Octavic theta series

Eberhard Freitag
Mathematisches Institut
Im Neuenheimer Feld 288
D69120 Heidelberg
`freitag@mathi.uni-heidelberg.de`

Riccardo Salvati Manni
Dipartimento di Matematica,
Piazzale Aldo Moro, 2
I-00185 Roma, Italy.
`salvati@mat.uniroma1.it`

2014

Introduction

In the paper [FS] we considered the even unimodular lattice $L = \Pi_{2,10}$ of signature $(2, 10)$. It can be realized as direct sum of the negative of the lattice E_8 and two hyperbolic planes. We considered t. A certain subgroup of index two $\mathrm{O}^+(L)$ acts biholomorphically on a ten dimensional tube domain $\mathcal{H}_{10}$. We denote the variables with respect to the standard embedding $\mathcal{H}_{10} \hookrightarrow \mathbb{C}^{10} = \Pi_{1,9} \otimes \mathbb{C}$ which we used in [FS] by z (compare Sect. 4). We recall one of the main results in [FS].

Theorem. *There exists a non vanishing modular form $f(z)$ of weight* 4 *(the singular weight) with respect to the full modular group $\mathrm{O}^+(L)$. It is uniquely determined up to a constant factor. The form $f(2z)$ belongs to the principal congruence subgroup of level two $\mathrm{O}^+(L)[2]$. The $\mathrm{O}^+(L)$-orbit of $f(2z)$ spans a* 715*-dimensional space which is the direct sum of $\mathbb{C}f(z)$ and a* 714*-dimensional irreducible space.*

The existence and uniqueness of $f(z)$ can already be derived from the paper [EK] of Eie and Krieg. In [FS] we obtained also the following result. The 715-dimensional space defines an everywhere regular birational embedding of the associated modular variety into $\mathbb{P}^{714}$. The image is contained in a certain system of quadrics. If one is very optimistic, one may conjecture that the ring of modular forms of weight divisible by 4 is generated by the 715-dimensional space and that the quadratic relations are the defining ones.

There is a close relation with the Borcherds Φ-function [Bo] which is a modular form of weight 4 with respect to the orthogonal group of the Enriques lattice $\sqrt{2}M$ where

$$M = U \oplus \sqrt{2}U \oplus (-\sqrt{2}E_8).$$

In fact, as it is explained in [Bo], the lattice $2L$ and the inverse image of any non-zero isotropic vector of $L/2L$ generate a copy of the Enriques lattice. Hence each non-zero isotropic vector α of $L/2L$ leads to a realization f_α of the Φ-function inside our 715-dimensional space. Their divisors are Heegner divisors $H_\alpha(-2)$ which belong to vectors of norm -2. They have an infinite product expansion. In the paper [FS] we proved that the 2079 forms f_α generate the 714-dimensional space.

Recently, in [KMY], it has been proved that the restriction of Φ to a subdomain isomorphic to the Siegel space $\mathbb{H}_2$ of degree 2 can be expressed as a theta series. As a by-product they showed that the eighth power of any even theta constant in genus two is expressed as an infinite product of Borcherds type.

So we have been asked by one of the authors whether Φ is related to theta series. In this paper we give an affirmative answer. We will construct a modular embedding of $\mathcal{H}_{10}$ into the Siegel half plane $\mathbb{H}_{16}$ of degree 16. This means that every substitution of $\mathrm{O}^+(L)$ extends to a Siegel modular substitution. Even more, we will show that it extends to a substitution of the theta group $\Gamma_{16,\vartheta}$ which is the group $\Gamma_{16}[1,2]$ in Igusa's notation. As a consequence, $f(z)$ can be constructed as the restriction of the simplest among all theta series.

$$\vartheta(Z) = \sum_{g\in\mathbb{Z}^n} e^{\pi \mathrm{i} Z[g]} \qquad (n = 16).$$

This modular embedding has also the property that the transformations in $\mathrm{O}^+(L)[2]$ extend to transformations in $\Gamma_{16}[2,4]$. The most natural modular forms on $\Gamma_n[2,4]$ are the theta series of second kind

$$\sum_{g\in\mathbb{Z}^n} e^{2\pi \mathrm{i} Z[g+a/2]}, \quad a \in \mathbb{Z}^n.$$

We will see that their restrictions span the 715-dimensional space.

We have seen already in [FS] that the forms of weight 4 are determined by their values at the zero dimensional cusp classes. Since one can compute these values in both pictures, one can make the identification of the orthogonal modular forms in [FS] and the restrictions of the thetas of second kind explicit.

The construction of the modular embedding rests on some results about the Clifford algebra of the lattice $-E_8$ which we will derive by means of the octavic multiplication. Similar constructions can be found in [FH].

1. The Clifford algebra

Let (V, q) be a quadratic space of positive dimension over a field K. We want to include characteristic two, so we recall that q means a map $q : V \to K$ with the properties

a) $q(ta) = t^2 q(a)$ for $t \in K$,
b) $(a, b) = q(a + b) - q(a) - q(b)$ is bilinear.

The Clifford algebra $\mathcal{C}(V)$ is an associative algebra with unit which contains V as sub-vector space and which is generated by V as algebra. The defining relations are

$$ab + ba = (a, b) \qquad (a, b \in V).$$

Of course K is embeddedd into $\mathcal{C}(V)$ by $t \mapsto t\,1_{\mathcal{C}(V)}$. This defines an embedding

$$K \oplus V \longrightarrow \mathcal{C}(V).$$

The main involution of $\mathcal{C}(V)$ is denoted by $a \mapsto a'$. This is an involutive antiisomorphism which acts on V as the negative of the identity,

$$(a + b)' = a' + b', \ (ab)' = b'a', \quad a' = -a \text{ for } a \in V.$$

The even part of $\mathcal{C}(V)$ is the subalgebra $\mathcal{C}^+(V)$ generated by the two-products ab, $a, b \in V$. It is invariant under the main involution. We are interested in the case $K = \mathbb{R}$, when the dimension of V is eight and q is negative definite. It is known in this case that there exists an isomorphism of involutive algebras

$$\mathcal{C}^+(\mathbb{R}^8) \cong M_8(\mathbb{R}) \times M_8(\mathbb{R}) \qquad (q(x) = -x_1^2 - \cdots - x_8^2).$$

