UNIFORM RESOLVENT ESTIMATES AND ABSENCE OF EIGENVALUES FOR LAMÉ OPERATORS WITH COMPLEX POTENTIALS

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ABSTRACT. We consider the 0-order perturbed Lamé operator $-\Delta^* + V(x)$. It is well known that if one considers the free case, namely $V = 0$, the spectrum of $-\Delta^*$ is purely continuous and coincides with the non-negative semi-axis. The first purpose of the paper is to show that, at least in part, this spectral property is preserved in the perturbed setting. Precisely, developing a suitable multipliers technique, we will prove the absence of point spectrum for Lamé operator with potentials which satisfy a variational inequality with suitable small constant. We stress that our result also covers complex-valued perturbation terms. Moreover the techniques used to prove the absence of eigenvalues enable us to provide uniform resolvent estimates for the perturbed operator under the same assumptions about V.

1. INTRODUCTION

This paper is concerned with operators of the form

$$-\Delta^* + V(x)$$

acting on the Hilbert space $[L^2(\mathbb{R}^d)]^d$ that is the Hilbert space of the vector fields with components in $L^2(\mathbb{R}^d)$.

$-\Delta^*$ denotes the Lamé operator of elasticity, that is a linear symmetric differential operator of second order that acts on smooth L^2 vector fields u on $\mathbb{R}^d$, for example $[C_c^\infty(\mathbb{R}^d)]^d$, in this way:

$$-\Delta^* u := -\mu(\Delta u_1, \Delta u_2, \dots, \Delta u_d) - (\lambda + \mu)\nabla \operatorname{div}(u_1, u_2, \dots, u_d),$$

where λ and μ are the so called Lamé's coefficients which obey the standard inequality

$$\mu > 0, \ \lambda > -\frac{2}{d}\,\mu; \tag{1}$$

this condition guarantees the very strong ellipticity of $-\Delta^*$ (please refer to subsection 2.2 for further details), moreover an explanation of the physical significance of (1) in $d = 3$ can be found in [20].

$V(x)$ is a notation for the multiplication operator by the *complex* matrix-valued potential $V(x)\colon \mathbb{R}^d \to \mathcal{M}_{d\times d}(\mathbb{C})$, this means that the context we are working in is a non self-adjoint setting.

The theory of non self-adjoint operators is very much less unified than the self-adjoint one. Nevertheless, recently there has been a growing interest in this context, indeed an increasing number of problems, particularly in physics, requires the analysis of non self-adjoint operators.

Just to quote a pair of them, first let us mention the very recent problem to investigate on the distribution, in the complex plane, of eigenvalues of the non self-adjoint Schrödinger operators. Roughly speaking this is the problem to find the correct analogue to the Lieb-Thirring inequalities (cf. [22]) for non self-adjoint operators, see, for example, [1, 21, 12, 17].

An other context, in which dealing with non self-adjoint operator turns out to be useful, is the study of complex resonances of self-adjoint Schrödinger operator (cf. [1, 6]). If one uses the techniques of complex scaling, then these resonances turn into eigenvalues of associated non self-adjoint Schrödinger operator.

Fanelli, Krejčiřík and Vega, in a very recent work [15], that primarily motivates our paper, improve the state of the art in the picture of spectral properties for non self-adjoint Schrödinger operators.

Precisely, using a generalized version of the Birman-Schwinger principle, they proved that in three dimensions the spectrum of Schrödinger operator $H := -\Delta + V$ is purely continuous and coincides with the non-negative semi-axis for all complex-valued potential satisfying the following form subordinated smallness condition:

$$\exists\, a < 1, \quad \forall\, \psi \in H^1(\mathbb{R}^d), \qquad \int_{\mathbb{R}^d} |V||\psi|^2 \le a \int_{\mathbb{R}^d} |\nabla\psi|^2. \tag{2}$$

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The relevance of this condition relies on the fact that first it seems to be new also in the self-adjoint setting, moreover if one is interested in the less strong result about the mere absence of eigenvalues, this is a weaker condition than the more classical ones (for example, the belonging to the Rollnick class) and, at the same time, it seems to recover potentials that are not typically covered by previous works treating those arguments.

In addition to this result, with a completely alternative approach, which substantially exploits a suitable development of the multipliers technique, they prove the absence of point spectrum in all dimensions $d \geq 3$, under a stronger hypothesis on the potential than the previous one.

Our first purpose goes in this direction. Precisely we want to investigate if some spectral properties are preserved under small perturbations of the operator we are dealing with, i.e. the Lamé operator.

The Helmholtz decomposition strongly comes into play. Indeed, making use of this tool, which is a standard way to decompose smooth vector fields into a sum of a divergence free vector field and a gradient, we can explicitly see the connection between the Lamé operator and the Laplacian. As a matter of fact, it is very easy to see that, for any $u = u_P + u_S$, the operator $-\Delta^*$ acts on u in this way:

$$-\Delta^* u = -\mu \Delta u_S - (\lambda + 2\mu)\Delta u_P,$$

where the component u_S is the divergence free vector field and the component u_P is the gradient.

The relation between the two operators, that the previous identity have highlighted, has played a fundamental role in order to motivate this paper. Indeed, our purpose is to prove a result that is the counterpart of that Fanelli, Krejčiřík and Vega in [15] have proved for the Laplacian.

Precisely we will prove the following result, which substantially guarantees that, in $d \geq 3$, the absence of eigenvalues is preserved under suitable smallness and complex perturbation of the operator.

Theorem 1.1. *Let $d \geq 3$. Assume that $\lambda, \mu \in \mathbb{R}$ satisfy (1) and that $V : \mathbb{R}^d \to \mathcal{M}_{d\times d}(\mathbb{C})$ is such that*

$$\forall\, u \in [H^1(\mathbb{R}^d)]^d, \qquad \int_{\mathbb{R}^d} |x|^2 |V|^2 |u|^2 \leq \Lambda^2 \int_{\mathbb{R}^d} |\nabla u|^2, \tag{3}$$

where Λ satisfies

$$\frac{\Lambda}{\min\{\mu, \lambda + 2\mu\}} \frac{4(2d-3)}{d-2}(C+1) + \frac{\Lambda^{\frac{3}{2}}}{\min\{\mu, \lambda+2\mu\}^{\frac{3}{2}}} \frac{4\sqrt{2}}{\sqrt{d-2}}(C+1)^{\frac{3}{2}} < 1, \tag{4}$$

and where $C > 0$ is a suitable constant. Then $\sigma_p(-\Delta^* + V) = \varnothing$.

Remark 1.1. Here and in the sequel $|V|$ represents the matrix norm $|V| := \left(\sum_{i=1}^d \sum_{j=1}^d |V_{ij}|^2\right)^{\frac{1}{2}}$.

Remark 1.2. The constant $C > 0$ that is in the statement of the above theorem is the which one that will appear in the elliptic regularity Lemma 3.3 we will state below.

Remark 1.3. Let us stress that for the physical case $d = 3$ (that is covered by our result), although we only prove the absence of the point spectrum, we expect that an analogous to the stronger result in [15] (Theorem 1) holds but this will be matter of other investigations. We just remark that it is not obvious how to refine the proof in [15] and to develop this in our context. This, primarily, is due to the fact that the Birman-Schwinger operator associated to Lamé, instead of Laplace operator, has a "lack of symmetry" in its structure which does not allow to apply directly the strategy under the Birman-Schwinger principle.

Remark 1.4. We underline that, actually, the most natural generalization of the condition (2) about V, which appears in [15], is not (3) but the following one:

$$\exists\, a < \min\{\mu, \lambda + 2\mu\}, \qquad \forall\, u \in [H^1(\mathbb{R}^d)]^d, \qquad \int_{\mathbb{R}^d} |V||u|^2 \leq a \int_{\mathbb{R}^d} |\nabla u|^2. \tag{5}$$

Moreover, it's not difficult to see that the condition (3), which is in our result, is stronger than (5) indeed, assuming that (3) holds and making use of the classical Hardy inequality that reads

$$\forall\, \psi \in H^1(\mathbb{R}^d), \qquad \int_{\mathbb{R}^d} \frac{|\psi(x)|^2}{|x|^2}\, dx \leq \frac{4}{(d-2)^2} \int_{\mathbb{R}^d} |\nabla \psi|^2, \tag{6}$$

one has

$$\forall\, u \in [H^1(\mathbb{R}^d)]^d \qquad \int_{\mathbb{R}^d} |V||u|^2 \leq \| |x||V||u| \| \left\| \frac{u}{|x|} \right\| \leq \frac{2\Lambda}{d-2} \|\nabla u\|^2,$$

as a consequence of the restriction (4) we have $\Lambda < \min\{\mu, \lambda + 2\mu\} \frac{d-2}{2}$ then $2\Lambda/(d-2) < \min\{\mu, \lambda + 2\mu\}$, hence (5) holds.

Remark 1.5. Here and in the sequel, we use the following definition of the L^2- norm of a vector field u, precisely

$$\|u\| := \|u\|_{[L^2(\mathbb{R}^d)]^d} = \left(\sum_{j=1}^d \|u_j\|^2_{L^2(\mathbb{R}^d)}\right)^{\frac{1}{2}}.$$

Remark 1.6. Let us observe that, in our theorem, we are assuming that λ and μ are constants, on the other hand, an interesting open problem, that is not object of this paper, is concerned with the validity of the similar results in the variable-coefficients setting, precisely when $\mu = \mu(x)$ and $\lambda = \lambda(x)$.

As a further application of the multipliers technique we have developed to prove Theorem 1.1, we are also able to perform uniform resolvent estimates for the operator $-\Delta^* + V$, which generalize the ones obtained, for the Helmholtz equation, by Barceló, Vega and Zubeldia in [5].

Precisely, in the last section of the paper we consider the resolvent equation

$$\Delta^* u - Vu + ku = f, \tag{7}$$

where $k = k_1 + ik_2$ is any complex constant, with $k_1 := \Re k$ and $k_2 := \Im k$, and $f\colon \mathbb{R}^d \to \mathbb{C}^d$ is a measurable function and we will prove, for solution of (7), the following result

Theorem 1.2. *Let $d \geq 3$, $\||\cdot|f\| < \infty$ and assume that V satisfies (3). Then, there exist $c > 0$ independent of k and f such that for any solution $u \in [H^1(\mathbb{R}^d)]^d$ of the equation (7) one has*

$$\||x|^{-1}u\| \leq c\||x|f\|. \tag{8}$$

Remark 1.7. We remark that the estimate (8) is already proved in [3]. On the other hand our integral-smallness assumption on the potential is weaker than the one required in that work. Indeed, to be more precise, the authors provide the uniform resolvent estimate (8) for the equation

$$\Delta^* u + ku - \delta V u = f,$$

assuming that $\||x|^2 V\|_{L^\infty} < \infty$ and $|\delta|$ sufficiently small to let the perturbation argument work.

Actually, in order to prove Theorem 1.2 we establish the following stronger result, which shows that a priori estimates for solutions of (7) hold.

Theorem 1.3. *Let $d \geq 3$, $\||\cdot|f\| < \infty$ and assume that V satisfies (3). Then, there exist $c > 0$ independent of k and f such that for any solution $u \in [H^1(\mathbb{R}^d)]^d$ of the equation (7) one has*

- *for $|k_2| \leq k_1$*

$$\|\nabla u_S^-\| \leq c\||x|f\|, \qquad \text{and} \qquad \|\nabla u_P^-\| \leq c\||x|f\|, \tag{9}$$

- *for $|k_2| > k_1$*

$$\|\nabla u\| \leq c\||x|f\|. \tag{10}$$

Remark 1.8. The vector fields u_S^- and u_P^-, which appear in the statement, are defined in (29) and (31) respectively.

