

# Revisiting the linear recursions with nonnegative coefficients problem

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## Abstract

The purpose of this paper is to state the correct formulation of a theorem proposed by M. Roitman and Z. Rubinstein on the characterization of linear recursions which imply a linear recursion with nonnegative coefficients. The authors present a counterexample to such a theorem and then state its correct formulation.

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## 1 Introduction

The purpose of this paper is to state the correct formulation of a theorem proposed by M. Roitman and Z. Rubinstein in [3] on the characterization of linear recursions which imply a linear recursion with nonnegative coefficients. In particular, we refer to Theorem 5 in [3]. Such a theorem asserts that three sets of conditions, namely (A), (B) and (C), are equivalent. Preliminarily we provide a counterexample that shows that condition (C) is not equivalent to condition (B). Then, we provide the correct formulation of the theorem by correcting condition (C) and by proving that this new formulation is equivalent to condition (B). Hereinafter, for the reader's convenience, we recall conditions (B) and (C) of Theorem 5 in [3]. These conditions are defined on a polynomial  $P(x) = x^k - a_1x^{k-1} - \dots - a_k$ , where  $a_1, \dots, a_k$  are complex numbers:

(B) *The polynomial  $P(x)$  divides a polynomial in  $\mathcal{Q}$ .*

(C) *In case the polynomial  $P(x)$  has a positive root  $r$ , then all conditions (1)-(4) below are satisfied:*

(1)  $r \geq |\alpha|$  for any root  $\alpha$  of  $P(x)$ ;

(2) if  $|\alpha| = r$  for some root  $\alpha$  of  $P(x)$ , then  $\alpha/r$  is a root of unity;

(3) all roots of  $P(x)$  with absolute value  $r$  are simple;

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(4) if  $P(r) = P(r\epsilon) = 0$ , where  $\epsilon^k = 1$  with  $k \geq 1$  minimal, then  $P(x)$  has no roots of the form  $s\omega$  where  $0 < s < r$  and  $\omega^k = 1$ .

where  $\mathcal{Q}$  denotes the set of all real polynomials of the form

$$c_n x^n - \sum_{i=0}^{n-1} c_i x^i \quad (1)$$

with  $n \geq 1$ ,  $c_n > 0$ , and  $c_i \geq 0$  for all  $i = 0, \dots, n-1$ .

## 2 Result

We start this section providing a counterexample to Theorem 5 in [1]. To this end, consider the monic polynomial

$$P(x) = (x-1)(x - e^{i\frac{\pi}{2}})(x - e^{-i\frac{\pi}{2}})(x - e^{i\frac{2\pi}{3}})(x - e^{-i\frac{2\pi}{3}})\left(x - \frac{e^{i\frac{\pi}{6}}}{2}\right)\left(x - \frac{e^{-i\frac{\pi}{6}}}{2}\right) \quad (2)$$

This polynomial satisfies conditions (C) of Theorem 5 in [3]. In particular, note that the fourth condition of (C) is fulfilled with  $k = 4$  for the roots  $e^{i\frac{\pi}{2}}$ ,  $e^{-i\frac{\pi}{2}}$ , and with  $k = 3$  for the roots  $e^{i\frac{2\pi}{3}}$ ,  $e^{-i\frac{2\pi}{3}}$ .

Assume condition (B) does hold. Hence, the polynomial  $P(x)$  divides a polynomial  $Q(x)$  in  $\mathcal{Q}$ , that is a polynomial as in (1) with, in particular,  $c_n = 1$ . As a consequence, the matrix

$$A = \begin{pmatrix} 0 & 1 & 0 & \cdots & 0 \\ 0 & 0 & 1 & & \vdots \\ \vdots & & \ddots & \ddots & 0 \\ 0 & & & 0 & 1 \\ c_0 & c_1 & \cdots & c_{n-2} & c_{n-1} \end{pmatrix}$$

