \gamma_n\}$ of generators of π , that satisfy the unique relation $[\alpha_1, \beta_1] \cdots [\alpha_{g(S)}, \beta_{g(S)}] \gamma_1 \cdots \gamma_n = \text{Id}$. In particular, γ_i is freely homotopic to a small loop that simply winds about the puncture p_i counterclockwise. We will also write

∂_i for the conjugacy class of γ_i in π (or, equivalently, for the free homotopy class of γ_i on \dot{S}).

We will denote by G a reductive real or complex algebraic group with finite center $Z = Z(G)$ and by \mathfrak{g} its Lie algebra. If c_i is a conjugacy class or a union of conjugacy classes in G , then we denote by \bar{c}_i its closure. Similarly, if \mathfrak{c}_i is a conjugacy class or a union of conjugacy classes in \mathfrak{g} , we denote by $\bar{\mathfrak{c}}_i$ its closure. We use the symbol \mathbf{c} for an n -uple (c_1, \dots, c_n) and \mathfrak{c} for an n -uple $(\mathfrak{c}_1, \dots, \mathfrak{c}_n)$, and similarly for their closures $\bar{\mathbf{c}}$ and $\bar{\mathfrak{c}}$.

If $\mathbf{r} = (r_1, \dots, r_n)$ is a string of n non-negative real numbers, then $\|\mathbf{r}\|_1$ will denote their sum $r_1 + r_2 + \dots + r_n$.

1.5.1. *Convention.* We identify $\mathbb{H} \subset \mathbb{CP}^1$ via $z \mapsto [1 : z]$, so that a matrix in $\mathrm{PSL}_2(\mathbb{R})$ acts on \mathbb{H} as

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix} \cdot z = \frac{c + dz}{a + bz}$$

Consider the transformations $R_\theta, T \in \mathrm{PSL}_2(\mathbb{R})$ defined as

$$R_\theta = \begin{pmatrix} \cos(\theta/2) & -\sin(\theta/2) \\ \sin(\theta/2) & \cos(\theta/2) \end{pmatrix}, \quad T = \begin{pmatrix} 1 & 0 \\ 1 & 1 \end{pmatrix}$$

Isometries of \mathbb{H} conjugate to R_θ (resp. to T , or to T^{-1}) in $\mathrm{PSL}_2(\mathbb{R})$ are called *rotations of angle θ* (resp. *positive unipotents*, or *negative unipotents*).

2. REPRESENTATIONS OF THE FUNDAMENTAL GROUP AND FLAT BUNDLES

2.1. **Representation spaces.** The set $\mathrm{Hom}(\pi, G)$ of homomorphisms $\pi \rightarrow G$ is denoted by $\mathrm{Rep}_b(\dot{S}, G)$ and it can be identified to the locus

$$\left\{ (A_1, B_1, \dots, A_{g(S)}, B_{g(S)}, C_1, \dots, C_n) \in G^{2g(S)+n} \mid [A_1, B_1] \cdots [A_{g(S)}, B_{g(S)}] C_1 \cdots C_n = \mathrm{Id} \right\}$$

inside $G^{2g(S)+n}$. The algebraic structure on $\mathrm{Rep}_b(\dot{S}, G)$ induced as a hypersurface in $G^{2g(S)+n}$ is independent of the choice of the generators.

The group G acts on $\mathrm{Rep}_b(\dot{S}, G)$ by conjugation, that is sending a homomorphism $\rho \in \mathrm{Rep}_b(\dot{S}, G)$ to $\mathrm{Ad}_g \circ \rho$. Given any other base-point $b' \in \dot{S}$, the isomorphism $\mathrm{Rep}_b(\dot{S}, G) \cong \mathrm{Rep}_{b'}(\dot{S}, G)$ depends on the choice of a path between b and b' , but it becomes canonical after factoring out the action of G .

Definition 2.1. The *representation space* $\mathrm{Rep}(\dot{S}, G)$ is the Hausdorffization of the topological quotient $\widehat{\mathrm{Rep}}(\dot{S}, G) := \widehat{\mathrm{Rep}_b(\dot{S}, G)}/G$.

If G is compact, then all G -orbits in $\mathrm{Rep}_b(\dot{S}, G)$ are closed and so the map $\widehat{\mathrm{Rep}}(\dot{S}, G) \rightarrow \mathrm{Rep}(\dot{S}, G)$ is a homeomorphism.

Remark 2.2. If G is a complex group, then $\mathrm{Rep}(\dot{S}, G)$ agrees with the GIT quotient $\mathrm{Rep}_b(\dot{S}, G)//G$. If G is real and $G_{\mathbb{C}}$ is its complexification, then $\mathrm{Rep}(\dot{S}, G)$ is a locally semialgebraic subset (i.e. locally defined by real algebraic equalities and inequalities) inside the real locus $\mathrm{Rep}(\dot{S}, G_{\mathbb{C}})(\mathbb{R})$, see for instance [30].

2.1.1. *Closed case.* For closed surfaces ($n = 0$ and so $\dot{S} = S$), the following result was proven by Rapinchuk, Benyash-Krivetz and Chernousov [51].

Theorem 2.3 (Irreducibility for representations in $\mathrm{SL}_N(\mathbb{C})$ and $\mathrm{PSL}_N(\mathbb{C})$). The algebraic varieties $\mathrm{Rep}_b(S, G)$ are irreducible for $G = \mathrm{SL}_N(\mathbb{C}), \mathrm{PSL}_N(\mathbb{C})$.

Problem 1 (Topology of representation spaces). Assuming $n = 0$, determine the topology of $\mathrm{Rep}_b(S, G)$ and $\mathrm{Rep}(S, G)$: in particular, enumerate the connected components (for the classical topology) of $\mathrm{Rep}(S, G)$.

We will see below that the above problem was almost completely solved by Hitchin [29] for $G = \mathrm{SL}_2(\mathbb{R}), \mathrm{PSL}_2(\mathbb{R})$.

2.1.2. *Punctured case.* Assume now $n > 0$. Since $\pi_1(\dot{S})$ is free on $2g(S) + n - 1$ generators, $\text{Rep}_b(\dot{S}, G)$ is isomorphic to $G^{2g(S)+n-1}$ and G acts diagonally by conjugation on each factor of $G^{2g(S)+n-1}$.

The situation becomes more interesting if we consider the relative situation.

Definition 2.4. Let $\mathbf{c} = (c_1, \dots, c_n)$, where c_i is a conjugacy class in G (or possibly a finite union of conjugacy classes). The *relative homomorphism space* $\text{Rep}_b(\dot{S}, G, \mathbf{c})$ is the locus in $\text{Rep}_b(\dot{S}, G)$ of homomorphisms $\rho : \pi \rightarrow G$ such that $\rho(\partial_i) \in c_i$.

Equivalently, $\text{Rep}_b(\dot{S}, G, \mathbf{c})$ is the preimage of $\mathbf{c} \subset G^n$ under the evaluation map $\text{ev} : \text{Rep}_b(\dot{S}, G) \rightarrow G^n$ that sends ρ to $(\rho(\gamma_1), \dots, \rho(\gamma_n))$.

If G is a complex group, then conjugacy classes and their closures are algebraic subvarieties and so $\text{Rep}_b(\dot{S}, G, \mathbf{c})$ is an algebraic subvariety of $\text{Rep}_b(\dot{S}, G)$. If G is real, then conjugacy classes and their closures are in general semialgebraic subsets and so $\text{Rep}_b(\dot{S}, G, \mathbf{c})$ is semialgebraic inside $\text{Rep}_b(\dot{S}, G)$.

Example 1. In the special case of $G = \text{PSL}_2(\mathbb{R})$ conjugacy classes consisting of hyperbolic or elliptic elements are algebraic and closed; on the other hand, the class $c \subset \text{PSL}_2(\mathbb{R})$ consisting of positive unipotent elements (or negative unipotent elements) is semi-algebraic and not closed; indeed, its classical closure \bar{c} consists of $c \cup \{\text{Id}\}$ and its Zariski closure is $c \cup \{\text{Id}\} \cup (-c)$.

The definition of $\text{Rep}_b(\dot{S}, G, \mathbf{c})$ as well as the induced (semi-)algebraic structure are independent of the choice of the loops $\gamma_1, \dots, \gamma_n$.

Definition 2.5. The *relative representation space* $\text{Rep}(\dot{S}, G, \mathbf{c})$ is the Hausdorffization of the topological quotient $\widehat{\text{Rep}}(\dot{S}, G, \mathbf{c}) := \widehat{\text{Rep}_b(\dot{S}, G, \mathbf{c})}/G$.

As in the absolute case, if G is a complex group, then $\text{Rep}(\dot{S}, G, \mathbf{c})$ agrees with the GIT quotient $\text{Rep}_b(\dot{S}, G, \mathbf{c})//G$ and it is an algebraic subvariety of $\text{Rep}(\dot{S}, G)$. If G is real, then $\text{Rep}(\dot{S}, G, \mathbf{c})$ is a locally semialgebraic subset of $\text{Rep}(\dot{S}, G)$.

Problem 2 (Topology of relative representation spaces). Assuming $n > 0$, determine the topology of $\text{Rep}_b(\dot{S}, G, \mathbf{c})$ and $\text{Rep}(\dot{S}, G, \mathbf{c})$: in particular, enumerate the connected components (for the classical topology) of $\text{Rep}(\dot{S}, G, \mathbf{c})$.

Same remarks and questions hold for representations with boundary values in $\bar{\mathbf{c}}$.

Remark 2.6. In light of the short exact sequence $0 \rightarrow Z \rightarrow G \rightarrow G/Z \rightarrow 0$, we can view $\text{Rep}_b(\dot{S}, G)$ as an $H^1(\dot{S}, Z)$ -bundle over $\text{Rep}_b(\dot{S}, G/Z)$. It is easy to see that, if $\tilde{c}_i \subset G$ is the preimage of the conjugacy class $c_i \subset G/Z$ under the projection $G \rightarrow G/Z$, then $\text{Rep}_b(\dot{S}, G, \tilde{\mathbf{c}})$ is an $H^1(\dot{S}, Z)$ -bundle over $\text{Rep}_b(\dot{S}, G/Z, \mathbf{c})$.

2.2. Flat G -bundles. Let \mathcal{G} be the sheaf of smooth functions with values in G and \underline{G} the subsheaf of locally constant functions.

It is well-known that there is a bijective correspondence between G -local systems on \dot{S} and principal G -bundles $\xi \rightarrow \dot{S}$ endowed with a flat connection $\nabla \in \Omega^1(\xi, \mathfrak{g})^G$. Indeed, for every flat G -bundle (ξ, ∇) , the sheaf $\underline{\xi}$ of parallel sections of ξ is a local system; vice versa, given a G -local system $\underline{\xi}$, the G -bundle whose sheaf of smooth sections is $\xi = \mathcal{G} \times_{\underline{G}} \underline{\xi}$ is endowed with a flat connection induced by the exterior differential $d : \mathcal{O} \rightarrow \Omega^1$. Such a construction also establishes a correspondence between *framed flat principal G -bundles* (ξ, ∇, τ) consisting of a flat G -bundle (ξ, ∇) on \dot{S} with a trivialization $\tau : \xi_b \xrightarrow{\sim} G$ at the base point $b \in \dot{S}$ and *framed G -local systems* $(\underline{\xi}, \tau')$ consisting of a G -local system $\underline{\xi}$ with a trivialization $\tau' : \underline{\xi}_b \xrightarrow{\sim} G$ at b . The isomorphisms τ and τ' are called *framings*.

Notation. We will denote by $\text{Fl}(\dot{S}, G)$ the set isomorphism classes of flat principal G -bundles on \dot{S} and by $\text{Fl}_b(\dot{S}, G)$ the set of isomorphism classes of b -framed flat principal G -bundles on \dot{S} .

We wish to recall the well-known correspondence between flat G -bundles and G -representations of π and to adapt it to the framed case.

Consider the trivial G -bundle $\tilde{\xi} := \tilde{S} \times G \rightarrow \tilde{S}$ with the framing $\tilde{\tau} : \tilde{\xi}_b = \{\tilde{b}\} \times G \xrightarrow{\sim} G$ given by the projection onto the second factor. The flat connection $\tilde{\nabla}$ on $\tilde{\xi}$ is simply given by the de Rham differential.

Given a representation $\rho \in \mathrm{Rep}_b(\dot{S}, G)$, the fundamental group π acts on $\tilde{\xi}$, and more precisely via deck transformations on the factor \tilde{S} and via $m_L \circ \rho$ on the factor G , where m_L is the action of G on G by left multiplication. As a consequence, $\tilde{\nabla}$ descends to a flat connection ∇ on the G -bundle $\xi_\rho := \tilde{\xi}/\pi$ on \dot{S} . Moreover, $\tilde{\tau}$ induces a framing $\tau : \xi_b \xrightarrow{\sim} G$ through the isomorphism $\tilde{\xi}_b \xrightarrow{\sim} \xi_b$. This construction determines an application

$$\Xi_b : \mathrm{Rep}_b(\dot{S}, G) \longrightarrow \mathrm{Fl}_b(\dot{S}, G).$$

Vice versa, given a framed flat G -bundle (ξ, ∇, τ) , the holonomy representation based at b descends to a homomorphism $\pi \rightarrow \mathrm{Aut}(\xi_b) \cong G$ by the flatness of ∇ , and so via τ to a homomorphism $\rho := \mathrm{hol}_b(\xi) : \pi \rightarrow G$. This construction determines an application

$$\mathrm{hol}_b : \mathrm{Fl}_b(\dot{S}, G) \longrightarrow \mathrm{Rep}_b(\dot{S}, G).$$

It is easy to check that Ξ_b and hol_b are set-theoretically inverse of each other.

Any two trivializations $\tau_1, \tau_2 : \xi_b \rightarrow G$ at b are related by a unique element $g \in G$, namely $\tau_2 = m_L(g) \circ \tau_1$. Hence, factoring out the action of G one obtains the applications $\Xi : \widehat{\mathrm{Rep}}(\pi, G) \rightarrow \widehat{\mathrm{Fl}}(\dot{S}, G)$ and $\widehat{\mathrm{hol}} : \widehat{\mathrm{Fl}}(\dot{S}, G) \rightarrow \widehat{\mathrm{Rep}}(\pi, G)$ which are set-theoretically inverse of each other.

Remark 2.7. Viewing $\mathrm{Fl}_b(\dot{S}, G)$ as a quotient of the space of flat connections on $G \times \dot{S} \rightarrow \dot{S}$, it is possible to endow $\mathrm{Fl}_b(\dot{S}, G)$ with the structure of (real or complex) analytic variety that makes hol_b and Ξ_b analytic isomorphisms. Since we do not want to go deeper in this direction, we can alternatively just put on $\mathrm{Fl}_b(\dot{S}, G)$ the analytic structure induced by hol_b and Ξ_b .

The correspondence in the relative case is dealt with analogously.

Notation. Denote by $\mathrm{Fl}_b(\dot{S}, G, \mathbf{c})$ the set of isomorphism classes of b -framed flat G -bundles on \dot{S} with holonomy along the path ∂_i in c_i , and by $\widehat{\mathrm{Fl}}(\dot{S}, G, \mathbf{c})$ the set of isomorphism classes of G -bundles on \dot{S} with holonomy along the i -th end in c_i . Analogous notation for flat bundles with boundary monodromy in $\bar{\mathbf{c}}$.

The same construction as above works in the relative case and we summarize the discussion in the following statement.

Lemma 2.8 (Equivalence between representations of π_1 and flat G -bundles). The applications

$$\begin{array}{ccc} & \xrightarrow{\mathrm{hol}_b} & \\ \mathrm{Rep}_b(\dot{S}, G, \mathbf{c}) & & \mathrm{Fl}_b(\dot{S}, G, \mathbf{c}) \\ & \xleftarrow{\Xi_b} & \end{array}$$

are set-theoretically inverse of each other. By factoring the G -action by conjugation on the representation and by left multiplication on the b -framing, we recover the correspondence

$$\begin{array}{ccc} & \xrightarrow{\widehat{\mathrm{hol}}} & \\ \widehat{\mathrm{Rep}}(\dot{S}, G, \mathbf{c}) & & \widehat{\mathrm{Fl}}(\dot{S}, G, \mathbf{c}) \\ & \xleftarrow{\Xi} & \end{array}$$

We call the composition $\widehat{\mathrm{Fl}}(\dot{S}, G, \mathbf{c}) \xrightarrow{\widehat{\mathrm{hol}}} \widehat{\mathrm{Rep}}(\dot{S}, G, \mathbf{c}) \rightarrow \mathrm{Rep}(\dot{S}, G, \mathbf{c})$ simply hol .

2.3. Smoothness. Given a representation $\rho : \pi \rightarrow G$, we denote by $\mathfrak{g}^\rho \subset \mathfrak{g}$ (resp. $G^\rho \subset G$) the subset of elements which are invariant under the adjoint action of π through ρ .

We can naturally identify $\text{Aut}(\xi_\rho)$ with the centralizer $Z_G(\rho(\pi)) = G^\rho$ of $\rho(\pi)$ in G and the space $T_{\text{Id}}\text{Aut}(\xi_\rho)$ of first-order automorphisms with $\mathfrak{g}^\rho \cong H^0(\dot{S}, \text{Ad}(\xi_\rho))$, where $\text{Ad}(\xi_\rho)$ is the flat \mathfrak{g} -bundle with monodromy $\text{Ad} \circ \rho$.

Definition 2.9. A representation ρ is *regular* if $\mathfrak{g}^\rho = \{0\}$; it is *Zariski-dense* if its image is Zariski-dense. We denote by $\text{Rep}_b^r(\dot{S}, G)$ (resp. by $\text{Rep}_b^{Zd}(\dot{S}, G)$) the subsets of $\text{Rep}_b(\dot{S}, G)$ of regular (resp. Zariski-dense) representations, and similarly by $\text{Rep}^r(\dot{S}, G)$ (resp. by $\text{Rep}^{Zd}(\dot{S}, G)$) the corresponding locus in $\text{Rep}(\dot{S}, G)$.

Since we are assuming G algebraic, ρ is regular if and only if $Z(\rho(\pi))$ is finite. A proof of the following statement can be found in [31] and in Section 5.3.4 of [37].

Lemma 2.10 (Proper action of G on Rep_b^r). The action on the subset $\text{Rep}_b^{Zd}(\dot{S}, G) \subseteq \text{Rep}_b^r(\dot{S}, G)$ is proper and with stabilizer Z .

A standard deformation theory argument shows that first-order deformations of ξ are parametrized by $H^1(\dot{S}, \text{Ad}(\xi))$. The proof of the following result can be found for instance in [22], [31], [37].

Proposition 2.11 (Tangent space to representation spaces). For every representation $\rho \in \text{Rep}_b(\dot{S}, G)$, we have

$$H^0(\dot{S}; \text{Ad}(\xi_\rho)) \cong \mathfrak{g}^\rho \quad \text{and} \quad H^2(\dot{S}; \text{Ad}(\xi_\rho)) \cong \begin{cases} (\mathfrak{g}^\vee)^\rho & \text{if } n = 0 \\ 0 & \text{otherwise} \end{cases}$$

by Poincaré duality. The centralizer G^ρ acts on $H^1(\dot{S}; \text{Ad}(\xi_\rho))$ by adjunction and the quotient can be identified to $T_{[\rho]}\text{Rep}(\dot{S}, G)$. Hence, the regular locus is an open dense orbifold in the representation spaces and

	$\text{Rep}_b^r(\dot{S}, G)$	$\text{Rep}^r(\dot{S}, G)$
dim	$(-\chi(\dot{S}) + 1) \dim(G)$	$-\chi(\dot{S}) \dim(G)$

for any $n \geq 0$.

The tangent space $T_{[\rho]}\text{Rep}(\dot{S}, G, \mathbf{c})$ in the relative case can be analyzed by means of the following exact sequence associated to the couple $(\dot{S}, \dot{\Delta})$

$$\begin{aligned} 0 &= H^0(\dot{S}, \dot{\Delta}; \text{Ad}(\xi)) \rightarrow H^0(\dot{S}; \text{Ad}(\xi)) \rightarrow H^0(\dot{\Delta}; \text{Ad}(\xi)) \rightarrow H^1(\dot{S}, \dot{\Delta}; \text{Ad}(\xi)) \rightarrow \\ &\rightarrow H^1(\dot{S}; \text{Ad}(\xi)) \rightarrow H^1(\dot{\Delta}; \text{Ad}(\xi)) \rightarrow H^2(\dot{S}, \dot{\Delta}; \text{Ad}(\xi)) \rightarrow H^2(\dot{S}; \text{Ad}(\xi)) = 0 \end{aligned}$$

where the Δ_i 's are disjoint open contractible neighbourhoods of the p_i 's and $\dot{\Delta} = \bigcup_{i=1}^n \dot{\Delta}_i$ is the union of the punctured disks $\dot{\Delta}_i = \Delta_i \setminus \{p_i\}$.