From this theorem, as a straightforward corollary, we easily obtain Theorem 1.2.

2. Preliminaries

We devote this preliminary section to recall some very well known facts about the Lamé operator which, actually, are interesting in their own right.

First we will recall some properties about the free Lamé operator and subsequently we will point out the differences with the perturbed setting.

2.1. Self-adjointness.

First of all we want to give a rigorous meaning to the Lamé operator as a self-adjoint operator, i.e. we want to build the self-adjoint extension of the operator $-\Delta^*$; in order to do that we proceed using a quadratic form approach.

Let us introduce the quadratic form associated with the operator $-\Delta^*$,

$$Q_0[u] = \int_{\mathbb{R}^d} q_0[u]\, dx,$$

where

$$q_0[u] = \lambda \left| \sum_{j=1}^{d} \frac{\partial u_j}{\partial x_j} \right|^2 + \frac{\mu}{2} \sum_{j,k=1}^{d} \left| \frac{\partial u_j}{\partial x_k} + \frac{\partial u_k}{\partial x_j} \right|^2, \qquad u \in [C_c^\infty(\mathbb{R}^d)]^d.$$

Remark 2.1. Since a complex setting will be needed once the perturbation V will come into play, from now on we assume u to be complex-valued; although this assumption is not yet necessary in the free framework ($V = 0$).

A straightforward computation, made explicit in section 2.2, shows that under the physical assumption (1) the quadratic form $q_0[u]$, and, thus, $-\Delta^*$ is positive.

Remark 2.2. The quadratic form clearly remains positive under the stronger condition $\lambda, \mu > 0$.

We recall that, since our form Q_0 is associated with a densely defined positive and symmetric operator, this form is closable.

Let $\overline{Q}_0$ be the closure of our form. Even though completely standard, for reason of completeness, we will show the closedness of our form $\overline{Q}_0$ with form domain the Sobolev space of H^1- vector fields. In order to do that we need to prove that $[H^1(\mathbb{R}^d)]^d$ equipped with the norm

$$\|u\|_{\overline{Q}_0} := \big(\overline{Q}_0[u] + \|u\|^2_{[L^2(\mathbb{R}^d)]^d}\big)^{\frac{1}{2}}$$

is complete. For this purpose we just have to prove that $\|u\|_{\overline{Q}_0}$ is equivalent to $\|u\|_{[H^1(\mathbb{R}^d)]^d} := \big(\|u\|^2_{[L^2(\mathbb{R}^d)]^d} + \|\nabla u\|^2_{[L^2(\mathbb{R}^d)]^d}\big)^{\frac{1}{2}}$. We need the following trivial chain of inequalities, for every $d \times d$ matrix ξ

$$\tfrac{1}{d}|\mathrm{Tr}(\xi)|^2 \le \big|\tfrac{1}{2}(\xi + \xi^T)\big|^2 \le |\xi|^2. \tag{11}$$

Calling ξ the Jacobian matrix, $\xi_i^j := \frac{\partial u_i}{\partial x_j}$ we can rewrite $\overline{Q}_0$ in terms of ξ in this way:

$$\overline{Q}_0[u] := 2\mu \int_{\mathbb{R}^d} \big|\tfrac{1}{2}(\xi + \xi^T)\big|^2 + \lambda \int_{\mathbb{R}^d} |\mathrm{Tr}(\xi)|^2.$$

Assuming λ and μ to satisfy (1) and using (11), we get

$$\overline{Q}_0[u] \le 2\mu \int_{\mathbb{R}^d} \big|\tfrac{1}{2}(\xi + \xi^T)\big|^2 + (2\mu + \lambda d) \int_{\mathbb{R}^d} \tfrac{1}{d}|\mathrm{Tr}(\xi)|^2 \le (4\mu + \lambda d) \int_{\mathbb{R}^d} |\xi|^2 = (4\mu + \lambda d)\|\nabla u\|^2_{[L^2(\mathbb{R}^d)]^d}.$$

Summing up we proved that

$$\|u\|^2_{\overline{Q}_0} \le (4\mu + \lambda d)\|\nabla u\|^2_{[L^2(\mathbb{R}^d)]^d} + \|u\|^2_{[L^2(\mathbb{R}^d)]^d} \le \max\{4\mu + \lambda d, 1\}\|u\|^2_{[H^1(\mathbb{R}^d)]^d}. \tag{12}$$

As a starting point for proving the opposite inequality, we show that there exists $c > 0$ such that

$$\overline{Q}_0[u] \ge c \int_{\mathbb{R}^d} \big|\tfrac{1}{2}(\xi + \xi^T)\big|^2 = c \sum_{j,k=1}^{d} \int_{\mathbb{R}^d} \big|\tfrac{1}{2}\big(\tfrac{\partial u_j}{\partial x_k} + \tfrac{\partial u_k}{\partial x_j}\big)\big|^2. \tag{13}$$

Since the Lamé parameters have to satisfy (1), there exists $c > 0$ such that $2\mu \ge c$ and $2\mu + \lambda d \ge c$, using this fact we obtain

$$\overline{Q}_0[u] - c \int_{\mathbb{R}^d} \big|\tfrac{1}{2}(\xi + \xi^T)\big|^2 \ge (2\mu - c) \int_{\mathbb{R}^d} \Big[\big|\tfrac{1}{2}(\xi + \xi^T)\big|^2 - \tfrac{1}{d}|\mathrm{Tr}(\xi)|^2\Big] \ge 0,$$

and this clearly gives the claim. In order to conclude we make use of the *Korn's inequality* that reads

$$\|u\|^2_{[H^1(\mathbb{R}^d)]^d} \le \tilde{c}\Big(\sum_{j=1}^{d} \int_{\mathbb{R}^d} |u_j|^2 + \sum_{j,k=1}^{d} \int_{\mathbb{R}^d} \Big|\tfrac{1}{2}\big(\tfrac{\partial u_j}{\partial x_k} + \tfrac{\partial u_k}{\partial x_j}\big)\Big|^2\Big), \tag{14}$$

for some constant $\tilde{c} \ge 0$.

Exploiting (13) and (14) we easily obtain

$$\|u\|^2_{\overline{Q}_0} \ge c \sum_{j,k=1}^{d} \int_{\mathbb{R}^d} \Big|\tfrac{1}{2}\big(\tfrac{\partial u_j}{\partial x_k} + \tfrac{\partial u_k}{\partial x_j}\big)\Big|^2 + \sum_{j=1}^{d} \int_{\mathbb{R}^d} |u_j|^2 \ge \tfrac{\min\{c,1\}}{\tilde{c}}\|u\|^2_{[H^1(\mathbb{R}^d)]^d}. \tag{15}$$

Observe that from (12) and (15) we have the anticipated equivalence, that is $\overline{Q}_0$ is closed.

Therefore, as $\overline{Q}_0$ is a densely defined lower semi-bounded (actually positive) closed form on an Hilbert space, then there is a canonical way to build from it a distinguished self-adjoint extension, called Friedrichs extension, of the symmetric operator $-\Delta^*$, that is the self-adjoint operator we are looking for and that, with abuse of notation, we again write as $-\Delta^*$.

2.2. **Ellipticity properties.**

The aim of this section is to point out how the notion of ellipticity has been generalized for systems, in particular for systems of second order differential operators. Let's consider the system of m second order constant coefficients operator L acting on $u\colon \mathbb{R}^n \to \mathbb{C}^m$ and defined as

$$Lu := -\sum_{\alpha=1}^{n}\sum_{\beta=1}^{n} A^{\alpha\beta}\partial_\alpha\partial_\beta u; \tag{16}$$

more precisely the j-th component of Lu is

$$[Lu]_{1\le j\le m} := -\sum_{\alpha=1}^{n}\sum_{\beta=1}^{n}\sum_{k=1}^{m} A_{jk}^{\alpha\beta}\partial_\alpha\partial_\beta u_k,$$

where $A = \big(A_{jk}^{\alpha\beta}\big)_{\substack{1\le\alpha,\beta\le n\\ 1\le j,k\le m}}$ is the coefficient tensor associated with L. Via Fourier transformation we associate to L its principal symbol $p(\xi)$ that is an $m\times m$ matrix of this form

$$p(\xi) := \sum_{\alpha=1}^{n}\sum_{\beta=1}^{n} A^{\alpha\beta}\xi_\alpha\xi_\beta.$$

By virtue of the previous definition, the classical notion of ellipticity for operators is naturally generalized to systems as the condition of *invertibility* of the symbol $p(\xi)$.

Now we introduce a stronger notion of ellipticity. An operator as in (16) is strongly elliptic if its characteristic tensor satisfies the so called *Legendre-Hadamard* condition, that is, if there exists $c>0$ such that

$$\Re\Big(\sum_{\alpha,\beta=1}^{n}\sum_{j,k=1}^{m} A_{jk}^{\alpha\beta}\xi_\alpha\xi_\beta\eta_j\overline{\eta}_k\Big) \ge c|\xi|^2_{\mathbb{R}^n}|\eta|^2_{\mathbb{C}^m} \qquad \forall\,\xi\in\mathbb{R}^n, \eta\in\mathbb{C}^m. \tag{17}$$

To conclude our survey about elliptic properties for systems of constant coefficients second order operators, we state the notion of very strong ellipticity: an operator as in (16) is very strongly elliptic if satisfies the *Legendre* condition, that is, if there exists $c>0$ such that

$$\Re\Big(\sum_{\alpha,\beta=1}^{n}\sum_{j,k=1}^{m} A_{jk}^{\alpha\beta}\tau_\beta^k\overline{\tau}_\alpha^j\Big) \ge c|\tau|^2_{\mathbb{C}^{n\times m}} \qquad \forall\,\tau\in\mathbb{C}^{n\times m}.$$

Remark 2.3. For many applications, however, the Legendre condition is too strong. This comes from the fact that the tensor A usually, for example in elasticity theory and in compressible fluids, has symmetries like

$$A_{jk}^{\alpha\beta} = A_{\alpha\beta}^{jk} = A_{j\beta}^{\alpha k} = A_{\alpha k}^{j\beta}. \tag{18}$$

These symmetries are called hyperelastic and mean that A only acts on the symmetric part of a matrix and yields again a symmetric matrix. For this situation the appropriate condition reads

$$\Re\Big(\sum_{\alpha,\beta=1}^{n}\sum_{j,k=1}^{m} A_{jk}^{\alpha\beta}\sigma_\beta^k\overline{\sigma}_\alpha^j\Big) \ge c|\sigma|^2_{\mathbb{C}^{m\times m}} \qquad \forall\,\sigma\in \mathrm{Sym}(\mathbb{C}^{m\times m}), \tag{19}$$

for some $c>0$, where $\mathrm{Sym}(\mathbb{C}^{m\times m})$ is the space of symmetric matrices.

Now we are in position to consider our Lamé system $-\Delta^*$. We want to write $-\Delta^*$ in the form of a general system of d second order constant coefficients operator, precisely we want to rewrite $-\Delta^*$ as in (16). Let's underline that two different coefficient tensors A may have the same associated operator L, indeed, it's easy to see that both the tensors

$$A_{jk}^{\alpha\beta} := (\lambda+\mu)\delta_{j\alpha}\delta_{k\beta} + \mu\delta_{\alpha\beta}\delta_{jk}$$

and

$$A_{jk}^{\alpha\beta} := \mu(\delta_{\alpha\beta}\delta_{jk} + \delta_{\alpha k}\delta_{\beta j}) + \lambda\delta_{\alpha j}\delta_{\beta k}$$

give the Lamé operator of elasticity (here we used the δ-Kronecker formalism).