Here the involution on $M_8(\mathbb{R}) \times M_8(\mathbb{R})$ is defined by $(A, B) \mapsto (A', B')$, where the dash now means matrix transposition. As a consequence, there exists an homomorphism of involutive algebras

$$\mathcal{C}^+(\mathbb{R}^8) \longrightarrow M_8(\mathbb{R}) \qquad (q(x) = -x_1^2 - \cdots - x_8^2).$$

It is important for us to get an explicit homomorphism, which preserves also a certain integral structure. This comes into the game if we consider a lattice (L, q) and $V = L \otimes_{\mathbb{Z}} \mathbb{R}$. Then we can consider the $\mathbb{Z}$-algebra $\mathcal{C}(L) \subset \mathcal{C}(V)$ generated by L and also the $\mathbb{Z}$-algebra $\mathcal{C}^+(L)$ generated by all ab, $a, b \in L$. Both algebras are invariant under the main involution. We are in particular interested in the lattice E_8 and its negative definite version. Sometimes we write simply $-E_8$ for it. We mentioned that there exists an involutive homomorphism $\mathcal{C}(-E_8 \otimes_{\mathbb{Z}} \mathbb{R}) \to M_8(\mathbb{R})$. We want that it induces a (surjective) homomorphism

$$\mathcal{C}(-E_8) \longrightarrow M_8(\mathbb{Z}).$$

We will use the octavic multiplication for this purpose.

2. Octaves

We consider $\mathbb{O} = \mathbb{R}^8$ with the standard basis

$$e_0 = (1,0,0,0,0,0,0,0), \ldots, e_7 = (0,0,0,0,0,0,0,1).$$

The octavic product is the bilinear map

$$\mathbb{O} \times \mathbb{O} \to \mathbb{O}, \quad (a,b) \longmapsto a * b,$$

defined by the multiplication table

e_0	e_1	e_2	e_3	e_4	e_5	e_6	e_7
e_1	$-e_0$	e_4	e_7	$-e_2$	e_6	$-e_5$	$-e_3$
e_2	$-e_4$	$-e_0$	e_5	e_1	$-e_3$	e_7	$-e_6$
e_3	$-e_7$	$-e_5$	$-e_0$	e_6	e_2	$-e_4$	e_1
e_4	e_2	$-e_1$	$-e_6$	$-e_0$	e_7	e_3	$-e_5$
e_5	$-e_6$	e_3	$-e_2$	$-e_7$	$-e_0$	e_1	e_4
e_6	e_5	$-e_7$	e_4	$-e_3$	$-e_1$	$-e_0$	e_2
e_7	e_3	e_6	$-e_1$	e_5	$-e_4$	$-e_2$	$-e_0$

It is known that this product has no zero divisors. We will call the elements of $\mathbb{O}$ from now on "octaves". An octave is integral, if it is in the $\mathbb{Z}$-module

$$\mathbb{O}(\mathbb{Z}) := \mathbb{Z}f_0 + \cdots + \mathbb{Z}f_7,$$

where

$$f_0 = e_0, \quad f_1 = e_1, \quad f_2 = e_2, \quad f_3 = e_3$$

and

$$f_4 = \frac{e_1 + e_2 + e_3 - e_4}{2}, \quad f_5 = \frac{-e_0 - e_1 - e_4 + e_5}{2},$$
$$f_6 = \frac{-e_0 + e_1 - e_2 + e_6}{2}, \quad f_7 = \frac{-e_0 + e_2 + e_4 + e_7}{2}.$$

This is a subring (closed under octavic multiplication). We also extend the octavic multiplication $\mathbb{C}$-linearly to $\mathbb{O}(\mathbb{C}) := \mathbb{C}^8$.

The conjugate of an octave $x \in \mathbb{O}$ is defined by

$$\overline{x_0e_0 + x_1e_1 + \cdots + x_7e_7} = x_0e_0 - x_1e_1 + \cdots - x_7e_7.$$

One has

$$\overline{x * y} = \bar{y} * \bar{x}.$$

Because the element e_0 is the unit element, we embed $\mathbb{R}$ into $\mathbb{O}$ by sending t do te_0. Sometimes we identify t with its image te_0. The norm is defined by

$$N : \mathbb{R}^8 \longrightarrow \mathbb{R}, \quad N(x) = \bar{x}x = x_0^2 + \cdots + x_7^2.$$

One has $N(xy) = N(x)N(y)$. The norm of an integral octave is an integer. We consider also the trace

$$\mathrm{tr} : \mathbb{R}^8 \longrightarrow \mathbb{R}, \quad \mathrm{tr}(x) = x + \bar{x} = 2x_0.$$

Sometimes $\mathrm{Re}(x) := x_0 = \mathrm{tr}(x)/2$ is called the *real part* of x. The traces of integral octaves are integral and the trace can be extended $\mathbb{C}$-linearly to $\mathbb{O}(\mathbb{C})$.

The norm is a quadratic form with associated bilinear form $\mathrm{tr}(\bar{x} * y)$. Hence $(\mathbb{O}(\mathbb{Z}), N)$ can be considered as an even lattice. Actually it is a copy of the E_8-lattice.

Since the octavic multiplication is not associative, one has sometimes to be a little careful. On can check

$$\mathrm{tr}((a * b) * c) = \mathrm{tr}(a * (b * c))$$

and then use the notation $\mathrm{tr}(a * b * c)$ for this expression. One also has

$$\mathrm{tr}(a * b * c) = \mathrm{tr}(b * c * a).$$

We consider now the Clifford algebra of $\mathbb{R}^8$ equipped with the negative definite form $-x_1^2 - \cdots - x_8^2$. We identify $\mathbb{R}^8 = \mathbb{O}$ and write $-\mathbb{O}$ to indicated that we consider the negative definit quadratic form $-N$. We embed $\mathbb{O}$ into the even part of the Clifford algebra $\mathcal{C}^+(-\mathbb{O})$,

$$\mathbb{O} \longrightarrow \mathcal{C}^+(-\mathbb{O}), \quad a \longmapsto e_0 a.$$

The algebra $\mathcal{C}^+(-\mathbb{O})$ is generated as algebra by the image of this map. Hence a homomorphism starting from this algebra is determined if we know its values at elements of the form $e_0 a$.