In fact, since G is reductive, the Lie algebra \mathfrak{g} has a non-degenerate invariant symmetric bilinear form, which induces an Ad -invariant isomorphism $\mathfrak{g} \cong \mathfrak{g}^\vee$. By Lefschetz duality, $H^2(\dot{S}, \dot{\Delta}; \text{Ad}(\xi)) \cong H^0(\dot{S}; \text{Ad}(\xi))^\vee$ and $H^1(\dot{S}, \dot{\Delta}; \text{Ad}(\xi)) \cong H^1(\dot{S}; \text{Ad}(\xi))^\vee$. Thus, we obtain the G^ρ -equivariant exact sequence

$$0 \rightarrow \mathfrak{g}^\rho \rightarrow H^0(\dot{\Delta}; \text{Ad}(\xi)) \xrightarrow{\epsilon^\vee} H^1(\dot{S}; \text{Ad}(\xi))^\vee \rightarrow H^1(\dot{S}; \text{Ad}(\xi)) \xrightarrow{\epsilon} H^1(\dot{\Delta}; \text{Ad}(\xi)) \rightarrow (\mathfrak{g}^\vee)^\rho \rightarrow 0$$

Cocycles in $\ker(\epsilon)$ induce first-order deformations ρ_t of ρ such that for every i the boundary value $\rho_t(\gamma_i)$ is conjugate to $\rho(\gamma_i)$ for all t . Thus, taking G^ρ -coinvariants, we obtain $T_{[\rho]}\text{Rep}(\dot{S}, G, \mathbf{c}) \cong \ker(\epsilon)_{G^\rho}$.

Combining the above computation with the properness in Lemma 2.10, we can conclude as follows.

Corollary 2.12 (Tangent space to relative representation spaces). For $n > 0$, the locus $\mathrm{Rep}^r(\dot{S}, G, \mathbf{c})$ is a smooth orbifold of dimension

	$\mathrm{Rep}_b^r(\dot{S}, G, \mathbf{c})$	$\mathrm{Rep}^r(\dot{S}, G, \mathbf{c})$
dim	$(-\chi(S) + 1) \dim(G) + \dim(\mathbf{c})$	$-\chi(S) \dim(G) + \dim(\mathbf{c})$

Moreover, the singular locus of $\mathrm{Rep}^r(\dot{S}, G, \bar{\mathbf{c}})$ consists of those $[\rho]$ with boundary values in the singular locus of $\bar{\mathbf{c}}$.

2.4. Flat vector bundles. Let $\mathbb{K} = \mathbb{R}, \mathbb{C}$ and consider first $G = \mathrm{GL}_N(\mathbb{K})$. To every flat principal G -bundle ξ on \dot{S} we can associate a vector bundle $V = \xi \times_G \mathbb{K}^N$ of rank N endowed with a natural flat connection ∇ in such a way that the monodromy of (V, ∇) coincides with that of ξ .

Vice versa, given a flat vector bundle (V, ∇) , we can construct the associate flat principal G -bundle ξ using the same locally constant transition functions as V , so that ξ has the same monodromy as V .

This establishes a correspondence

$$\mathrm{Fl}(\dot{S}, \mathbb{K}^N) \begin{array}{c} \xrightarrow{\quad} \\ \xleftarrow{\quad} \end{array} \mathrm{Fl}(\dot{S}, \mathrm{GL}_N(\mathbb{K})).$$

where $\mathrm{Fl}(\dot{S}, \mathbb{K}^N)$ is the set of isomorphism classes of flat vector bundles of rank N . An analogous correspondence holds for framed flat bundle, or for flat bundles with monodromy at the punctures in prescribed conjugacy classes.

For $G = \mathrm{SL}_N(\mathbb{K})$, the correspondence is between flat principal SL_N -bundles and flat vector bundles V of rank N endowed with a trivialization of their determinant $\det(V) = \Lambda^N V$.

Similarly, flat PGL_N -bundles correspond to flat $\mathbb{K}\mathbb{P}^{N-1}$ -bundles. Such a $\mathbb{K}\mathbb{P}^{N-1}$ -bundle \mathbb{P} need not be a projectivization of a flat vector bundle, since its monodromy need not lift to $\mathrm{GL}_N(\mathbb{K})$.

In the real case, $\mathrm{PSL}_{2N+1}(\mathbb{R}) = \mathrm{PGL}_{2N+1}(\mathbb{R})$; whereas flat $\mathrm{PSL}_{2N}(\mathbb{R})$ -bundles correspond to flat orientable $\mathbb{R}\mathbb{P}^{2N-1}$ -bundles.

2.5. The case of PSL_2 . Assume now that $G/Z = \mathrm{PSL}_2(\mathbb{C})$, namely that $G = \mathrm{SL}_2(\mathbb{C}), \mathrm{PSL}_2(\mathbb{C})$, and so $\mathfrak{g} = \mathfrak{sl}_2(\mathbb{C})$. Then to each flat principal G -bundle $\xi \rightarrow \dot{S}$ we can associate a flat $\mathbb{C}\mathbb{P}^1$ -bundle $\mathbb{P} := \xi \times_G \mathbb{C}\mathbb{P}^1$ on \dot{S} .

Remark 2.13. The image of ρ is Zariski-dense if and only if ρ is irreducible, namely if and only if no point of $\mathbb{C}\mathbb{P}^1$ is fixed by $\rho(\pi)$. The same holds for $G/Z = \mathrm{PSU}_2 \subset \mathrm{PSL}_2(\mathbb{C})$. In case $G/Z = \mathrm{PSL}_2(\mathbb{R})$, such Zariski-density can be expressed in terms of non-existence of fixed points in \mathbb{H} .

An easy consequence of Corollary 2.12 is the following.

Corollary 2.14 (Smoothness for PSL_2). Let G/Z be $\mathrm{PSL}_2(\mathbb{C}), \mathrm{PSU}_2, \mathrm{PSL}_2(\mathbb{R})$ and let $\mathbf{c} = (c_1, \dots, c_n)$ an n -uple of conjugacy classes of elements of G .

- (a) If $G/Z = \mathrm{PSU}_2$, then $\mathrm{Rep}_b^r(\dot{S}, G, \mathbf{c})$ and $\mathrm{Rep}^r(\dot{S}, G, \mathbf{c})$ are smooth orbifolds.
- (b) If $G/Z = \mathrm{PSL}_2(\mathbb{C}), \mathrm{PSL}_2(\mathbb{R})$, then $\rho \in \mathrm{Rep}_b^r(\dot{S}, G, \bar{\mathbf{c}})$ and $[\rho] \in \mathrm{Rep}^r(\dot{S}, G, \bar{\mathbf{c}})$ are singular points if and only if there exists i such that $\rho(\partial_i) \in Z$ and c_i consists of non-central unipotent elements.

Condition (b) in the above lemma can be easily understood by remembering that the only non-closed conjugacy class in $\mathrm{PSL}_2(\mathbb{C})$ is the class c of non-trivial unipotent elements, whose closure contains the identity as a singular point of $\bar{\mathbf{c}}$.

2.6. Euler number of $\mathrm{PSL}_2(\mathbb{R})$ -representations. Let $\widetilde{\mathrm{PSL}}_2(\mathbb{R})$ be the universal cover of $\mathrm{PSL}_2(\mathbb{R})$ and let $\mathbb{Z} \cdot \zeta \subset \widetilde{\mathrm{PSL}}_2(\mathbb{R})$ be its center, where $\zeta = \exp(R)$ and

$$R = \pi \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} \in \mathfrak{sl}_2(\mathbb{R})$$

is an infinitesimal generator of the subgroup of (counterclockwise) rotations that fix $i \in \mathbb{H}$. We define the *rotation number* $\mathrm{rot} : \widetilde{\mathrm{PSL}}_2(\mathbb{R}) \rightarrow \mathbb{R}$ as follows.

If $\tilde{g} \in \widetilde{\mathrm{PSL}}_2(\mathbb{R})$ is elliptic, then \tilde{g} is conjugate to $\exp(r \cdot R)$ for a unique $r \in \mathbb{R}$. In this case, we define $\mathrm{rot}(\tilde{g}) := r$, so that $\mathrm{rot}(\tilde{g}) \in \mathbb{Z} \iff \tilde{g} \in \mathbb{Z} \cdot \zeta$.

If $\tilde{g} \in \widetilde{\mathrm{PSL}}_2(\mathbb{R})$ is not elliptic, then there exists a unique $r \in \mathbb{Z}$ such that \tilde{g} can be connected to $r \cdot \zeta$ through a continuous path of non-elliptic elements. In this case, we define $\mathrm{rot}(\tilde{g}) := r$.

Define also a “fractional” rotation number as $\{\mathrm{rot}\} : \mathrm{PSL}_2(\mathbb{R}) \rightarrow [0, 1)$ by requiring that $\mathrm{rot}(\tilde{g}) - \{\mathrm{rot}\}(g) \in \mathbb{Z}$, where \tilde{g} is any lift of g to $\widetilde{\mathrm{PSL}}_2(\mathbb{R})$.

It can be easily seen that rot and $\{\mathrm{rot}\}$ are invariant under conjugation by elements of $\mathrm{PSL}_2(\mathbb{R})$ and that the rotation number rot is continuous but $\{\mathrm{rot}\}$ is not (see also [10] for more properties of the rotation number).

Here we adopt a result by Burger-Iozzi-Wienhard [10] as a definition of Euler number for a representation $\rho : \pi_1(\dot{S}) \rightarrow \mathrm{PSL}_2(\mathbb{R})$.

Definition 2.15. Assume $n > 0$. The *Euler number* of $\rho : \pi \rightarrow \mathrm{PSL}_2(\mathbb{R})$ is

$$\mathrm{eu}(\rho) := - \sum_{i=1}^n r_i \in \mathbb{R}$$

where $\tilde{\rho} : \pi \rightarrow \widetilde{\mathrm{PSL}}_2(\mathbb{R})$ is any lift of ρ and $r_i := \mathrm{rot}(\tilde{\rho}(\gamma_i)) \in \mathbb{R}$.

Notice that $\mathrm{eu}(\rho) + \|\{\mathbf{r}\}\|_1 \in \mathbb{Z}$, where $\{r_i\} := \{\mathrm{rot}\}(\rho(\gamma_i)) \in [0, 1)$.

Remark 2.16. If a representation is obtained as a composition $\rho' : \pi_1(S \setminus \{p_1, \dots, p_n\}, b) \rightarrow \pi_1(S \setminus \{p_1, \dots, p_k\}, b) \xrightarrow{\rho} \mathrm{PSL}_2(\mathbb{R})$ with $0 \leq k < n$, then $\{r_{k+1}\} = \dots = \{r_n\} = 0$ and $\mathrm{eu}(\rho') = \mathrm{eu}(\rho)$. This allows to coherently define the Euler number in the case of an unpunctured surface.

Remark 2.17. Suppose that $[\rho]$ is a singular point of $\mathrm{Rep}(\dot{S}, \mathrm{PSL}_2(\mathbb{R}))$. Then ρ must fix a point of \mathbb{RP}^1 or it must be Abelian. In the former case, there exists a lift $\tilde{\rho}$ whose action on the universal cover $\widetilde{\mathbb{RP}^1}$ fixes a point. Hence, $\mathrm{rot} \circ \tilde{\rho} = 0$ and so $\mathrm{eu}(\rho) = 0$ and $\{r_j\} = 0$ for all j . In the latter case, ρ Abelian implies $\mathrm{eu}(\rho) = 0$ and so $\|\{\mathbf{r}\}\|_1 \in \mathbb{Z}$.

Being invariant under conjugation by elements of $\mathrm{PSL}_2(\mathbb{R})$, the Euler number descends to a continuous map

$$\mathrm{eu} : \mathrm{Rep}(\dot{S}, \mathrm{PSL}_2(\mathbb{R})) \longrightarrow \mathbb{R}$$

and we call $\mathrm{Rep}(\dot{S}, \mathrm{PSL}_2(\mathbb{R}))_e$ the preimage $\mathrm{eu}^{-1}(e)$. If $n = 0$, then eu is integral and so it is constant on connected components of $\mathrm{Rep}(S, \mathrm{PSL}_2(\mathbb{R}))$. Moreover, the conjugation by an element in $\mathrm{PGL}_2(\mathbb{R}) \setminus \mathrm{PSL}_2(\mathbb{R})$ induces the isomorphism $\mathrm{Rep}(S, \mathrm{PSL}_2(\mathbb{R}))_e \cong \mathrm{Rep}(S, \mathrm{PSL}_2(\mathbb{R}))_{-e}$.

For closed surfaces the topology of $\mathrm{Rep}(S, \mathrm{PSL}_2(\mathbb{R}))_e$ with $e \neq 0$ is completely determined.

Theorem 2.18 (Topology of representation spaces of closed surfaces in $\mathrm{PSL}_2(\mathbb{R})$). Assume $n = 0$ and $g(S) \geq 2$.

- (a) Every $\rho \in \mathrm{Rep}_b(S, \mathrm{PSL}_2(\mathbb{R}))$ satisfies $|\mathrm{eu}(\rho)| \leq -\chi(S)$ (Milnor [43], Wood [66]). If $\mathrm{eu}(\rho) = -\chi(S)$, then ρ is the monodromy of a hyperbolic metric (Goldman [25]).
- (b) $\mathrm{Rep}(S, \mathrm{PSL}_2(\mathbb{R}))_e \neq \emptyset$ if and only if

$$e \in \mathbb{Z} \cap \left[\chi(S), -\chi(S) \right]$$

and, in this case, it is also connected (Goldman [25]).

(c) If $e \neq 0$, then $\mathrm{Rep}(S, \mathrm{PSL}_2(\mathbb{R}))_e$ is smooth.

For $e > 0$ the manifold $\mathrm{Rep}(S, \mathrm{PSL}_2(\mathbb{R}))_e$ is real-analytically diffeomorphic to a complex vector bundle of rank $-\frac{3}{2}\chi(S) - m$ over $\mathrm{Sym}^m(S)$, where $m = -\chi(S) - e$ (Hitchin [29]).

Let now $n > 0$. Given an n -uple $\mathbf{c} = (c_1, \dots, c_n)$ of conjugacy classes in $\mathrm{PSL}_2(\mathbb{R})$, we call $\{r_i\}$ the rotation number of any element in c_i . Moreover, we denote by P_{hyp} (resp. P_{ell} , P_+ , P_- , P_0) the subset of points $p_i \in P$ such that c_i is hyperbolic (resp. elliptic, positive unipotent, negative unipotent, the identity) and we let $s_- = \#P_-$ (resp. $s_0 = \#P_0$).

The restriction

$$\mathrm{eu}_{\mathbf{c}} : \mathrm{Rep}(\dot{S}, \mathrm{PSL}_2(\mathbb{R}), \mathbf{c}) \longrightarrow \mathbb{R}$$

of eu has the property that $\mathrm{eu}_{\mathbf{c}} + \|\{\mathbf{r}\}\|_1 \in \mathbb{Z}$, where $\|\{\mathbf{r}\}\|_1$ only depends on \mathbf{c} . Thus, $\mathrm{eu}_{\mathbf{c}}$ is also constant on connected components of $\mathrm{Rep}(\dot{S}, \mathrm{PSL}_2(\mathbb{R}), \mathbf{c})$.

As above, we denote by $\mathrm{Rep}(\dot{S}, \mathrm{PSL}_2(\mathbb{R}), \mathbf{c})_e$ the preimage $\mathrm{eu}_{\mathbf{c}}^{-1}(e)$ and we observe that the conjugation by an element of $\mathrm{PGL}_2(\mathbb{R}) \setminus \mathrm{PSL}_2(\mathbb{R})$ induces the isomorphism $\mathrm{Rep}(\dot{S}, \mathrm{PSL}_2(\mathbb{R}), \mathbf{c})_e \cong \mathrm{Rep}(\dot{S}, \mathrm{PSL}_2(\mathbb{R}), \mathbf{c}^{-1})_{-e}$, where $c_i^{-1} = \{g^{-1} \in \mathrm{PSL}_2(\mathbb{R}) \mid g \in c_i\}$.

In analogy with Theorem 2.18 in the closed case, the following result holds in the punctured case.

Theorem 2.19 (Topology of relative representation spaces of punctured surfaces in $\mathrm{PSL}_2(\mathbb{R})$). Assume $n > 0$ and let \mathbf{c} and $\{\mathbf{r}\}$ be as above.

- (a) The image of $\mathrm{eu} : \mathrm{Rep}(\dot{S}, \mathrm{PSL}_2(\mathbb{R})) \rightarrow \mathbb{R}$ is the interval $[\chi(\dot{S}), -\chi(\dot{S})]$. If $\mathrm{eu}(\rho) = -\chi(\dot{S})$, then all $\rho(\partial_i)$'s are hyperbolic or positive unipotent elements for all i and ρ is the monodromy of a hyperbolic metric with geodesic boundary components and cusps (Burger-Iozzi-Wienhard [10]).
- (b) Assume $e > 0$ and fix conjugacy classes $c_1, \dots, c_n \subset \mathrm{PSL}_2(\mathbb{R})$. Then $\mathrm{Rep}(\dot{S}, \mathrm{PSL}_2(\mathbb{R}), \mathbf{c})_e \neq \emptyset$ if and only if

$$e + \|\{\mathbf{r}\}\|_1 + s_0 + s_- \in \mathbb{Z} \cap (0, -\chi(\dot{S})]$$

and, in this case, it is smooth and connected.

Assume now that e satisfies the hypotheses in (b).

- (c) The component $\mathrm{Rep}(\dot{S}, \mathrm{PSL}_2(\mathbb{R}), \mathbf{c})_e$ is real-analytically diffeomorphic to the complement of s_- nowhere-parallel affine subbundles of codimension 1 inside a holomorphic affine bundle of rank $-\frac{3}{2}\chi(S) + n - m + s_-$ over $\mathrm{Sym}^{m-(s_0+s_-)}(S \setminus (P_{hyp} \cup P_+))$, where $m = -\chi(\dot{S}) - e - \|\{\mathbf{r}\}\|_1$.
- (d) The locus $\mathrm{Rep}(\dot{S}, \mathrm{PSL}_2(\mathbb{R}), \bar{\mathbf{c}})_e$ is homeomorphic to an affine holomorphic bundle of rank $-\frac{3}{2}\chi(S) + n - m + s_-$ over $\mathrm{Sym}^{m-(s_0+s_-)}(S \setminus P_{hyp})$.
- (e) If c_i is the class of positive unipotents and \mathbf{c}^0 is obtained from \mathbf{c} by replacing c_i with $\{\mathrm{Id}\}$, then $\mathrm{Rep}(\dot{S}, \mathrm{PSL}_2(\mathbb{R}), \bar{\mathbf{c}}^0)_e$ includes in $\mathrm{Rep}(\dot{S}, \mathrm{PSL}_2(\mathbb{R}), \bar{\mathbf{c}})_e$ as the preimage over $p_i + \mathrm{Sym}^{m-(s_0+s_-)-1}(S \setminus P_{hyp}) \subset \mathrm{Sym}^{m-(s_0+s_-)}(S \setminus P_{hyp})$.
If c_i is the class of negative unipotents and \mathbf{c}^0 is obtained from \mathbf{c} by replacing c_i with $\{\mathrm{Id}\}$, then $\mathrm{Rep}(\dot{S}, \mathrm{PSL}_2(\mathbb{R}), \bar{\mathbf{c}}^0)_e$ includes in $\mathrm{Rep}(\dot{S}, \mathrm{PSL}_2(\mathbb{R}), \bar{\mathbf{c}})_e$ as an affine subbundle over $\mathrm{Sym}^{m-(s_0+s_-)}(S \setminus P_{hyp})$ of codimension 1.

Claims (c-d-e) are consequence of Theorem 4.14, which in turn relies on Proposition 3.16 and Corollary 3.17. Claim (b) easily follows from (c-d-e).

The case of some $c_i = \{\mathrm{Id}\}$ can be also dealt with as in Remark 2.16.