The rest of the section will be used to highlight under which conditions about Lamé parameters λ and μ, our operator is elliptic (in sense of the three definitions stated above).

First of all we want to point out that the characteristic matrix (or principal symbol) of $-\Delta^*$ is invertible. This guarantees the ellipticity of the operator in the most classical sense. A straightforward computation shows that the principal symbol of $-\Delta^*$, is a matrix of this form:

$$p(\xi) := \mu|\xi|^2 I + (\lambda+\mu)\xi\otimes\xi, \qquad \forall\,\xi\in\mathbb{R}^d;$$

where $\xi\otimes\xi$ is the dyadic product of ξ and ξ that is defined as $(v\otimes w)_{jk} = v_j w_k$ for all $v=(v_1,v_2,\dots,v_d)$ and $w=(w_1,w_2,\dots,w_d)$, and I denotes the $d\times d$ identity matrix. For our convenience we rewrite $p(\xi)$ in this way:

$$p(\xi) = \mu|\xi|^2\left(I + \frac{\lambda+\mu}{\mu} l(\xi)\right),$$

now the matrix $l(\xi) := \frac{\xi}{|\xi|}\otimes\frac{\xi}{|\xi|}$ is idempotent, i.e $l^2(\xi) = l(\xi)$, and therefore $p(\xi)$ is invertible and it is quite easy to find its inverse, that is

$$p^{-1}(\xi) := \frac{1}{\mu|\xi|^2}\left(I - \frac{\lambda+\mu}{\lambda+2\mu} l(\xi)\right).$$

Remark 2.4. Let's underline that everything makes sense if μ, $\lambda+2\mu\neq 0$, however our condition (1) guarantees that the previous assumption is fulfilled.

Now let's see that assuming $\mu>0$ and $\lambda+2\mu>0$, the operator $-\Delta^*$ satisfies the Legendre-Hadamard condition(17). Using one of the two possible definitions of the tensor associated with the Lamé operator, we easily have

$$\sum_{\alpha,\beta=1}^d\sum_{j,k=1}^d A^{\alpha\beta}_{jk}\xi_\alpha\xi_\beta\eta_j\overline{\eta}_k = \sum_{\alpha,\beta=1}^d(\lambda+\mu)\,\xi_\alpha\eta_\alpha\xi_\beta\overline{\eta}_\beta + \sum_{\alpha,j=1}^d\mu\,\xi_\alpha^2|\eta_j|^2 = (\lambda+\mu)|\xi\cdot\eta|^2 + \mu|\xi|^2|\eta|^2.$$

At this point, we just need to prove that there exists a constant $c>0$ such that

$$(\lambda+\mu)|\xi\cdot\eta|^2 + \mu|\xi|^2|\eta|^2 \geq c\,|\xi|^2|\eta|^2.$$

It's not difficult to see ([23] Lemma 3.1) that the previous is equivalent to

$$\min\{\mu,\lambda+2\mu\}>0,$$

which proves the claim.

Remark 2.5. We observe that the Legendre-Hadamard condition can be reformulated in terms of the characteristic matrix saying that our operator is strongly elliptic if the symbol $p(\xi)$ is a positive defined matrix, that is, if there exists $c>0$ such that

$$\langle p(\xi)\eta,\eta\rangle = \mu|\xi|^2|\eta|^2 + (\lambda+\mu)|\xi\cdot\eta|^2 \geq c|\xi|^2|\eta|^2.$$

Now we analyze the very strong ellipticity. Considering the second form of the tensor A, one can observe that the hyperelasticity condition (18) holds. This means that we just need to prove that $-\Delta^*$ satisfies (19).

Considering the second form of the tensor A, assuming $\mu>0$ and $2\mu+d\lambda>0$, by the same argument exploited in the previous section for proving the closedness of the quadratic form $\overline{Q}_0$, it is easy to see the validity of the condition (19), indeed for all $\sigma\in\operatorname{Sym}(\mathbb{C}^{d\times d})$, we have

$$\sum_{\alpha,\beta=1}^d\sum_{j,k=1}^d A^{\alpha\beta}_{jk}\sigma^k_\beta\overline{\sigma}^j_\alpha = 2\mu\sum_{\alpha,j=1}^d|\sigma^j_\alpha|^2 + \lambda|\sum_{\alpha=1}^d\sigma^\alpha_\alpha|^2 = 2\mu|\sigma|^2 + \lambda|\operatorname{Tr}(\sigma)|^2 \geq c|\sigma|^2.$$

Remark 2.6. Note that if one assume A to satisfy (19), that is if $-\Delta^*$ is very strongly elliptic, immediately follows the positivity of the quadratic form Q_0 associated with the operator. Let's consider our quadratic form Q_0, we want to write Q_0 in terms of the simmetric matrix $\sigma = (\sigma^k_j)_{1\leq j,k\leq d} := \frac{1}{2}\big(\frac{\partial u_j}{\partial x_k} + \frac{\partial u_k}{\partial x_j}\big)$. A straightforward computation gives

$$Q_0[u] = \lambda\int_{\mathbb{R}^d}|\sum_{j=1}^d\frac{1}{2}\Big(\frac{\partial u_j}{\partial x_j} + \frac{\partial u_j}{\partial x_j}\Big)|^2 + 2\mu\sum_{j,k=1}^d\int_{\mathbb{R}^d}|\frac{1}{2}\Big(\frac{\partial u_j}{\partial x_k} + \frac{\partial u_k}{\partial x_j}\Big)|^2 = 2\mu\int_{\mathbb{R}^d}|\sigma|^2 + \lambda\int_{\mathbb{R}^d}|\operatorname{Tr}(\sigma)|^2,$$

since $\sigma\in\operatorname{Sym}(\mathbb{C}^{d\times d})$ and $-\Delta^*$ satisfies (19), we conclude that $Q_0[u]\geq c\int_{\mathbb{R}^d}|\sigma|^2$ which, in particular, gives the positivity. Summing up, if $-\Delta^*$ is very strongly elliptic, that is if $\mu>0$ and $2\mu+d\lambda>0$, our quadratic form is positive.

Remark 2.7. Let's underline that if one assume for the Lamé operator just to be strongly elliptic, the quadratic form associated with the operator is still positive under weaker conditions about the Lamé parameters. Indeed by Plancherel theorem we can see that

$$\begin{aligned} Q_0[u] &:= \lambda\|\sum_{j=1}^{d}\frac{\partial u_j}{\partial x_j}\|_{L^2}^2 + \frac{\mu}{2}\sum_{j,k=1}^{d}\|\frac{\partial u_j}{\partial x_k} + \frac{\partial u_k}{\partial x_j}\|_{L^2}^2 = \lambda\|\sum_{j=1}^{d}\xi_j\hat{u}_j\|_{L^2}^2 + \frac{\mu}{2}\sum_{j,k=1}^{d}\|\xi_k\hat{u}_j + \xi_j\hat{u}_k\|_{L^2}^2 \\ &= \lambda\int_{\mathbb{R}^d}|\sum_{j=1}^{d}\xi_j\hat{u}_j|^2 + \mu\sum_{j,k=1}^{d}\int_{\mathbb{R}^d}\xi_k^2|\hat{u}_j|^2 + \mu\int_{\mathbb{R}^d}|\xi\cdot\hat{u}|^2 = (\lambda+\mu)\int_{\mathbb{R}^d}|\xi\cdot\hat{u}|^2 + \mu\int_{\mathbb{R}^d}|\xi|^2|\hat{u}|^2. \end{aligned}$$

Now, if $\mu > 0$ and $\lambda + 2\mu > 0$, the Lamé operator satisfies (17), therefore there exists $c > 0$ such that

$$(\lambda+\mu)\int_{\mathbb{R}^d}|\xi\cdot\hat{u}|^2 + \mu\int_{\mathbb{R}^d}|\xi|^2|\hat{u}|^2 \geq c\int_{\mathbb{R}^d}|\xi|^2|\hat{u}|^2,$$

and clearly this provides the positivity of the quadratic form.

2.3. **Helmholtz decomposition.**

Now our purpose is to understand the action of $-\Delta^*$ on smooth vector fields. In order to do that we are going to make use of the well known *Helmholtz decomposition*, which is a standard way to decompose a vector field into a sum of a gradient and a divergence free vector field. To be more precise, we have that every smooth vector field u sufficiently rapidly decaying at infinity, can be uniquely decomposed as

$$u = u_S + u_P,$$

where $\operatorname{div} u_S = 0$ and $u_P = \nabla\varphi$, for some smooth scalar function φ.

We summarize in the following lemma some useful properties about the two components of the Helmholtz decomposition which will be needed in the proof of our main results.

Lemma 2.1. *Let u be a smooth vector field sufficiently rapidly decaying at infinity. Let u_S and u_P be the two components of the Helmholtz decomposition. Then*

- *u_S and u_P are L^2-orthogonal.*
- *u_S and u_P are H^1-orthogonal.*

Proof. The proof of both the sentences makes use of an integration by part argument. The first one immediately follows from the assumption about u_S and u_P.

Let us now consider the second sentence. In order to simplify the notation we call $F := u_S$ and $G := u_P$, thus F is the divergence free vector field and G is the gradient.

$$\begin{aligned} (\nabla F, \nabla G) &= \int_{\mathbb{R}^d}\nabla\overline{F}\cdot\nabla G = \sum_{j=1}^{d}\int_{\mathbb{R}^d}\nabla\overline{F}_j\cdot\nabla G_j = -\sum_{j=1}^{d}\int_{\mathbb{R}^d}\overline{F}_j\Delta G_j = -\sum_{j=1}^{d}\int_{\mathbb{R}^d}\overline{F}_j\Delta(\partial_j\varphi) \\ &= \sum_{j=1}^{d}\int_{\mathbb{R}^d}\partial_j\overline{F}_j\Delta\varphi = \int_{\mathbb{R}^d}\operatorname{div}\overline{F}\,\Delta\varphi = 0. \end{aligned}$$

□

Using the Helmholtz decomposition, a straightforward computation shows that for any $u = u_S + u_P$, the operator $-\Delta^*$ acts on u in this way

$$-\Delta^* u = -\mu\Delta u_S - (\lambda+2\mu)\Delta u_P,$$

where, again, u_S is the divergence free vector field and u_P is the gradient.

The quadratic form associated with the operator $-\Delta^*$, explicitly written in the Helmholtz decomposition, is

$$Q_0[u] = \int_{\mathbb{R}^d} q_0[u]\,dx,$$

with

$$q_0[u] = \mu|\nabla u_S|^2 + (\lambda+2\mu)|\nabla u_P|^2 \qquad \text{and} \qquad \mathscr{D}(Q_0) = [H^1(\mathbb{R}^d)]^d,$$

where $|\nabla v|^2$, when $v = (v_1, v_2, \dots, v_d)$ is a vector field, denotes $\sum_{j=1}^{d}|\nabla v_j|^2$.

We observe that here, with an abuse of notation, we have used the same symbol Q_0 for the quadratic form associated with $-\Delta^*$, both when its action is written explicitly using the Helmholtz decomposition and when the operator is defined in its classical way.