2.1 Proposition. *For an octave a we define the 8×8-matrix*

$$P(a) := (\mathrm{tr}(\bar{e}_i * a * e_k)/2).$$

There is a unique involutive homomorphism of algebras

$$\mathcal{C}^+(-\mathbb{O}) \longrightarrow M_8(\mathbb{R}), \quad e_0 a \longmapsto P(a).$$

Proof. Let be $P_i = P(e_i)$. The matrix P_0 is the negative unit matrix. The remaining defining relations are $P_1^2 = \cdots = P_7^2 = P_0$ and $P_i P_j = -P_j P_i$ for $1 \le i < j \le 7$. They are easy to check. Hence we obtain a homomorphismus. It is involutive because the $P_1, \dots, P_7$ are skew-symmetric. □

The integral part $\mathcal{C}^+(-\mathbb{O}(\mathbb{Z}))$ of the Clifford algebra is the abelian group generated by the products (including the empty product)

$$f_{i_1} \cdots f_{i_m}, \quad 0 \le i_1 < \cdots < i_m \le 7, \quad m \equiv 0 \bmod 2.$$

From the Clifford relations follows $(e_0 f_i)(e_0 f_j) = f_i f_j + (e_0, f_i) e_0 f_j$. Hence another basis is given by the

$$(f_0 f_{j_1}) \cdots (f_0 f_{j_m}), \quad 1 \le j_1 < \cdots < j_m \le 7.$$

Hence we obtain

2.2 Corollary. *The image of $\mathcal{C}^+(-\mathbb{O}(\mathbb{Z}))$ in $M_8(\mathbb{R})$ is the abelian group generated by the matrices (the empty product included)*

$$P(f_{j_1})\cdots P(f_{j_m}), \quad 1 \le j_1 < \cdots < j_m \le 7.$$

3. A spin group

We consider now the 12-dimensional vector space V with the basis

$$h_1, h_2, h_3, h_4,\ e_0, \dots, e_7.$$

We equip it with the bilinear form defined by the Gram matrix

$$\begin{pmatrix} 0 & 1 & & & & & \\ 1 & 0 & & & & & \\ & & 0 & 1 & & & \\ & & 1 & 0 & & & \\ & & & & -2 & & \\ & & & & & \ddots & \\ & & & & & & -2 \end{pmatrix}.$$

By means of the subspaces

$$H_1(\mathbb{R}) := \mathbb{R}h_1 + \mathbb{R}h_2,\ H_2(\mathbb{R}) := \mathbb{R}h_3 + \mathbb{R}h_4,\ V_0 := \mathbb{R}e_0 + \cdots + \mathbb{R}e_7$$

we obtain an orthogonal decomposition

$$V = H_1(\mathbb{R}) \oplus H_2(\mathbb{R}) \oplus V_0.$$

The following description of the structure of the Clifford algebra is taken from [FH].

Let $\mathfrak{A}$ by an involutive algebra, i.e. an unital associative $\mathbb{R}$-algebra which is equipped with an involution $a \mapsto a'$. We extend this involution to matrices with entries from $\mathfrak{A}$ by the formula

$$(m_{ij})^* := (m'_{ji}).$$

3.1 Lemma. *There exists a commutative diagram of algebras*

$$\begin{array}{ccc} \mathcal{C}(V) & \overset{\sim}{\longrightarrow} & M_4(\mathcal{C}(V_0)) \\ \cup & & \cup \\ \mathcal{C}^+(V) & \overset{\sim}{\longrightarrow} & M_4(\mathcal{C}^+(V_0)). \end{array}$$

The main involution of $\mathcal{C}(V)$ corresponds to the involution

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix} \longmapsto \begin{pmatrix} d^* & -b^* \\ -c^* & a^* \end{pmatrix}.$$

(The blocks a, b, c, d are 2×2-matrices). The diagram is defined by the assignments

$$h_1 \longmapsto e_0 \begin{pmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{pmatrix},$$

$$h_2 \longmapsto e_0 \begin{pmatrix} 0 & 0 & 0 & 0 \\ -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & -1 \\ 0 & 0 & 0 & 0 \end{pmatrix},$$

$$h_3 \longmapsto e_0 \begin{pmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix},$$

$$h_4 \longmapsto e_0 \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ -1 & 0 & 0 & 0 \end{pmatrix},$$

$$e_0 \longmapsto e_0 \begin{pmatrix} -1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix},$$

$$e_i \longmapsto e_i \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}, \quad 1 \le i \le 7.$$

Proof of lemma 3.1. The images of the basis elements satisfy the defining relations of the Clifford algebra. By the universal property of the Clifford algebra this extends to a homomorphism $\mathcal{C}(V) \to M_4(\mathcal{C}(V_0))$. One verifies that this homomorphism is surjective. A dimension argument shows that it is an isomorphism. The rest is clear. □

The *spin-group* $\mathrm{Spin}(V)$ of a quadratic space V consists of all $g \in \mathcal{C}^+(V)$ with the properties

$$g'g = 1 \quad \text{and} \quad gVg^{-1} = V.$$

The resulting transformations of V

$$x \mapsto gxg^{-1}$$

are orthogonal. This defines a two to one homomorphism

$$\mathrm{Spin}(V) \longrightarrow \mathrm{O}(V).$$

The image is the spinor kenel (i.e. the connected component of the orthogonal group). If $L \subset V$ is a lattice we can define the integral spin group

$$\mathrm{Spin}(L) := \mathrm{Spin}(V) \cap \mathcal{C}^+(L).$$

Using the description of 3.1 of the Clifford algebra of our 12-dimensional space V we obtain a relation between its spin group and a symplectic group:

The (Hermitian) symplectic group (of degree two) $\mathrm{Sp}(2, \mathfrak{A})$ of an involutive algebra $\mathfrak{A}$ consists of all 4×4-matrices M with entries from $\mathfrak{A}$, such that

$$M^* I M = I, \qquad I = \begin{pmatrix} 0 & e \\ -e & 0 \end{pmatrix} \qquad (e = \text{unit matrix}).$$

From 3.1 we obtain

3.2 Lemma. *The restriction of the homomorphism* 3.1 *defines an embedding*

$$\mathrm{Spin}(V) \longrightarrow \mathrm{Sp}(2, \mathcal{C}^+(V_0)).$$

Using the realization $V_0 = -\mathbb{O}$ and combining with the homomorphism 2.1 we obtain an injective homomorphim

$$J_0 : \mathrm{Spin}(V) \longrightarrow \mathrm{Sp}(16, \mathbb{R}).$$

Here $\mathrm{Sp}(16, \mathbb{R})$ *denotes the standard symplectic group of degree* 16 *(*32×32-*matrices).*

4. An embedding into the Siegel half plane

Again we consider the vector space $V = \mathbb{R}^{12} = \mathbb{R}^4 \times V_0$ with the basis $h_1, \dots, h_4,\ e_0, \dots, e_7$ equipped with the quadratic form q of signature $(2, 10)$. We denote the corresponding orthogonal half plane by $\mathcal{H}_{10}$. It can be defined a the set of triples $(z_1, z_2, \mathfrak{z})$, where z_1, z_2 are in the usual upper half plane and where $\mathfrak{z} \in V_0(\mathbb{C})$ such that $y_1 y_2 + q(\mathfrak{y}) > 0$. We recall that q is negative definite on V_0. There is an embedding $\mathcal{H}_{10} \to \mathbb{P}(V(\mathbb{C}))$ such that the image is one connected component of the subset $\tilde{\mathcal{H}}_{10}$ defined in $V(\mathbb{C})$ by $(z, z) = 0$ and $(z, \bar{z}) > 0$.

From this we obtain an action of a subgroup of index two $\mathrm{O}^+(V)$ of $\mathrm{O}(V)$. The image of $\mathrm{Spin}(V)$ is the connected component of $\mathrm{O}(V)$. hence we have a natural map $\mathrm{Spin}(V) \to \mathrm{O}^+(V)$. We use this map to define an action of $\mathrm{Spin}(V)$ on $\mathcal{H}_{10}$. Our next goal is to construct an embedding of $\mathcal{H}_{10}$ into the Siegel half plane $\mathbb{H}_{16}$ of degree 16 which is compatible with the homomorphism $\mathrm{Spin}(V) \to \mathrm{Sp}(16,\mathbb{R})$ and the standard action of the symplectic group one the Siegel half plane. For this purpose it is convenient to push $\mathcal{H}_{10}$ into the even part of the Clifford algebra, more precisely we consider the map

$$\mathcal{H}_{10} \longrightarrow M_2(\mathcal{C}^+(V_0(\mathbb{C})), \quad (z_1, z_2, \mathfrak{z}) \longmapsto \begin{pmatrix} z_1 & e_0\mathfrak{z} \\ (e_0\mathfrak{z})' & z_2 \end{pmatrix}.$$

We explain the notations: The real bilinear form $(\cdot,\cdot)$ extends to a $\mathbb{C}$-bilinear form on the complexification $V_0(\mathbb{C})$ of V_0. Its (complex) Clifford algebra is $\mathcal{C}(V_0(\mathbb{C}))$. It can be identified with $\mathcal{C}(V_0) \otimes_{\mathbb{R}} \mathbb{C}$. The same is true for the even part $\mathcal{C}^+$. The main involution extends to a $\mathbb{C}$-linear involution of $\mathcal{C}^+(V_0(\mathbb{C}))$, which is denoted by the same letter. Now we consider the homomorphism $\mathcal{C}^+(V_0) \to M_8(\mathbb{R})$. We extend in the natural way to a map

$$M_2(\mathcal{C}^+(V_0(\mathbb{C})) \longrightarrow M_{16}(\mathbb{C}).$$

4.1 Proposition. *The image of $\mathcal{H}_{10}$ under the maps*

$$\mathcal{H}_{10} \longrightarrow M_2(\mathcal{C}^+(V_0(\mathbb{C})) \longrightarrow M_{16}(\mathbb{C})$$

is contained in the Siegel half plane of degree 16. This is an embedding $j_0 : \mathcal{H}_{10} \to \mathbb{H}_{16}$ which is compatible with the homomorphism $J_0 : \mathrm{Spin}(V) \to \mathrm{Sp}(16,\mathbb{R})$ in the sense that the diagram

$$\begin{array}{ccc} \mathrm{Spin}(V) \times \mathcal{H}_{10} & \xrightarrow{(J_0,j_0)} & \mathrm{Sp}(16,\mathbb{R}) \times \mathbb{H}_{16} \\ \downarrow & & \downarrow \\ \mathcal{H}_{10} & \xrightarrow{j_0} & \mathbb{H}_{16} \end{array}$$

commutes.

5. A modular embedding

We have to modify the embedding (J_0, j_0) slightly because we want to have that the integral structures are preserved. We identify now $V_0 = \mathbb{R}^8$ with the octaves $\mathbb{O}$ and we consider the lattice $\mathbb{O}(\mathbb{Z})$. In the quadratic space $V = \mathbb{R}^4 \times (-\mathbb{O})$ we consider the lattice $\mathbb{Z}^4 \times (-\mathbb{O}(\mathbb{Z}))$ which is even and unimodular and of signature $(2, 10)$. Hence we denote it simply by

$$\mathrm{II}_{2,10} := \mathbb{Z}^4 \times (-\mathbb{O}(\mathbb{Z})).$$

5.1 Lemma. *The image of $\mathcal{C}^+(\mathrm{II}_{2,10})$ under the homomorphism 3.1 is $M_4(\mathcal{C}^+(-\mathbb{O}(\mathbb{Z})))$.*