2.7. Hyperbolic metrics. Let $\ell = (\ell_1, \dots, \ell_n)$ with $\ell_i = \sqrt{-1}\vartheta_i$ and $\vartheta_i > 0$ for $i = 1, \dots, k$ and $\ell_i \geq 0$ for $i = k+1, \dots, n$. We are interested in isotopy classes of *hyperbolic metrics of boundary type* ℓ on \dot{S} , i.e. metrics on curvature -1 on \dot{S} , whose completion has a conical

singularity of angle ϑ_i at p_i for $i = 1, \dots, k$ and a boundary component of length ℓ_i (resp. a cusp if $\ell_i = 0$) instead of the puncture p_i for $i = k + 1, \dots, n$.

We assume that the quantity

$$e_\ell := -\chi(\dot{S}) - \sum_{i=1}^k \frac{\vartheta_i}{2\pi} = -\chi(S) - \sum_{i=1}^k \left(\frac{\vartheta_i}{2\pi} - 1 \right)$$

is positive, since hyperbolic metrics of boundary type ℓ have total area $e_\ell > 0$ by Gauss-Bonnet. In this case, denote by $\mathcal{Y}(\dot{S}, \ell)$ the space of isotopy classes of hyperbolic metrics on \dot{S} of boundary type ℓ .

Surfaces of curvature -1 are locally isometric to portions of the hyperbolic plane \mathbb{H} . Given a metric h of curvature -1 on \dot{S} , consider the pull-back \tilde{h} on \tilde{S} . Since \tilde{S} is simply-connected, local isometries into \mathbb{H} glue to give a global *developing map* $\text{dev}_h : \tilde{S} \rightarrow \mathbb{H}$, which is a local isometry. Moreover, π acts on \mathbb{H} via a *monodromy* homomorphism $\rho_h : \pi \rightarrow \text{Iso}_+(\mathbb{H}) \cong \text{PSL}_2(\mathbb{R})$ and dev_h is π -equivariant. Notice that dev_h is well-defined up to post-composition with an isometry of \mathbb{H} , and so also ρ_h is well-defined only as an element of $\text{Rep}(\dot{S}, \text{PSL}_2(\mathbb{R}))$.

We also observe that ρ_h arises as a monodromy of a flat principal $\text{PSL}_2(\mathbb{R})$ -bundle as follows. Pull the trivial $\text{PSL}_2(\mathbb{R})$ -bundle over \mathbb{H} back via dev_h to a (trivializable) $\tilde{\xi}_h \rightarrow \tilde{S}$. By ρ_h -equivariance, it descends to a flat principal $\text{PSL}_2(\mathbb{R})$ -bundle $\xi_h \rightarrow \dot{S}$ with $\text{hol}_{\xi_h} = \rho_h$.

Then we have a composition of real-analytic maps

$$\begin{array}{ccc} \mathcal{Y}(\dot{S}, \ell) & \xrightarrow{\Xi_\ell} & \text{Fl}(\dot{S}, \text{PSL}_2(\mathbb{R}), \mathbf{c}_\ell) \xrightarrow{\text{hol}} \text{Rep}(\dot{S}, \text{PSL}_2(\mathbb{R}), \mathbf{c}_\ell) \\ h \vdash & \longrightarrow & \xi_h \vdash \longrightarrow \rho_h \end{array}$$

where the correspondence between ℓ_i and c_i is dictated by the following table

conjugacy class c_ℓ	ℓ
Id	$2\pi\sqrt{-1} \cdot \mathbb{N}_+$
positive unipotents	0
hyperbolics g with $ \text{Tr}(g) = 2 \cosh(\ell/2) $	\mathbb{R}_+
elliptics g with $\{\text{rot}\}(g) = \{\ell/(2\pi\sqrt{-1})\}$	$2\pi\sqrt{-1} \cdot (\mathbb{R}_+ \setminus \mathbb{N}_+)$

In fact, a cusp at p_i corresponds to positive unipotent monodromy along γ_i .

Proposition 2.20 (Uniformization components). Let $\vartheta_1, \dots, \vartheta_k > 0$ and $\ell_{k+1}, \dots, \ell_n \geq 0$ and call $\ell = (\sqrt{-1}\vartheta_1, \dots, \sqrt{-1}\vartheta_k, \ell_{k+1}, \dots, \ell_n)$. If $e_\ell > 0$, then the image of $\text{hol} \circ \Xi_\ell$ is contained inside $\text{Rep}(\dot{S}, \text{PSL}_2(\mathbb{R}), \mathbf{c}_\ell)_{e_\ell}$.

Proof. After the above discussion, it is enough to notice that the Euler number eu can be identified to the Toledo invariant (see for instance [10]) and so $\text{eu}(\text{hol}_h) = \frac{1}{2\pi} \text{Area}(h)$. \square

2.7.1. *Questions.* Still little is known in general about the image of

$$\text{hol} \circ \Xi_\ell : \mathcal{Y}(\dot{S}, \ell) \longrightarrow \text{Rep}(\dot{S}, \text{PSL}_2(\mathbb{R}), \mathbf{c}_\ell)_{e_\ell}$$

beside the fact that it is open for the classical topology and so its complement \mathcal{C}_ℓ is closed.

Question 1. For which ℓ is the image of $\text{hol} \circ \Xi_\ell$ the whole $\text{Rep}(\dot{S}, \text{PSL}_2(\mathbb{R}), \mathbf{c}_\ell)_{e_\ell}$?

Question 2. For which ℓ is \mathcal{C}_ℓ of zero measure?

Question 3. For which ℓ is \mathcal{C}_ℓ a countable union of proper (semi-)algebraic subsets with no internal part?

Question 4. For which ℓ does the mapping class group of (S, P) act ergodically on $\text{Rep}(\dot{S}, \text{PSL}_2(\mathbb{R}), \mathbf{c}_\ell)_{e_\ell}$?

Question 1 has affirmative answer if all the angles are smaller than π . Roughly speaking, a possible argument is as follows.

Since the angles are smaller than π , simple closed geodesics on \hat{S} avoid the conical points and so their lengths are detected by the traces of the monodromy. Moreover, any pair of pants decomposition of \hat{S} gives rise to Fenchel-Nielsen coordinates (and so $\mathcal{Y}(\hat{S}, \ell)$ is diffeomorphic to $(\mathbb{R}_+ \times \mathbb{R})^{3g-3+n}$) and this quickly leads to the injectivity of $\mathrm{hol} \circ \Xi_\ell$. The collar lemma allows to prove properness, and one concludes that $\mathrm{hol} \circ \Xi_\ell$ is indeed a diffeomorphism. In this case, the action of the mapping class group is properly discontinuous and with finite stabilizers, and so very far from being ergodic.

Concerning Question 4, Goldman has conjectured that the mapping class group of a closed surface S acts ergodically on the components $\mathrm{Rep}(S, \mathrm{PSL}_2(\mathbb{R}))_e$ with $e \neq 0, \pm\chi(S)$. A positive answer to Question 4 would immediately imply a positive answer to Question 2.

3. PARABOLIC HIGGS BUNDLES

In this section, S will denote a compact connected Riemann surface endowed with complex structure I .

3.1. Parabolic structures. Let E be a holomorphic vector bundle on S , which we constantly identify with the locally-free sheaf of its sections.

Definition 3.1. Let E be a holomorphic vector bundle on S . A *parabolic structure* on E over (S, P) is a filtration $\mathbb{R} \ni w \mapsto E_w \subset E(\infty \cdot P)$ of the sheaf $E(\infty \cdot P)$ of sections which are meromorphic at P such that

- (a) $E_w \supseteq E_{w'}$ if $w \leq w'$ (*decreasing*)
- (b) for every $w \in \mathbb{R}$ there exists $\varepsilon > 0$ such that $E_{w-\varepsilon} = E_w$ (*left-continuous*)
- (c) $E_0 = E$ and $E_{w+1} = E_w(-P)$ (*normalized*).

We will denote by E_\bullet the datum of the bundle E and the given parabolic structure.

Notation. Denote by E_{p_i} the space of regular germs at p_i of sections of E and by $E_{p_i}(\infty \cdot p_i)$ the space of germs at p_i of sections of E which are meromorphic at p_i . Given a parabolic structure on E , the induced filtration on $E_{p_i}(\infty \cdot p_i)$ is denoted by $w \mapsto E_{p_i, w}$.

By definition, the *jumps* in the filtration at p_i occur at those weights w such that $E_{p_i, w} \supsetneq E_{p_i, w+\varepsilon}$ for all $\varepsilon > 0$. Thus, a parabolic structure is equivalent to the datum of a weights

$$0 \leq w_1(p_i) < w_2(p_i) < \cdots < w_{b_i}(p_i) < w_{b_i+1}(p_i) = 1$$

for some $b_i \in [1, \mathrm{rk}(E) + 1]$ and a filtration

$$E_{p_i} = E_{p_i, w_1(p_i)} \supsetneq \cdots \supsetneq E_{p_i, w_{b_i}(p_i)} \supsetneq E_{p_i, w_{b_i+1}(p_i)} = E_{p_i}(-p_i)$$

for each $p_i \in P$.

Notation. We use the symbol \mathbf{w} to denote the collection of $(w_k(p_i), m_k(p_i))_{i=1}^n$, where $m_k(p_i) = \dim(E_{p_i, w_k(p_i)}/E_{p_i, w_{k+1}(p_i)})$ and we will say that the parabolic bundle E_\bullet is of *type* \mathbf{w} . We will also write $\|\mathbf{w}\|_1 = (\|w(p_1)\|_1, \dots, \|w(p_n)\|_1)$, where $\|w(p_i)\|_1 = \sum_{k=1}^{b_i} m_k(p_i)w_k(p_i)$, and we will say that \mathbf{w} is *integral* if $\|\mathbf{w}\|_1 \in \mathbb{N}^n$.

The (*parabolic*) *degree* of E_\bullet is defined as $\mathrm{deg}(E_\bullet) := \mathrm{deg}(E) + \sum_{p_i \in P} \|w(p_i)\|_1$.

Every holomorphic bundle E can be endowed with a *trivial parabolic structure*, by choosing $b_i = 1$ and $w_1(p_i) = 0$ for all $i = 1, \dots, n$. This provides an embedding on the category of holomorphic bundles on S inside the category of parabolic bundles on S .

Direct sums, homomorphisms and tensor products of parabolic bundles are defined as

$$\begin{aligned} (E \oplus E')_{p_i, w} &:= E_{p_i, w} \oplus E'_{p_i, w} \\ \text{Hom}(E_\bullet, E'_\bullet) &:= \left\{ f \in \text{Hom}(E, E') \mid w_j(p_i) > w'_k(p_i) \implies f(E_{p_i, w_j(p_i)}) \subseteq E'_{p_i, w'_k(p_i)} \right\} \\ (E \otimes E')_{p_i, w''} &:= \left(\bigcup_{w+w'=w''} E_{p_i, w} \otimes E'_{p_i, w'} \right) \subset (E \otimes E')_{p_i}(\infty \cdot p_i). \end{aligned}$$

and we will write $\mathbf{w} \otimes \mathbf{w}'$ for the type of a parabolic bundle $E_\bullet \otimes E'_\bullet$ obtained by tensoring E_\bullet of type \mathbf{w} with E'_\bullet of type \mathbf{w}' .

It is also possible to define a $\mathcal{H}om$ -sheaf just by letting $\mathcal{H}om(E_\bullet, E'_\bullet)(U) := \text{Hom}(E_\bullet|_U, E'_\bullet|_U)$ with $\mathcal{H}om(E_\bullet, E'_\bullet)_{p_i, w} = \{\text{germs at } p_i \text{ of morphisms } E_\bullet \rightarrow E'_{\bullet+w}\}$ and also a dual $E_\bullet^\vee := \mathcal{H}om(E_\bullet, \mathcal{O}_S)$.

We will say that a homomorphism is *injective* if it is so as a morphism of sheaves, namely if it is injective at the general point of S , and *properly injective* if it is injective but not an isomorphism. A *parabolic sub-bundle* of E_\bullet is just a sub-bundle $F \subseteq E$, endowed with the induced filtration $F_w := F \cap E_w$; the quotient bundle E/F can be also endowed with a natural parabolic structure by letting $(E/F)_w$ be the image of E_w under the natural projection $E \rightarrow E/F$.

Since $H^0(U, E_\bullet) = \text{Hom}(\mathcal{O}_U, E_\bullet|_U) = \text{Hom}(\mathcal{O}_U, E|_U) = H^0(U, E)$, sections of E_\bullet are sections of E and so the same holds for higher cohomology groups.

In order to understand parabolic structures, it is enough to localize the analysis and consider bundles on a disk with parabolic structure at the origin. The typical setting is the following.

Example 2 (Flat vector bundles on a punctured disk). Let $N > 0$ be an integer and let $\hat{\Delta} = \Delta \setminus \{p\}$ with $p = 0$. Let $\mathbb{H} \rightarrow \hat{\Delta}$ be the universal cover, defined as $u \mapsto z = \exp(2\pi\sqrt{-1} \cdot u)$, and let $b \in \hat{\Delta}$ be a base-point. Call $\tilde{V} := \mathbb{H} \times \mathbb{C}^N \rightarrow \mathbb{H}$ the trivial vector bundle and endow it with the natural connection $\tilde{\nabla}$ that can be expressed as $[\tilde{\nabla}]_{\tilde{V}} = d$ with respect to the canonical basis $\tilde{\mathcal{V}} = \{\tilde{v}_1, \dots, \tilde{v}_N\}$ of sections of \tilde{V} and a natural holomorphic structure $\bar{\partial}^{\tilde{V}}$. Given $T = \exp(-2\pi\sqrt{-1} \cdot M) \in \text{GL}_N(\mathbb{C})$, one can lift the natural action of $\pi_1(\hat{\Delta}, b) = \langle \gamma \rangle$ on \mathbb{H} to an action on \tilde{V} by letting $\gamma \cdot (u, v) = (u+1, T(v))$. The induced bundle $\dot{V} := \tilde{V}/\pi_1(\hat{\Delta}, b)$ inherits a flat connection ∇ that can be written as $[\nabla]_{\dot{V}} = d$ with respect to the basis $\mathcal{V} = \{v_1, \dots, v_N\}$ of flat $\bar{\partial}^{\dot{V}}$ -holomorphic multi-sections of \dot{V} . Moreover, chosen the standard determination of $\log(z)$ on $\hat{\Delta}$,

$$\begin{pmatrix} v'_1 \\ \vdots \\ v'_N \end{pmatrix} := \exp(\log(z) \cdot M) \begin{pmatrix} v_1 \\ \vdots \\ v_N \end{pmatrix}$$

defines a basis $\mathcal{V}' = \{v'_1, \dots, v'_N\}$ of univalent $\bar{\partial}^{\dot{V}}$ -holomorphic sections of \dot{V} such that $[\nabla]_{\mathcal{V}'} = d + M \frac{dz}{z}$. Notice that ∇ and so $\text{Res}_p(\nabla) \in \text{End}(V|_p)$ are not uniquely defined by T , since $\exp(-2\pi\sqrt{-1} \cdot \bullet) : \mathfrak{gl}_N(\mathbb{C}) \rightarrow \text{GL}_N(\mathbb{C})$ is not injective: in particular, the eigenvalues of $\text{Res}_p(\nabla)$ are only well-defined in \mathbb{C}/\mathbb{Z} .

3.1.1. *Rank 1.* A parabolic structure at P on a line bundle $L \rightarrow S$ is just the datum of a weight $w(p_i) \in [0, 1)$ for each $p_i \in P$, so that it makes sense to write $L_\bullet = L(\sum_i w(p_i)p_i)$. The parabolic degree of L_\bullet is simply $\text{deg}(L_\bullet) = \text{deg}(L) + \sum_i w(p_i)$.

Notation. If $L_\bullet = L(\sum_i w(p_i)p_i)$ is a parabolic line bundle, we denote by $[L_\bullet]$ its *integral part*, namely the underlying line bundle L with trivial parabolic structure and by $\{L_\bullet\} := L_\bullet \otimes [L_\bullet]^\vee = \mathcal{O}_S(\sum_i w(p_i)p_i)$ its *fractional part*.

We incidentally remark that integral parabolic structures in rank 1 are trivial.

Example 3 (Unitary line bundles on a punctured disk). Keep the notation as in Example 2 and let $N = 1$, $v = v_1$ and $T = \exp(-2\pi\sqrt{-1} \cdot \lambda) \in \mathrm{U}_1$ with $\lambda \in [0, 1)$. With respect to $\mathcal{V}' = \{v'\}$, the flat connection

$$[\nabla]_{\mathcal{V}'} = d + \lambda \frac{dz}{z}$$

on \dot{V} with $[\mathrm{Res}_p(\nabla)]_{\mathcal{V}'} = \lambda$ has monodromy $\langle T \rangle$ and $v = z^{-\lambda}v'$ is a flat holomorphic multi-section of \dot{V} . Let $V \rightarrow \Delta$ be the extension of \dot{V} defined by requiring that v' is a generator. We can put on V a ∇ -invariant metric H by prescribing that $\|v\|_H = c > 0$ is constant, namely

$$\|v'\|_H := c|z|^\lambda$$

and so such an invariant norm has a zero of order $\lambda \in [0, 1)$ at 0. We can then put on $E := V$ the same holomorphic structure as V and take $e = v'$ as a holomorphic generator of E , and define the filtration E_w by

$$H^0(U, E_w) = \left\{ s \in H^0(U, E) \mid \begin{array}{l} \text{on every neighbourhood } U' \Subset U \text{ of } p \\ |z|^{\varepsilon-w} \|s\|_H \text{ is bounded for all } \varepsilon > 0 \end{array} \right\}$$

so that $H^0(U, E) = H^0(U, E_w)$ for every $w \in (-1 + \lambda, \lambda]$ and the jumps occur at weights in $\lambda + \mathbb{Z}$.

The determinant $\det(E_\bullet)$ of a parabolic bundle E_\bullet of rank N is the parabolic line bundle defined in the standard way as a quotient of $E_\bullet^{\otimes N}$ by the alternating action of \mathfrak{S}_N . If E_\bullet is of type \mathbf{w} , then

$$\det(E_\bullet) \cong \det(E) \otimes \left(\bigotimes_{i=1}^n \mathcal{O}_S(\|w(p_i)\|_{1p_i}) \right)$$

In particular, if \mathbf{w} is integral, then $\det(E_\bullet)$ has trivial parabolic structure.

3.1.2. Rank 2 of integral type. Among all parabolic bundles of rank 2 we focus on those of integral type because of their relation with SL_2 -bundles.

Remark 3.2. A parabolic structure E_\bullet on a bundle E of rank 2 on (S, P) is of integral type \mathbf{w} (or, briefly, just *integral*) if and only if the following condition holds for every $p_i \in P$:

- (1) either $b_i = 1$ (*degenerate case*) and $w_1(p_i) \in \{0, \frac{1}{2}\}$;
- (2) or $b_i = 2$ (*non-degenerate case*) and $(0, \frac{1}{2}) \ni w_1(p_i) < w_2(p_i) = 1 - w_1(p_i) \in (\frac{1}{2}, 1)$.

If E_\bullet is integral of rank 2, then $\deg(E_\bullet) = \deg(E) + \#\{p_i \in P \mid w_1(p_i) > 0\}$.

We remark that, if E_\bullet is non-degenerate at p_i (i.e. $b_i = 2$), then giving $E_{p_i, w_2(p_i)}$ is equivalent to giving a line $L_i \subset E|_{p_i}$. Indeed, knowing $E_{p_i, w_2(p_i)}$, the line L_i can be recovered as the kernel of $E|_{p_i} \rightarrow (E_{p_i}/E_{p_i, w_2(p_i)})$. Vice versa, given L_i , the germ $E_{p_i, w_2(p_i)}$ is the kernel of $E_{p_i} \rightarrow E|_{p_i}/L_i$.

Example 4 (Type of parabolic line sub-bundle). Let E_\bullet be an integral parabolic bundle of rank 2 and let $F \subset E$ be a sub-bundle of rank 1. Then the jump of the induced parabolic structure on F at p_i occurs at $w_F(p_i)$, where

	E_\bullet degenerate at p_i	E_\bullet non-degenerate at p_i
$w_F(p_i)$	$w_1(p_i)$	$w_1(p_i)$ if $F _{p_i} \neq L_i$ $1 - w_1(p_i)$ if $F _{p_i} = L_i$

The following example illustrates how parabolic structures on holomorphic vector bundles arise from unitary representations: since the argument is local, we will only deal with the case of a bundle over a disk. A complete treatment can be found in [42].