Now we are in position to consider the perturbed operator

$$-\Delta^* + V(x),$$

where $V\colon \mathbb{R}^d \to \mathcal{M}_{d\times d}(\mathbb{C})$ is the perturbation term.

Clearly, in the Helmholtz decomposition, this operator acts on a smooth vector fields u in this way

$$(-\Delta^* + V)u = -\mu\Delta u_S - (\lambda + 2\mu)\Delta u_P + Vu.$$

The corresponding perturbed quadratic form associated with this operator is

$$Q_{\text{pert}}[u] = Q_0[u] + Q_V[u] = Q_0[u] + \int_{\mathbb{R}^d} q_V[u]\,dx,$$

where

$$q_V[u] = \overline{Vu}\cdot u \qquad \text{and} \qquad \mathscr{D}(Q_V) = \left\{u \in [L^2(\mathbb{R}^d)]^d : \int_{\mathbb{R}^d} |V||u|^2 < \infty\right\}.$$

We assume the smallness condition (5) about V. It's not difficult to see that, as a consequence of the constrictions on a, Q_V is relatively bounded with respect to Q_0 with bound less than one.

Let us suppose, for a moment, that our potential V is real-valued. As a consequence, the sesquilinear form, associated with the quadratic form Q_V, is symmetric. By virtue of these remarks, we are able to build from Q_{pert} an associated self-adjoint operator on $[L^2(\mathbb{R}^d)]^d$ exploiting the well known forms counterpart of the Kato-Rellich perturbation result for operators, namely the KLMN theorem (see for example [28], Thm X.17, or [30], Thm 10.21).

If one is dealing with complex-valued potentials, as our setting, instead of real-valued ones, the scenario turns out to be quite different. In fact, assuming now that V is a complex-valued potential, the sesquilinear form Q_V is no more symmetric and, as a consequence, we clearly can't expect to be able to build from Q_{pert} a self-adjoint extension of $-\Delta^* + V$. Nevertheless, even though we are dealing with non symmetric forms, we can obtain useful information about the operator $-\Delta^* + V$ by exploiting the theory about sectorial forms (resp. operators). Precisely we can use the representation theorem (see [19], Thm. VI.2.1) to build an m-sectorial operator from a densely defined, sectorial and closed form.

3. Absence of eigenvalues: proof of Theorem 1.1

We devote this section to the proof of Theorem 1.1 we stated in the introduction.

We recall that our strategy wants to be built in analogy to that one in the recent work [15] of Fanelli, Krejčiříc and Vega, who established the analogous result for the Laplace operator.

First of all, to this end, starting from the eigenvalue equation associated with the perturbed Laplacian, they provided three integral identities which had a crucial role in the proof of their main result; in order to do that they re-adapted to a non self-adjoint setting the standard technique of Morawetz multipliers. This tool was introduced in [24] for the Klein-Gordon equation and then it was developed in several other contexts. For example, about the Helmholtz equation's framework, let us mention the seminal works of Perthame and Vega [25], [26] which are concerned with the purely electric case and then [16, 14, 32, 2, 5, 33], which extend the technique in an electromagnetic setting. We should also quote [9] for an adaptation of multipliers method on exterior domains.

Now we are in position to begin the proof of our result.

The eigenvalue problem for the perturbed Lamé operator is

$$\Delta^* u + ku = Vu, \tag{20}$$

where k is any complex constant (throughout the paper we will denote by $k_1 = \Re k$ and by $k_2 = \Im k$).

Just to simplify the notations, we start assuming that u is a solution of this more general problem

$$\Delta^* u + ku = f, \tag{21}$$

where $f\colon \mathbb{R}^d \to \mathbb{C}^d$ is a measurable function.

Clearly, we can identify the problem (21) with (20) by setting $f = Vu$.

As we said above, the Helmholtz decomposition has been a fundamental tool for our purposes, according to this, writing $u = u_S + u_P$ and $f = f_S + f_P$, the resolvent equation (21) associated to the Lamé operator can be re-written as

$$\mu\Delta u_S + (\lambda + 2\mu)\Delta u_P + ku_S + ku_P = f_S + f_P, \tag{22}$$

where, again, the S component is the divergence free vector field and the P component is the gradient.

Let us observe that the equation written in this form is very far to be easy to handle, indeed the two components has the same frequency of oscillation k but different speed of propagation μ and $\lambda+2\mu$ respectively, and therefore the first attempt one would like to try is splitting the previous equation into a system of two decoupled equations involving separately the two components u_S and u_P. This attempt is going to work indeed, as a consequence of Lemma 2.1 which guarantees the L^2-orthogonality of the S and P components of the Helmholtz decomposition and of their gradients, we are allowed to reduce our "intertwining" equation into a system of two decoupled equations, precisely one has the following result

Lemma 3.1. *Let $u = u_S + u_P$ be a solution to equation (22), then the two components of the Helmholtz decomposition, u_S and u_P respectively, satisfies this two unrelated problems*

$$\begin{cases} \mu\Delta u_S + k u_S = f_S \\ (\lambda+2\mu)\Delta u_P + k u_P = f_P. \end{cases} \tag{23}$$

Proof. As we have already said we basically are going to use the L^2 and H^1-orthogonality of u_S and u_P.

Since u is a solution to (22), clearly we have

$$\|\mu\Delta u_S + (\lambda+2\mu)\Delta u_P + k u_S + k u_P - f_S - f_P\|^2 = 0,$$

or more explicitly

$$\int_{\mathbb{R}^d} \overline{(\mu\Delta u_S + (\lambda+2\mu)\Delta u_P + k u_S + k u_P - f_S - f_P)} \cdot (\mu\Delta u_S + (\lambda+2\mu)\Delta u_P + k u_S + k u_P - f_S - f_P) = 0.$$

A straightforward computation allows us to write the previous as

$$\|\mu\Delta u_S + k u_S - f_S\|^2 + \|(\lambda+2\mu)\Delta u_P + k u_P - f_P\|^2 + 2\Re \int \overline{(\mu\Delta u_S + k u_S - f_S)} \cdot ((\lambda+2\mu)\Delta u_P + k u_P - f_P) = 0.$$

In order to obtain the thesis is just needed to show that the third term is zero. Let us consider

$$I := \int \overline{(\mu\Delta u_S + k u_S - f_S)} \cdot ((\lambda+2\mu)\Delta u_P + k u_P - f_P),$$

we can write this explicitly and we have

$$\begin{aligned} I = \mu(\lambda+2\mu) &\int_{\mathbb{R}^d} \Delta\overline{u}_S \cdot \Delta u_P + \mu\, k \int_{\mathbb{R}^d} \Delta\overline{u}_S \cdot u_P - \mu \int_{\mathbb{R}^d} \Delta\overline{u}_S f_P + (\lambda+2\mu)\,\overline{k} \int_{\mathbb{R}^d} \overline{u}_S \cdot \Delta u_P \\ &+ |k|^2 \int_{\mathbb{R}^d} \overline{u}_S \cdot u_P - \overline{k} \int_{\mathbb{R}^d} \overline{u}_S \cdot f_P - (\lambda+2\mu) \int_{\mathbb{R}^d} \overline{f}_S \cdot \Delta u_P - k \int_{\mathbb{R}^d} \overline{f}_S \cdot u_P + \int_{\mathbb{R}^d} \overline{f}_S \cdot f_P. \end{aligned} \tag{24}$$

The L^2-orthogonality of the S component and P component gives immediately that the first two and the last two integrals in the second row of (24) vanish. Thus one gets

$$I = \mu(\lambda+2\mu) \underbrace{\int_{\mathbb{R}^d} \Delta\overline{u}_S \cdot \Delta u_P}_{I_1} + \mu\, k \underbrace{\int_{\mathbb{R}^d} \Delta\overline{u}_S \cdot u_P}_{I_2} - \mu \underbrace{\int_{\mathbb{R}^d} \Delta\overline{u}_S f_P}_{I_3} + (\lambda+2\mu)\,\overline{k} \underbrace{\int_{\mathbb{R}^d} \overline{u}_S \cdot \Delta u_P}_{I_4} - (\lambda+2\mu) \underbrace{\int_{\mathbb{R}^d} \overline{f}_S \cdot \Delta u_P}_{I_5}.$$

We are going to consider the five integrals separately. In order to simplify the details, again we use the notation adopted in Lemma 2.1, that is $F := u_S$ and $G := u_P$, thus F is the divergence free vector field and G is the gradient.

$$I_1 = \int_{\mathbb{R}^d} \Delta\overline{F} \cdot \Delta G = \sum_{j=1}^{d} \int_{\mathbb{R}^d} \Delta\overline{F}_j\, \Delta G_j = \sum_{j=1}^{d} \int_{\mathbb{R}^d} \Delta\overline{F}_j\, \Delta\partial_j\varphi = -\int_{\mathbb{R}^d} \Delta \operatorname{div} \overline{F}\, \Delta\varphi = 0.$$

Now we see I_2, we omit the details for I_3, I_4 and I_5, indeed they could be handle in the same manner.

$$I_2 = \int_{\mathbb{R}^d} \Delta\overline{F} \cdot G = \sum_{j=1}^{d} \int_{\mathbb{R}^d} \Delta\overline{F}_j\, G_j = -\sum_{j=1}^{d} \int_{\mathbb{R}^d} \nabla\overline{F}_j \cdot \nabla G_j = (\nabla F, \nabla G) = 0.$$

Putting the previous altogether we obtain that $I = 0$ and consequently we have

$$\|\mu\Delta u_S + k u_S - f_S\|^2 + \|(\lambda+2\mu)\Delta u_P + k u_P - f_P\|^2 = 0$$

which obviously implies (23). $\square$

As a starting point for the proof of Theorem 1.1, we consider the weak formulation of (23)

$$\forall\, v \in [H^1(\mathbb{R}^d)]^d, \qquad \begin{cases} -\mu(\nabla v, \nabla u_S) + k(v, u_S) = (v, f_S) \\ -(\lambda + 2\mu)(\nabla v, \nabla u_P) + k(v, u_P) = (v, f_P). \end{cases} \tag{25}$$

Following [5] we divide the proof of our result into two cases depending on the relation between real and imaginary part of the eigenvalue k: $|k_2| \leq k_1$ and $|k_2| > k_1$.

Let us start by the more technical case $|k_2| \leq k_1$

Case $|k_2| \leq k_1$: For the purpose of letting the proof more understandable, we will point out in the following lemma what Fanelli, Krejčiříc and Vega have proved in their paper [15] as the main tool to guarantee the absence of eigenvalues for the perturbed Laplace operator.

Lemma 3.2. *Let $u\colon \mathbb{R}^d \to \mathbb{C}$ be a solution to*

$$\Delta u + ku = f,$$

where k is any complex constants, we write $k_1 = \Re k$ and $k_2 = \Im k$ and $f\colon \mathbb{R}^d \to \mathbb{C}$ is a measurable function. If one sets

$$u^-(x) := e^{-i\,\mathrm{sgn}(k_2) k_1^{\frac{1}{2}} |x|}\, u(x)$$

the following estimate holds

$$\|\nabla u^-\|^2 + \frac{d-3}{d-1}\frac{|k_2|}{k_1^{\frac{1}{2}}}\int_{\mathbb{R}^d} |x||\nabla u^-|^2 \leq \frac{2(2d-3)}{d-2}\||x|f\|\|\nabla u^-\| + \frac{\sqrt{2}}{\sqrt{d-2}}\||x|f\|^{\frac{3}{2}}\|\nabla u^-\|^{\frac{1}{2}}. \tag{26}$$

Proof. As we have already mentioned in the introduction, this result is based on an appropriate use of the multipliers technique and we refer to [15] for the details of the proof. □

Remark 3.1. Let us just underline that the main tool of the proof is an integration by parts argument, therefore in order to make the calculations rigorous they need to assume u and f sufficiently smooth and then the result will be obtained by a standard density argument.