We want to modify the homomorphism 2.1 in such a way that the image of $\mathcal{C}^+(\mathbb{O}(\mathbb{Z}))$ consists of integral matrices. For this we consider the matrix F whose rows are the vectors $f_0,\dots,f_7$. Then $S := 2FF'$ is an even unimodular matrix. We consider the symplectic matrix

$$M = \begin{pmatrix} \sqrt{2}F & 0 & 0 & 0 \\ 0 & \sqrt{2}F & 0 & 0 \\ 0 & 0 & (\sqrt{2}F)'^{-1} & 0 \\ 0 & 0 & 0 & (\sqrt{2}F)'^{-1} \end{pmatrix}.$$

We combine the map $\mathcal{H}_{10} \to \mathbb{H}_{16}$ defined a above with the substitution $Z \mapsto M(Z)$. Because of $(\sqrt{2}F)P(a)(\sqrt{2}F)' = (\operatorname{tr}(\bar{f}_i a f_j))$ we obtain:

5.2 Definition. *For an octave a we define the matrix $Q(a) = \big(\operatorname{tr}(\bar{f}_i a f_j)\big)$. This gives a map*

$$Q : \mathbb{O}(\mathbb{Z}) \longrightarrow M_8(\mathbb{Z}).$$

We also consider

$$j : \mathcal{H}_{10} \longrightarrow \mathbb{H}_{16}, \quad (z_1, z_2, \mathfrak{z}) \longmapsto \begin{pmatrix} z_0 S & Q(\mathfrak{z}) \\ Q(\mathfrak{z})' & z_2 S \end{pmatrix}$$

and we consider the homomorphism

$$J : \operatorname{Spin}(V) \longrightarrow \operatorname{Sp}(16, \mathbb{R}),$$

which is the composition of the homomorphism described in 4.1 and the inner automorphism $N \longmapsto M^{-1}NM$ of $\operatorname{Sp}(16,\mathbb{R})$.

The advantage of this modified embedding is that it is modular, i.e. it preserves the modular groups:

5.3 Proposition. *The diagram*

$$\begin{array}{ccc} \operatorname{Spin}(V) \times \mathcal{H}_{10} & \xrightarrow{(J,j)} & \operatorname{Sp}(16,\mathbb{R}) \times \mathbb{H}_{16} \\ \downarrow & & \downarrow \\ \mathcal{H}_{10} & \xrightarrow{j} & \mathbb{H}_{16} \end{array}$$

commutes. The homomorphism J maps the group $\operatorname{Spin}(\mathrm{II}_{2,10})$ *into the Siegel modular group* $\operatorname{Sp}(16,\mathbb{Z})$.

Proof. We have to determine the image of the *integral* spin-group $\operatorname{Spin}(\mathrm{II}_{2,10})$ in the symplectic group. We have to consider the map $\mathcal{C}^+(-\mathbb{O}(\mathbb{Z})) \to M_8(\mathbb{R})$ (see 2.1 and 2.2) . We denote its image by $\mathcal{A}$. We have to consider 4×4-blocks with entries in $\mathcal{A}$ and conjugate this with M. The resulting matrix consists of blocs of the form

$$FAF^{-1}, \quad 2FAF', \quad \frac{1}{2}F'^{-1}AF^{-1}, \quad F'^{-1}AF' \qquad (A \in \mathcal{A}).$$

We have to show that they are integral. Because $S = 2FF'$ is even unimodular, they all are integral if one is integral. Hence its remains to remind that $Q(a) = 2FP(a)F'$ is integral for all integral octaves (s.5.2). □

6. The theta group

In a first step we investigate the principal congruence subgroups of level two. First of all we recall the principal congruence subgroup of level l in the symplectic case. It consists of all integral symplectic matrices of degree $2n$ such that $\left(\begin{smallmatrix} A & B \\ C & D \end{smallmatrix}\right)$ is congruent to $\left(\begin{smallmatrix} 1_n & 0 \\ 0 & 1_n \end{smallmatrix}\right) \bmod l$. We denote it by $\Gamma_n[l]$.

There are two ways to define the principal congruence subgroups of level two in $\mathrm{Spin}(\mathrm{II}_{2,10})$. Firstly one can consider of $\mathrm{O}(\mathrm{II}_{2,10})[2]$, which is the subgroup of $\mathrm{O}(\mathrm{II}_{2,10})$ acting trivial on $\mathrm{II}_{2,10}/2\mathrm{II}_{2,10}$. We denote its inverse image in $\mathrm{Spin}(\mathrm{II}_{2,10})$ by $\mathrm{Spin}(\mathrm{II}_{2,10})[2]$. Secondly one can consider the reduction of the Clifford algebra mod 2, which is nothing else but the Clifford algebra of the quadratic space $\mathbb{F}_2^{12}$. The reduction homomorphism

$$\mathcal{C}(\mathrm{II}_{2,10}) \longrightarrow \mathcal{C}(\mathbb{F}_2^{12}) = \mathcal{C}(\mathrm{II}_{2,10}) \otimes_{\mathbb{Z}} \mathbb{F}_2$$

induces a homomorphism

$$\mathrm{Spin}(\mathrm{II}_{2,10}) \longrightarrow \mathrm{Spin}(\mathbb{F}_2^{12}).$$

Since we are in characteristic two now, the Spin group is a subgroup of the orthogonal group. More precisely $\mathrm{Spin}(\mathbb{F}_2^{12})$ is the simple subgroup of index two of the orthogonal group $\mathrm{O}(\mathbb{F}_2^{12})$. We see that the kernel of the reduction homomorphism is $\mathrm{Spin}(\mathrm{II}_{2,10})[2]$. This shows that every element of this group can be written in the form $e+2a$, where a is an element of the integral Clifford algebra $\mathcal{C}(\mathrm{II}_{2,10})$. Now the same proof as in the second part of 5.3 shows:

6.1 Remark. *The image of* $\mathrm{Spin}(\mathrm{II}_{2,10})[2]$ *under the homomorphism* J *is contained in the principal congruence subgroup of level two* $\Gamma_{16}[2]$.