Example 5 (Rank 2 special unitary vector bundles on a punctured disk). Keep the notation as in Example 2 and let $N = 2$ and $T \in \mathrm{SU}_2$. Up to conjugation, we can assume that T is diagonal and that $T(v_1) = \exp(-2\pi\sqrt{-1} \cdot \lambda)v_1$ and $T(v_2) = \exp(2\pi\sqrt{-1} \cdot \lambda)v_2$ with $\lambda \in [0, 1)$,

and so T acts on \mathbb{CP}^1 as a positive rotation of angle $4\pi\lambda$ that fixes $[1 : 0]$. The connection ∇ on \dot{V} defined as

$$[\nabla]_{\mathcal{V}'} = d + \begin{pmatrix} \lambda & 0 \\ 0 & -\lambda \end{pmatrix} \frac{dz}{z} \quad \text{with } [\text{Res}_p(\nabla)]_{\mathcal{V}'} = \begin{pmatrix} \lambda & 0 \\ 0 & -\lambda \end{pmatrix}$$

with respect to the basis $\mathcal{V}' = \{v'_1 = z^\lambda v_1, v'_2 = z^{-\lambda} v_2\}$ has monodromy $\langle T \rangle$. Put on \dot{V} a ∇ -invariant metric H by prescribing that v_1, v_2 are orthogonal with $\|v_i\|_H = c_i > 0$ and define the univalent sections e_1, e_2 of \dot{V} as

$$e_1 = v'_1, \quad e_2 = \begin{cases} v'_2 & \text{if } \lambda = 0 \\ z \cdot v'_2 & \text{if } \lambda > 0 \end{cases}$$

and extend \dot{V} to a vector bundle $V \rightarrow \Delta$ with generators e_1, e_2 . The holomorphic vector bundle $E := V$ endowed with the same holomorphic structure as V has a pointwise orthogonal basis $\mathcal{E} = \{e_1, e_2\}$ of holomorphic sections, that satisfy

$$\|e_1\|_H = c_1 |z|^\lambda, \quad \|e_2\|_H = \begin{cases} c_2 & \text{if } \lambda = 0 \\ c_2 |z|^{1-\lambda} & \text{if } \lambda > 0 \end{cases}$$

If $\lambda = 0$, then $\|e_i\|_H = c_i$ and we have a degenerate parabolic structure with $w_1 = 0$. Similarly, if $\lambda = \frac{1}{2}$, then $\|e_i\|_H = c_i |z|^{\frac{1}{2}}$ and $w_1 = \frac{1}{2}$.

Assume now that $\lambda > 0$ but $\lambda \neq \frac{1}{2}$.

If $\lambda \in (0, \frac{1}{2})$, then we let $w_1 = \lambda < w_2 = 1 - \lambda$ and $L = \mathbb{C}e_2 \times \{p\} \subset E|_p$. Since $|z|^\lambda > |z|^{1-\lambda}$, a section $s = f_1(z)e_1 + f_2(z)e_2$ satisfies

$$\text{ord}_p \|s\|_H = \text{ord}_p (|f_1 z^\lambda| + |f_2 z^{1-\lambda}|) = \begin{cases} w_1 = \lambda & \text{if } f_1(p) \neq 0, \text{ i.e. if } s(p) \notin L \\ w_2 = 1 - \lambda & \text{if } f_1(p) = 0, \text{ i.e. if } s(p) \in L. \end{cases}$$

If $\lambda \in (\frac{1}{2}, 1)$, then we let $w_1 = 1 - \lambda < w_2 = \lambda$ and $L = \mathbb{C}e_1 \times \{p\} \subset E|_p$. In both cases, E_w is defined by

$$H^0(U, E_w) = \{s \in H^0(U, E) \mid \|s\|_H \cdot |z|^{\varepsilon-w} \text{ bounded near } p \text{ for all } \varepsilon > 0\}$$

and in particular, $H^0(U, E_{w_2}) = \{s \in H^0(U, E) \mid s(p) \in L\}$ and

$$H^0(U, E_w) = \begin{cases} H^0(U, E) & \text{for } w \in [0, w_1] \\ H^0(U, E_{w_2}) & \text{for } w \in (w_1, w_2) \\ H^0(U, E(-p)) & \text{for } w \in (w_2, 1). \end{cases}$$

Observe that such parabolic structure is integral.

3.2. Higgs bundles.

Definition 3.3. A *parabolic Higgs bundle* on (S, P) is a couple (E_\bullet, Φ) , where E_\bullet is a holomorphic parabolic vector bundle of rank N and $\Phi \in H^0(S, K(P) \otimes \text{End}(E_\bullet))$. The *residue* of Φ at p_i is the induced endomorphism $\text{Res}_{p_i}(\Phi) \in \text{End}(E_\bullet|_{p_i})$ of the filtered vector space $E_\bullet|_{p_i}$.

Here is the motivating example in rank $N = 1$ on the punctured disk.

Example 6 (Line bundles on a punctured disk). Keep the notation as in Example 2 and let $N = 1$ and $T = \exp[-2\pi\sqrt{-1}(\lambda + i\nu)] \in \text{GL}_1(\mathbb{C}) = \mathbb{C}^*$ with $\lambda \in [0, 1)$ and $\nu \in \mathbb{R}$. We can assume that $\nu \neq 0$, as the case $\nu = 0$ has already been discussed in Example 3. Such a monodromy T is induced by a flat connection ∇ that can be written as

$$[\nabla]_{\mathcal{V}'} = d + (\lambda + i\nu) \frac{dz}{z} \quad \text{with } [\text{Res}_p(\nabla)]_{\mathcal{V}'} = \lambda + i\nu$$

with respect to $\mathcal{V}' = \{v' = v'_1\}$. The metric H on \dot{V} defined by

$$\|v'\|_H := c|z|^\lambda$$

is harmonic with respect to ∇ , since $i\partial\bar{\partial}\log\|v'\|_H^2 = 0$. Thus,

$$[\nabla]_{\mathcal{V}'} = [\nabla^H]_{\mathcal{V}'} + \Phi + \bar{\Phi} = \left(d + \lambda \frac{dz}{z} + i\frac{\nu}{2} \frac{dz}{z} + i\frac{\nu}{2} \frac{d\bar{z}}{\bar{z}} \right) + \left(i\frac{\nu}{2} \frac{dz}{z} \right) + \left(-i\frac{\nu}{2} \frac{d\bar{z}}{\bar{z}} \right)$$

where ∇^H is a connection on \dot{V} compatible with the metric H and $\mathrm{Res}_p(\Phi) = i\nu/2$. Extend \dot{V} to the bundle $V = \mathbb{C}v' \times \Delta \rightarrow \Delta$ and put on the complex line bundle $E := V$ the holomorphic structure given by $\bar{\partial}^E := \bar{\partial}^V - \bar{\Phi}$, so that ∇^H is a Chern connection on $(E, \bar{\partial}^E, H)$. Holomorphic sections of E are generated then by $e = \exp(-i\nu \log|z|)v'$. Since $\|e\|_H = \|v'\|_H = c|z|^\lambda$, the filtration E_\bullet is as in Example 3, and so that the jumps occur at $\lambda + \mathbb{Z}$.

The following computation is borrowed from [53].

Example 7 (Flat SL_2 -vector bundles on a punctured disk). Keep the notation as in Example 2 and let $N = 2$ and $T \in \mathrm{SL}_2(\mathbb{C})$. If T is diagonalizable, then the bundle \dot{E} splits and we are reduced to the rank 1 case of Example 6. Thus, up to conjugation, we can assume that

$$[T]_{\mathcal{V}'} = \frac{1}{2} \begin{pmatrix} 1 & -1 \\ 0 & 1 \end{pmatrix}, \quad [\nabla]_{\mathcal{V}'} = d + \frac{1}{2} \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} \frac{dz}{z}$$

with respect to $\mathcal{V}' = \{v'_1, v'_2\}$ and so $[\mathrm{Res}_p(\nabla)]_{\mathcal{V}'} = \frac{1}{2} \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}$. Put on \dot{V} the metric H defined by

$$[H]_{\mathcal{V}'} = \begin{pmatrix} 2|\log|z||^{-1} & -1 \\ -1 & |\log|z|| \end{pmatrix}$$

which is harmonic for ∇ and extend \dot{V} to a complex bundle $V \rightarrow S$ with basis \mathcal{V}' . Thus, $\nabla = \nabla^H + \Phi + \bar{\Phi}^H$, where

$$[\Phi]_{\mathcal{V}'} = \frac{1}{2} \begin{pmatrix} -|\log|z||^{-1} & -1 \\ -|\log|z||^{-2} & |\log|z||^{-1} \end{pmatrix} \frac{dz}{z}, \quad [\bar{\Phi}^H]_{\mathcal{V}'} = \frac{1}{2} \begin{pmatrix} 0 & 0 \\ |\log|z||^{-2} & 0 \end{pmatrix} \frac{d\bar{z}}{\bar{z}}.$$

and ∇^H is compatible with H . Let now $\dot{E} := \dot{V}$ as a complex vector bundle and notice that $\mathcal{E} = \{e_1 := v'_1 + \frac{v'_2}{|\log|z||}, e_2 := v'_2\}$ is a set of generators for \dot{E} , which are holomorphic with respect to the operator $\bar{\partial}^E := \bar{\partial}^V - \bar{\Phi}^H$. Thus, we can extend \dot{E} to a $E \rightarrow \Delta$ by requiring that e_1, e_2 are generators. A quick calculation gives

$$[\Phi]_{\mathcal{E}} = \frac{1}{2} \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} \frac{dz}{z}, \quad [\bar{\Phi}^H]_{\mathcal{E}} = \frac{1}{2} \begin{pmatrix} 0 & 0 \\ |\log|z||^{-2} & 0 \end{pmatrix} \frac{d\bar{z}}{\bar{z}}$$

and so $\Phi \in H^0(\Delta, K(p) \otimes \mathrm{End}_0(\dot{E}))$ is a traceless Higgs field with nonzero nilpotent residue at p and $\bar{\Phi}^H$ is its H -adjoint. From

$$\frac{c}{|\log|z||^{\frac{1}{2}}} \leq \|e_j\|_H \leq c' |\log|z||^{\frac{1}{2}}$$

it follows that $w_1 = 0$, the parabolic structure is integral degenerate and the jumps occur at \mathbb{Z} .

A similar computation shows that, for $T' = -T$, the norm satisfies

$$\frac{c|z|^{\frac{1}{2}}}{|\log|z||^{\frac{1}{2}}} \leq \|e_j\|_H \leq c' |\log|z||^{\frac{1}{2}} |z|^{\frac{1}{2}}$$

and so we would still obtain an integral degenerate parabolic structure with $w_1 = \frac{1}{2}$ and a traceless Higgs field with nonzero nilpotent residue, but the jumps would occur at $\frac{1}{2} + \mathbb{Z}$.

A morphism of parabolic Higgs bundles $f : (E_\bullet, \Phi) \rightarrow (E'_\bullet, \Phi')$ is a map $f : E_\bullet \rightarrow E'_\bullet$ of parabolic bundle that makes the following diagram

$$\begin{array}{ccc} E & \xrightarrow{\Phi} & E \otimes K(P) \\ \downarrow f & & \downarrow f \otimes 1 \\ E' & \xrightarrow{\Phi'} & E' \otimes K(P) \end{array}$$

commutative. We will say that f is injective (resp. properly injective) if $f : E \rightarrow E'$ is. A parabolic Higgs sub-bundle of (E_\bullet, Φ) is a sub-bundle $F_\bullet \subseteq E_\bullet$ such that $\Phi(F) \subseteq F \otimes K(P)$; the map $\Psi : (E/F)_\bullet \rightarrow (E/F)_\bullet \otimes K(P)$ induced by Φ makes $((E/F)_\bullet, \Psi)$ into a parabolic Higgs quotient bundle.

3.3. Stability. The *slope* of the parabolic bundle E_\bullet on (S, P) is defined as $\mu(E_\bullet) := \frac{\deg(E_\bullet)}{\text{rk}(E)}$.

Definition 3.4. A parabolic Higgs bundle (E_\bullet, Φ) on (S, P) is *stable* (resp. *semi-stable*) if $\mu(F_\bullet) < \mu(E_\bullet)$ (resp. $\mu(F_\bullet) \leq \mu(E_\bullet)$) for every properly injective $(F_\bullet, \Psi) \rightarrow (E_\bullet, \Phi)$. A direct sum of stable parabolic Higgs bundles with the same μ is said *polystable*.

Remark 3.5. It is well-known that stable bundles are *simple*, i.e. their endomorphisms are multiples of the identity, and so the group of their automorphism is \mathbb{C}^* . The same argument works for parabolic Higgs bundles. Indeed, if $f : (E_\bullet, \Phi) \rightarrow (E_\bullet, \Phi)$ is a non-zero homomorphism, then $\mu(E_\bullet) \leq \mu(\text{Im}(f)) \leq \mu(E_\bullet)$ by semistability of (E_\bullet, Φ) . This forces $\text{Im}(f) = E_\bullet$ because (E_\bullet, Φ) is stable. Now pick a point $q \in S$ and let $\lambda \in \mathbb{C}$ be an eigenvalue of $f_q : E|_q \rightarrow E|_q$. The endomorphism $(f - \lambda \cdot \text{Id}) \in \text{End}(E_\bullet, \Phi)$ is not surjective and so it vanishes by the above argument. It follows that $f = \lambda \cdot \text{Id}$.

Semi-stable parabolic Higgs bundles have the Jordan-Hölder property: if (E_\bullet, Φ) is semi-stable, then there exists a filtration

$$\{0\} = E_\bullet^0 \subsetneq E_\bullet^1 \subsetneq E_\bullet^2 \subsetneq \dots \subsetneq E_\bullet$$

by parabolic sub-Higgs-bundles such that the Higgs bundle structure $\text{Gr}^s(E_\bullet, \Phi)$ induced on the quotient $E_\bullet^s/E_\bullet^{s-1}$ is stable and with slope $\mu(\text{Gr}^s(E_\bullet)) = \mu(E_\bullet)$. It can be checked that, though the filtration is not canonical, the associated graded object

$$\text{Gr}(E_\bullet, \Phi) = \bigoplus_s \text{Gr}^s(E_\bullet, \Phi)$$

is. As for vector bundles, two parabolic Higgs bundles (E_\bullet, Φ) , (E'_\bullet, Φ') are called *S-equivalent* if $\text{Gr}(E_\bullet, \Phi) \cong \text{Gr}(E'_\bullet, \Phi')$. Thus, every semistable object is *S-equivalent* to a unique polystable one, up to isomorphism.

3.4. Moduli spaces of parabolic SL_n -Higgs bundles. Fix a type \mathbf{w} , an N -uple \mathbf{c} of conjugacy classes in $\mathfrak{sl}_N(\mathbb{C})$ and a base-point $b \in \dot{S}$. Fix also a holomorphic parabolic line bundle \mathcal{D}_\bullet of type $\|\mathbf{w}\|_1$.

Definition 3.6. A *parabolic Higgs bundle of rank N with determinant \mathcal{D}_\bullet* on (S, P) of type \mathbf{w} is a triple (E_\bullet, η, Φ) , where E_\bullet is a holomorphic parabolic vector bundle of rank N and type \mathbf{w} , endowed with an isomorphism $\eta : \det(E_\bullet) \xrightarrow{\sim} \mathcal{D}_\bullet$ and a *Higgs field* $\Phi \in H^0(S, K(P) \otimes \text{End}_0(E_\bullet))$. An isomorphism $(E_\bullet, \eta, \Phi) \rightarrow (E'_\bullet, \eta', \Phi')$ of parabolic Higgs bundles of rank N with determinant \mathcal{D}_\bullet is a map $f : E_\bullet \rightarrow E'_\bullet$ which is an isomorphism of parabolic Higgs bundles and such that $\eta = \eta' \circ \det(f)$.

By Remark 3.5, an automorphism f of (E_\bullet, η, Φ) must satisfy $\det(f_x) = 1$ at all $x \in S$. Hence, if (E_\bullet, Φ) is simple, then $\text{Aut}(E_\bullet, \eta, \Phi) = \mu_N \cdot \text{Id}$, where $\mu_N \subset \mathbb{C}^*$ is the cyclic subgroup of N -th roots of unity.

Denote by $\mathrm{Higgs}_b^{ss}(S, N, \mathbf{w}, \mathcal{D}_\bullet, \mathbf{c})$ the set of isomorphism classes of quadruples $(E_\bullet, \eta, \Phi, \tau)$ such that

- (E_\bullet, η, Φ) is a semistable parabolic Higgs bundle on (S, P) of rank N , type \mathbf{w} and with determinant \mathcal{D}_\bullet .
- $\mathrm{Res}_{p_i}(\Phi) \in \mathbf{c}_i$ for all $i = 1, \dots, n$
- $\tau : E|_b \xrightarrow{\sim} \mathbb{C}^N$ is a framing at b .

We denote by $\mathrm{Higgs}_b^s \subseteq \mathrm{Higgs}_b^{ps} \subseteq \mathrm{Higgs}_b^{ss}$ the stable and polystable loci.

The following two results are due to Simpson [53], Konno [34] and Yokogawa [67].

Theorem 3.7 (Moduli space of framed semi-stable parabolic Higgs bundles). The space $\mathrm{Higgs}_b^{ss}(S, N, \mathbf{w}, \mathcal{D}_\bullet, \mathbf{c})$ is a normal quasi-projective variety and a fine moduli space of b -framed semi-stable Higgs bundles on (S, P) of rank N , type \mathbf{w} and with determinant \mathcal{D}_\bullet . Moreover, the stable locus $\mathrm{Higgs}_b^s(S, N, \mathbf{w}, \mathcal{D}_\bullet, \mathbf{c})$ is smooth.

The group $\mathrm{GL}_N(\mathbb{C})$ acts by post-composition on the b -framing, and so it acts on $\mathrm{Higgs}_b^{ss}(S, N, \mathbf{w}, \mathcal{D}_\bullet, \mathbf{c})$: we denote by $\widehat{\mathrm{Higgs}}(S, N, \mathbf{w}, \mathcal{D}_\bullet; \mathbf{c})$ the set-theoretic quotient and by $\mathrm{Higgs}(S, N, \mathbf{w}, \mathcal{D}_\bullet, \mathbf{c})$ its Hausdorffization.

Theorem 3.8 (Moduli space of stable parabolic Higgs bundles). The Hausdorff quotient $\mathrm{Higgs}(S, N, \mathbf{w}, \mathcal{D}_\bullet, \mathbf{c})$ is a normal quasi-projective variety, whose points are in bijection with S -equivalence classes of semi-stable Higgs bundles on (S, P) of rank N , type \mathbf{w} and with determinant \mathcal{D}_\bullet . The open locus $\mathrm{Higgs}^s(S, N, \mathbf{w}, \mathcal{D}_\bullet, \mathbf{c})$ is an orbifold and a fine moduli space of stable objects.

Let F_\bullet be a line bundle on S with parabolic structure of type \mathbf{w}_F (which is trivial at b) and fix a trivialization of F_\bullet at the basepoint $b \in \dot{S}$. Then $E_\bullet \mapsto F_\bullet \otimes E_\bullet$ induces a $\mathrm{GL}_N(\mathbb{C})$ -equivariant isomorphism

$$\mathrm{Higgs}_b^{ss}(S, N, \mathbf{w}, \mathcal{D}_\bullet, \mathbf{c}) \xrightarrow{\sim} \mathrm{Higgs}_b^{ss}(S, N, \mathbf{w}_F \otimes \mathbf{w}, F_\bullet^{\otimes N} \otimes \mathcal{D}_\bullet, \mathbf{c})$$

that preserves polystable and stable locus. Thus, it induces an isomorphism

$$\mathrm{Higgs}(S, N, \mathbf{w}, \mathcal{D}_\bullet, \mathbf{c}) \cong \mathrm{Higgs}(S, N, \mathbf{w}_F \otimes \mathbf{w}, F_\bullet^{\otimes N} \otimes \mathcal{D}_\bullet, \mathbf{c})$$

that preserves the stable locus.

Remark 3.9. Let $L_\bullet = L(\sum_{i=1}^n w(p_i)p_i)$ be a parabolic line bundle on (S, P) and let $N \geq 2$. Chosen $p_0 \in S$, there exist an integer r and a line bundle Q such that $L \cong Q^{\otimes N} \otimes \mathcal{O}_S(r \cdot p_0)$ with $0 \leq r < N - 1$. Thus, $L_\bullet \cong Q(\frac{r}{N}p_0 + \sum_{i=1}^n \frac{w(p_i)}{N}p_i)^{\otimes N}$. Hence, if $n > 0$, then we can choose $p_0 = p_1$ for instance, and so L_\bullet admits an N -root which is a line bundle on S with parabolic structure at P . If $n = 0$, then L_\bullet admits an N -th root which is a line bundle (possibly) with parabolic structure at p_0 .