At this point, the next step is, in some sense, obliged. In fact the most natural way to proceed is to use directly the estimate which appears in Lemma 26 for our two decoupled equations (23).

In order to do that we have to make the two equations independent of Lamé's coefficients, for this purpose we need to re-define appropriately k and f (differently in each equations). Precisely calling

$$k_S := \frac{k}{\mu}, \qquad g_S := \frac{f_S}{\mu} \tag{27}$$

and on the counterpart

$$k_P := \frac{k}{\lambda + 2\mu}, \qquad g_P := \frac{f_P}{\lambda + 2\mu}, \tag{28}$$

we have that u_S and u_P satisfies

$$\begin{cases} \Delta u_S + k_S\, u_S = g_S \\ \Delta u_P + k_P\, u_P = g_P, \end{cases}$$

that written in components clearly are

$$\begin{cases} \Delta (u_S)_j + k_S\,(u_S)_j = (g_S)_j \\ \Delta (u_P)_j + k_P\,(u_P)_j = (g_P)_j, \end{cases}$$

for all $j = 1, \dots, d$.

First let us handle the equation for u_S.

Setting

$$u_S^-(x) = e^{-i\,\mathrm{sgn}(k_{S,2}) k_{S,1}^{\frac{1}{2}}|x|} u_S(x), \tag{29}$$

where $k_{S,1} := \Re(k_S)$ and $k_{S,2} = \Im(k_S)$ and exploiting Lemma 26, we have that

$$\|\nabla (u_S^-)_j\|^2 + \frac{d-3}{d-1}\frac{|k_{S,2}|}{k_{S,1}^{\frac{1}{2}}}\int_{\mathbb{R}^d} |x||\nabla (u_S^-)_j|^2 \leq \frac{2(2d-3)}{d-2}\||x|(g_S)_j\|\|\nabla (u_S^-)_j\| + \frac{\sqrt{2}}{\sqrt{d-2}}\||x|(g_S)_j\|^{\frac{3}{2}}\|\nabla (u_S^-)_j\|^{\frac{1}{2}}.$$

Summing on $j=1,\dots,d$ and using finite-dimensional Cauchy-Schwartz and Hölder inequalities in the last two terms respectively, we get

$$\|\nabla u_S^-\|^2+\frac{d-3}{d-1}\frac{|k_{S,2}|}{k_{S,1}^{\frac12}}\int_{\mathbb{R}^d}|x||\nabla u_S^-|^2\leq \frac{2(2d-3)}{d-2}\||x|g_S\|\|\nabla u_S^-\|+\frac{\sqrt{2}}{\sqrt{d-2}}\||x|g_S\|^{\frac32}\|\nabla u_S^-\|^{\frac12}.$$

Going back to our old notation, i.e. recalling (27), it is easy to obtain

$$\|\nabla u_S^-\|^2+\frac{1}{\sqrt{\mu}}\frac{d-3}{d-1}\frac{|k_2|}{k_1^{\frac12}}\int_{\mathbb{R}^d}|x||\nabla u_S^-|^2\leq \frac{1}{\mu}\frac{2(2d-3)}{d-2}\||x|f_S\|\|\nabla u_S^-\|+\frac{1}{\mu^{\frac32}}\frac{\sqrt{2}}{\sqrt{d-2}}\||x|f_S\|^{\frac32}\|\nabla u_S^-\|^{\frac12}. \tag{30}$$

Now we can provide the same estimate for the P component. Clearly, we mostly omit the details in fact these are the same we have already shown for the divergence free vector field u_S.

We define the vector field u_P^- in the same way as u_S^-, precisely

$$u_P^-(x):=e^{-i\,\mathrm{sgn}(k_{P,2})k_{P,1}^{\frac12}|x|}\,u_P(x), \tag{31}$$

where $k_{P,1}:=\Re(k_P)$ and $k_{p,2}=\Im(k_P)$.

Now proceeding in the same way as the previous case we get

$$\begin{aligned}\|\nabla u_P^-\|^2+\frac{1}{\sqrt{\lambda+2\mu}}\frac{d-3}{d-1}\frac{|k_2|}{k_1^{\frac12}}\int_{\mathbb{R}^d}|x||\nabla u_P^-|^2\leq{}& \frac{1}{\lambda+2\mu}\frac{2(2d-3)}{d-2}\||x|f_P\|\|\nabla u_P^-\|\\ &+\frac{1}{(\lambda+2\mu)^{\frac32}}\frac{\sqrt{2}}{\sqrt{d-2}}\||x|f_P\|^{\frac32}\|\nabla u_P^-\|^{\frac12}.\end{aligned} \tag{32}$$

In order to complete the argument and to obtain the thesis, the following elliptic regularity lemma will be useful in the immediate sequel.

Lemma 3.3. *Let $f\in[C_c^\infty(\mathbb{R}^d)]^d$ be a smooth-compactly supported vector field in $\mathbb{R}^d$, and let $\psi\colon\mathbb{R}^d\to\mathbb{C}$ be a smooth solution to*

$$\Delta\psi=\operatorname{div} f. \tag{33}$$

Then for any $s\in(-d,d)$ the following estimate holds

$$\||x|^s\nabla\psi\|\leq c\,Q_2(|x|^s)^2\||x|^s f\|,$$

for some constant $c>0$ only depending on the dimension d and $Q_2(|x|^s)$ the A_2 characteristic of the weight $|x|^s$ recalled below.

Proof. The proof of this result basically relies on the very well known theorem about Calderón-Zygmund operator, which ensures that if T is an operator of Calderón-Zygmund type, which, roughly speaking, is a class of integral operator whose kernel has a singularity of the size $|x-y|^{-d}$ asymptotically as $|x-y|$ goes to zero, then for any weight w in the A_p class, with $1<p<\infty$, T is bounded on the weighted space $L^p(w)$ (see for example [13], Thm. 7.11). Actually we are interested on a particular Calderón-Zygmund operator, i.e. the well known Riesz transform defined for any $f\in L^2(\mathbb{R}^d)$, via the Fourier transform, by

$$\widehat{R_jf}(\xi)=-i\frac{\xi_j}{|\xi|}\hat f(\xi),\qquad \forall\, j=1,2,\dots,d. \tag{34}$$

In particular we will use the following result of Petermichl which is concerned with the sharp bound for the operator norm of the Riesz transform in $L^2(w)$. He proved [27] that for all $j=1,2,\dots d$, if $w\in A_2$ then

$$\|R_jf\|_{L^2(w)}\leq c\,Q_2(w)\|f\|_{L^2(w)}, \tag{35}$$

where $Q_2(w)$ is the A_2 characteristic of the weight w defined as

$$Q_2(w):=\sup_Q\Big(\frac{1}{|Q|}\int_Q w\Big)\Big(\frac{1}{|Q|}\int_Q w^{-1}\Big),$$

with Q any cube in $\mathbb{R}^d$ and c a constant only depending upon the dimension.

We are interested in the weighted boundedness of the Riesz transform because our operator $T(f):=\nabla\psi$, with ψ a solution to (33), can be written in terms of Riesz transform in this way, for all $j=1,2,\dots,d$

$$\partial_j\psi=-(d-2)\sum_{k=1}^d R_jR_kF_k.$$

Indeed, using the fundamental solution of $-\Delta$ in $\mathbb{R}^d$, we can write

$$\psi = (-\Delta)^{-1}\operatorname{div} F = -\frac{1}{\omega_d}|x|^{-(d-2)} * \operatorname{div} F = -\frac{1}{\omega_d}\sum_{k=1}^{d}\partial_k |x|^{-(d-2)} * F_k,$$

where ω_d is the surface area of the d-dimensional unit sphere, precisely, making use of the gamma function, it is $\omega_d := \frac{2\pi^{d/2}}{\Gamma\left(\frac{d}{2}\right)}$. We are interested in the partial derivatives of ψ, that is

$$\partial_j \psi(x) = -\frac{1}{\omega_d}\sum_{k=1}^{d}\partial_j\partial_k |x|^{-(d-2)} * F_k(x).$$

Consequently, using that the Fourier transform of the homogeneous function $|x|^{-\alpha}$ is $\widehat{|x|^{-\alpha}}(\xi) := c_{d,\alpha}|\xi|^{\alpha-d}$, with $c_{d,\alpha} := \frac{2^{\frac{d}{2}-\alpha}\Gamma\left(\frac{d-\alpha}{2}\right)}{\Gamma\left(\frac{\alpha}{2}\right)}$ and that $\widehat{f*g} = (2\pi)^{d/2}\hat{f}\hat{g}$, we get

$$\widehat{\partial_j\psi}(\xi) = \frac{(2\pi)^{d/2}c_{d,d-2}}{\omega_d}\sum_{k=1}^{d}\xi_j\xi_k|\xi|^{-2}\hat{F}_k(\xi).$$

From the definition of the Riesz transform (34), given in terms of Fourier transformation, it is straightforward to see that, for all $j,k = 1,2\dots,d$

$$\widehat{R_jR_k f}(\xi) = -\frac{\xi_j\xi_k}{|\xi|^2}\hat{f}(\xi),$$

this clearly means that

$$\widehat{\partial_j\psi}(\xi) = -\frac{(2\pi)^{d/2}c_{d,d-2}}{\omega_d}\sum_{k=1}^{d}\widehat{R_jR_kF_k}(\xi).$$

Now, using the linearity of the Fourier transform, antitrasforming the previous identity and making explicit the constants, we obtain our claim.

Moreover it can be proved that if $-d < s < d$ then $|x|^s$ belongs to A_2 class. Calling $w := |x|^s$, the statement of our lemma is equivalent to find a constant C such that

$$\|\nabla\psi\|_{[L^2(\mathbb{R}^d)(w)]^d} \leq C\|F\|_{[L^2(\mathbb{R}^d)(w)]^d}.$$

Putting all the previous facts together we get

$$\begin{aligned}\|\nabla\psi\|_{[L^2(\mathbb{R}^d)(w)]^d} &:= \Big(\sum_{j=1}^{d}\|\partial_j\psi\|^2_{L^2(\mathbb{R}^d)(w)}\Big)^{\frac{1}{2}} \leq d(d-2)\Big(\sum_{j,k=1}^{d}\|R_jR_kF_k\|^2_{L^2(\mathbb{R}^d)(w)}\Big)^{\frac{1}{2}}\\ &\leq d\sqrt{d}(d-2)c^2Q_2(w)^2\Big(\sum_{k=1}^{d}\|F_k\|^2_{L^2(\mathbb{R}^d)(w)}\Big)^{\frac{1}{2}} = d\sqrt{d}(d-2)c^2Q_2(w)^2\|F\|_{[L^2(\mathbb{R}^d)(w)]^d}.\end{aligned}$$

This concludes the proof.

Remark 3.2. Let's underline that the constant c that appears in the previous equation is the one stated in the Petermichl result.