There is a rather involved refinement of the second statement of 5.3: Recall that the Igusa group $\Gamma_n[l, 2l]$ consists of all $\left(\begin{smallmatrix} A & B \\ C & D \end{smallmatrix}\right)$ in $\Gamma_n[l]$ such that AB'/l and CD'/l have even diagonal. The group $\Gamma_n[1,2]$ is the so-called theta group. We claim now:

6.2 Proposition. *The image of the integral spin-group* $\mathrm{Spin}(\mathrm{II}_{2,10})$ *under the homomorphism* J *is contained in the theta-group* $\Gamma_{16}[1,2]$.

Proof. It is enough to check the images for a system of elements whose images in $\mathrm{Spin}(\mathbb{F}_2^{12})$ generate this group. One can take the following system of elements from $M_4(\mathcal{C}^+(-\mathbb{O}(\mathbb{Z}))$

$$\begin{pmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ -1 & 0 & 0 & 0 \\ 0 & 0 & -1 & 0 \end{pmatrix}, \quad \begin{pmatrix} 1 & 0 & h_1 & \mathfrak{h} \\ 0 & 0 & \mathfrak{h}' & h_2 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}, \quad \begin{pmatrix} 1 & \mathfrak{h} & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & -\mathfrak{h}' & 1 \end{pmatrix}$$

where $h_1, h_2 \in \mathbb{Z}$ and $\mathfrak{h} \in e_0\mathbb{O}(\mathbb{Z})$. It is easy to check that they are contained in $\mathrm{Spin}(\mathrm{II}_{2,10})$. It remains to show that their images under J are contained in $\Gamma_{16}[1,2]$. From the definition of J one sees that this means that the matrix $Q(a)$ (see 5.2) has even diagonal for $a \in \mathbb{O}(\mathbb{Z})$. One has to check this for $a = f_i$. But for $i > 0$ this matrix is skew-symmetric and for $i = 0$ one obtains a Gram-matrix for E_8 which is an even lattice. □

There is an improvement of 6.1.

6.3 Lemma. *The image of* $\mathrm{Spin}(\mathrm{II}_{2,10})[2]$ *under the homomorphism* J *is contained in the Igusa group* $\Gamma_{16}[2,4]$.

Proof. We consider

$$N = \begin{pmatrix} \sqrt{2}E & 0 \\ 0 & \sqrt{2}^{-1}E \end{pmatrix} \in \mathrm{Spin}(V).$$

It has the effect N(Z)=2Z for $Z \in \mathcal{H}_{10}$. We conjugate an arbitrary element $M \in \mathrm{Spin}(L)[2]$ with N. We claim that the element $N^{-1}MN$ is still in in $\mathrm{Spin}(\mathrm{II}_{2,10})$. For this is sufficient to show that $N^{-1}MN$ is in the Clifford algebra $\mathcal{C}(\mathrm{II}_{2,10})$ if $M \in 2\mathcal{C}(\mathrm{II}_{2,10})$. This is sufficient to check for generators of the $\mathbb{Z}$-algebra $\mathcal{C}(\mathrm{II}_{2,10})$, which is easy. The replacement $M \mapsto N^{-1}MN$ has the effect $AB' \mapsto AB'/2$. Using 6.2 this leads to $M \in \mathrm{Spin}(L)[2,4]$. □

We need the relation between the spin- and the orthogonal group.

6.4 Lemma. *The image of the natural homomorphism*

$$\mathrm{Spin}(\mathrm{II}_{2,10}) \longrightarrow \mathrm{O}(\mathrm{II}_{2,10})$$

is the subgroup of index four $\mathrm{SO}^+(\mathrm{II}_{2,10})$.

It is easy to show that elements of the group $\mathrm{O}^+(\mathrm{II}_{2,10})[2]$ have determinant one. So we can reformulate the results about the spin group as follows:

6.5 Lemma. *The homomorphism* J *induces homomorphisms*

$$\mathrm{SO}^+(\mathrm{II}_{2,10}) \longrightarrow \Gamma_{16}[1,2]/\pm E,$$
$$\mathrm{O}^+(\mathrm{II}_{2,10})[2] \longrightarrow \Gamma_{16}[2,4]/\pm E.$$

7. Theta series

We consider the standard theta series on $\mathbb{H}_{16}$

$$\vartheta_\varphi(Z) := \sum_{g\in\mathbb{Z}^{16}} \phi(g)e^{\pi \mathrm{i} Z[g]/2}, \quad \varphi : (\mathbb{Z}/2\mathbb{Z})^{16} \longrightarrow \mathbb{C}.$$

Another way to define them is to use the basis

$$\sum_{g\in\mathbb{Z}^{16}} e^{2\pi \mathrm{i} Z[g+m/2]}, \quad m \in (\mathbb{Z}/2\mathbb{Z})^{16}.$$

We restrict them to $\mathcal{H}_{10}$ using the modular embedding j. For sake of simplicity we use the notation

$$\vartheta_\varphi(z) := \vartheta_\varphi(j(z)).$$

With the notation $g = (g_1, g_2)'$, $h_i = g_{i1}f_1 + \cdots + g_{i8}f_8$ and $\varphi(h) = \phi(g)$ we obtain for the restriction

$$\vartheta_\varphi(z) = \sum_{h\in\mathbb{O}(\mathbb{Z})^2} \varphi(h)e^{\pi \mathrm{i}\{N(h_1)z_1+N(h_2)z_2+\mathrm{tr}(\bar h_1 * \mathfrak{z} * h_2)\}}.$$

We recall the notion of an orthogonal modular form of weight $k \in \mathbb{Z}$ with respect to a subgroup $\Gamma \subset \mathrm{O}^+(\mathrm{II}_{2,10})$ of finite index and with respect to a character $v : \Gamma \to \mathbb{C}^\bullet$. It is a function $f : \tilde{\mathcal{H}}_{10} \to \mathbb{C}$ with the properties

$$f(\gamma z) = v(\gamma)f(z),$$
$$f(tz) = t^{-k}f(z).$$

The vector space of holomorphic forms is denoted by $[\Gamma, k, v]$ and by $[\Gamma, k]$ when v is trivial. These are finite dimensional spaces.