is a nonnegative matrix with characteristic polynomial  $Q(x)$ . From the Frobenius theorem [1, 2] it follows that  $Q(x)$  has all the 12-th roots of unity among its roots, together with  $e^{i\frac{\pi}{6}}/2$ ,  $e^{-i\frac{\pi}{6}}/2$ . In particular, the polynomial  $Q(x)$  has the roots  $e^{i\frac{2\pi}{12}} = e^{i\frac{\pi}{6}}$ ,  $e^{-i\frac{2\pi}{12}} = e^{-i\frac{\pi}{6}}$ . Hence, the polynomial  $Q(x)$  satisfies condition (B) by definition, but does not satisfy the fourth condition of (C) since it has roots of the form  $s\omega$  where  $0 < s = 0.5 < 1$  and  $\omega^{12} = 1$ .

We provide now the correct formulation of Theorem 5, which differs from the original one in the fourth condition of (C).

**Theorem 1** *Let  $a_1, \dots, a_k$  be given complex numbers, and let  $P(x)$  be the polynomial  $x^k - a_1 x^{k-1} - \dots - a_k$ . Then conditions (A), (B) and (C) below are equivalent:*

(A) *Any infinite sequence  $(u_n)_{n \geq 0}$  of complex numbers which satisfies the recursion*

$$u_{n+k} = a_1 u_{n+k-1} + a_2 u_{n+k-2} + \dots + a_k u_n$$

*for  $n \geq 0$  satisfies also a linear recursion with nonnegative coefficients.*

(B) The polynomial  $P(x)$  divides a polynomial in  $\mathcal{Q}$ .

(C) In case the polynomial  $P(x)$  has a positive root  $r$ , then all conditions (1)-(4) below are satisfied:

- (1)  $r \geq |\alpha|$  for any root  $\alpha$  of  $P(x)$ ;
- (2) if  $|\alpha| = r$  for some root  $\alpha$  of  $P(x)$ , then  $\alpha/r$  is a root of unity;
- (3) all roots of  $P(x)$  with absolute value  $r$  are simple;
- (4) if  $d$  is the minimal integer such that  $\epsilon^d = 1$  for all roots of unity  $\epsilon$  which satisfy  $P(r\epsilon) = 0$ , then  $P(x)$  has no roots of the form  $s\omega$  where  $0 < s < r$  and  $\omega^d = 1$ .

**Proof.** It suffices to prove that condition (B) is equivalent to conditions (C).

(C)  $\Rightarrow$  (B). Let

$$P(x) = \prod_{i=1}^k (x - \alpha_i)$$

and consider the polynomial

$$P_1(x) = \prod_{|\alpha_i| < r} (x - \alpha_i^d)$$

with  $d$  as in condition (4). From condition (4), the polynomial  $P_1(x)$  has no positive real roots. By Lemma 4 in [3], there exists a monic polynomial  $q(x)$  such that

$$Q(x) = (x - r^d) P_1(x)q(x) \in \mathcal{Q}$$

The polynomial  $Q(x^d)$  is in  $\mathcal{Q}$  and includes all the roots of  $P(x)$ . Hence condition (B) holds.

(B)  $\Rightarrow$  (C). Assume the polynomial  $P(x)$  has a positive root  $r$  and let  $Q(x)$  the polynomial as in (1), with  $c_n = 1$ , divisible by  $P(x)$ . Then, the matrix

$$A = \begin{pmatrix} 0 & 1 & 0 & \cdots & 0 \\ 0 & 0 & 1 & & \vdots \\ \vdots & & \ddots & \ddots & 0 \\ 0 & & & 0 & 1 \\ c_0 & c_1 & \cdots & c_{n-2} & c_{n-1} \end{pmatrix}$$

is a nonnegative matrix with characteristic polynomial  $Q(x)$ . The matrix  $A$  can be written in a two blocks form where the first block has all zero eigenvalues and the second block, which is always present, is irreducible. Then, conditions (C) follows directly from the Frobenius theorem [1, 2] applied to the irreducible block.

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**Remark 1** The polynomial  $P(x)$  in (2) satisfies condition (C) of Theorem 5 in [3], but it does not satisfy conditions (A) and (B), since it does not satisfy condition (C) of Theorem 1, the correct version of Theorem 5.

## References

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