□

Let us introduce a trivial decomposition of our f :

$$f = f - \nabla\psi + \nabla\psi,$$

where ψ is the unique solution of (33); as a consequence we have $\operatorname{div}(f - \nabla\psi) = 0$. By the uniqueness of the Helmholtz decomposition, it follows that $f_S = f - \nabla\psi$, $f_P = \nabla\psi$. Substituting these in (30) and (32) respectively, one gets the two following estimates

$$\begin{aligned}\|\nabla u_S^-\|^2 + \frac{1}{\sqrt{\mu}}\frac{d-3}{d-1}\frac{|k_2|}{k_1^{\frac{1}{2}}}\int_{\mathbb{R}^d}|x||\nabla u_S^-|^2 \leq \frac{1}{\mu}\frac{2(2d-3)}{d-2}(\||x|f\| + \||x|\nabla\psi\|)\|\nabla u_S^-\|\\ + \frac{1}{\mu^{\frac{3}{2}}}\frac{\sqrt{2}}{\sqrt{d-2}}(\||x|f\| + \||x|\nabla\psi\|)^{\frac{3}{2}}\|\nabla u_S^-\|^{\frac{1}{2}};\end{aligned}$$

and

$$\|\nabla u_P^-\|^2 + \frac{1}{\sqrt{\lambda+2\mu}}\frac{d-3}{d-1}\frac{|k_2|}{k_1^{\frac{1}{2}}}\int_{\mathbb{R}^d}|x||\nabla u_P^-|^2 \leq \frac{1}{\lambda+2\mu}\frac{2(2d-3)}{d-2}\||x|\nabla\psi\|\|\nabla u_P^-\| + \frac{1}{(\lambda+2\mu)^{\frac{3}{2}}}\frac{\sqrt{2}}{\sqrt{d-2}}\||x|\nabla\psi\|^{\frac{3}{2}}\|\nabla u_P^-\|^{\frac{1}{2}}.$$

Using the elliptic regularity result 3.3 we obtain respectively

$$\|\nabla u_S^-\|^2 + \frac{1}{\sqrt{\mu}}\frac{d-3}{d-1}\frac{|k_2|}{k_1^{\frac{1}{2}}}\int_{\mathbb{R}^d}|x||\nabla u_S^-|^2 \leq \frac{1}{\mu}\frac{2(2d-3)}{d-2}(C+1)\||x|f\|\|\nabla u_S^-\| + \frac{1}{\mu^{\frac{3}{2}}}\frac{\sqrt{2}}{\sqrt{d-2}}(C+1)^{\frac{3}{2}}\||x|f\|^{\frac{3}{2}}\|\nabla u_S^-\|^{\frac{1}{2}};$$

and

$$\|\nabla u_P^-\|^2 + \frac{1}{\sqrt{\lambda+2\mu}}\frac{d-3}{d-1}\frac{|k_2|}{k_1^{\frac{1}{2}}}\int_{\mathbb{R}^d}|x||\nabla u_P^-|^2 \leq \frac{1}{\lambda+2\mu}\frac{2(2d-3)}{d-2}C\||x|f\|\|\nabla u_P^-\| + \frac{1}{(\lambda+2\mu)^{\frac{3}{2}}}\frac{\sqrt{2}}{\sqrt{d-2}}C^{\frac{3}{2}}\||x|f\|^{\frac{3}{2}}\|\nabla u_P^-\|^{\frac{1}{2}}.$$

Recalling that, at the beginning, $f = Vu$ and using (3) one has

$$\||x|f\| = \||x|Vu\| \leq \||x|Vu_S\| + \||x|Vu_P\| \leq \Lambda\|\nabla u_S^-\| + \Lambda\|\nabla u_P^-\|.$$

By virtue of the previous inequality and using the convexity of the function $g(x) = |x|^p$ for $p \geq 1$ (in the inequality for the S component), we have

$$\|\nabla u_S^-\|^2 + \frac{1}{\sqrt{\mu}}\frac{d-3}{d-1}\frac{|k_2|}{k_1^{\frac{1}{2}}}\int_{\mathbb{R}^d}|x||\nabla u_S^-|^2 \leq \frac{\Lambda}{\mu}\frac{2(2d-3)}{d-2}(C+1)\|\nabla u_S^-\|^2 + \frac{\Lambda^{\frac{3}{2}}}{\mu^{\frac{3}{2}}}\frac{2}{\sqrt{d-2}}(C+1)^{\frac{3}{2}}\|\nabla u_S^-\|^2 + \frac{\Lambda}{\mu}\frac{2(2d-3)}{d-2}(C+1)\|\nabla u_S^-\|\|\nabla u_P^-\| + \frac{\Lambda^{\frac{3}{2}}}{\mu^{\frac{3}{2}}}\frac{2}{\sqrt{d-2}}(C+1)^{\frac{3}{2}}\|\nabla u_S^-\|^{\frac{1}{2}}\|\nabla u_P^-\|^{\frac{3}{2}};$$

and

$$\|\nabla u_P^-\|^2 + \frac{1}{\sqrt{\lambda+2\mu}}\frac{d-3}{d-1}\frac{|k_2|}{k_1^{\frac{1}{2}}}\int_{\mathbb{R}^d}|x||\nabla u_P^-|^2 \leq \frac{\Lambda}{\lambda+2\mu}\frac{2(2d-3)}{d-2}C\|\nabla u_P^-\|^2 + \frac{\Lambda^{\frac{3}{2}}}{(\lambda+2\mu)^{\frac{3}{2}}}\frac{2}{\sqrt{d-2}}C^{\frac{3}{2}}\|\nabla u_P^-\|^2 + \frac{\Lambda}{\lambda+2\mu}\frac{2(2d-3)}{d-2}C\|\nabla u_S^-\|\|\nabla u_P^-\| + \frac{\Lambda^{\frac{3}{2}}}{(\lambda+2\mu)^{\frac{3}{2}}}\frac{2}{\sqrt{d-2}}C^{\frac{3}{2}}\|\nabla u_S^-\|^{\frac{3}{2}}\|\nabla u_P^-\|^{\frac{1}{2}}.$$

Summing these two inequality together and majoring C with $C+1$, we obtain

$$\begin{aligned}
&\|\nabla u_S^-\|^2 + \|\nabla u_P^-\|^2 + \frac{1}{\sqrt{\mu}}\frac{d-3}{d-1}\frac{|k_2|}{k_1^{\frac{1}{2}}}\int_{\mathbb{R}^d}|x||\nabla u_S^-|^2 + \frac{1}{\sqrt{\lambda+2\mu}}\frac{d-3}{d-1}\frac{|k_2|}{k_1^{\frac{1}{2}}}\int_{\mathbb{R}^d}|x||\nabla u_P^-|^2\\
&\leq \frac{\Lambda}{\mu}\frac{2(2d-3)}{d-2}(C+1)\|\nabla u_S^-\|^2 + \frac{\Lambda}{\lambda+2\mu}\frac{2(2d-3)}{d-2}(C+1)\|\nabla u_P^-\|^2\\
&+\frac{\Lambda^{\frac{3}{2}}}{\mu^{\frac{3}{2}}}\frac{2}{\sqrt{d-2}}(C+1)^{\frac{3}{2}}\|\nabla u_S^-\|^2 + \frac{\Lambda^{\frac{3}{2}}}{(\lambda+2\mu)^{\frac{3}{2}}}\frac{2}{\sqrt{d-2}}(C+1)^{\frac{3}{2}}\|\nabla u_P^-\|^2\\
&+\frac{\Lambda}{\mu}\frac{2(2d-3)}{d-2}(C+1)\|\nabla u_S^-\|\|\nabla u_P^-\| + \frac{\Lambda}{\lambda+2\mu}\frac{2(2d-3)}{d-2}(C+1)\|\nabla u_S^-\|\|\nabla u_P^-\|\\
&+\frac{\Lambda^{\frac{3}{2}}}{\mu^{\frac{3}{2}}}\frac{2}{\sqrt{d-2}}(C+1)^{\frac{3}{2}}\|\nabla u_S^-\|^{\frac{1}{2}}\|\nabla u_P^-\|^{\frac{3}{2}} + \frac{\Lambda^{\frac{3}{2}}}{(\lambda+2\mu)^{\frac{3}{2}}}\frac{2}{\sqrt{d-2}}(C+1)^{\frac{3}{2}}\|\nabla u_S^-\|^{\frac{3}{2}}\|\nabla u_P^-\|^{\frac{1}{2}}.
\end{aligned}$$

Making use of the Young's inequality, which state that for all non-negative real numbers a and b holds

$$ab \leq \frac{a^p}{p} + \frac{b^q}{q},$$

where p, q are determined by $\frac{1}{p}+\frac{1}{q}=1$, one gets

$$\|\nabla u_S^-\|\|\nabla u_P^-\| \leq \frac{1}{2}\|\nabla u_S^-\|^2+\frac{1}{2}\|\nabla u_P^-\|^2,$$

$$\|\nabla u_S^-\|^{\frac{1}{2}}\|\nabla u_P^-\|^{\frac{3}{2}} \leq \frac{1}{4}\|\nabla u_S^-\|^2+\frac{3}{4}\|\nabla u_P^-\|^2 \qquad \text{and} \qquad \|\nabla u_S^-\|^{\frac{3}{2}}\|\nabla u_P^-\|^{\frac{1}{2}} \leq \frac{3}{4}\|\nabla u_S^-\|^2+\frac{1}{4}\|\nabla u_P^-\|^2.$$

Using the latter in the former and the fact that $\mu, \lambda+2\mu \geq \min\{\mu, \lambda+2\mu\}$, we have

$$\left(1-\frac{\Lambda}{\min\{\mu,\lambda+2\mu\}}\frac{4(2d-3)}{d-2}(C+1)-\frac{\Lambda^{\frac{3}{2}}}{\min\{\mu,\lambda+2\mu\}^{\frac{3}{2}}}\frac{4}{\sqrt{d-2}}(C+1)^{\frac{3}{2}}\right)(\|\nabla u_S^-\|^2+\|\nabla u_P^-\|^2)$$
$$+\frac{1}{\sqrt{\mu}}\frac{d-3}{d-1}\frac{|k_2|}{k_1^{\frac{1}{2}}}\int_{\mathbb{R}^d}|x||\nabla u_S^-|^2+\frac{1}{\sqrt{\lambda+2\mu}}\frac{d-3}{d-1}\frac{|k_2|}{k_1^{\frac{1}{2}}}\int_{\mathbb{R}^d}|x||\nabla u_P^-|^2 \leq 0.$$

Since the two term in the second row are positive, the last inequality becomes

$$\left(1-\frac{\Lambda}{\min\{\mu,\lambda+2\mu\}}\frac{4(2d-3)}{d-2}(C+1)-\frac{\Lambda^{\frac{3}{2}}}{\min\{\mu,\lambda+2\mu\}^{\frac{3}{2}}}\frac{4}{\sqrt{d-2}}(C+1)^{\frac{3}{2}}\right)(\|\nabla u_S^-\|^2+\|\nabla u_P^-\|^2) \leq 0.$$

Clearly, by virtue of (4), the term in parenthesis is strictly positive then it follows that u_S^-, u_P^- and thus u_S, u_P are identically equal to zero and, as a consequence of the Helmholtz decomposition, u is identically equal to zero as well.

We treat now the simpler case $|k_2| > k_1$.