By the above remark, we can choose an N -th root F_\bullet of \mathcal{D}_\bullet^\vee and so we have established an isomorphism $\mathrm{Higgs}^s(S, N, \mathbf{w}, \mathcal{D}_\bullet, \mathbf{c}) \rightarrow \mathrm{Higgs}^s(S, N, \mathbf{w}_F \otimes \mathbf{w}, \mathcal{O}_S, \mathbf{c})$.

For rank 2 integral parabolic structures the remark specializes to the following.

Corollary 3.10 (Odd and even rank 2 integral parabolic structures). Let $N = 2$ and fix an integral parabolic type \mathbf{w} and a line bundle \mathcal{D} with trivial parabolic structure. Also fix an auxiliary point $p_0 \in \dot{S}$ different from b and let \mathbf{w}_0 be the parabolic type of $\mathcal{O}_S(\frac{1}{2}p_0)$. Then the moduli space $\mathrm{Higgs}^s(S, 2, \mathbf{w}, \mathcal{D}, \mathbf{c})$ is isomorphic to either of the following:

- (1) $\mathrm{Higgs}^s(S, 2, \mathbf{w}_0 \otimes \mathbf{w}, \mathcal{O}_S, \mathbf{c})$, if $\deg(\mathcal{D})$ is odd;
- (2) $\mathrm{Higgs}^s(S, 2, \mathbf{w}, \mathcal{O}_S, \mathbf{c})$, if $\deg(\mathcal{D})$ is even.

3.5. Involution and fixed locus. We now restrict to the case of rank $N = 2$, and we remark that every element $X \in \mathfrak{sl}_2(\mathbb{C})$ is in the same $\mathrm{Ad}_{\mathrm{SL}_2(\mathbb{C})}$ -orbit as $-X$.

Fix integral \mathbf{w} , an n -uple of conjugacy classes \mathbf{c} in $\mathfrak{sl}_2(\mathbb{C})$ and a line bundle \mathcal{D} with trivial parabolic structure and let $d_0 = \deg(\mathcal{D})$.

Following Hitchin, consider the involution $\sigma_b : \text{Higgs}_b^{ss}(S, 2, \mathbf{w}, \mathcal{D}, \mathbf{c}) \rightarrow \text{Higgs}_b^{ss}(S, 2, \mathbf{w}, \mathcal{D}, \mathbf{c})$ defined as $\sigma(E_\bullet, \eta, \Phi, \tau) := (E_\bullet, \eta, -\Phi, \tau)$, and let σ be the induced map on $\text{Higgs}(S, 2, \mathbf{w}, \mathcal{D}, \mathbf{c})$.

Lemma 3.11 (σ -fixed locus). Let (E_\bullet, η, Φ) be a polystable parabolic Higgs bundle of rank 2, type \mathbf{w} , determinant \mathcal{D} and with residues at p_i in \mathbf{c}_i . The point $[E_\bullet, \eta, \Phi]$ is fixed under σ if and only if there exists an isomorphism $\iota : (E_\bullet, \eta, \Phi) \xrightarrow{\sim} (E_\bullet, \eta, -\Phi)$, which happens if and only if one of the following conditions is satisfied:

- (a) $\Phi = 0$ and E_\bullet is polystable;
- (b) $E_\bullet \cong (L_\bullet^\vee \otimes \mathcal{D}) \oplus L_\bullet$ with $\deg(\mathcal{D}) \leq 2 \deg(L_\bullet)$ and

$$0 \neq \Phi = \begin{pmatrix} 0 & \phi \\ \psi & 0 \end{pmatrix}, \quad \iota = \pm \begin{pmatrix} i & 0 \\ 0 & -i \end{pmatrix}$$

with respect to this decomposition, where

$$\begin{aligned} 0 \neq \phi &\in \text{Hom}(L_\bullet, L_\bullet^\vee \otimes \mathcal{D}) \otimes K(P) \\ \psi &\in \text{Hom}(L_\bullet^\vee \otimes \mathcal{D}, L_\bullet) \otimes K(P). \end{aligned}$$

If $\deg(\mathcal{D}) < 2 \deg(L_\bullet)$, then (E_\bullet, Φ) is necessarily stable. If $\deg(\mathcal{D}) = 2 \deg(L_\bullet)$, then (E_\bullet, Φ) is polystable if and only if $\psi \neq 0$ too.

Furthermore, if (E_\bullet, η, Φ) is stable with $\Phi \neq 0$ and $[E_\bullet, \eta, \Phi]$ is fixed by σ , then

- (b1) the isomorphism ι is unique up to $\{\pm 1\}$;
- (b2) if $\mathcal{D} \not\cong L_\bullet^{\otimes 2}$, then the decomposition $E_\bullet \cong (L_\bullet^\vee \otimes \mathcal{D}) \oplus L_\bullet$ is unique;
- (b3) if $\mathcal{D} \cong L_\bullet^{\otimes 2}$ and so $E_\bullet \cong L_\bullet \oplus L_\bullet$, then ϕ, ψ are not proportional.

Finally, if (E_\bullet, η, Φ) is strictly polystable with $\Phi \neq 0$ and $[E_\bullet, \eta, \Phi]$ is fixed by σ , then $(E_\bullet, \Phi) \cong \left(L_\bullet \oplus L_\bullet, \begin{pmatrix} \phi & 0 \\ 0 & -\phi \end{pmatrix} \right)$ with $0 \neq \phi \in H^0(S, K(P))$ and $\iota = \pm \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$.

Proof. The argument is essentially the same as in [29], Sec. 10.

In case (a), the Higgs bundle $(E_\bullet, 0)$ is polystable. So we assume $\Phi \neq 0$ and we want to show that (b) holds.

Since $\iota \circ \Phi \circ \iota^{-1} = -\Phi$, it is easy to check that Φ must vanish at all points Q where ι is a unipotent automorphism. Moreover, a quick computation shows that ι must have eigenvalues $\pm i$ at $\dot{S} \setminus Q$. Since $\Phi \neq 0$ except at a finite number of points, ι has everywhere eigenvalues $\pm i$ with eigenbundles L_\bullet and $L_\bullet^\vee \otimes \mathcal{D}$, and so $\iota = \pm \begin{pmatrix} i & 0 \\ 0 & -i \end{pmatrix}$ and $\Phi = \begin{pmatrix} 0 & \phi \\ \psi & 0 \end{pmatrix}$.

Suppose first $\deg(\mathcal{D}) < 2 \deg(L_\bullet)$.

Then L_\bullet cannot be preserved by Φ by semi-stability, and so $\phi \neq 0$. Moreover, $\deg(\mathcal{D})/2 > \deg(L_\bullet^\vee \otimes \mathcal{D})$ implies a line sub-bundle $L'_\bullet \subset E_\bullet$ with $\deg(L'_\bullet) > \mu(E_\bullet) = \deg(\mathcal{D})/2$ must necessarily be $L'_\bullet = L_\bullet$. It follows that the projection $L'_\bullet \rightarrow L_\bullet^\vee \otimes \mathcal{D}$ necessarily vanishes and so (E_\bullet, Φ) is stable.

Suppose now $\deg(\mathcal{D}) = 2 \deg(L_\bullet)$.

Then $\phi = 0$ would imply $\psi = 0$ by polystability (and vice versa): hence, we must have $\phi, \psi \neq 0$.

About the second part of the statement, given $\iota, \iota' : (E_\bullet, \eta, \Phi) \rightarrow (E_\bullet, \eta, -\Phi)$ isomorphisms of stable Higgs bundles with determinant \mathcal{D} , the composition $(\iota^{-1} \circ \iota') \in \text{Aut}(E_\bullet, \eta, \Phi)$ and so $\iota^{-1} \circ \iota' = \pm \text{Id}$ because (E_\bullet, η, Φ) is simple. Property (b2) easily follows from (b1) and (b). As for (b3), a destabilizing sub-bundle (necessarily isomorphic to L_\bullet) exists if and only if $0 \neq \phi, \psi \in H^0(S, K(P))$ are proportional.

Concerning the final claim, it is enough to observe that ϕ vanishes only on finitely many points of S and the involution ι must exchange the two eigenspaces of Φ/ϕ away from those finitely many points. \square

Denote by $\mathrm{Higgs}_b^{ps}(S, 2, \mathbf{w}, \mathcal{D}, \mathbf{c})(\mathbb{R})$ the set of $(E_\bullet, \eta, \Phi, \tau, \{\pm\iota\})$ such that $(E_\bullet, \eta, \Phi, \tau)$ is a polystable parabolic Higgs bundle of rank 2 of type \mathbf{w} with determinant \mathcal{D} and residues in \mathbf{c} , and $\iota : (E_\bullet, \eta, \Phi) \rightarrow (E_\bullet, \eta, -\Phi)$ is an isomorphism. We denote by $\mathrm{Higgs}_b^s(S, 2, \mathbf{w}, \mathcal{D}, \mathbf{c})(\mathbb{R})$ the locus of stable objects and by $\mathrm{Higgs}^s(S, 2, \mathbf{w}, \mathcal{D}, \mathbf{c})(\mathbb{R})$ its quotient by $\mathrm{SL}_2(\mathbb{C})$.

By the above lemma, $\mathrm{Higgs}^s(S, 2, \mathbf{w}, \mathcal{D}, \mathbf{c})(\mathbb{R})$ can be identified to the locus in $\mathrm{Higgs}_b^s(S, 2, \mathbf{w}, \mathcal{D}, \mathbf{c})$ fixed by σ , and so we will denote a point just by (E_\bullet, η, Φ) .

If $\bar{\mathbf{c}} \supseteq \mathbf{o} = \{0\}^n$, then $\mathrm{Higgs}^s(S, 2, \mathbf{w}, \mathcal{D}, \bar{\mathbf{c}})(\mathbb{R})$ contains the locus of $\Phi = 0$, namely

$$\mathrm{Bun}^s(S, 2, \mathbf{w}, \mathcal{D}) := \{(E_\bullet, \eta) \text{ stable parabolic rank 2 bundle of type } \mathbf{w} \text{ with } \eta : \det(E_\bullet) \xrightarrow{\sim} \mathcal{D}\}$$

Definition 3.12. We say that the couple (\mathbf{w}, \mathbf{c}) is *compatible (with the σ -involution)* if

- $0 < w_1(p_i) < 1/2 \implies \mathbf{c}_i = \{0\}$
- $w_1(p_i) = 0; 1/2 \implies \det(\mathbf{c}_i) \geq 0$
- $s_0 + \chi(\dot{S}) < 0$.

Notation. Given \mathbf{w} , denote by J_{deg}^{ev} (resp. J_{deg}^{odd}) the set of indices $j \in \{1, 2, \dots, n\}$ for which $w_1(p_j) = 0$ (resp. $w_1(p_j) = \frac{1}{2}$) and let $J_{deg} = J_{deg}^{ev} \cup J_{deg}^{odd}$.

Given \mathbf{c} , we define $J_0 = \{j \in J_{deg} \mid \mathbf{c}_j = \{0\}\}$, $J_{nil} = \{j \in J_{deg} \mid \mathbf{c}_j \text{ nilpotent}\}$ and $J_{inv} = \{j \in J_{deg} \mid \det(\mathbf{c}_j) \neq 0\}$.

Finally, call $s = \#J_{deg}$, $s_0 = \#J_0$ and $s_{inv} = \#J_{inv}$, and also $s^{ev} = \#J_{deg}^{ev}$ and $s^{odd} = \#J_{deg}^{odd}$.

Definition 3.13. Given \mathcal{D} and compatible (\mathbf{w}, \mathbf{c}) , we say that the couple $(d, \mathbf{a}) \in \mathbb{Z} \times \{0, 1\}^n$ is *admissible* if

- (a) $a_j = 0$ for all $j \in J_{deg}$;
- (b) $e(d, \mathbf{a}, \mathbf{w}) := 2d - d_0 + 2 \sum_{i=1}^n (a_i + (-1)^{a_i} w_1(p_i)) \geq 0$
- (c) $2d \leq d_0 - \chi(S) - \|\mathbf{a}\|_1 + s^{ev} - s_0$.

For every admissible (d, \mathbf{a}) , we define the following locus in $\mathrm{Higgs}^s(S, 2, \mathbf{w}, \mathcal{D}, \mathbf{c})(\mathbb{R})$

$$\mathrm{Higgs}^s(S, 2, \mathbf{w}, \mathcal{D}, \mathbf{c})(\mathbb{R})_{d, \mathbf{a}} := \left\{ \begin{array}{l} E_\bullet \cong (L_\bullet^\vee \otimes \mathcal{D}) \oplus L_\bullet \text{ with } \Phi = \begin{pmatrix} 0 & \phi \\ \psi & 0 \end{pmatrix} \text{ and } \mathrm{Res}_{p_i}(\Phi) \in \mathbf{c}_i \\ 0 \neq \phi \in H^0(S, \mathcal{D}KL_\bullet^{-2}(P)), \quad \psi \in H^0(S, \mathcal{D}^\vee KL_\bullet^2(P)) \\ \deg(L) = d, \quad w_L(p_i) = \begin{cases} w_1(p_i) & \text{if } a_i = 0 \\ 1 - w_1(p_i) & \text{if } a_i = 1 \end{cases} \end{array} \right\}$$

Thus, $e(d, \mathbf{a}, \mathbf{w})$ can be rewritten as $e(d, \mathbf{a}, \mathbf{w}) = 2d - d_0 + 2 \sum_{i=1}^n w_L(p_i)$ and the admissibility constraints (b-c) read

$$-2 \sum_{i=1}^n w_L(p_i) \leq 2d - d_0 \leq -\chi(S) - \|\mathbf{a}\|_1 + s^{ev} - s_0.$$

Moreover, the condition $e(d, \mathbf{a}, \mathbf{w}) \geq 0$ is equivalent to $\deg(L_\bullet) \geq \deg(\mathcal{D} \otimes L_\bullet^\vee)$; thus, in view of Lemma 3.11, it is understood that we also require $\psi \neq 0$, if $e(d, \mathbf{a}, \mathbf{w}) = 0$.

We can rephrase our analysis as follows.

Proposition 3.14 (Partition of the σ -fixed locus). The space $\mathrm{Higgs}^s(S, 2, \mathbf{w}, \mathcal{D}, \bar{\mathbf{c}})$ can be decomposed into the disjoint union of the following loci

$$\begin{array}{ll} \mathrm{Bun}^s(S, 2, \mathbf{w}, \mathcal{D}) & \text{if } \mathbf{o} \in \bar{\mathbf{c}} \\ \mathrm{Higgs}^s(S, 2, \mathbf{w}, \mathcal{D}, \bar{\mathbf{c}})(\mathbb{R})_{d, \mathbf{a}} & \text{for admissible } (d, \mathbf{a}). \end{array}$$

3.6. Topology of the σ -fixed locus. Let \mathbf{c}_i be conjugacy classes and $\bar{\mathbf{c}}_i$ be their closures in $\mathfrak{sl}_2(\mathbb{C})$. Throughout this section, we will assume that (\mathbf{w}, \mathbf{c}) are compatible, that (d, \mathbf{a}) is admissible and that $e(d, \mathbf{a}, \mathbf{w}) > 0$.

We begin with some simple observations.

Lemma 3.15. Let $(E_\bullet, \eta, \Phi, \{\pm\iota\}) \in \mathrm{Higgs}^s(S, 2, \mathbf{w}, \mathcal{D}, \bar{\mathbf{c}})(\mathbb{R})_{d, \mathbf{a}}$ and $E_\bullet = (L_\bullet^\vee \otimes \mathcal{D}) \oplus L_\bullet$. Then

- (i) for $i \notin J_{deg}$ (\mathbf{w} non-degenerate at p_i), $L_{\bullet}^{-2}(P + \mathbf{a}P)$ has parabolic twist $1 + a_i - 2w_L(p_i) \in [0, 1)$ and $L_{\bullet}^2(-\mathbf{a}P)$ has twist $2w_L(p_i) - a_i \in [0, 1)$ at p_i ;
- (ii) for $j \in J_{deg}$ (\mathbf{w} degenerate at p_j), L^2 and L^{-2} have trivial parabolic structure at p_j .

Thus, $\deg[\mathcal{D}L_{\bullet}^{-2}(P)] = d_0 - 2d - \|\mathbf{a}\|_1 + s^{ev}$ and $\deg[\mathcal{D}^{\vee}L_{\bullet}^2] = 2d - d_0 + \|\mathbf{a}\|_1 + s^{odd} \geq 1 - n$. As a consequence,

$$\begin{aligned} m &:= \deg[\mathcal{D}L_{\bullet}^{-2}K(P)] = d_0 + 2g - 2 - 2d - \|\mathbf{a}\|_1 + s^{ev} \geq 0 \\ &\quad \deg[\mathcal{D}^{\vee}L_{\bullet}^2K(P)] = -d_0 + 2g - 2 + n + 2d + \|\mathbf{a}\|_1 + s^{odd} \geq 0 \\ m' &:= h^0(S, \mathcal{D}^{\vee}L_{\bullet}^2K(P)) = -d_0 + g - 1 + n + 2d + \|\mathbf{a}\|_1 + s^{odd} \geq 0 \end{aligned}$$

since $\deg([\mathcal{D}^{\vee}L_{\bullet}^2(P)]) > 0$. Thus, $m + m' = 3g - 3 + n + s$.

Proof. Parts (i) and (ii) are elementary computations. The bound for $\deg[\mathcal{D}^{\vee}L_{\bullet}^2]$ follows by observing that $0 < e(d, \mathbf{a}, \mathbf{w}) = 2d - d_0 + 2\sum_{i=1}^n w_L(p_i) \leq 2d - d_0 + \|\mathbf{a}\|_1 + n$ because $w_L(p_i) \leq a_i + 1$. Thus, m' can be calculated by Riemann-Roch. \square

Since we are assuming $e(d, \mathbf{a}, \mathbf{w}) > 0$, a point of $\text{Higgs}^s(S, 2, \mathbf{w}, \mathcal{D}, \bar{\mathbf{c}})(\mathbb{R})_{d, \mathbf{a}}$ can be identified with a triple $(L_{\bullet}, \phi, \psi)$ that satisfies certain conditions, up to isomorphism. In fact, the map $L_{\bullet} \rightarrow L_{\bullet}$ of multiplication by $\lambda \in \mathbb{C}^*$ induces an isomorphism of $(L_{\bullet}, \phi, \psi)$ with $(L_{\bullet}, \lambda^{-2}\phi, \lambda^2\psi)$. Hence, such a point can be identified with the triple $(L_{\bullet}, Q, -\phi\psi)$, where Q is an effective divisor in the linear system $|\mathcal{D}[L_{\bullet}^{-2}]K(P)|$ and $-\phi\psi = \det(\Phi) \in H^0(S, K^2(2P))$. Moreover, L_{\bullet}^{-2} can be reconstructed up to isomorphism from Q : so, given $(Q, \det(\Phi))$, there are exactly 2^{2g} choices for L_{\bullet} and the set of such choices is a $\text{Pic}^0(S)[2]$ -torsor.

Consider then the residues of $\det(\Phi)$.

- Suppose that $i \notin J_{deg}$ and so \mathbf{w} is non-degenerate at p_i .
Then necessarily $\text{Res}_{p_i}(\Phi) = 0$. Thus, we will assume that $\mathbf{c}_i = \{0\}$ for all $i \notin J_{deg}$.
- Suppose that $j \in J_{deg}$ and so \mathbf{w} is degenerate at p_j .
Then the condition on the residue at p_j is not automatically satisfied. For j in J_{nil} or J_{inv} , the elements in $\bar{\mathbf{c}}_j$ are detected by their determinant $\det(\mathbf{c}_j)$: thus it is enough to require that $(-\phi\psi)(p_j) = \det(\mathbf{c}_j)$. For $j \in J_0$, we must require that $\phi(p_j) = 0$ and that $\text{ord}_{p_j}(\phi\psi) > \text{ord}_{p_j}(\phi)$.