A modular form f is determined by the function

$$F(z) := F(z_0, z_2, \mathfrak{z}) := f(1, *, z_0, z_2, \mathfrak{z})$$

It satisfies the transformation formula

$$F(\gamma(z_0, z_2, \mathfrak{z})) = a(\gamma, (z_0, z_2, \mathfrak{z}))^k F(z_0, z_2, \mathfrak{z}).$$

Here $\gamma(z_0, z_2, \mathfrak{z})$ and $a(\gamma, (z_0, z_2, \mathfrak{z}))$ are defined as follows: Consider

$$\gamma((1, *, z_0, z_2, \mathfrak{z}) = t(1, *, w_0, w_2, \mathfrak{w})$$

and define

$$a(\gamma, (z_0, z_2, \mathfrak{z})) = t^{-1}, \quad \gamma(z_0, z_2, \mathfrak{z}) = (w_0, w_2, \mathfrak{w}).$$

With these notations we obtain

7.1 Proposition. *The theta series $\vartheta_\varphi(z)$ are modular forms of weight 4 for the group* $\mathrm{O}^+(\mathrm{II})[2]$.

Let f be a modular form on some congruence group Γ with trivial multiplier system. We defined in [FS] the value of f at a non-zero rational isotropic vector in $\mathbb{R}^4 \times \mathbb{O}$ and this value is Γ-invariant. We always can restrict to the case, where the first component of the isotropic vector is different from 0 because for any congruence group Γ there is a Γ-equivalent with this property. Now let

$$R = \begin{pmatrix} r_1 & \mathfrak{r} \\ \bar{\mathfrak{r}} & r_2 \end{pmatrix}, \quad r_1, r_2 \in \mathbb{Q},\ \mathfrak{r} \in \mathbb{O}(\mathbb{Q}),$$

be a rational octavic Hermitian matrix. Then we can define the isotropic element $r = (1, *, r_1, r_2, \mathfrak{r})$. We use the notation $f(R)$ for this value.

7.2 Lemma. *Let R be a rational octavic Hermitian matrix. The value of $\vartheta_\varphi(s, Z)$ at R is*

$$\vartheta_\varphi(s, R) = s^{-8} N^{-16} \sum_{g \bmod N} \varphi(g) e^{\pi \mathrm{i} s R[g]}.$$

Here N denotes a natural number such that $\varphi(g)e^{\pi \mathrm{i} s R[g]}$ is periodic with respect to $N\mathbb{O}(\mathbb{Z}) \times N\mathbb{O}(\mathbb{Z})$.

Proof. Again one can use the modular embedding into the Siegel case of degree 16. □

Recall that we defined the value of a modular form as follows. One takes for the representative of an isotropic line a primitive vector in the lattice $\mathrm{II}_{2,10}$. This is well-defined up to sign. Hence we can define the value at a cusp only for forms of even weight. When we start with an rational Hermitian matrix R the isotropic vector $r = (1, *, R)$ is usually integral and hence has to be normalized. One can around the problem of treating this normalizing factor as follows. Consider the easiest case

$$\vartheta(z) := \vartheta_1(1, 2z) = \sum_{h \in \mathbb{O}(\mathbb{Z})^2} e^{2\pi \mathrm{i}\{N(h_1)z_1 + N(h_2)z_2 + \mathrm{tr}(\bar{h}_1 * \mathfrak{z} * h_2)\}}.$$

This series has been introduced by Eie and Krieg and they proved that it is an additive lift. From this it follows that ϑ is a modular form with respect to the full $\mathrm{O}^+(\mathrm{II}_{2,10})$. This implies that the values at all cusps is the same, namely one. Introducing the Gauss sum

$$G(R) = N^{-16} \sum_{g \bmod N} e^{\pi \mathrm{i} R[g]}$$

we obtain now:

7.3 Lemma. *Let R be a rational octavic Hermitian matrix. The value of $\vartheta_\varphi(s,Z)$ at the associated cups class is*

$$\frac{\vartheta_\varphi(s,R)}{G(2R)} = \frac{\displaystyle\sum_{g \bmod N} \varphi(g) e^{\pi \mathrm{i} s R[g]}}{s^8 \displaystyle\sum_{g \bmod N} e^{2\pi \mathrm{i} R[g]}}.$$

By the way, it is not clear from this formula (but follows from our deduction) that this expression depends only on the cusp class. This has to do with rules concerning Gauss sums as reciprocity laws.

We want to compute the values at the cusps in the case $s = 1/2$ and where $\varphi(g)$ only depends on g mod 2. Theta series of this species sometimes are called "'of second kind"'.

7.4 Lemma. *The theta series of second kind*

$$\sum_{h \in \mathbb{O}(\mathbb{Z})^2} \varphi(h) e^{\pi \mathrm{i}\{N(h_1)z_1 + N(h_2)z_2 + \mathrm{tr}(\bar h_1 * \mathfrak{z} * h_2)\}}, \quad \varphi : (\mathbb{O}(\mathbb{Z})/2\mathbb{O}(\mathbb{Z}))^2 \longrightarrow \mathbb{C},$$

are modular forms on the congruence group of level two $\mathrm{O}^+(\mathrm{II}_{2,10})[2]$ *(with trivial multipliers).*

Proof. We recall the construction of the two-fold covering of the orthogonal group. One has to consider the Clifford algebra (over $\mathbb{Z}$) of the lattice $\mathrm{II}_{2,10}$. The spin group $\mathrm{Spin}(\mathrm{II}_{2,10})$ is a subgroup of its unit group. There is a natural homomorphism

$$\mathrm{Spin}(\mathrm{II}_{2,10}) \longrightarrow \mathrm{SO}^+(\mathrm{II}_{2,10}).$$

It is known that this homomorphism is surjective. The modular embedding which we mentioned already should be understood as a homomorphism

$$\mathrm{Spin}(\mathrm{II}_{2,10}) \longrightarrow \mathrm{Sp}(16,\mathbb{Z}).$$

One can reduce this homomorphism mod 2. The spin group and the group O^+ agree over the filed of two elements. Hence we get a commutative diagram

$$\begin{array}{ccc} \mathrm{Spin}(\mathrm{II}_{2,10}) & \longrightarrow & \mathrm{Sp}(16,\mathbb{Z}) \\ \downarrow & & \downarrow \\ \mathrm{O}^+(\mathbb{F}_2^{12}) & \longrightarrow & \mathrm{Sp}(16,\mathbb{F}_2) \end{array}.$$