Case $|k_2| > k_1$: let $u \in [H^1(\mathbb{R}^d)]^d$ be a solution of (21), i.e. a solution of (25).Choosing $v := \pm u_S$ in the first of (25) and $v := \pm u_P$ in the second of (25), taking real and imaginary parts of the resulting identities and summing these two identities, we obtain respectively for u_S and u_P

$$(k_1 \pm k_2)\int_{\mathbb{R}^d}|u_S|^2 = \mu\int_{\mathbb{R}^d}|\nabla u_S|^2+\Re\int_{\mathbb{R}^d}\overline{u}_S\cdot f_S \pm \Im\int_{\mathbb{R}^d}\overline{u}_S\cdot f_S,$$

and

$$(k_1 \pm k_2)\int_{\mathbb{R}^d}|u_P|^2 = (\lambda+2\mu)\int_{\mathbb{R}^d}|\nabla u_P|^2+\Re\int_{\mathbb{R}^d}\overline{u}_P\cdot f_P \pm \Im\int_{\mathbb{R}^d}\overline{u}_P\cdot f_P.$$

Taking the sum of the previous and making use of the H^1- orthogonality of u_S and u_P, one has

$$(k_1 \pm k_2)\int_{\mathbb{R}^d}|u|^2 = \mu\int_{\mathbb{R}^d}|\nabla u|^2+(\lambda+\mu)\int_{\mathbb{R}^d}|\nabla u_P|^2+\Re\int_{\mathbb{R}^d}\overline{u}\cdot f \pm \Im\int_{\mathbb{R}^d}\overline{u}\cdot f. \tag{36}$$

Now we want to estimate the last two terms on the right hand side of (36), in order to obtain the bound we are going to make use only of the Schwarz's inequality, the classical Hardy's inequality (6) and the assumption (3). Indeed, recalling that $f := Vu$, one has

$$\int_{\mathbb{R}^d}|u||f| \leq \frac{2}{d-2}\Lambda\|\nabla u\|^2,$$

using the following trivial chains of inequalities

$$\Re\int_{\mathbb{R}^d}\overline{u}\cdot f \geq -\left|\int_{\mathbb{R}^d}\overline{u}\cdot f\right| \geq -\int_{\mathbb{R}^d}|u||f|, \qquad \text{and} \qquad \pm\Im\int_{\mathbb{R}^d}\overline{u}\cdot f \geq -\left|\int_{\mathbb{R}^d}\overline{u}\cdot f\right| \geq -\int_{\mathbb{R}^d}|u||f|,$$

we easily obtain

$$(k_1 \pm k_2)\int_{\mathbb{R}^d}|u|^2 \geq \left(\mu-\frac{4}{d-2}\Lambda\right)\|\nabla u\|^2+(\lambda+\mu)\|\nabla u_P\|^2.$$

Let us recall that, to make the quadratic form associated to the Lamé operator positive , we have assumed for the Lamé coefficients the condition (1); under this hypothesis immediately follows that $\lambda+\mu > 0$ thus we obtain

$$(k_1 \pm k_2)\int_{\mathbb{R}^d}|u|^2 \geq \left(\mu-\frac{4}{d-2}\Lambda\right)\|\nabla u\|^2.$$

It's easy to see that any Λ verifying (4), necessarily satisfies $\frac{4}{d-2}\Lambda < \mu$, therefore one gets

$$(k_1 \pm k_2)\int_{\mathbb{R}^d}|u|^2 \geq 0.$$

Thus from the last inequality follows that $k_1 \pm k_2 \geq 0$, unless u is identically equal to zero.

It is a straightforward exercise to prove that, under conditions (3) and (4), the possible eigenvalues of $-\Delta^*+V$ have to be included in the right complex plane, that is $k_1 > 0$. Noticing that we are assuming $|k_2| > k_1 > 0$, which implies that the inequality $k_1 \pm k_2 \geq 0$ cannot hold, we obtain $u = 0$.

4. Uniform resolvent estimate: proof of Theorem 1.2

The aim of this section is to investigate about uniform resolvent estimate for the solution $u\colon \mathbb{R}^d \to \mathbb{C}^d$ of (7).

Just to quote a pair of papers on this topic, in a context of Helmholtz equation, we recall Burq, Planchon, Stalker and Tahvildar-Zadeh [7, 8] and the work of Barceló, Vega and Zubeldia [5] which generalizes the previous to electromagnetic Hamiltonians. Whereas, for this kind of estimate in an elasticity setting, we can cite [4].

As we have already mentioned in the introduction, we are going to prove a stronger result than Theorem 1.2, which establishes the validity of a priori estimates; our theorem will follows as a corollary making use of the Hardy's inequality only.

In view of the previous comment, we can now start with the proof of Theorem 1.3

Proof of Theorem 1.3. Since the estimates are different according to the relation between the real and imaginary part of the frequency, that is when $|k_2| \leq k_1$ or the contrary, we treat the two cases separately.

As a starting point, we will easily show that this kind of estimates holds in the free framework, that is in the setting in which $V = 0$. Secondly we prove the estimates in the perturbed case, assuming about V the same integral-smallness condition of Theorem 1.1.

Case $|k_2| \leq k_1$: We consider the case $V = 0$. In this framework our equation (7) reduces to the one we considered in Theorem 1.1, precisely (21). Throughout the proof of Theorem 1.1, taking into account the Helmholtz decomposition, we proved for this equation the two estimates (30) and (32) respectively for the S and P component of the solution u of (21) that we are going to rewrite in order to clarify our argument. One had

$$\|\nabla u_S^-\|^2 + \frac{1}{\sqrt{\mu}}\frac{d-3}{d-1}\frac{|k_2|}{k_1^{\frac{1}{2}}}\int_{\mathbb{R}^d}|x||\nabla u_S^-|^2 \leq \frac{1}{\mu}\frac{2(2d-3)}{d-2}\||x|f_S\|\|\nabla u_S^-\| + \frac{1}{\mu^{\frac{3}{2}}}\frac{\sqrt{2}}{\sqrt{d-2}}\||x|f_S\|^{\frac{3}{2}}\|\nabla u_S^-\|^{\frac{1}{2}}, \tag{37}$$

and

$$\begin{aligned}\|\nabla u_P^-\|^2 + \frac{1}{\sqrt{\lambda+2\mu}}\frac{d-3}{d-1}\frac{|k_2|}{k_1^{\frac{1}{2}}}\int_{\mathbb{R}^d}|x||\nabla u_P^-|^2 \leq{}& \frac{1}{(\lambda+2\mu)}\frac{2(2d-3)}{d-2}\||x|f_P\|\|\nabla u_P^-\| \\ &+ \frac{1}{(\lambda+2\mu)^{\frac{3}{2}}}\frac{\sqrt{2}}{\sqrt{d-2}}\||x|f_P\|^{\frac{3}{2}}\|\nabla u_P^-\|^{\frac{1}{2}}.\end{aligned}$$

Let us consider the first inequality only, the details for the second one will be similar.

We want to estimate the right hand side of the inequality, to this end, let ε, $\delta > 0$, making use of the Young's inequality one has

$$\||x|f_S\|\|\nabla u_S^-\| \leq \frac{1}{2\varepsilon^2}\||x|f_S\|^2 + \frac{\varepsilon^2}{2}\|\nabla u_S^-\|^2 \qquad \text{and} \qquad \||x|f_S\|^{\frac{3}{2}}\|\nabla u_S^-\|^{\frac{1}{2}} \leq \frac{3}{4\delta^{\frac{4}{3}}}\||x|f_S\|^2 + \frac{\delta^4}{4}\|\nabla u_S^-\|^2.$$

Putting this two in (37) and observing that the quantity $\frac{1}{\sqrt{\mu}}\frac{|k_2|}{k_1^{\frac{1}{2}}}\frac{d-3}{d-1}\int_{\mathbb{R}^d}|x||\nabla u_S^-|^2$ is positive, we get

$$\begin{aligned}\|\nabla u_S^-\|^2 \leq{}& \frac{1}{\mu}\frac{1}{\varepsilon^2}\frac{2d-3}{d-2}\||x|f_S\|^2 + \varepsilon^2\frac{1}{\mu}\frac{2d-3}{d-2}\|\nabla u_S^-\|^2 + \frac{3}{4\delta^{\frac{4}{3}}}\frac{1}{\mu^{\frac{3}{2}}}\frac{\sqrt{2}}{\sqrt{d-2}}\||x|f_S\|^2 \\ &+ \frac{\delta^4}{4}\frac{1}{\mu^{\frac{3}{2}}}\frac{\sqrt{2}}{\sqrt{d-2}}\|\nabla u_S^-\|^2.\end{aligned}$$

Thus it may be concluded that

$$\left(1 - \varepsilon^2\frac{1}{\mu}\frac{2d-3}{d-2} - \frac{\delta^4}{4}\frac{1}{\mu^{\frac{3}{2}}}\frac{\sqrt{2}}{\sqrt{d-2}}\right)\|\nabla u_S^-\|^2 \leq \left(\frac{1}{\mu}\frac{1}{\varepsilon^2}\frac{2d-3}{d-2} + \frac{3}{4\delta^{\frac{4}{3}}}\frac{1}{\mu^{\frac{3}{2}}}\frac{\sqrt{2}}{\sqrt{d-2}}\right)\||x|f_S\|^2.$$

The same calculations done for the P component give

$$\left(1 - \varepsilon^2\frac{1}{\lambda+2\mu}\frac{2d-3}{d-2} - \frac{\delta^4}{4}\frac{1}{(\lambda+2\mu)^{\frac{3}{2}}}\frac{\sqrt{2}}{\sqrt{d-2}}\right)\|\nabla u_P^-\|^2 \leq \left(\frac{1}{\lambda+2\mu}\frac{1}{\varepsilon^2}\frac{2d-3}{d-2} + \frac{3}{4\delta^{\frac{4}{3}}}\frac{1}{(\lambda+2\mu)^{\frac{3}{2}}}\frac{\sqrt{2}}{\sqrt{d-2}}\right)\||x|f_P\|^2.$$

Now, since $\mu, \lambda+2\mu \geq \min\{\mu, \lambda+2\mu\}$ and choosing ε, δ small enough, one can write

$$\|\nabla u_S^-\| \leq D_{\varepsilon,\delta} \||x| f_S\|,$$

and

$$\|\nabla u_P^-\| \leq D_{\varepsilon,\delta} \||x| f_P\|,$$

where

$$D_{\varepsilon,\delta} = \left(\frac{\frac{1}{\min\{\mu,\lambda+2\mu\}} \frac{1}{\varepsilon^2} \frac{2d-3}{d-2} + \frac{3}{4\delta^{\frac{4}{3}}} \frac{1}{\min\{\mu,\lambda+2\mu\}^{\frac{3}{2}}} \frac{\sqrt{2}}{\sqrt{d-2}}}{1 - \varepsilon^2 \frac{1}{\min\{\mu,\lambda+2\mu\}} \frac{2d-3}{d-2} - \frac{\delta^4}{4} \frac{1}{\min\{\mu,\lambda+2\mu\}^{\frac{3}{2}}} \frac{\sqrt{2}}{\sqrt{d-2}}} \right)^{\frac{1}{2}}.$$

At the end, using the trivial Helmholtz decomposition of $f = f - \nabla\psi + \nabla\psi$ and the elliptic regularity Lemma 3.3, one easily concludes

$$\|\nabla u_S^-\| \leq c\||x| f\| \qquad \text{and} \qquad \|\nabla u_P^-\| \leq c\||x| f\|,$$

where $c := (C+1) D_{\varepsilon,\delta}$. Let us remark that $c > 0$ does not depend on the frequency k and on f.

Now we can prove our result in the perturbed setting, i.e. $V \neq 0$.