Consider the space $\mathcal{X} = \text{Sym}^m(S \setminus P_{inv}) \times H^0(S, K^2(P + P_{deg}))$, where $P_{deg} = \sum_{j \in J_{deg}} p_j$ and $P_{inv} = \sum_{j \in J_{inv}} p_j$, and the loci

$$\begin{aligned} \mathcal{Q} &= \{(Q, q) \in \mathcal{X} \mid Q \leq \text{div}(q)\} \\ \bar{\mathcal{R}}_i &= \{(Q, q) \in \mathcal{X} \mid \text{Res}_{p_i}(q) = \det(\mathbf{c}_i)\} \quad \text{for all } i = 1, \dots, n. \end{aligned}$$

Moreover, for $j \in J_0 \cup J_{nil}$, the locus $\bar{\mathcal{R}}_j$ can be split into

$$\begin{aligned} \bar{\mathcal{R}}_j^- &= \{(Q, q) \in \mathcal{X} \mid p_j \in Q\} && \text{corresponding to } \phi(p_j) = 0 \\ \bar{\mathcal{R}}_j^+ &= \{(Q, q) \in \mathcal{X} \mid \text{ord}_{p_j}(q) > \text{mult}_{p_j}(Q)\} && \text{corresponding to } \psi(p_j) = 0 \end{aligned}$$

and we call $\mathcal{R}_j^0 := \bar{\mathcal{R}}_j^+ \cap \bar{\mathcal{R}}_j^-$ for all $j \in J_0$. Finally, for every $\varepsilon : J_{nil} \rightarrow \{+, -\}$ we denote by $P_{\pm}(\varepsilon)$ the subsets of points p_j such that $\varepsilon(j) = \pm$ and let $s_{\pm}(\varepsilon) = \#P_{\pm}(\varepsilon)$. Then we define

$$\bar{\mathcal{R}}^{\varepsilon} := \left(\bigcap_{j \in J_{inv}} \bar{\mathcal{R}}_j \right) \cap \left(\bigcap_{j \in J_{nil}} \bar{\mathcal{R}}_j^{\varepsilon_j} \right) \cap \left(\bigcap_{j \in J_0} \mathcal{R}_j^0 \right) \quad \text{and} \quad \bar{\mathcal{R}} = \bigcup_{\varepsilon} \bar{\mathcal{R}}^{\varepsilon}.$$

and we call $\mathcal{R}^{\varepsilon} := \bar{\mathcal{R}}^{\varepsilon} \setminus \bigcup_{j \in J_{nil}} \mathcal{R}_j^0$ and $\mathcal{R} := \bigcup_{\varepsilon} \mathcal{R}^{\varepsilon}$.

The locally closed subvariety \mathcal{Q} has codimension m in \mathcal{X} and it is isomorphic to a holomorphic vector bundle of rank m' over $\text{Sym}^m(S \setminus P_{inv})$ and so the above discussion leads to the following conclusion.

Proposition 3.16 (Topology of σ -fixed components). The loci in \mathcal{X} defined above satisfy the following properties.

- (a) The locus $\mathcal{Q} \cap \overline{\mathcal{R}}^\varepsilon$ is a holomorphic affine bundle of rank $m' - [s_{inv} + s_0 + s_+(\varepsilon)]$ over $\mathrm{Sym}^{m-[s_0+s_-(\varepsilon)]}(S \setminus P_{inv})$. The locus $\mathcal{Q} \cap \mathcal{R}^\varepsilon$ is obtained from $\mathcal{Q} \cap \overline{\mathcal{R}}^\varepsilon$ by first restricting the affine bundle over $\mathrm{Sym}^{m-[s_0+s_-(\varepsilon)]}(S \setminus (P_+(\varepsilon) \cup P_{inv}))$ and then removing $s_-(\varepsilon)$ nowhere-parallel affine subbundles of codimension 1.
- (b) The locus $\mathcal{Q} \cap \overline{\mathcal{R}}$ is connected, has pure codimension $s + s_0 + m$ in \mathcal{X} and consists of the irreducible components $\mathcal{Q} \cap \overline{\mathcal{R}}^\varepsilon$ of dimension $3g - 3 + n - s_0$. The locus $\mathcal{Q} \cap \mathcal{R}$ is the disjoint union of all $\mathcal{Q} \cap \mathcal{R}^\varepsilon$.
- (c) The morphism

$$\begin{array}{ccc} \mathrm{Higgs}^s(S, 2, \mathbf{w}, \mathcal{D}, \overline{\mathbf{c}})(\mathbb{R})_{d, \mathbf{a}} & \longrightarrow & \mathcal{X} \\ (E_\bullet, \eta, \Phi) & \longmapsto & (\mathrm{div}(\phi), \det(\Phi)) \end{array}$$

is a $\mathrm{Pic}^0(S)[2]$ -torsor over $\mathcal{Q} \cap \overline{\mathcal{R}}$ and $\mathrm{Higgs}^s(S, 2, \mathbf{w}, \mathcal{D}, \mathbf{c})(\mathbb{R})_{d, \mathbf{a}}$ is a $\mathrm{Pic}^0(S)[2]$ -torsor over $\mathcal{Q} \cap \mathcal{R}$.

- (d) The restriction $\mathrm{Higgs}^s(S, 2, \mathbf{w}, \mathcal{D}, \overline{\mathbf{c}})(\mathbb{R})_{d, \mathbf{a}}^\varepsilon$ of the $\mathrm{Pic}^0(S)[2]$ -torsor in (c) over the component $\mathcal{Q} \cap \mathcal{R}^\varepsilon$ is connected, unless $\mathcal{Q} \cap \overline{\mathcal{R}}^\varepsilon$ is an affine space (i.e. $m - [s_0 + s_-(\varepsilon)] = 0$): in this case it is necessarily trivial.

We stress that, by definition, $\mathcal{Q} \cap \mathcal{R}^\varepsilon = \emptyset$ if $m - [s_0 + s_-(\varepsilon)] < 0$ or $m' - [s_{inv} + s_0 + s_+(\varepsilon)] < 0$.

Corollary 3.17 (Topology of $\mathrm{Pic}^0(S)[2]$ -quotient of σ -fixed irreducible components). The quotient $\mathrm{Higgs}^s(S, 2, \mathbf{w}, \mathcal{D}, \overline{\mathbf{c}})(\mathbb{R})_{d, \mathbf{a}}^\varepsilon / \mathrm{Pic}^0(S)[2]$ is isomorphic to a holomorphic affine bundle of rank $m' - [s - s_-(\varepsilon)]$ over $\mathrm{Sym}^{m-[s_0+s_-(\varepsilon)]}(S \setminus P_{inv})$ and $\mathrm{Higgs}^s(S, 2, \mathbf{w}, \mathcal{D}, \mathbf{c})(\mathbb{R})_{d, \mathbf{a}}^\varepsilon / \mathrm{Pic}^0(S)[2]$ is isomorphic to the complement of $s_-(\varepsilon)$ affine codimension 1 subbundles inside a holomorphic affine bundle of rank $m' - [s - s_-(\varepsilon)]$ over $\mathrm{Sym}^{m-[s_0+s_-(\varepsilon)]}(S \setminus (P_+(\varepsilon) \cup P_{inv}))$.

Proof of Proposition 3.16. Considering the above discussion, we are only left to prove (d).

The action of $\mathrm{Pic}^0(S)[2]$ on $\mathrm{Higgs}^s(S, 2, \mathbf{w}, \mathcal{D}, \overline{\mathbf{c}})(\mathbb{R})_{d, \mathbf{a}}^\varepsilon$ is given by $A \cdot (E_\bullet, \eta, \Phi) \mapsto (E_\bullet \otimes A, \eta, \Phi)$ for $A \in \mathrm{Pic}^0(S)[2]$.

If $m - [s_0 + s_-(\varepsilon)] = 0$, then $\mathcal{Q} \cap \overline{\mathcal{R}}^\varepsilon$ is an affine space and so the torsor is trivial.

Assume now $m - [s_0 + s_-(\varepsilon)] > 0$ and fix $A \in \mathrm{Pic}^0(S)[2]$. We want to show that every (E_\bullet, η, Φ) in $\mathrm{Higgs}^s(S, 2, \mathbf{w}, \mathcal{D}, \overline{\mathbf{c}})(\mathbb{R})_{d, \mathbf{a}}^\varepsilon$ can be connected to $(E_\bullet \otimes A, \eta, \Phi)$ by a continuous path.

Consider the map $f : \mathrm{Higgs}^s(S, 2, \mathbf{w}, \mathcal{D}; \mathbf{c})(\mathbb{R})_{d, \mathbf{a}}^\varepsilon \rightarrow \mathrm{Pic}^d(S) \times \mathrm{Sym}^{m-s_0(\varepsilon)-s_-(\varepsilon)}(S \setminus P_{inv})$ defined by $f(E_\bullet, \eta, \Phi) = (L, [\phi])$, where $E_\bullet = (L_\bullet^\vee \otimes \mathcal{D}) \oplus L_\bullet$ and $\Phi = \begin{pmatrix} 0 & \phi \\ \psi & 0 \end{pmatrix}$. Since the knowledge of \mathbf{w} and \mathbf{a} allows to reconstruct the parabolic bundle L_\bullet out of L , the image $\tilde{\mathcal{S}}$ of f can be identified to the locus of couples (L, Q) such that $Q_{fix} + Q \in |\mathcal{D}L_\bullet^{-2}K(P)|$, where $Q_{fix} := \sum_{j \in J_0} p_j + \sum_{\varepsilon_j = -} p_j$. Note that $\tilde{\mathcal{S}}$ is an étale cover over $\mathrm{Sym}^{m-s_0(\varepsilon)-s_-(\varepsilon)}(S \setminus P_{inv})$ of degree 2^{2g} and f is a fibration with fiber $\mathbb{C}^{m'-[s-s_-(\varepsilon)]}$. Thus, it is enough to show that $\tilde{\mathcal{S}}$ is connected.

Now fix $(L, Q) \in \tilde{\mathcal{S}}$ and let $B : [0, 1] \rightarrow \mathrm{Pic}^0(S)$ be a continuous path from \mathcal{O}_S to A , so that $B^{-2} : [0, 1] \rightarrow \mathrm{Pic}^0(S)$ is a closed path. Since $m - [s_0 + s_-(\varepsilon)] > 0$, we can choose a point $x \in Q$ and we can consider the map

$$\begin{array}{ccc} S \setminus P_{inv} & \longrightarrow & \mathrm{Pic}^0(S) \\ y & \longmapsto & \mathcal{O}_S(y - x) \end{array}$$

which induces a surjection $\pi_1(S \setminus P_{inv}) \twoheadrightarrow \pi_1(\mathrm{Pic}^0(S))$. Thus, there exists a path $Y : [0, 1] \rightarrow S \setminus P_{inv}$ based at $Y(0) = Y(1) = x$ which is mapped to a path homotopic to B^{-2} . Define $Q(t) := Q - x + Y(t)$ and let $A(t)$ be the unique continuous path in $\mathrm{Pic}^0(S)$ such that $A(0) = \mathcal{O}_S$ and $A(t)^2 = \mathcal{O}_S(x - Y(t))$. By definition, $Q_{fix} + Q(t) \in |\mathcal{D}(A(t) \otimes L_\bullet)^{-2}K(P)|$.

Since the path A^{-2} is homotopic to B^{-2} , we have $A(1) = B(1) = A$ and so the path $t \mapsto (L \otimes A(t), Q(t))$ joins (L, Q) and $(L \otimes A, Q)$. \square

Provided $e(d, \mathbf{a}, \mathbf{w})$ remains positive, Proposition 3.16 shows that the isomorphism class of the moduli space $\text{Higgs}^s(S, 2, \mathbf{w}, \mathcal{D}, \bar{\mathbf{c}})(\mathbb{R})_{d, \mathbf{a}}^{\varepsilon}$ remains constant as a parabolic weight $w_1(p_i)$ is varied within the interval $(0, 1/2)$ and \mathbf{c}_i is kept equal to $\{0\}$. Moreover, if $w_1(p_i) \in (0, 1/2)$ is pushed to $w'_1(p_i) = 0$; $1/2$ and the class \mathbf{c}_i is switched to $\mathbf{c}'_i = \{\text{nilpotents}\}$, then the moduli space is isomorphic to some $\text{Higgs}^s(S, 2, \mathbf{w}', \mathcal{D}, \bar{\mathbf{c}}')(\mathbb{R})_{d', \mathbf{a}'}^{\varepsilon'}$, where $\varepsilon'_i = +$ if $e(d', \mathbf{a}', \mathbf{w}') > e(d, \mathbf{a}, \mathbf{w}) > 0$, and $\varepsilon'_i = -$ if $0 < e(d', \mathbf{a}', \mathbf{w}') < e(d, \mathbf{a}, \mathbf{w})$. More precisely, we have the following.

Corollary 3.18 (Varying the parabolic weights). Fix \mathbf{c} , d , \mathbf{a} and \mathbf{w} such that $e = e(d, \mathbf{a}, \mathbf{w}) > 0$, and assume that \mathbf{w} is non-degenerate at p_i and $\mathbf{c}_i = \{0\}$. Then $\text{Higgs}^s(S, 2, \mathbf{w}, \mathcal{D}, \bar{\mathbf{c}})(\mathbb{R})_{d, \mathbf{a}}^{\varepsilon}$ is isomorphic to

- (a) $\text{Higgs}^s(S, 2, \mathbf{w}', \mathcal{D}, \bar{\mathbf{c}})(\mathbb{R})_{d', \mathbf{a}'}^{\varepsilon'}$ with $d' = d$, $\mathbf{a}' = \mathbf{a}$, $\varepsilon' = \varepsilon$ and for every \mathbf{w}' that differs from \mathbf{w} only on the i -th entrance and such that $0 < w'_1(p_i) < 1/2$ (see also Nakajima [47]);
- (b) $\text{Higgs}^s(S, 2, \mathbf{w}', \mathcal{D}, \bar{\mathbf{c}}')(\mathbb{R})_{d', \mathbf{a}'}^{\varepsilon'}$ where \mathbf{w}' , \mathbf{c}' , \mathbf{a}' differ from \mathbf{w} , \mathbf{c} , \mathbf{a} only on the i -th entrance, $\mathbf{c}'_i = \{\text{nilpotents}\}$, $\mathbf{a}'_i = 0$ and either of the following hold:

a_i	$w'_1(p_i)$	ε'_i	d'	e'
0	0	-	d	$e - 2w_1(p_i)$
0	1/2	+	d	$e + (1 - 2w_1(p_i))$
1	0	+	$d + 1$	$e + 2w_1(p_i)$
1	1/2	-	d	$e - (1 - 2w_1(p_i))$

as long as $e' = e(d', \mathbf{a}', \mathbf{w}') > 0$.

Similarly, since $\text{Res}_{p_i}(q) = \det(\mathbf{c}_i)$ is an affine equation in $H^0(S, K^2(P + P_{deg}))$, we also have the following result.

Corollary 3.19 (Varying the quadratic residue of $\det(\Phi)$). Let (\mathbf{w}, \mathbf{c}) , d , \mathbf{a} be such that $e(d, \mathbf{a}, \mathbf{w}) > 0$ and assume that $\det(\mathbf{c}_i) > 0$. Then $\text{Higgs}^s(S, 2, \mathbf{w}, \mathcal{D}, \bar{\mathbf{c}})(\mathbb{R})_{d, \mathbf{a}}^{\varepsilon}$ is isomorphic to $\text{Higgs}^s(S, 2, \mathbf{w}, \mathcal{D}, \bar{\mathbf{c}}')(\mathbb{R})_{d, \mathbf{a}}^{\varepsilon}$, where \mathbf{c}' differs from \mathbf{c} only on the i -th entrance and $\det(\mathbf{c}'_i) > 0$.

Notice that compact components may also occur, but only in a few limited cases. If $g = 0$ and $n = 3 + s_0$, then $\text{Higgs}^s(S, 2, \mathbf{w}, \mathcal{D}, \bar{\mathbf{c}})(\mathbb{R})_{d, \mathbf{a}}^{\varepsilon}$ consists of a single point (if it is nonempty) and so it is compact. Biquard-Tholozan remarked that the other cases of compact components with non-degenerate parabolic structure correspond to representations that Deroin-Tholozan [14] call ‘‘super-maximal’’ via Theorem 4.14. I would like to thank Nicolas Tholozan for drawing my attention to this point.

Corollary 3.20 (Compact components). Assume $e = e(d, \mathbf{a}, \mathbf{w}) > 0$ and $(g, n) \neq (0, 3 + s_0)$. The locus $\text{Higgs}^s(S, 2, \mathbf{w}, \mathcal{D}, \bar{\mathbf{c}})(\mathbb{R})_{d, \mathbf{a}}^{\varepsilon}$ is compact if and only if

$$(\star) \begin{cases} g = 0 \\ s_-(\varepsilon) = s_{inv} = 0 \\ e(d, \mathbf{a}, \mathbf{w}) = 1 - \sum_{\substack{i \notin J_{deg} \\ a_i = 0}} (1 - 2w_1(p_i)) - \sum_{\substack{i \notin J_{deg} \\ a_i = 1}} 2w_1(p_i) \in (0, 1]. \end{cases}$$

In this case, Higgs bundles $[E, \Phi] \in \text{Higgs}^s(S, 2, \mathbf{w}, \mathcal{D}, \bar{\mathbf{c}})(\mathbb{R})_{d, \mathbf{a}}^{\varepsilon}$ have $\psi = 0$ and so nilpotent Φ , and the whole component is isomorphic to $\mathbb{C}\mathbb{P}^{n-3-s_0}$.

The locus $\text{Higgs}^s(S, 2, \mathbf{w}, \mathcal{D}, \bar{\mathbf{c}})(\mathbb{R})_{d, \mathbf{a}}^{\varepsilon}$ is compact if and only if the condition (\star) is satisfied and $s_+(\varepsilon) = 0$. Moreover, in this case such component is again isomorphic to $\mathbb{C}\mathbb{P}^{n-3-s_0}$.

Proof. Since $e = e(d, \mathbf{a}, \mathbf{w}) = -d_0 + 2d + \sum_{i=1}^n 2w_L(p_i)$, we have

$$\begin{aligned} m' - s + s_-(\varepsilon) &= (-d_0 + 2d) + g - 1 + n + \|\mathbf{a}\|_1 + s^{\mathrm{odd}} - s + s_-(\varepsilon) = \\ &= e - \sum_{i=1}^n 2w_L(p_i) + g - 1 + n + \|\mathbf{a}\|_1 - s^{\mathrm{ev}} + s_-(\varepsilon) = \\ &= g - 1 + e + \sum_{i \notin J_{deg}} (1 + a_i - 2w_L(p_i)) + s_-(\varepsilon) \geq e + g - 1 \end{aligned}$$

In Corollary 3.17, $\mathrm{Higgs}^s(S, 2, \mathbf{w}, \mathcal{D}, \bar{\mathbf{c}})(\mathbb{R})_{d, \mathbf{a}}^\varepsilon$ is presented as a fibration over a symmetric product and the fiber is an open subset of an affine space. Thus, it is compact if and only if the fiber is 0-dimensional and the base is a symmetric product of a compact surface (i.e. $s_{inv} = 0$). For the fiber to be 0-dimensional, we must have $m' \leq s - s_-(\varepsilon)$. Since $1 + a_i - 2w_L(p_i) > 0$ for $i \notin J_{deg}$ and $e > 0$, this implies that $g = 0$, $s_-(\varepsilon) = 0$ and

$$0 < e = 1 - \sum_{i \notin J_{deg}} (1 + a_i - 2w_L(p_i)) \leq 1.$$

Vice versa, if the above numerical conditions are satisfied, it is immediate to check that the fiber is 0-dimensional and indeed it consists of a single point. In this case, such component is isomorphic to $\mathrm{Sym}^{n-3-s_0}(\mathbb{CP}^1) \cong \mathbb{CP}^{n-3-s_0}$.

Similarly, the open component $\mathrm{Higgs}^s(S, 2, \mathbf{w}, \mathcal{D}, \mathbf{c})(\mathbb{R})_{d, \mathbf{a}}^\varepsilon$, fibers over $\mathrm{Sym}^{n-3-s_0}(S \setminus (P_{inv} \cup P_+))$. The numerical conditions for the 0-dimensionality of the fiber are the same; for the base to be compact we need $s_{inv} = s_+(\varepsilon) = 0$. Again the component will be isomorphic to \mathbb{CP}^{n-3-s_0} . \square

4. HITCHIN-SIMPSON CORRESPONDENCE AND TOPOLOGY

Fix a complex structure I on the compact surface S and let \mathcal{O}_S be the sheaf of I -holomorphic functions on S .

4.1. Closed case. Let S be compact and unpunctured ($n = 0$).

Since locally constant functions on S are I -holomorphic, a flat \mathbb{C} -vector bundle V on S can be naturally given an I -holomorphic structure $\bar{\partial}^V$. In particular, if (ξ, ∇) is a flat GL_N -bundle, then $V := \xi \times_{\mathrm{GL}_N} \mathbb{C}^N$ is a I -holomorphic bundle endowed with a flat connection, which we will still denote by ∇ by a little abuse.