This shows that a two fold covering of $\mathrm{O}^+(\mathrm{II}_{2,10})[2]$ appears as subgroup of a Siegel modular group. This reduces 7.4 to a Siegel analogue case which is assumed to be known. □

We want to compute the values of the theta series of second kind explicitely. Recall that the cups in the level two case are given by non zero isotropic vectors from $\mathbb{F}_2^{12}$. We can take a representative in $\mathrm{II}_{2,10}$ such that the first coordinate is one or two. In terms of the octavic Hermitian matrices R this means that it is sufficient to assume that R either contains only entries 0 and 1 or 0 and 1/2. We prefer to write $R/2$ instead of R with an integral R. Then we have:

Let R be an octavic Hermitian matrix, such that R either contains 0 and 1 or 0 and 2. The value of the second kind theta function $\sum_{g \text{ integral}} \varphi(g) e^{\pi \mathrm{i} Z[g]}$ at the cusp class defined by $R/2$ is

$$\frac{\displaystyle\sum_{g \bmod 4} \varphi(g) e^{\pi \mathrm{i} R[g]/2}}{\displaystyle\sum_{g \bmod 2} e^{\pi \mathrm{i} R[g]}}.$$

We are going to compute

$$G(\varphi, R) = \sum_{g \bmod 4} \varphi(g) e^{\pi \mathrm{i} R[g]/2}$$

in the cases where R either contains only 0 and 2 or 0 and 1. This sum can be computed easily:

$$\begin{aligned} G(\varphi, R) &= \sum_{g \bmod 2} \varphi(g) \sum_{h \bmod 2} e^{\pi \mathrm{i} R[g+2h]/2} \\ &= \sum_{g \bmod 2} \varphi(g) e^{\pi \mathrm{i} R[g]/2} \sum_{h \bmod 2} e^{\pi \mathrm{i} \operatorname{tr}(\bar{h}' R g)}. \end{aligned}$$

The last sum is a sum a sum over the values of a character, which is not zero if and only if the character is trivial. This means that Rg is a even (contained in $2(\mathbb{O}(\mathbb{Z}) \times \mathbb{O}(\mathbb{Z}))$). The result is

$$\sum_{g \bmod 2,\ Rg \text{ even}} \varphi(g) e^{\pi \mathrm{i} R[g]/2}.$$

Using this formula, the values at the cusps can be calculated. This can be done for all the characteristic functions φ. Since the weight 4 is the singular weight, we can prove now by calculation:

7.5 Proposition. *The space of modular forms $\vartheta_\varphi(z)$ has dimension 715.*

It follows that this space decomposes into the trivial one-dimensional and the 714-dimensional irreducible representation. By the uniqueness of $f(z)$ we obtain that the invariant form $\vartheta(z)$ is contained in the space generated by the $\vartheta_\varphi(z)$. Obviously $\vartheta(2z)$ is nothing else but $\vartheta_\varphi(z)$, where φ is the characteristic function of the zero element of $(\mathbb{O}/2\mathbb{O})^2$. It can be seen form the values at the cusps that $\vartheta(2z)$ is not contained in the 714-dimensional space. Thus the theorem in the introduction now follows.

7.6 Theorem. *The space of theta functions of second kind $\vartheta_\varphi(z)$ agrees with the 715-dimensional additive lift space described in [FS]. It contains the full invariant form $\vartheta(z)$.*

8. Enriques surfaces

Denote by U the unimodular lattice $\mathbb{Z}\times\mathbb{Z}$ with quadratic form $(x,x)=2x_1x_2$. Kondo investigated in [Ko] the case of the lattice

$$M = U \oplus \sqrt{2}U \oplus (-\sqrt{2}E_8).$$

This case is related to the moduli space of marked Enriques surfaces, i.e. Enriques surfaces with a choice of level 2 structure of the Picard lattice.

The group $\mathrm{O}^+(\mathrm{II}_{2,10})[2]$ is related to the lattice

$$L = \sqrt{2}\mathrm{II}_{2,10} \cong \sqrt{2}U \oplus \sqrt{2}U + \oplus(-\sqrt{2}E_8),$$

cf. [FS]. It can be embedded into Kondo's lattice by means of

$$\sqrt{2}U \longrightarrow U, \quad \sqrt{2}(x_1,x_2) \longmapsto (x_1, 2x_2).$$

Hence, according to [FS], we have

8.1 Proposition. *The embedding of $L = \sqrt{2}U \oplus \sqrt{2}U \oplus (-\sqrt{2}E_8) \cong \sqrt{2}II_{2,10}$ into $M = U \oplus \sqrt{2}U \oplus (-\sqrt{2}E_8)$ defines an embedding of Kondo's 186-dimensional space of modular forms of weight four into our 714-dimensional space.*

Hence as immediate consequence of the previous section we have

8.2 Theorem. *Kondo's 186-dimensional space of modular forms of weight four is contained in the space of theta functions of second kind $\vartheta_\varphi(z)$.*

References

[Bo] Borcherds, R.: *The moduli space of Enriques surfaces and the fake monster Lie superalgebra.* Topology . **35**, 699-710 (1996)

[EK] Eie, M., Krieg, A.: *The Maaß Space on the Half-Plane of Cayley Numbers of Degree Two,* Math. Z. **210**, 113-128 (1992)

[FH] Freitag, E., Hermann, C.F.: *Some modular varieties in low dimension,* Advances in Math. **152**, 203-287 (2000)

[FS] Freitag, E., Salvati-Manni, R.: *Modular forms for the even unimodular lattice of signature (2,10),* J. Algebraic Geom. **16**, 753-791 (2007)

[KMY] Kawaguchi, S., Mukai, S., Yoshikawa, K.: *Resultants and the Borcherds* Φ*-function,* arXiv 1308.6454v1

[Ko] Kondo, S.: *The moduli space of Enriques surfaces and Borcherds products,* J. Algebraic Geometry **11**, 601-627 (2002)