First of all we define $g := Vu$ and $h := f + g$. Thus, with this notation, u solves the following equation

$$\Delta^* u + ku = h; \tag{38}$$

again we have these estimates for the two components of the solution:

$$\|\nabla u_S^-\|^2 \leq \frac{1}{\mu} \frac{2(2d-3)}{d-2} \||x| h_S\| \|\nabla u_S^-\| + \frac{1}{\mu^{\frac{3}{2}}} \frac{\sqrt{2}}{\sqrt{d-2}} \||x| h_S\|^{\frac{3}{2}} \|\nabla u_S^-\|^{\frac{1}{2}},$$

and

$$\|\nabla u_P^-\|^2 \leq \frac{1}{\lambda+2\mu} \frac{2(2d-3)}{d-2} \||x| h_P\| \|\nabla u_P^-\| + \frac{1}{(\lambda+2\mu)^{\frac{3}{2}}} \frac{\sqrt{2}}{\sqrt{d-2}} \||x| h_P\|^{\frac{3}{2}} \|\nabla u_P^-\|^{\frac{1}{2}}.$$

We only consider the first one. Clearly, since $h_S = f_S + g_S$, it can be rewritten as

$$\begin{aligned} \|\nabla u_S^-\|^2 \leq{}& \frac{1}{\mu} \frac{2(2d-3)}{d-2} \||x| f_S\| \|\nabla u_S^-\| + \frac{1}{\mu^{\frac{3}{2}}} \frac{2}{\sqrt{d-2}} \||x| f_S\|^{\frac{3}{2}} \|\nabla u_S^-\|^{\frac{1}{2}} \\ &+ \frac{1}{\mu} \frac{2(2d-3)}{d-2} \||x| g_S\| \|\nabla u_S^-\| + \frac{1}{\mu^{\frac{3}{2}}} \frac{2}{\sqrt{d-2}} \||x| g_S\|^{\frac{3}{2}} \|\nabla u_S^-\|^{\frac{1}{2}}. \end{aligned}$$

Since in the free case we have already bounded the terms in which f appears, now let us only consider the terms involving g.

We introduce the trivial decomposition of $g = g - \nabla\phi + \nabla\phi$, where, as usual, ϕ is the unique solution of the elliptic problem $\Delta\phi = \operatorname{div} g$. Following the strategy in Theorem 1.1 about the absence of eigenvalues and, in particular, recalling that formerly $g = Vu$ and that V satisfies (3), one can show

$$\begin{aligned} \|\nabla u_S^-\|^2 \leq{}& \frac{1}{\mu} \frac{2(2d-3)}{d-2} \||x| f_S\| \|\nabla u_S^-\| + \frac{1}{\mu^{\frac{3}{2}}} \frac{2}{\sqrt{d-2}} \||x| f_S\|^{\frac{3}{2}} \|\nabla u_S^-\|^{\frac{1}{2}} \\ &+ \frac{\Lambda}{\mu} \frac{2(2d-3)}{d-2} (C+1) \|\nabla u_S^-\|^2 + \frac{\Lambda^{\frac{3}{2}}}{\mu^{\frac{3}{2}}} \frac{2\sqrt{2}}{\sqrt{d-2}} (C+1)^{\frac{3}{2}} \|\nabla u_S^-\|^2 \\ &+ \frac{\Lambda}{\mu} \frac{2(2d-3)}{d-2} (C+1) \|\nabla u_S^-\| \|\nabla u_P^-\| + \frac{\Lambda^{\frac{3}{2}}}{\mu^{\frac{3}{2}}} \frac{2\sqrt{2}}{\sqrt{d-2}} (C+1)^{\frac{3}{2}} \|\nabla u_S^-\|^{\frac{1}{2}} \|\nabla u_P^-\|^{\frac{3}{2}}. \end{aligned}$$

For the P component we have the following analogue estimate

$$\begin{aligned} \|\nabla u_P^-\|^2 \leq{}& \frac{1}{\lambda+2\mu} \frac{2(2d-3)}{d-2} \||x| f_P\| \|\nabla u_P^-\| + \frac{1}{(\lambda+2\mu)^{\frac{3}{2}}} \frac{2}{\sqrt{d-2}} \||x| f_P\|^{\frac{3}{2}} \|\nabla u_P^-\|^{\frac{1}{2}} \\ &+ \frac{\Lambda}{\lambda+2\mu} \frac{2(2d-3)}{d-2} C \|\nabla u_P^-\|^2 + \frac{\Lambda^{\frac{3}{2}}}{(\lambda+2\mu)^{\frac{3}{2}}} \frac{2\sqrt{2}}{\sqrt{d-2}} C^{\frac{3}{2}} \|\nabla u_P^-\|^2 \\ &+ \frac{\Lambda}{\lambda+2\mu} \frac{2(2d-3)}{d-2} C \|\nabla u_S^-\| \|\nabla u_P^-\| + \frac{\Lambda^{\frac{3}{2}}}{(\lambda+2\mu)^{\frac{3}{2}}} \frac{2\sqrt{2}}{\sqrt{d-2}} C^{\frac{3}{2}} \|\nabla u_S^-\|^{\frac{3}{2}} \|\nabla u_P^-\|^{\frac{1}{2}}. \end{aligned}$$

Now estimating the terms involving f as in the free case, summing these inequalities, and using the Young's inequality, we obtain

$$\Bigg(1-\frac{\Lambda}{\min\{\mu,\,\lambda+2\mu\}}\frac{4(2d-3)}{d-2}(C+1)-\frac{\Lambda^{\frac{3}{2}}}{\min\{\mu,\,\lambda+2\mu\}^{\frac{3}{2}}}\frac{4\sqrt{2}}{\sqrt{d-2}}(C+1)^{\frac{3}{2}}$$

$$-\varepsilon^2\frac{1}{\min\{\mu,\,\lambda+2\mu\}}\frac{2d-3}{d-2}-\frac{\delta^4}{2}\frac{1}{\min\{\mu,\,\lambda+2\mu\}^{\frac{3}{2}}}\frac{1}{\sqrt{d-2}}\Bigg)(\|\nabla u_S^-\|^2+\|\nabla u_P^-\|^2)$$

$$\leq\left(\frac{1}{\min\{\mu,\lambda+2\mu\}}\frac{1}{\varepsilon^2}\frac{2d-3}{d-2}+\frac{3}{2\delta^{\frac{4}{3}}}\frac{1}{\min\{\mu,\lambda+2\mu\}^{\frac{3}{2}}}\frac{1}{\sqrt{d-2}}\right)(\||x|f_S\|^2+\||x|f_P\|^2)$$

Since V satisfies (3) and assuming ε,δ sufficiently small, the constant in the left hand side of the previous inequality is positive, thus we can write

$$(\|\nabla u_S^-\|^2+\|\nabla u_P^-\|^2)\leq D_{\varepsilon,\delta}^2(\||x|f_S\|^2+\||x|f_P\|^2),$$

where, obviously, $D_{\varepsilon,\delta}^2$ is the ratio between the two constants which respectively appear on the right and on the left hand side of the last but one inequality.

Using now the trivial Helmholtz decomposition of $f=f-\nabla\psi+\nabla\psi$ and the elliptic regularity Lemma 3.3, one easily has

$$\|\nabla u_S^-\|^2+\|\nabla u_P^-\|^2\leq c^2\||x|f\|^2,$$

where

$$c^2:=2(C+1)^2D_{\varepsilon,\delta}^2$$

does not depend on the frequency k and on f. Moreover, it is clear that the following hold

$$\|\nabla u_S^-\|\leq c\||x|f\| \qquad \text{and} \qquad \|\nabla u_P^-\|\leq c\||x|f\|.$$

Now we can treat the less technical case.

Case $|k_2|>k_1$.: First we consider the free setting.

As the previous case, our equation (7) becomes the one we have considered in Theorem 1.1, precisely (21). Choosing $v=u_S$ in the first of (25) and $v=u_P$ in the second and taking the real part of the resulting identity one obtains respectively

$$k_1\int_{\mathbb{R}^d}|u_S|^2-\mu\int_{\mathbb{R}^d}|\nabla u_S|^2=\Re\int_{\mathbb{R}^d}\overline{u}_S\cdot f_S$$

and

$$k_1\int_{\mathbb{R}^d}|u_P|^2-(\lambda+2\mu)\int_{\mathbb{R}^d}|\nabla u_P|^2=\Re\int_{\mathbb{R}^d}\overline{u}_P\cdot f_P.$$

Taking the sum of the previous and making use of the L^2 and H^1- orthogonality of u_S and u_P, one has

$$k_1\int_{\mathbb{R}^d}|u|^2-\mu\int_{\mathbb{R}^d}|\nabla u|^2-(\lambda+\mu)\int_{\mathbb{R}^d}|\nabla u_P|^2=\Re\int_{\mathbb{R}^d}\overline{u}\cdot f.$$

Starting again from the weak formulation (25), choosing $v=\frac{k_2}{|k_2|}u_S$ in the first and $v=\frac{k_2}{|k_2|}u_P$ in the second, taking the imaginary part and then summing the resulting identities, one obtains

$$|k_2|\int_{\mathbb{R}^d}|u|^2\leq\int_{\mathbb{R}^d}|u||f|.$$

Using the latter in the former (here we need the assumption $|k_2|>k_1$) and observing the positivity of the term $(\lambda+\mu)\int_{\mathbb{R}^d}|\nabla u_P|^2$, we have

$$\mu\int_{\mathbb{R}^d}|\nabla u|^2\leq 2\int_{\mathbb{R}^d}|u||f|.$$

From the Cauchy Schwarz and the Hardy's inequalities follows

$$\mu\|\nabla u\|^2\leq\frac{4}{d-2}\||x|f\|\|\nabla u\|.$$

Thus, it may be concluded that

$$\|\nabla u\|<\frac{1}{\mu}\frac{4}{d-2}\||x|f\|.$$

We now proceed to show the a priori estimates in the perturbed context. Exploiting the same notation we have used in the case $|k_2| \leq k_1$, again u solves the equation (38). As a consequence of the estimates we have just proved for the free case, recalling that $h = f + g$, one easily obtains

$$\|\nabla u\| < \frac{1}{\mu}\frac{4}{d-2}\||x|h\| \leq \frac{1}{\mu}\frac{4}{d-2}\||x|f\| + \frac{1}{\mu}\frac{4}{d-2}\||x|g\|.$$

Writing now explicitly g as Vu, by assumption (3) we have

$$\|\nabla u\| < \frac{1}{\mu}\frac{4}{d-2}\||x|f\| + \frac{\Lambda}{\mu}\frac{4}{d-2}\|\nabla u\|$$

or, more explicitly

$$\left(1 - \frac{\Lambda}{\mu}\frac{4}{d-2}\right)\|\nabla u\| < \frac{1}{\mu}\frac{4}{d-2}\||x|f\|.$$

The condition (4) about Λ guarantees the positivity of the parenthesis of the left hand side and then the theorem is proved.

□

Finally, we are in position to prove the uniform resolvent estimate we are looking for.

Proof of Theorem (1.2). First of all, we consider the case $|k_2| \leq k_1$, as a consequence of (9), making use of the Hardy's inequality, it is not difficult to show that the following chain of inequalities holds

$$\begin{aligned}\||x|^{-1}u\| &\leq \||x|^{-1}u_S^-\| + \||x|^{-1}u_P^-\| \leq \frac{2}{d-2}(\|\nabla u_S^-\| + \|\nabla u_P^-\|)\\ &\leq \frac{4c}{d-2}\||x|f\|.\end{aligned}$$

Assuming $|k_2| > k_1$, using (10) and again the Hardy's inequality, we have

$$\||x|^{-1}u\| \leq \frac{2c}{d-2}\||x|f\|.$$ □

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