The point of departure is then the following classical result.

Theorem 4.1 (Narasimhan-Seshadri [48]). The map that sends a flat U_N -bundle (ξ, ∇) on S to the I -holomorphic vector bundle $E := \xi \times_{U_N} \mathcal{O}_S^{\oplus N}$ induces a real-analytic homeomorphism

$$\mathrm{Fl}^{\mathrm{irr}}(S, U_N) \xrightarrow{\sim} \mathrm{Bun}^s(S, N)_0$$

between the moduli space of irreducible flat U_N -principal bundles on S and the moduli space of stable I -holomorphic vector bundles of rank N and degree 0 on (S, I) . Such homeomorphism restricts to

$$\begin{aligned} \mathrm{Fl}^{\mathrm{irr}}(S, \mathrm{SU}_N) &\xrightarrow{\sim} \mathrm{Bun}^s(S, N, \mathcal{O}_S) \\ (\xi, \nabla) &\longmapsto (E, \eta) \end{aligned}$$

where $E = \xi \times_{\mathrm{SU}_N} \mathcal{O}_S^{\otimes N}$ and $\eta : \det(E) \xrightarrow{\sim} \mathcal{O}_S$ sends the unit volume element to 1.

Such correspondence can be lifted to bundles endowed with a trivialization τ at the base-point b ; moreover, just by taking direct sums it can be extended to a correspondence between completely decomposable b -framed flat bundles and polystable b -framed I -holomorphic bundles.

Corollary 4.2. There is a real-analytic homeomorphism

$$\begin{aligned} \mathrm{Fl}_b^{dec}(S, \mathrm{U}_N) &\xrightarrow{\sim} \mathrm{Bun}_b^{ps}(S, N)_0 \\ (\xi, \nabla, \tau) &\longmapsto (E = \xi \times_{\mathrm{U}_N} \mathcal{O}_S^{\oplus N}, \tau' = \tau \otimes_{\mathrm{U}_N} \mathbb{C}^N) \end{aligned}$$

which restricts to

$$\begin{aligned} \mathrm{Fl}_b^{dec}(S, \mathrm{SU}_N) &\xrightarrow{\sim} \mathrm{Bun}_b^{ps}(S, N, \mathcal{O}_S) \\ (\xi, \nabla, \tau) &\longmapsto (E = \xi \times_{\mathrm{SU}_N} \mathcal{O}_S^{\oplus N}, \eta, \tau'). \end{aligned}$$

In the proof by Narasiman-Seshadri, surjectivity is achieved by continuity method. In particular, it does not provide a way to construct (ξ, ∇) starting from a stable E . The following important result fills such a gap.

Theorem 4.3 (Donaldson [15]). A holomorphic vector bundle E of rank N and degree 0 on (S, I) admits a flat invariant metric if and only if E is polystable. Moreover, such a metric is unique up to automorphisms of E .

In the case of bundles with monodromy not contained in U_N , no invariant Hermitian metric is available. In order to codify all information in term of holomorphic structures on (S, I) , the idea is to replace invariant metrics by harmonic metrics.

Definition 4.4. A *harmonic metric* on a flat GL_N -bundle (ξ, ∇) on (S, I) is the Hermitian metric $H = h h^T$ on the flat bundle $V = \xi \times_{\mathrm{GL}_N} \mathbb{C}^N$ associated to a section $h : S \rightarrow \xi \times_{\mathrm{GL}_N} (\mathrm{GL}_N/\mathrm{U}_N)$ that minimizes the energy with respect to the natural metric on the symmetric space $\mathrm{GL}_N/\mathrm{U}_N$.

Existence and uniqueness of harmonic metrics and was first proven by Donaldson [16] in the rank 2 case. A more general existence theorem is due to Corlette: here we recall the statement for Riemann surfaces only.

Theorem 4.5 (Corlette [12]). A flat GL_N -bundle (ξ, ∇) on (S, I) has a harmonic metric H if and only if ξ is reductive (i.e. the closure of the image of hol_ξ in $\mathrm{GL}_N(\mathbb{C})$ is reductive). Moreover, such an H is unique up to automorphisms of ξ .

Given a hermitian metric H on the flat vector bundle $V = \xi \times_{\mathrm{GL}_N} \mathbb{C}^N$, the connection ∇ decomposes as

$$\nabla = \nabla^H + \Phi + \bar{\Phi}^H$$

where Φ is an $\mathrm{End}(V)$ -valued $(1, 0)$ -form, $\bar{\Phi}^H$ is its H -adjoint and ∇^H is compatible with H . We can define a holomorphic structure on $E = V$ by letting $\bar{\partial}^E = \bar{\partial}^V - \bar{\Phi}^H$, so that ∇^H is a Chern connection for the holomorphic Hermitian bundle (E, H) .

Harmonicity of the metric H is then equivalent to the $\bar{\partial}^E$ -holomorphicity of $\Phi \in C^\infty(S, K \otimes \mathrm{End}(E))$. Thus, once we find a harmonic metric H on (V, ∇) , we can produce a Higgs bundle (E, Φ) of degree $\deg(E) = \deg(V) = 0$.

Conversely, given a Higgs bundle (E, Φ) on (S, I) and a Hermitian metric H on E , we can consider the underlying complex vector bundle V endowed with the connection $\nabla = \nabla^H + \Phi + \bar{\Phi}^H$, where ∇^H is the Chern connection on (E, H) and $\bar{\Phi}^H$ is the H -adjoint of Φ . Harmonicity of the metric H on (V, ∇) is equivalent to the flatness of ∇ and the following theorem provides the wished counterpart to Theorem 4.5 (proven before by Hitchin [29] in the rank 2 case).

Theorem 4.6 (Simpson [52]). A GL_N -Higgs bundle (E, Φ) on (S, I) supports a metric H such that the induced (V, ∇) is flat if and only if (E, Φ) is polystable. Moreover, such a metric is unique up to automorphisms of (E, Φ) .

Considering the SL_N -bundles correspond to vector bundles with trivialized determinant, we summarize the above results as follows.

Corollary 4.7 (Simpson [54] [55]). There are smooth diffeomorphisms

$$\begin{aligned} \mathrm{Fl}_b^{\mathrm{red}}(S, \mathrm{GL}_N) &\longrightarrow \mathrm{Higgs}_b^{\mathrm{ps}}(S, N)_0 \\ \mathrm{Fl}_b^{\mathrm{red}}(S, \mathrm{SL}_N) &\longrightarrow \mathrm{Higgs}_b^{\mathrm{ps}}(S, N, \mathcal{O}_S) \end{aligned}$$

which induce a correspondence

$$\begin{aligned} \mathrm{Fl}^{\mathrm{Zd}}(S, \mathrm{GL}_N) &\longrightarrow \mathrm{Higgs}^s(S, N)_0 \\ \mathrm{Fl}^{\mathrm{Zd}}(S, \mathrm{SL}_N) &\longrightarrow \mathrm{Higgs}^s(S, N, \mathcal{O}_S) \end{aligned}$$

between the space of flat bundles with Zariski-dense monodromy and the space of stable I -holomorphic Higgs bundles.

Again, the case $N = 2$ of above corollary is due to Hitchin [29], whose construction also implies that the involved diffeomorphisms are real-analytic. Simpson showed that in general the above correspondence does not continuously extend over the whole semi-stable locus (see [55], pp.38–39).

We remind that flat U_N -bundles are always decomposable and so reductive; in this case, Zariski-density of the monodromy is equivalent to irreducibility.

4.2. Punctured case. Given a flat U_N -bundle (ξ, ∇) on the punctured surface \hat{S} , we can as before produce a I -holomorphic vector bundle $\dot{E} = \xi \times_{\mathrm{U}_N} \mathcal{O}_{\hat{S}}^{\oplus N}$ on \hat{S} which carries an invariant Hermitian metric H . Moreover, such \dot{E} admits a unique extension $E \rightarrow S$ (*Deligne extension* [13]) such that the induced ∇ has real residues at p_i with eigenvalues $0 \leq w_1(p_i) < w_2(p_i) < \dots < w_{b_i}(p_i) < 1$ for all $p_i \in P$ and algebraic multiplicities $m_1(p_i), \dots, m_{b_i}(p_i)$.

Notation. Given an endomorphism f of a complex vector space and a $w \in \mathbb{R}$, we denote by $\mathrm{Eig}_w(f)$ the direct sum of generalized eigenspaces of f corresponding to eigenvalues with real part w . We also denote by $\mathrm{Eig}_{\geq w}(f)$ the direct sum of all $\mathrm{Eig}_{w'}(f)$ with $w' \geq w$.

Equipping each vector space $E|_{p_i}$ with the flag

$$E|_{p_i} = \mathrm{Eig}_{\geq w_1(p_i)}(\mathrm{Res}_{p_i}(\nabla)) \supseteq \mathrm{Eig}_{\geq w_2(p_i)}(\mathrm{Res}_{p_i}(\nabla)) \supseteq \dots \supseteq \mathrm{Eig}_{\geq w_{b_i}(p_i)}(\mathrm{Res}_{p_i}(\nabla)) \supseteq \{0\}$$

defines a parabolic structure on E at P of type \mathbf{w} .

If a local section s near p_i satisfies $0 \neq s(p_i) \in \mathrm{Eig}_w(\mathrm{Res}_{p_i}(\nabla))$, then $\mathrm{ord}_{p_i} \|s\|_H = w$ and so the parabolic structure just defined corresponds to the filtration

$$E_w = \{s \mid \|s\|_H \cdot |z_i|^{\varepsilon - w} \text{ bounded near } p_i \text{ for all } \varepsilon > 0 \text{ and all } p_i \in P\} \subset E(\infty \cdot P)$$

where z_i is a local holomorphic coordinate on S centered at p_i .

Thus, $\mathrm{Res}_{p_i}(\nabla)$ belongs to the conjugacy class $\mathrm{Cl}(M_i) \subset \mathfrak{u}_N$ of

$$M_i = \left(\begin{array}{c|c|c} w_1(M_i)\mathrm{Id}_{m_1(M_i)} & 0 & 0 \\ \hline 0 & \ddots & 0 \\ \hline 0 & 0 & w_{b_i}(M_i)\mathrm{Id}_{m_{b_i}(M_i)} \end{array} \right)$$

where $w_k(M_i) = w_k(p_i)$ and $m_k(M_i) = m_k(p_i)$, and the monodromy of ∇ along ∂_i belongs to the conjugacy class $c_i = \mathrm{Cl}(\exp[-2\pi\sqrt{-1}M_i]) \subset \mathrm{U}_N$ of $\exp(-2\pi\sqrt{-1}M_i)$.

Notation. If $c_i = \mathrm{Cl}(C_i) \subset \mathrm{U}_N$, then there exists a unique matrix $M_i \in \sqrt{-1} \cdot \mathfrak{u}_N$ with real eigenvalues in $[0, 1)$ such that $\exp(-2\pi\sqrt{-1}M_i) = C_i$. We will write $w_k(c_i) := w_k(M_i)$ and $m_k(c_i) := m_k(M_i)$ and $\mathbf{w}(c)$ for the collection of all $w_k(M_i)$ and $m_k(M_i)$.

We can now state the analogue of Theorem 4.1 for punctured surfaces, whose proof is again by continuity method.

Theorem 4.8 (Mehta-Seshadri [42]). The map that sends a flat U_N -bundle (ξ, ∇) on \dot{S} to the Deligne extension E_\bullet of the I -holomorphic vector bundle $\hat{E} = \xi \times_{U_N} \mathcal{O}_S^{\oplus N}$ induces a real-analytic homeomorphism

$$\mathrm{Fl}^{irr}(\dot{S}, U_N, \mathbf{c}) \xrightarrow{\sim} \mathrm{Bun}^s(S, \mathbf{w}(\mathbf{c}), N)_0$$

between the moduli space of irreducible flat U_N -principal bundles on \dot{S} with monodromy along ∂_i in c_i and the moduli space of stable I -holomorphic parabolic vector bundles of rank N , type $\mathbf{w}(\mathbf{c})$ and degree 0 on (S, I) .

Such homeomorphism restricts to

$$\begin{aligned} \mathrm{Fl}^{irr}(\dot{S}, \mathrm{SU}_N, \mathbf{c}) &\xrightarrow{\sim} \mathrm{Bun}^s(S, \mathbf{w}(\mathbf{c}), N, \mathcal{O}_S) \\ (\xi, \nabla) &\longmapsto (E_\bullet, \eta) \end{aligned}$$

where $\eta : \det(E_\bullet) \rightarrow \mathcal{O}_S$ sends the unit volume element to 1.

As seen before, the statement extends to b -framed polystable parabolic bundles.

Corollary 4.9. There are real-analytic homeomorphisms

$$\begin{aligned} \mathrm{Fl}_b^{dec}(\dot{S}, U_N, \mathbf{c}) &\xrightarrow{\sim} \mathrm{Bun}_b^{ps}(S, \mathbf{w}(\mathbf{c}), N)_0 \\ \mathrm{Fl}_b^{dec}(\dot{S}, \mathrm{SU}_N, \mathbf{c}) &\xrightarrow{\sim} \mathrm{Bun}_b^{ps}(S, \mathbf{w}(\mathbf{c}), N, \mathcal{O}_S). \end{aligned}$$

In both cases, irreducible flat bundles correspond to stable I -holomorphic bundles.

The counterpart to Theorem 4.3 for punctured surfaces was proven by Biquard using analytic methods.

Theorem 4.10 (Biquard [3]). A holomorphic vector bundle E_\bullet of rank N and degree 0 on (S, I) with parabolic structure at P of type \mathbf{w} admits a flat invariant metric if and only if E_\bullet is polystable. Moreover, such a metric is unique up to automorphisms of E_\bullet .

The above achievements (both in the closed and punctured case) culminate in the more general correspondence between flat GL_N -bundles on punctured surfaces and parabolic Higgs bundles proven by Simpson [53]. Here we describe how one direction works, namely how to go from flat bundles to parabolic Higgs bundles.

Given a GL_N -bundle ξ on \dot{S} , the induced flat vector bundle $\hat{V} = \xi \times_{\mathrm{GL}_N} \mathbb{C}^N$ can be extended to $V \rightarrow S$ in such a way that the real parts of the eigenvalues $\lambda_k(p_i) + i\nu_k(p_i)$ of $\mathrm{Res}_{p_i}(\nabla)$ satisfy $0 \leq \lambda_1(p_i) < \dots < \lambda_{b_i}(p_i) < 1$. Moreover, if ξ has reductive monodromy, an adaptation of Corlette's theorem in the noncompact case (see also Labourie [36]) ensure the existence (and uniqueness up to isomorphism) of a *tame* harmonic metric H on the flat vector bundle $\hat{V} = \xi \times_{\mathrm{GL}_N} \mathbb{C}^N$. This means that ∇ on $E := V$ can be decomposed as $\nabla = \nabla^H + \Phi + \bar{\Phi}^H$, where ∇^H is compatible with H as before, Φ and its H -adjoint $\bar{\Phi}^H$ have *at worst simple poles at P* and the *Higgs field* $\Phi \in C^\infty(S, K(P) \otimes \mathrm{End}(E_\bullet))$ is holomorphic with respect to $\bar{\partial}^E = \bar{\partial}^V - \bar{\Phi}^H$.

Furthermore, E can be endowed with a parabolic structure at P of type \mathbf{w} defined by the filtration

$$E|_{p_i} = \mathrm{Eig}_{\geq w_1(p_i)}(\mathrm{Res}_{p_i}(\nabla)) \supseteq \mathrm{Eig}_{\geq w_2(p_i)}(\mathrm{Res}_{p_i}(\lambda)) \supseteq \dots \supseteq \mathrm{Eig}_{\geq w_{b_i}(p_i)}(\mathrm{Res}_{p_i}(\nabla)) \supseteq \{0\}$$

where $w_k(p_i) = \lambda_k(p_i)$ and $m_k(p_i) = \dim \mathrm{Eig}_{\lambda_k(p_i)}(\mathrm{Res}_{p_i}(\nabla))$.

Notation. Given a matrix $M = D + 2M^0 \in \mathfrak{gl}_N(\mathbb{C})$ in Jordan form, with D diagonal and M^0 nilpotent, we call $M' = \mathrm{Re}(D) + M^0$ and $M'' = \sqrt{-1} \mathrm{Im}(D) + M^0$.

It can be checked that, if $\mathrm{Res}_{p_i}(\nabla)$ belongs to $\mathrm{Cl}(M_i) \subset \mathfrak{sl}_N$ with M_i in Jordan form and so the monodromy along ∂_i belongs to $c_i = \mathrm{Cl}(\exp(-2\pi\sqrt{-1}M_i))$, then $\mathrm{Res}_{p_i}(\Phi)$ belongs to $\mathfrak{c}_i = \mathrm{Cl}(M_i''/2) \subset \mathfrak{sl}_N$. As before, we will denote by $\mathbf{w}(\mathbf{c})$ the collection of all $w_k(M_i)$

and $m_k(M_i)$.

Summarizing our discussion, the correspondence preserves generalized eigenspaces of $\mathrm{hol}_\xi(\gamma_i)$, of $\mathrm{Res}_{p_i}(\nabla)$ and of $\mathrm{Res}_{p_i}(\Phi)$; inside a single generalized eigenspace it works as illustrated in the table below (borrowed from [53]), where ς is a local holomorphic section of E that does not vanish at p_i .

	(E_\bullet, Φ)	(V, ∇)	monodromy	$\mathrm{ord}_{p_i} \ \varsigma\ _H$
jump at p_i	λ	λ		λ
residue eigenvalue at p_i	$\sqrt{-1}\nu/2$	$\lambda + \sqrt{-1}\nu$	$\exp[-2\pi\sqrt{-1}(\lambda + \sqrt{-1}\nu)]$	

Remark 4.11. The order of growth of $\|\varsigma\|_H$ near p_i may have logarithmic factors as in Example 7 (see [53]). More precisely, if $\varsigma(p_i)$ takes values in a subspace of $V|_{p_i}$ corresponding to a Jordan block of $\mathrm{Res}_{p_i}(\nabla)$ of size m and eigenvalue $\lambda + \sqrt{-1}\nu$, then

$$\|\varsigma\|_H \sim |z_i|^\lambda |\log |z_i||^{l - \frac{m+1}{2}}$$

where z_i is a local coordinate on S at p_i and $l > 0$ is the smallest integer such that $[\mathrm{Res}_{p_i}(\nabla) - (\lambda + \sqrt{-1}\nu)\mathrm{Id}]^l s(p_i) = 0$.

The above set-theoretic correspondence can be promoted to a real-analytic one. Similarly to the closed case treated by Hitchin and Simpson, the moduli spaces of flat bundles and of Higgs bundles are different holomorphic manifestations of the same hyperkähler manifold: the moduli space of harmonic bundles. This is proven by Konno [34] for Higgs fields with nilpotent residues and by Biquard-Boalch [4] in general.

Theorem 4.12 (Simpson [53], Konno [34], Biquard-Boalch [4]). Let $M_i \in \mathfrak{gl}_N$ be a matrix in Jordan form and let $c_i = \mathrm{Cl}(\exp(-2\pi\sqrt{-1}M_i))$ be a conjugacy class in GL_N and $\mathfrak{c}_i = \mathrm{Cl}(M_i''/2) \subset \mathfrak{gl}_N$ for $i = 1, \dots, n$.

There is a real-analytic diffeomorphism

$$\mathrm{Fl}_b^{\mathrm{red}}(\dot{S}, \mathrm{GL}_N, \mathfrak{c}) \xrightarrow{\sim} \mathrm{Higgs}_b^{\mathrm{ps}}(S, \mathbf{w}, N, \mathfrak{c})_0$$

between the moduli space of b -framed flat GL_N -bundles (ξ, ∇, τ) on \dot{S} with reductive monodromy and $\mathrm{hol}_\xi(\partial_i) \in c_i$ and the moduli space of b -framed polystable Higgs bundles $(E_\bullet, \eta, \Phi, \tau)$ on (S, I) of parabolic type $\mathbf{w} = \mathbf{w}(\mathfrak{c})$ at P and degree 0 with $\mathrm{Res}_{p_i}(\Phi) \in \mathfrak{c}_i$. Under such diffeomorphism, flat bundles with Zariski-dense monodromy correspond to stable parabolic Higgs bundles, and so the induced

$$\mathrm{Fl}^{\mathrm{Zd}}(\dot{S}, \mathrm{GL}_N, \mathfrak{c}) \xrightarrow{\sim} \mathrm{Higgs}^s(S, \mathbf{w}, N, \mathfrak{c})_0$$

is a real-analytic diffeomorphism too. Similarly, via

$$\mathrm{Fl}_b^{\mathrm{red}}(\dot{S}, \mathrm{SL}_N, \mathfrak{c}) \xrightarrow{\sim} \mathrm{Higgs}_b^{\mathrm{ps}}(S, \mathbf{w}, N, \mathcal{O}_S, \mathfrak{c})$$

$$\mathrm{Fl}^{\mathrm{Zd}}(\dot{S}, \mathrm{SL}_N, \mathfrak{c}) \xrightarrow{\sim} \mathrm{Higgs}^s(S, \mathbf{w}, N, \mathcal{O}_S, \mathfrak{c})$$

SL_N -bundles correspond to parabolic Higgs bundles (E_\bullet, Φ) endowed with a trivialization $\eta : \det(E_\bullet) \xrightarrow{\sim} \mathcal{O}_S$.

4.3. Correspondence and the real locus in rank 2. Following Hitchin [29], consider a point (E_\bullet, η, Φ) in $\mathrm{Higgs}^s(S, \mathbf{w}, 2, \mathcal{O}_S, \mathfrak{c})$ fixed by the involution σ that sends Φ to $-\Phi$.

By Lemma 3.11, (E_\bullet, η, Φ) may belong to $\mathrm{Bun}^s(S, \mathbf{w}, 2, \mathcal{O}_S)$ or to some $\mathrm{Higgs}^s(S, \mathbf{w}, 2, \mathcal{O}_S, \mathfrak{c})(\mathbb{R})_{d,\mathfrak{a}}$.

In the former case, $\Phi = 0$ and so the point corresponds to a flat SU_2 -bundle by Theorem 4.8.

In the latter case, $\Phi \neq 0$ and we assume $e(d, \mathfrak{a}, \mathbf{w}) > 0$. Then Lemma 3.11 provides a splitting $E_\bullet = L_\bullet^\vee \oplus L_\bullet$ and an automorphism ι of E_\bullet that preserves the splitting and that sends Φ to $-\Phi$. Moreover, such splitting and ι are essentially unique, since (E_\bullet, Φ) is stable. Let $\dot{V} := \dot{E}$ and let ∇ (resp. ∇') be the flat connection on \dot{V} associated to (E_\bullet, Φ) (resp.

$(E_\bullet, -\Phi)$). It is easy to see that the flat vector bundles (\dot{V}, ∇) and (\dot{V}, ∇') support the same tame harmonic metric H , which means that $\nabla = \nabla^H + \Phi + \bar{\Phi}^H$ and $\nabla' = \nabla^H - \Phi - \bar{\Phi}^H$. Since $\iota^*(-\Phi) = \Phi$ and $\iota^*(\nabla') = \nabla$, the automorphism ι preserves ∇^H and so is an H -isometry; as a consequence, L_\bullet and L_\bullet^\vee are H -orthogonal and H induces the identification $\bar{L}_\bullet \cong L_\bullet^\vee$. Since the anti-linear involution

$$T : L_\bullet^\vee \oplus L_\bullet \longrightarrow L_\bullet^\vee \oplus L_\bullet \\ (\alpha, \beta) \longmapsto (\bar{\beta}, \bar{\alpha})$$

satisfies $T \circ \Phi = \bar{\Phi}^H \circ T$ and so commutes with ∇ , the monodromy of ∇ preserves $\dot{V}(\mathbb{R}) := \text{Fix}(T) \subset \dot{V}$ and so it defines a representation $\rho_{\mathbb{R}}$ that takes values in $\text{SL}_2(\mathbb{R})$. Moreover, $\dot{V}(\mathbb{R}) \hookrightarrow \dot{V} \cong \dot{L} \oplus \bar{\dot{L}} \cong \dot{V}(\mathbb{R}) \otimes \mathbb{C}$ and we identify $\dot{V}(\mathbb{R})$ to \dot{L} .

As a consequence, if $\rho_{\mathbb{R}}(\gamma_i)$ is elliptic and $\{\text{rot}\}(\rho_{\mathbb{R}}(\gamma_i)) = \{r_i\}$, then $\{r_i\} = \text{deg}_{p_i}\{L_\bullet^{-2}\}$. Notice that the power 2 appears because $\text{SL}_2(\mathbb{R}) \rightarrow \text{PSL}_2(\mathbb{R})$ is a cover of degree 2.

A version of the following can be found for instance in Section 3.6 of [11].

Lemma 4.13 (Euler number as a first Chern class). The parabolic degree of L_\bullet and the Euler number of ρ satisfy $\text{eu}(\rho) = 2 \text{deg}(L_\bullet) = 2 \text{deg}(L) + 2\|\mathbf{w}_L\|_1$.

The above discussion then leads to the following result, the bound on d being a consequence of Proposition 3.16(a) and Lemma 4.13.

Theorem 4.14 (Correspondence for $\text{SL}_2(\mathbb{R})$). For every $i = 1, \dots, n$, let

- c_i be a conjugacy class in $\text{SL}_2(\mathbb{R})$
- \mathbf{c}_i be a conjugacy class in $\mathfrak{sl}_2(\mathbb{C})$
- $w_1(p_i) \in [0, 1/2]$ and $\varepsilon : J_{\text{nil}} = \{j \mid \mathbf{c}_j \text{ nilpotent}\} \rightarrow \{+, -\}$
- $a_i \in \{0, 1\}$

that match according to the following table

	a	\mathbf{c}	\mathbf{c}	w_L
deg.	0	0	$(-1)^{2w_1} \text{Id}$	$w_1 = 0; \frac{1}{2}$
deg. $\varepsilon = \pm 1$	0	$\begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}$	$(-1)^{2w_1} \begin{pmatrix} 1 & 0 \\ \varepsilon & 1 \end{pmatrix}$	$w_1 = 0; \frac{1}{2}$
deg. $\ell > 0$	0	$\begin{pmatrix} \sqrt{-1}\ell/8\pi & 0 \\ 0 & -\sqrt{-1}\ell/8\pi \end{pmatrix}$	$(-1)^{2w_1} \begin{pmatrix} \exp(\ell/2) & 0 \\ 0 & \exp(-\ell/2) \end{pmatrix}$	$w_1 = 0; \frac{1}{2}$
non-deg.	$\{0, 1\}$	0	$\begin{pmatrix} \cos(2\pi w_L) & \sin(2\pi w_L) \\ -\sin(2\pi w_L) & \cos(2\pi w_L) \end{pmatrix}$	$a + (-1)^a w_1$

TABLE 1

Let $d \in \mathbb{Z}$ such that

$$-\|\mathbf{w}_L\|_1 < d \leq g - 1 + \frac{s^{ev} - \|\mathbf{a}\|_1 - s_0 - s_-(\varepsilon)}{2}$$

where $s^{ev} = \#\{j \mid w_1(p_j) = 0\}$, $s_0 = \#\{i \mid c_i = \{\text{Id}\}\}$ and $s_-(\varepsilon) = \#\{j \in J_{\text{nil}} \mid \varepsilon_j = -\}$. Then there are real-analytic diffeomorphisms

$$\text{Rep}(\dot{S}, \text{SL}_2(\mathbb{R}), \mathbf{c})_e \xrightarrow{\sim} \text{Higgs}(S, \mathbf{w}, 2, \mathcal{O}_S, \mathbf{c})(\mathbb{R})_{d, \mathbf{a}}^\varepsilon \\ \text{Rep}(\dot{S}, \text{PSL}_2(\mathbb{R}), \mathbf{c})_e \xrightarrow{\sim} \text{Higgs}(S, \mathbf{w}, 2, \mathcal{O}_S, \mathbf{c})(\mathbb{R})_{d, \mathbf{a}}^\varepsilon / \text{Pic}^0(S)[2]$$

where $e = 2d + 2 \sum_{i=1}^n w_L(p_i) > 0$ is the Euler number of the associated oriented \mathbb{RP}^1 -bundle. Moreover, the above maps extend to homeomorphisms

$$\text{Rep}(\dot{S}, \text{SL}_2(\mathbb{R}), \bar{\mathbf{c}})_e \xrightarrow{\sim} \text{Higgs}(S, \mathbf{w}, 2, \mathcal{O}_S, \bar{\mathbf{c}})(\mathbb{R})_{d, \mathbf{a}}^\varepsilon \\ \text{Rep}(\dot{S}, \text{PSL}_2(\mathbb{R}), \bar{\mathbf{c}})_e \xrightarrow{\sim} \text{Higgs}(S, \mathbf{w}, 2, \mathcal{O}_S, \bar{\mathbf{c}})(\mathbb{R})_{d, \mathbf{a}}^\varepsilon / \text{Pic}^0(S)[2]$$

for the closures $\bar{\mathbf{c}}$ of \mathbf{c} and $\bar{\mathbf{c}}$ of \mathbf{c} .

Proof. We are only left to prove the last assertion, namely that the map $\mathrm{Rep}(\dot{S}, \mathrm{SL}_2(\mathbb{R}), \bar{\mathbf{c}})_e \rightarrow \mathrm{Higgs}(S, \mathbf{w}, 2, \mathcal{O}_S, \bar{\mathbf{c}})(\mathbb{R})_{d, \mathbf{a}}^\varepsilon$ is a homeomorphism. We already know that it is bijective and we want to show that it is continuous and proper.

Endow \dot{S} with the unique I -conformal hyperbolic metric of finite area and its universal cover \tilde{S} with the pull-back metric. Let S° be the compact subsurface obtained from S by removing small open disk neighborhoods of the marked points and let \tilde{S}° be its preimage inside the universal cover \tilde{S} and $F \subset \tilde{S}$ be a fundamental domain for the action of $\pi_1(\dot{S})$. We recall that, for every equivariant harmonic map $\tilde{h} : \tilde{S} \rightarrow \mathbb{H}^2 \cong \mathrm{SL}_2(\mathbb{R})/\mathrm{SO}_2(\mathbb{R})$, we have

$$(\star) \quad \left| \nabla \tilde{h} \right|^2 \leq C \cdot \mathcal{E}_{S^\circ}(\tilde{h})$$

at every point of $\tilde{S}^\circ \cap F$, where C depends only on the diameter of $\tilde{S}^\circ \cap F$ and $\mathcal{E}_{S^\circ}(\tilde{h})$ is the energy of the restriction of \tilde{h} to $\tilde{S}^\circ \cap F$.

Consider a sequence $\rho^{(k)}$ of representations in $\mathrm{Rep}(\dot{S}, \mathrm{SL}_2(\mathbb{R}), \bar{\mathbf{c}})_e$ and let $(E_\bullet^{(k)}, \eta^{(k)}, \Phi^{(k)})$ be the corresponding Higgs bundle. By Lemma 3.11, the bundle $E^{(k)}$ is isomorphic to $(L_\bullet^{(k)})^\vee \oplus L_\bullet^{(k)}$, with $L^{(k)}$ of fixed degree d .

Suppose now that $\rho^{(k)} \rightarrow \rho$ and let $(E_\bullet = L_\bullet^\vee \oplus L_\bullet, \eta, \Phi)$ the Higgs bundle determined ρ . By Proposition 2.6.1 of [35], there are $\rho^{(k)}$ -equivariant harmonic maps $\tilde{h}^{(k)} : \tilde{S} \rightarrow \mathbb{H}^2$ that are locally equi-bounded and equi-Lipschitz. Thus, $\tilde{h}^{(k)}$ locally Lipschitz converges to the unique ρ -equivariant harmonic map \tilde{h} . Thus, $\bar{\partial}^{L^{(k)}} \rightarrow \bar{\partial}^L$ and $\Phi^{(k)} \rightarrow \Phi$ uniformly on the compact subsets of \dot{S} . Since the parabolic structure is fixed, this implies that $L^{(k)} \rightarrow L$ and so $\Phi^{(k)} \rightarrow \Phi$. This proves continuity of the correspondence.

Suppose finally that $\rho^{(k)}$ is divergent. We want to show that $\|\Phi^{(k)}|_{S^\circ}\|^2$ is divergent. By contradiction, up to extracting a subsequence, $\mathcal{E}_{S^\circ}(\tilde{h}^{(k)}) = 2\|\Phi^{(k)}|_{S^\circ}\|^2$ would be bounded. By the locally uniform bound (\star) , the $\tilde{h}^{(k)}|_{\tilde{S}^\circ \cap F}$ would be equi-Lipschitz and so $\rho^{(k)}$ would have a convergent subsequence. \square

Notice that c_i determines a_i , $w_1(p_i)$ and ε_i (and the sign ε_i , if \mathbf{c}_i is nilpotent) and vice versa. Thus we can also draw the following conclusion.

Corollary 4.15 (Components of $\mathrm{PSL}_2(\mathbb{R})$ -representations). Connected components of $\mathrm{Rep}(\dot{S}, \mathrm{PSL}_2(\mathbb{R}), \mathbf{c})$ with Euler number $e = 2d + 2\|\mathbf{w}_L\|_1 > 0$ are classified by the integers d such that $-\|\mathbf{w}_L\|_1 < d \leq g - 1 + \frac{1}{2}(s^{ev} - \|\mathbf{a}\|_1 - s_0 - s_-(\varepsilon))$.

The topology of $\mathrm{Higgs}(S, \mathbf{w}, 2, \mathcal{O}_S, \mathbf{c})(\mathbb{R})_{d, \mathbf{a}}$ and of its quotient by $\mathrm{Pic}^0(S)[2]$ is described in Proposition 3.16.

4.4. Uniformization components. Let $\ell = (\ell_1, \dots, \ell_n)$ with $\ell_i = \sqrt{-1}\vartheta_i$ and $\vartheta_i > 0$ for $i = 1, \dots, k$ and $\ell_i \geq 0$ for $i = k + 1, \dots, n$. Call $s_0 = \#\{i \in \{1, \dots, k\} \mid \vartheta_i \in 2\pi\mathbb{N}_+\}$.

A consequence of the above work is the following result stated in the introduction.

Corollary 4.16 (Topology of uniformization components). Assume $e_\ell > 0$ and consider the monodromy map

$$\mathrm{hol} \circ \Xi_\ell : \mathcal{Y}(\dot{S}, \ell) \longrightarrow \mathrm{Rep}(\dot{S}, \mathrm{PSL}_2(\mathbb{R}), \mathbf{c}_\ell)_{e_\ell}.$$

The space $\mathrm{Rep}(\dot{S}, \mathrm{PSL}_2(\mathbb{R}), \mathbf{c}_\ell)_{e_\ell}$ is real-analytically diffeomorphic to a holomorphic affine bundle of rank $3g - 3 + n - m$ over $\mathrm{Sym}^{m-s_0}(S \setminus \{p_{k+1}, \dots, p_n\})$, with $m = \sum_{1 \leq i \leq k} \lfloor \frac{\vartheta_i}{2\pi} \rfloor$.

Proof. The monodromy map takes values in $\mathrm{Rep}(\dot{S}, \mathrm{PSL}_2(\mathbb{R}), \mathbf{c}_\ell)_{e_\ell}$ by Proposition 2.20. The result then follows from Theorem 4.14 and Proposition 3.16, remembering that cusps correspond to positive unipotents. \square

In [14] Derooin-Tholozan show that “super-maximal” components of $\mathrm{PSL}_2(\mathbb{R})$ -representations consist of monodromies of hyperbolic metrics. Derooin told me that this result can be recovered using Corollary 3.20 and the analysis carried out in Section 3.6 as follows.

Corollary 4.17. Fix $\mathbf{c} = (c_1, \dots, c_n)$ and $e > 0$ and assume $n > 3 + s_0$. Then $\mathrm{Rep}(\dot{S}, \mathrm{PSL}_2(\mathbb{R}), \bar{\mathbf{c}})_e$ is compact if and only if

$$\begin{cases} g = 0 \\ \text{no } c_i \text{ is hyperbolic or negative unipotent} \\ e = 1 - \|\{\mathbf{r}\}\|_1 \in (0, 1]. \end{cases}$$

Moreover, in this case every representation in $\mathrm{Rep}(\dot{S}, \mathrm{PSL}_2(\mathbb{R}), \bar{\mathbf{c}})_e$ is the monodromy of some hyperbolic metric on \dot{S} .

Proof. By Corollary 3.20, representations lying in compact components cannot have hyperbolic or negative unipotent boundary monodromy.

Fix a complex structure I on S .

Let \mathcal{D} be a line bundle of degree $d_0 = 0$ if $n - 1$ is even, or of degree $d_0 = 1$ if $n - 1$ is odd. Fix $d = (d_0 + 1 - n)/2$. Let $a_1 = \dots = a_n = 0$ and let $w_1(p_i) = \frac{1}{2}(1 - \{r_i\})$. Finally, pick the conjugacy classes \mathbf{c}_i to be $\{0\}$ if c_i is elliptic, and to be nilpotent if c_i is positive unipotent (and, in this case, we set $\varepsilon_i = +$).

By Theorem 4.14, the component $\mathrm{Rep}(\dot{S}, \mathrm{PSL}_2(\mathbb{R}), \bar{\mathbf{c}})_e$ is homeomorphic to $\mathrm{Higgs}(S, \mathbf{w}, 2, \mathcal{O}_S, \bar{\mathbf{c}})_{d, \mathbf{a}}^\varepsilon$ and so the first claim follows from Corollary 3.20.

As for the last claim, remember that all Higgs bundles $[E, \Phi]$ in the compact $\mathrm{Higgs}(S, \mathbf{w}, 2, \mathcal{O}_S, \bar{\mathbf{c}})_{d, \mathbf{a}}^\varepsilon$ have $\psi = 0$ and so Φ is nilpotent. The harmonic section h of $\xi_\rho \times_{\mathrm{SL}_2(\mathbb{R})} (\mathrm{SL}_2(\mathbb{R})/\mathrm{SO}_2(\mathbb{R}))$ constructed by Donaldson corresponds to a ρ -equivariant harmonic map $\tilde{h} : \tilde{S} \rightarrow \mathbb{H}^2 \cong \mathrm{SL}_2(\mathbb{R})/\mathrm{SO}_2(\mathbb{R})$. Since the quadratic differential $\det(\Phi)$ pulls back to \tilde{S} to the Hopf differential of \tilde{h} , the vanishing $\det(\Phi) = 0$ implies that \tilde{h} is conformal. Hence, the pull-back via \tilde{h} of the hyperbolic metric on \mathbb{H}^2 descends to a conformal hyperbolic metric on (\dot{S}, I) , possibly with extra conical points of angles in $2\pi\mathbb{N}_+$, and with monodromy ρ . \square

The case $(g, n) = (0, 3 + s_0)$ of the pair of pants can be done by hands.

The following result by McOwen [41] and Troyanov [58] is a version of Koebe’s uniformization theorem [32] [33] for hyperbolic surfaces with conical singularities.

Theorem 4.18 (Uniformization with conical singularities). For every $\vartheta_1, \dots, \vartheta_n \geq 0$ such that $e_{\sqrt{-1}\vartheta} > 0$, there exists exactly one metric on \dot{S} with conical singularity of angle $2\pi\vartheta_i$ at p_i (or with a cusp at p_i , if $\vartheta_i = 0$) and which is I -conformal.

Mimicking Hitchin’s computation [29] in the case of a closed surface, we then have the following expected consequence.

Corollary 4.19 (Uniformization Higgs bundles). Assume $k = n$ and let $\delta_i = \frac{\vartheta_i}{2\pi} - 1$. Then the monodromy representation $\mathrm{hol}_h : \pi \rightarrow \mathrm{PSL}_2(\mathbb{R})$ of the unique hyperbolic metric h in $\mathcal{Y}(\dot{S}, \ell)$ with conical singularities of angle $2\pi\vartheta_i$ at p_i which is I -conformal corresponds to the $\mathrm{Pic}^0(S)[2]$ -equivalence class of the parabolic Higgs bundle (E_\bullet, Φ) , with

$$E_\bullet = L_\bullet^\vee \oplus L_\bullet, \quad \Phi = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}$$

where $L_\bullet = B(-\frac{1}{2}\delta \cdot P)$ and $B^2 \cong K_S$, and so $L_\bullet^{-2}K(P) = T_S(\delta \cdot P) \otimes K_S(P)$.

Proof. It is enough to notice that the harmonic metric the \mathbb{R}^2 -bundle $\dot{V} \rightarrow \dot{S}$ with monodromy hol_h is provided by the (equivariant) developing map $(\tilde{S}, \tilde{h}) \rightarrow \mathbb{H}^2 = \mathrm{SL}_2(\mathbb{R})/\mathrm{SO}_2(\mathbb{R})$ itself, and then follow Hitchin’s computation. \square

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